Article

Sheaf Mereology

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Abstract

Classical mereology is often developed in a parts-first manner: one begins by stipulating which parts there are, and fusions are then freely generated from those parts. Many find this unsatisfying, since it appears to license "mere Cambridge" fusions. This motivates a fusions-first alternative, on which fusions are taken as primary and parts are understood relative to them. In this paper, we develop a fusions-first framework for mereology based on sheaf theory. Mathematically, sheaf theory is a natural fit for this role, since one of its central features is a fully developed notion of coherent gluing. We introduce the relevant mathematical background and show how increasingly tighter sheaf-theoretic structures — presheaves, monopresheaves, and sheaves — can be used to model how pieces glue together in different sorts of part–whole complexes. We show that within this framework, certain principles such as extensionality and supplementation are not assumed, but instead emerge — or fail to emerge — as structural features of the fusions being modeled.

Keywords: mereology; restricted composition; fusions and integral wholes; presheaves; monopresheaves; sheaves; point-free topology; frames and locales

1. Introduction

Classical mereology ([1], [2], [3], [4]) is often developed in a parts-first manner. One begins by stipulating a primitive parthood relation and then one adopts a principle of unrestricted composition: for any selection of parts, there exists a fusion of exactly those parts.

This leads to a large number of fusions. Not only do familiar entities such as Socrates count as fusions, but so does any arbitrary collection of parts. For instance, a pencil in Boston together with a certain Highland Scotsman fuse into a whole. Or, to take one of Lewis's examples ([5, p. 80], [6, p. 48]), there are trout-turkeys: their front halves are trout and their back halves are turkeys. Any selection of parts, however heterogeneous, fuses into a whole.

Many find this counterintuitive. Unrestricted composition appears to make no distinction between genuine unities and gerrymandered or "mere Cambridge" fusions. On the face of it, so the objection goes, Socrates seems to be a genuine fusion in a way that the pencil and the Scot — or trout-turkeys — are not.

As stated, this objection begs the question against the classical mereologist. To deny that the pencil and the Scot form a fusion is simply to reject unrestricted composition. From within the classical framework, there is no principled basis for excluding such cases.

Of course, there is more substantive debate about unrestricted composition (regarding vagueness, e.g. [7], [8], [9], [10]; regarding composition as identity, e.g. [11], [12], [13], [14]; and so on). But that need not detain us. The persistence of the naive objection reveals a

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widely shared intuition that there is more to composition than the mere selection of parts. Genuine fusion seems to involve some form of connection or coherence: the constituents of a whole must be related in the right way, held or glued together rather than merely collected.

This suggests a different methodological strategy. Instead of starting with a parthood relation and freely generating fusions, one might take a fusions-first approach: begin by stipulating which combinations of entities count as genuine fusions — those in which appropriate gluing conditions are satisfied — and then define parthood in terms of those fusions.

To be sure, plenty of philosophers take a fusions-first approach (at least in spirit) by adopting some form or other of *restricted* composition (e.g., [15], [16], [9], [17], [18], [19]). Many of these approaches tend to proceed by developing a mereological theory from scratch.

From a mathematical perspective, sheaf theory ([20], [21], [22], [23]) already provides a natural framework for articulating the idea. Central to sheaf theory is a precise notion of gluing: local pieces may be combined into a global whole only when they satisfy explicit compatibility and coherence conditions. The theory therefore already comes equipped with a principled account of when and how wholes are formed from parts.

Sheaf theory is well established in topology, algebraic geometry, and topos theory, but it has seen comparatively little use in philosophical mereology. This paper develops a sheaf-theoretic framework for modeling part-whole complexes. It is an honest fusions-first approach to mereology.

The proposal has clear affinities with mereotopology (e.g. [24], [25], [26], [27]), which likewise aims to identify genuine fusions by articulating notions such as connection and continuity. Yet despite its openness to topological ideas, mereotopology has largely not drawn on the technical resources of sheaf theory. One aim of this paper is therefore to help bridge the gap between mathematical techniques that are well suited to modeling glued wholes and philosophical discussions of parthood and composition.

There is another, independent reason to adopt the sheaf-theoretic approach. Classical mereology treats the collection of available parts as having a rich algebraic structure — typically something close to a Boolean algebra. At the same time, it draws ontological conclusions directly from that algebraic structure. For instance, consider the bottom element. Formally, this is very natural as the least element of a Boolean algebra, but many find it ontologically suspect. So, many classicists remove it from the algebra (as Tarski noticed long ago, [28], [29]; cf. [4, §4.5]).

This reflects a deeper confusion between two distinct questions: what *combinations* of parts are formally available within a parts-space, and which of those combinations are ontologically *realized*. Classical mereology attempts to resolve ontological worries by altering the algebra itself, rather than by distinguishing the algebraic framework from the entities that inhabit it.

The sheaf-theoretic framework avoids this confusion from the outset. The sheaf-theoretic approach requires that one first specify a background structure of parts — a parts-space that determines all the ways in which regions or parts can combine. This structure is fixed independently of ontological considerations. Only once this background is in place does one specify which entities, if any, occupy the various part-locations. Some regions of the parts-space may be inhabited; others may remain empty. The absence of an occupant is an ontological fact, not a defect in the underlying algebra.

In this respect, the framework separates possibility from actuality, and algebra from ontology. The parts-space records the space of possible decompositions and recombinations, while the sheaf data records how, and where, material or other entities are instantiated within that space. Ontological sparsity is expressed by empty fibers, not by removing elements from the algebra.

A close relative is so-called slot mereology, which distinguishes between part "slots" and "fillers" ([30], [31], [32], [33]). In its original formulation, slot mereology requires only that the slots form a multiset. More recently, [34] enriched the slots with further algebraic structure. Even so, this background structure is not equipped with a notion of covering, and thus does not by itself encode admissible decompositions. By contrast, the sheaf-theoretic approach already treats the background parts-space as a geometric *space*, richly structured with algebraic combinations and coverings.

This bears on the relation between our approach and mereotopology too. Topological structure is often used in mereotopology to model spatial or material features. Sheaf theory accommodates such applications but is not limited to them. In our approach, we generalize beyond topological spaces and instead employ structures known as locales ([35], [36], [37], [22, §IX], [38, ch. 1]). These are point-free spaces whose structure is given entirely by the lattice of their regions. Moreover, Alexander Grothendieck long ago generalized sheaf theory far beyond spatial settings. This makes sheaf theory well suited for modeling part-whole complexes that are not straightforwardly spatial or material, while still retaining a precise notion of gluing and coherence.

Another benefit of the sheaf-theoretic approach is that it is highly flexible. It provides a hierarchy of structures — called presheaves, monopresheaves, and sheaves — corresponding to increasingly stronger requirements on how local pieces fit together. This allows part-whole complexes to be modeled more loosely or more strictly, depending on the phenomena at hand. As a result, the framework supports an intuitive mereology while not building in substantive principles such as extensionality or supplementation by default. These principles may be imposed when appropriate, but they are not forced by the formalism.

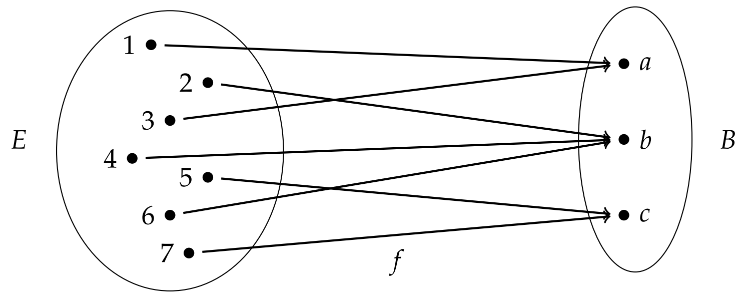
The plan of the paper is as follows. In Section 2, we introduce the elements of sheaf theory needed for the remainder of the paper. We assume only familiarity with set-theoretic notation and first-order logic. No prior knowledge of topology or sheaf theory is required, and we provide worked examples to illustrate the core ideas. In Section 3, we demonstrate the flexibility of the framework by modeling a range of different part-whole complexes. In Section 4, we translate familiar mereological principles into the sheaf-theoretic setting, making explicit what those principles amount to within the framework. We conclude with a brief summary and discussion of directions for further work.

2. Sheaf Theory

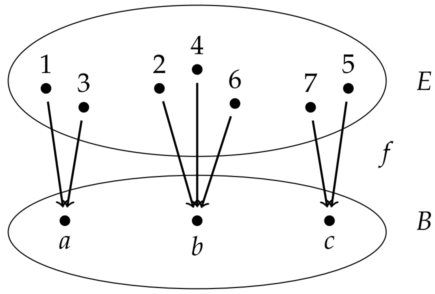
In this section, we introduce the parts of sheaf theory needed for the sequel.

2.1. Fibers

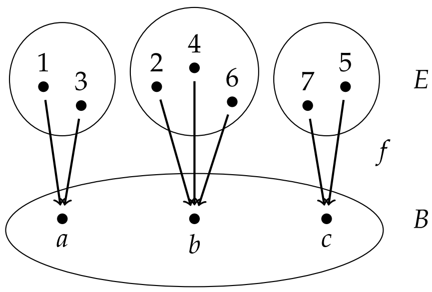
Suppose we have a map (function) $f : E \to B$ that looks something like this:



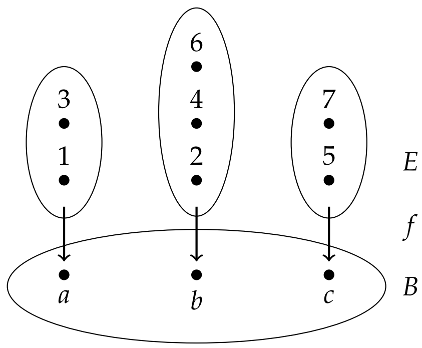
It is sometimes convenient to turn the diagram sideways and group together points in the domain that get sent to the same target, like so:



That makes the pre-images very easy to see. For any point in B, its pre-image is just the group of points sitting "over" it:



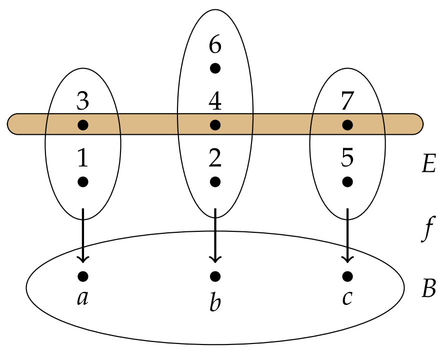
If we stack the points in each pre-image vertically, one on top of the other, we can then think of each pre-image as a kind of "stalk" growing over its base point:



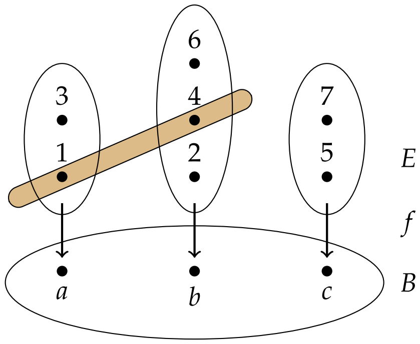
This gives rise to the idea of the "fibers" of a map. The fibers of a map are just its pre-images. For instance, the fiber over b is $\{2, 4, 6\}$.

Definition 1 (Fibers)**.** *Given a map $f : E \to B$ and a point $y \in B$, the fiber over y is its pre-image $f^{-1}(y) = \{x \mid f(x) = y\}$. B is called the base space of f, and y the base point of the fiber.*

We can take a cross-section of one or more fibers by selecting a point from each of the fibers in question. For instance, we can take 3, 4, and 7 as a cross-section of the fibers $f^{-1}(a)$, $f^{-1}(b)$, and $f^{-1}(c)$:



We can also take cross-sections local to only some of the fibers. For instance, we can take 1 and 4 as a cross-section of $f^{-1}(a)$ and $f^{-1}(b)$:



Definition 2 (Sections)**.** *Given a map $f : E \to B$ and a subset of base points $C \subseteq B$, a section of f (over C) is a choice of one element from each fiber over each base point $x \in C$.*

Remark 1. *Since each point in a fiber amounts to a section over the fiber's base, the elements of a fiber are often just called the sections of the fiber.*

2.2. Spaces

In the above examples, the base B was a set. We often want to consider bases that have more structure, e.g., bases that have spatial structure.

In traditional topology, spaces are built out of the points of the space. Given a set of points, a topology on that set specifies which points belong in which regions of the space.

Definition 3 (Topology)**.** *Let X be a non-empty set, thought of as the points of the space. A topology on X is a collection T of subsets of X, thought of as the regions of the space and called the open sets (or just the opens) of T, that satisfy the following conditions:*

(T1) The empty set and the whole set are open:

$$\varnothing \in T, X \in T.$$

(T2) Arbitrary unions of opens are open:

$$\text{if } \{U_i\}_{i \in I} \subseteq T, \text{ then } \bigcup_{i \in I} U_i \in T.$$

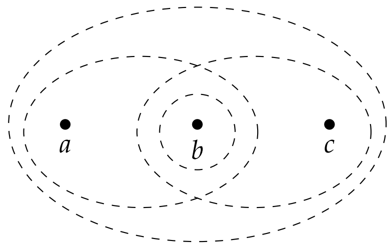
(T3) Finite intersections of opens are open:

$$\text{if } U_1, \ldots, U_n \in T, \text{ then } \bigcap_{i=1}^{n} U_i \in T.$$

These conditions encode the way that spatial regions are put together. For instance, it ensures that if two regions overlap, then their overlapping area is a region too.

Remark 2. *The regions of a topology, ordered by inclusion, form a complete lattice. Since the topology includes arbitrary unions, the join of this lattice is set union, but since the topology includes only finite intersections, the meet of this lattice is the* interior *of set intersection.*

Example 1. *Let $X = \{a, b, c\}$. One possible topology is $T = \{\varnothing, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. If we draw dashed circles around each of the opens (regions), we get:*



There are two regions $\{a,b\}$ and $\{b,c\}$ that overlap at b (so $\{b\}$ is a region in T too). There is also the full region $\{a,b,c\}$, which is the union of the smaller regions.

We can draw T as a Hasse diagram, which shows that the regions form a lattice:

$$\begin{tikzcd}[column sep=small] & \{a,b,c\} \arrow[dl, dash] \arrow[dr, dash] & \\ \{a,b\} \arrow[dr, dash] & & \{b,c\} \arrow[dl, dash] \\ & \{b\} \arrow[d, dash] & \\ & \varnothing & \end{tikzcd}$$

The lattice structure suggests that much of what is important about a space is not so much its points, but rather its opens/regions. This leads to the idea that topology-like reasoning can be done without the points. So, we can generalize: take a topology, and drop the points. That leaves just the opens/regions, which we call a frame (or locale).

Definition 4 (Frames/locales)**.** *A frame (synonymously, a locale) $\mathbb{L}$ is a partially ordered set L (whose elements are called opens or regions) that satisfies the following conditions:*

(L1) L is a complete lattice:

- *Every subset $S \subseteq L$ has a join, denoted $\bigvee S$.*
- *Every finite subset $F \subseteq L$ has a meet, denoted $\bigwedge F$.*

(L2) Finite meets distribute over arbitrary joins:

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i), \text{ for all } a \in L \text{ and all families } \{b_i\}_{i \in I} \subseteq L.$$

Define $V \preccurlyeq U$ (read "V is included in U") by $a = a \wedge b$.

Remark 3. *The fact that $V \preccurlyeq U$ is equivalent to $a = a \wedge b$ means we can deal with the opens of a frame algebraically (via $\wedge$ and $\vee$ operations), or order-theoretically (via the $\preccurlyeq$ relation), whichever is more convenient.*

Remark 4. *The category of locales is defined as the dual/opposite of the category of frames, and so frames and locales are quite literally the very same objects. In practice, frames are often used for algebraic purposes, and locales are used for (generalized) spatial purposes. Here, we will have no reason to distinguish these two roles, and so we will use the names "frame" and "locale" interchangeably.*

By definition, every locale has a lowest element, which is the join of no regions at all. It represents the absence of any regions whatever. Hence, we typically denote it with the symbol "$\perp$." Dually, every locale has a highest element, which is the join of all of the regions. Hence, when convenient we can denote it with the symbol "$\top$."

2.3. Presentations of locales

Locales have presentations much like groups and other algebraic structures. To give the presentation of a locale, specify a set of generators and relations.

Definition 5 (Presentations). *A presentation $\langle G, R\rangle$ of a locale $\mathbb{L}$ is comprised of:*

(P1) A set of generators $G = \{U_k, U_m, \ldots\}$.
(P2) A set of relations $R \subseteq G \times G$ on those generators.

The locale $\mathbb{L}$ presented by $\langle G, R\rangle$ is the smallest one freely generated from G which satisfies R.

Remark 5. *Every locale has a presentation, and a locale can have multiple presentations.*

To calculate the locale that corresponds to a presentation, start with the generators, then take all finite meets and all arbitrary joins that satisfy R (and of course L1 and L2).

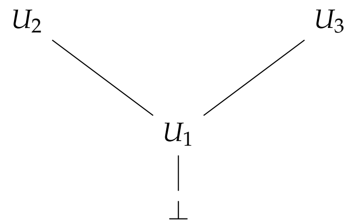
Example 2. *Let a locale $\mathbb{L}$ be given by the presentation $\langle G, R\rangle$ where:*

- $G = \{\perp, U_1, U_2, U_3\}$.
- $R = \{\perp \preccurlyeq U_1, U_1 \preccurlyeq U_2, U_1 \preccurlyeq U_3\}$.

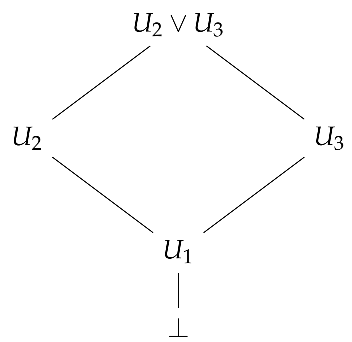
There are four generators ($\perp$, U_1, U_2, and U_3), and $\perp$ is below U_1 while U_1 is a sub-region of U_2 and U_3. Since U_1 is a sub-region of both U_2 and U_3, U_1 is their meet:

- $U_1 = U_2 \wedge U_3$.

At this point, we have generated this much of the locale:



R says nothing to constrain joins, so we need to join everything we can. In this case, we need to join U_2 and U_3:



There are no further joins or meets that aren't already represented in the picture. For instance, all further non-trivial meets are already accounted for:

- $U_1 \wedge \perp = \perp$.
- $U_2 \wedge U_1 = U_1$ *and* $U_3 \wedge U_1 = U_1$.
- $U_2 \wedge \perp = \perp$ *and* $U_3 \wedge \perp = \perp$.
- $(U_2 \vee U_3) \wedge U_2 = U_2$ *and* $(U_2 \vee U_3) \wedge U_3 = U_3$.
- $(U_2 \vee U_3) \wedge U_1 = U_1$.

- $(U_2 \vee U_3) \wedge \perp = \perp$.

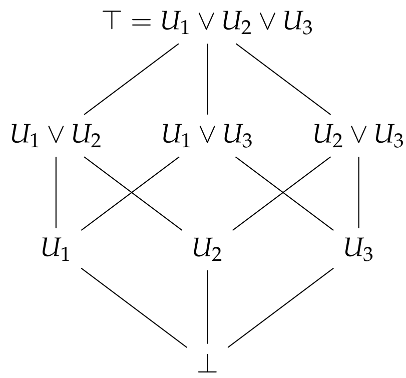
Similarly, all other non-trivial joins are also already accounted for:

- $\perp \vee U_1 = U_1$.
- $\perp \vee U_2 = U_2$ *and* $\perp \vee U_3 = U_3$.
- $\perp \vee (U_2 \vee U_3) = U_2 \vee U_3$.
- $U_1 \vee U_2 = U_2$ *and* $U_1 \vee U_3 = U_3$.
- $U_2 \vee (U_2 \vee U_3) = U_2 \vee U_3$ *and* $U_2 \vee (U_3 \vee U_3) = U_2 \vee U_3$.

Example 3. *Let* $\mathbb{L} = \langle G, R \rangle$ *be given by:*

- $G = \{U_1, U_2, U_3\}$.
- $R = \varnothing$.

We have three generators (U_1, U_2, and U_3), and there are no relations restricting how those generators are related. Thus, the locale that is freely generated from this presentation is isomorphic to the power set of three elements:



A presentation provides the most "minimal" information from which the rest of the locale is generated.

2.4. Presheaves

Above we considered the fibers of a map $f : E \to B$, where E and B were sets. We can also consider fibers over locales, where the fibers respect the locale's structure. This is called a presheaf. A presheaf is an assignment of data to each of a locale's regions that is "stable under restriction," i.e., that respects zooming in and out.

Definition 6 (Presheaf)**.** *Let* $\mathbb{L}$ *be a locale, and let* $\mathsf{Arr}(\mathbb{L})$ *be* $\{\langle A, B\rangle \mid A \preccurlyeq B \in \mathbb{L}\}$. *A presheaf on* $\mathbb{L}$ *is a pair* $\langle F, \{\rho^B_A\}_{\langle A,B\rangle \in \mathsf{Arr}(\mathbb{L})}\rangle$, *where:*

- *F assigns to each region $U \in L$ some data $F(U)$.*
- $\{\rho^B_A\}_{\langle A,B\rangle \in \mathsf{Arr}(\mathbb{L})}$ *is a family of maps* $\rho^B_A : F(B) \to F(A)$ *(called restriction maps), each of which specifies how to restrict the data over* $F(B)$ *down to the data over* $F(A)$.

All together, $\langle F, \{\rho^B_A\}_{\langle A,B\rangle \in \mathsf{Arr}(\mathbb{L})}\rangle$ *must satisfy the following conditions:*

(R1) Restrictions preserve identity:

$$\rho^U_U = \mathsf{id}_U \text{ (the identity on } U\text{), for every } U \in \mathbb{L}.$$

(R2) Restrictions compose:

$$\text{If } A \preccurlyeq B \text{ and } B \preccurlyeq C, \text{ then } \rho^C_A = \rho^B_A \circ \rho^C_B.$$

Since F assigns data $F(U)$ to each region $U \in \mathbb{L}$, we can think of the $F(U)$s as the "fibers" over $\mathbb{L}$, and the restriction maps as "zoom in" maps that go from bigger fibers down to smaller fibers.

Remark 6. *For the category-theoretically inclined, a presheaf is just a set-valued contravariant functor $F : \mathbb{L}^{\mathsf{op}} \to \mathbb{S}\mathsf{et}$. To each region B of $\mathbb{L}$, F assigns to it a set $F(B)$. The contravariance comes from the fact that, to each arrow $B \preccurlyeq C$ of $\mathbb{L}$, F assigns a restriction map that goes the other way (i.e., that restricts the data from $F(C)$ down to $F(B)$). (R1) and (R2) are automatically satisfied by the fact that F is a functor.*

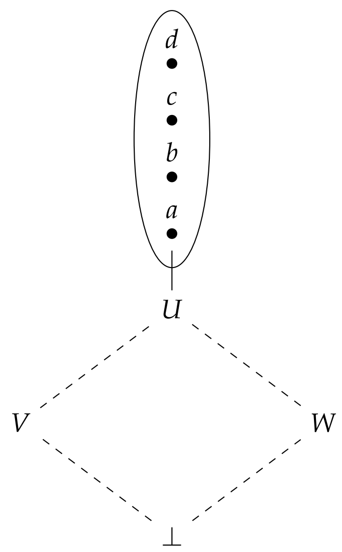
Example 4. *Let $\mathbb{L}$ be a locale $\{\bot, W, V, U\}$ with the following structure:*

$$\begin{tikzcd}[column sep=small] & U \arrow[dl, dash] \arrow[dr, dash] & \\ V \arrow[dr, dash] & & W \arrow[dl, dash] \\ & \bot & \end{tikzcd}$$

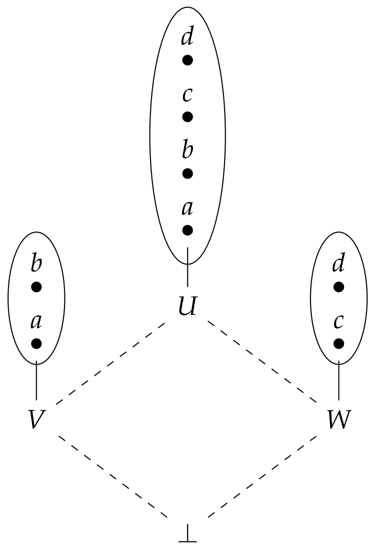
Next, let's define a presheaf F as follows:

- $F(U) = \{a, b, c, d\}$, $F(V) = \{a, b\}$, $F(W) = \{c, d\}$, $F(\bot) = \{*\}$.
- *Define ρ^U_V as the projection (send a to a, b to b, and the rest can go anywhere), and similarly for ρ^U_W. Let $\rho^U_\bot$, $\rho^V_\bot$, and $\rho^W_\bot$ send their data to $\{*\}$, and let the rest be identities.*

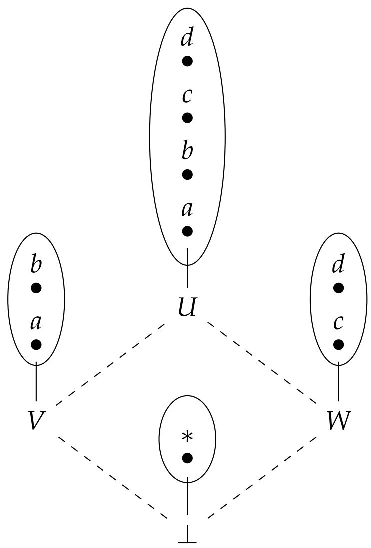
We can see F's assignments as fibers over $\mathbb{L}$ by drawing them over the regions they are assigned to. For instance, over U we have $F(U)$, i.e., $\{a, b, c, d\}$:



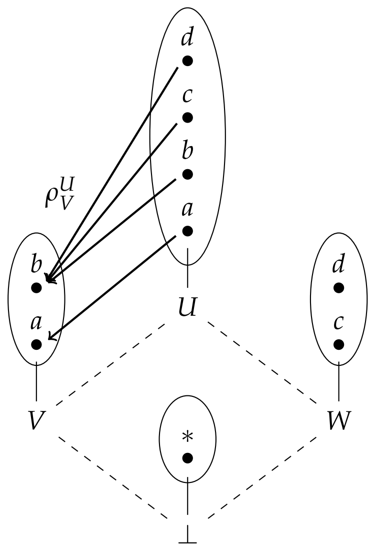
Similarly, over V and W, we have $F(V) = \{a, b\}$ and $F(W) = \{c, d\}$:



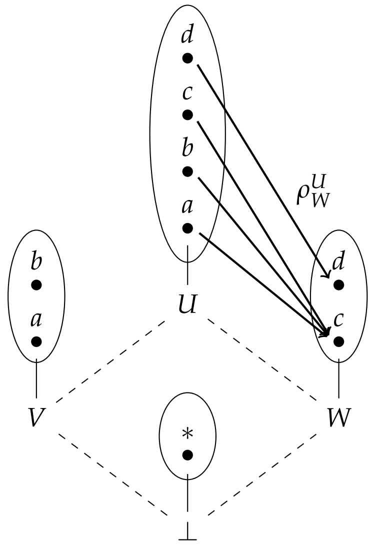
Finally, over $\perp$*, we have a singleton set:*



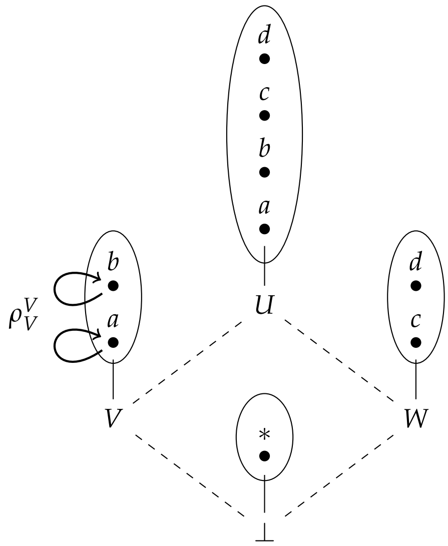
The restriction maps show how to "zoom in" on the data over each region. For instance, ρ^U_V *shows how to restrict the data in the fiber over* U *down to the data in the fiber over* V*:*



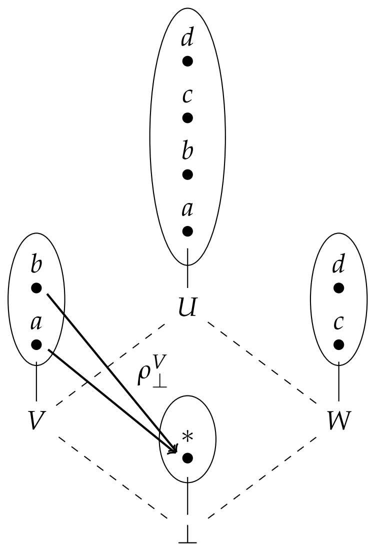
It's similar for the fiber over W:



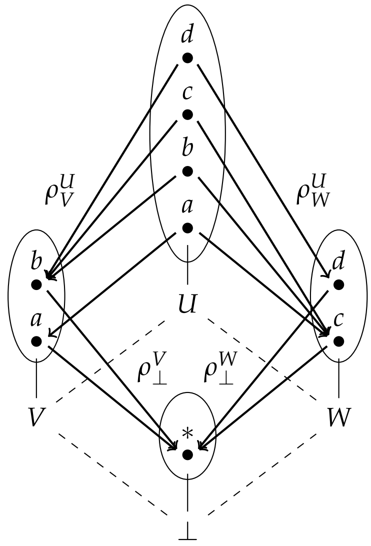
Restricting a fiber to itself is just the identity on the fiber:



The other restriction maps restrict down to the singleton set. For instance:



All of this makes it clear that the structure of the presheaf data that sits in the fibers over $\mathbb{L}$ mimics (respects) the structure of the base locale:



2.5. Gluing

The definition of a presheaf requires only that the data be stable under restriction (zooming in on a region). It does not require that the data fit together across different regions (fibers).

In some cases though, certain sections in different fibers turn out to be compatible (i.e., there is a coherent way to patch them together). When this occurs, those compatible sections can be glued together to form sections that stretch across fibers.

To get at this idea, let's first define a cover. A cover of a region U is a selection of sub-regions that covers U in its entirety. The chosen sub-regions don't leave any part of U exposed.

Definition 7 (Cover)**.** *Let $\mathbb{L}$ be a topology or a locale, and let U be a region of $\mathbb{L}$. A cover of U is a family $\{U_i\}_{i\in I} \subseteq \mathbb{L}$ such that:*

$$U = \bigvee_{i\in I}\{U_i\}.$$

In other words, a cover of U is a family of regions that join together to form U.

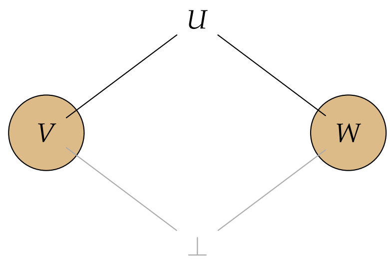
Example 5. *Take the topology from Example 1: $T = \{\varnothing, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$. A cover of $\{a,b,c\}$ is $\{a,b\}$ and $\{b,c\}$, because altogether, $\{a,b\}$ and $\{b,c\}$ cover all of the points in $\{a,b,c\}$.*

Another cover of $\{a,b,c\}$ is $\{\{a,b\},\{b,c\},\{b\}\}$. Although $\{b\}$ is redundant here, this choice of sub-regions still entirely covers $\{a,b,c\}$ as required.

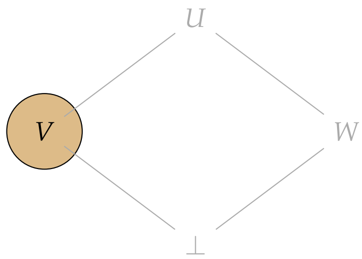
Example 6. *In the context of locales, where there are no points, a cover of U is just a selection of sub-regions of U that join together to form U. Take the locale from Example 4:*

$$\begin{tikzcd}[column sep=small]
 & U \arrow[dl, dash] \arrow[dr, dash] & \\
V \arrow[dr, dash] & & W \arrow[dl, dash] \\
 & \perp &
\end{tikzcd}$$

A cover of U is $\{V, W\}$, since $U = \bigvee\{V, W\}$:



A cover of V is just $\{V\}$:

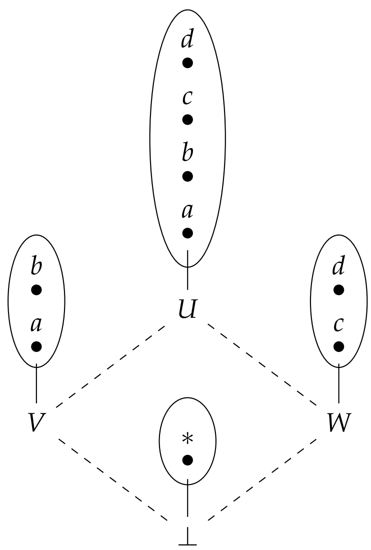


Remark 7. *A cover over the least element of a locale (or a topology) is empty (the empty set), because there are no regions (or points) to cover.*

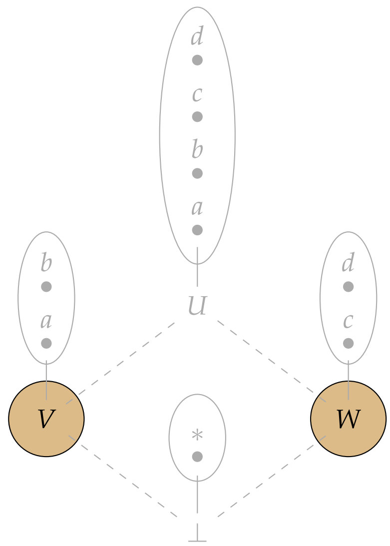
Given a presheaf F over a locale $\mathbb{L}$, if we have a cover $\{U_i\}_{i \in I}$ of some portion of $\mathbb{L}$, there is a corresponding family of fibers $\{F(U_i)\}_{i \in I}$ over that cover. We can pick one section (i.e., one element) from each such fiber to get a slice of elements that spans all of the fibers over that cover. Let us call such a choice a selection of patch candidates.

Definition 8 (Patch candidates)**.** *Given a presheaf F and a cover $\{U_i\}_{i \in I}$ with a corresponding family of fibers $\{F(U_i)\}_{i \in I}$, a selection of patch candidates $\{s_i\}_{i \in I}$ is a choice of one section s_i from each $F(U_i)$.*

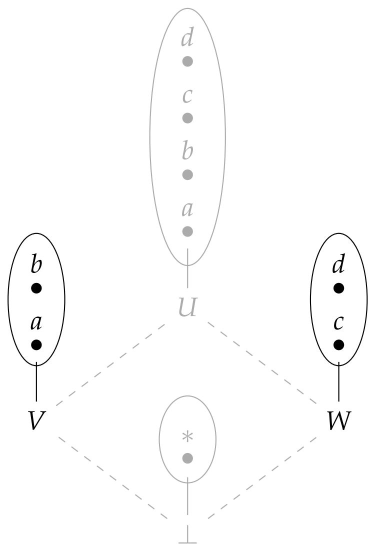
Example 7. *Take the presheaf from Example 4:*



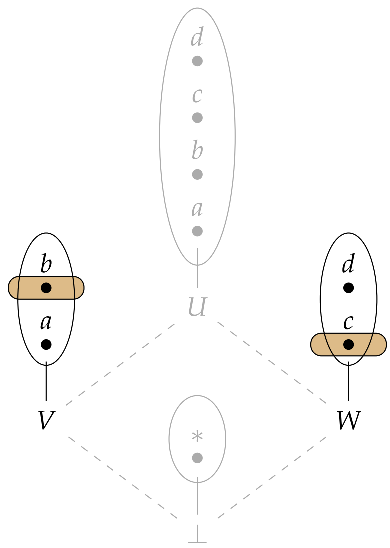
Let $\{V, W\}$ be the cover of interest:



Over this cover, we have a corresponding family of fibers:



A selection of patch candidates is a choice of one section (element) from each fiber. For instance, we might pick b from $F(V)$ and c from $F(W)$:



Similarly, we might pick $\{a, d\}$, $\{b, d\}$, or $\{a, c\}$, each of which is a valid selection of patch candidates.

Example 8. *Consider the empty cover. Since there are no sub-regions below the least element of a locale, there are no patch candidates we could choose for the empty cover either. Hence, any selection of patch candidates for the empty cover is $\varnothing$.*

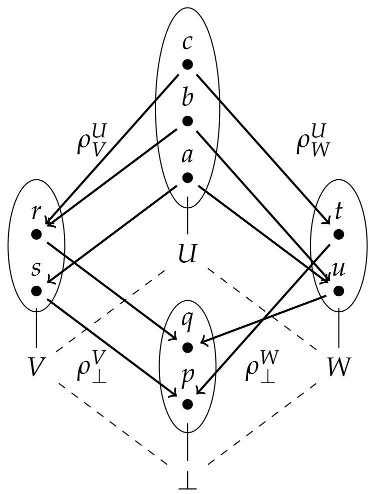
A selection of patch candidates might fit together, or they might not. We say they are compatible if they fit together, i.e., if they agree on overlaps. To check this, take any pair of patch candidates, and check if they restrict to the same data on their overlap.

Definition 9 (Compatible patch candidates)**.** *Given two fibers $F(U_i)$ and $F(U_j)$ and a patch candidate from each, $s_i \in F(U_i)$ and $s_j \in F(U_j)$, s_i and s_j are compatible if they restrict to the same data on their overlap $U_i \wedge U_j$:*

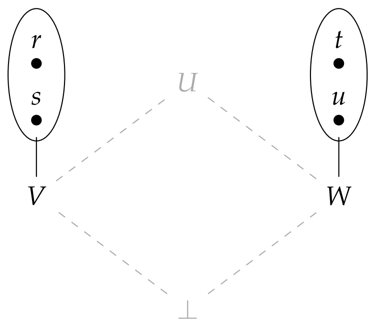
$$\rho^{U_i}_{U_i \wedge U_j}(s_i) = \rho^{U_j}_{U_i \wedge U_j}(s_j).$$

A selection of patch candidates $\{s_i\}_{i \in I}$ is compatible if all of its members are pair-wise compatible.

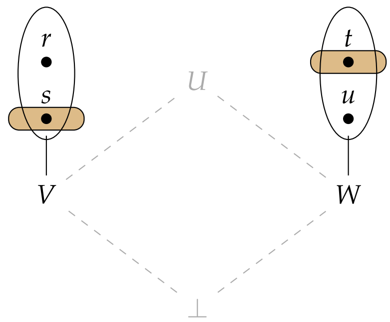
Example 9. *Consider the following presheaf F:*



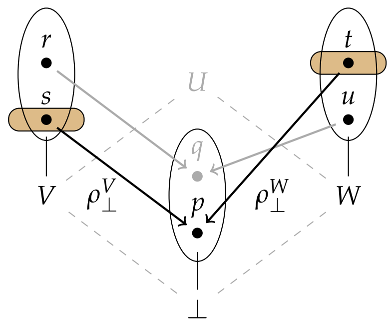
Take the cover $\{V, W\}$ and its corresponding fibers:



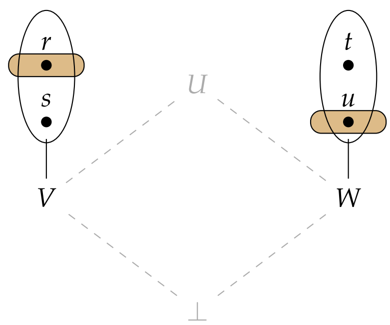
Suppose we pick $\{s, t\}$ for patch candidates:



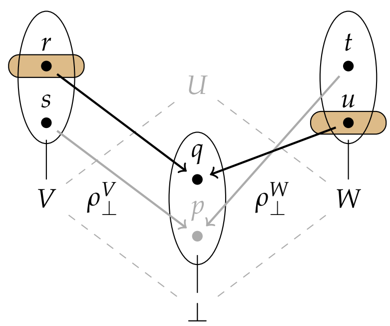
Is this selection compatible? We have to check if they agree on their overlap. The overlap $V \wedge W$ is $\perp$. Where does $\rho^V_\perp$ send our chosen patch candidate s? It sends it to p, since $\rho^V_\perp(s) = p$. Where does $\rho^W_\perp$ send our other chosen patch candidate t? It also sends it to p, since $\rho^W_\perp(t) = p$. On the overlap $\perp$ then, $\rho^V_\perp(s) = \rho^W_\perp(t)$, so s and t are compatible. This is easy to see in the diagram, since s and t both get sent to the same place:



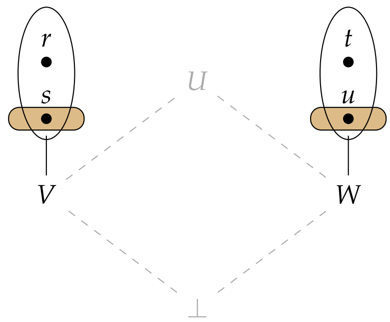
Now suppose we pick $\{r, u\}$ for patch candidates:



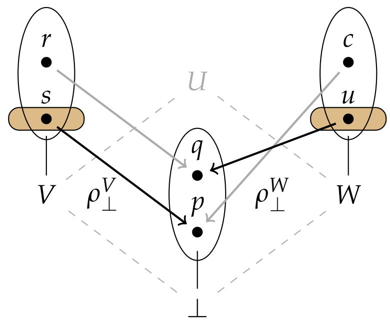
These are also compatible. They agree on their overlap (both restrict to q):



Finally, suppose we pick $\{s, u\}$ for patch candidates:



These are not compatible. They do not agree on their overlap:



Example 10. *Consider the empty cover. Since any selection of patch candidates for the empty cover is empty, compatibility is satisfied vacuously.*

When selected patch candidates $s_i, \dots, s_k$ across a cover of U are compatible, we say those patches glue together if if there's a section s in $F(U)$ that restricts down to exactly those patches.

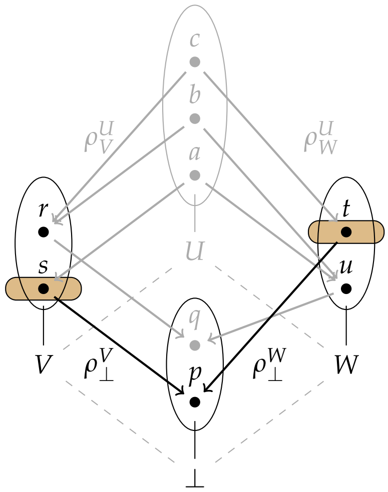
Definition 10 (Gluing)**.** *Given a presheaf F and a selection of compatible patch candidates $\{s_i\}_{i \in I}$ for a cover $\{U_i\}_{i \in I}$, $\{s_i\}_{i \in I}$ glue together only if there is a section $s \in F(U)$ that restricts down to s_i on each fiber $F(U_i)$ of the cover, i.e., only if s is such that:*

$$\rho^U_{U_i}(s) = s_i, \text{ for each } i \in I.$$

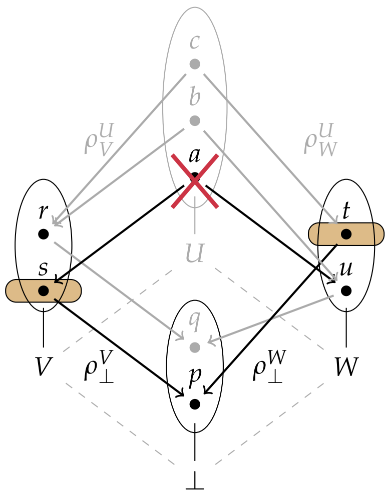
A selection of patches $\{s_i\}_{i \in I}$ glues uniquely if there is one and only one such section $s \in F(U)$ that is glued from them.

Remark 8. *As a matter of terminology, if a section $s \in F(U)$ is glued from patches $\{s_i\}_{i \in I}$, we say that s is a global section of the cover, and each s_i is a local section of the cover. We may also say variously that s is a* gluing *of those patches, that s is* composed *of those patches, that those patches* compose *s, or that gluing those patches* yields *s.*

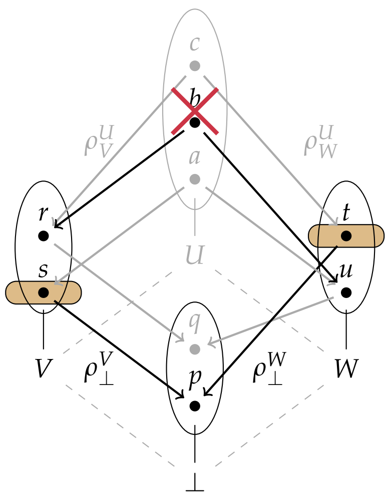
Example 11. *Take the presheaf from Example 9, and consider the cover $\{V, W\}$ again. Take the selection of patches $\{s, t\}$, which are compatible because they agree on overlap:*



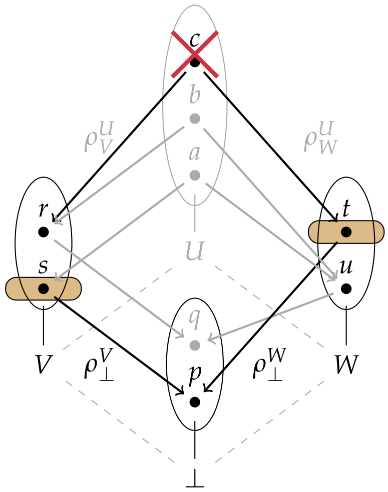
Even though s and t are compatible, they do not glue together, because there is no section in $F(U)$ that restricts down to them. Consider $a \in F(U)$ first. It restricts to $s \in F(V)$ on the left, but it does not restrict to $t \in F(W)$ on the right:



As for $b \in F(U)$, it restricts to neither $s \in F(V)$ on the left nor $t \in F(W)$ on the right:

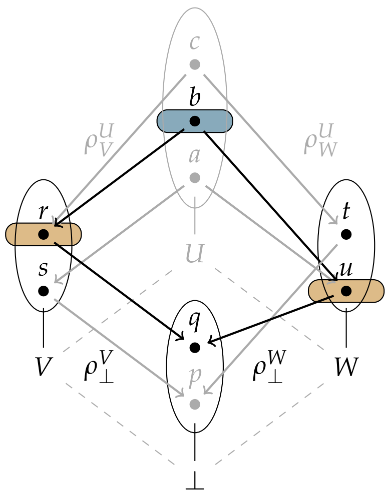


Finally, $c \in F(U)$ restricts to $t \in F(W)$ on the right, but not to $s \in F(V)$ on the left:



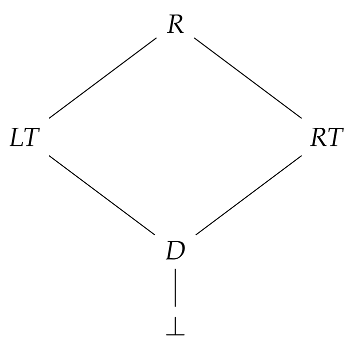
Thus, none of a, b, or c in $F(U)$ are glued from $\{s,t\}$, because none of them decompose into s on the left and t on the right.

Now suppose we pick $\{r,u\}$ for patch candidates. These do glue together, because there is a section in $F(U)$ (namely $b \in F(U)$) that restricts down to $r \in F(V)$ on the left and $u \in F(W)$ on the right:



Example 12. *Consider an example that glues together behaviors. Imagine a toy robot that looks something like a small tank: it has tracks on the left and right sides, and the two tracks are connected by a single drive controller. The controller either drives at a constant speed, or it sits idle. When it drives, it turns both tracks at the same speed.*

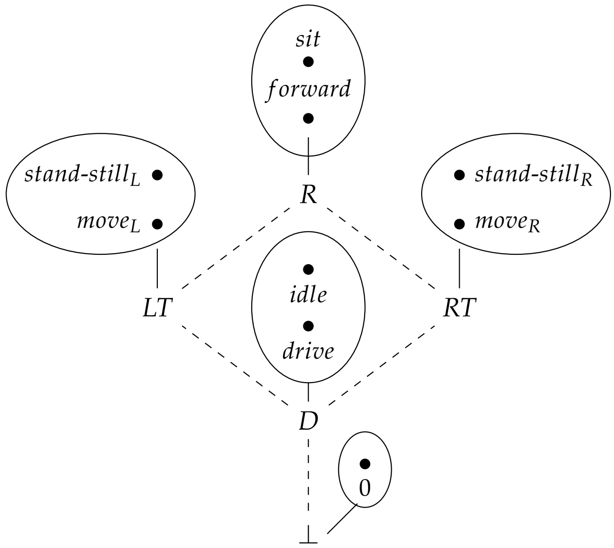
Let's represent the robot as a locale. Let LT and RT be the left and right track assemblies respectively, let D be the drive controller that is shared by LT and RT, and let R be the whole robot (the join of LT and RT). As a picture:



For a presheaf, let's assign to each region the behaviors that are locally observable at that region:

- *The drive controller D can either drive or sit idle.*
- *The left track assembly can either* $move_L$ *or* $stand\text{-}still_L$.
- *The right track assembly can also either* $move_R$ *or* $stand\text{-}still_R$.
- *The entire robot can either move* $forward$ *or sit stationary.*
- *For the fiber over* $\perp$, *where there are no regions that could carry any behaviors to begin with, assign the special symbol zero.*

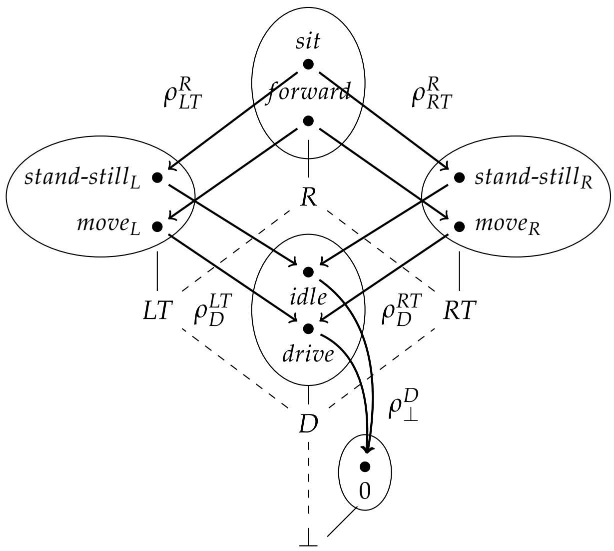
In a picture:



For the restriction maps, let's say that they restrict the observable behavior of a larger region to the observable behavior of the smaller region. For instance, if you are observing the whole robot moving forward ($forward$*), and you then "zoom in" on the left track assembly, you'll see those tracks rotating (*$move_L$*).*

- $\rho^R_{LT}(sit) = stand\text{-}still_L$, $\rho^R_{LT}(forward) = move_L$.
- $\rho^R_{RT}(sit) = stand\text{-}still_R$, $\rho^R_{RT}(forward) = move_R$.
- $\rho^{LT}_D(stand\text{-}still_L) = idle$, $\rho^{LT}_D(move_L) = drive$.
- $\rho^{RT}_D(stand\text{-}still_R) = idle$, $\rho^{RT}_D(move_R) = drive$.
- $\rho^D_\perp(idle) = \rho^D_\perp(drive) = 0$.

In a picture:



Now take the cover $\{LT, RT\}$ of R. The patch candidates $\{move_L, move_R\}$ are compatible, because they agree on overlap (they both restrict down to drive). But they also glue uniquely, yielding $forward$. In other words, the robot's forward motion is patched together precisely from the two pieces of its cover, namely the left tracks rotating ($move_L$) and the right tracks rotating ($move_R$).

Similarly, the Robot's sitting still (sit) behavior is also glued from the two pieces of its cover, namely the left track assembly standing still ($stand\text{-}still_L$) and the right track assembly standing still ($stand\text{-}still_R$).

Thus, there are two global sections of R's behavior: moving forwards (patched together from its left and right motions), or standing still (patched together from its left and right lack of motion).

2.6. Monopresheaves

In a presheaf, compatible patch candidates can be glued together to form sections that span multiple fibers. However, nothing said so far prevents there being multiple gluings from the same patch candidates. In other words, nothing requires gluings to be extensional.

If we want to work only with extensional gluings, then we can impose a restriction that says gluings must be unique: i.e., if a selection of patch candidates can glue, they form at most one gluing. Presheaves where this obtains are called monopresheaves.

Definition 11 (Monopresheaves)**.** *A presheaf F is a monopresheaf iff it satisfies the following gluing-uniqueness condition:*

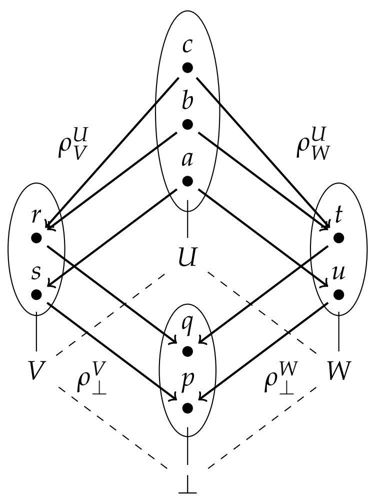
(G1) For every cover $\{U_i\}_{i \in I}$ of a region U and every selection of compatible patch candidates $\{s_i\}_{i \in I}$ for that cover, if there is a gluing $s \in F(U)$ of $\{s_i\}_{i \in I}$, then it is unique.

Equivalently, given $s, t \in F(U)$:

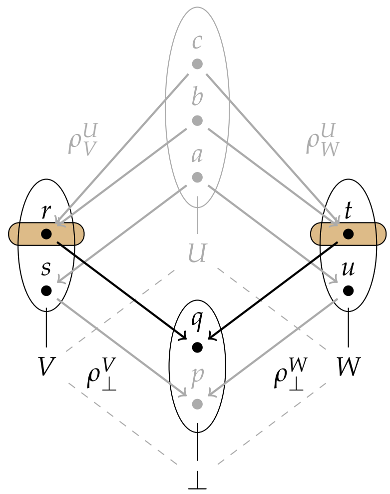
(G1) For every cover $\{U_i\}_{i \in I}$ of U, if $\rho^U_{U_i}(s) = \rho^U_{U_i}(t)$ for each U_i, then $s = t$.*

Remark 9. *Monopresheaves are also called separated presheaves. The "mono" part of the name comes from category theory: every joint restriction to a covering family is a monomorphism, i.e., there can be at most one section restricting to a given selection of patch candidates.*

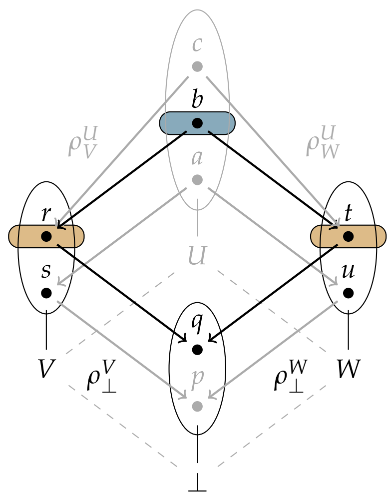
Example 13. *By way of counter-example, consider the following presheaf:*



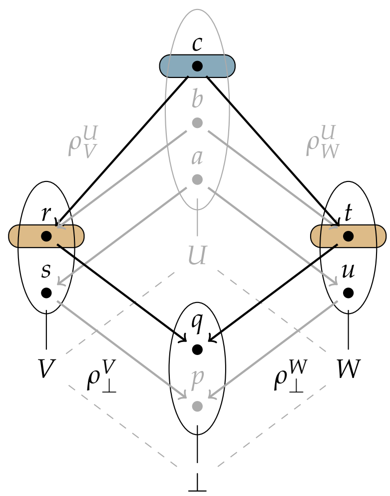
Take the cover $\{V, W\}$ and selection of patch candidates $\{r, t\}$, which are compatible because they agree on overlap:



Do r and t glue? That is to say, is there a section in $F(U)$ that decomposes exactly to r and t? In this case, $b \in F(U)$ is a gluing of r and t:



However, b is not a unique gluing, since c is also a gluing of r and t:



Thus, this is not a monopresheaf, since gluable patch candidates don't glue uniquely.

Example 14. *On the other hand, the presheaf from Example 9 is a monopresheaf, for whenever patch candidates glue together in that presheaf, they do so uniquely.*

2.7. Sheaves

The definition of a monopresheaf requires only that if compatible patch candidates glue, they do so uniquely. It does not require that compatible patch candidates always do glue together. Patch candidates in a monopresheaf need not glue.

If we want to work with monopresheaves where all gluable patch candidates do in fact glue together, then we can work with sheaves. A sheaf is a monopresheaf that satisfies an extra existence requirement: whenever patch candidates *can* glue, they *do* glue.

Definition 12 (Sheaf)**.** *A monopresheaf F is a sheaf iff it satisfies the following gluing-existence condition:*

(G2) For every cover $\{U_i\}_{i\in I}$ of a region U and every selection of patch candidates $\{s_i\}_{i\in I}$ for that cover, if $\{s_i\}_{i\in I}$ are compatible, then there exists a unique gluing $s \in F(U)$ of $\{s_i\}_{i\in I}$.

Example 15. *The presheaf from Example 9 fails to be sheaf, because as we saw in Example 11, there is a compatible selection of patch candidates (namely, $\{s, t\}$) which fails to glue. To be a sheaf, every compatible selection of patch candidates must glue.*

There is a subtlety regarding what sheaves look like over the least element of a locale. Note that the gluing condition is formulated as an implication. That is to say, it says that, for every cross-section of patch candidates, *if* that cross-section can glue, *then* it glues in exactly one way.

Next, consider the fact that the cover over the least region of a locale is an empty cover. Since there are no patch candidates that need to be checked for compatibility, there is nothing that needs to be done to get a "selection of gluable patch candidates." Hence, the antecedent of the gluing condition is satisfied vacuously over the least element of the locale.

But since the empty cover satisfies the antecedent of the gluing condition vacuously, it follows that if a presheaf is to qualify as a sheaf, it must ensure that the consequent is satisfied over the empty cover as well. In other words, it must assign a unique glued section (a singleton set) to the least region of the locale. So, even though a *presheaf* or *monopresheaf* may assign a larger set of data to the least element of a locale, a *sheaf* always assigns a singleton to that region.

Example 16. *The presheaf from Example 12 is a sheaf. Note that the bottom fiber is a singleton. This ensures that* all *gluable selections of patch candidates (including the empty one) glue uniquely.*

If we consider presheaves, monopresheaves, and sheaves together, we see that we have a hierarchy of increasingly strict gluing requirements. (1) Presheaves have no gluing requirements. (2) Monopresheaves have a uniqueness requirement: gluings need not exist, but when they do, they are unique. (3) Sheaves have both a uniqueness and an existence requirement: gluings exist whenever possible, and they are unique.

2.8. The bottom fiber

The bottom element of a locale represents no regions at all. Thus, it plays a special role. Since it represents the *absence* of any regions, the data that we assign to its fiber is of a different kind than the data we assign to other fibers.

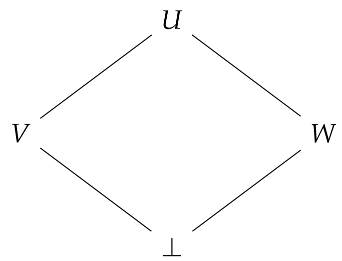
The other fibers sit over genuine regions in the parts space, so the data we assign to them plays a kind of ontological role: it's the "stuff" that occupies that region. By contrast, the fiber over $\perp$ cannot play this role. Since $\perp$ represents no region at all, its fiber cannot represent material occupancy. Instead, it plays a structural role.

Since any compatibility check between patch candidates ultimately factors through restriction maps that ultimately land in the fiber over $\perp$, agreement at $\perp$ functions as a final anchor point. Given this fact, we can make a general observation: there are as many kinds or modes of gluing as there are anchor points over $\perp$.

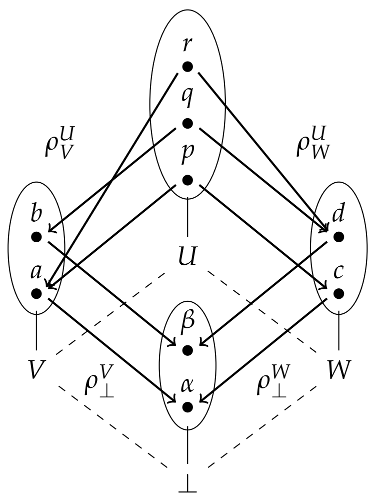
If there is a single point in the fiber over $\perp$, then there will be only one kind of gluing that occurs throughout the presheaf. This is most evident in a sheaf, which as we saw requires a singleton over $\perp$. This makes sense, because in a sheaf, gluing must be consistent and uniform throughout, and so all gluing has to anchor to a single point.

If we move to presheaves, and hence relax our gluing constraints, then we can have multiple points in the fiber over $\perp$. These will correspond to multiple kinds or modes of gluing. This also makes sense, since multiple gluings can only exist in a structure with weaker gluing conditions than we find in a sheaf.

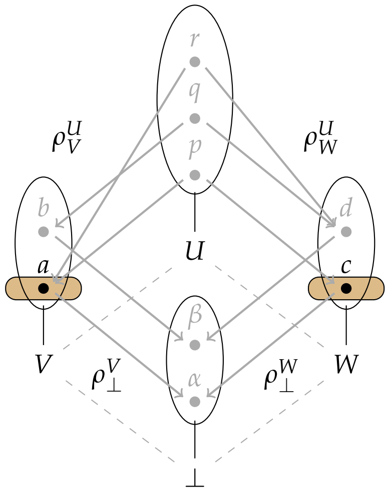
Example 17. *Consider the following simple locale:*



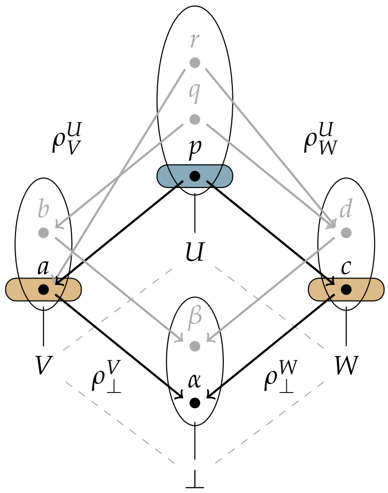
Now consider a presheaf with more than one anchor over $\perp$:



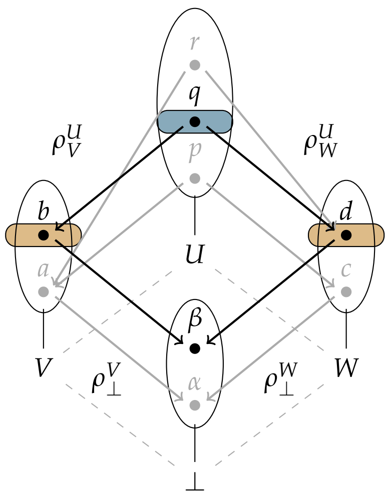
For the cover $\{V, W\}$ of U, take patch candidates a and c:



These glue at p, and are anchored to α:



Similarly, the patch candidates b and d glue at q, and anchor at β:



The patch candidates a and d cannot glue, because their restrictions to $\perp$ do not agree. Here is another place where sheaf theory controls coherence: it prevents gluing in more than one mode at the same time. Glued sections must be glued consistently (in whatever mode they anchor to over $\perp$).

3. Modeling Part-Whole Complexes as Presheaves

As noted in Section 1, the central claim of this paper is that we can model part-whole complexes as presheaves over locales, with varying gluing conditions. In particular, the locale provides the abstract parts space of "regions" that the pieces can occupy, the presheaf assigns actual pieces to those regions, and the gluings determine which pieces fuse.

We can thus define the core mereological concepts of part and whole in sheaf-theoretic terms. Regarding wholes, we can identify fusion with gluing: to say that some pieces fuse or form a "fusion" is just to say that they are glued together. Regarding parts, to say that a piece is a "part" is just to say that it is a part of a fusion. In other words, the parts of a fusion are just the pieces from which it is glued together.

Definition 13 (Fusions and parts)**.** *We say that a section $s \in F(U)$ is a fusion iff there exists a cover $\{U_i\}_{i\in I}$ of U and a selection of patch candidates $\{s_i\}_{i\in I}$ such that:*

$$\rho^U_{U_i}(s) = s_i, \quad \textit{for each } U_i.$$

Given $t \in F(V)$ and $s \in F(U)$ with $V \preccurlyeq U$ and $V \neq \bot$, we say t is a part of s, denoted $t \sqsubseteq s$, iff s is a fusion and:

$$\rho^U_V(s) = t.$$

Remark 10. *$V \neq \bot$, for as we saw in Section 2.8, $\bot$ represents no regions at all, and hence cannot be occupied by parts. The elements in the fiber over $\bot$ are structural anchors, not parts.*

Because fusions do not freely arise here, but rather only exist where parts are explicitly glued together, sheaf theory thus provides a systematic framework with which to model a large variety of part-whole complexes in a "fusions-first" manner. In the rest of this section, we illustrate with examples. In each case, we construct a custom presheaf designed to model a particular part-whole complex. Our choices of presheaves should be interpreted as modeling choices. One could construct different presheaves, and each can be evaluated on its own merits.

Example 18. *Consider a building with a west room, an east room, and a hallway between them. For simplicity, let us consider only the floors of the building (ignore walls, ceilings, and so on). The ambient locale is given by the presentation*

- $\mathbb{L} = \langle G, R\rangle = \langle \{WR, H, ER\}, \varnothing\rangle$

where

- *WR = west room*
- *H = hallway*
- *ER = east room*

As a Hasse diagram:

$$\begin{tikzcd}
& WR \vee H \vee ER \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
WR \vee H \arrow[d, dash] \arrow[dr, dash] & WR \vee ER \arrow[dl, dash] \arrow[dr, dash] & H \vee ER \arrow[dl, dash] \arrow[d, dash] \\
WR \arrow[dr, dash] & H \arrow[d, dash] & ER \arrow[dl, dash] \\
& \bot &
\end{tikzcd}$$

For the presheaf F, let it assign to each region whatever materials (if any) cover its floor uniformly. Let us say that the west room's and hallway's floors are each covered uniformly by wood, while the east room's floor is covered uniformly by tiles:

- $F(WR) = \{wood\}$
- $F(H) = \{wood\}$
- $F(ER) = \{tile\}$

Since the west room's and hallway's floors are covered uniformly by wood, their join is too:

- $F(WR \vee H) = \{wood\}$

Since none of the other regions are covered uniformly by the same material, we assign nothing to them:

- $F(WR \vee ER) = \varnothing$
- $F(H \vee ER) = \varnothing$
- $F(WR \vee H \vee ER) = \varnothing$

Finally, for the fiber over bottom, where there are no regions to cover with materials, let us assign the special symbol zero:

- $F(\bot) = \{0\}$

For the restrictions, let us say that the materials that cover a larger region are restricted down to the materials that cover the smaller region. Hence, the non-empty fibers restrict by identity:

- $\rho^{WR \vee H}_{WR}(wood) = \rho^{WR \vee H}_{H}(wood) = wood$

The empty fibers restrict via the empty function (there is nothing to restrict):

- $\rho^{WR \vee H \vee ER}_{WR \vee H} = \rho^{WR \vee H \vee ER}_{WR \vee ER} = \rho^{WR \vee H \vee ER}_{H \vee ER} =$ *empty function*
- $\rho^{WR \vee ER}_{WR} = \rho^{WR \vee ER}_{ER} =$ *empty function*
- $\rho^{H \vee ER}_{H} = \rho^{H \vee ER}_{ER} =$ *empty function*

Finally, fibers restrict to bottom via the constant function:

- $\rho^{WR}_{\bot}(wood) = \rho^{H}_{\bot}(wood) = \rho^{ER}_{\bot}(tile) = 0$

In this building, there are two maximal fusions:

- *The flooring of the west room and the hallway glue into one piece that covers both.*
- *The flooring that covers the east room.*

Thus, the flooring of this building is really a collection of two independent fusions: the wooden floor that covers the west room and hallway, and the tiled floor that covers the east room. That implies:

- *To separate the floors of the west room and hallway, you would have to use a saw to cut them, since they are fused. They are not merely sitting next to each other. Rather, they make up a single (fused) piece.*
- *By contrast, to separate the hallway and the east room, you would not need to cut them, since they are not fused. They simply happen to be sitting next to each other.*

The parts of the fusions are clear:

- *The wooden floor that covers the west room and the hallway has two parts: the wooden floor that covers the west room, and the wooden floor that covers the hallway.*
- *The tiled floor of the east room has no parts (in this locale), since it is not the fusion of other fusions.*

This is particular example fails to be a sheaf, because everything that can glue does not glue. In particular, $F(WR \vee H \vee ER)$ is covered by $\{WR, H, ER\}$, and the patch candidates $\{wood, wood, tile\}$ are compatible (they pair-wise restrict to 0). However, there is nothing in $F(WR \vee H \vee ER)$ that is *glued together from those patch candidates (indeed, $F(WR \vee H \vee ER)$ is empty). Hence, this is not a sheaf. Rather, it is a monopresheaf.*

But this is precisely what one would expect when modeling two discrete pieces of flooring that happen to sit next to each other. The west room and hallway do glue together here, as expected. But the maximal wood and tile pieces do not glue together, also as expected (there is a boundary between them, where the wood ends and the tile begins).

In the previous example, none of the regions overlapped. The presheaf was free to glue or not glue pieces as it saw fit. The story is different if there are overlaps in the locale itself. Overlaps in the locale require overlaps in the presheaf, wherever you want gluings.

Example 19. *Consider the floor of a single room. Let us say that the regions of interest are its west half, its east half, and a six inch span where they overlap.*

The ambient locale of this kind of space can be given by the presentation

- $L = \langle G, R \rangle = \langle \{\perp, WH, O, EH\}, \{\perp \preccurlyeq O, O \preccurlyeq WH, O \preccurlyeq EH\} \rangle$

where

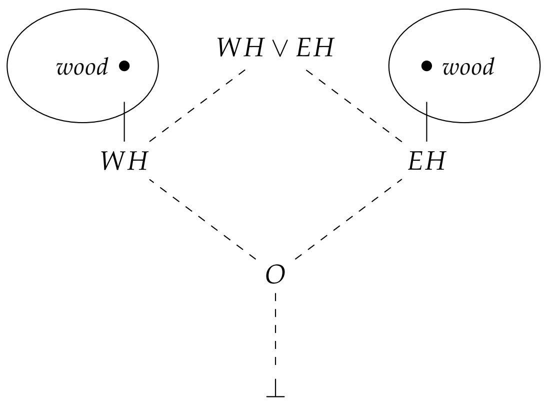
- WH = *west half*
- O = *overlap*
- EH = *east half*

For a presheaf F, let us say that it behaves much like in the previous example: it assigns to each region the materials (if any) that cover the floor uniformly.

For instance, let us assign wood to both halves:

- $F(WH) = \{wood\}$
- $F(EH) = \{wood\}$

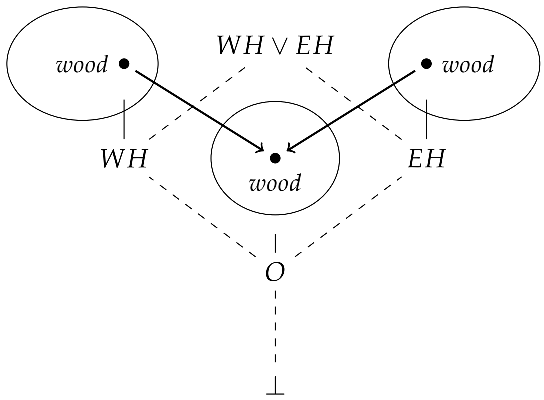
In a picture:



By construction $O = WH \wedge EH$, and the two halves restrict to the same material there:

- $F(O) = \{wood\}$

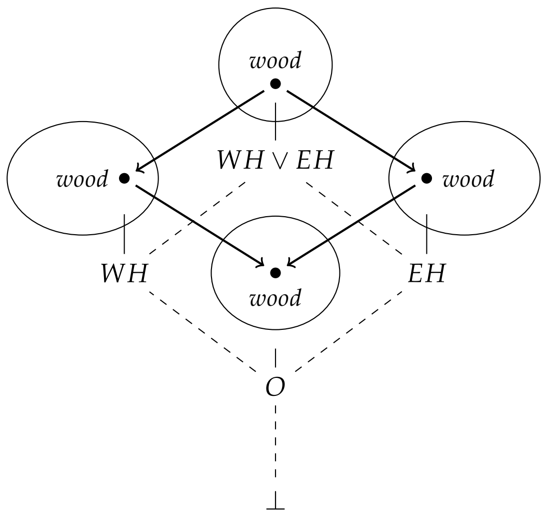
Thus:



For the join, the west and east halves glue, since they're made from the same flooring materials and agree on their overlap:

- $F(WH \vee EH) = \{wood\}$

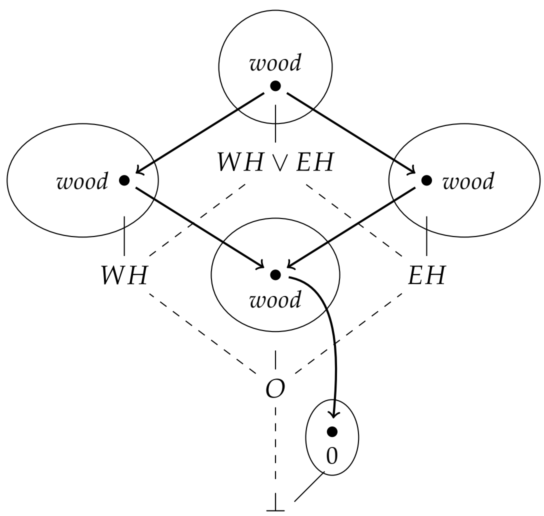
Thus:



Finally, for the fiber over bottom, where there are no regions to cover with materials, assign the special symbol zero:

- $F(\perp) = \{0\}$

In a picture:



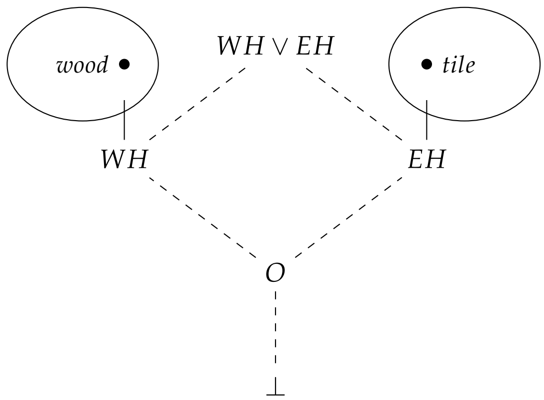
The maximal fusion here is a single piece of wooden flooring (namely, wood $\in F(WH \vee EH)$*) that covers the whole room. Its parts are the west and east halves, and (transitively) their overlap. The west and east halves themselves have a shared part, the strip of overlap.*

This particular example is a sheaf: the parts glue together coherently in a single manner across the entire parts space. This is exactly as one would expect when modeling a floor that is covered uniformly in its entirety by wood flooring: as you restrict down to smaller parts of the room, you get smaller pieces of wood flooring. In contrast to Example 18, here the regions have a nontrivial overlap. By the sheaf condition, agreement on that overlap forces a unique fusion of the parts, just as expected.

Example 20. *To illustrate a failed attempt to build a sheaf, let us take the locale and gluing condition from Example 19, but suppose that we assign different flooring materials to the east and west halves of the room:*

- $F(WH) = \{wood\}$
- $F(EH) = \{tile\}$

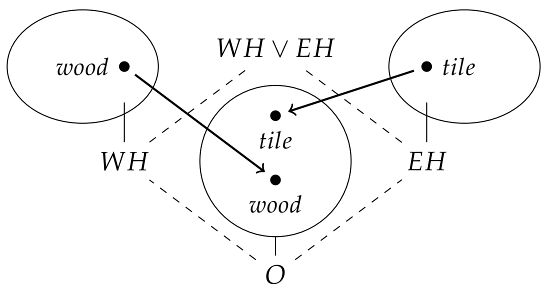
As a picture:



Next, at the overlap, allow both wood and tile:

- $F(O) = \{wood, tile\}$
- $\rho_O^{WH}(wood) = wood$
- $\rho_O^{EH}(tile) = tile$

Thus, as a picture:

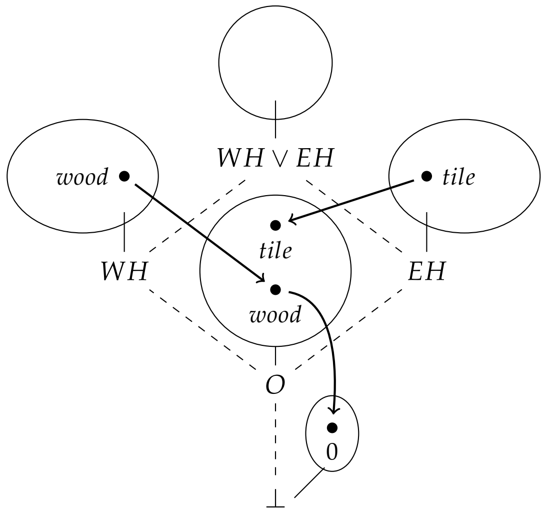


Note, however, that the wood and tile in the west and east halves cannot glue, because they do not agree on their overlap. Hence, there cannot be a single kind of flooring material that uniformly covers the maximal join $WH \vee EH$*. This illustrates how the sheaf condition is a very strong condition, but also a helpful one: it requires and manages coherent gluing at all levels. Because it requires that pieces glue together coherently at every level of "zoom," it prevents us from ever putting together an incoherent part-whole sheaf in the first place.*

It is worth spelling the failure out explicitly. If we want to model this room as a sheaf, then we are requiring coherent, unique gluing across all of the regions: wherever pieces can coherently glue together, they must do so. But here, since WH and EH disagree on their overlap, there is no way to make them cohere into a single piece.

Intuitively, this makes sense. If the western and eastern halves of a room were truly floored with different materials, then they could not overlap. Imagine if two builders started at opposite ends of the room: one flooring with wood and the other flooring with tile. When they reach the mid-point, they'd realize they made a mistake. In such a scenario, it would be impossible to complete the original vision of having a single, uniform flooring across the entire room.

However, what if we weaken our requirements and consider this as a presheaf? In a presheaf, nothing disallows the two halves from restricting differently on an overlap. The full presheaf looks like this:



Interpreted as a presheaf, there is a sensible interpretation of this structure. Nothing sits in the fiber over $WH \vee EH$, since a single coherent piece of flooring cannot be glued from wood and tile. The wood and tile from the two halves each extend into the overlap though, but since they don't agree, one of them must sit on top of the other in that overlapping area. It is like when two area rugs overlap: one sits on top of the other.

As a final point, note that this fails to be monopresheaf, because there are two sections in $F(O)$ that restrict to $\perp$. The gluings are trivial here, but nonetheless, gluing is not unique precisely over $\perp$. Hence, this is a presheaf, but not a monopresheaf. This is a limit case where presheaves allow multiple assemblies of the same parts, but monopresheaves do not.

The previous two examples were spatial. But parts come in non-spatial guises too, and sheaf theory can model them just as well.

Example 21. *Suppose we say that human society (under some description) consists of the mesh of a specified set of relationships between the people that participate in that society.*

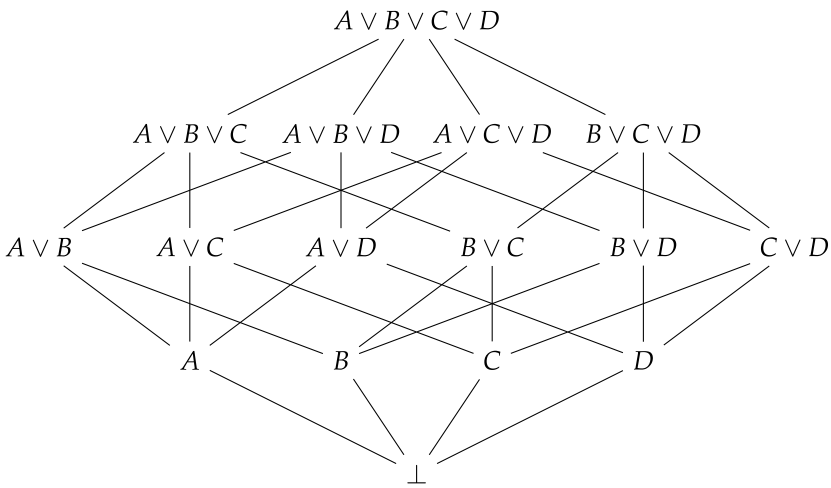
Let P be the population in question (a finite set of individual people), and let the regions of our locale be subsets of such individuals. Then the ambient locale is given by the presentation:

- $\mathbb{L} = \langle G, R \rangle = \langle P, \varnothing \rangle$

For concreteness, suppose:

- $P = \{A, B, C, D\}$*, with A short for Alice, B for Bob, C for Carol, and D for Denny.*

Then the Hasse diagram is isomorphic to the powerset of P:



Let us next define a presheaf F that models the mesh of a selected set of relationships over P. To do that, let us first specify a set R that picks out the (binary, symmetric) relationships of interest:

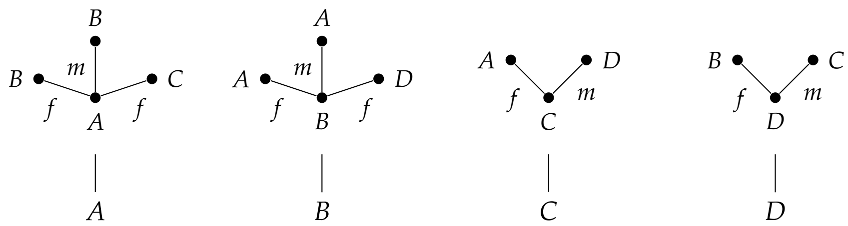
- $R = \{f, m, \ldots\}$*, with f short for being friends, m for being married, etc.*

For convenience, if $U, V \in P$, $r \in R$, and U and V stand in relationship r, we will write $r(U, V)$.

For the generators, let us fix a choice of local data by assigning to each person the relations they stand in, e.g.:

- $F(A) = \{\langle\{f(A,B), f(A,C), m(A,B)\}\rangle\}$
- $F(B) = \{\langle\{f(B,A), m(B,A), f(B,D)\}\rangle\}$
- $F(C) = \{\langle\{f(C,A), m(C,D)\}\rangle\}$
- $F(D) = \{\langle\{f(D,B), m(D,C)\}\rangle\}$

To visualize this data, we can picture each fiber as a mini-graph:



For example, in the fiber over A:

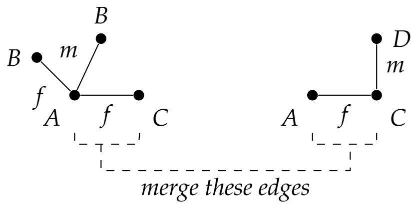
- *The f-labeled edge from A to B represents $f(A,B)$: A and B are friends.*
- *The m-labeled edge from A to B represents $m(A,B)$: A and B are married.*
- *The f-labeled edge from A to C represents $f(A,C)$: A and C are friends.*

In the bottom fiber, over the region of no people to carry relationships, assign the special symbol 0:

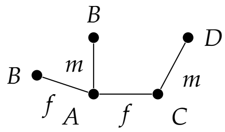
- $F(\perp) = \{0\}$

Next, let us extend the above data to binary joins by merging mini-graphs along shared edges, wherever the components share exactly the same edges. To see how this works, consider (for example) the mini-graphs over A and C:

Can these be merged? The answer is yes, because they share exactly one edge, namely the one labeled f. If you rotate the graphs sideways a bit, you can see how they can be merged along $f(A, C)$:



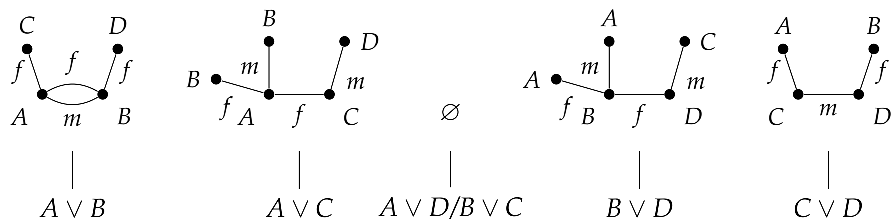
Merging along $f(A, C)$ yields the following graph:



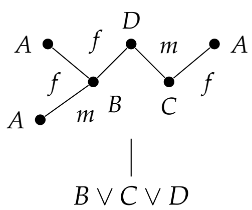
By merging all binary joins in this fashion, we get:

- $F(A \vee B) = \{\langle \{f(A,B), m(A,B), f(A,C)\}, \{f(B,A), m(B,A), f(B,D)\}\rangle\}$
- $F(A \vee C) = \{\langle \{f(B,A), m(B,A), f(B,D)\}\rangle, \{f(C,A), m(C,D)\}\}$
- $F(A \vee D) = \varnothing$
- $F(B \vee C) = \varnothing$
- $F(B \vee D) = \{\langle \{f(B,A), m(B,A), f(B,D)\}, \{f(D,B), m(D,C)\}\rangle\}$
- $F(C \vee D) = \{\langle \{f(C,A), m(C,D)\}, \{f(D,B), m(D,C)\}\rangle\}$

As pictures:



Having merged the graphs from joins of two regions, we must next merge the graphs of joins from three regions. For instance, take $B \vee C \vee D$. We can merge the graphs of $B \vee C$ trivially (because they share no edges), we can merge the graphs of $C \vee D$ along their shared f-edge, and we can merge the graphs of $B \vee D$ along their shared f-edge. That yields:



By merging the graphs of all joins of three regions in this fashion, we get:

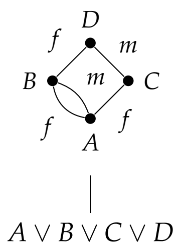
- $F(A \vee B \vee C) = \left\{ \begin{array}{l} \langle \quad \{f(A,B), f(A,C), m(A,B)\}, \\ \qquad \{f(B,A), m(B,A), f(B,D)\}, \\ \qquad \{f(C,A), m(C,D)\} \quad \rangle \end{array} \right\}$
- $F(A \vee B \vee D) = \left\{ \begin{array}{l} \langle \quad \{f(A,B), f(A,C), m(A,B)\}, \\ \qquad \{f(B,A), m(B,A), f(B,D)\}, \\ \qquad \{f(D,B), m(D,C)\} \quad \rangle \end{array} \right\}$

- $F(A \vee C \vee D) = \left\{ \begin{array}{l} \langle \quad \{f(A,B), f(A,C), m(A,B)\}, \\ \quad\ \ \{f(C,A), m(C,D)\}, \\ \quad\ \ \{f(D,B), m(D,C)\} \quad \rangle \end{array} \right\}$
- $F(B \vee C \vee D) = \left\{ \begin{array}{l} \langle \quad \{f(B,A), m(B,A), f(B,D)\}, \\ \quad\ \ \{f(C,A), m(C,D)\}, \\ \quad\ \ \{f(D,B), m(D,C)\} \quad \rangle \end{array} \right\}$

At the top-most join of the locale, if we merge the graphs of all four regions, we get:

- $F(A \vee B \vee C \vee D) = \left\{ \begin{array}{l} \langle \quad \{f(A,B), f(A,C), m(A,B)\}, \\ \quad\ \ \{f(B,A), m(B,A), f(B,D)\}, \\ \quad\ \ \{f(C,A), m(C,D)\}, \\ \quad\ \ \{f(D,B), m(D,C)\} \quad \rangle \end{array} \right\}$

As a picture:



In effect, the fiber over each region U is a consistent assignment of relationships involving exactly the people in U, obtained by restriction from the full society. A restriction from $F(U)$ to $F(V)$ in effect forget any edges and vertices not involving people in V. It truly restricts the relationships mesh from those people in U to only those in V.

This is in fact a sheaf. The sheaf condition requires that the relationship meshes over two opens agrees precisely on their overlap: when you restrict the meshes down to their overlap, you get the same sub-mesh.

The result is a sheaf: a fused mesh of relationships over the population, which is glued together from smaller meshes over smaller subsets of the population.

- *Each fiber is a part of the whole (human society), and its data encodes the internal (relational) structure of that part.*
- *Mereological overlap is then modeled by shared relationships: two parts overlap if their relational graphs intersect coherently.*
- *Regions that are not covered by a mesh (as in $F(A \vee D) = \varnothing$ and $F(B \vee C) = \varnothing$) reflect mereological separation: the regions in question are simple not related, so there is nothing to glue.*

For another example, consider processes. A process (or more generally any sequence of events, states, etc.) can be seen as a part-whole complex too.

Example 22. *Imagine a scenario where something can do one of two things repeatedly: at each step, it can do one thing ("option a") or another thing ("option b"), and then repeat the choice again.*

To model this, fix a finite alphabet $\Sigma = \{a, b\}$, with "a" for "option a" and "b" for "option b." Then let Σ^* be the set of all finite sequences (words) over Σ, with ϵ denoting the empty sequence. For instance, the sequence aab represents the sequence of length 3 that picks "option a" first, then "option a" again, and then finally "option b."

Let us say that $\Sigma^{\leqslant n}$ is the set of all finite sequences less than length n, and let us say that $\Sigma^{=n}$ is the set of finite sequences of exactly length n. Hence:

- $\Sigma^{=0} = \{\epsilon\}$.

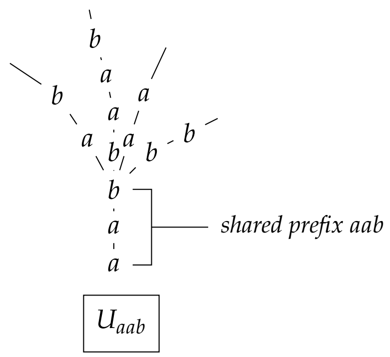
- $\Sigma^{=1} = \Sigma^{\leqslant 1} = \{\epsilon, a, b\}$.
- $\Sigma^{\leqslant 2} = \{\epsilon, a, b, aa, bb, ab, ba\}$.
- $\Sigma^{=2} = \{aa, bb, ab, ba\}$.
- *Etc.*

Given sequences $w, v \in \Sigma^{\leqslant n}$ *with* $length(w) \leqslant length(v)$, *let us write* $w \subseteq v$ *to denote that* w *is a prefix of* v, *as in* $aab \subseteq aabc$.

Next, define a topology over $\Sigma^{\leqslant n}$ *by setting the open sets to be sequences that share a prefix:*

- $U_w = \{v \in \Sigma^{\leqslant n} \mid w \subseteq v\}$.

So U_w *consists of all sequences that continue* w. *For instance, if* $w = aab$, *then we might picture* U_w *as a kind of bouquet or bundle of sequences that are all bound at their shared stem (aab) but then branch out in different directions:*



We can form a locale from this topology. Let $\mathbb{L}$ *be the locale given by the presentation* $\langle G, R\rangle$, *where:*

- $G = \{U_w \mid w \in \Sigma^n\}$, *i.e., each open is a generator.*
- $R = \{U_w \preccurlyeq U_v \mid v \subseteq w\}$, *i.e., bouquets with longer prefixes are lower.*

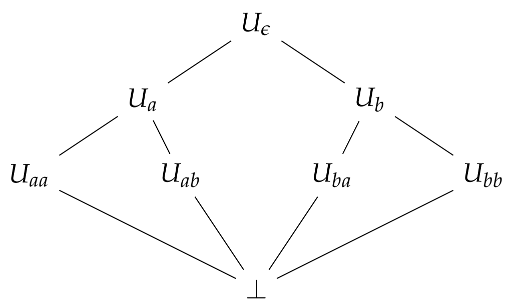
For example, given $\Sigma^{\leqslant 2}$, *we have the following generators:*

- $G = \{U_\epsilon, U_a, U_b, U_{aa}, U_{bb}, U_{ab}, U_{ba}\}$.

Here are some of the relations:

- $U_{aa} \preccurlyeq U_a$ *and* $U_{ab} \preccurlyeq U_a$, *since "a" is a prefix of aa and ab.*
- $U_{bb} \preccurlyeq U_b$ *and* $U_{ba} \preccurlyeq U_b$, *since "b" is a prefix of bb and ba.*
- *Every generator is lower than* U_ϵ, *since* ϵ *(the empty sequence) is a prefix of every sequence.*

The Hasse diagram looks like this:



Think of moving upwards in this locale as forgetting information about (or alternatively, as committing less to) the history of the sequence. For example, think of U_{ab} *as a region where we know that "a" happened first and then "b" happened, but think of* U_a *as a region where we know only that "a" happened first and we don't know what happened after that. The top element is* U_ϵ, *which means we don't know anything about the sequence of actions. The* $\perp$ *element indicates not that we know nothing, but that there is no sequence at all.*

Notice that implication moves upwards: U_{ab} implies U_a because if I know (at U_{ab}) that "a" happened first and then "b" happened, then I certainly know that "a" happened first.

This particular locale is interesting because it models the "process space" of any 2-stage sequence that can make one of two choices at each stage. Let us now assign some actual processes to this ambient space, using a presheaf.

Imagine a machine m that can run multiple concurrent processes, all of whom share the same memory. For simplicity, let us suppose that the machine has two registers ($R = \{r_1, r_2\}$), each of which can hold one bit (1 or 0). So, at any point in time the machine's memory state $S : \{0,1\} \times \{0,1\}$ can be one of the following:

- $S = \{\langle 0,0\rangle, \langle 1,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle\}$, with initial state $s_0 = \langle 0,0\rangle$.

We can think of the concurrent processes of interest as a selection of programs that we want to run on the machine all at the same time. In terms of behavior, let us say that each program-run reads a word from its input stream, one character at a time, and in response to each character, it takes one of the following actions A: it writes a value (1 or 0) to one of the registers, it writes (possibly distinct) values to both registers, or it does nothing and leaves the registers as they are:

- $A = \{\{r1 \mapsto v\}, \{r2 \mapsto v\}, \{r1 \mapsto v, r2 \mapsto w\}, \varnothing\}$, where $v, w \in \{0,1\}$.

We can define a process (program trace) as a map from n-length words to n-length sequences of write actions, where we require that such maps agree on prefixes (since a process responding to ab and aa would do the same thing on the first a). This way, a program trace records for each input stream the sequence of write actions that result. For concreteness, here are two such traces:

- $f : \Sigma^{=2} \to A \times A$
  - $f(aa) = \langle \{r1 \mapsto 1\}, \{r1 \mapsto 0\}\}\rangle$
  - $f(ab) = \langle \{r1 \mapsto 1\}, \{r2 \mapsto 0\}\}\rangle$
  - $f(bb) = \langle \{\{r1 \mapsto 0\}, \{r2 \mapsto 1\}\}\rangle$
  - $f(ba) = \langle \{\{r1 \mapsto 0\}, \{r1 \mapsto 1\}\}\rangle$
- $g : \Sigma^{=2} \to A \times A$
  - $g(aa) = \langle \{r2 \mapsto 0\}, \{r2 \mapsto 0\}\}\rangle$
  - $g(ab) = \langle \{r2 \mapsto 0\}, \varnothing\}\rangle$
  - $g(bb) = \langle \{\{r1 \mapsto 1\}, \varnothing\}\rangle$
  - $g(ba) = \langle \{\{r1 \mapsto 1\}, \varnothing\}\rangle$

Let us say that we now want to run f and g concurrently on the machine. For each word $w \in \Sigma^{=2}$, let us write $\mathcal{R}(w)$ for

$$(f(w), g(w)),$$

i.e., the joint run of f and g on w. For instance:

$$\mathcal{R}(aa) = (f(aa), g(aa)) = (\langle \{r1 \mapsto 1\}, \{r1 \mapsto 0\}\}\rangle, \langle \{r2 \mapsto 0\}, \{r2 \mapsto 0\}\}\rangle).$$

Next, let us say that two joint runs $\mathcal{R}(w)$ and $\mathcal{R}(v)$ are prefix-compatible iff:

- For every stage k such that w and v share their first k letters, the combined writes of f and g at stage k do not assign different values to the same register.

For instance, $\mathcal{R}(aa)$ and $\mathcal{R}(ab)$ are prefix-compatible because at stage 1 (at letter a), their writes do not conflict:

- $f(aa)$ and $f(ab)$ write 1 to $r1$, while $g(aa)$ and $g(ab)$ write 0 to $r2$, and this is no conflict because they write to different registers.

By contrast, $\mathcal{R}(bb)$ and $\mathcal{R}(ba)$ are not prefix-compatible because at stage 1 (at letter b) their writes do conflict:

- $f(bb)$ and $f(ba)$ write 0 to $r1$, while $g(bb)$ and $g(ba)$ write 1 to $r1$, so they attempt to write conflicting values to the same register.

For a region U_w with $|w| \leq 2$, let $\mathsf{ext}(U_w)$ be the set of length-2 words that extend w. For instance, $\mathsf{ext}(U_a) = \{aa, ab\}$, since aa and ab extend a, and $\mathsf{ext}(U_{ab}) = \{ab\}$, since ab is already a fully extended length-2 word.

Now let us define a presheaf F as follows. For each region U_w, let us provide an execution model that describes how f and g execute w and then continue on. More specifically, we need to assign prefix-compatible program runs for the extensions of w. The idea is that $F(U_w)$ will present a coherent description of how the machine will behave, as f and g jointly execute w and then continue on to their continuations.

At the most specified regions (aa, ab, bb, and ba), everything is fully specified, so the joint runs are fully determined already:

- $F(U_{aa}) = \{aa \mapsto \mathcal{R}(aa)\}$
- $F(U_{ab}) = \{ab \mapsto \mathcal{R}(ab)\}$
- $F(U_{bb}) = \{bb \mapsto \mathcal{R}(bb)\}$
- $F(U_{ba}) = \{ba \mapsto \mathcal{R}(ba)\}$

The more complicated case involves a less specified region, e.g. U_a. At this region, we know that the first step a happened, but we don't yet know whether the second step will be a or b. So, from the point of view of U_a, both continuations are possible.

Thus, what we need for $F(U_a)$ is a coherent description of how the machine will behave no matter which continuation actually occurs next. We must therefore:

- Pick a joint run of aa.
- Pick a joint run of ab.
- But these two choices must be prefix-compatible, since they are supposed to represent the same joint run that has two different futures.

Are there any joint runs of f and g on aa and ab that are prefix-compatible at a? Yes:

- $aa \mapsto \mathcal{R}(aa)$
- $ab \mapsto \mathcal{R}(ab)$

For as we saw earlier, $\mathcal{R}(aa)$ and $\mathcal{R}(ab)$ are prefix-compatible. Hence, there is a coherent execution model for f and g at U_a:

- $F(U_a) = \{aa \mapsto \mathcal{R}(aa), ab \mapsto \mathcal{R}(ab)\}$

Now consider U_b. The continuations of b are bb and ba. Are there any joint runs of f and g on bb and ba that are prefix-compatible? Here, the answer is no, for as we saw above, $\mathcal{R}(bb)$ and $\mathcal{R}(ba)$ are not prefix-compatible. Hence, there is no execution model for f and g at U_b:

- $F(U_b) = \varnothing$

What about U_ϵ? This would have to be an execution model of a joint run that is prefix compatible with the execution models at both U_a and U_b. Since there is no possible execution model at U_b, no execution model is possible for U_ϵ:

- $F(U_\epsilon) = \varnothing$

Formally, we can summarize the above description of F as follows. Let $\mathcal{R} = \{\mathcal{R}(v) \mid v \in \Sigma^{=2}\}$. Then:

$$F(U_w) = \left\{ s : \mathsf{ext}(w) \to \mathcal{R} \mid \begin{array}{l} \textit{(i) } s(v) = \mathcal{R}(v) \textit{ for all } v \in \mathsf{ext}(w), \\ \textit{(ii) } \{s(v) \mid v \in \mathsf{ext}(w)\} \textit{ is pairwise prefix-compatible} \end{array} \right\}$$

For the bottom fiber, where no processes occur, assign the special symbol zero:

- $F(\bot) = \{0\}$.

To notate the restriction of a function $f : B \to D$ to a smaller domain $C \subseteq B$, write $f|_C$. Then, whenever $U_v \preccurlyeq U_w$, the restriction maps are straightforward restrictions:

$$\rho^{U_w}_{U_v}(s) = s|_{\mathsf{ext}(U_v)} \,.$$

Hence, for example:

- $\rho^{U_a}_{U_{ab}}(\{aa \mapsto \mathcal{R}(aa), ab \mapsto \mathcal{R}(ab)\}) = \{ab \mapsto \mathcal{R}(ab)\}$.
- $\rho^{U_a}_{U_{aa}}(\{aa \mapsto \mathcal{R}(aa), ab \mapsto \mathcal{R}(ab)\}) = \{aa \mapsto \mathcal{R}(aa)\}$.

The maximal fusions in this presheaf occur over U_a: the execution model in U_a's fiber is glued from those of U_{aa} and U_{ab}, exactly as one would expect, since f and g can run concurrently without conflict at a. By contrast, there is no gluing over U_b, since f and g cannot run concurrently without conflict at b.

This is not a sheaf, because it is missing gluings (e.g., over U_b). However, it is a monopresheaf, because when gluings exist (over U_a), they are unique.

This relevance of this example is t hat it illustrates how sheaf theory can model processes, concurrency, and resource conflicts. Here the processes are programs running on a simple machine, but they could just as easily be biological processes competing for resources, etc.

Whatever the concrete details may be, this example captures how local behaviors integrate and extend over larger regions of the process space. One might naively think that the "parts" of such systems are the processes. But there is a different way to slice it: if you want to talk about the integrity of the "whole" of a concurrent system, you need to talk about how that involves coherent, integrated behavior that is functionally united locally across the various "regions" and "stages" of the system's evolution.

Sheaf theory works in continuous environments too.

Example 23. *Suppose we want to model the inhabitants of an apartment building over time. Let $T = (0, 10) \subseteq \mathbb{R}$ be an open interval representing a period of time (a span of 10 years, say).*

Let T have the standard Euclidean topology, and let $\mathbb{L}$ be the locale of opens of the topology. This is the ambient locale we want to work with.

Next, let P be the set of people who at some point or other lived in the building:

$$P = \{A, B, C, \ldots\}, \textit{ with } A \textit{ short for Alice, } B \textit{ for Bob, } C \textit{ for Carol, and so on.}$$

Let I assign to each person the set of intervals during which they lived in the building. For instance:

- $I(A) = \{(1, 3), (4, 7)\}$
- $I(B) = \{(4, 9)\}$
- $I(C) = \{(6, 10)\}$

Let F be a presheaf given as follows. For each time span $U \in \mathbb{L}$:

$$F(U) = \{p \in P \mid U \subseteq V, \textit{ for some } V \in I(p)\}$$

In other words, $F(U)$ is the set of people who live in the building for the entire duration of U. For $V \preccurlyeq U$, the restriction maps are just inclusion:

$$\rho_V^U(J) = J,$$

since if the set of people J lived in the building throughout the span U, then they most certainly lived there during the smaller interval V.

This is a sheaf, since any compatible selection of patch candidates glues uniquely. For any cover $\{U_i\}_{i \in I}$ of U:

$$F(U) = \bigcap_i F(U_i).$$

A fusion in this sheaf is a glued section, and its parts are the patch candidates it is glued from. For example, take $U = (6,7)$, with cover $\{U_1 = (6, 6.5), U_2 = (6.4, 7)\}$. Any selection of compatible patch candidates from this cover glue uniquely.

This is a particularly simple sheaf, but the sheaf's strong gluing conditions tell us why the parts glue together so straightforwardly here. Gluing is thoroughly integrated throughout the structure.

Example 24. *For another continuous example, let's model a lump of clay through time.*

Let $T = (0, 10)$ be a span of time, and let $\mathbb{L}$ be the locale of the opens of T with its standard Euclidean topology again.

For simplicity, let us assume that 3-dimensional space is just $\mathbb{R}^3$. Then fix a function

$$\phi : T \to \wp(\mathbb{R}^3)$$

that, for each $t \in T$,

$$\phi(t) \subseteq \mathbb{R}^3$$

is the open region of space occupied by the clay at time t.

If we think of $\phi(t)$ as a snapshot of the clay at t, then at some ts, $\phi(t)$ might be shaped like a statue, at other times like a lump, at still other times like two disconnected lumps, and so on.

With that background fixed, let us now define a presheaf F that assigns material parts to each interval of T. In particular, for each interval $U \in \mathbb{L}$:

$$F(U) = \{S \subseteq \mathbb{R}^3 \mid S \text{ is open and } \forall t \in U, S \subseteq \phi(t)\}.$$

In other words, F(U) is the set of the clay's material parts that persist through the duration of U.

Restriction is inclusion again, since persistence through a larger time span U implies persistence through a smaller span $V \preccurlyeq U$:

$$\rho_V^U(S) = S.$$

This is a sheaf, since compatible local section glue by intersection. The fibers of this sheaf are typically infinite: each fiber includes the whole lump of clay, all of is sub-regions, and these regions are themselves spatially continuous.

Given a time span U, if

$$\bigcap_{t \in U} \phi(t)$$

is connected, then there will be a maximal connected section $S \in F(U)$ such that all other sections in $F(U)$ sit inside it. This models a single lump of clay through time.

By contrast, if

$$\bigcap_{t \in U} \phi(t)$$

has two or more connected components, then there will not be a maximal connected section $S \in F(U)$ *that all other sections of* $F(U)$ *fit inside. This models a fragmented lump of clay through time.*

What exactly does the sheaf give us here that ϕ *doesn't? What* ϕ *gives us is a time-indexed history of spatial occupation. It is like a movie: each frame is just a picture. It says nothing about what the material parts are, or which parts persist through time. That information has to be imposed on top of* ϕ*.*

F *provides that extra information explicitly.* F *provides precisely what the material parts are that persist through time, along with a principled notion of fusion. In short,* ϕ *gives us what points in space are occupied at each instant, whereas* F *gives us which parts persist over regions of time, and how they compose.*

Adopting weaker gluing conditions allows one to model more exotic part-whole complexes that fail extensionality.

Example 25. *Take the example of the lump of clay from Example 24. Examples like this can be enriched to model further features.*

For instance, suppose we model Socrates in the fashion of Example 24. But then we might want to enrich the example and say that the fibers carry not just persistent material parts, but rather that they carry persistent material parts tagged with an accidental mode (e.g., a posture, like seated or standing).

Let $\mathsf{Posture} = \{seated, standing, \ldots, \} \cup \{unspecified\}$*. Then for every* $U \in \mathbb{L}$*, we can say that* $F(U)$ *is a set of pairs* $\langle S, \pi \rangle$ *such that* $\forall t \in U$, $S \subseteq \phi(t)$*, and* $\pi \in \mathsf{Posture}$*. Intuitively, these pairs are the material parts in standing-Socrates, seated-Socrates, and Socrates-simpliciter that persist in the given mode throughout* U*. Such a presheaf would then have multiple, distinct persistent material fusions (e.g., Socrates-simpliciter and seated-Socrates) that simultaneously inhabit the same regions of space and differ only by their accidental mode.*

Note that in the sheaf-theoretic setting, we can model part-whole complexes where gluing need not be unique (via presheaves). In other words, if we choose to give up the stricter gluing conditions of monopresheaves and sheaves, we can then use presheaves to model situations where multiple fusions are glued from the same parts (cf. the models of [39]). For instance, we can model seated-/standing-Socrates, Athena and Lumpl, matter-and-substantial-form, a soul-and-its-powers, etc.

Sheaf-theoretic mereology is rare in the literature, but there are a few notable nearby cases. We conclude this section by distinguishing two kindred approaches from one genuine instance of the framework developed here.

Example 26. *Thomas Mormann's work on trope sheaves [40] provides an explicit philosophical application of sheaf theory to the construction of complex entities. Although Mormann does not frame this work in mereological terms, and does not define parts and fusions via restriction and gluing of sections, the motivations and mathematical setting are clearly allied with our approach.*

In later work on what he calls "structural mereology" ([41], [42], [43]), Mormann frequently employs presheaf- and sheaf-like constructions (often without using that terminology). In these cases, however, parts and wholes are defined via the category-theoretic notion of a subobject, and fusions are given by unions (joins) of subobjects. Since mereological structure then lives in the lattice of an object's subobjects rather than in the fibers of the presheaf, this yields a distinct mereological theory.

Thus, while Mormann's structural mereology is a close conceptual relative of sheaf-theoretic mereology, it is not an instance of the gluing-based framework developed here.

Example 27. *A genuine instance of our framework appears in the theory of "behavioral mereology" developed by Fong, Myers, and Spivak ([44]; cf. [45]). On this approach, a system is defined to be a sheaf of behaviors. Global sections represent complete system behaviors (wholes), while restrictions of sections to smaller regions represent parts of the system.*

In this setting, composition of parts is given by the sheaf-theoretic gluing operation, exactly as in the framework presented here. Parts are not subobjects but fiber-level data: they are local behaviors that arise as restrictions of global ones. So this is an exact instance of our framework.

Interestingly, Fong, Myers, and Spivak impose an additional constraint, requiring restriction maps to be surjective. The effect is that a part of a system consists precisely of those behaviors observable when attention is restricted to a smaller region. In this sense, a part is legitimate only insofar as it is genuinely realized by the whole system.

This yields a distinctly Aristotelian conception of parthood (cf. [46], [47], [48]). Famously, Aquinas ([49], [50], [51]) held that a hand is a part of a body only insofar as it remains functionally integrated with it; a severed hand is no longer a hand. Interpreting a human organism as a system, the hand is exactly that portion of the organism's behavior observable at the hand. Once detached, it is no longer a surjective restriction of the whole system's behavior.

This example illustrates the flexibility of the sheaf-theoretic setting. By imposing additional constraints — such as surjectivity of restrictions — one can recover substantively different metaphysical views of parthood within a single mathematical framework.

4. Classical Mereological Notions in the Sheaf-theoretic Setting

In this section, we show how classical mereological notions translate to the sheaf-theoretic setting.

4.1. Standard Definitions

Recall the definitions of fusion and part.

Definition 14 (Fusions and parts)**.** *We say that a section $s \in F(U)$ is a fusion iff there exists a cover $\{U_i\}_{i \in I}$ of U and a selection of patch candidates $\{s_i\}_{i \in I}$ such that:*

$$\rho^U_{U_i}(s) = s_i, \quad \text{for each } U_i.$$

Given $t \in F(V)$ and $s \in F(U)$ with $V \preccurlyeq U$ and $V \neq \bot$, we say t is a part of s, denoted $t \sqsubseteq s$, iff s is a fusion and

$$\rho^U_V(s) = t.$$

Morally, the overlap of two fusions is a shared part. In this context, it is a section used in two gluings.

Definition 15 (Overlap)**.** *Let $s \in F(V)$ and $t \in F(U)$ be fusions. We say that s and t overlap, denoted $s \sqcap t$, iff*

- *there exists a region W such that $W \preccurlyeq (V \wedge U)$ and $W \neq \bot$,*
- *a section $r \in F(W)$*

such that $r \sqsubseteq F(V)$ and $r \sqsubseteq F(U)$.

Note that overlap is not merely order-theoretic: it is not "just" a shared region. In this setting, two fusions can have a shared region without having a shared part in that region. Parts are only those sections that comprise a gluing.

If fusions overlap, the regions they occupy overlap.

Theorem 1 (Regional overlap)**.** *Let $s \in F(V)$ and $t \in F(U)$ be fusions. Then:*

$$s \sqcap t \implies V \wedge U \neq \perp .$$

Proof. Suppose $s \sqcap t$. By definition, there exists a region W such that $\perp \prec W \preccurlyeq (V \wedge U)$ with a section $r \in F(W)$ that is a part of both s and t. Since W is strictly larger than $\perp$ and $W \preccurlyeq V \wedge U$, then $V \wedge U \neq \perp$. $\square$

The converse is not true. Fusions can fail to overlap either because they occupy disjoint regions, or because they don't have a part in a shared region. So geometric overlap is sufficient but not necessary for mereological disjointness.

A proper part is a part that is not identical to its fusion.

Definition 16 (Proper part)**.** *Let $s \in F(V)$ and $t \in F(U)$. We say that s is a proper part of t, denoted $s \sqsubset t$, iff*

- *$s \sqsubseteq t$,*
- *$V \neq U$.*

Theorem 2 (Nothing is a proper part of itself)**.** *For any part s, $s \not\sqsubset s$.*

Proof. By reflexivity, $s \sqsubseteq s$. But $s \in F(U)$ for some U, and $U = U$, so $s \not\sqsubset s$. $\square$

4.2. Partial Ordering

In the sheaf-theoretic setting, parthood is a partial order.

Theorem 3 (Reflexivity)**.** *For any part s, $s \sqsubseteq s$.*

Proof. For any U, $\{U\}$ is its trivial cover. For any $s \in F(U)$, $\{s\}$ is a trivial selection of patch candidates for that trivial cover $\{U\}$. Further, $\rho^U_U(s) = s$, since restricting to the same region is an identity. Hence, s is a fusion of itself, and $s \sqsubseteq s$, as required. $\square$

Theorem 4 (Transitivity)**.** *For any parts s, t, u, if $s \sqsubseteq t$ and $t \sqsubseteq u$, then $s \sqsubseteq u$.*

Proof. Suppose $s \sqsubseteq t$ and $t \sqsubseteq u$, with $s \in F(W)$, $t \in F(V)$, and $u \in F(U)$. Then $\rho^U_V(u) = t$, and $\rho^V_W(t) = s$. By transitivity of restriction, $\rho^U_W(u) = s$, and hence $s \sqsubseteq u$. $\square$

Theorem 5 (Antisymmetry)**.** *For any parts s, t, if $s \sqsubseteq t$ and $t \sqsubseteq s$, then $s = t$.*

Proof. Suppose $s \sqsubseteq t$ and $t \sqsubseteq s$, with $s \in F(U)$ and $t \in F(V)$. Since $s \sqsubseteq t$ and $t \in F(V)$, there is a region W such that $s \in F(W)$, $W \preccurlyeq V$, and $\rho^V_W(t) = s$. But since we already have that $s \in F(U)$, it must be that $U = W$. Substituting U for W in $W \preccurlyeq V$ and $\rho^V_W(t) = s$ yields $U \preccurlyeq V$ and $\rho^V_U(t) = s$.

Conversely, since $t \sqsubseteq s$, by a similar argument, there is a region Z such that $V = Z$, and substituting V for Z yields $V \preccurlyeq U$ and $\rho^U_V(s) = t$.

Since $U \preccurlyeq V$ and $V \preccurlyeq U$, it must be that $U = V$. If we then substitute U for V in $\rho^V_U(t) = s$ and $\rho^U_V(s) = t$, we get $\rho^U_U(t) = s$ and $\rho^U_U(s) = t$. But ρ^U_U is the identity, so $t = s$, as required. $\square$

4.3. Extensionality

In the sheaf-theoretic setting, extensionality says that fusions are identical when they are glued from the same patch candidates. Formally:

Definition 17 (Extensionality). *We say that extensionality holds in a presheaf F iff, for all fusions s, t in F:*

$$(\forall r, r \sqsubseteq t \iff r \sqsubseteq s) \implies s = t.$$

If extensionality holds, then equal gluings must live in the same fiber.

Theorem 6 (Equality in fibers). *If $s \in F(V)$ and $t \in F(U)$ are gluings and $s = t$, then $U = V$.*

Proof. Suppose $s \in F(V)$, $t \in F(U)$, and $s = t$. Since s is a fusion, there exists a cover $\{V_i\}$ and selection of patch candidates $\{s_i\}_{i \in I}$ such that $\rho^V_{V_i} = s_i$ for every $i \in I$. But since $s = t$, if we substitute t for s, we get $\rho^U_{V_i}(t) = s_i$, for every $i \in I$.

Since t restricts to each region U_i in the cover, it follows that $U_i \preccurlyeq V$, for all $i \in I$. But since $\{U_i\}_{i \in I}$ is a cover of U, U is their join:

$$U = \bigvee_{i \in I} U_i.$$

Since every U_i is below V, it follows that the join of the cover's components is also below V, for the join of any collection of regions is their least upper bound, hence, V is guaranteed to be no lower than that join. Hence $U \preccurlyeq V$.

Going the other way, by a similar argument, we can show that $V \preccurlyeq U$. Then, by antisymmetry, $V = U$. □

Extensionality can fail in presheaves.

Theorem 7 (Extensionality in presheaves). *It is not the case that extensionality holds in every presheaf.*

Proof. In the presheaf from Example 13, b and c are glued from the same parts, yet $b \neq c$. □

By contrast, monopresheaves and sheaves have extensional gluings.

Theorem 8 (Extensionality in monopresheaves and sheaves). *Let F be a presheaf over a locale. If F is a monosheaf or a sheaf, then extensionality holds in F.*

Proof. Let F be a monoprsheaf, and let $s, t \in F(U)$ be fusions such that

$$\forall r, r \sqsubseteq s \iff r \sqsubseteq t.$$

Then for any cover $\{U_i\}_{i \in I}$ of U, every patch r_i used to glue s is also used to glue t. Thus, $\rho^U_{U_i}(s) = \rho^U_{U_i}(t)$, for each U_i in the cover.

By the definition of monopresheaves, it follows that $s = t$. Since a sheaf is a monopresheaf with extra conditions, the same argument shows that extensionality holds for sheaves. □

4.4. Supplementation

In the sheaf-theoretic setting, weak supplementation says that fusions are glued from more than one patch candidate.

Definition 18 (Weak supplementation). *We say that weak supplementation holds in a presheaf F iff, for any $s \in F(V)$ and $t \in F(U)$:*

$$s \sqsubset t \implies \exists W \in \mathbb{L}, r \in F(W)(r \sqsubset t \textit{ and } \neg(r \sqcap s)).$$

In the sheaf-theoretic setting, weak supplementation need not hold. Consider a simple locale $\mathbb{L}$ with $\bot \prec V \prec U$. Then define a presheaf F such that $F(\bot) = \{0\}$, $F(V) = \{s\}$, and $F(U) = \{t\}$. This is (trivially) a presheaf, monopresheaf, and a sheaf, yet weak supplementation fails.

Strong supplementation says that if a section s lives in a fiber outside of a fusion t, it is disjoint from one of the patch candidates from which t is glued.

Definition 19 (Strong supplementation)**.** *We say that strong supplementation holds in a presheaf F iff, for any fusions $s \in F(V)$ and $t \in F(U)$:*

$$s \not\sqsubseteq t \implies \exists W \in \mathbb{L}, r \in F(W)(r \sqsubset t \textit{ and } \neg(r \sqcap s)).$$

This is a strictly stronger condition than weak supplementation, so it fails in the sheaf-theoretic setting too.

It may be tempting to suppose that mereological supplementation imposes a corresponding supplementation condition on the underlying locale. However, this inference is invalid in the sheaf-theoretic setting. Two fusions may fail to overlap either because their regions are disjoint or because, despite regional overlap, no part of either fusion occupies the overlapping region.

4.5. Unrestricted fusion

There is a sense in which unrestricted fusion does not make sense in the sheaf-theoretic setting. Consider the following.

- In the classical-setting, everything in the domain is already a part. By contrast, in the sheaf-theoretic setting, not every section of a fiber is a part. Only those sections that glue are parts.
- In the classical setting, you can collect together any plurality of parts as candidates for a fusion. By contrast, in the sheaf-theoretic setting, you can't pick just any selection of patch candidates. You must select them from the regions in a cover.
- In the classical setting, once you select a plurality of parts, no further coherence or compatibility conditions need to be met before fusing them. By contrast, in the sheaf-theoretic setting, a selection of patch candidates can be glued only if they are compatible.

If one really wanted to translate unrestricted fusion into the sheaf-theoretic setting, it would have to be stated along the following lines:

Definition 20 (Sheaf-theoretic unrestricted fusion)**.** *Given a presheaf F over a local $\mathbb{L}$, for any region $U \in \mathbb{L}$ and selection of sections $\{s_F(U_i)\}_{i \in I}$ satisfying*

- *$\{U_i\}_{i \in I}$ covers U*
- *$\{s_F(U_i)\}_{i \in I}$ are compatible*

There exists a section $s \in F(U)$ such that $\rho^U_{U_i}(s) = s_i$ for each i.

But that is just the sheaf condition. In the sheaf-theoretic setting, unrestricted fusion thus amounts to requiring that all part-whole complexes are sheaves. That is a strong requirement. As we have seen, many natural part-whole complexes are more naturally modeled with the looser gluing conditions of monopresheaves and presheaves.

5. Conclusion

The foregoing makes clear that sheaf theory provides a rich and natural framework for modeling mereology. Because it is built around a notion of coherent gluing, it comes equipped from the outset with a notion of fusion, while parts may be understood as the local patch candidates from which those fusions are assembled. Moreover, presheaves, monopresheaves, and sheaves naturally correspond to increasingly stronger gluing conditions, yielding a structured spectrum of mereological possibilities within a single mathematical setting.

A chief virtue of the framework is its separation of concerns. The algebra of regions (the base locale) captures the structural relations among parts, while the fibers encode the ontological "stuff" that inhabits those regions. Gluing is not forced by axioms but rather arises from the presheaf assignment, allowing fusions, persistence, overlap, and supplementation to be analyzed as structural features of the chosen presheaf. In this way, the framework is both flexible and principled: by varying the presheaf assignment or the strength of its gluing conditions, one obtains a unified family of mereological models rather than a collection of ad hoc theories.

A natural next step is to enrich this picture with modalities. In a sheaf-theoretic context, modalities arise as closure operators—also known as local operators or Lawvere–Tierney topologies. These internal modal operators provide a principled way to distinguish, for example, necessary from merely possible fusions, stable from transient fusions, or coarse-grained from fine-grained notions of parthood, further extending the expressive power of the framework without abandoning its structural foundations.

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