Article

Sheaf Mereology

Firstname Lastname [1], Firstname Lastname [2] and Firstname Lastname [2,*]

[1] Affiliation 1; e-mail@e-mail.com
[2] Affiliation 2; e-mail@e-mail.com
* Correspondence: e-mail@e-mail.com; Tel.: (optional; include country code; if there are multiple corresponding authors, add author initials) +xx-xxxx-xxx-xxxx (F.L.)

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1. Classical Mereological Notions in the Sheaf-theoretic Setting

In this section, we show how classical mereological notions translate to the sheaf-theoretic setting.

1.1. Standard Definitions

Recall the definitions of fusion and part.

Definition 1 (Fusions and parts)**.** *We say that a section $s \in F(U)$ is a fusion iff there exists a cover $\{U_i\}_{i \in I}$ of U and a selection of patch candidates $\{s_i\}_{i \in I}$ such that:*

$$\rho^U_{U_i}(s) = s_i, \quad \textit{for each } U_i.$$

Given $t \in F(V)$ and $s \in F(U)$ with $V \preccurlyeq U$ and $V \neq \perp$, we say t is a part of s, denoted $t \sqsubseteq s$, iff s is a fusion and

$$\rho^U_V(s) = t.$$

Morally, the overlap of two fusions is a shared part. In this context, it is a section used in two gluings.

Definition 2 (Overlap)**.** *Let $s \in F(V)$ and $t \in F(U)$ be fusions. We say that s and t overlap, denoted $s \sqcap t$, iff*

- *there exists a region W such that $W \preccurlyeq (V \wedge U)$ and $W \neq \perp$,*
- *a section $r \in F(W)$*

such that $r \sqsubseteq F(V)$ and $r \sqsubseteq F(U)$.

Note that overlap is not merely order-theoretic: it is not "just" a shared region. In this setting, two fusions can have a shared region without having a shared part in that region. Parts are only those sections that comprise a gluing.

If fusions overlap, the regions they occupy overlap.

Theorem 1 (Regional overlap)**.** *Let $s \in F(V)$ and $t \in F(U)$ be fusions. Then:*

$$s \sqcap t \implies V \wedge U \neq \perp.$$

Proof. Suppose $s \sqcap t$. By definition, there exists a region W such that $\perp \prec W \preccurlyeq (V \wedge U)$ with a section $r \in F(W)$ that is a part of both s and t. Since W is strictly larger than $\perp$ and $W \preccurlyeq V \wedge U$, then $V \wedge U \neq \perp$. □

The converse is not true. Fusions can fail to overlap either because they occupy disjoint regions, or because they don't have a part in a shared region. So geometric overlap is sufficient but not necessary for mereological disjointness.

A proper part is a part that is not identical to its fusion.

Definition 3 (Proper part)**.** *Let $s \in F(V)$ and $t \in F(U)$. We say that s is a proper part of t, denoted $s \sqsubset t$, iff*

- *$s \sqsubseteq t$,*
- *$V \neq U$.*

Theorem 2 (Nothing is a proper part of itself)**.** *For any part s, $s \not\sqsubset s$.*

Proof. By reflexivity, $s \sqsubseteq s$. But $s \in F(U)$ for some U, and $U = U$, so $s \not\sqsubset s$. □

1.2. Partial Ordering

In the sheaf-theoretic setting, parthood is a partial order.

Theorem 3 (Reflexivity). *For any part s, $s \sqsubseteq s$.*

Proof. For any U, $\{U\}$ is its trivial cover. For any $s \in F(U)$, $\{s\}$ is a trivial selection of patch candidates for that trivial cover $\{U\}$. Further, $\rho^U_U(s) = s$, since restricting to the same region is an identity. Hence, s is a fusion of itself, and $s \sqsubseteq s$, as required. □

Theorem 4 (Transitivity). *For any parts s, t, u, if $s \sqsubseteq t$ and $t \sqsubseteq u$, then $s \sqsubseteq u$.*

Proof. Suppose $s \sqsubseteq t$ and $t \sqsubseteq u$, with $s \in F(W)$, $t \in F(V)$, and $u \in F(U)$. Then $\rho^U_V(u) = t$, and $\rho^V_W(t) = s$. By transitivity of restriction, $\rho^U_W(u) = s$, and hence $s \sqsubseteq u$. □

Theorem 5 (Antisymmetry). *For any parts s, t, if $s \sqsubseteq t$ and $t \sqsubseteq s$, then $s = t$.*

Proof. Suppose $s \sqsubseteq t$ and $t \sqsubseteq s$, with $s \in F(U)$ and $t \in F(V)$. Since $s \sqsubseteq t$ and $t \in F(V)$, there is a region W such that $s \in F(W)$, $W \preccurlyeq V$, and $\rho^V_W(t) = s$. But since we already have that $s \in F(U)$, it must be that $U = W$. Substituting U for W in $W \preccurlyeq V$ and $\rho^V_W(t) = s$ yields $U \preccurlyeq V$ and $\rho^V_U(t) = s$.

Conversely, since $t \sqsubseteq s$, by a similar argument, there is a region Z such that $V = Z$, and substituting V for Z yields $V \preccurlyeq U$ and $\rho^U_V(s) = t$.

Since $U \preccurlyeq V$ and $V \preccurlyeq U$, it must be that $U = V$. If we then substitute U for V in $\rho^V_U(t) = s$ and $\rho^U_V(s) = t$, we get $\rho^U_U(t) = s$ and $\rho^U_U(s) = t$. But ρ^U_U is the identity, so $t = s$, as required. □

1.3. Extensionality

In the sheaf-theoretic setting, extensionality says that fusions are identical when they are glued from the same patch candidates. Formally:

Definition 4 (Extensionality). *We say that extensionality holds in a presheaf F iff, for all fusions s, t in F:*

$$(\forall r, r \sqsubseteq t \iff r \sqsubseteq s) \implies s = t.$$

If extensionality holds, then equal gluings must live in the same fiber.

Theorem 6 (Equality in fibers). *If $s \in F(V)$ and $t \in F(U)$ are gluings and $s = t$, then $U = V$.*

Proof. Suppose $s \in F(V)$, $t \in F(U)$, and $s = t$. Since s is a fusion, there exists a cover $\{V_i\}$ and selection of patch candidates $\{s_i\}_{i \in I}$ such that $\rho^V_{V_i} = s_i$ for every $i \in I$. But since $s = t$, if we substitute t for s, we get $\rho^U_{V_i}(t) = s_i$, for every $i \in I$.

Since t restricts to each region U_i in the cover, it follows that $U_i \preccurlyeq V$, for all $i \in I$. But since $\{U_i\}_{i \in I}$ is a cover of U, U is their join:

$$U = \bigvee_{i \in I} U_i.$$

Since every U_i is below V, it follows that the join of the cover's components is also below V, for the join of any collection of regions is their least upper bound, hence, V is guaranteed to be no lower than that join. Hence $U \preccurlyeq V$.

Going the other way, by a similar argument, we can show that $V \preccurlyeq U$. Then, by antisymmetry, $V = U$. □

Extensionality can fail in presheaves.

Theorem 7 (Extensionality in presheaves)**.** *It is not the case that extensionality holds in every presheaf.*

Proof. In the presheaf from **??**, b and c are glued from the same parts, yet $b \neq c$. □

By contrast, monopresheaves and sheaves have extensional gluings.

Theorem 8 (Extensionality in monopresheaves and sheaves)**.** *Let F be a presheaf over a locale. If F is a monosheaf or a sheaf, then extensionality holds in F.*

Proof. Let F be a monoprsheaf, and let $s, t \in F(U)$ be fusions such that

$$\forall r, r \sqsubseteq s \iff r \sqsubseteq t.$$

Then for any cover $\{U_i\}_{i \in I}$ of U, every patch r_i used to glue s is also used to glue t. Thus, $\rho^U_{U_i}(s) = \rho^U_{U_i}(t)$, for each U_i in the cover.

By the definition of monopresheaves, it follows that $s = t$. Since a sheaf is a monopresheaf with extra conditions, the same argument shows that extensionality holds for sheaves. □

1.4. Supplementation

In the sheaf-theoretic setting, weak supplementation says that fusions are glued from more than one patch candidate.

Definition 5 (Weak supplementation)**.** *We say that weak supplementation holds in a presheaf F iff, for any $s \in F(V)$ and $t \in F(U)$:*

$$s \sqsubset t \implies \exists W \in \mathbb{L}, r \in F(W)(r \sqsubset t \textit{ and } \neg(r \sqcap s)).$$

In the sheaf-theoretic setting, weak supplementation need not hold. Consider a simple locale $\mathbb{L}$ with $\bot \prec V \prec U$. Then define a presheaf F such that $F(\bot) = \{0\}$, $F(V) = \{s\}$, and $F(U) = \{t\}$. This is (trivially) a presheaf, monopresheaf, and a sheaf, yet weak supplementation fails.

Strong supplementation says that if a section s lives in a fiber outside of a fusion t, it is disjoint from one of the patch candidates from which t is glued.

Definition 6 (Strong supplementation)**.** *We say that strong supplementation holds in a presheaf F iff, for any fusions $s \in F(V)$ and $t \in F(U)$:*

$$s \not\sqsubseteq t \implies \exists W \in \mathbb{L}, r \in F(W)(r \sqsubset t \textit{ and } \neg(r \sqcap s)).$$

This is a strictly stronger condition than weak supplementation, so it fails in the sheaf-theoretic setting too.

It may be tempting to suppose that mereological supplementation imposes a corresponding supplementation condition on the underlying locale. However, this inference is invalid in the sheaf-theoretic setting. Two fusions may fail to overlap either because their regions are disjoint or because, despite regional overlap, no part of either fusion occupies the overlapping region.

1.5. Unrestricted fusion

There is a sense in which unrestricted fusion does not make sense in the sheaf-theoretic setting. Consider the following.

- In the classical-setting, everything in the domain is already a part. By contrast, in the sheaf-theoretic setting, not every section of a fiber is a part. Only those sections that glue are parts.
- In the classical setting, you can collect together any plurality of parts as candidates for a fusion. By contrast, in the sheaf-theoretic setting, you can't pick just any selection of patch candidates. You must select them from the regions in a cover.
- In the classical setting, once you select a plurality of parts, no further coherence or compatibility conditions need to be met before fusing them. By contrast, in the sheaf-theoretic setting, a selection of patch candidates can be glued only if they are compatible.

If one really wanted to translate unrestricted fusion into the sheaf-theoretic setting, it would have to be stated along the following lines:

Definition 7 (Sheaf-theoretic unrestricted fusion)**.** *Given a presheaf F over a local $\mathbb{L}$, for any region $U \in \mathbb{L}$ and selection of sections $\{s_F(U_i)\}_{i \in I}$ satisfying*

- *$\{U_i\}_{i \in I}$ covers U*
- *$\{s_F(U_i)\}_{i \in I}$ are compatible*

There exists a section $s \in F(U)$ such that $\rho^U_{U_i}(s) = s_i$ for each i.

But that is just the sheaf condition. In the sheaf-theoretic setting, unrestricted fusion thus amounts to requiring that all part-whole complexes are sheaves. That is a strong requirement. As we have seen, many natural part-whole complexes are more naturally modeled with the looser gluing conditions of monopresheaves and presheaves.

Todo list