*Article*

# Sheaf Mereology

**Firstname Lastname [1], Firstname Lastname [2] and Firstname Lastname [2,*]**

[1] Affiliation 1; e-mail@e-mail.com
[2] Affiliation 2; e-mail@e-mail.com
* Correspondence: e-mail@e-mail.com; Tel.: (optional; include country code; if there are multiple corresponding authors, add author initials) +xx-xxxx-xxx-xxxx (F.L.)

**Abstract**

A single paragraph of about 200 words maximum. For research articles, abstracts should give a pertinent overview of the work. We strongly encourage authors to use the following style of structured abstracts, but without headings: (1) Background: place the question addressed in a broad context and highlight the purpose of the study; (2) Methods: describe briefly the main methods or treatments applied; (3) Results: summarize the article's main findings; (4) Conclusions: indicate the main conclusions or interpretations. The abstract should be an objective representation of the article, it must not contain results which are not presented and substantiated in the main text and should not exaggerate the main conclusions.

**Keywords:** mereology; fusions and integral wholes; sheaves; point-free topology; frames and locales; toposes; modality; merology logic; categorical logic

## 1. Introduction

Standard presentations of mereology tend to take what we might call a "parts-first" approach. You start by taking the parthood relation as primitive, and then you proceed by stipulating axioms that govern the relation. The goal is to choose your axioms well enough that the resulting models that satisfy your theory align nicely with the actual part-whole complexes that we encounter in the world around us.

By most standard accounts (e.g., [1], [2], or [3]), partisans of the parts-first approach have more or less agreed on a common (minimal) "core" known as "classical mereology." First, classicists adopt the following principles that govern the ordering of the parts:

- Parthood is reflexive, antisymmetric, and transitive (i.e., it is a partial order).

Second, classicists adopt the following decomposition principle that governs how wholes decompose:

- Wholes decompose into more than one proper part (i.e., parthood obeys some form of so-called "supplementation," "remainder," or "complementation" principle which says that no whole consists of only a single proper part — there must be some remainder or relative complement).

Third, classicists adopt the following principle that governs how parts fuse into wholes:

- Any collection of parts whatever forms a fusion (i.e. unrestricted fusion).

Classicists also require (either as an explicit axiom or as a consequence of the other axioms) some version of extensionality:

consult Contoir's article on this and maybe cite it

Received:
Revised:
Accepted:
Published:

**Citation:** Lastname, F.; Lastname, F.; Lastname, F. Title. *Philosophies* **2025**, *1*, 0. https://doi.org/

- If wholes have the same parts, then they're the same wholes.

With the above axioms fixed, the classicist can then define a number of other useful notions in the obvious ways, e.g.:

- Overlap and underlap
- Complement/difference
- Etc.

At first sight, most of the classicist's principles can feel deeply intuitive. However, philosophers have objected to virtually all of them. Take for instance that parthood is transitive: if $x$ is a part of $y$ and $y$ is a part of $z$, then surely $x$ is a part of $z$. But there appear to be counter-examples. For example, my appendix is a part of me, and I am a member of the orchestra, but my appendix is not a member of the orchestra.

A standard response is to point out that this sort of objection exploits an ambiguity: we utilize different, more specialized notions of "parthood" when we talk about the integration of the parts of biological organisms vs. those of orchestras. My appendix is a part of me under one description (as a part of a biological organism), while I am a part of the orchestra under another (as a member of a musical ensemble).

Defenders of the classical axioms have said that the fact that we can partition the general notion of parthood into more specialized versions only shows that the above axioms do in fact characterize a general notion of parthood, characterized precisely by the above classical notions.

**cite** Simons, Varzi, etc

However, one can't help but feel that there is something circular about this response, since it turns on the assumption that the different notions of functional unity are species (or determinations, or partitions, or what have you) of a more generic relation. But the existence of that generic relation hasn't been established, and there is no reason to think that mereological pluralism isn't correct — namely, that there are many parthood relations, not one.

**cite** Fine, etc.

Another common objection to the classical approach revolves around composition principles. In particular, if we adopt unrestricted fusion, as the classical mereologist does, then we seem to get too many fusions. For instance, take the pencil on the table in front of me and your left knee. Are we really to believe that there is a fusion of that pencil and your left knee? Such a fusion would have two parts that live quite far apart (possibly even on different sides of the globe).

Defenders of the classical approach do have a response though: just because we may not have a word or concept that names the pencil+knee fusion, that doesn't mean it doesn't exist.

**cite** Varzi, etc.

Indeed, just as Moore attempt to show that extramental things exist by holding up his two hands and saying, "Here is one hand, here is the other," so too might one try to show that the pencil+knee fusion exists by saying, "there is the pencil, there is the knee."

Again though, one can't help but feel that there is something circular about this response. To appeal to the pencil+knee's fusion after its fused-ness has been questioned just brings the principle under scrutiny back into the mix. A better response would provide an independent reason to think that pencil+knee qualifies as more than a "mere Cambridge" fusion.

Whether these objections/responses constitute any real conceptual clarification or are inexorably stuck in semantic circularities is not something we want to decide here. We mention these points only because they illustrate something else: they illustrate that we seem to have intuitions not just about parts, but also about fusions. In the two objections just mentioned, we seem to have certain conceptions of integrated fusions somewhere in the back of our minds, and those seem to be driving the objections.

For example, the reason it seems wrong to say my appendix is not part of the orchestra is because we seem to think that biological organisms are integrated in a different way than orchestras. Similarly, the reason we can so easily think that a pencil and a hand don't fuse is because they don't integrate in one of the ways that we ambiently accept as legitimate.

That leads to the following question: if we have ambient intuitions about which collections count as fusions and which ones don't, then why not take a "fusions-first" approach to mereology? Instead of taking parthood as the primitive relation, and then try to work up to a notion of fusions, why not take fusions as the primitive relation, and then work backwards to parts?

Such an approach is far less common in the technical mereological literature. That raises yet another question: why has the "fusions-first" approach been so neglected? One proposal is that it might seem to be too unwieldy. One might say that there are just too many different ways that things can integrate into wholes/fusions, and so it is a hopeless task to try and enumerate them and offer any sort of a unifying taxonomy.

**cite** Simons

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Another reason might be that such a view would be inelegant and perhaps would even fail to qualify as an "explanation" of the part-whole phenomenon altogether. If you ask me to explain why various $X$s seem to exhibit the same (or sufficiently similar) properties, it would be quite dissatisfying if I said, "that's easy, there is no unifying explanation."

remove/rewrite these last two paragraphs.

The "fusions-first" approach is not so uncommon in the literature as I just made it out to be. Mereotopology exists as a branch of hereology, for all intensive purposes, precisely because it is the fusions-first response to the classical parts-first approach. Cite Casati and Varzi, chapter 3 and others.

Fortunately, a "fusions-first" approach need not be as doomed as it may seem. In this paper, we claim that there is a satisfying "fusions-first" approach to mereology, and we present it in what follows. To accomplish this, we will build a bridge between category theory and philosophy. In particular, we will take well-known techniques used to manage the gluing-together of parts in algebraic geometry and topos theory, and we will apply those techniques to the realm of mereology.

### 1.1. *From Parts to Sheaves*

To begin, we want to suggest that it is useful to draw a distinction between what one might call the algebra of parts on the one hand, and the integrity or gluing-together of the parts on the other. To get a sense of what this distinction means, and why it is useful, fix a part-whole complex to analyze (a statue of Dion, let's say), and let an enumeration of its parts be given. The classical principle of unrestricted fusion says that any combination of those parts glues into a fusion. In essence, this generates all possible combinations of parts. As such, it nearly yields a complete lattice, with overlap and underlap serving as the meet and join operators.

However, it only *nearly* yields a lattice because mereologists have been reluNctant to allow a bottom element. Since mereologists are ontologists, they find a null element to be ontologically suspect. And indeed, what could an empty thing that is part of all other things possibly *be*? So, instead of admitting it into their mereological systems, classical mereologists have simply omitted it altogether. David Lewis ([4]) even went so far as to formulate a version of set theory that had no empty set.

Yet despite their suspicion of a null element, classical mereologists have not shied away from allowing all possible fusions to exist, as noted already. Since the bottom element of a lattice is the empty join, we can put the point like this: classical mereologists are ontologically conservative about empty joins, and ontologically permissive about non-empty joins. As Tarski pointed out long ago, the principles of classical mereology thus yield a boolean algebra, with the bottom element removed

**cite**

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But it is difficult to see the motivation here. On the one hand, if you want to be ontologically conservative, then why allow so many fusions? If we are going to be suspicious of an empty join, then wouldn't we also be suspicious of the fusion of (say) Dion's left hand and right knee? Conversely, if we are happy to admit the existence of entities like the fusion of Dion's left hand and right knee, then why not an empty join?

One way to diagnose the problem is to say that we, as classical mereologists, have confused the algebra of parts with the integrity of the wholes. We have defined the algebra of parts in a combinatorial way, but then at the same time, we tried to make that algebra do ontological work. But this inevitably pulls us in two directions. So, we end up letting the algebraic aspects of our parthood relation do ontological work (creating any fusion whatever), until it goes too far (e.g. the null element), at which point we try to pull back on the ontological reigns.

For another point of tension, consider extensionality. The classicist's axiom says if $x$ and $y$ have the same parts, then $x = y$. This is ontologically conservative: "no difference without a difference maker"

**cite** Lewis

. However, this flattens all structure, and so it judges that "tip" and "pit" cannot be different words, since they have the same parts, after flattening. But that of course feels wrong. These two words have a different ordering of letters, so why would we neglect that in determining their identity? Here too we don't want the combinatorics to do any ontological work, even though we're happy to let the join operation freely generate entities.

We can free ourselves from these sorts of tensions if we separate the algebra of parts from the integrity or gluing of the parts. Let us think of the lattice of parts merely as the abstract "parts space," i.e., as the set of all *possible* combinations of the given parts into larger pieces. Moreover, let us be clear that this does not do any ontological work. A "parts space" is just an abstract description of the various combinations of parts that could be. Think of it as a kind of mold that has slots that could be filled in with actual pieces. To specify an *actual* part-whole complex that occupies that parts space, we need to take a second step and fill in certain of those slots with actual stuff, and specify which of those pieces glue together into bigger pieces.

Once we have made this distinction, we can let the algebra of parts be an algebra, and we can even allow a bottom element without worry. As a component of the abstract parts space, the bottom element is no more a real thing than the join of any other arbitrary regions of the parts space. At the same time, when we specify which pieces really occupy the parts space, we can be as ontologically conservative or as permissive as we like. We have the freedom to provide gluing conditions that are as fine-grained as we need. For instance, we can say that certain pieces glue together, while others do not (e.g., Dion's right knee glues directly to his right femur, but not directly to his left hand). Moreover, we can let the identity conditions be determined by the gluing conditions, and so maintain structured extensionality (a difference in fusions comes from different parts, or different gluings).

Once we make the distinction between the background algebra of parts and the foreground integrity of the fusion, our task takes on a distinctive shape: now we find ourselves trying to coherently glue pieces together over an ambient space. And that is something known well to algebraic geometers and topos theorists: it is the task of constructing a sheaf over a space. For the algebraic geometer and topos theorist, sheaf theory provides a systematic framework for gluing together pieces over a space in such a way that the gluing is done coherently and consistently against the ambient structure of the underlying space. It stands to reason, then, that the mathematician's sheaf-theoretic techniques can be used profitably in mereology.

### *1.2. The Central Thesis*

In this paper, we want to build a bridge between sheaf theory and mereology by importing some of those sheaf-theoretic techniques into the mereological setting. The central claim of this paper is thus: part-whole complexes can be usefully modeled as sheaves over locales. The key ideas are as follows:

- An algebra of parts tells us all the ways that parts can combine to form bigger wholes. In this sense, an algebra of parts generates the ambient "parts space" of an object, i.e. the abstract lattice-theoretic structure of *possible* combinations. But not all possible combinations actually glue together to form *actual* fusions. In many cases, we want to allow that only some of the parts glue together into an integrated whole/fusion.
- So, we then require a separate step where we, the mereologists, have to "fill in" the abstract parts space with actual parts: we have to specify which bits of stuff fill in or occupy which slots in that ambient parts space, and we have to specify how those various bits of stuff glue together to form integrated fusions.
- It is tempting to try to model the ambient parts space as a topology. However, topologies have points, and it is not clear that all of the part-whole complexes that we might wish to consider are usefully modeled with points. This limitation is easy to overcome if we generalize and move to the point-free setting: instead of a topology, we choose to model the ambient parts space as a locale (a point-free generalization of a topology).
- We use a sheaf to specify which bits of stuff inhabit an ambient locale and also to stipulate how those bits glue together. A sheaf is precisely an assignment of data to a topology or locale that coherently glues that data together. So, we choose to model the actual part-whole complex as a sheaf over the ambient locale.
- To specify a part-whole complex, then, we (the mereologists) simply need to define a sheaf over the given locale. The "data" that we assign to the ambient locale are the bits of actual stuff that inhabit that parts space, and the gluing condition specifies how those pieces glue together.
- This yields a straightforward procedure that can be used to model any part-whole complex: first, specify the ambient locale, i.e., the abstract space of parts that the part-whole complex in question inhabits; second, fill in that ambient space with actual pieces and say how they glue together; third, let the sheaf framework do the rest of the work. Then the glued sections of the sheaf turn out to be the fusions, whose parts are the smaller sections each fusion is glued from. This is an honest "fusions-first" approach.

There are two important benefits that come along for free when we take this approach.

- The sheaves over a locale form a topos. A topos is a special kind of category that you can do "parts"-like logic in. Indeed, every topos comes equipped with just such an internal logic. It turns out that this internal logic corresponds exactly to the correct mereological logic that governs the part-whole complexes that can be formed over that locale. So, there is no need to manually create a mereological logic to reason about the part-whole complexes that occupy the ambient locale. We get that for free.

- Modalities are natural operators that occur in sheaves, where they are easily defined and managed. These modalities interact correctly with the internal logic of the topos (and in fact are part of that internal logic). So we get mereological modalities for free too.

To our mind, the fact that these benefits come for free offers a compelling reason to adopt a sheaf-theoretic approach to part-whole complexes.

### 1.3. Literature

The literature on mereology is vast. What we might think of as formal mereology (axiomatized systems) goes back at least to Leśniewski's system called "Mereology" [cite] and Leonard and Goodman's "Calculus of Individuals" ([5] and [6]), along with other contributions by Whitehead [cite], Tarski ([7] and [8]), Rescher ([9]), and others. Surveys of the resulting literature and ideas can be found in the now standard works by Simons and others (e.g. [10], [1], [2], or [3]).

Topological concepts have been used in mereology for a long time (see the historical coverage and discussion in [1]). So-called "mereotopology" explicitly aims to characterize mereological questions in topological ways, especially using notions like boundaries, interiors, and connectedness. A standard introduction to modern mereotopology is [11].

However, despite its heavy reliance on topology, mereotopology has not (to our knowledge) utilized sheaf-theory in any significant way (nor has classical mereology).

discuss Spivak's behavioral mereology/temporal type theory

([12], [13]), Moltmann's trope sheaves, Moltmann's mereology.

For sheaves, see [14], [15], [16], or [17]. For locales, see [18], [19], [20], [21].

For toposes, see [22], [23], [24], [15], [25].

discuss the idea of "deriving" the logic from the underlying structure, rather than "inventing" it axiomatically. Perhaps Moltmann's mereology is to be cited here.

discuss how mereological concepts are used in certain "fusion-first" approaches, e.g., Peter Simons and using "connectedness." Discuss Van Inwagen's special composition question ([26]).

discuss non-boolean approaches. Discuss boolean algebra stuff from Protow, the survey "Logic in Heyting Algebras," Moltmann's Heyting mereology.

### 1.4. Contributions

The contributions of this paper are as follows:

- We demonstrate a viable "fusions-first" approach to mereology.
- We separate the algebra of parts from the integrity of fusions.
- We build a bridge between mereological techniques of mathematics and philosophy. In particular:
  - We utilize sheaves to systematically manage coherence and gluing over parts spaces.
  - We acquire the correct mereological logics for free from the internal language of the underlying topos.

### 1.5. Plan of the Paper

The plan of this paper is as follows.

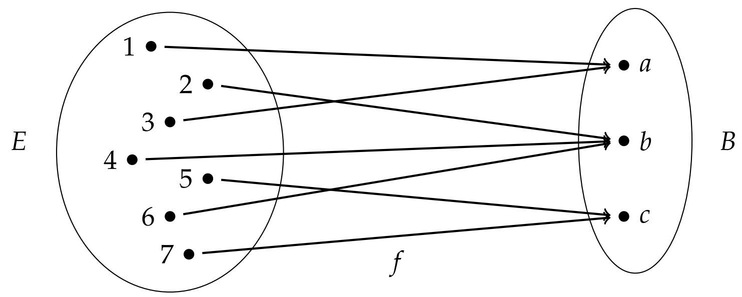
- In Section 2, we introduce the relevant parts of sheaf theory that will be used in the rest of the paper.
- In Section 3, we define part and whole in sheaf-theoretic terms, and we show how to model different kinds of part-whole complexes as sheaves.
- In **??**, we show how mereological modalities arise naturally in sheaves.
- In Section 4, we discuss what classical mereological notions look like in the sheaf-theoretic setting.

## 2. Sheaf Theory

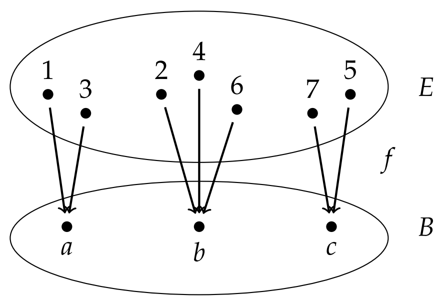
In this section, we introduce the parts of sheaf theory needed for the sequel.

### 2.1. Fibers

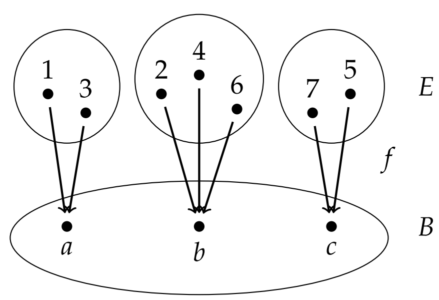
Suppose we have a map (function) $f : E \to B$ that looks something like this:



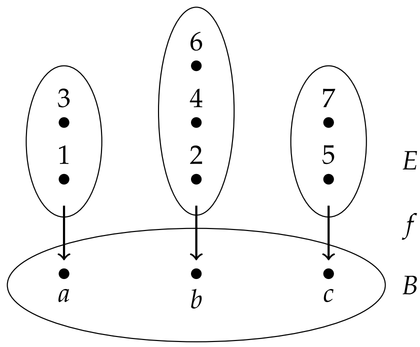

It is sometimes convenient to turn the diagram sideways and group together points in the domain that get sent to the same target, like so:




That makes the pre-images very easy to see. For any point in $B$, its pre-image is just the group of points sitting "over" it:



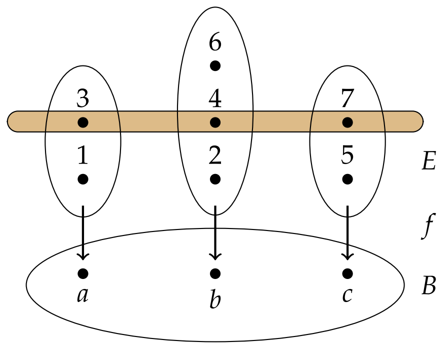

If we stack the points in each pre-image vertically, one on top of the other, we can then think of each pre-image as a kind of "stalk" growing over its base point:



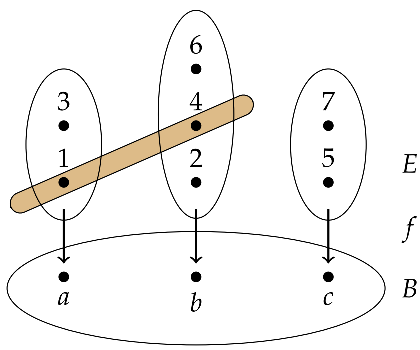

This gives rise to the idea of the "fibers" of a map. The fibers of a map are just its pre-images. For instance, the fiber over $b$ is $\{2, 4, 6\}$.

**Definition 1** (Fibers)**.** *Given a map $f : E \to B$ and a point $y \in B$, the fiber over $y$ is its pre-image $f^{-1}(y) = \{x \mid f(x) = y\}$. $B$ is called the base space of $f$, and $y$ the base point of the fiber.*

We can take a cross-section of one or more fibers by selecting a point from each of the fibers in question. For instance, we can take 3, 4, and 7 as a cross-section of the fibers $f^{-1}(a)$, $f^{-1}(b)$, and $f^{-1}(c)$:




We can also take cross-sections local to only some of the fibers. For instance, we can take 1 and 4 as a cross-section of $f^{-1}(a)$ and $f^{-1}(b)$:




**Definition 2** (Sections)**.** *Given a map $f : E \to B$ and a subset of base points $C \subseteq B$, a section of $f$ (over $C$) is a choice of one element from each fiber over each base point $x \in C$.*

**Remark 1.** *Since each point in a fiber amounts to a section over the fiber's base, the elements of a fiber are often just called the sections of the fiber.*

*2.2. Spaces*

In the above examples, the base $B$ was a set. We often want to consider bases that have more structure, e.g., bases that have spatial structure.

In traditional topology, spaces are built out of the points of the space. Given a set of points, a topology on that set specifies which points belong in which regions of the space.

**Definition 3** (Topology)**.** *Let $X$ be a non-empty set, thought of as the points of the space. A topology on $X$ is a collection $T$ of subsets of $X$, thought of as the regions of the space and called the open sets (or just the opens) of $T$, that satisfy the following conditions:*

*(T1) The empty set and the whole set are open:*

$$\varnothing \in T, X \in T.$$

*(T2) Arbitrary unions of opens are open:*

$$\text{if } \{U_i\}_{i \in I} \subseteq T, \text{ then } \bigcup_{i \in I} U_i \in T.$$

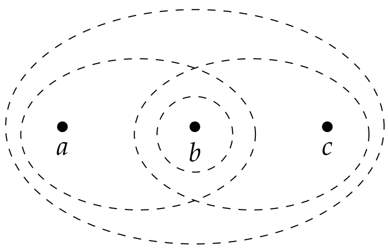
*(T3) Finite intersections of opens are open:*

$$\text{if } U_1, \ldots, U_n \in T, \text{ then } \bigcap_{i=1}^{n} U_i \in T.$$

These conditions encode the way that spatial regions are put together. For instance, it ensures that if two regions overlap, then their overlapping area is a region too.

**Remark 2.** *The regions of a topology, ordered by inclusion, form a complete lattice. Since the topology includes arbitrary unions, the join of this lattice is set union, but since the topology includes only finite intersections, the meet of this lattice is the* interior *of set intersection.*

**Example 1.** *Let $X = \{a, b, c\}$. One possible topology is $T = \{\varnothing, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. If we draw dashed circles around each of the opens (regions), we get:*




*There are two regions $\{a, b\}$ and $\{b, c\}$ that overlap at $b$ (so $\{b\}$ is a region in $T$ too). There is also the full region $\{a, b, c\}$, which is the union of the smaller regions.*

*We can draw $T$ as a Hasse diagram, which shows that the regions form a lattice:*

$$\begin{tikzcd}[column sep=small]
 & \{a,b,c\} \arrow[dl, dash] \arrow[dr, dash] & \\
\{a,b\} \arrow[dr, dash] & & \{b,c\} \arrow[dl, dash] \\
 & \{b\} \arrow[d, dash] & \\
 & \varnothing &
\end{tikzcd}$$

The lattice structure suggests that much of what is important about a space is not so much its points, but rather its opens/regions. This leads to the idea that topology-like reasoning can be done without the points. So, we can generalize: take a topology, and drop the points. That leaves just the opens/regions, which we call a frame (or locale).

**Definition 4** (Frames/locales). *A frame (synonymously, a locale) $\mathbb{L}$ is a partially ordered set $L$ (whose elements are called opens or regions) that satisfies the following conditions:*

*(L1) $L$ is a complete lattice:*

- *Every subset $S \subseteq L$ has a join, denoted $\bigvee S$.*
- *Every finite subset $F \subseteq L$ has a meet, denoted $\bigwedge F$.*

*(L2) Finite meets distribute over arbitrary joins:*

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i), \text{ for all } a \in L \text{ and all families } \{b_i\}_{i \in I} \subseteq L.$$

*Define $V \preccurlyeq U$ (read "$V$ is included in $U$") by $a = a \wedge b$.*

**Remark 3.** *The fact that $V \preccurlyeq U$ is equivalent to $a = a \wedge b$ means we can deal with the opens of a frame algebraically (via $\wedge$ and $\vee$ operations), or order-theoretically (via the $\preccurlyeq$ relation), whichever is more convenient.*

**Remark 4.** *The category of locales is defined as the dual/opposite of the category of frames, and so frames and locales are quite literally the very same objects. In practice, frames are often used for algebraic purposes, and locales are used for (generalized) spatial purposes. Here, we will have no reason to distinguish these two roles, and so we will use the names "frame" and "locale" interchangeably.*

By definition, every locale has a lowest element, which is the join of no regions at all. It represents the absence of any regions whatever. Hence, we typically denote it with the symbol "$\perp$." Dually, every locale has a highest element, which is the join of all of the regions. Hence, when convenient we can denote it with the symbol "$\top$."

### *2.3. Presentations of locales*

Locales have presentations much like groups and other algebraic structures. To give the presentation of a locale, specify a set of generators and relations.

**Definition 5** (Presentations). *A presentation $\langle G, R \rangle$ of a locale $\mathbb{L}$ is comprised of:*

*(P1) A set of generators $G = \{U_k, U_m, \ldots\}$.*
*(P2) A set of relations $R \subseteq G \times G$ on those generators.*

*The locale $\mathbb{L}$ presented by $\langle G, R \rangle$ is the smallest one freely generated from $G$ which satisfies $R$.*

**Remark 5.** *Every locale has a presentation, and a locale can have multiple presentations.*

To calculate the locale that corresponds to a presentation, start with the generators, then take all finite meets and all arbitrary joins that satisfy $R$ (and of course L1 and L2).

**Example 2.** *Let a locale $\mathbb{L}$ be given by the presentation $\langle G, R \rangle$ where:*

- $G = \{\perp, U_1, U_2, U_3\}$.
- $R = \{\perp \preccurlyeq U_1, U_1 \preccurlyeq U_2, U_1 \preccurlyeq U_3\}$.

*There are four generators ($\perp$, $U_1$, $U_2$, and $U_3$), and $\perp$ is below $U_1$ while $U_1$ is a sub-region of $U_2$ and $U_3$. Since $U_1$ is a sub-region of both $U_2$ and $U_3$, $U_1$ is their meet:*

- $U_1 = U_2 \wedge U_3$.

*At this point, we have generated this much of the locale:*

$$\begin{tikzcd}
U_2 & & U_3 \\
& U_1 \arrow[ul, dash] \arrow[ur, dash] & \\
& \bot \arrow[u, dash] &
\end{tikzcd}$$

*$R$ says nothing to constrain joins, so we need to join everything we can. In this case, we need to join $U_2$ and $U_3$:*

$$\begin{tikzcd}
& U_2 \vee U_3 & \\
U_2 \arrow[ur, dash] & & U_3 \arrow[ul, dash] \\
& U_1 \arrow[ul, dash] \arrow[ur, dash] & \\
& \bot \arrow[u, dash] &
\end{tikzcd}$$

*There are no further joins or meets that aren't already represented in the picture. For instance, all further non-trivial meets are already accounted for:*

- $U_1 \wedge \bot = \bot$.
- $U_2 \wedge U_1 = U_1$ *and* $U_3 \wedge U_1 = U_1$.
- $U_2 \wedge \bot = \bot$ *and* $U_3 \wedge \bot = \bot$.
- $(U_2 \vee U_3) \wedge U_2 = U_2$ *and* $(U_2 \vee U_3) \wedge U_3 = U_3$.
- $(U_2 \vee U_3) \wedge U_1 = U_1$.
- $(U_2 \vee U_3) \wedge \bot = \bot$.

*Similarly, all other non-trivial joins are also already accounted for:*

- $\bot \vee U_1 = U_1$.
- $\bot \vee U_2 = U_2$ *and* $\bot \vee U_3 = U_3$.
- $\bot \vee (U_2 \vee U_3) = U_2 \vee U_3$.
- $U_1 \vee U_2 = U_2$ *and* $U_1 \vee U_3 = U_3$.
- $U_2 \vee (U_2 \vee U_3) = U_2 \vee U_3$ *and* $U_2 \vee (U_3 \vee U_3) = U_2 \vee U_3$.

**Example 3.** *Let $\mathbb{L} = \langle G, R \rangle$ be given by:*

- $G = \{U_1, U_2, U_3\}$.
- $R = \varnothing$.

*We have three generators ($U_1$, $U_2$, and $U_3$), and there are no relations restricting how those generators are related. Thus, the locale that is freely generated from this presentation is isomorphic to the power set of three elements:*

$$\begin{tikzcd}
 & \top = U_1 \vee U_2 \vee U_3 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
U_1 \vee U_2 \arrow[d, dash] \arrow[dr, dash] & U_1 \vee U_3 \arrow[dl, dash] \arrow[dr, dash] & U_2 \vee U_3 \arrow[dl, dash] \arrow[d, dash] \\
U_1 \arrow[dr, dash] & U_2 \arrow[d, dash] & U_3 \arrow[dl, dash] \\
 & \bot &
\end{tikzcd}$$

A presentation provides the most "minimal" information from which the rest of the locale is generated.

### 2.4. Presheaves

Above we considered the fibers of a map $f : E \to B$, where $E$ and $B$ were sets. We can also consider fibers over locales, where the fibers respect the locale's structure. This is called a presheaf. A presheaf is an assignment of data to each of a locale's regions that is "stable under restriction," i.e., that respects zooming in and out.

**Definition 6** (Presheaf)**.** *Let $\mathbb{L}$ be a locale, and let $\mathsf{Arr}(\mathbb{L})$ be $\{\langle A, B\rangle \mid A \preccurlyeq B \in \mathbb{L}\}$. A presheaf on $\mathbb{L}$ is a pair $\langle F, \{\rho^B_A\}_{\langle A,B\rangle \in \mathsf{Arr}(\mathbb{L})}\rangle$, where:*

- *$F$ assigns to each region $U \in L$ some data $F(U)$.*
- *$\{\rho^B_A\}_{\langle A,B\rangle \in \mathsf{Arr}(\mathbb{L})}$ is a family of maps $\rho^B_A : F(B) \to F(A)$ (called restriction maps), each of which specifies how to restrict the data over $F(B)$ down to the data over $F(A)$.*

*All together, $\langle F, \{\rho^B_A\}_{\langle A,B\rangle \in \mathsf{Arr}(\mathbb{L})}\rangle$ must satisfy the following conditions:*

*(R1) Restrictions preserve identity:*

$$\rho^U_U = \mathsf{id}_U \text{ (the identity on } U\text{), for every } U \in \mathbb{L}.$$

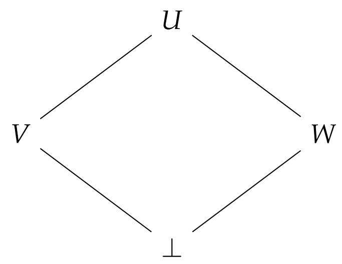
*(R2) Restrictions compose:*

$$\text{If } A \preccurlyeq B \text{ and } B \preccurlyeq C, \text{ then } \rho^C_A = \rho^B_A \circ \rho^C_B.$$

Since $F$ assigns data $F(U)$ to each region $U \in \mathbb{L}$, we can think of the $F(U)$s as the "fibers" over $\mathbb{L}$, and the restriction maps as "zoom in" maps that go from bigger fibers down to smaller fibers.

**Remark 6.** *For the category-theoretically inclined, a presheaf is just a set-valued contravariant functor $F : \mathbb{L}^{\mathsf{op}} \to \mathbb{S}\mathsf{et}$. To each region $B$ of $\mathbb{L}$, $F$ assigns to it a set $F(B)$. The contravariance comes from the fact that, to each arrow $B \preccurlyeq C$ of $\mathbb{L}$, $F$ assigns a restriction map that goes the other way (i.e., that restricts the data from $F(C)$ down to $F(B)$). (R1) and (R2) are automatically satisfied by the fact that $F$ is a functor.*

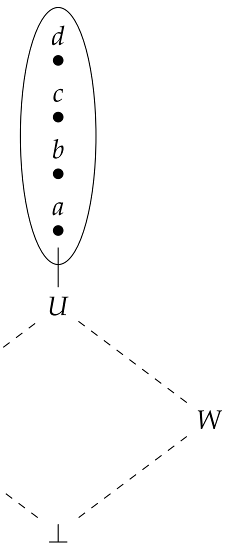
**Example 4.** *Let $\mathbb{L}$ be a locale $\{\bot, W, V, U\}$ with the following structure:*



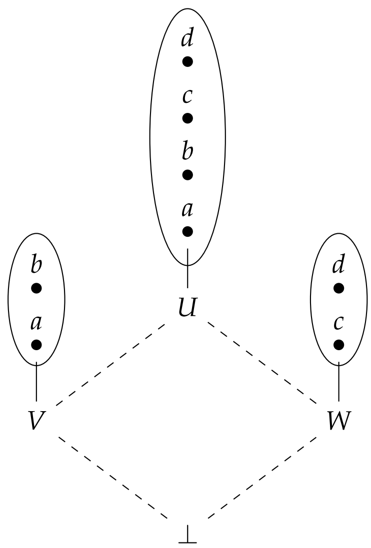

*Next, let's define a presheaf F as follows:*

- $F(U) = \{a, b, c, d\}$, $F(V) = \{a, b\}$, $F(W) = \{c, d\}$, $F(\perp) = \{*\}$.
- *Define* $\rho^U_V$ *as the projection (send a to a, b to b, and the rest can go anywhere), and similarly for* $\rho^U_W$. *Let* $\rho^U_\perp$, $\rho^V_\perp$, *and* $\rho^W_\perp$ *send their data to* $\{*\}$, *and let the rest be identities.*

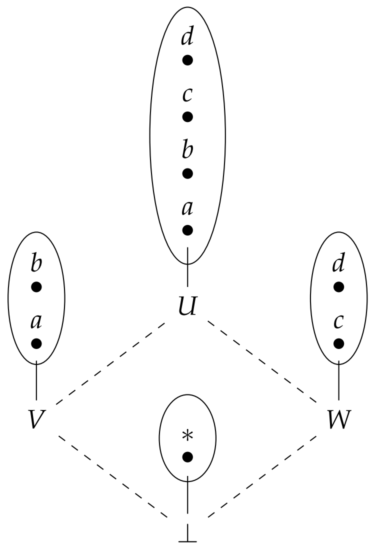
*We can see F's assignments as fibers over* $\mathbb{L}$ *by drawing them over the regions they are assigned to. For instance, over U we have* $F(U)$, *i.e.,* $\{a, b, c, d\}$:



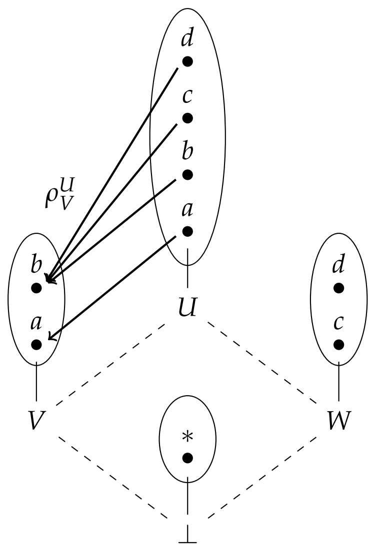

*Similarly, over V and W, we have* $F(V) = \{a, b\}$ *and* $F(W) = \{c, d\}$:



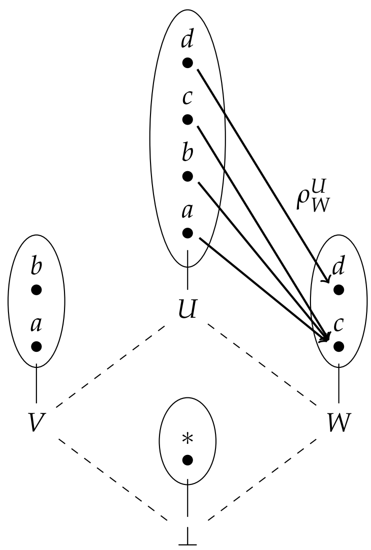

*Finally, over* $\perp$, *we have a singleton set:*



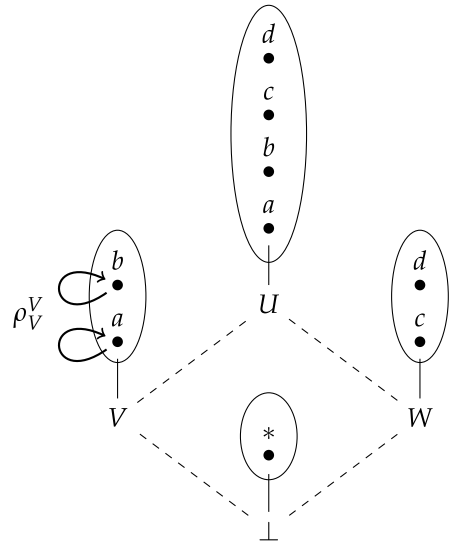

*The restriction maps show how to "zoom in" on the data over each region. For instance, $\rho^U_V$ shows how to restrict the data in the fiber over $U$ down to the data in the fiber over $V$:*



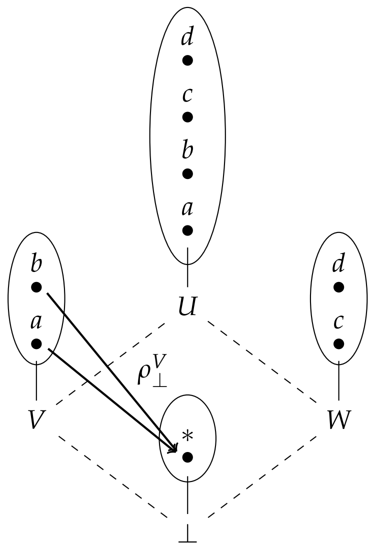

*It's similar for the fiber over $W$:*



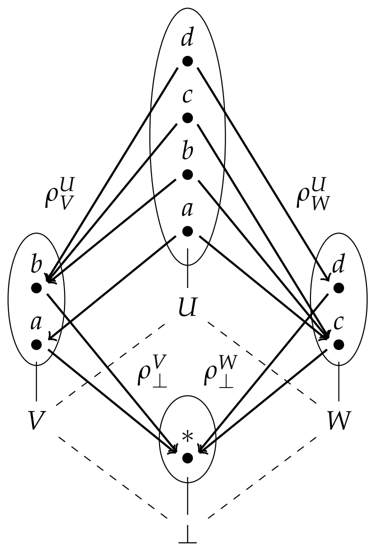
*Restricting a fiber to itself is just the identity on the fiber:*




*The other restriction maps restrict down to the singleton set. For instance:*




*All of this makes it clear that the structure of the presheaf data that sits in the fibers over $\mathbb{L}$ mimics (respects) the structure of the base locale:*



### 2.5. Gluing

The definition of a presheaf requires only that the data be stable under restriction (zooming in on a region). It does not require that the data fit together across different regions (fibers).

In some cases though, certain sections in different fibers turn out to be compatible (i.e., there is a coherent way to patch them together). When this occurs, those compatible sections can be glued together to form sections that stretch across fibers.

To get at this idea, let's first define a cover. A cover of a region $U$ is a selection of sub-regions that covers $U$ in its entirety. The chosen sub-regions don't leave any part of $U$ exposed.

**Definition 7** (Cover)**.** *Let $\mathbb{L}$ be a topology or a locale, and let $U$ be a region of $\mathbb{L}$. A cover of $U$ is a family $\{U_i\}_{i \in I} \subseteq \mathbb{L}$ such that:*

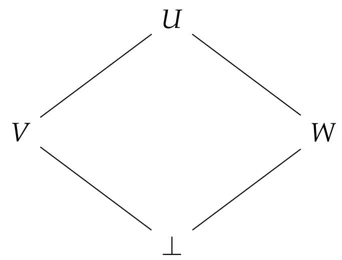
$$U = \bigvee_{i \in I} \{U_i\}.$$

*In other words, a cover of $U$ is a family of regions that join together to form $U$.*

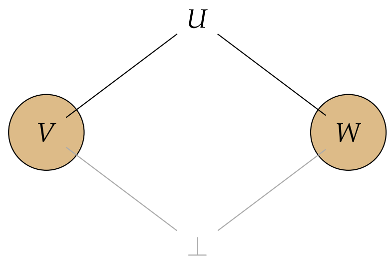
**Example 5.** *Take the topology from Example 1: $T = \{\varnothing, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$. A cover of $\{a,b,c\}$ is $\{a,b\}$ and $\{b,c\}$, because altogether, $\{a,b\}$ and $\{b,c\}$ cover all of the points in $\{a,b,c\}$.*

*Another cover of $\{a,b,c\}$ is $\{\{a,b\},\{b,c\},\{b\}\}$. Although $\{b\}$ is redundant here, this choice of sub-regions still entirely covers $\{a,b,c\}$ as required.*

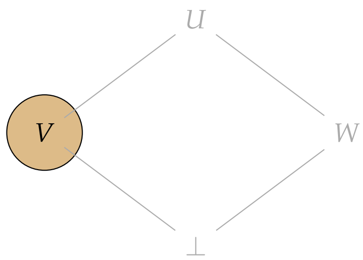
**Example 6.** *In the context of locales, where there are no points, a cover of $U$ is just a selection of sub-regions of $U$ that join together to form $U$. Take the locale from Example 4:*




*A cover of $U$ is $\{V, W\}$, since $U = \bigvee\{V, W\}$:*




*A cover of $V$ is just $\{V\}$:*

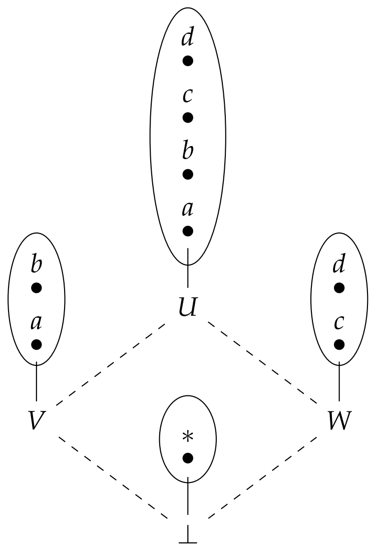


**Remark 7.** *A cover over the least element of a locale (or a topology) is empty (the empty set), because there are no regions (or points) to cover.*

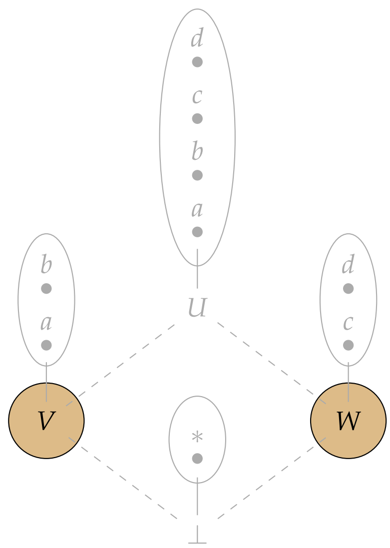
Given a presheaf $F$ over a locale $\mathbb{L}$, if we have a cover $\{U_i\}_{i \in I}$ of some portion of $\mathbb{L}$, there is a corresponding family of fibers $\{F(U_i)\}_{i \in I}$ over that cover. We can pick one section (i.e., one element) from each such fiber to get a slice of elements that spans all of the fibers over that cover. Let us call such a choice a selection of patch candidates.

**Definition 8** (Patch candidates). *Given a presheaf $F$ and a cover $\{U_i\}_{i \in I}$ with a corresponding family of fibers $\{F(U_i)\}_{i \in I}$, a selection of patch candidates $\{s_i\}_{i \in I}$ is a choice of one section $s_i$ from each $F(U_i)$.*

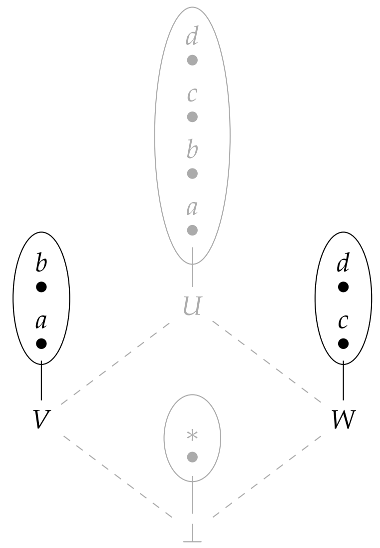
**Example 7.** *Take the presheaf from Example 4:*



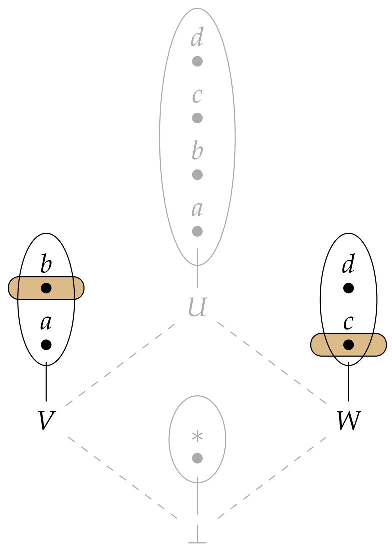

*Let $\{V, W\}$ be the cover of interest:*




*Over this cover, we have a corresponding family of fibers:*




*A selection of patch candidates is a choice of one section (element) from each fiber. For instance, we might pick b from $F(V)$ and c from $F(W)$:*




*Similarly, we might pick $\{a, d\}$, $\{b, d\}$, or $\{a, c\}$, each of which is a valid selection of patch candidates.*

**Example 8.** *Consider the empty cover. Since there are no sub-regions below the least element of a locale, there are no patch candidates we could choose for the empty cover either. Hence, any selection of patch candidates for the empty cover is $\varnothing$.*

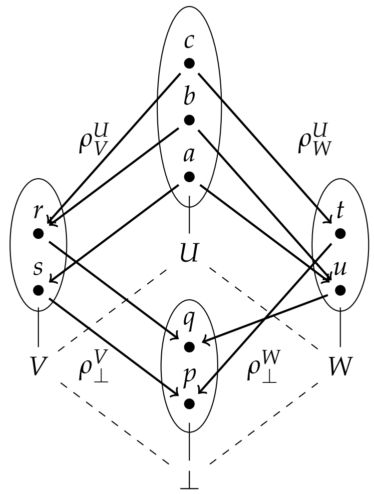
A selection of patch candidates might fit together, or they might not. We say they are compatible if they fit together, i.e., if they agree on overlaps. To check this, take any pair of patch candidates, and check if they restrict to the same data on their overlap.

**Definition 9** (Compatible patch candidates)**.** *Given two fibers $F(U_i)$ and $F(U_j)$ and a patch candidate from each, $s_i \in F(U_i)$ and $s_j \in F(U_j)$, $s_i$ and $s_j$ are compatible if they restrict to the same data on their overlap $U_i \wedge U_j$:*

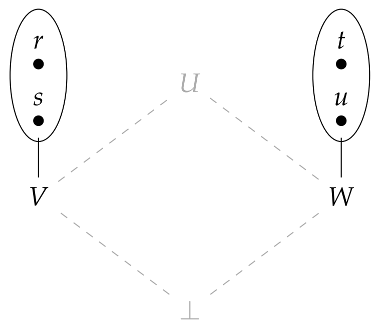
$$\rho^{U_i}_{U_i \wedge U_j}(s_i) = \rho^{U_j}_{U_i \wedge U_j}(s_j).$$

*A selection of patch candidates $\{s_i\}_{i \in I}$ is compatible if all of its members are pair-wise compatible.*

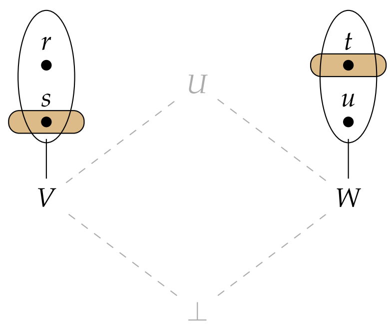
**Example 9.** *Consider the following presheaf $F$:*



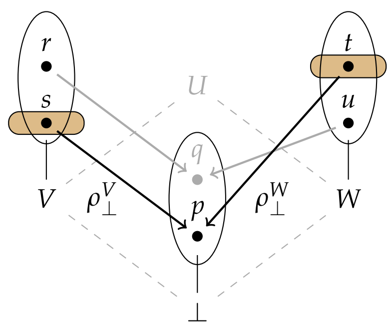

*Take the cover $\{V, W\}$ and its corresponding fibers:*



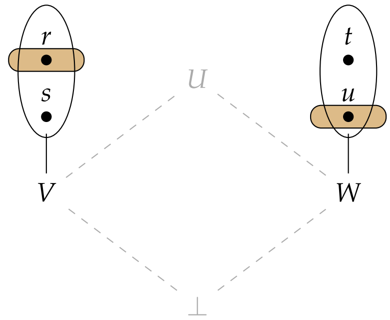

*Suppose we pick $\{s, t\}$ for patch candidates:*



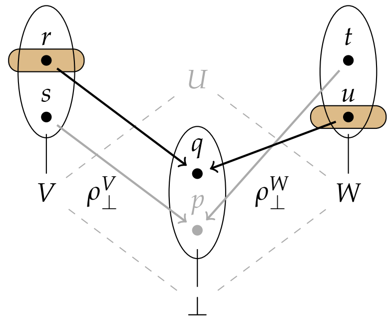

*Is this selection compatible? We have to check if they agree on their overlap. The overlap $V \wedge W$ is $\perp$. Where does $\rho^V_\perp$ send our chosen patch candidate $s$? It sends it to $p$, since $\rho^V_\perp(s) = p$. Where does $\rho^W_\perp$ send our other chosen patch candidate $t$? It also sends it to $p$, since $\rho^W_\perp(t) = p$. On the overlap $\perp$ then, $\rho^V_\perp(s) = \rho^W_\perp(t)$, so $s$ and $t$ are compatible. This is easy to see in the diagram, since $s$ and $t$ both get sent to the same place:*



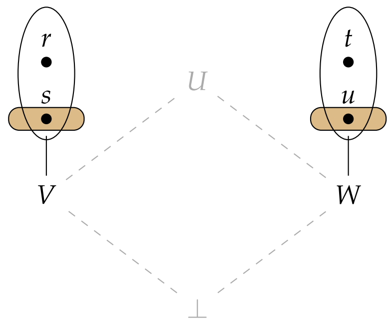
*Now suppose we pick $\{r, u\}$ for patch candidates:*



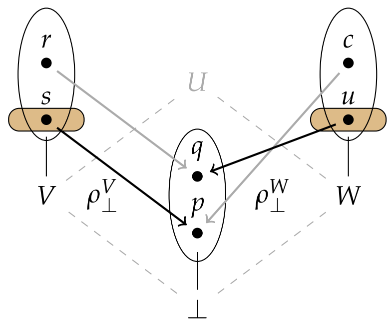

*These are also compatible. They agree on their overlap (both restrict to $q$):*




*Finally, suppose we pick $\{s, u\}$ for patch candidates:*




*These are not compatible. They do not agree on their overlap:*




**Example 10.** *Consider the empty cover. Since any selection of patch candidates for the empty cover is empty, compatibility is satisfied vacuously.*

When selected patch candidates $s_i, \ldots, s_k$ across a cover of $U$ are compatible, we say those patches glue together if if there's a section $s$ in $F(U)$ that restricts down to exactly those patches.

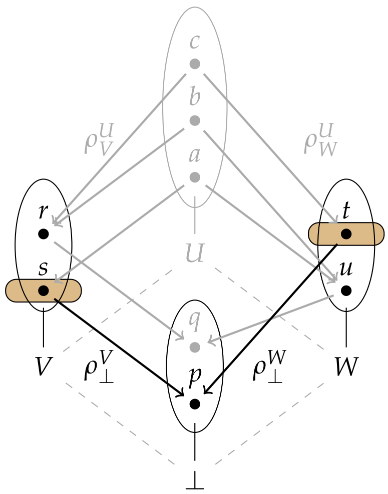
**Definition 10** (Gluing). *Given a presheaf $F$ and a selection of compatible patch candidates $\{s_i\}_{i \in I}$ for a cover $\{U_i\}_{i \in I}$, $\{s_i\}_{i \in I}$ glue together only if there is a section $s \in F(U)$ that restricts down to $s_i$ on each fiber $F(U_i)$ of the cover, i.e., only if $s$ is such that:*

$$\rho^{U}_{U_i}(s) = s_i, \text{ for each } i \in I.$$

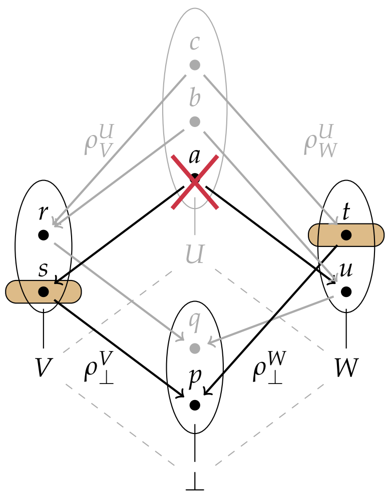
*A selection of patches $\{s_i\}_{i \in I}$ glues uniquely if there is one and only one such section $s \in F(U)$ that is glued from them.*

**Remark 8.** *As a matter of terminology, if a section $s \in F(U)$ is glued from patches $\{s_i\}_{i \in I}$, we say that $s$ is a global section of the cover, and each $s_i$ is a local section of the cover. We may also say variously that $s$ is a* gluing *of those patches, that $s$ is* composed *of those patches, that those patches* compose *$s$, or that gluing those patches* yields *$s$.*

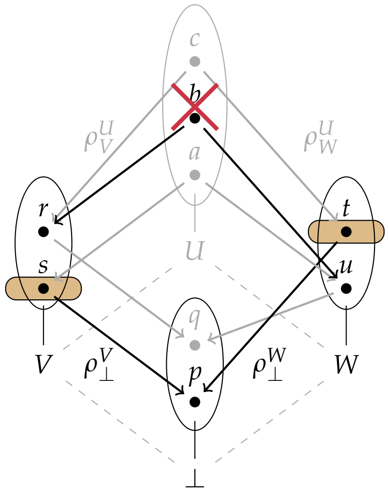
**Example 11.** *Take the presheaf from Example 9, and consider the cover $\{V, W\}$ again. Take the selection of patches $\{s, t\}$, which are compatible because they agree on overlap:*



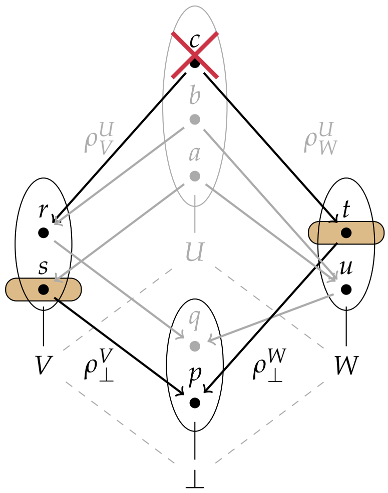

*Even though $s$ and $t$ are compatible, they do not glue together, because there is no section in $F(U)$ that restricts down to them. Consider $a \in F(U)$ first. It restricts to $s \in F(V)$ on the left, but it does not restrict to $t \in F(W)$ on the right:*




*As for $b \in F(U)$, it restricts to neither $s \in F(V)$ on the left nor $t \in F(W)$ on the right:*

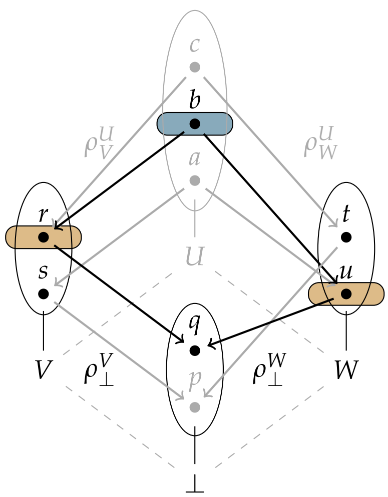



*Finally, $c \in F(U)$ restricts to $t \in F(W)$ on the right, but not to $s \in F(V)$ on the left:*



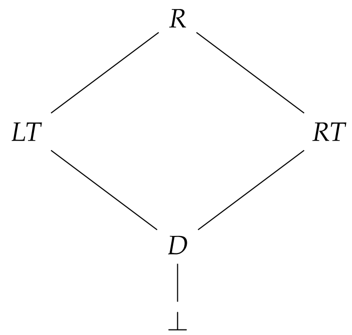

*Thus, none of $a$, $b$, or $c$ in $F(U)$ are glued from $\{s,t\}$, because none of them decompose into $s$ on the left and $t$ on the right.*

*Now suppose we pick $\{r,u\}$ for patch candidates. These do glue together, because there is a section in $F(U)$ (namely $b \in F(U)$) that restricts down to $r \in F(V)$ on the left and $u \in F(W)$ on the right:*



**Example 12.** *Consider an example that glues together behaviors. Imagine a toy robot that looks something like a small tank: it has tracks on the left and right sides, and the two tracks are connected by a single drive controller. The controller either drives at a constant speed, or it sits idle. When it drives, it turns both tracks at the same speed.*

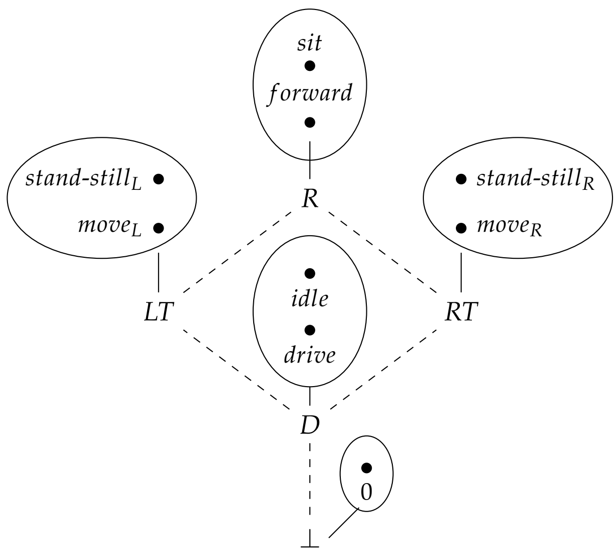
*Let's represent the robot as a locale. Let $LT$ and $RT$ be the left and right track assemblies respectively, let $D$ be the drive controller that is shared by $LT$ and $RT$, and let $R$ be the whole robot (the join of $LT$ and $RT$). As a picture:*




*For a presheaf, let's assign to each region the behaviors that are locally observable at that region:*

- *The drive controller $D$ can either drive or sit idle.*
- *The left track assembly can either $move_L$ or $stand\text{-}still_L$.*
- *The right track assembly can also either $move_R$ or $stand\text{-}still_R$.*
- *The entire robot can either move $forward$ or sit stationary.*
- *For the fiber over $\perp$, where there are no regions that could carry any behaviors to begin with, assign the special symbol zero.*

*In a picture:*

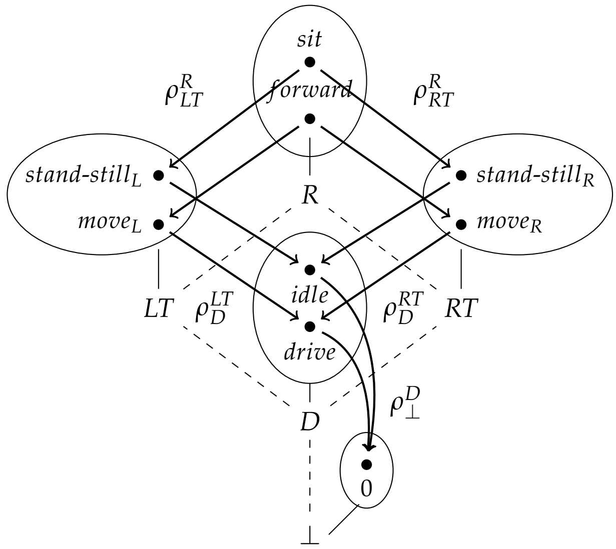



*For the restriction maps, let's say that they restrict the observable behavior of a larger region to the observable behavior of the smaller region. For instance, if you are observing the whole robot moving forward ($forward$), and you then "zoom in" on the left track assembly, you'll see those tracks rotating ($move_L$).*

- $\rho^R_{LT}(sit) = stand\text{-}still_L$, $\rho^R_{LT}(forward) = move_L$.
- $\rho^R_{RT}(sit) = stand\text{-}still_R$, $\rho^R_{RT}(forward) = move_R$.
- $\rho^{LT}_D(stand\text{-}still_L) = idle$, $\rho^{LT}_D(move_L) = drive$.

- $\rho^{RT}_{D}(\textit{stand-still}_R) = \textit{idle}$, $\rho^{RT}_{D}(\textit{move}_R) = \textit{drive}$.
- $\rho^{D}_{\perp}(\textit{idle}) = \rho^{D}_{\perp}(\textit{drive}) = 0$.

*In a picture:*




*Now take the cover $\{LT, RT\}$ of R. The patch candidates $\{move_L, move_R\}$ are compatible, because they agree on overlap (they both restrict down to drive). But they also glue uniquely, yielding $forward$. In other words, the robot's forward motion is patched together precisely from the two pieces of its cover, namely the left tracks rotating ($move_L$) and the right tracks rotating ($move_R$).*

*Similarly, the Robot's sitting still (sit) behavior is also glued from the two pieces of its cover, namely the left track assembly standing still (stand-still$_L$) and the right track assembly standing still (stand-still$_R$).*

*Thus, there are two global sections of R's behavior: moving forwards (patched together from its left and right motions), or standing still (patched together from its left and right lack of motion).*

### *2.6. Monopresheaves*

In a presheaf, compatible patch candidates can be glued together to form sections that span multiple fibers. However, nothing said so far prevents there being multiple gluings from the same patch candidates. In other words, nothing requires gluings to be extensional.

If we want to work only with extensional gluings, then we can impose a restriction that says gluings must be unique: i.e., if a selection of patch candidates can glue, they form at most one gluing. Presheaves where this obtains are called monopresheaves.

**Definition 11** (Monopresheaves)**.** *A presheaf F is a monopresheaf iff it satisfies the following gluing-uniqueness condition:*

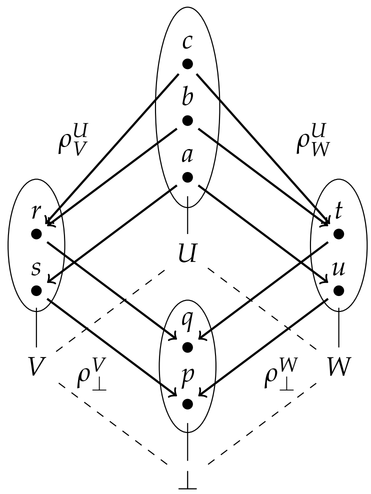
*(G1) For every cover $\{U_i\}_{i\in I}$ of a region U and every selection of compatible patch candidates $\{s_i\}_{i\in I}$ for that cover, if there is a gluing $s \in F(U)$ of $\{s_i\}_{i\in I}$, then it is unique.*

*Equivalently, given $s, t \in F(U)$:*

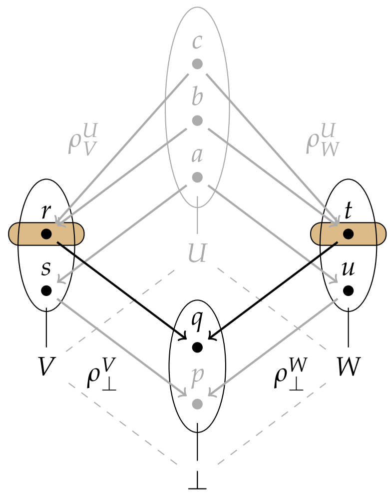
*(G1*)For every cover $\{U_i\}_{i\in I}$ of U, if $\rho^U_{U_i}(s) = \rho^U_{U_i}(t)$ for each $U_i$, then $s = t$.*

**Remark 9.** *Monopresheaves are also called separated presheaves. The "mono" part of the name comes from category theory: every joint restriction to a covering family is a monomorphism, i.e., there can be at most one section restricting to a given selection of patch candidates.*

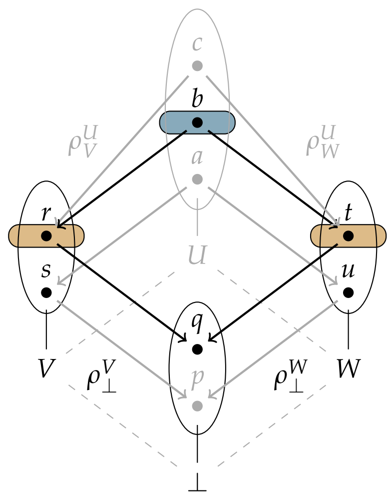
**Example 13.** *By way of counter-example, consider the following presheaf:*



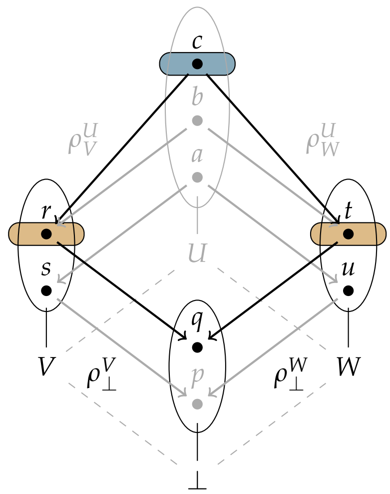
*Take the cover* $\{V, W\}$ *and selection of patch candidates* $\{r, t\}$*, which are compatible because they agree on overlap:*



*Do r and t glue? That is to say, is there a section in* $F(U)$ *that decomposes exactly to r and t? In this case,* $b \in F(U)$ *is a gluing of r and t:*



*However, b is not a unique gluing, since c is also a gluing of r and t:*




*Thus, this is not a monopresheaf, since gluable patch candidates don't glue uniquely.*

**Example 14.** *On the other hand, the presheaf from Example 9 is a monopresheaf, for whenever patch candidates glue together in that presheaf, they do so uniquely.*

### *2.7. Sheaves*

The definition of a monopresheaf requires only that if compatible patch candidates glue, they do so uniquely. It does not require that compatible patch candidates always do glue together. Patch candidates in a monopresheaf need not glue.

If we want to work with monopresheaves where all gluable patch candidates do in fact glue together, then we can work with sheaves. A sheaf is a monopresheaf that satisfies an extra existence requirement: whenever patch candidates *can* glue, they *do* glue.

**Definition 12** (Sheaf)**.** *A monopresheaf $F$ is a sheaf iff it satisfies the following gluing-existence condition:*

(G2) *For every cover $\{U_i\}_{i\in I}$ of a region $U$ and every selection of patch candidates $\{s_i\}_{i\in I}$ for that cover, if $\{s_i\}_{i\in I}$ are compatible, then there exists a unique gluing $s \in F(U)$ of $\{s_i\}_{i\in I}$.*

**Example 15.** *The presheaf from Example 9 fails to be sheaf, because as we saw in Example 11, there is a compatible selection of patch candidates (namely, $\{s, t\}$) which fails to glue. To be a sheaf, every compatible selection of patch candidates must glue.*

There is a subtlety regarding what sheaves look like over the least element of a locale. Note that the gluing condition is formulated as an implication. That is to say, it says that, for every cross-section of patch candidates, *if* that cross-section can glue, *then* it glues in exactly one way.

Next, consider the fact that the cover over the least region of a locale is an empty cover. Since there are no patch candidates that need to be checked for compatibility, there is nothing that needs to be done to get a "selection of gluable patch candidates." Hence, the antecedent of the gluing condition is satisfied vacuously over the least element of the locale.

But since the empty cover satisfies the antecedent of the gluing condition vacuously, it follows that if a presheaf is to qualify as a sheaf, it must ensure that the consequent is satisfied over the empty cover as well. In other words, it must assign a unique glued section (a singleton set) to the least region of the locale. So, even though a *presheaf* or *monopresheaf* may assign a larger set of data to the least element of a locale, a *sheaf* always assigns a singleton to that region.

**Example 16.** *The presheaf from Example 12 is a sheaf. Note that the bottom fiber is a singleton. This ensures that* all *gluable selections of patch candidates (including the empty one) glue uniquely.*

If we consider presheaves, monopresheaves, and sheaves together, we see that we have a hierarchy of increasingly strict gluing requirements. (1) Presheaves have no gluing requirements. (2) Monopresheaves have a uniqueness requirement: gluings need not exist, but when they do, they are unique. (3) Sheaves have both a uniqueness and an existence requirement: gluings exist whenever possible, and they are unique.

### *2.8. The bottom fiber*

The bottom element of a locale represents no regions at all. Thus, it plays a special role. Since it represents the *absence* of any regions, the data that we assign to its fiber is of a different kind than the data we assign to other fibers.

The other fibers sit over genuine regions in the parts space, so the data we assign to them plays a kind of ontological role: it's the "stuff" that occupies that region. By contrast, the fiber over $\perp$ cannot play this role. Since $\perp$ represents no region at all, its fiber cannot represent material occupancy. Instead, it plays a structural role.

Since any compatibility check between patch candidates ultimately factors through restriction maps that ultimately land in the fiber over $\perp$, agreement at $\perp$ functions as a final anchor point. Given this fact, we can make a general observation: there are as many kinds or modes of gluing as there are anchor points over $\perp$.

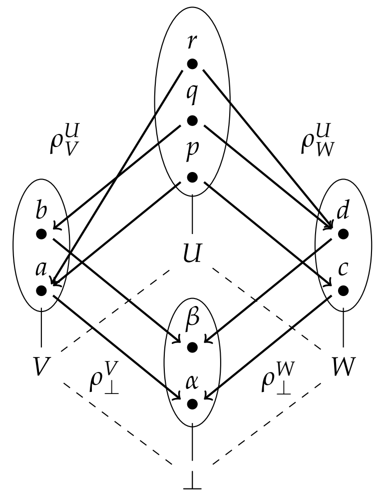
If there is a single point in the fiber over $\perp$, then there will be only one kind of gluing that occurs throughout the presheaf. This is most evident in a sheaf, which as we saw requires a singleton over $\perp$. This makes sense, because in a sheaf, gluing must be consistent and uniform throughout, and so all gluing has to anchor to a single point.

If we move to presheaves, and hence relax our gluing constraints, then we can have multiple points in the fiber over $\perp$. These will correspond to multiple kinds or modes of gluing. This also makes sense, since multiple gluings can only exist in a structure with weaker gluing conditions than we find in a sheaf.

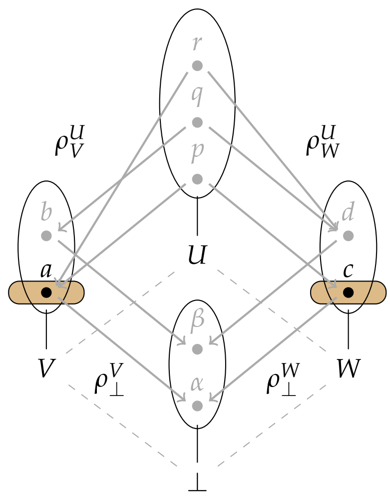
**Example 17.** *Consider the following simple locale:*

$$\begin{tikzcd}
 & U \arrow[dl, dash] \arrow[dr, dash] & \\
V \arrow[dr, dash] & & W \arrow[dl, dash] \\
 & \perp &
\end{tikzcd}$$

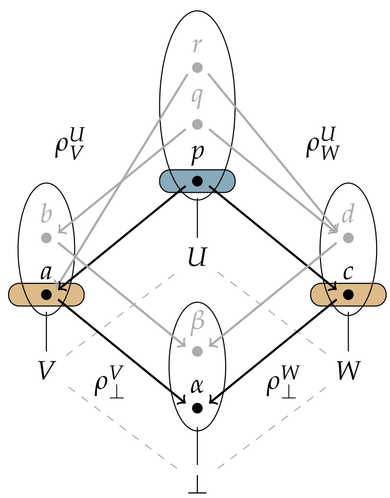
*Now consider a presheaf with more than one anchor over* $\perp$*:*



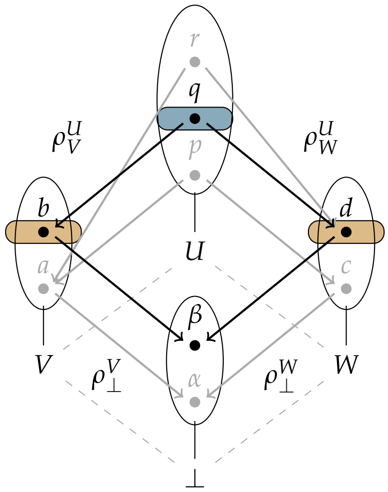

*For the cover* $\{V, W\}$ *of* $U$*, take patch candidates* $a$ *and* $c$*:*




*These glue at* $p$*, and are anchored to* $\alpha$*:*




*Similarly, the patch candidates* $b$ *and* $d$ *glue at* $q$*, and anchor at* $\beta$*:*




*The patch candidates a and d cannot glue, because their restrictions to $\perp$ do not agree. Here is another place where sheaf theory controls coherence: it prevents gluing in more than one mode at the same time. Glued sections must be glued consistently (in whatever mode they anchor to over $\perp$).*

## 3. Modeling Part-Whole Complexes as Presheaves

As noted in Section 1, the central claim of this paper is that we can model part-whole complexes as presheaves over locales, with varying gluing conditions. In particular, the locale provides the abstract parts space of "regions" that the pieces can occupy, the presheaf assigns actual pieces to those regions, and the gluings determine which pieces fuse.

We can thus define the core mereological concepts of part and whole in sheaf-theoretic terms. Regarding wholes, we can identify fusion with gluing: to say that some pieces fuse or form a "fusion" is just to say that they are glued together. Regarding parts, to say that a piece is a "part" is just to say that it is a part of a fusion. In other words, the parts of a fusion are just the pieces from which it is glued together.

**Definition 13** (Fusions and parts)**.** *We say that a section $s \in F(U)$ is a fusion iff there exists a cover $\{U_i\}_{i \in I}$ of $U$ and a selection of patch candidates $\{s_i\}_{i \in I}$ such that:*

$$\rho^U_{U_i}(s) = s_i, \quad \textit{for each } U_i.$$

*Given $t \in F(V)$ and $s \in F(U)$ with $V \preccurlyeq U$ and $V \neq \perp$, we say $t$ is a part of $s$, denoted $t \sqsubseteq s$, iff $s$ is a fusion and:*

$$\rho^U_V(s) = t.$$

Because fusions do not freely arise here, but rather only exist where parts are explicitly glued together, sheaf theory thus provides a systematic framework with which to model a large variety of part-whole complexes in a "fusions-first" manner. In the rest of this section, we illustrate with examples. In each case, we construct a custom presheaf designed to model a particular part-whole complex. Our choices of presheaves should be interpreted as modeling choices. One could construct different presheaves, and each can be evaluated on its own merits.

**Example 18.** *Consider a building with a west room, an east room, and a hallway between them. For simplicity, let us consider only the floors of the building (ignore walls, ceilings, and so on). The ambient locale is given by the presentation*

- $\mathbb{L} = \langle G, R \rangle = \langle \{WR, H, ER\}, \varnothing \rangle$

*where*

- $WR$ = *west room*
- $H$ = *hallway*
- $ER$ = *east room*

*As a Hasse diagram:*

$$
\begin{tikzcd}
& WR \vee H \vee ER \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
WR \vee H \arrow[d, dash] \arrow[dr, dash] & WR \vee ER \arrow[dl, dash] \arrow[dr, dash] & H \vee ER \arrow[dl, dash] \arrow[d, dash] \\
WR \arrow[dr, dash] & H \arrow[d, dash] & ER \arrow[dl, dash] \\
& \perp &
\end{tikzcd}
$$

*For the presheaf $F$, let it assign to each region whatever materials (if any) cover its floor uniformly. Let us say that the west room's and hallway's floors are each covered uniformly by wood, while the east room's floor is covered uniformly by tiles:*

- $F(WR) = \{wood\}$
- $F(H) = \{wood\}$
- $F(ER) = \{tile\}$

*Since the west room's and hallway's floors are covered uniformly by wood, their join is too:*

- $F(WR \vee H) = \{wood\}$

*Since none of the other regions are covered uniformly by the same material, we assign nothing to them:*

- $F(WR \vee ER) = \varnothing$
- $F(H \vee ER) = \varnothing$
- $F(WR \vee H \vee ER) = \varnothing$

*Finally, for the fiber over bottom, where there are no regions to cover with materials, let us assign the special symbol zero:*

- $F(\perp) = \{0\}$

*For the restrictions, let us say that the materials that cover a larger region are restricted down to the materials that cover the smaller region. Hence, the non-empty fibers restrict by identity:*

- $\rho_{WR}^{WR \vee H}(wood) = \rho_{H}^{WR \vee H}(wood) = wood$

*The empty fibers restrict via the empty function (there is nothing to restrict):*

- $\rho_{WR \vee H}^{WR \vee H \vee ER} = \rho_{WR \vee ER}^{WR \vee H \vee ER} = \rho_{H \vee ER}^{WR \vee H \vee ER} =$ *empty function*
- $\rho_{WR}^{WR \vee ER} = \rho_{ER}^{WR \vee ER} =$ *empty function*
- $\rho_{H}^{H \vee ER} = \rho_{ER}^{H \vee ER} =$ *empty function*

*Finally, fibers restrict to bottom via the constant function:*

- $\rho_{\perp}^{WR}(wood) = \rho_{\perp}^{H}(wood) = \rho_{\perp}^{ER}(tile) = 0$

*In this building, there are two maximal fusions:*

- *The flooring of the west room and the hallway glue into one piece that covers both.*
- *The flooring that covers the east room.*

*Thus, the flooring of this building is really a collection of two independent fusions: the wooden floor that covers the west room and hallway, and the tiled floor that covers the east room. That implies:*

- *To separate the floors of the west room and hallway, you would have to use a saw to cut them, since they are fused. They are not merely sitting next to each other. Rather, they make up a single (fused) piece.*
- *By contrast, to separate the hallway and the east room, you would not need to cut them, since they are not fused. They simply happen to be sitting next to each other.*

*The parts of the fusions are clear:*

- *The wooden floor that covers the west room and the hallway has two parts: the wooden floor that covers the west room, and the wooden floor that covers the hallway.*
- *The tiled floor of the east room has no parts (in this locale), since it is not the fusion of other fusions.*

*This is particular example fails to be a sheaf, because everything that can glue does not glue. In particular,* $F(WR \vee H \vee ER)$ *is covered by* $\{WR, H, ER\}$*, and the patch candidates* $\{wood, wood, tile\}$ *are compatible (they pair-wise restrict to* $0$*). However, there is nothing in* $F(WR \vee H \vee ER)$ *that is glued together from those patch candidates (indeed,* $F(WR \vee H \vee ER)$ *is empty). Hence, this is not a sheaf. Rather, it is a monopresheaf.*

*But this is precisely what one would expect when modeling two discrete pieces of flooring that happen to sit next to each other. The west room and hallway do glue together here, as expected. But the maximal wood and tile pieces do not glue together, also as expected (there is a boundary between them, where the wood ends and the tile begins).*

In the previous example, none of the regions overlapped. The presheaf was free to glue or not glue pieces as it saw fit. The story is different if there are overlaps in the locale itself. Overlaps in the locale require overlaps in the presheaf, wherever you want gluings.

**Example 19.** *Consider the floor of a single room. Let us say that the regions of interest are its west half, its east half, and a six inch span where they overlap.*

*The ambient locale of this kind of space can be given by the presentation*

- $L = \langle G, R \rangle = \langle \{\bot, WH, O, EH\}, \{\bot \preccurlyeq O, O \preccurlyeq WH, O \preccurlyeq EH\} \rangle$

*where*

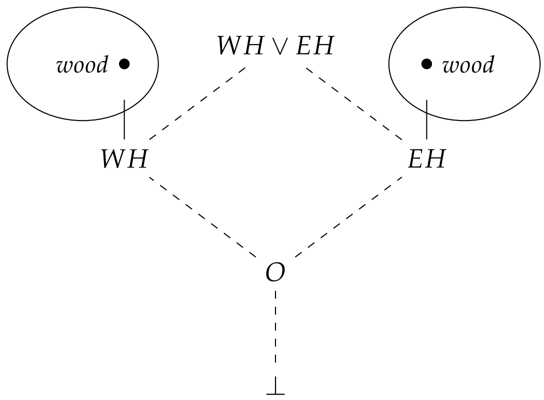
- $WH$ *= west half*
- $O$ *= overlap*
- $EH$ *= east half*

*For a presheaf F, let us say that it behaves much like in the previous example: it assigns to each region the materials (if any) that cover the floor uniformly.*

*For instance, let us assign wood to both halves:*

- $F(WH) = \{wood\}$
- $F(EH) = \{wood\}$

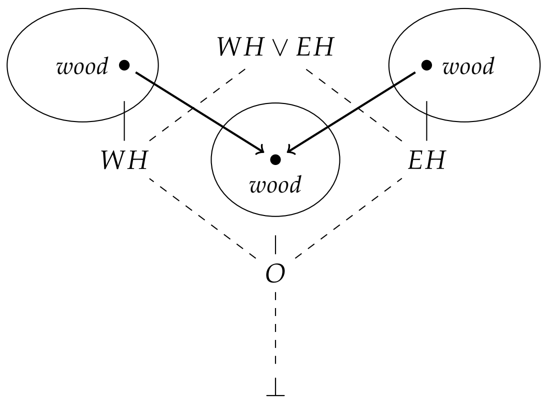
*In a picture:*




*By construction $O = WH \wedge EH$, and the two halves restrict to the same material there:*

- $F(O) = \{wood\}$

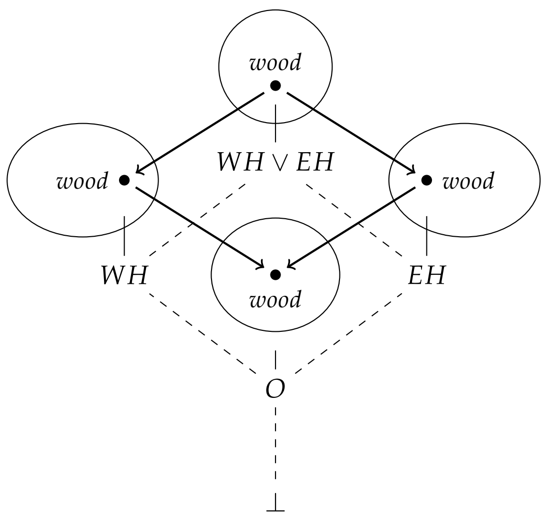
*Thus:*




*For the join, the west and east halves glue, since they're made from the same flooring materials and agree on their overlap:*

- $F(WH \vee EH) = \{wood\}$

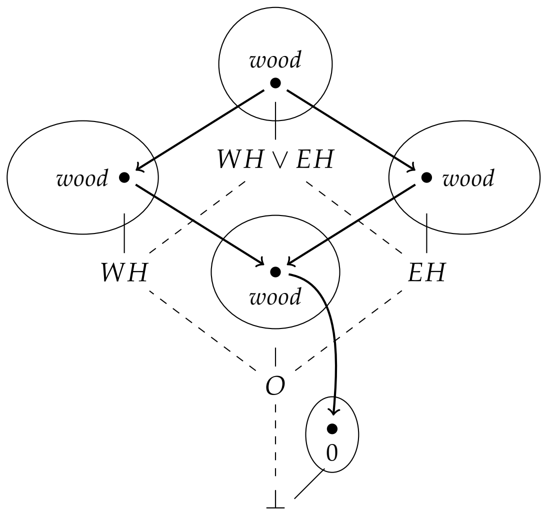
*Thus:*




*Finally, for the fiber over bottom, where there are no regions to cover with materials, assign the special symbol zero:*

- $F(\perp) = \{0\}$

*In a picture:*



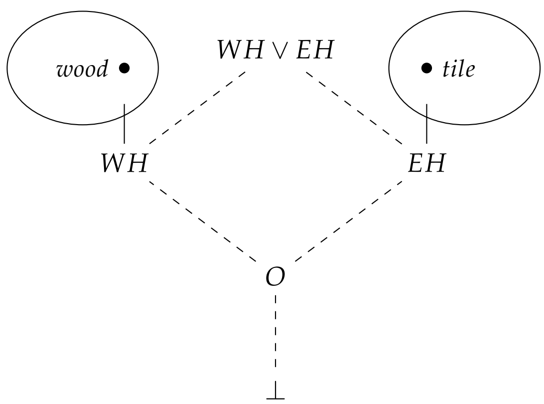

*The maximal fusion here is a single piece of wooden flooring (namely, wood* $\in F(WH \vee EH)$*) that covers the whole room. Its parts are the west and east halves, and (transitively) their overlap. The west and east halves themselves have a shared part, the strip of overlap.*

*This particular example is a sheaf: the parts glue together coherently in a single manner across the entire parts space. This is exactly as one would expect when modeling a floor that is covered uniformly in its entirety by wood flooring: as you restrict down to smaller parts of the room, you get smaller pieces of wood flooring. In contrast to Example 18, here the regions have a nontrivial overlap. By the sheaf condition, agreement on that overlap forces a unique fusion of the parts, just as expected.*

**Example 20.** *To illustrate a failed attempt to build a sheaf, let us take the locale and gluing condition from Example 19, but suppose that we assign different flooring materials to the east and west halves of the room:*

- $F(WH) = \{wood\}$
- $F(EH) = \{tile\}$

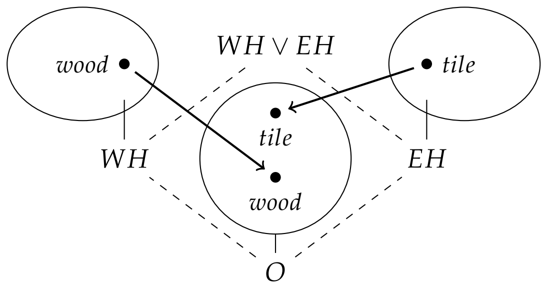
*As a picture:*




*Next, at the overlap, allow both wood and tile:*

- $F(O) = \{wood, tile\}$
- $\rho_O^{WH}(wood) = wood$
- $\rho_O^{EH}(tile) = tile$

*Thus, as a picture:*

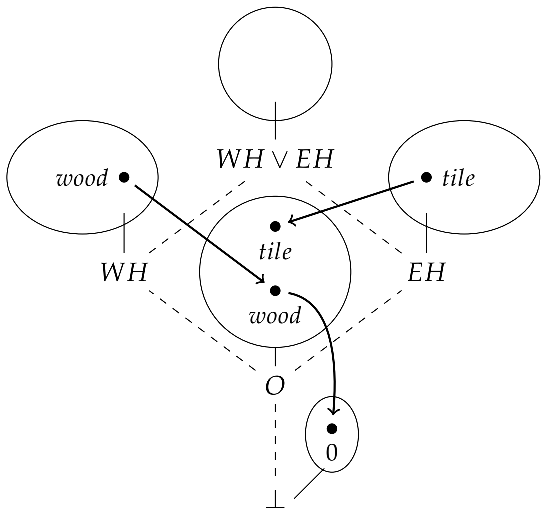



*Note, however, that the wood and tile in the west and east halves cannot glue, because they do not agree on their overlap. Hence, there cannot be a single kind of flooring material that uniformly covers the maximal join $WH \vee EH$. This illustrates how the sheaf condition is a very strong condition, but also a helpful one: it requires and manages coherent gluing at all levels. Because it requires that pieces glue together coherently at every level of "zoom," it prevents us from ever putting together an incoherent part-whole sheaf in the first place.*

*It is worth spelling the failure out explicitly. If we want to model this room as a sheaf, then we are requiring coherent, unique gluing across all of the regions: wherever pieces can coherently glue together, they must do so. But here, since $WH$ and $EH$ disagree on their overlap, there is no way to make them cohere into a single piece.*

*Intuitively, this makes sense. If the western and eastern halves of a room were truly floored with different materials, then they could not overlap. Imagine if two builders started at opposite ends of the room: one flooring with wood and the other flooring with tile. When they reach the mid-point, they'd realize they made a mistake. In such a scenario, it would be impossible to complete the original vision of having a single, uniform flooring across the entire room.*

*However, what if we weaken our requirements and consider this as a presheaf? In a presheaf, nothing disallows the two halves from restricting differently on an overlap. The full presheaf looks like this:*




*Interpreted as a presheaf, there is a sensible interpretation of this structure. Nothing sits in the fiber over $WH \vee EH$, since a single coherent piece of flooring cannot be glued from wood and tile. The wood and tile from the two halves each extend into the overlap though, but since they don't agree, one of them must sit on top of the other in that overlapping area. It is like when two area rugs overlap: one sits on top of the other.*

*As a final point, note that this fails to be monopresheaf, because there are two sections in $F(O)$ that restrict to $\perp$. The gluings are trivial here, but nonetheless, gluing is not unique precisely in the overlap. Hence, this is a presheaf, but not a monopresheaf. This illustrates how presheaves allow multiple assemblies of parts, but monopresheaves do not.*

The previous two examples were spatial. But parts come in non-spatial guises too, and sheaf theory can model them just as well.

**Example 21.** *Suppose we say that human society (under some description) consists of the mesh of a specified set of relationships between the people that participate in that society.*

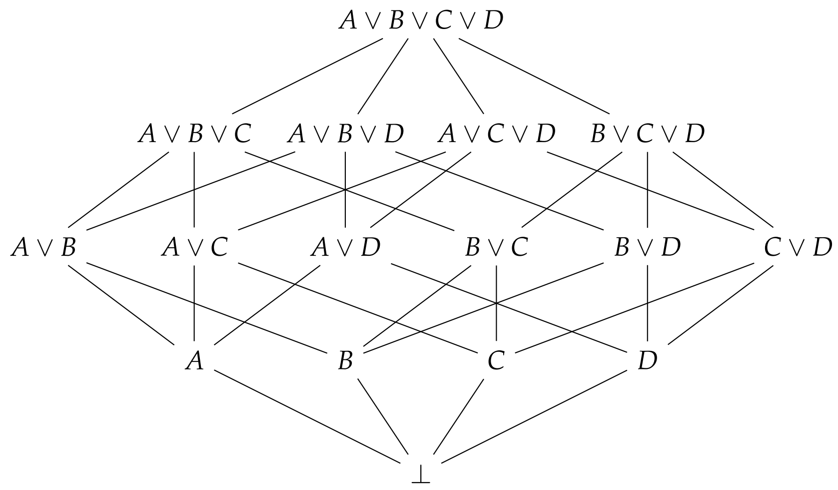
*Let $P$ be the population in question (a finite set of individual people), and let the regions of our locale be subsets of such individuals. Then the ambient locale is given by the presentation:*

- $\mathbb{L} = \langle G, R \rangle = \langle P, \varnothing \rangle$

*For concreteness, suppose:*

- $P = \{A, B, C, D\}$, *with $A$ short for Alice, $B$ for Bob, $C$ for Carol, and $D$ for Denny.*

*Then the Hasse diagram is isomorphic to the powerset of $P$:*




*Let us next define a presheaf $F$ that models the mesh of a selected set of relationships over $P$. To do that, let us first specify a set $R$ that picks out the (binary, symmetric) relationships of interest:*

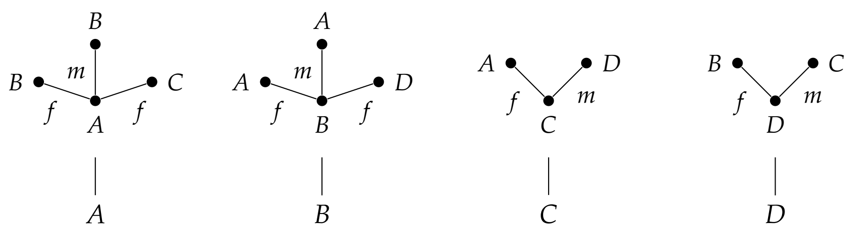
- $R = \{f, m, \ldots\}$, *with $f$ short for being friends, $m$ for being married, etc.*

*For convenience, if $U, V \in P$, $r \in R$, and $U$ and $V$ stand in relationship $r$, we will write $r(U, V)$.*

*For the generators, let us fix a choice of local data by assigning to each person the relations they stand in, e.g.:*

- $F(A) = \{\langle\{f(A,B), f(A,C), m(A,B)\}\rangle\}$
- $F(B) = \{\langle\{f(B,A), m(B,A), f(B,D)\}\rangle\}$
- $F(C) = \{\langle\{f(C,A), m(C,D)\}\rangle\}$
- $F(D) = \{\langle\{f(D,B), m(D,C)\}\rangle\}$

*To visualize this data, we can picture each fiber as a mini-graph:*




*For example, in the fiber over $A$:*

- *The $f$-labeled edge from $A$ to $B$ represents $f(A, B)$: $A$ and $B$ are friends.*
- *The $m$-labeled edge from $A$ to $B$ represents $m(A, B)$: $A$ and $B$ are married.*

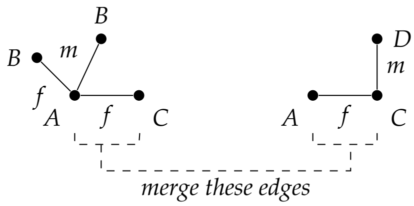
- *The $f$-labeled edge from $A$ to $C$ represents $f(A, C)$: $A$ and $C$ are friends.*

*In the bottom fiber, over the region of no people to carry relationships, assign the special symbol* $0$:

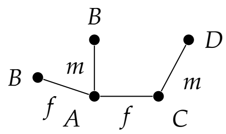
- $F(\bot) = \{0\}$

*Next, let us extend the above data to binary joins by merging mini-graphs along shared edges, wherever the components share exactly the same edges. To see how this works, consider (for example) the mini-graphs over $A$ and $C$:*

*Can these be merged? The answer is yes, because they share exactly one edge, namely the one labeled $f$. If you rotate the graphs sideways a bit, you can see how they can be merged along $f(A, C)$:*



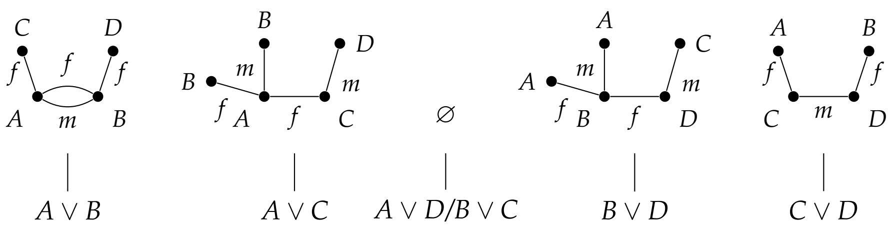

*Merging along $f(A, C)$ yields the following graph:*



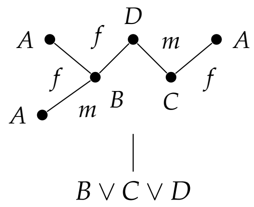
*By merging all binary joins in this fashion, we get:*

- $F(A \vee B) = \{\langle\{f(A,B), m(A,B), f(A,C)\}, \{f(B,A), m(B,A), f(B,D)\}\rangle\}$
- $F(A \vee C) = \{\langle\{f(B,A), m(B,A), f(B,D)\}\rangle, \{f(C,A), m(C,D)\}\}$
- $F(A \vee D) = \varnothing$
- $F(B \vee C) = \varnothing$
- $F(B \vee D) = \{\langle\{f(B,A), m(B,A), f(B,D)\}, \{f(D,B), m(D,C)\}\rangle\}$
- $F(C \vee D) = \{\langle\{f(C,A), m(C,D)\}, \{f(D,B), m(D,C)\}\rangle\}$

*As pictures:*




*Having merged the graphs from joins of two regions, we must next merge the graphs of joins from three regions. For instance, take $B \vee C \vee D$. We can merge the graphs of $B \vee C$ trivially (because they share no edges), we can merge the graphs of $C \vee D$ along their shared $f$-edge, and we can merge the graphs of $B \vee D$ along their shared $f$-edge. That yields:*



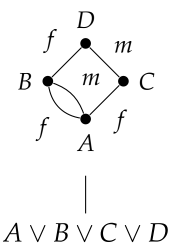

*By merging the graphs of all joins of three regions in this fashion, we get:*

- $$F(A \vee B \vee C) = \left\{ \begin{array}{l} \langle \quad \{f(A,B), f(A,C), m(A,B)\}, \\ \quad\ \ \{f(B,A), m(B,A), f(B,D)\}, \\ \quad\ \ \{f(C,A), m(C,D)\} \quad \rangle \end{array} \right\}$$
- $$F(A \vee B \vee D) = \left\{ \begin{array}{l} \langle \quad \{f(A,B), f(A,C), m(A,B)\}, \\ \quad\ \ \{f(B,A), m(B,A), f(B,D)\}, \\ \quad\ \ \{f(D,B), m(D,C)\} \quad \rangle \end{array} \right\}$$
- $$F(A \vee C \vee D) = \left\{ \begin{array}{l} \langle \quad \{f(A,B), f(A,C), m(A,B)\}, \\ \quad\ \ \{f(C,A), m(C,D)\}, \\ \quad\ \ \{f(D,B), m(D,C)\} \quad \rangle \end{array} \right\}$$
- $$F(B \vee C \vee D) = \left\{ \begin{array}{l} \langle \quad \{f(B,A), m(B,A), f(B,D)\}, \\ \quad\ \ \{f(C,A), m(C,D)\}, \\ \quad\ \ \{f(D,B), m(D,C)\} \quad \rangle \end{array} \right\}$$

*At the top-most join of the locale, if we merge the graphs of all four regions, we get:*

- $$F(A \vee B \vee C \vee D) = \left\{ \begin{array}{l} \langle \quad \{f(A,B), f(A,C), m(A,B)\}, \\ \quad\ \ \{f(B,A), m(B,A), f(B,D)\}, \\ \quad\ \ \{f(C,A), m(C,D)\}, \\ \quad\ \ \{f(D,B), m(D,C)\} \quad \rangle \end{array} \right\}$$

*As a picture:*




*In effect, the fiber over each region $U$ is a consistent assignment of relationships involving exactly the people in $U$, obtained by restriction from the full society. A restriction from $F(U)$ to $F(V)$ in effect forget any edges and vertices not involving people in $V$. It truly restricts the relationships mesh from those people in $U$ to only those in $V$.*

*This is in fact a sheaf. The sheaf condition requires that the relationship meshes over two opens agrees precisely on their overlap: when you restrict the meshes down to their overlap, you get the same sub-mesh.*

*The result is a sheaf: a fused mesh of relationships over the population, which is glued together from smaller meshes over smaller subsets of the population.*

- *Each fiber is a part of the whole (human society), and its data encodes the internal (relational) structure of that part.*
- *Mereological overlap is then modeled by shared relationships: two parts overlap if their relational graphs intersect coherently.*
- *Regions that are not covered by a mesh (as in $F(A \vee D) = \varnothing$ and $F(B \vee C) = \varnothing$) reflect mereological separation: the regions in question are simple not related, so there is nothing to glue.*

For another example, consider processes. A process (or more generally any sequence of events, states, etc.) can be seen as a part-whole complex too.

**Example 22.** *Imagine a scenario where something can do one of two things repeatedly: at each step, it can do one thing ("option a") or another thing ("option b"), and then repeat the choice again.*

*To model this, fix a finite alphabet $\Sigma = \{a, b\}$, with "a" for "option a" and "b" for "option b." Then let $\Sigma^*$ be the set of all finite sequences (words) over $\Sigma$, with $\epsilon$ denoting the empty sequence. For instance, the sequence aab represents the sequence of length 3 that picks "option a" first, then "option a" again, and then finally "option b."*

*Let us say that $\Sigma^{\leqslant n}$ is the set of all finite sequences less than length $n$, and let us say that $\Sigma^{=n}$ is the set of finite sequences of exactly length $n$. Hence:*

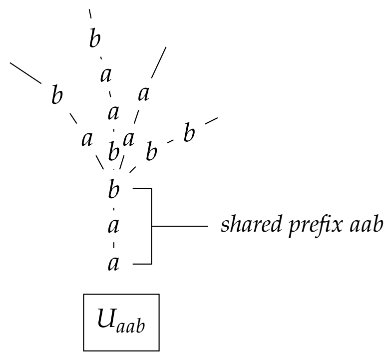
- $\Sigma^{=0} = \{\epsilon\}$.
- $\Sigma^{=1} = \Sigma^{\leqslant 1} = \{\epsilon, a, b\}$.
- $\Sigma^{\leqslant 2} = \{\epsilon, a, b, aa, bb, ab, ba\}$.
- $\Sigma^{=2} = \{aa, bb, ab, ba\}$.
- *Etc.*

*Given sequences $w, v \in \Sigma^{\leqslant n}$ with $length(w) \leqslant length(v)$, let us write $w \subseteq v$ to denote that $w$ is a prefix of $v$, as in $aab \subseteq aabc$.*

*Next, define a topology over $\Sigma^{\leqslant n}$ by setting the open sets to be sequences that share a prefix:*

- $U_w = \{v \in \Sigma^{\leqslant n} \mid w \subseteq v\}$.

*So $U_w$ consists of all sequences that continue $w$. For instance, if $w = aab$, then we might picture $U_w$ as a kind of bouquet or bundle of sequences that are all bound at their shared stem (aab) but then branch out in different directions:*




*We can form a locale from this topology. Let $\mathbb{L}$ be the locale given by the presentation $\langle G, R\rangle$, where:*

- $G = \{U_w \mid w \in \Sigma^n\}$*, i.e., each open is a generator.*
- $R = \{U_w \preccurlyeq U_v \mid v \subseteq w\}$*, i.e., bouquets with longer prefixes are lower.*

*For example, given $\Sigma^{\leqslant 2}$, we have the following generators:*

- $G = \{U_\epsilon, U_a, U_b, U_{aa}, U_{bb}, U_{ab}, U_{ba}\}$.

*Here are some of the relations:*

- $U_{aa} \preccurlyeq U_a$ *and* $U_{ab} \preccurlyeq U_a$*, since "a" is a prefix of aa and ab.*
- $U_{bb} \preccurlyeq U_b$ *and* $U_{ba} \preccurlyeq U_b$*, since "b" is a prefix of bb and ba.*
- *Every generator is lower than $U_\epsilon$, since $\epsilon$ (the empty sequence) is a prefix of every sequence.*

*The Hasse diagram looks like this:*

$$
\begin{tikzcd}[arrows=dash]
 & & & U_\epsilon \arrow[dll] \arrow[drr] & & & \\
 & U_a \arrow[dl] \arrow[dr] & & & & U_b \arrow[dl] \arrow[dr] & \\
U_{aa} \arrow[drrr] & & U_{ab} \arrow[dr] & & U_{ba} \arrow[dl] & & U_{bb} \arrow[dlll] \\
 & & & \perp & & &
\end{tikzcd}
$$

*Think of moving upwards in this locale as forgetting information about (or alternatively, as committing less to) the history of the sequence. For example, think of $U_{ab}$ as a region where we know that "a" happened first and then "b" happened, but think of $U_a$ as a region where we know only that "a" happened first and we don't know what happened after that. The top element is $U_\epsilon$, which means we don't know anything about the sequence of actions. The $\perp$ element indicates not that we know nothing, but that there is no sequence at all.*

*Notice that implication moves upwards: $U_{ab}$ implies $U_a$ because if I know (at $U_{ab}$) that "a" happened first and then "b" happened, then I certainly know that "a" happened first.*

*This particular locale is interesting because it models the "process space" of any 2-stage sequence that can make one of two choices at each stage. Let us now assign some actual processes to this ambient space, using a presheaf.*

*Imagine a machine $m$ that can run multiple concurrent processes, all of whom share the same memory. For simplicity, let us suppose that the machine has two registers ($R = \{r_1, r_2\}$), each of which can hold one bit (1 or 0). So, at any point in time the machine's memory state $S : \{0,1\} \times \{0,1\}$ can be one of the following:*

- $S = \{\langle 0,0\rangle, \langle 1,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle\}$, *with initial state* $s_0 = \langle 0,0\rangle$.

*We can think of the concurrent processes of interest as a selection of programs that we want to run on the machine all at the same time. In terms of behavior, let us say that each program-run reads a word from its input stream, one character at a time, and in response to each character, it takes one of the following actions $A$: it writes a value (1 or 0) to one of the registers, it writes (possibly distinct) values to both registers, or it does nothing and leaves the registers as they are:*

- $A = \{\{r1 \mapsto v\}, \{r2 \mapsto v\}, \{r1 \mapsto v, r2 \mapsto w\}, \varnothing\}$, *where* $v, w \in \{0,1\}$.

*We can define a process (program trace) as a map from n-length words to n-length sequences of write actions, where we require that such maps agree on prefixes (since a process responding to ab and aa would do the same thing on the first a). This way, a program trace records for each input stream the sequence of write actions that result. For concreteness, here are two such traces:*

- $f : \Sigma^{=2} \to A \times A$
  - $f(aa) = \langle\{r1 \mapsto 1\}, \{r1 \mapsto 0\}\}\rangle$
  - $f(ab) = \langle\{r1 \mapsto 1\}, \{r2 \mapsto 0\}\}\rangle$
  - $f(bb) = \langle\{\{r1 \mapsto 0\}, \{r2 \mapsto 1\}\}\rangle$
  - $f(ba) = \langle\{\{r1 \mapsto 0\}, \{r1 \mapsto 1\}\}\rangle$
- $g : \Sigma^{=2} \to A \times A$
  - $g(aa) = \langle\{r2 \mapsto 0\}, \{r2 \mapsto 0\}\}\rangle$
  - $g(ab) = \langle\{r2 \mapsto 0\}, \varnothing\}\rangle$
  - $g(bb) = \langle\{\{r1 \mapsto 1\}, \varnothing\}\rangle$
  - $g(ba) = \langle\{\{r1 \mapsto 1\}, \varnothing\}\rangle$

*Let us say that we now want to run $f$ and $g$ concurrently on the machine. For each word $w \in \Sigma^{=2}$, let us write $\mathcal{R}(w)$ for*

$$(f(w), g(w)),$$

*i.e., the joint run of $f$ and $g$ on $w$. For instance:*

$$\mathcal{R}(aa) = (f(aa), g(aa)) = (\langle\{r1 \mapsto 1\}, \{r1 \mapsto 0\}\}\rangle, \langle\{r2 \mapsto 0\}, \{r2 \mapsto 0\}\}\rangle).$$

*Next, let us say that two joint runs $\mathcal{R}(w)$ and $\mathcal{R}(v)$ are prefix-compatible iff:*

- *For every stage $k$ such that $w$ and $v$ share their first $k$ letters, the combined writes of $f$ and $g$ at stage $k$ do not assign different values to the same register.*

*For instance, $\mathcal{R}(aa)$ and $\mathcal{R}(ab)$ are prefix-compatible because at stage 1 (at letter $a$), their writes do not conflict:*

- *$f(aa)$ and $f(ab)$ write 1 to $r1$, while $g(aa)$ and $g(ab)$ write 0 to $r2$, and this is no conflict because they write to different registers.*

*By contrast, $\mathcal{R}(bb)$ and $\mathcal{R}(ba)$ are not prefix-compatible because at stage 1 (at letter $b$) their writes do conflict:*

- *$f(bb)$ and $f(ba)$ write 0 to $r1$, while $g(bb)$ and $g(ba)$ write 1 to $r1$, so they attempt to write conflicting values to the same register.*

*For a region $U_w$ with $|w| \leq 2$, let $\mathsf{ext}(U_w)$ be the set of length-2 words that extend $w$. For instance, $\mathsf{ext}(U_a) = \{aa, ab\}$, since $aa$ and $ab$ extend $a$, and $\mathsf{ext}(U_{ab}) = \{ab\}$, since $ab$ is already a fully extended length-2 word.*

*Now let us define a presheaf $F$ as follows. For each region $U_w$, let us provide an execution model that describes how $f$ and $g$ execute $w$ and then continue on. More specifically, we need to assign prefix-compatible program runs for the extensions of $w$. The idea is that $F(U_w)$ will present a coherent description of how the machine will behave, as $f$ and $g$ jointly execute $w$ and then continue on to their continuations.*

*At the most specified regions ($aa$, $ab$, $bb$, and $ba$), everything is fully specified, so the joint runs are fully determined already:*

- $F(U_{aa}) = \{aa \mapsto \mathcal{R}(aa)\}$
- $F(U_{ab}) = \{ab \mapsto \mathcal{R}(ab)\}$
- $F(U_{bb}) = \{bb \mapsto \mathcal{R}(bb)\}$
- $F(U_{ba}) = \{ba \mapsto \mathcal{R}(ba)\}$

*The more complicated case involves a less specified region, e.g. $U_a$. At this region, we know that the first step $a$ happened, but we don't yet know whether the second step will be $a$ or $b$. So, from the point of view of $U_a$, both continuations are possible.*

*Thus, what we need for $F(U_a)$ is a coherent description of how the machine will behave no matter which continuation actually occurs next. We must therefore:*

- *Pick a joint run of $aa$.*
- *Pick a joint run of $ab$.*
- *But these two choices must be prefix-compatible, since they are supposed to represent the same joint run that has two different futures.*

*Are there any joint runs of $f$ and $g$ on $aa$ and $ab$ that are prefix-compatible at $a$? Yes:*

- $aa \mapsto \mathcal{R}(aa)$
- $ab \mapsto \mathcal{R}(ab)$

*For as we saw earlier, $\mathcal{R}(aa)$ and $\mathcal{R}(ab)$ are prefix-compatible. Hence, there is a coherent execution model for $f$ and $g$ at $U_a$:*

- $F(U_a) = \{aa \mapsto \mathcal{R}(aa), ab \mapsto \mathcal{R}(ab)\}$

*Now consider $U_b$. The continuations of $b$ are $bb$ and $ba$. Are there any joint runs of $f$ and $g$ on $bb$ and $ba$ that are prefix-compatible? Here, the answer is no, for as we saw above, $\mathcal{R}(bb)$ and $\mathcal{R}(ba)$ are not prefix-compatible. Hence, there is no execution model for $f$ and $g$ at $U_b$:*

- $F(U_b) = \varnothing$

*What about $U_\epsilon$? This would have to be an execution model of a joint run that is prefix compatible with the execution models at both $U_a$ and $U_b$. Since there is no possible execution model at $U_b$, no execution model is possible for $U_\epsilon$:*

- $F(U_\epsilon) = \varnothing$

*Formally, we can summarize the above description of F as follows. Let $\mathcal{R} = \{\mathcal{R}(v) \mid v \in \Sigma^{=2}\}$. Then:*

$$F(U_w) = \left\{ s : \mathsf{ext}(w) \to \mathcal{R} \;\middle|\; \begin{array}{l} \text{(i) } s(v) = \mathcal{R}(v) \text{ for all } v \in \mathsf{ext}(w), \\ \text{(ii) } \{s(v) \mid v \in \mathsf{ext}(w)\} \text{ is pairwise prefix-compatible} \end{array} \right\}$$

*For the bottom fiber, where no processes occur, assign the special symbol zero:*

- $F(\bot) = \{0\}$.

*To notate the restriction of a function $f : B \to D$ to a smaller domain $C \subseteq B$, write $f|_C$. Then, whenever $U_v \preccurlyeq U_w$, the restriction maps are straightforward restrictions:*

$$\rho^{U_w}_{U_v}(s) = s|_{\mathsf{ext}(U_v)}\,.$$

*Hence, for example:*

- $\rho^{U_a}_{U_{ab}}(\{aa \mapsto \mathcal{R}(aa), ab \mapsto \mathcal{R}(ab)\}) = \{ab \mapsto \mathcal{R}(ab)\}$.
- $\rho^{U_a}_{U_{aa}}(\{aa \mapsto \mathcal{R}(aa), ab \mapsto \mathcal{R}(ab)\}) = \{aa \mapsto \mathcal{R}(aa)\}$.

*The maximal fusions in this presheaf occur over $U_a$: the execution model in $U_a$'s fiber is glued from those of $U_{aa}$ and $U_{ab}$, exactly as one would expect, since $f$ and $g$ can run concurrently without conflict at $a$. By contrast, there is no gluing over $U_b$, since $f$ and $g$ cannot run concurrently without conflict at $b$.*

*This is not a sheaf, because it is missing gluings (e.g., over $U_b$). However, it is a monopresheaf, because when gluings exist (over $U_a$), they are unique.*

*This relevance of this example is t hat it illustrates how sheaf theory can model processes, concurrency, and resource conflicts. Here the processes are programs running on a simple machine, but they could just as easily be biological processes competing for resources, etc.*

*Whatever the concrete details may be, this example captures how local behaviors integrate and extend over larger regions of the process space. One might naively think that the "parts" of such systems are the processes. But there is a different way to slice it: if you want to talk about the integrity of the "whole" of a concurrent system, you need to talk about how that involves coherent, integrated behavior that is functionally united locally across the various "regions" and "stages" of the system's evolution.*

Sheaf theory works in continuous environments too.

**Example 23.** *Suppose we want to model the inhabitants of an apartment building over time. Let $T = (0, 10) \subseteq \mathbb{R}$ be an open interval representing a period of time (a span of 10 years, say).*

*Let $T$ have the standard Euclidean topology, and let $\mathbb{L}$ be the locale of opens of the topology. This is the ambient locale we want to work with.*

*Next, let $P$ be the set of people who at some point or other lived in the building:*

$$P = \{A, B, C, \ldots\}, \text{ with } A \text{ short for Alice, } B \text{ for Bob, } C \text{ for Carol, and so on.}$$

*Let $I$ assign to each person the set of intervals during which they lived in the building. For instance:*

- $I(A) = \{(1,3), (4,7)\}$
- $I(B) = \{(4,9)\}$
- $I(C) = \{(6,10)\}$

*Let $F$ be a presheaf given as follows. For each time span $U \in \mathbb{L}$:*

$$F(U) = \{p \in P \mid U \subseteq V, \text{ for some } V \in I(p)\}$$

*In other words, $F(U)$ is the set of people who live in the building for the entire duration of $U$.*

*For $V \preccurlyeq U$, the restriction maps are just inclusion:*

$$\rho^U_V(J) = J,$$

*since if the set of people $J$ lived in the building throughout the span $U$, then they most certainly lived there during the smaller interval $V$.*

*This is a sheaf, since any compatible selection of patch candidates glues uniquely. For any cover $\{U_i\}_{i \in I}$ of $U$:*

$$F(U) = \bigcap_i F(U_i).$$

*A fusion in this sheaf is a glued section, and its parts are the patch candidates it is glued from. For example, take $U = (6,7)$, with cover $\{U_1 = (6, 6.5), U_2 = (6.4, 7)\}$. Any selection of compatible patch candidates from this cover glue uniquely.*

*This is a particularly simple sheaf, but the sheaf's strong gluing conditions tell us why the parts glue together so straightforwardly here. Gluing is thoroughly integrated throughout the structure.*

**Example 24.** *For another continuous example, let's model a lump of clay through time.*

*Let $T = (0, 10)$ be a span of time, and let $\mathbb{L}$ be the locale of the opens of $T$ with its standard Euclidean topology again.*

*For simplicity, let us assume that 3-dimensional space is just $\mathbb{R}^3$. Then fix a function*

$$\phi : T \to \wp(\mathbb{R}^3)$$

*that, for each $t \in T$,*

$$\phi(t) \subseteq \mathbb{R}^3$$

*is the open region of space occupied by the clay at time $t$.*

*If we think of $\phi(t)$ as a snapshot of the clay at $t$, then at some $t$s, $\phi(t)$ might be shaped like a statue, at other times like a lump, at still other times like two disconnected lumps, and so on.*

*With that background fixed, let us now define a presheaf $F$ that assigns material parts to each interval of $T$. In particular, for each interval $U \in \mathbb{L}$:*

$$F(U) = \{S \subseteq \mathbb{R}^3 \mid S \text{ is open and } \forall t \in U, S \subseteq \phi(t)\}.$$

*In other words, F(U) is the set of the clay's material parts that persist through the duration of $U$.*

*Restriction is inclusion again, since persistence through a larger time span* $U$ *implies persistence through a smaller span* $V \preccurlyeq U$*:*

$$\rho^U_V(S) = S.$$

*This is a sheaf, since compatible local section glue by intersection. The fibers of this sheaf are typically infinite: each fiber includes the whole lump of clay, all of is sub-regions, and these regions are themselves spatially continuous.*

*Given a time span* $U$*, if*

$$\bigcap_{t \in U} \phi(t)$$

*is connected, then there will be a maximal connected section* $S \in F(U)$ *such that all other sections in* $F(U)$ *sit inside it. This models a single lump of clay through time.*

*By contrast, if*

$$\bigcap_{t \in U} \phi(t)$$

*has two or more connected components, then there will not be a maximal connected section* $S \in F(U)$ *that all other sections of* $F(U)$ *fit inside. This models a fragmented lump of clay through time.*

*What exactly does the sheaf give us here that* $\phi$ *doesn't? What* $\phi$ *gives us is a time-indexed history of spatial occupation. It is like a movie: each frame is just a picture. It says nothing about what the material parts are, or which parts persist through time. That information has to be imposed on top of* $\phi$*.*

$F$ *does exactly that, and makes it explicit.* $F$ *assigns to the locale precisely what the material parts are that persist through time, along with a principled notion of fusion. In short,* $\phi$ *gives us what points in space are occupied at each instant, whereas* $F$ *gives us which parts persist over regions of time, and how they compose.*

*Examples like this can be enriched to model further features. For instance, we might allow multiple overlapping material parts in the same fiber, each tagged by a different modifier of some kind.*

*With the lump of clay, each fiber only carried persistent material parts. But we might let the fibers carry persistent material parts tagged with an accidental mode (e.g., a posture, like seated or standing).*

*Let* $\mathsf{Posture} = \{seated, standing, \ldots,\} \cup \{unspecified\}$*. Then for every* $U \in \mathbb{L}$*, we can say that* $F(U)$ *is a set of pairs* $\langle S, \pi \rangle$ *such that* $\forall t \in U$*,* $S \subseteq \phi(t)$*, and* $\pi \in \mathsf{Posture}$*. Intuitively, these pairs are the material parts in standing-Socrates, seated-Socrates, and Socrates-simpliciter that persist in the given mode throughout* $U$*. This presheaf can have multiple, distinct persistent material fusions (e.g., Socrates-simpliciter and seated-Socrates) that simultaneously inhabit the same regions of space and differ only by their accidental mode.*

TODO:

- Note Spivak et al's behavioral mereology is an example of a $\mathcal{G}$-sheaf (and check the details to make sure that's really true).
- Mormann's "structural mereology" is basically just our thesis. Add examples from his similarity structures.

## 4. Classical Mereological Notions in the Sheaf-theoretic Setting

In this section, we provide a discussion of what classical notions of mereology look like in the sheaf-theoretic setting.

- *Cambridge fusions*. Sheaves handle Cambridge fusions correctly.
- *Mere collections*. The collection of all dogs. Is that a "whole"? Well, we could build a sheaf whose atomic regions are filled with dogs, none of which glue. Then we have a collection of dogs, but no glued object. That matches exactly the intuition: yes, we have a "collection" (we built a sheaf for it, after all), but the internals of that sheaf reveal that it's *merely* a collection, i.e., that its parts are not glued.
- *Co-habitating fusions*. Sheaves allow multiple fusions to occupy the same locale, without being glued. For instance, in the sheaf of real-valued functions over real number line, there are many functions that glue together, and occupy the same locale.
- *Non-boolean algebra*. The parts space is Heyting, not Boolean. We're not saddled with such a strong complement operation. You can pick a locale that is Boolean if you need it, but this framework doesn't require it. In fact, the positive logic of a locale is "geometric logic."
- *Reflexivity, antisymmetry, and transitivity*. These are guaranteed. Locally, of course, you may not have transitivity. But globally, it's a theorem. [Check that.]
- *Distributivity*.

  do the glued sections of a sheaf have to be distrubitive? Only inside what glues (since we glue pairwise, so every $i \vee j$ of the cover.
- *An empty element*. There is a need for a bottom element in the *algebra* of parts, but a sheaf need not contain any such thing. There is no need here to try and construct awkward mathematical structures that do algebra on parts but yet don't have a bottom element because our ontological intuitions tell us there can be no such thing. That confuses two issues: algebra and integrity. So here we separate those cleanly, and the algebra can do algebra while the sheaf can do integrity. [In a sheaf you CAN'T assign an empty element to bottom, for coherence, so the bottom element is special...need to say more about that and figure it out.]
- *Supplementation principles*. Sheaves don't constrain one way or another. [Is that really true? Maybe it's better to say that it doesn't force any supplementation principles, which might provide a reason to call into question whether supplementation is another one of those ideas that is about integrity of parts but has been confused with the algebra of parts.]
- *Ordering of parts*. Consider that "pit" and "tip" have the same parts but are different words. These differences can be handled by different sheaves over a 3-stage prefix-ordered locale as in the example of concurrent processes. Note that we retain extensionality.
- *Extensionality*. Classical mereology's notion of extensionality essentially flattens any structure and is thus overly aggressive. This is why extensionality is so controversial. The sheaf-theoretic perspective retains extensionality, but is much more nuanced. [Here too, I suspect that mereological discussions of extensionality have confused the algebra of parts and the integrity of wholes.]
- *Gunk and atoms*. You can model continuity and gunky parts if you so desire. You just need a sober space to do it.

  check that we can model continuity in the locale in this way.

  can you do continuity only in the sheaf data, without an underlying continuous decompositon in the locale? I would think that if you can't infinitely decompose into smaller opens around a point in the locale, you couldn't do such a thing in the sheaf data?
- *Priority of wholes*. The framework is agnostic as to whether you take an Aristotelian-Thomistic approach

**cite** Aquinas, Arlig, and that guy who wrote that recent book defending the Aristotelian view

.

- *The whole is greater than its parts*. The framework is agnostic as to whether you want to be a Scotist and say that the whole is something over and above its parts (cite Cross) or an Ockhamist who says it is not

  **cite** Normore, Arlig

  .

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