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A Modified Gurson Model: Formulation and Implementation

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Abstract

In this report a modified Gurson modeled is presented. It can be used to model ductile behavior up to and including material failure. The formulation incorporates the Gurson failure surface, including void nucleation, growth, and coalescence, with a J_2 yield surface with user-defined hardening behavior. Aspects of the formulation and implementation will be discussed.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

The purpose of this work is to present a modified Gurson constitutive model for use in capturing the behavior of ductile materials in the failure regime. The Gurson model has been used extensively in metal plasticity, starting with [CITE Gurson], and the modified Gurson model described here was recently proposed by [CITE Nahshon-Hutchinson 2008], and extended to include void nucleation in [CITE Nahshon-Xue 2009]. In this treatment with incorporate a hyperelastic strain energy potential to define the underlying model of elasticity, and investigate a fully implicit Newton algorithm for integration of the evolution equations associated with the state variables used to define the constitutive response.

Chapter 2

Model Formulation

2.0.1 Yield function

The yield function of Gurson model is given in terms of two stress measures, i.e., the mean stress σ_m and the effective stress σ_e

$$F(\sigma_m, \sigma_e, \sigma_M, f^*) = \left(\frac{\sigma_e}{\sigma_M}\right)^2 + 2q_1 f^* \cosh\left(\frac{3q_2}{2}\frac{\sigma_m}{\sigma_M}\right) - \left(1 + q_3 f^{*2}\right) \tag{2.1}$$

where $\sigma_m = 1/3\sigma_{ii}$ is the mean stress, $\sigma_e = \sqrt{3/2 s_{ij} s_{ij}}$ is the effective stress with $s_{ij} = \sigma_{ij} - 1/3\sigma_{kk}\delta_{ij}$ being the deviatoric stress tensor. q_1, q_2 and q_3 are fitting parameters introduced by [CITE Tvergaard]. σ_M is the effective stress of the undamaged matrix material and f^* is a function which accounts for damage or softening of the material due to void coalescence:

$$f^* = \begin{cases} f, & f \leq f_c \\ f_c + \frac{\bar{f}_f - f_c}{f_f - f_c}\left(f - f_c\right), & f_c < f < f_f \\ \bar{f}_f, & f \geq f_f \end{cases} \tag{2.2}$$

where f_c is the critical value of the volume fraction, f_f is the value of the volume fraction at failure and $\bar{f}_f = (q_1 + \sqrt{q_1^2 - q_3})/q_3$.

The current state of the material is then characterized by the mean stress σ_m, the effective stress σ_e and two internal variables σ_M and f. The evolution equations for the two internal variables will be given in the following sections. In what follows, the subscription M refers to the undamaged matrix material.

2.0.2 Evolution of void volume fraction

The change of void volume fraction $\dot{f}$ due to plastic deformation consists of two parts, the void growth $\dot{f}_g$ and the void nucleation $\dot{f}_n$, i.e.,

$$\dot{f} = \dot{f}_g + \dot{f}_n \tag{2.3}$$

The modified Gurson model extends the expression of the void growth $\dot{f}_g$ by adding the dependence on the third stress invariant J_3. The evolution equation is given by

$$\dot{f}_g = (1-f)\dot{\epsilon}^p_{ii} + k_w \frac{f\omega(\boldsymbol{\sigma})}{\sigma_e} s_{ij}\dot{\epsilon}^p_{ii} \tag{2.4}$$

where the parameter k_w is introduced by [CITE Nahshon-Hutchinson] to set the magnitude of the damage growth rate in shear states. The function $\omega(\boldsymbol{\sigma})$ includes the effect of third stress invariant on void growth as

$$\omega(\boldsymbol{\sigma}) = 1 - \left(\frac{27J_3}{2\sigma_e^3}\right)^2 \tag{2.5}$$

The plastic strain controlled void nucleation evolution is written [CITE Chu and Needleman]

$$\dot{f}_n = A(\epsilon^p_M)\dot{\epsilon}^p_M \tag{2.6}$$

where ϵ^p_M is the effective plastic strain in the matrix. $A(\epsilon^p_M)$ is proposed to have the following form

$$A(\epsilon^p_M) = \begin{cases} \frac{f_N}{s_N\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\epsilon^p_M - \epsilon_N}{s_N}\right)^2\right], & \sigma_m \geq 0 \\ 0, & \sigma_m < 0 \end{cases} \tag{2.7}$$

where the nucleation strain follows a normal distribution with a mean value ϵ_N and a standard deviation s_N with the volume fraction of the nucleated voids given by f_N.

2.0.3 Evolution of effective matrix stress

Plastic work in the matrix is taken to be a relative fraction of the macroscopic plastic work such that

$$(1-f)\sigma_M\dot{\epsilon}^p_M = \sigma_{ij}\dot{\epsilon}^p_{ij} \tag{2.8}$$

This provides the evolution equation for effective plastic strain in the matrix, i.e.,

$$\dot{\epsilon}^p_M = \frac{\sigma_{ij}\dot{\epsilon}^p_{ij}}{(1-f)\sigma_M} \tag{2.9}$$

The evolution of the effective matrix stress σ_M is then given by

$$\dot{\sigma}_M = \frac{h_M\sigma_{ij}\dot{\epsilon}^p_{ij}}{(1-f)\sigma_M} \tag{2.10}$$

where h_M is the hardening modulus of the matrix defined in terms of the equivalent tensile stress-plastic strain in uniaxial tension.

$$h_M = \frac{d\sigma_M}{d\epsilon^p_M} \tag{2.11}$$

Given a specific stress-strain relation for the matrix material, the hardening modulus could be derived from the above definition. The next section provides an example for a specific stress-strain relation.

2.0.4 Stress-strain relation for the matrix material

A variety of models for the effective matrix stress-strain may be used. As an example, we consider a simple linear-power law function as follows:

$$\sigma_M = \begin{cases} E\epsilon_M, & \sigma_M < \sigma_Y \\ \sigma_Y \left(\frac{E\epsilon_M}{\sigma_Y}\right)^N, & \sigma_M \geq \sigma_Y \end{cases} \tag{2.12}$$

where σ_Y is the initial yield stress of the matrix material and ϵ_M is the total effective strain in the matrix. Given the above stress-strain law, the hardening modulus of the matrix h_M can be obtained as follows: upon yielding of the matrix material ($\sigma_M \geq \sigma_Y$), the plastic strain in the matrix is given by

$$\begin{aligned} \epsilon_M^p &= \epsilon_M - \epsilon_M^e \\ &= \left(\frac{\sigma_Y}{E}\right)\left(\frac{\sigma_M}{\sigma_Y}\right)^{\frac{1}{N}} - \frac{\sigma_M}{E} \end{aligned} \tag{2.13}$$

then, given definition in (**??**)

$$h_M = \frac{d\sigma_M}{d\epsilon_M^p} = \frac{1}{d\epsilon_M^p / d\sigma_M} \tag{2.14}$$

considering (**??**), the hardening modulus of the matrix is obtained as

$$h_M = \frac{E}{\frac{1}{N}\left(\frac{\sigma_M}{\sigma_Y}\right)^{\frac{1-N}{N}} - 1} \tag{2.15}$$

2.0.5 Flow rule and generalized Hook's law

To complete the formulation for the modified Gurson model, at small strain, an additive decomposition of the strain rate is assumed as:

$$\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}^e + \dot{\boldsymbol{\epsilon}}^p \tag{2.16}$$

The plastic flow equation is given

$$\dot{\boldsymbol{\epsilon}}^p = \dot{\gamma}\frac{\partial F}{\partial \boldsymbol{\sigma}} \tag{2.17}$$

where $\dot{\gamma}$ is the plastic multiplier. For linear isotropic elasticity, the generalized Hook's law is written as

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{c}^e : \dot{\boldsymbol{\epsilon}}^e = \boldsymbol{c}^e : (\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}^p) \tag{2.18}$$

where $\boldsymbol{c}^e$ is the fourth-order elasticity tensor.

2.1 Integration algorithm for the modified Gurson model

The systems of equations of the modified Gurson model to be solved consist of the following:

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{c}^e : (\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}^p) \tag{2.19}$$

$$\dot{\boldsymbol{\epsilon}}^p = \dot{\gamma}\frac{\partial F}{\partial \boldsymbol{\sigma}} \tag{2.20}$$

$$\dot{f} = \dot{f}_g + \dot{f}_n \tag{2.21}$$

$$\dot{\sigma}_M = \frac{h_M \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p}{(1-f)\sigma_M} \tag{2.22}$$

$$F(\boldsymbol{\sigma}, \sigma_M, f^*) = 0 \tag{2.23}$$

Given a strain increment $\Delta\boldsymbol{\epsilon}$ of the bulk material, and its stress state and internal variables at time t_n, i.e., $\boldsymbol{\sigma}_{t_n}$, $(\sigma_M)_{t_n}$ and f_{t_n}, the goal of the integration algorithm is to solve for the values of these variables at time t_{n+1}. Using the implicit Euler scheme to write the discrete form of (**??**) to (**??**) as

$$\Delta\boldsymbol{\sigma} = \boldsymbol{c}^e : (\Delta\boldsymbol{\epsilon} - \Delta\boldsymbol{\epsilon}^p) \tag{2.24}$$

$$\Delta\boldsymbol{\epsilon}^p = \Delta\gamma \left(\frac{\partial F}{\partial \boldsymbol{\sigma}}\right)_{t_{n+1}} \tag{2.25}$$

$$\Delta f = \Delta f_g + \Delta f_n \tag{2.26}$$

$$\Delta\sigma_M = \frac{h_M}{(1-f_{t_{n+1}})\sigma_{Mt_{n+1}}}\boldsymbol{\sigma} : \Delta\boldsymbol{\epsilon}^p \tag{2.27}$$

$$F(\boldsymbol{\sigma}_{t_{n+1}}, \sigma_{Mt_{n+1}}, f^*_{t_{n+1}}) = 0 \tag{2.28}$$

where the increment of the plastic strain in the matrix is obtained from (**??**) as

$$\Delta\epsilon^p_M = \frac{\boldsymbol{\sigma}_{t_{n+1}} : \Delta\boldsymbol{\epsilon}^p}{(1-f_{t_{n+1}})\sigma_{Mt_{n+1}}} \tag{2.29}$$

and the incremental form of void growth and nucleation is written as

$$
\begin{aligned}
\Delta f_g &= (1 - f_{t_{n+1}})\,\mathrm{tr}\,(\Delta \boldsymbol{\epsilon}^p) + k_w \frac{f \omega(\boldsymbol{\sigma}_{t_{n+1}})}{\sigma_{e t_{n+1}}} \boldsymbol{s}_{t_{n+1}} : \Delta \boldsymbol{\epsilon}^p && (2.30) \\
\Delta f_n &= A(\epsilon^p_{M t_{n+1}}) \Delta \epsilon^p_M && (2.31) \\
& && (2.32)
\end{aligned}
$$

A Newton type iterative scheme is used to solve the system of nonlinear equations (**??**) to (**??**). For symmetric Cauchy stress tensor, the unknown vector consists of the following

$$
\boldsymbol{X} = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}, \sigma_M, f, \Delta\gamma\}^T \tag{2.33}
$$

using return mapping algorithm, the trial stress state $\boldsymbol{\sigma}^{\mathrm{tr}}$ is defined based on the assumption that the increment is elastic

$$
\boldsymbol{\sigma}^{\mathrm{tr}} = \boldsymbol{\sigma}_{t_n} + \boldsymbol{c}^e : \Delta \boldsymbol{\epsilon} \tag{2.34}
$$

Then, the residual vector is formed as

$$
\boldsymbol{R}(\boldsymbol{X}) = \left\{ \begin{array}{c}
\sigma_{11} - \sigma_{11}^{\mathrm{tr}} + \Delta\gamma c^e_{11ij} \left(\partial F / \partial \sigma\right)_{ij} \\
\sigma_{22} - \sigma_{22}^{\mathrm{tr}} + \Delta\gamma c^e_{22ij} \left(\partial F / \partial \sigma\right)_{ij} \\
\sigma_{33} - \sigma_{33}^{\mathrm{tr}} + \Delta\gamma c^e_{33ij} \left(\partial F / \partial \sigma\right)_{ij} \\
\sigma_{23} - \sigma_{23}^{\mathrm{tr}} + \Delta\gamma c^e_{23ij} \left(\partial F / \partial \sigma\right)_{ij} \\
\sigma_{13} - \sigma_{13}^{\mathrm{tr}} + \Delta\gamma c^e_{13ij} \left(\partial F / \partial \sigma\right)_{ij} \\
\sigma_{12} - \sigma_{12}^{\mathrm{tr}} + \Delta\gamma c^e_{12ij} \left(\partial F / \partial \sigma\right)_{ij} \\
f - f_{t_n} - \Delta f \\
\sigma_M - \sigma_{M t_n} - \Delta \sigma_M \\
F(\boldsymbol{\sigma}, \sigma_M, f)
\end{array} \right\} = \mathbf{0} \tag{2.35}
$$

where Δf and $\Delta \sigma_M$ are given as

$$
\begin{aligned}
\Delta f &= \Delta\gamma \left[(1 - f) \left(\partial F / \partial \sigma\right)_{ii} + k_w \frac{f \omega(\boldsymbol{\sigma})}{\sigma_e} s_{ij} \left(\partial F / \partial \sigma\right)_{ij} + A(\epsilon^p_M) \Delta \epsilon^p_M \right] && (2.36) \\
\Delta \sigma_M &= \Delta\gamma \frac{h_M}{(1 - f)\sigma_M} \sigma_{ij} \left(\partial F / \partial \sigma\right)_{ij} && (2.37)
\end{aligned}
$$

where the subscript t_{n+1} has been left off to simplify notation. The integration algorithm is summarized in the following box

Box 1. Integration algorithm for modified Gurson model

GIVEN: $\boldsymbol{\sigma}_{t_n}$, $\boldsymbol{\epsilon}_{t_n}$, f_{t_n}, σ_{Mt_n} and $\Delta\boldsymbol{\epsilon}$.
FIND: $\boldsymbol{\sigma}_{t_{n+1}}$, $f_{t_{n+1}}$ and $\sigma_{Mt_{n+1}}$.
STEP 1. Compute the trial stress state $\boldsymbol{\sigma}^{\text{tr}} = \boldsymbol{\sigma}_{t_n} + \boldsymbol{c}^e : \Delta\boldsymbol{\epsilon}$.
STEP 2. Check yielding: $F^{\text{tr}}(\boldsymbol{\sigma}^{\text{tr}}, f_{t_n}, \sigma_{Mt_n}) > 0$?
No, set $\boldsymbol{\sigma}_{t_{n+1}} = \boldsymbol{\sigma}^{\text{tr}}$, $f_{t_{n+1}} = f_{t_n}$, $\sigma_{Mt_{n+1}} = \sigma_{Mt_n}$ and exit.
STEP 3. Yes, local Newton loop.
3.1 Initialize $\boldsymbol{X}^k$ (**??**) and iteration count $k = 0$.
3.2 Assemble residual $\boldsymbol{R}(\underset{\sim}{X}^k)$ (**??**).
3.3 Check convergence: $\parallel \underset{\sim}{R} \parallel < tol$?
Yes, converged and go to STEP 4.
3.4 No, compute local Jacobian matrix $\underset{\sim}{J} = \partial\boldsymbol{R}/\partial\boldsymbol{X}$.
3.5 Solve system of equations $\boldsymbol{J} \cdot \delta\boldsymbol{X}^k = \boldsymbol{R}$ for $\delta\boldsymbol{X}$.
3.6 Update $\boldsymbol{X}^{k+1} = \boldsymbol{X}^k - \delta\boldsymbol{X}$ and go to 3.2
STEP 4. Update $\sigma_{t_{n+1}}$, $f_{t_{n+1}}$, $\sigma_{Mt_{n+1}}$, and exit.

2.2 Benchmark material point simulations

In this section, the numerical implementation of the modified Gurson model is verified through three benchmark material point tests, i.e., hydrostatic tension, plane strain tension and simple shear tests, where the analytical solutions or benchmark results from literature can be obtained.

2.2.1 Hydrostatic tension test

In this test, the material is subjected to a uniform volumetric expansion under strain controlled loading, with incremental size $\Delta\epsilon_{ii} = 1.0e - 3$. The void volume fraction function is taken as $f^* = f$ for all values of f to facilitate the comparison with analytical solution. Also, for hydrostatic tension, the value $\omega(\boldsymbol{\sigma}) = 0$, which means the void growth only depends on the volumetric part of the plastic strain (c.f. equation (**??**)), and the results are independent of k_w.

The material properties are summarized in the following table

Parameter	Value
E/σ_Y	300
ν	0.2524
f_0	0.04
k_w	0.0
N	0.1
q_1	1.5
q_2	1.0
q_3	2.25
e_N	0.3
s_N	0.1
f_N	0.04

Table 2.1. Material properties used for modified Gurson model in hydrostatic tension test.

For matrix material, the following stress-strain relation is used

$$\frac{\sigma_M}{\sigma_Y} = \left(\frac{\sigma_M}{\sigma_Y} + \frac{3G}{\sigma_Y}\epsilon_M^p\right)^N \tag{2.38}$$

where ϵ_M^p is the plastic strain in the matrix material.

The results of the hydrostatic tension tests are shown in the following figures. The analytical solution has been obtained in [CITE].

2.2.2 Plane strain tension test

In the plane strain tension test, the material properties and stress-strain law for matrix material are the same as in previous section, except the initial void ration $f_0 = 0$ in order to compare with available benchmark results from [CITE]. Also, the dependence on the third stress invariant through $\omega(\boldsymbol{\sigma})$ is studied by varying the parameter k_w in equation (**??**). The stepsize for the axial strain increment is taken to be $\Delta\epsilon_a = 5.0e - 3$. The stress-strain behavior as well as the void growth are shown in the following figures.

2.2.3 Simple shear test

In the simple shear test, to facilitate comparison with analytical results, it is assumed that $f^* = f$ for all values of f and void nucleation is neglected. Elasticity is also neglected when deriving analytical solution. The set of material properties for simple shear tests are shown in Table **??**. The matrix stress-strain law is taken to be linear-power law type as in equation (**??**). To study the material behavior in shear dominated state, the parameter k_w is varied from 0 to 5. The step size for the simple shear is taken to be $\Delta\epsilon_s = 1.0e - 2$.

Parameter	Value
E	200 GPa
σ_Y	200 MPa
ν	0.2524
f_0	0.005
N	0.1
q_1	1.1
q_2	1.0
q_3	1.0

Table 2.2. Material properties for the modified Gurson model in simple shear test.

Results of the simple shear tests are shown in the following figures. The case $k_w = 0$ corresponds to the original Gurson model, where the void growth only depends on the volumetric plastic strain.

Chapter 3

Results

Chapter 4

Conclusions

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