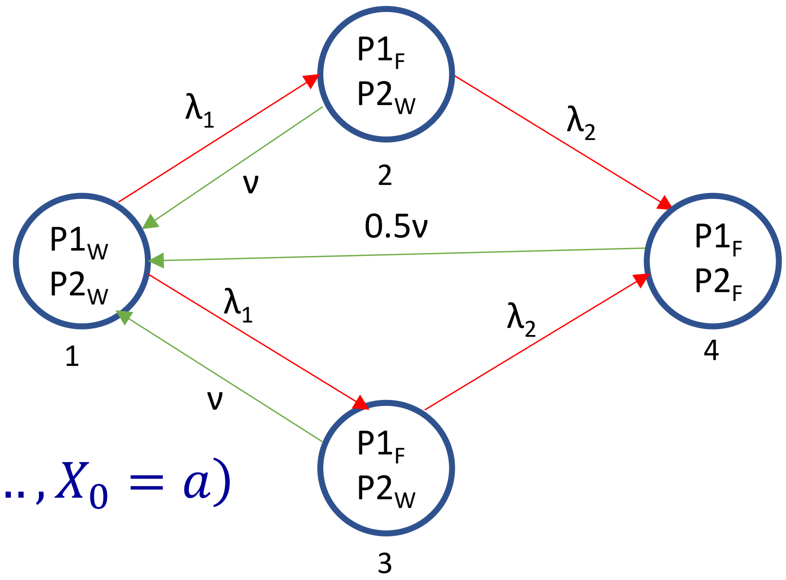
Modelling
Dependencies and Complexities in Engineering Systems

Markov Analysis



Characteristics

- State – based method
    - States represent the system states
- Memoryless property

$$P(X_{t+dt} = k \mid X_t = j, X_{t-dt} = i, X_{t-2dt} = h, \dots, X_0 = a)$$

$$= P(X_{t+dt} = k \mid X_t = j)$$

- Exponential distribution for state residence times (constant transition rates)

$$(\dot{P_1}, \dot{P_2}, \dot{P_3}, \dots, \dot{P_n}) = (P_1, P_2, P_3, \dots, P_n) \begin{bmatrix} -\lambda_{1,1} & \cdots & \lambda_{1,n} \\ \vdots & \ddots & \vdots \\ \lambda_{n,1} & \cdots & -\lambda_{n,n} \end{bmatrix}$$

Solution

- Numerical Methods

Outputs

- The probability of being in each state at time t.

Markov Modelling Procedure

Model Development - Markov State Transition Diagram

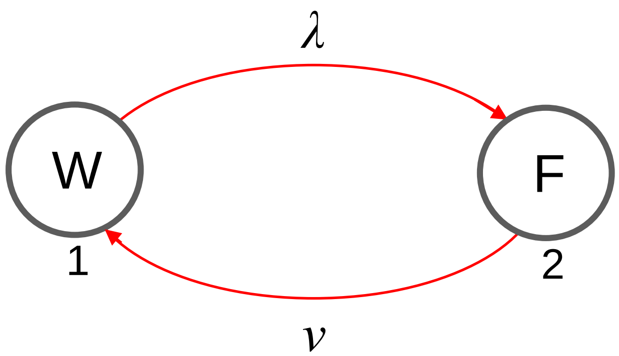
- Identify all possible states.
- List all transitions between states (failures/repairs).

Model Analysis

- Develop one equation for each state on the diagram (state equations).
- Solve equations to find probability of being in each state.

Single Component Failure Model

States:
- Working (W)
- Failed (F)



Transitions:
- Failure (W $\rightarrow$ F)
- Repair (F $\rightarrow$ W)

Outputs:

$P_F(t)$ = probability of component failed at time t unavailability

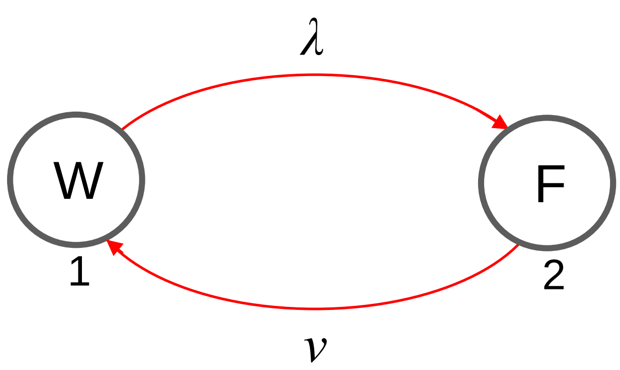
$P_W(t)$ = probability of component working at time t availability

Derive the Transition Rate Matrix

Rate of change of state i probability =

- (rate of leaving state i) x P(residing in state i)

$$+\sum_{\substack{j=1 \\ j\neq i}}^{n} (rate\ of\ arriving\ in\ state\ i\ from\ state\ j)\ \times P(residing\ in\ state\ j)$$



$$\frac{dP_W(t)}{dt} = -\lambda P_W(t) + \nu P_F(t)$$

$$\frac{dP_F(t)}{dt} = \lambda P_W(t) - \nu P_F(t)$$

Derive the Transition Rate Matrix (A)

$$\frac{dP_W(t)}{dt} = -\lambda P_W(t) + \nu P_F(t)$$

$$\frac{dP_F(t)}{dt} = \lambda P_W(t) - \nu P_F(t)$$

Denote: $\frac{dP_W(t)}{dt}$ $by\ \dot{P}_W$

$\frac{dP_F(t)}{dt}$ $by\ \dot{P}_F$

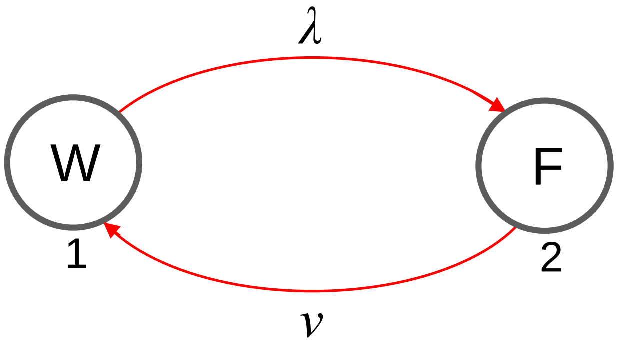
Therefore in Matrix form:

$$[\dot{P}_W(t) \quad \dot{P}_F(t)] = [P_W \quad P_F].\begin{bmatrix} -\lambda & \lambda \\ \nu & -\nu \end{bmatrix}$$

$$\dot{\boldsymbol{P}} = \boldsymbol{P}.[\boldsymbol{A}]$$

$$[\boldsymbol{A}] = \begin{bmatrix} -\lambda & \lambda \\ \nu & -\nu \end{bmatrix}$$

Transition Rate Matrix



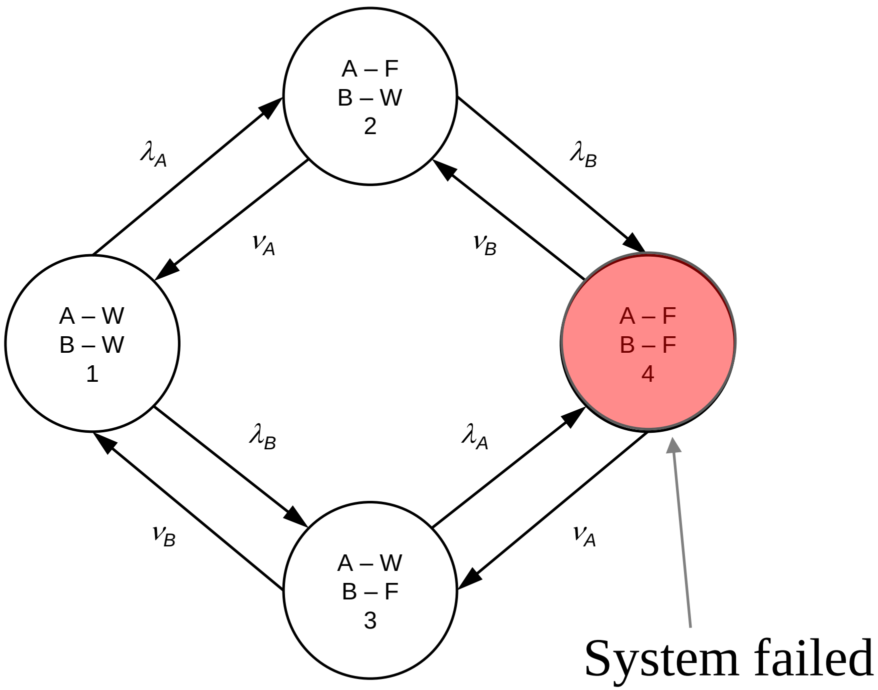
$$[\boldsymbol{A}] = \begin{bmatrix} -\lambda & \lambda \\ \nu & -\nu \end{bmatrix}$$

Rules:

- The dimension of the matrix is equal to the number of states.
- Element *i*, *j* (*i*th row, *j*th col) represents the transition rate from state *i* to state *j*.
- A diagonal element *i*,*i* is the total transition rate out of state *i* (always negative). (All rows sum to zero).

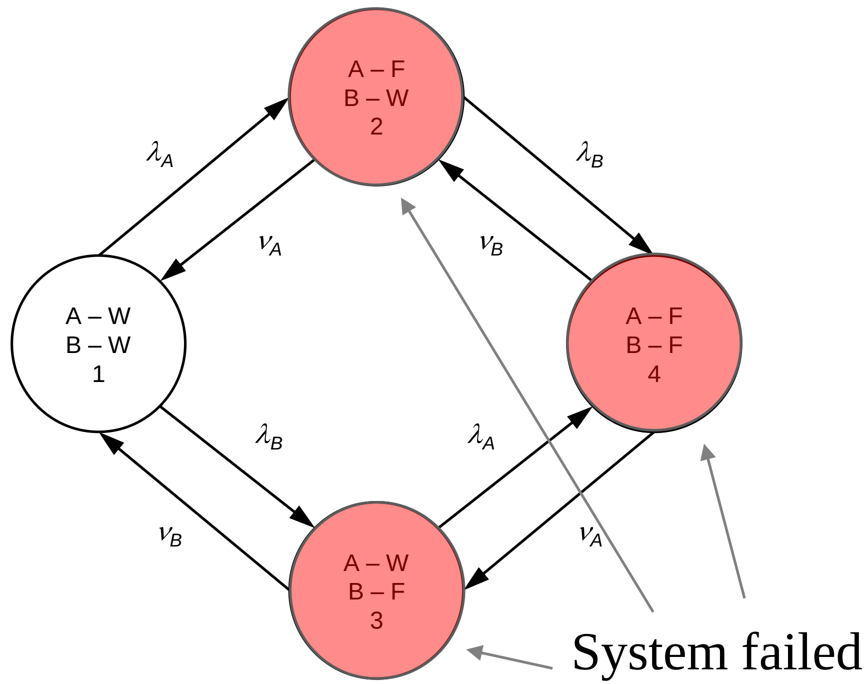
Example – 2 Component System

Two component parallel system (availability model)



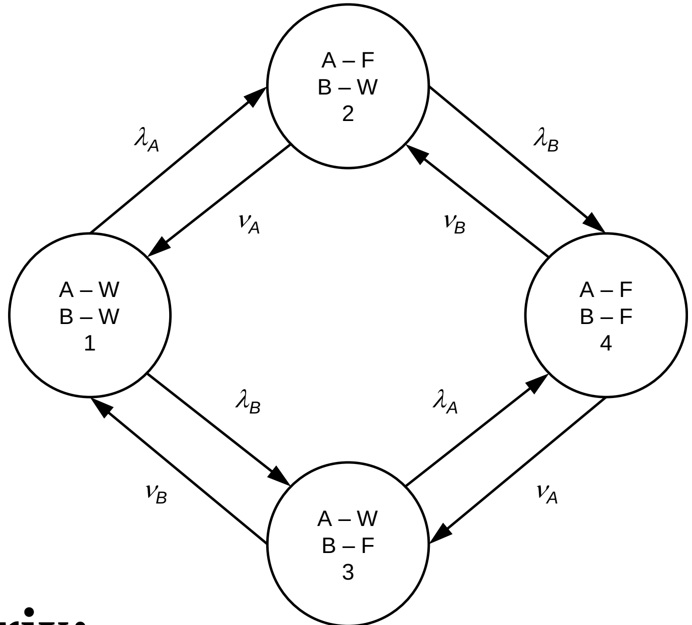
$$Q_{SYS} = P_4$$

Two component series system (availability model)



$$Q_{SYS} = P_2 + P_3 + P_4$$

Example – Availability Model

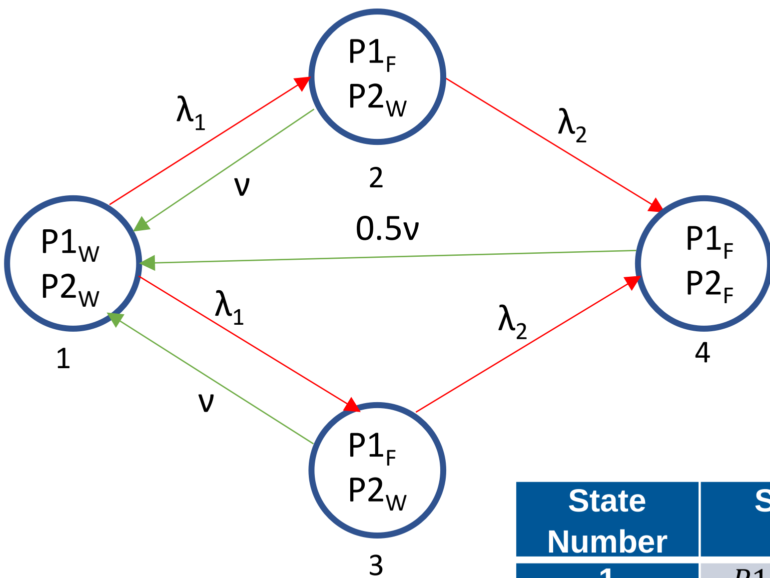


Transition rate matrix:

$$[A] = \begin{array}{cc} & \text{to:} \\ & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \text{from: } \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} -(\lambda_A + \lambda_B) & \lambda_A & \lambda_B & 0 \\ \nu_A & -(\lambda_B + \nu_A) & 0 & \lambda_B \\ \nu_B & 0 & -(\lambda_A + \nu_B) & \lambda_A \\ 0 & \nu_B & \nu_A & -(\nu_A + \nu_B) \end{bmatrix} \end{array}$$

Dependency Example

Pumps P1 and P2 operate together to provide a flow. Should one pump fail then the second can deliver the required flow on its own. However, when one fails it puts an extra load on the other and increases its failure rate from λ_1 to λ_2.



Pump Failure:

λ_1 = 2.0 x 10^{-5} per hour Normal Load

λ_2 = 5.0 x 10^{-3} per hour Full Load

Pump Repair:

$\nu = 0.041667$ (MTTF = 24hrs)

$\nu_2 = 0.5\,\nu$.

State Number	State	State Probability	Intensity Expression	State Intensity
1	$P1_W P2_W$	0.99743518	$w1 = (Q2 + Q3).\nu + Q4.0.5\nu$	7.12456×10^{-5}
2	$P1_F P2_W$	0.00042747	$w2 = Q1.\lambda_1$	1.99487×10^{-5}
3	$P1_W P2_F$	0.00042747	$w3 = Q1.\lambda_1$	1.99487×10^{-5}
4	$P1_F P2_F$	0.00170988	$w4 = (Q2 + Q3).\lambda_2$	4.2747×10^{-6}