MIDTERM
CS 111
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1 Setup and Initial Conditions

The objective of this midterm project was to simulate the motion of two billiard balls of radius r on an $L \times 2L$ table for a total of $t = 5$ seconds. In our implementation, we have taken $L = 1$ and $r = 0.075$. We have also taken $\Delta t = 0.03$ and have imposed a dampening coefficient $\alpha = 0.8$ for wall collisions. Moreover, the initial conditions for the balls are as follows:

$$\vec{x}_{r,i} = [0.1, 0.1], \quad \vec{v}_{r,i} = [3, 4]$$

$$\vec{x}_{b,i} = [0.5, 1], \quad \vec{v}_{b,i} = [0, 0],$$

where $\vec{x}_{r,i}, \vec{x}_{b,i} \in \mathbb{R}^2$ denote the initial positions of the center of the red and blue balls, respectively (similarly for $\vec{v}_{r,i}, \vec{v}_{b,i} \in \mathbb{R}^2$). Note that $\|\vec{v}_{r,i}\| = 5$ and $\|\vec{v}_{b,i}\| = 0$, as required. Finally, to draw each billiard ball we use the $draw_disc_color.m$ function, in which a parametrization given by polar coordinates is used.

2 Ball Mechanics and Equations of Motion

To simulate the motion of the two billiard balls, we use a general Euler scheme to update the position function for the center of each billiard ball. With the exception of two cases, we update the position at each incremental time step t_i according to the rule

$$\vec{x}_{t_{i+1}} = \vec{x}_{t_i} + \vec{v}_{t_i} \cdot \Delta t.$$

2.1 Case 1: Wall Collision

In this case, we are concerned with how to update the position $\vec{x}_{t_{i+1}}$ of the billiard balls in case either one should leave the box $[r, 1 - r] \times [r, 2 - r]$.

If so, we must find the precise Δt_{wall} that puts the center of the ball at the boundary of the box (given by *getdt.m*)

$$\Delta t_{wall} = \frac{w - r - \vec{x}_{t_i} \cdot e_i}{\vec{v}_{t_i} \cdot e_i},$$

where w denotes the location of the wall that the ball is about to hit and e_i[1] is the axis along which the ball will hit said wall. Using the new Δt_{wall}, we re-update the position and velocities as follows (letting $i' = i \bmod 2 + 1$):

$$\vec{x}_{t_{i+1}} = \vec{x}_{t_i} + \vec{v}_{t_i} \cdot \Delta t_{wall}$$

$$\vec{v}_{t_{i+1}} = \alpha \left(\langle -e_i, \vec{v}_{t_i} \rangle e_i + \langle e_{i'}, \vec{v}_{t_i} \rangle e_{i'} \right).$$

2.2 Case 2: Ball Collision

In this second case, we are concerned with how to update the positions $\vec{x}_{r,t_{i+1}}, \vec{x}_{b,t_{i+1}}$ of the red and blue balls, respectively, in case they are about to collide with each other (i.e. when $||\vec{x}_{r,t_{i+1}} - \vec{x}_{b,t_{i+1}}|| \leq 2r$). If so, we must calculate the precise Δt_{ball} that puts the center of each ball at distance $2r$ from the other (taken from lecture and given by *getdtBall.m*):

$$\Delta t_{ball} = \frac{||\vec{x}_{r,t_i} - \vec{x}_{b,t_i}|| - 2r}{||\vec{v}_{r,t_i} - \vec{v}_{b,t_i}||}.$$

Using this new Δt_{ball} we update the position and velocity functions as follows. Take a normal vector $\vec{n} = \vec{x}_{r,t_i} - \vec{x}_{b,t_i}$, and construct the tangent vector $\vec{t} = \left[1, -\frac{\vec{n} \cdot e_1}{\vec{n} \cdot e_2}\right]$ (note: by construction, $\langle \vec{n}, \vec{t} \rangle = 0$). We then follow the procedure outlined in the Billiards Simulation document on Gauchospace to update the position and velocity of each billiard ball (where the transformation from the (x, y)-coordinate system to the (n, t)-coordinate system is given by the invertible matrix $A = [\vec{n}; \vec{t}]$).

[1] e_i denotes the i-th coordinate vector of the standard basis for $\mathbb{R}^2$

3 Figures

The sequence of figures below depicts a ball-to-ball collision. From left to right, we have before, during and after the collision: