Article

Sheaf Mereology

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1. Classical Mereological Notions in the Sheaf-theoretic Setting

In this section, we provide a discussion of what classical notions of mereology look like in the sheaf-theoretic setting.

1.1. Standard Definitions

Recall the definitions of fusion and part.

Definition 1 (Fusions and parts)**.** *We say that a section $s \in F(U)$ is a fusion iff there exists a cover $\{U_i\}_{i \in I}$ of U and a selection of patch candidates $\{s_i\}_{i \in I}$ such that:*

$$\rho^U_{U_i}(s) = s_i, \quad \textit{for each } U_i.$$

Given $t \in F(V)$ and $s \in F(U)$ with $V \preccurlyeq U$ and $V \neq \perp$, we say t is a part of s, denoted $t \sqsubseteq s$, iff s is a fusion and

$$\rho^U_V(s) = t.$$

Morally, the overlap of two fusions is a shared part. In this context, it is a section used in two gluings.

Definition 2 (Overlap)**.** *Let $s \in F(V)$ and $t \in F(U)$ be fusions. We say that s and t overlap, denoted $s \sqcap t$, iff*

- *there exists a region W such that $W \preccurlyeq (V \wedge U)$ and $W \neq \perp$,*
- *a section $r \in F(W)$*

such that $r \sqsubseteq F(V)$ and $r \sqsubseteq F(U)$.

Note that overlap is not merely order-theoretic: it is not "just" a shared region. In this setting, two fusions can have a shared region without having a shared part in that region. Parts are only those sections that comprise a gluing.

Disjointness in fusions amounts to disjointness in the regions they occupy:

Theorem 1 (Regional disjointness)**.** *Let $s \in F(V)$ and $t \in F(U)$ be fusions. Then:*

$$\neg(s \sqcap t) \implies V \wedge U = \perp.$$

Proof.

Admitted.

□

A proper part is a part that is not identical to its fusion.

Definition 3 (Proper part)**.** *Let $s \in F(V)$ and $t \in F(U)$. We say that s is a proper part of t, denoted $s \sqsubset t$, iff*

- *$s \sqsubseteq t$,*
- *$V \neq U$.*

Theorem 2 (Nothing is a proper part of itself)**.** *For any part s, $s \not\sqsubset s$.*

Proof. By reflexivity, $s \sqsubseteq s$. But $s \in F(U)$ for some U, and $U = U$, so $s \not\sqsubset s$. □

1.2. Partial Ordering

In the sheaf-theoretic setting, parthood is a partial order.

Theorem 3 (Reflexivity). *For any part s, $s \sqsubseteq s$.*

Proof. For any U, $\{U\}$ is its trivial cover. For any $s \in F(U)$, $\{s\}$ is a trivial selection of patch candidates for that trivial cover $\{U\}$. Further, $\rho^U_U(s) = s$, since restricting to the same region is an identity. Hence, s is a fusion of itself, and $s \sqsubseteq s$, as required. □

Theorem 4 (Transitivity). *For any parts s, t, u, if $s \sqsubseteq t$ and $t \sqsubseteq u$, then $s \sqsubseteq u$.*

Proof. Suppose $s \sqsubseteq t$ and $t \sqsubseteq u$, with $s \in F(W)$, $t \in F(V)$, and $u \in F(U)$. Then $\rho^U_V(u) = t$, and $\rho^V_W(t) = s$. By transitivity of restriction, $\rho^U_W(u) = s$, and hence $s \sqsubseteq u$. □

Theorem 5 (Antisymmetry). *For any parts s, t, if $s \sqsubseteq t$ and $t \sqsubseteq s$, then $s = t$.*

Proof. Suppose $s \sqsubseteq t$ and $t \sqsubseteq s$, with $s \in F(U)$ and $t \in F(V)$. Since $s \sqsubseteq t$ and $t \in F(V)$, there is a region W such that $s \in F(W)$, $W \preccurlyeq V$, and $\rho^V_W(t) = s$. But since we already have that $s \in F(U)$, it must be that $U = W$. Substituting U for W in $W \preccurlyeq V$ and $\rho^V_W(t) = s$ yields $U \preccurlyeq V$ and $\rho^V_U(t) = s$.

Conversely, since $t \sqsubseteq s$, by a similar argument, there is a region Z such that $V = Z$, and substituting V for Z yields $V \preccurlyeq U$ and $\rho^U_V(s) = t$.

Since $U \preccurlyeq V$ and $V \preccurlyeq U$, it must be that $U = V$. If we then substitute U for V in $\rho^V_U(t) = s$ and $\rho^U_V(s) = t$, we get $\rho^U_U(t) = s$ and $\rho^U_U(s) = t$. But ρ^U_U is the identity, so $t = s$, as required. □

1.3. Extensionality

In the sheaf-theoretic setting, extensionality says that fusions are identical when they are glued from the same patch candidates. Formally:

Definition 4 (Extensionality). *We say that extensionality holds in a presheaf F iff, for all fusions s, t in F:*

$$(\forall r, r \sqsubseteq t \iff r \sqsubseteq s) \implies s = t.$$

If extensionality holds, then equal gluings must live in the same fiber.

Theorem 6 (Equality in fibers). *If $s \in F(V)$ and $t \in F(U)$ are gluings and $s = t$, then $U = V$.*

Proof. Suppose $s \in F(V)$, $t \in F(U)$, and $s = t$. Since s is a fusion, there exists a cover $\{V_i\}$ and selection of patch candidates $\{s_i\}_{i \in I}$ such that $\rho^V_{V_i} = s_i$ for every $i \in I$. But since $s = t$, if we substitute t for s, we get $\rho^U_{V_i}(t) = s_i$, for every $i \in I$.

Since t restricts to each region U_i in the cover, it follows that $U_i \preccurlyeq V$, for all $i \in I$. But since $\{U_i\}_{i \in I}$ is a cover of U, U is their join:

$$U = \bigvee_{i \in I} U_i.$$

Since every U_i is below V, it follows that the join of the cover's components is also below V, for the join of any collection of regions is their least upper bound, hence, V is guaranteed to be no lower than that join. Hence $U \preccurlyeq V$.

Going the other way, by a similar argument, we can show that $V \preccurlyeq U$. Then, by antisymmetry, $V = U$. $\square$

Extensionality can fail in presheaves.

Theorem 7 (Extensionality failure in presheaves). *It is not the case that extensionality holds in every presheaf.*

Proof. In the presheaf from **??**, b and c are glued from the same parts, yet $b \neq c$. $\square$

By contrast, monopresheaves and sheaves have extensional gluings.

Theorem 8 (Extensionality in monopresheaves and sheaves). *Let F be a presheaf over a locale. If F is a monosheaf or a sheaf, then extensionality holds in F.*

Proof. Let F be a monoprsheaf, and let $s, t \in F(U)$ be fusions such that

$$\forall r, r \sqsubseteq s \iff r \sqsubseteq t.$$

Then for any cover $\{U_i\}_{i \in I}$ of U, every patch r_i used to glue s is also used to glue t. Thus, $\rho^U_{U_i}(s) = \rho^U_{U_i}(t)$, for each U_i in the cover.

By the definition of monopresheaves, it follows that $s = t$. Since a sheaf is a monopresheaf with extra conditions, the same argument shows that extensionality holds for sheaves. $\square$

1.4. Supplementation

Supplementation is the idea that fusions are not made from a single proper part. If you remove a proper part from a fusion, there should be at least one other proper part left over.

There are weaker and stronger formulations. In this setting, weak supplementation is the claim that if s is a proper part of t, then t has another part r disjoint from s.

Definition 5 (Weak supplementation). *We say that weak supplementation holds in a presheaf F iff, for any part $s \in F(V)$ and fusion $t \in F(U)$:*

$$s \sqsubset t \implies \exists r \in F(W)(r \sqsubseteq t \textit{ and } \neg(r \sqcap s)).$$

Note that disjointness of fusions r and s implies disjointness of regions W and V, while $s \sqsubset t$ means that $V \prec U$. Thus, for weak supplementation to hold, there must be another region W disjoint from V. This is purely a requirement on the available regions.

Definition 6 (Regional supplementation). *We say that a locale $\mathbb{L}$ is regionally supplemented iff, for all $U \in \mathbb{L}$ and all $V \prec U$ with $V \neq \perp$, there exists a $W \neq \perp$ such that*

$$W \preccurlyeq U \quad \textit{and} \quad W \wedge V = \perp .$$

In other words, in order for weak supplementation to hold in a presheaf, the underlying locale must be regionally supplemented.

Regional supplementation holds in Boolean locales.

Cite Johnstone or Goldblatt for Boolean locales, e.g. complement pushing to top and bottom via join and meet.

Theorem 9 (Boolean locales are regionally supplemented). *Let $\mathbb{L}$ be a locale. If $\mathbb{L}$ is Boolean, then $\mathbb{L}$ is regionally supplemented.*

Proof. Suppose $\mathbb{L}$ is Boolean, and fix a $V \prec U$ with $V \neq \bot$. Since $\mathbb{L}$ is Boolean, V has a complement $\neg V$ satisfying

$$V \wedge \neg V = \bot \quad \text{and} \quad V \vee \neg V = \top.$$

Define $W = U \wedge \neg V$. Then:

1. In a locale, $a \wedge b \preccurlyeq a$. Let $a = U$ and $b = \neg V$. Then $U \wedge \neg V \preccurlyeq U$. Substituting W yields $W \preccurlyeq U$.
2. $W \wedge V = (U \wedge \neg V) \wedge V = U \wedge (\neg V \wedge V) = U \wedge \bot = \bot$.
3. In a Boolean locale, $a \wedge b = \bot \iff a \preccurlyeq \neg b$. If we assume for contradiction that $U \wedge \neg V = \bot$, it therefore follows that $U \preccurlyeq \neg(\neg V)$. But since $\neg\neg a = a$ in a Boolean locale, $U \preccurlyeq V$. That contradicts the assumption $V \prec U$. Hence, $W = U \wedge \neg V \neg = \bot$.

Thus, W witnesses regional supplementation. □

So, Boolean locales are regionally supplemented. It goes the other direction too.

Theorem 10 (Regionally supplemented locales are Boolean)**.** *Let $\mathbb{L}$ be a locale. If $\mathbb{L}$ is regionally supplemented, then $\mathbb{L}$ is Boolean.*

Proof. To prove that $\mathbb{L}$ is Boolean, it suffices to show that every $V \in \mathbb{L}$ has a complement.

We construct a candidate complement. Suppose $\mathbb{L}$ is regionally supplemented, and fix a $V \in \mathbb{L}$. Next, define:

$$\mathcal{D}_V = \{X \in \mathbb{L} \mid X \wedge V = \bot\},$$

i.e., the set of all regions disjoint from V. This set is nonempty, since it at least contains $\bot$. Then, define the complement of V as the largest element of D_V, which is obtained by taking their join:

$$\neg V = \bigvee \mathcal{D}_V.$$

This join exists because $\mathbb{L}$ is a complete lattice.

Having constructed a candidate complement of V, we next show that is the complement by showing that it satisfies the Boolean complement laws, namely that $V \wedge \neg V = \bot$ and $V \vee \neg V = \top$.

First, we show that $V \wedge \neg V = \bot$. We can do this by the distributivity of locales:

$$V \wedge \neg V = V \wedge \bigvee_{X \in \mathcal{D}_V} X = \bigvee_{X \in \mathcal{D}_V} (V \wedge X) = \bigvee \bot = \bot\,.$$

In other words, $\neg V$ is disjoint from V.

Next, we must show that $V \vee \neg V = \top$. Assume, for contradiction, that $V \vee \neg V \prec \top$. Let U be $V \vee \neg V$. Then $V \prec U \prec \top$.

Regional supplementation says that, since $V \prec U$, there exists a $W \neq \bot$ such that $W \preccurlyeq U$ and $W \wedge V = \bot$.

But $W \wedge V = \bot$ implies that $W \in \mathcal{D}_V$. So, $W \preccurlyeq \bigvee \mathcal{D}_V = \neg V$. In a locale, if $a \preccurlyeq b$ and $a \preccurlyeq c$, then $a \preccurlyeq b \wedge c$. Here, we have that $W \preccurlyeq U$ and $W \preccurlyeq \neg V$. Thus, $W \preccurlyeq U \wedge \neg V$. But then:

$$W \preccurlyeq U \wedge \neg V = (V \vee \neg V) \wedge \neg V = \neg V.$$

That implies that $W \preccurlyeq \neg V \preccurlyeq U$, so W was already below $\neg V$.

Now observe that $\neg V \preccurlyeq V \vee \neg V = U$, so adding W below $\neg V$ cannot enlarge U. But regional supplementation requires that $W \neq \perp$ and $W \preccurlyeq U$ strictly witnessing supplementation inside U. This contradicts the assumption that $\neg V$ already collected *all* elements disjoint from V.

Formally, if $V \vee \neg V \prec \top$, regional supplementation produces a non-bottom disjoint region below U, but by construction all such regions are already $\preccurlyeq \neg V$, forcing $U = V \vee \neg \top$, which is a contradiction. Hence, $V \vee \neg V = \top$.

Since $V \wedge \neg V = \perp$ and $V \vee \neg V = \top$, $\neg V$ is a Boolean complement. Since V was chosen arbitrarily, this holds for all V in $\mathbb{L}$, hence $\mathbb{L}$ is a Boolean locale. □

Thus, regional supplementation is equivalent to the Booleanness of the locale, and we can conclude that a presheaf can satisfy weak supplementation only if it has the right kind of geometry, namely a Boolean geometry.

Theorem 11 (Booleanness = regional supplementation)**.** *Let $\mathbb{L}$ be a locale. Then the following are equivalent:*

1. *$\mathbb{L}$ is regionally supplemented.*
2. *$\mathbb{L}$ is Boolean.*
3. *For any presheaf F over $\mathbb{L}$, F can satisfy weak supplementation only if $\mathbb{L}$ is regionally supplemented or Boolean.*

Tighten up 3. What's that really mean? Maybe keep it outside the theorem, and keep it meta.

Proof. Immediate. □

This strongly suggests that classical supplementation is not merely a mereological axiom. It presupposes a Boolean geometry of parts. This explains why supplementation can fail in topological contexts (e.g. in $Sh([0,1])$), since topological contexts are typically non-Boolean, and it explains why supplementation tacitly assumes a Boolean background.

Todo list