**Referee: 1   Comments to the Author**

This is a very nice and very interesting paper. It presents a simple, yet very efficient and useful, idea for numerical analysis of material instabilities. The idea and its context are well presented, and comprehensive testing is provided on small strains and finite strains examples.

*We thank the reviewer for the very valuable comments that help greatly improve the quality of this manuscript. Responses to individual comments are as follows.*

A couple minor remarks appeared during the reading:

(1) in section 2.1, it would be interesting that the authors motivate their specific choice for the interpolation of F (eq. 2.10)

*The motivation on the choice of F has been clarified and added after eq. (A.10) (previously (2.10)) in the appendix section*

*“The implicit assumption in this interpolation is that the deformation gradient F belongs to a Lie group and thus requires the exponential and logarithmic maps for its proper interpolation, whereas the internal variables Z belong to a space that admits direct linear interpolation [40]. In order to obtain an explicit scheme, we choose α = 0.”*

*[40] A. Mota, W. Sun, J. T. Ostien, J. W. Foulk III, and K. N. Long. Lie-group interpolation and variational recovery for internal variables. Computational Mechanics, 52(6):1281– 1299, Dec 2013. ISSN 0178-7675.*

(2) in the uniaxial tension test at finite strains, two global minima appear: with the cartesian parametrization, they are separated by a quite small ridge. Can the authors comment on numerical difficulties (if any) associated to that 'weak' separation of two different solutions?

*This work adopts a two-stage approach (initial sampling + Newton iterative solve). In the first sampling stage, the objective function is checked at every sampling grid and the minimum value is picked as the initial guess for the following Newton iterative solve. Though the code could detect multiple global minima up to the machine precision, in the case of multiple global minima, it only returns the first of the multiple minima. For the purpose of identifying when bifurcation happens, such implementation is sufficient. This comment has been added after Figure 13.*

*“It should be pointed out that, for the Cartesian parametrization, the two global minima are separated by a very small ridge. Though the current algorithm could detect multiple global minima, it only returns the first of the multiple minima. For the purpose of identifying when bifurcation happens, such implementation is sufficient.”*

**Referee: 2   Comments to the Author**

The work considers the problem of the bifurcation of the constitutive behaviour of dissipative materials, identified as a material instability. It mainly focuses on analyzing the incidence, of the parameterization of the normal to the strain discontinuity path, in the performance of one of the possible strategies for the numerical resolution of the material bifurcation problem.

The problem is the classical one in non-linear solids mechanics, consisting of finding the polarization vectors of the “acoustic” tensor, by minimization of its determinant. The considered numerical strategy is based on two stages: a) sampling, in the exploration domain determined by every parametrization in the tridimensional space, in order to (possibly) fall on the convex set around the minimum, and b) and iterative (possibly Newton-like) procedure, in the proposed algorithm, to set its exact value. The performance of different parameterizations (including the one proposed by the authors and termed “Cartesian parametrization”) is then analyzed and compared in terms of the robustness and the associate computational cost.

This is an interesting piece of work, focusing a subject that clearly falls into the scope of IJNME. The paper is well written, clearly structured and related works are appropriately referenced.

*We thank the reviewer for those detailed and very valuable comments that help greatly improve the quality of this manuscript. Responses to individual comments are as follows.*

Before sending a revised version of the work I suggest the authors to consider the following issues:

(1) It seems to me that section 2.1 “Variational constitutive updates” is here superfluous. I do not claim it is wrong, I just claim that the proposed algorithm could be retrieved directly from the chosen constitutive equation (2.13) and, thus, that the derivations in equations 2.1 to 2.12 could be skipped (or sent to an Appendix) in order to not distract the reader’s attention from the main topic of the work.

*The “Variational constitutive updates” section has been moved to an Appendix.*

(2) It is clear that the proposed parametrization (Cartesian parametrization) is the one best performing out of the four compared. However, it is still requiring the two-stages (space-sampling + iteration) strategy. The sampling stage shows necessary, since even with the authors’ proposed parametrization, the algorithm is not converging in some cases (see last row in Table 4). I assume that this stage is also responsible for a large part of the computational cost in many cases. This would make interesting for the reader, that the work elucidated (either by providing some additional information or by performing some additional numerical simulations) the following aspects:

•Split the run time into every stage (at least for some representative cases)

•Provide some comparison, in terms of robustness and computational cost, with strategies skipping the sampling stage (the one of reference [25] for instance), in order to elucidate if further research on the topic should be addressed on skipping the first stage rather than on improving the efficiency of the second.  In any case, I think that some additional considerations on the single/double stage strategies for the numerical solution of the material bifurcation problem, and their associated benefits/disadvantages, should be made in the work (may be in the “Conclusions” section ?).

*We have split the total computation time into two stages: the time for the initial sampling stage and the time for the iterative Newton’s solve. The results for the finite deformation anisotropic damage model example are reported in the updated Table 4. Also, it is interesting to see that, except for the very dense sampling case (interval=0.05), most of the computation time is spent on the iterative Newton’s solve and that is where the proposed Cartesian parametrization excels. Those discussions are added in the revised manuscript above Table 4.*

*As the reviewer pointed out all parametrizations, including the proposed Cartesian one, require a two-stage solution approach (i.e., initial sampling and iterative Newton’s solve). When the intial sampling stage is skipped, see results in Tables 2 and 5, the Cartesian parametrization remains the most efficient and robust (in terms of success rate). In addition, we have performed analysis using Oliver’s method and found that the Oliver’s method is able to correctly detect material bifurcation for the small deformation example and is much more efficient (averaged run time of merely 10 microseconds out of 1000 trials, each with a random initial guess for the eigenvector). However, Oliver’s method cannot correctly detect bifurcation time nor bifurcation direction for the finite deformation example. The authors are also investigating a single stage stochastic-based approach, which will be published in a future work.*

*The following comments have been added at the end of Section 5.1 and Section 5.2*

*At the end of Section 5.1*

*“This first example illustrates the performance of different parametrizations on a very simple small deformation material model with tangents possessing both major and minor symmetries. For such material models, an alternative single stage approach as in Oliver et al. [25] could also be applied, which is based on the solution of a coupled eigenvalue problem in terms of the acoustic tensor. A preliminary analysis shows that this approach is able to correctly detect bifurcation and is more computationally efficient than the two-stage approach advanced herein for this type of material models. Our focus, however, is in the full finite-deformation case and therefore more complex material tangents that do not posses minor symmetries will be investigated next.”*

*Ate the end of Section 5.2*

*“We note that for this finite deformation example, the single-stage approach of Oliver et al. [25] cannot correctly detect the bifurcation time nor the bifurcation direction. The Cartesian parametrization thus provides a valuable tool in numerical bifurcation analysis.”*

(3) The presented numerical examples do not challenge very much the proposed algorithm. In my opinion the challenge of the material bifurcation problem does not lie on the large-strain character of the kinematics or on the complexity of the considered constitutive problem, but rather on the complexity of the stress/strain field that the material is subjected to. In spite that the proposed approach seems to be 3D, and general for symmetric/unsymmetric models, the considered tangent operators are symmetric (see eqs. 5.5 and 5.10) and, in addition, “in-plane” solutions are enforced (in all cases the z-component of the normals are zero). In my opinion, the benefits of the proposed approach in terms of robustness would be much more convincing if an additional numerical example with a more general character (unsymmetrical model and full 3D solution) was included.

*We first would like to acknowledge the reviewer’s, correctly stated, viewpoint that ultimately the benefit of the current research will manifest as tools and capabilities to analyze general 3D material behavior in the failure regime. To that end, the contribution of this paper lies in developing the understanding impact of the par on the illustrated landscapes. Even with idealized models and loadings, the landscapes of the minimization procedure are quite complex. of the character of the parameterizations for idealized models and loadings necessary to evaluate different parameterizations used within the bifurcation detection algorithm. Focusing on the evaluation of the character of, the numerical examples were intentionally picked as idealized models and boundary value problems. Even so the landscapes produced were sufficiently rich and diverse such that meaningful, quantified evaluations of the parameterizations could be made. At this time we feel that additional complexity in the boundary value problems studied would provide no further insight into the evaluation of the proposed methods in terms of the comparisons. However, as stated above, the long term vision of this work is to employ these methods to detect material bifurcation in general loadings, and this is precisely the future direction of our research.*

*Conclusion bullet: Idealized models and loadings provided sufficient complexity to evaluate the proposed parameterizations…*

*Put a future work blurb after the bullets.*

*Put a foreshadowing statement in the intro…*

(4) Some (minor) comments/suggestions are:

• Similarly to eq. (3.3), in eq. (3.7) I missed the restriction: “subjected to x^2+y^1=1”

• line 206 the intervals [-pi/2,pi/2] should be [-1,1].

*For tangent parametrization eq.(3.7), x, y are points on the tangent plane with* ***u****=[x, y, 0]T representing the position vector from north pole (Q) to point P on the tangent plane. Therefore, it is not subjected to x^2+y^1=1*. *The range [-pi/2, pi/2] guarantees that the position vector would cover all points on the tangent plane. The confusion might come from the use of canonical basis x,y to parametrize* ***n****. This has been clarified in the revised manuscript.*

*“Note that though the normal vector $~n$ is a unit vector, the position vector on the tangent plane $~u =[x,y,0]^T$ is not subjected to the constraint $x^2+y^2+z^2 = 1$ and the range $[-\pi / 2, \pi / 2]$ is sufficient for the position vector to cover all points on the plane.”*

• In eq. (5.1) and line 282, superscript “e” and the name “elastic“ to denote strain tensor is misleading and I think it should be reduced to “strain”, since it suggests that the total strain of the model is split into inelastic and inelastic counterparts (which is not the case).

*The superscript ``e’’ on strain tensor has been removed.*