Article

# Sheaf Mereology

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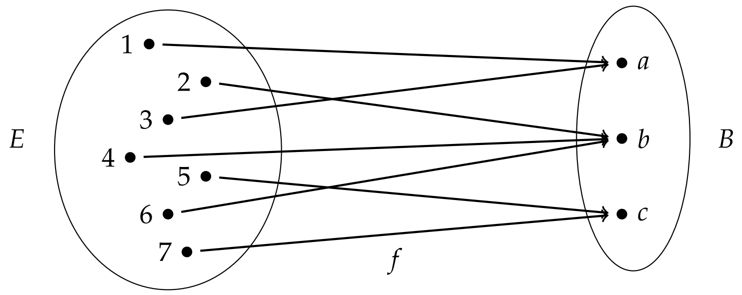
# 1. Sheaf Theory

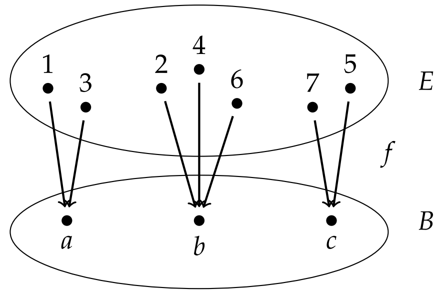
In this section, we introduce the parts of sheaf theory needed for the sequel.

### *1.1. Fibers*

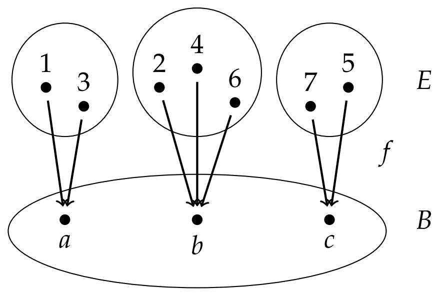
Suppose we have a map (function) $f : E \to B$ that looks something like this:



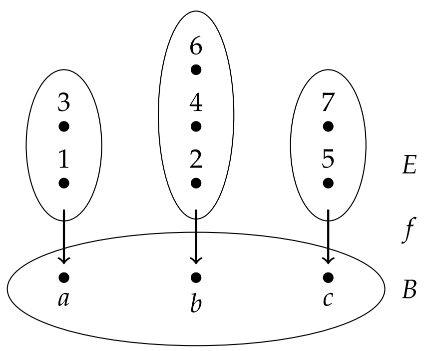

It is sometimes convenient to turn the diagram sideways and group together points in the domain that get sent to the same target, like so:




That makes the pre-images very easy to see. For any point in $B$, its pre-image is just the group of points sitting "over" it:



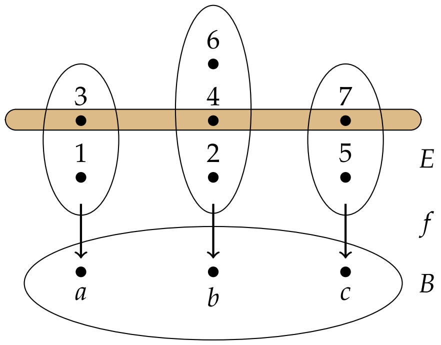

If we stack the points in each pre-image vertically, one on top of the other, we can then think of each pre-image as a kind of "stalk" growing over its base point:



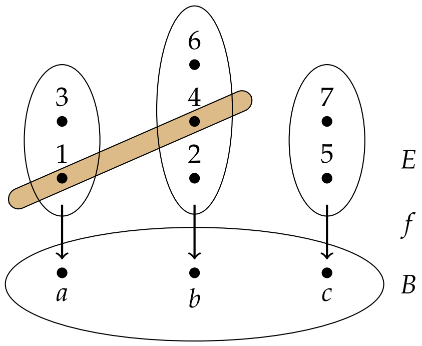

This gives rise to the idea of the "fibers" of a map. The fibers of a map are just its pre-images. For instance, the fiber over $b$ is $\{2, 4, 6\}$.

**Definition 1** (Fibers)**.** *Given a map $f : E \to B$ and a point $y \in B$, the fiber over $y$ is its pre-image $f^{-1}(y) = \{x \mid f(x) = y\}$. $B$ is called the base space of $f$, and $y$ the base point of the fiber.*

We can take a cross-section of one or more fibers by selecting a point from each of the fibers in question. For instance, we can take 3, 4, and 7 as a cross-section of the fibers $f^{-1}(a)$, $f^{-1}(b)$, and $f^{-1}(c)$:




We can also take cross-sections local to only some of the fibers. For instance, we can take 1 and 4 as a cross-section of $f^{-1}(a)$ and $f^{-1}(b)$:




**Definition 2** (Sections)**.** *Given a map $f : E \to B$ and a subset of base points $C \subseteq B$, a section of $f$ (over $C$) is a choice of one element from each fiber over each base point $x \in C$.*

**Remark 1.** *Since each point in a fiber amounts to a section over the fiber's base, the elements of a fiber are often just called the sections of the fiber.*

### *1.2. Spaces*

In the above examples, the base $B$ was a set. We often want to consider bases that have more structure, e.g., bases that have spatial structure.

In traditional topology, spaces are built out of the points of the space. Given a set of points, a topology on that set specifies which points belong in which regions of the space.

**Definition 3** (Topology)**.** *Let $X$ be a non-empty set, thought of as the points of the space. A topology on $X$ is a collection $T$ of subsets of $X$, thought of as the regions of the space and called the open sets (or just the opens) of $T$, that satisfy the following conditions:*

*(T1) The empty set and the whole set are open:*

$$\varnothing \in T, X \in T.$$

*(T2) Arbitrary unions of opens are open:*

$$\textit{if } \{U_i\}_{i \in I} \subseteq T, \textit{ then } \bigcup_{i \in I} U_i \in T.$$

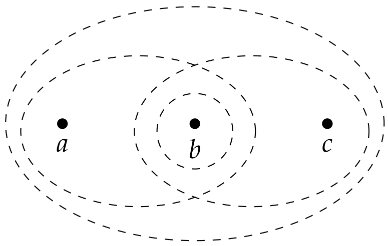
(T3) *Finite intersections of opens are open:*

$$\textit{if } U_1, \ldots, U_n \in T, \textit{ then } \bigcap_{i=1}^{n} U_i \in T.$$

These conditions encode the way that spatial regions are put together. For instance, it ensures that if two regions overlap, then their overlapping area is a region too.

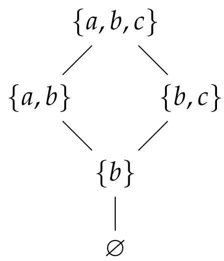
**Remark 2.** *The regions of a topology, ordered by inclusion, form a complete lattice. Since the topology includes arbitrary unions, the join of this lattice is set union, but since the topology includes only finite intersections, the meet of this lattice is the* interior *of set intersection.*

**Example 1.** *Let $X = \{a, b, c\}$. One possible topology is $T = \{\varnothing, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. If we draw dashed circles around each of the opens (regions), we get:*




*There are two regions $\{a, b\}$ and $\{b, c\}$ that overlap at $b$ (so $\{b\}$ is a region in $T$ too). There is also the full region $\{a, b, c\}$, which is the union of the smaller regions.*

*We can draw $T$ as a Hasse diagram, which shows that the regions form a lattice:*




The lattice structure suggests that much of what is important about a space is not so much its points, but rather its opens/regions. This leads to the idea that topology-like reasoning can be done without the points. So, we can generalize: take a topology, and drop the points. That leaves just the opens/regions, which we call a frame (or locale).

**Definition 4** (Frames/locales)**.** *A frame (synonymously, a locale) $\mathbb{L}$ is a partially ordered set $L$ (whose elements are called opens or regions) that satisfies the following conditions:*

(L1) *$L$ is a complete lattice:*

- *Every subset $S \subseteq L$ has a join, denoted $\bigvee S$.*
- *Every finite subset $F \subseteq L$ has a meet, denoted $\bigwedge F$.*

(L2) *Finite meets distribute over arbitrary joins:*

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i), \textit{ for all } a \in L \textit{ and all families } \{b_i\}_{i \in I} \subseteq L.$$

*Define $V \preccurlyeq U$ (read "$V$ is included in $U$") by $a = a \wedge b$.*

**Remark 3.** *The fact that $V \preccurlyeq U$ is equivalent to $a = a \wedge b$ means we can deal with the opens of a frame algebraically (via $\wedge$ and $\vee$ operations), or order-theoretically (via the $\preccurlyeq$ relation), whichever is more convenient.*

**Remark 4.** *The category of locales is defined as the dual/opposite of the category of frames, and so frames and locales are quite literally the very same objects. In practice, frames are often used for algebraic purposes, and locales are used for (generalized) spatial purposes. Here, we will have no reason to distinguish these two roles, and so we will use the names "frame" and "locale" interchangeably.*

### *1.3. Presentations of locales*

Locales have presentations much like groups and other algebraic structures. To give the presentation of a locale, specify a set of generators and relations.

**Definition 5** (Presentations)**.** *A presentation $\langle G, R \rangle$ of a locale $\mathbb{L}$ is comprised of:*

*(P1) A set of generators $G = \{U_k, U_m, \ldots\}$.*
*(P2) A set of relations $R \subseteq G \times G$ on those generators.*

*The locale $\mathbb{L}$ presented by $\langle G, R \rangle$ is the smallest one freely generated from $G$ which satisfies $R$.*

**Remark 5.** *Every locale has a presentation, and a locale can have multiple presentations.*

To calculate the locale that corresponds to a presentation, start with the generators, then take all finite meets and all arbitrary joins that satisfy $R$ (and of course L1 and L2).

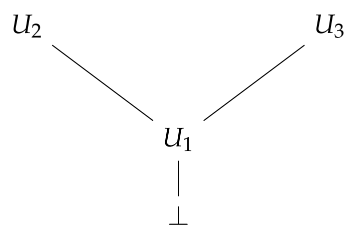
**Example 2.** *Let a locale $\mathbb{L}$ be given by the presentation $\langle G, R \rangle$ where:*

- $G = \{\perp, U_1, U_2, U_3\}$.
- $R = \{\perp \preccurlyeq U_1, U_1 \preccurlyeq U_2, U_1 \preccurlyeq U_3\}$.

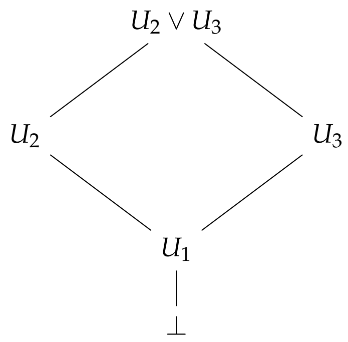
*There are four generators ($\perp$, $U_1$, $U_2$, and $U_3$), and $\perp$ is below $U_1$ while $U_1$ is a sub-region of $U_2$ and $U_3$. Since $U_1$ is a sub-region of both $U_2$ and $U_3$, $U_1$ is their meet:*

- $U_1 = U_2 \wedge U_3$.

*At this point, we have generated this much of the locale:*




*$R$ says nothing to constrain joins, so we need to join everything we can. In this case, we need to join $U_2$ and $U_3$:*




*There are no further joins or meets that aren't already represented in the picture. For instance, all further non-trivial meets are already accounted for:*

- $U_1 \wedge \perp = \perp$.
- $U_2 \wedge U_1 = U_1$ *and* $U_3 \wedge U_1 = U_1$.
- $U_2 \wedge \perp = \perp$ *and* $U_3 \wedge \perp = \perp$.
- $(U_2 \vee U_3) \wedge U_2 = U_2$ *and* $(U_2 \vee U_3) \wedge U_3 = U_3$.
- $(U_2 \vee U_3) \wedge U_1 = U_1$.
- $(U_2 \vee U_3) \wedge \perp = \perp$.

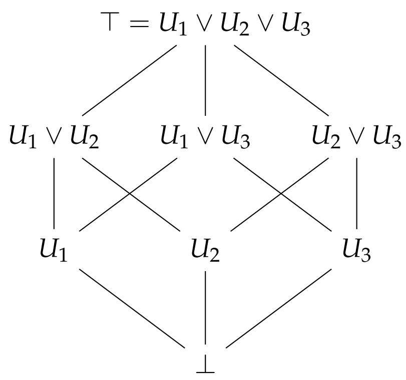
*Similarly, all other non-trivial joins are also already accounted for:*

- $\perp \vee U_1 = U_1$.
- $\perp \vee U_2 = U_2$ *and* $\perp \vee U_3 = U_3$.
- $\perp \vee (U_2 \vee U_3) = U_2 \vee U_3$.
- $U_1 \vee U_2 = U_2$ *and* $U_1 \vee U_3 = U_3$.
- $U_2 \vee (U_2 \vee U_3) = U_2 \vee U_3$ *and* $U_2 \vee (U_3 \vee U_3) = U_2 \vee U_3$.

**Example 3.** *Let* $\mathbb{L} = \langle G, R \rangle$ *be given by:*

- $G = \{U_1, U_2, U_3\}$.
- $R = \varnothing$.

*We have three generators ($U_1$, $U_2$, and $U_3$), and there are no relations restricting how those generators are related. Thus, the locale that is freely generated from this presentation is isomorphic to the power set of three elements:*




A presentation provides the most "minimal" information from which the rest of the locale is generated. At first glance, it might be tempting to think that these generators are "atomic" or indivisible regions — the fundamental pieces of the locale.

But this intuition is misleading. A generator in a presentation can sometimes be expressed as the nontrivial meet or join of other generators. In this sense, some generators are actually composite rather than truly atomic.

The genuinely indivisible generators — the true primitives — are exactly those that cannot be built nontrivially from other generators using meet and join. These are the elements that serve as the irreducible building blocks of the locale.

To make this precise, we first define what it means to be built trivially or nontrivially from other generators via meet and join.

**Definition 6** (Trivial meet-constructions). *Given a presentation $\mathbb{L} = \langle G, R\rangle$, a meet construction of generators, i.e.*

$$U = \bigwedge_{i=1}^{n} U_i$$

*with each $U_i \in G$, is a trivial meet-construction if the meet is equal to one of its factors, i.e. if*

$$U = U_i \text{ for some } i.$$

**Definition 7** (Trivial join-constructions). *Given a presentation $\mathbb{L} = \langle G, R\rangle$, a join construction of generators, i.e.*

$$U = \bigvee_{i=1}^{n} U_i$$

*with each $U_i \in G$, is a trivial join-construction if the join is equal to one of its factors, i.e. if*

$$U = U_i \text{ for some } i.$$

Intuitively, trivial meet- and join-constructions do not genuinely combine multiple pieces. The result is already contained in the inputs.

**Definition 8** (Meet-primitive generators). *Given a presentation $\mathbb{L} = \langle G, R\rangle$, a region $U \in G$ is meet-primitive (primitive for short) if every meet-construction*

$$U = \bigwedge_{i=1}^{n} U_i$$

*with each $U_i \in G$ is trivial.*

Intuitively, a generator is meet-primitive if it cannot be assembled from larger ones by taking meets.

**Definition 9** (Join-primitive generators). *Given a presentation $\mathbb{L} = \langle G, R\rangle$, a region $U \in G$ is join-primitive (primitive for short) if every join-construction*

$$U = \bigwedge_{i=1}^{n} U_i$$

*with each $U_i \in G$ is trivial.*

Intuitively, a generator is join-primitive if it cannot be assembled from smaller ones by taking joins.

**Remark 6.** *In the technical literature, meet-primitive generators are often called meet-irreducible.*

**Example 4.** *Take the locale from Example 2. $U_1$ is the overlap of $U_2$ and $U_3$, and $U_2$ and $U_3$ are strictly larger regions than $U_1$, so $U_1$ is not primitive.*

*By contrast, $U_2$ and $U_3$ are primitive, because they cannot be obtained as the meet of two strictly larger regions. Similarly, $\bot$ is primitive, because it is not the meet of two strictly larger regions either (it is the meet of only one strictly larger region, namely $U_1$).*

We can see the minimal meet-primitive generators of a locale as its atomic regions.

**Definition 10** (Atomic regions). *Given a presentation $\mathbb{L} = \langle G, R \rangle$, the atomic regions of $\mathbb{L}$, denoted $\mathsf{Atoms}(\mathbb{L})$, are the minimal primitive generators of $\mathbb{L}$, i.e. those generators $g \in G$ that satisfy the following two conditions:*

*(A1)* Meet-primitive. *$g$ is meet-primitive.*
*(A2)* Minimality. *There is no strictly smaller meet-primitive $h$ with $h \preccurlyeq g$.*

use "primitive" or "fundamental" instead of "atomic"? An atomic thing can't be broken down, but a meet-irreducible element does have a part!

### *1.4. Presheaves*

Above we considered the fibers of a map $f : E \to B$, where $E$ and $B$ were sets. We can also consider fibers over locales, where the fibers respect the locale's structure. This is called a presheaf. A presheaf is an assignment of data to each of a locale's regions that is "stable under restriction," i.e., that respects zooming in and out.

**Definition 11** (Presheaf). *Let $\mathbb{L}$ be a locale, and let $\mathsf{Arr}(\mathbb{L})$ be $\{\langle A, B \rangle \mid A \preccurlyeq B \in \mathbb{L}\}$. A presheaf on $\mathbb{L}$ is a pair $\langle F, \{\rho^B_A\}_{\langle A,B \rangle \in \mathsf{Arr}(\mathbb{L})} \rangle$, where:*

- *$F$ assigns to each region $U \in L$ some data $F(U)$.*
- *$\{\rho^B_A\}_{\langle A,B \rangle \in \mathsf{Arr}(\mathbb{L})}$ is a family of maps $\rho^B_A : F(B) \to F(A)$ (called "restriction maps"), each of which specifies how to restrict the data over $F(B)$ down to the data over $F(A)$.*

*All together, $\langle F, \{\rho^B_A\}_{\langle A,B \rangle \in \mathsf{Arr}(\mathbb{L})} \rangle$ must satisfy the following conditions:*

*(R1) Restrictions preserve identity:*

$$\rho^U_U = \mathsf{id}_U \text{ (the identity on } U\text{), for every } U \in \mathbb{L}.$$

*(R2) Restrictions compose:*

$$\text{If } A \preccurlyeq B \text{ and } B \preccurlyeq C, \text{ then } \rho^C_A = \rho^B_A \circ \rho^C_B.$$

Since $F$ assigns data $F(U)$ to each region $U \in \mathbb{L}$, we can think of the $F(U)$s as the "fibers" over $\mathbb{L}$, and the restriction maps as "zoom in" maps that go from bigger fibers down to smaller fibers.

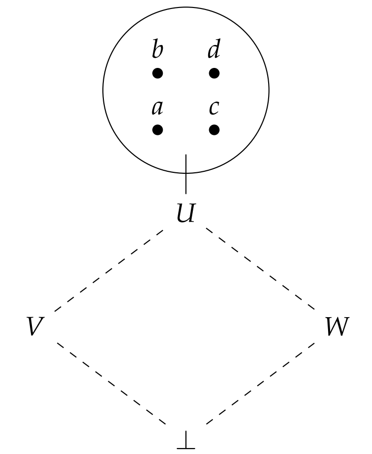
**Example 5.** *Let $\mathbb{L}$ be a locale $\{\bot, W, V, U\}$ with the following structure:*

$$\begin{tikzcd}
 & U \arrow[dl, dash] \arrow[dr, dash] & \\
V \arrow[dr, dash] & & W \arrow[dl, dash] \\
 & \bot &
\end{tikzcd}$$

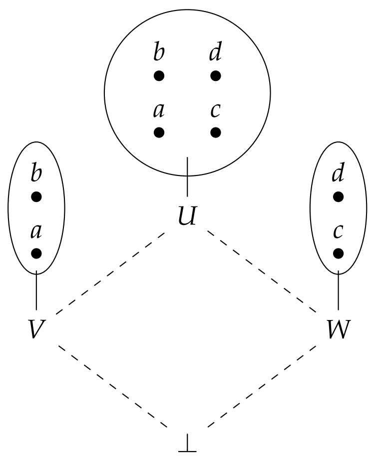
*Next, let's define a presheaf $F$ as follows:*

- $F(U) = \{a, b, c, d\}$, $F(V) = \{a, b\}$, $F(W) = \{c, d\}$, $F(\perp) = \{*\}$.
- *Define* $\rho^U_V$ *as the projection (send a to a, b to b, and the rest can go anywhere), and similarly for* $\rho^U_W$. *Let* $\rho^U_\perp$, $\rho^V_\perp$, *and* $\rho^W_\perp$ *send their data to* $\{*\}$, *and let the rest be identities.*

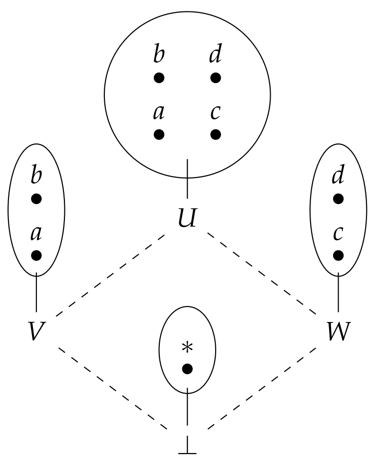
*We can see F's assignments as fibers over* $\mathbb{L}$ *by drawing them over the regions they are assigned to. For instance, over U we have* $F(U)$, *i.e.,* $\{a, b, c, d\}$:



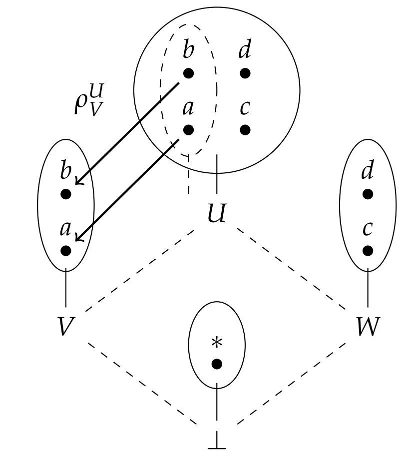

*Similarly, over V and W, we have* $F(V) = \{a, b\}$ *and* $F(W) = \{c, d\}$:



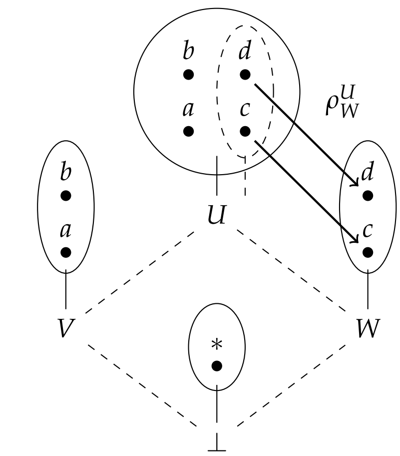

*Finally, over* $\perp$, *we have a singleton set:*



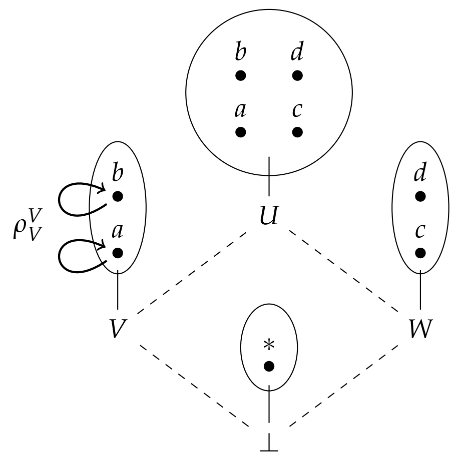

*The restriction maps show how to "zoom in" on the parts (sub-fibers) of any given fiber. For instance, we can see that the fiber over V is contained in the fiber over U. The restriction map just projects that sub-fiber out, thereby showing us how to "zoom in" on that sub-fiber:*



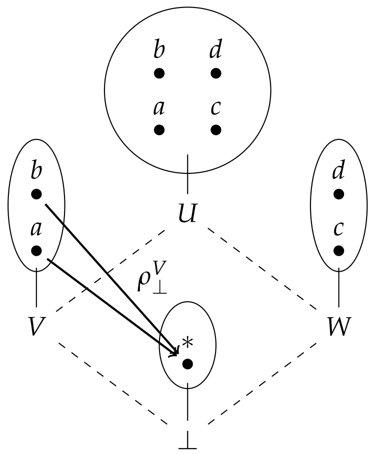

*It's similar for the fiber over W:*



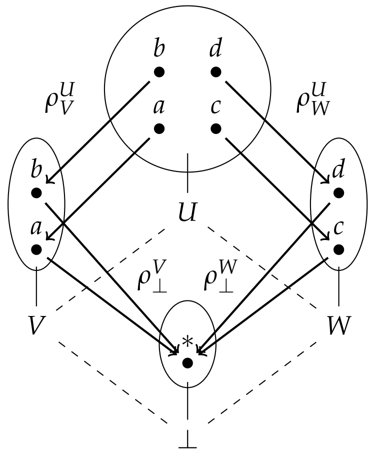

*Restricting a fiber to itself is just the identity on the fiber:*




*The other restriction maps restrict down to the singleton set. For instance:*




*All of this makes it clear that the structure of the presheaf data that sits in the fibers over $\mathbb{L}$ mimics (respects) the structure of the base locale:*




**Remark 7.** *In line with Remark 1, the elements of each fiber $F(U)$ are usually just called the "sections" of $F(U)$. For instance, $c$ is a section of $F(W)$, as is $d$.*

### *1.5. Sheaves*

The definition of a presheaf requires only that the data be stable under restriction (zooming in on a region). It does not require that the data fit together across different regions (fibers).

A sheaf is a presheaf with an added gluing condition: whenever you have compatible data on overlapping fibers, there must be a unique way to glue it together into data over the union. In other words, the data in the fibers must agree on overlap and combine coherently.

To get at this idea, let's first define a cover. A cover of a region $U$ is a selection of sub-regions that covers $U$ in its entirety. The chosen sub-regions don't leave any part of $U$ exposed.

**Definition 12** (Cover)**.** *Let $\mathbb{L}$ be a topology or a locale, and let $U$ be a region of $\mathbb{L}$. A cover of $U$ is a family $\{U_i\}_{i\in I} \subseteq \mathbb{L}$ such that:*

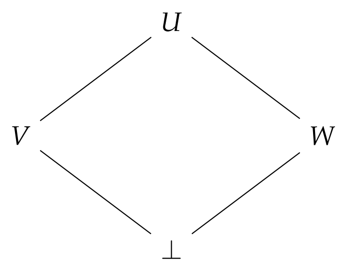
$$U = \bigvee_{i\in I} \{U_i\}.$$

*In other words, a cover of $U$ is a family of regions that join together to form $U$.*

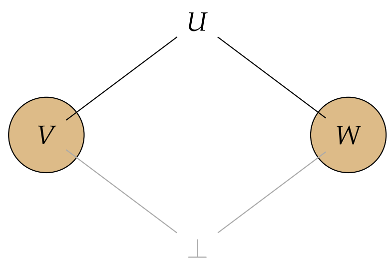
**Example 6.** *Take the topology from Example 1: $T = \{\varnothing, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$. A cover of $\{a,b,c\}$ is $\{a,b\}$ and $\{b,c\}$, because altogether, $\{a,b\}$ and $\{b,c\}$ cover all of the points in $\{a,b,c\}$.*

*Another cover of $\{a, b, c\}$ is $\{\{a, b\}, \{b, c\}, \{b\}\}$. Although $\{b\}$ is redundant here, this choice of sub-regions still entirely covers $\{a, b, c\}$ as required.*

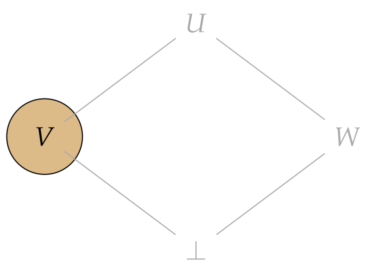
**Example 7.** *In the context of frames, where there are no points, a cover of $U$ is just a selection of sub-regions of $U$ that together join together to form $U$. Take the locale from Example 5:*




*A cover of $U$ is $\{V, W\}$, since $U = \bigvee\{V, W\}$:*




*A cover of $V$ is just $\{V\}$:*



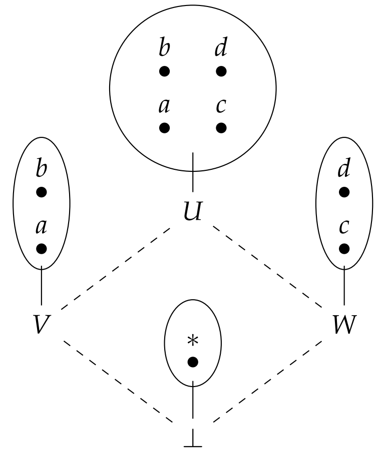

**Remark 8.** *A cover over the least element of a locale (or a topology) is empty (the empty set), because there are no regions (or points) to cover.*

Given a presheaf $F$ over a locale $\mathbb{L}$, if we have a cover $\{U_i\}_{i \in I}$ of some portion of $\mathbb{L}$, there is a corresponding family of fibers $\{F(U_i)\}_{i \in I}$ over that cover. We can pick one section (i.e., one element) from each such fiber to get a slice of elements that spans all of the fibers over that cover. Let us call such a choice a selection of patch candidates.

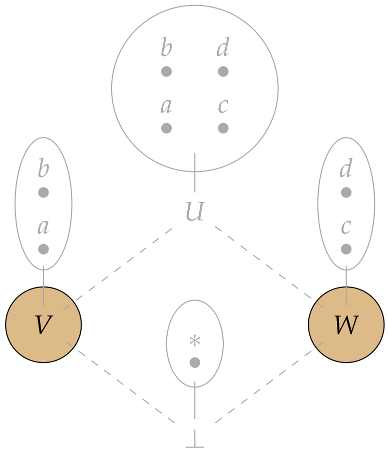
**Definition 13** (Patch candidates)**.** *Given a presheaf $F$ and a cover $\{U_i\}_{i \in I}$ with a corresponding family of fibers $\{F(U_i)\}_{i \in I}$, a selection of patch candidates $\{s_i\}_{i \in I}$ is a choice of one section $s_i$ from each $F(U_i)$:*

$$\{s_i\}_{i \in I} = \{s_i \mid s_i \in F(U_i) \text{ for each } F(U_i) \in \{F(U_i)\}_{i \in I}\}.$$

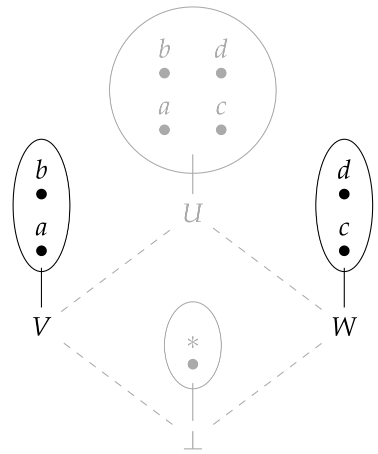
**Example 8.** *Take the presheaf from Example 5:*



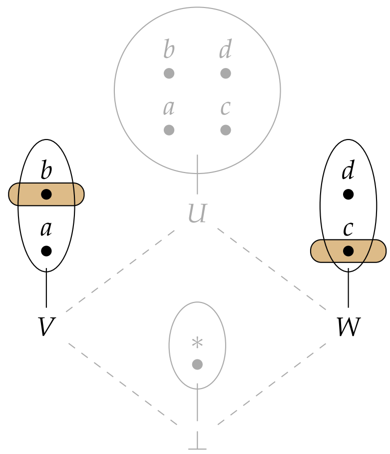

*Let $\{V, W\}$ be the cover of interest:*




*Over this cover, we have a corresponding family of fibers:*




*A selection of patch candidates is a choice of one section (element) from each fiber. For instance, we might pick $b$ from $F(V)$ and $c$ from $F(W)$:*




*Similarly, we might pick $\{a, d\}$, $\{b, d\}$, or $\{a, c\}$, each of which is a valid selection of patch candidates.*

**Remark 9.** *Consider the empty cover. Since there are no sub-regions below the least element of a locale, there are no patch candidates we could choose for the empty cover either. Hence, any selection of patch candidates for the empty cover is $\varnothing$.*

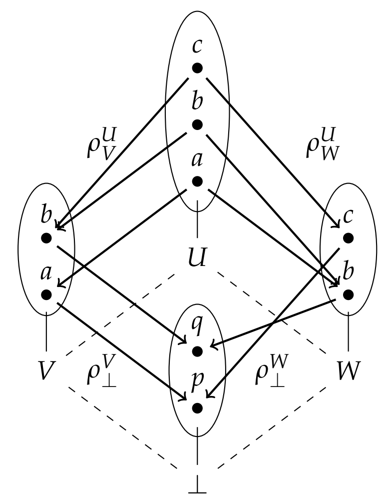
A selection of patch candidates might fit together, or they might not. We say they are compatible if they fit together, i.e., if they agree on overlaps. To check this, take any pair of patch candidates, and check if they restrict to the same data on their overlap.

**Definition 14** (Compatible patch candidates)**.** *Given two fibers $F(U_i)$ and $F(U_j)$ and a patch candidate from each, $s_i \in F(U_i)$ and $s_j \in F(U_j)$, $s_i$ and $s_j$ are compatible if they restrict to the same data on their overlap $U_i \wedge U_j$:*

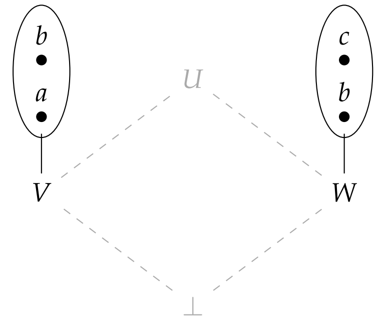
$$\rho^{U_i}_{U_i \wedge U_j}(s_i) = \rho^{U_j}_{U_i \wedge U_j}(s_j).$$

*A selection of patch candidates $\{s_i\}_{i \in I}$ is compatible if all of its members are pair-wise compatible.*

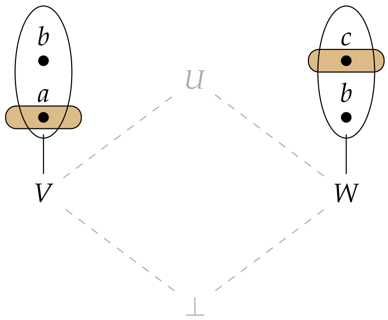
**Example 9.** *Consider the following presheaf $F$:*



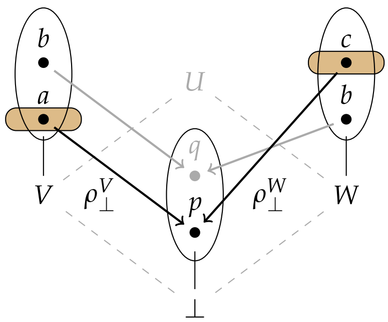

*Take the cover $\{V, W\}$ and its corresponding fibers:*



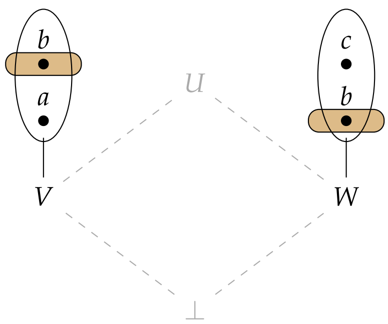

*Suppose we pick* $\{a, c\}$ *for patch candidates:*



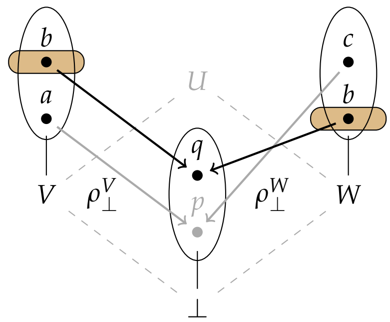

*Is this selection compatible? We have to check if they agree on their overlap. The overlap* $V \wedge W$ *is* $\perp$. *Where does* $\rho^V_\perp$ *send our chosen patch candidate* $a$? *It sends it to* $p$, *since* $\rho^V_\perp(a) = p$. *Where does* $\rho^W_\perp$ *send our other chosen patch candidate* $b$? *It also sends it to* $p$, *since* $\rho^W_\perp(c) = p$. *On the overlap* $\perp$ *then,* $\rho^V_\perp(a) = \rho^W_\perp(c)$, *so* $a$ *and* $c$ *are compatible. This is easy to see in the diagram, since* $a$ *and* $b$ *both get sent to the same place:*



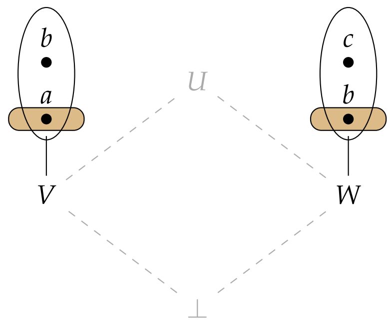

*Now suppose we pick* $\{b, b\}$ *for patch candidates:*



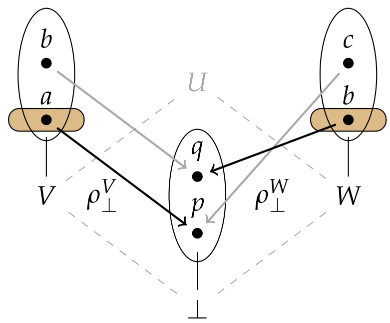

*These are also compatible. They agree on their overlap (both restrict to* $q$*):*




*Finally, suppose we pick $\{a, b\}$ for patch candidates:*




*These are not compatible. They do not agree on their overlap:*




**Remark 10.** *Consider the empty cover. Since any selection of patch candidates for the empty cover is empty, compatibility is satisfied vacuously.*

*As an analogy, if you ask your class to turn off all cell phones but nobody brought a cell phone to class, then your request is satisfied vacuously: there is simply nothing that needs to be done to make it happen. It's similar with the empty cover: since there are no patch candidates to check, compatibility is achieved vacuously.*

If a selection of patch candidates $s_i, \ldots, s_k$ across a cover of $U$ is compatible, we say those patches glue together if if there's a section $s$ in $F(U)$ that restricts down to exactly those patches.

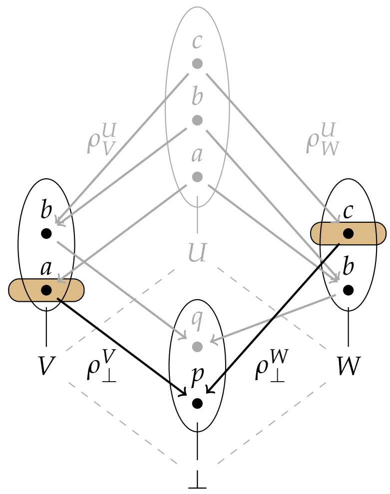
**Definition 15** (Gluing). *Given a presheaf $F$ and a selection of compatible patch candidates $\{s_i\}_{i \in I}$ for a cover $\{U_i\}_{i \in I}$, $\{s_i\}_{i \in I}$ glue together only if there is a section $s \in F(U)$ that restricts down to $s_i$ on each fiber $F(U_i)$ of the cover, i.e., only if $s$ is such that:*

$$\rho^U_{U_i}(s) = s_i, \textit{ for each } i \in I.$$

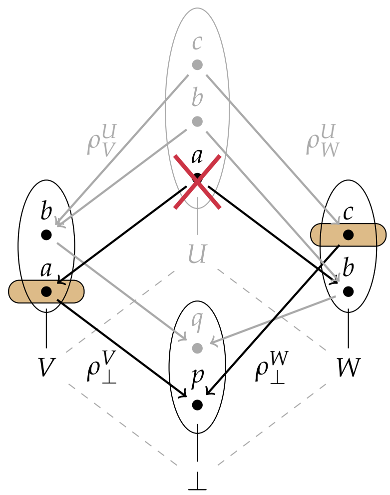
*As a matter of terminology, if a section $s \in F(U)$ is glued from patches $\{s_i\}_{i \in I}$, we say that $s$ is a global section of the cover, and each $s_i$ is a local section of the cover. We may also say variously that* *s is* glued from *those patches, that s is* composed *of those patches, that those patches* compose *s, or that gluing those patches* yields *s.*

*A selection of patches* $\{s_i\}_{i \in I}$ *glues uniquely if there is one and only one such section* $s \in F(U)$ *that is glued from them.*

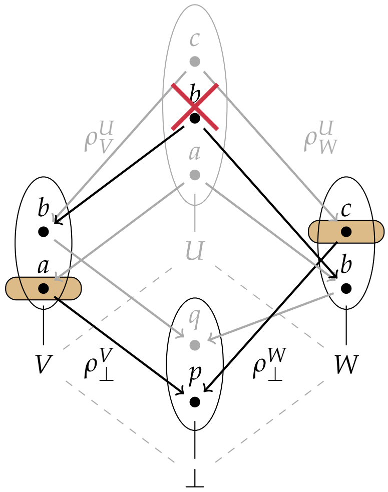
**Example 10.** *Take the presheaf from Example 9, and consider the cover* $\{V, W\}$ *again. Take the the selection of patches* $\{a, c\}$*, which are compatible because they agree on overlap:*



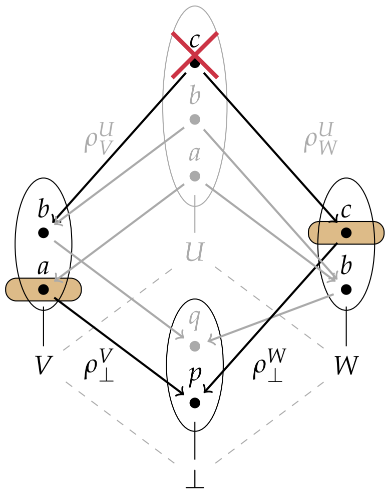

*Even though a and c are compatible, they do not glue together, because there is no section in* $F(U)$ *that restricts down to them. Consider* $a \in F(U)$ *first. It restricts to* $a \in F(V)$ *on the left, but it does not restrict to* $c \in F(W)$ *on the right:*




*As for* $b \in F(U)$*, it restricts to neither* $a \in F(V)$ *on the left nor* $c \in F(W)$ *on the right:*

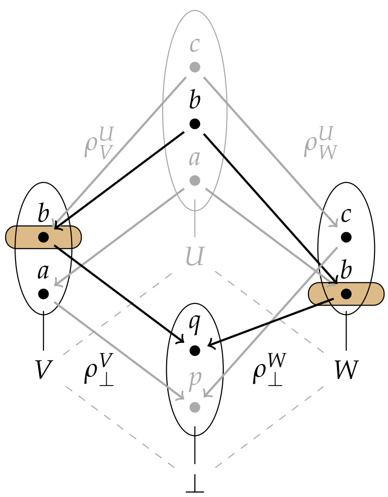



*Finally, $c \in F(U)$ restricts to $c \in F(W)$ on the right, but not to $a \in F(V)$ on the left:*



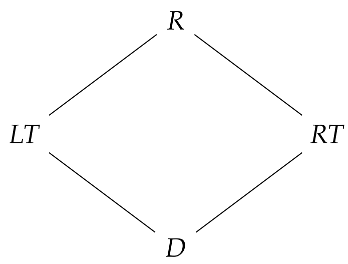

*Thus, none of $a$, $b$, or $c$ in $F(U)$ are glued from $\{a, c\}$, because none of them decompose into $a$ on the left and $c$ on the right.*

*Now suppose we pick $\{b, b\}$ for patch candidates. These do glue together (trivially), because there is a section in $F(U)$ (namely $b \in F(U)$) that restricts down to $b \in F(V)$ on the left and $b \in F(W)$ on the right:*



**Example 11.** *Consider an example that glues together behaviors. Imagine a toy robot that looks something like a small tank: it has tracks on the left and right sides, and the two tracks are connected by a single drive controller. The controller either drives at a constant speed, or it sits idle. When it drives, it turns both tracks at the same speed.*

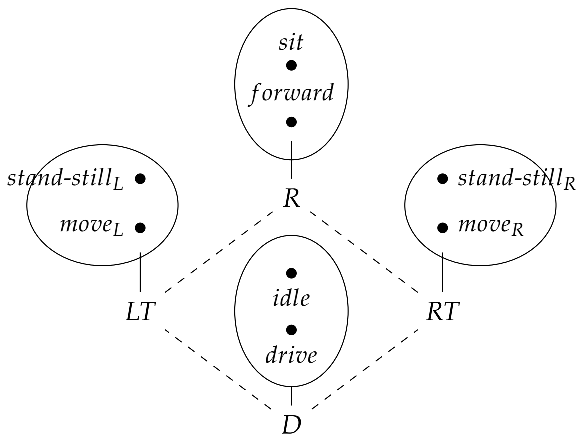
*Let's represent the robot as a locale. Let $LT$ and $RT$ be the left and right track assemblies respectively, let $D$ be the drive controller that is shared by $LT$ and $RT$, and let $R$ be the whole robot (the join of $LT$ and $RT$). As a picture:*




*For a presheaf, let's assign to each region the behaviors that are locally observable at that region:*

- *The drive controller $D$ can either drive (turn) or sit idle.*
- *The left track assembly can each either $move_L$ or $stand\text{-}still_L$.*
- *The right track assembly can also either $move_R$ or $stand\text{-}still_R$.*
- *The entire robot can either move $forward$ or sit stationary.*

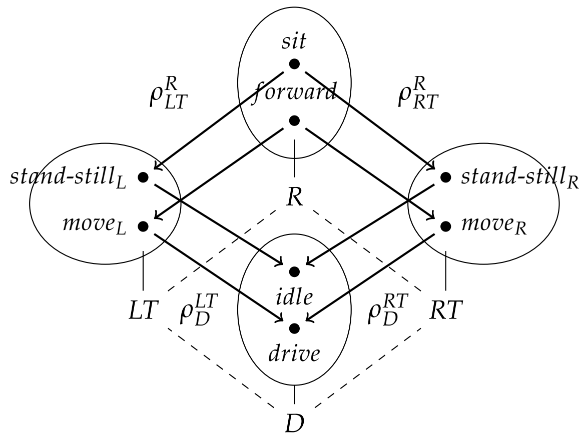
*In a picture:*




*For the restriction maps, let's say that they restrict the observable behavior of a larger region to the observable behavior of the smaller region. For instance, if you are observing the whole robot moving forward ($forward$), and you then "zoom in" on the left track assembly, you'll see those tracks rotating ($move_L$).*

- $\rho^R_{LT}(sit) = stand\text{-}still_L$, $\rho^R_{LT}(forward) = move_L$.
- $\rho^R_{RT}(sit) = stand\text{-}still_R$, $\rho^R_{RT}(forward) = move_R$.
- $\rho^{LT}_D(stand\text{-}still_L) = idle$, $\rho^{LT}_D(move_L) = drive$.
- $\rho^{RT}_D(stand\text{-}still_R) = idle$, $\rho^{RT}_D(move_R) = drive$.

*In a picture:*




*Now take the cover $\{LT, RT\}$ of R. The patch candidates $\{move_L, move_R\}$ are compatible, because they agree on overlap (they both restrict down to drive). But they also glue uniquely, yielding $forward$. In other words, the robot's forward motion is patched together precisely from the two pieces of its cover, namely the left tracks rotating ($move_L$) and the right tracks rotating ($move_R$).*

*Similarly, the Robot's sitting still (sit) behavior is also glued from the two pieces of its cover, namely the left track assembly standing still (stand-still$_L$) and the right track assembly standing still (stand-still$_R$).*

*Thus, there are two global sections of R's behavior: moving forwards (patched together from its left and right motions), or standing still (patched together from its left and right lack of motion).*

We can now state what it is to be a sheaf. A sheaf is a presheaf that satisfies a special gluing condition: namely, that for every cover, every compatible selection of patch candidates glues together uniquely.

Define separated presheaf here.

**Definition 16** (Sheaf). *A presheaf F is a sheaf iff it satisfies the following condition (called "the gluing condition"):*

(G0) *For every cover $\{U_i\}_{i\in I}$ of a region U and every selection of patch candidates $\{s_i\}_{i\in I}$ for that cover, if $\{s_i\}_{i\in I}$ are compatible, then there exists a unique gluing $s \in F(U)$ of $\{s_i\}_{i\in I}$.*

**Remark 11.** *There is a subtlety regarding what sheaves look like over the least element of a locale. Note that the gluing condition is formulated as an implication. That is to say, it says that, for every cross-section of patch candidates,* if *that cross-section can glue,* then *it glues in exactly one way.*

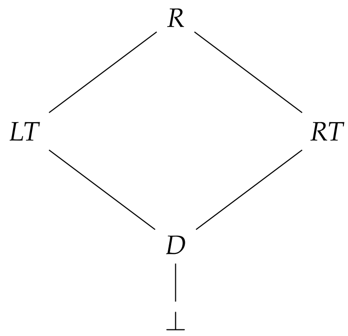
*Next, consider the fact that the cover over the least region of a locale is an empty cover. Since there are no patch candidates that need to be checked for compatibility, there is nothing that needs to be done to get a "selection of gluable patch candidates." Hence, the antecedent of the gluing condition is satisfied vacuously over the least element of the locale.*

*But since the empty cover satisfies the antecedent of the gluing condition vacuously, it follows that if a presheaf is to qualify as a sheaf, it must ensure that the consequent is satisfied over the empty cover as well. In other words, it must assign a unique glued section (a singleton set) to the least region of the locale. So, even though a* presheaf *may assign a larger set of data to the least element of a locale, a* sheaf *always assigns a singleton to that region.*

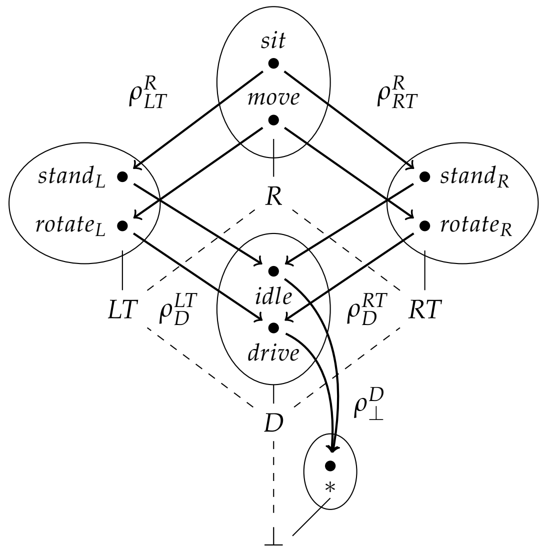
**Example 12.** *The presheaf from Example 9 fails to be sheaf, because as we saw in Example 10, there is a compatible selection of patch candidates (namely, $\{a, c\}$) which fails to glue. To be a sheaf, every compatible selection of patch candidates must glue.*

**Example 13.** *The presheaf from Example 11 fails to be a sheaf, because it does not assign a singleton to the lowest region of the underlying locale. In that example $D$ is the lowest region of the locale, and $F(D) = \{p, q\}$, a set containing two elements. Hence, $F$ fails to be a sheaf.*

*However, suppose we add a distinct bottom element to the locale:*




*If we assign a singleton set to $\bot$ (so that $F(\bot) = \{*\}$, say), then $F$ looks like this:*




*This modification ensures that $F$ qualifies as a sheaf, since it ensures that* all *gluable selections of patch candidates (including the empty one) glue uniquely.*

### *1.6. A Canonical Sheaf Construction*

Not every presheaf is a sheaf, since some presheaves fail the gluing condition. However, there is a canonical procedure called "sheafification" that turns any presheaf into a sheaf. To sheafify a presheaf, add any missing sections that glue, then quotient sections that are locally indistinguishable. The result is guaranteed to be a sheaf, by construction.

For our purposes, there is a simplified version of sheafification that we can use to construct sheaves that model part-whole complexes in a natural way. Given a presentation of a locale, the recipe to build such a sheaf over it goes like this:

1. Assign local data to atomic regions.
2. Specify a gluing condition.
3. Recursively glue more and more pieces together until you can't glue any more.

Let's make this more precise. Given a presented locale, we can uniquely write each region as the join of its atomic regions.

**Definition 17** (Atomic indices)**.** *Let $\mathbb{L} = \langle G, R \rangle$ be a presented locale with $G = \{U_1, \ldots, U_n\}$. Let $\mathbb{A} \subseteq \{0, \ldots, n\}$ be the indices of the atomic regions of $G$.*

*For any $U \in \mathbb{L}$, define its atomic support (denoted $I(U)$, or just $I$ for short) as:*

$$I(U) = \{i \in \mathbb{A} \mid U_i \preccurlyeq U\}$$

*Then $U$ can be written canonically as $U_{I(U)}$, the join of its atomic supports:*

$$U_{I(U)} = \bigvee_{i \in I(U)} U_i.$$

Say something along the lines that there are many kinds of sheafs, and the fibers can have many different shapes. For instance, a standard example of a sheaf over a topological space is the set of real-valued functions defined over each region of that topology.

We're going to focus on a simple kind of sparse sheaf. We will build this sheaf by assigning to atomic regions some basic data: to each atomic region $U_k$, we assign some piece of data $\{\langle b_k \rangle\}$. What $b_k$ is doesn't matter for our purposes at the moment. It just needs to be some piece of atomic data.

Then, we will glue together a selection of patch candidates $\{\langle b_1 \rangle\}$, $\{\langle b_2 \rangle\}$, and so on of atomic pieces of data to form $\{\langle b_1, b_2, \ldots \rangle\}$. To decide when to glue such tuples together, we will define a gluing condition.

Talk about how we are going to imagine that each "$b_k$" represents something that occupies the region $U_k$. E.g., it's a chunk of matter, or a mechanical gadget, or whatever. But it is the part or stuff in that atomic region. The larger whole will then be built up by fusing together these parts according to the gluing condition.

A gluing condition is a family of predicates that say when a selection of patch candidates glue.

**Definition 18** (Gluing condition)**.** *A gluing condition $\mathcal{G}$ is a family of predicates*

$$\mathcal{G}_U : \prod_{i \in I(U)} F(U_i) \to \{\text{true}, \text{false}\},$$

*i.e. for each region $U \in \mathbb{L}$ a predicate over the Cartesian product of the fibers of the atomic support of $U$, that collectively satisfy the following coherence conditions:*

*(G1)* Local data is glued. *If $U_k \in \mathsf{Atoms}(\mathbb{L})$ and $F(U_k) = \{\langle b_k \rangle\}$, then*

$$\mathcal{G}_{U_k}(\langle b_k \rangle) = \text{true}.$$

*(G2)* Downward stability. *If $\mathcal{G}_U(\langle b_i \rangle_{i \in I(U)}) = \text{true}$, then for each $V \preccurlyeq U$, we must have:*

$$\mathcal{G}_V(\rho^U_V(\langle b_i \rangle_{i \in I(U)})) = \text{true}.$$

*(G3)* Upward stability. *Given a selection of patch candidates $\langle b_i \rangle_{i \in I(U)}$, if $\mathcal{G}_{U_i \vee U_j}(\langle b_i, b_j \rangle) =$ true for each $i, j \in I(U)$, then we must have:*

$$\mathcal{G}_U(\langle b_i \rangle_{i \in I(U)}) = \text{true}.$$

A sheaf can be generated from a gluing condition by starting with some local data on the atomic regions and then gluing all pieces together that satisfy the gluing condition.

**Definition 19** ($\mathcal{G}$-sheaves)**.** *Given a gluing condition $\mathcal{G}$ and local data $F(U_k) = \{\langle b_k \rangle\}$ for each atomic region $U_k$, define for each region $U$:*

$$F(U) = \{\langle b_i \rangle_{i \in I(U)} \in \prod_{i \in I(U)} F(U_i) \mid \mathcal{G}_U(\langle b_i \rangle_{i \in I(U)}) = \mathsf{true}\}.$$

*For $V \preccurlyeq U$, define the restriction map*

$$\rho^U_V : F(U) \to F(V) \quad \textit{as} \quad \rho^U_V(\langle b_i \rangle_{i \in I(U)}) = \langle b_i \rangle_{i \in I(V)}.$$

*Set $F(\perp) = \{\langle\rangle\}$, the empty tuple.*

**Remark 12.** *Alternatively, given some local data and a gluing condition, define a presheaf over the given locale, call it $F_\wp$, that assigns all combinations of local data to each region:*

$$F_\wp(U) = \prod_{i \in I(U)} F(U_i).$$

*Then filter by the gluing condition. That produces the same sheaf.*

We must check that Definition 18 really defines a sheaf.

**Theorem 1** ($\mathcal{G}$-sheaves are presheaves)**.** *Given a gluing condition $\mathcal{G}$ and an assignment of local data to the atomic regions of the underlying locale, the corresponding $\mathcal{G}$-sheaf is a presheaf.*

**Proof.** We must show that restrictions preserve identities and composition.

- *Identities.* $\rho^U_U$ projects to the same index set, so $\rho^U_U = \mathsf{id}_{F(U)}$.
- *Composition.* $\rho^U_V$ restricts to the fiber over $V$, and $\rho^V_W$ restricts to the fiber over $W$, so $\rho^V_W \circ \rho^U_V = \rho^U_W$.

We must also show that the restrictions are well defined.

- For $V \preccurlyeq U$, if $\langle b_i \rangle_{i \in I(U)} \in F(U)$, then by (G2) $\langle b_i \rangle_{i \in I(V)}$ satisfies $\mathcal{G}_V$, so $\rho^U_V$ is well-defined. □

**Theorem 2** ($\mathcal{G}$-sheaves are sheaves)**.** *Given a gluing condition $\mathcal{G}$ and an assignment of local data to the atomic regions of the underlying locale, the corresponding $\mathcal{G}$-sheaf satisfies the gluing condition G0.*

**Proof.** We must show that every gluable selection of patch candidates $\langle b_i \rangle_{i \in I(U)}$ glues to yield a unique section in $F(U)$. Assume that we have a compatible selection of patch candidates $\langle b_i \rangle_{i \in I(U)}$. Then:

- Existence: we assumed the patch candidates are compatible. By (G3) then, $\langle b_i \rangle_{i \in I(U)} \in F(U)$.
- Uniqueness: let $s = \langle b_i \rangle_{i \in I(U)} \in F(U)$. If another section $t = \langle b_i \rangle_{i \in I(U)} \in F(U)$ were glued from the same components, then $s = t$, since both restrict to the same supports. □

Throughout the rest of this paper, we will use $\mathcal{G}$-sheaves to model part-whole complexes, but that is only for simplicity of exposition. Any sheaf over a locale would do just as well.

## Todo list