Article

Sheaf Mereology

Firstname Lastname [1], Firstname Lastname [2] and Firstname Lastname [2,*]

[1] Affiliation 1; e-mail@e-mail.com
[2] Affiliation 2; e-mail@e-mail.com
* Correspondence: e-mail@e-mail.com; Tel.: (optional; include country code; if there are multiple corresponding authors, add author initials) +xx-xxxx-xxx-xxxx (F.L.)

Received:
Revised:
Accepted:
Published:

Citation: Lastname, F.; Lastname, F.; Lastname, F. Title. *Philosophies* **2025**, *1*, 0. https://doi.org/

1. Modeling Part-Whole Complexes as Presheaves

As noted in **??**, the central claim of this paper is that we can model part-whole complexes as presheaves over locales, with varying gluing conditions. In particular, the locale provides the abstract parts space of "regions" that the pieces can occupy, the presheaf assigns actual pieces to those regions, and the gluings determine which pieces fuse.

We can thus define the core mereological concepts of part and whole in sheaf-theoretic terms. Regarding wholes, we can identify fusion with gluing: to say that some pieces fuse or form a "fusion" is just to say that they are glued together. Regarding parts, to say that a piece is a "part" is just to say that it is a part of a fusion. In other words, the parts of a fusion are just the pieces from which it is glued together.

Definition 1 (Fusions and parts)**.** *We say that a section $s \in F(U)$ is a fusion iff there exists a cover $\{U_i\}_{i \in I}$ of U and a selection of patch candidates $\{s_i\}_{i \in I}$ such that:*

$$\rho^U_{U_i}(s) = s_i, \quad \textit{for each } U_i.$$

Given $t \in F(V)$ and $s \in F(U)$ with $V \preccurlyeq U$ and $V \neq \perp$, we say t is a part of s, denoted $t \sqsubseteq s$, iff s is a fusion and:

$$\rho^U_V(s) = t.$$

Remark 1. *$V \neq \perp$, for as we saw in* **??**, *$\perp$ represents no regions at all, and hence cannot be occupied by parts. The elements in the fiber over $\perp$ are structural anchors, not parts.*

Because fusions do not freely arise here, but rather only exist where parts are explicitly glued together, sheaf theory thus provides a systematic framework with which to model a large variety of part-whole complexes in a "fusions-first" manner. In the rest of this section, we illustrate with examples. In each case, we construct a custom presheaf designed to model a particular part-whole complex. Our choices of presheaves should be interpreted as modeling choices. One could construct different presheaves, and each can be evaluated on its own merits.

Example 1. *Consider a building with a west room, an east room, and a hallway between them. For simplicity, let us consider only the floors of the building (ignore walls, ceilings, and so on). The ambient locale is given by the presentation*

- $\mathbb{L} = \langle G, R \rangle = \langle \{WR, H, ER\}, \varnothing \rangle$

where

- *WR = west room*
- *H = hallway*
- *ER = east room*

As a Hasse diagram:

$$\begin{tikzcd}
& WR \vee H \vee ER \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
WR \vee H \arrow[d, dash] \arrow[dr, dash] & WR \vee ER \arrow[dl, dash] \arrow[dr, dash] & H \vee ER \arrow[dl, dash] \arrow[d, dash] \\
WR \arrow[dr, dash] & H \arrow[d, dash] & ER \arrow[dl, dash] \\
& \perp &
\end{tikzcd}$$

For the presheaf F, let it assign to each region whatever materials (if any) cover its floor uniformly. Let us say that the west room's and hallway's floors are each covered uniformly by wood, while the east room's floor is covered uniformly by tiles:

- $F(WR) = \{wood\}$
- $F(H) = \{wood\}$
- $F(ER) = \{tile\}$

Since the west room's and hallway's floors are covered uniformly by wood, their join is too:

- $F(WR \vee H) = \{wood\}$

Since none of the other regions are covered uniformly by the same material, we assign nothing to them:

- $F(WR \vee ER) = \varnothing$
- $F(H \vee ER) = \varnothing$
- $F(WR \vee H \vee ER) = \varnothing$

Finally, for the fiber over bottom, where there are no regions to cover with materials, let us assign the special symbol zero:

- $F(\bot) = \{0\}$

For the restrictions, let us say that the materials that cover a larger region are restricted down to the materials that cover the smaller region. Hence, the non-empty fibers restrict by identity:

- $\rho_{WR}^{WR \vee H}(wood) = \rho_{H}^{WR \vee H}(wood) = wood$

The empty fibers restrict via the empty function (there is nothing to restrict):

- $\rho_{WR \vee H}^{WR \vee H \vee ER} = \rho_{WR \vee ER}^{WR \vee H \vee ER} = \rho_{H \vee ER}^{WR \vee H \vee ER} =$ *empty function*
- $\rho_{WR}^{WR \vee ER} = \rho_{ER}^{WR \vee ER} =$ *empty function*
- $\rho_{H}^{H \vee ER} = \rho_{ER}^{H \vee ER} =$ *empty function*

Finally, fibers restrict to bottom via the constant function:

- $\rho_{\bot}^{WR}(wood) = \rho_{\bot}^{H}(wood) = \rho_{\bot}^{ER}(tile) = 0$

In this building, there are two maximal fusions:

- *The flooring of the west room and the hallway glue into one piece that covers both.*
- *The flooring that covers the east room.*

Thus, the flooring of this building is really a collection of two independent fusions: the wooden floor that covers the west room and hallway, and the tiled floor that covers the east room. That implies:

- *To separate the floors of the west room and hallway, you would have to use a saw to cut them, since they are fused. They are not merely sitting next to each other. Rather, they make up a single (fused) piece.*
- *By contrast, to separate the hallway and the east room, you would not need to cut them, since they are not fused. They simply happen to be sitting next to each other.*

The parts of the fusions are clear:

- *The wooden floor that covers the west room and the hallway has two parts: the wooden floor that covers the west room, and the wooden floor that covers the hallway.*
- *The tiled floor of the east room has no parts (in this locale), since it is not the fusion of other fusions.*

This is particular example fails to be a sheaf, because everything that can glue does not glue. In particular, $F(WR \vee H \vee ER)$ is covered by $\{WR, H, ER\}$, and the patch candidates $\{wood, wood, tile\}$ are compatible (they pair-wise restrict to 0). However, there is nothing in $F(WR \vee H \vee ER)$ that is *glued together from those patch candidates (indeed, $F(WR \vee H \vee ER)$ is empty). Hence, this is not a sheaf. Rather, it is a monopresheaf.*

But this is precisely what one would expect when modeling two discrete pieces of flooring that happen to sit next to each other. The west room and hallway do glue together here, as expected. But the maximal wood and tile pieces do not glue together, also as expected (there is a boundary between them, where the wood ends and the tile begins).

In the previous example, none of the regions overlapped. The presheaf was free to glue or not glue pieces as it saw fit. The story is different if there are overlaps in the locale itself. Overlaps in the locale require overlaps in the presheaf, wherever you want gluings.

Example 2. *Consider the floor of a single room. Let us say that the regions of interest are its west half, its east half, and a six inch span where they overlap.*

The ambient locale of this kind of space can be given by the presentation

- $L = \langle G, R \rangle = \langle \{\bot, WH, O, EH\}, \{\bot \preccurlyeq O, O \preccurlyeq WH, O \preccurlyeq EH\} \rangle$

where

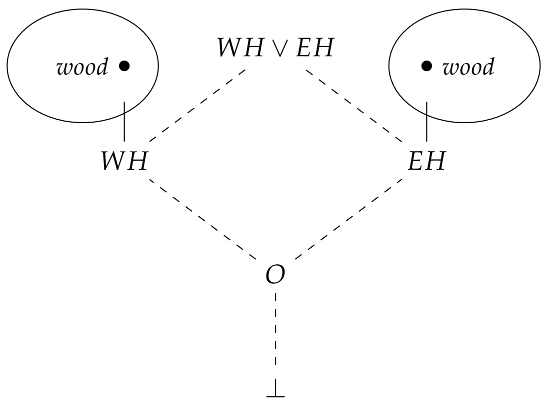
- WH = *west half*
- O = *overlap*
- EH = *east half*

For a presheaf F, let us say that it behaves much like in the previous example: it assigns to each region the materials (if any) that cover the floor uniformly.

For instance, let us assign wood to both halves:

- $F(WH) = \{wood\}$
- $F(EH) = \{wood\}$

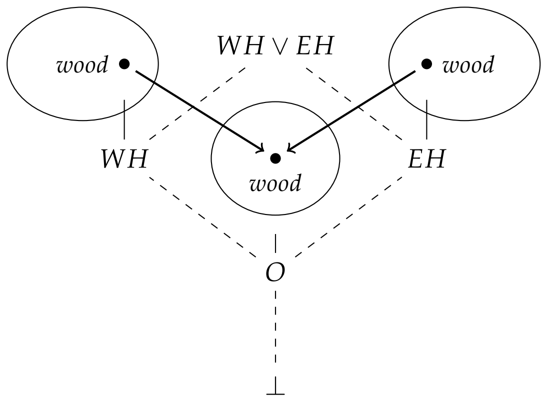
In a picture:



By construction $O = WH \wedge EH$, and the two halves restrict to the same material there:

- $F(O) = \{wood\}$

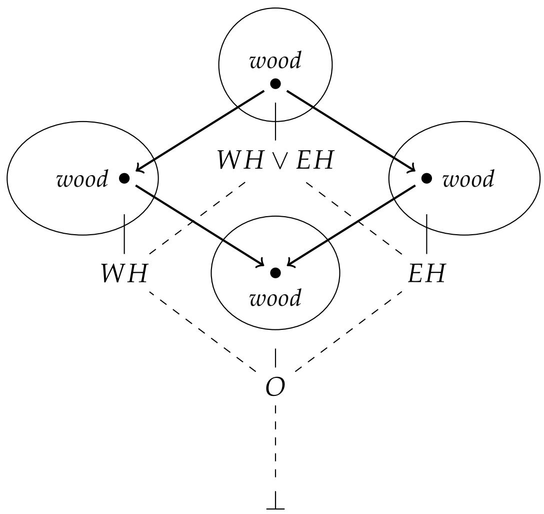
Thus:



For the join, the west and east halves glue, since they're made from the same flooring materials and agree on their overlap:

- $F(WH \vee EH) = \{wood\}$

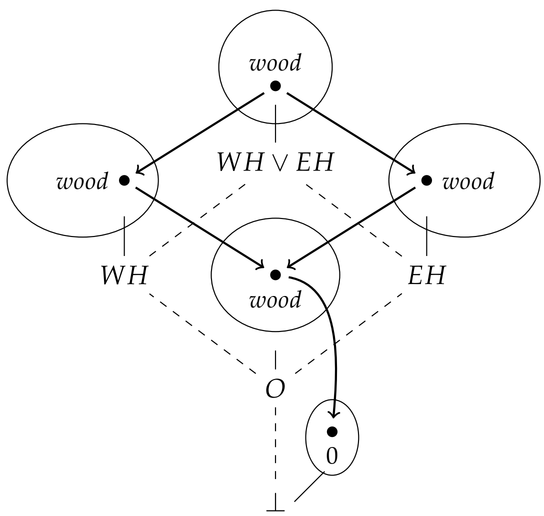
Thus:



Finally, for the fiber over bottom, where there are no regions to cover with materials, assign the special symbol zero:

- $F(\perp) = \{0\}$

In a picture:



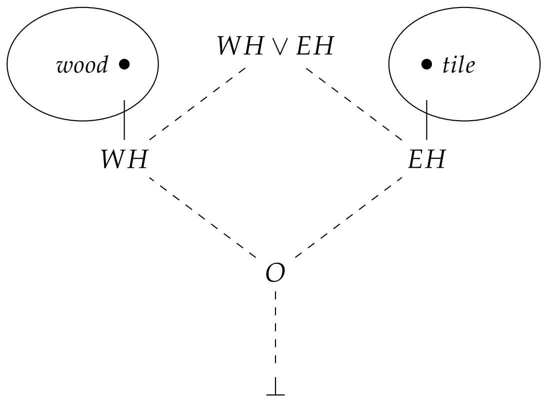
The maximal fusion here is a single piece of wooden flooring (namely, wood $\in F(WH \vee EH)$) that covers the whole room. Its parts are the west and east halves, and (transitively) their overlap. The west and east halves themselves have a shared part, the strip of overlap.

This particular example is a sheaf: the parts glue together coherently in a single manner across the entire parts space. This is exactly as one would expect when modeling a floor that is covered uniformly in its entirety by wood flooring: as you restrict down to smaller parts of the room, you get smaller pieces of wood flooring. In contrast to Example 1, here the regions have a nontrivial overlap. By the sheaf condition, agreement on that overlap forces a unique fusion of the parts, just as expected.

Example 3. *To illustrate a failed attempt to build a sheaf, let us take the locale and gluing condition from Example 2, but suppose that we assign different flooring materials to the east and west halves of the room:*

- $F(WH) = \{wood\}$
- $F(EH) = \{tile\}$

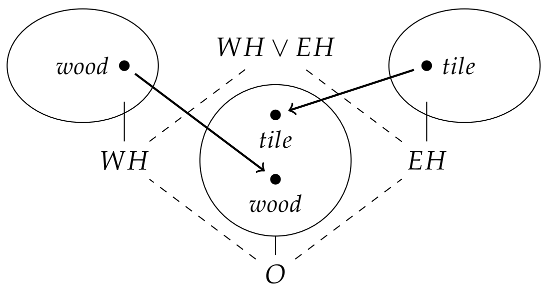
As a picture:



Next, at the overlap, allow both wood and tile:

- $F(O) = \{wood, tile\}$
- $\rho^{WH}_{O}(wood) = wood$
- $\rho^{EH}_{O}(tile) = tile$

Thus, as a picture:

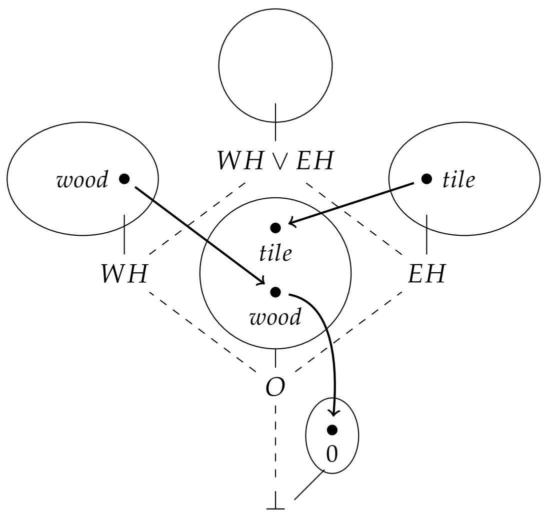


Note, however, that the wood and tile in the west and east halves cannot glue, because they do not agree on their overlap. Hence, there cannot be a single kind of flooring material that uniformly covers the maximal join $WH \vee EH$. This illustrates how the sheaf condition is a very strong condition, but also a helpful one: it requires and manages coherent gluing at all levels. Because it requires that pieces glue together coherently at every level of "zoom," it prevents us from ever putting together an incoherent part-whole sheaf in the first place.

It is worth spelling the failure out explicitly. If we want to model this room as a sheaf, then we are requiring coherent, unique gluing across all of the regions: wherever pieces can coherently glue together, they must do so. But here, since WH and EH disagree on their overlap, there is no way to make them cohere into a single piece.

Intuitively, this makes sense. If the western and eastern halves of a room were truly floored with different materials, then they could not overlap. Imagine if two builders started at opposite ends of the room: one flooring with wood and the other flooring with tile. When they reach the mid-point, they'd realize they made a mistake. In such a scenario, it would be impossible to complete the original vision of having a single, uniform flooring across the entire room.

However, what if we weaken our requirements and consider this as a presheaf? In a presheaf, nothing disallows the two halves from restricting differently on an overlap. The full presheaf looks like this:



Interpreted as a presheaf, there is a sensible interpretation of this structure. Nothing sits in the fiber over $WH \vee EH$, since a single coherent piece of flooring cannot be glued from wood and tile. The wood and tile from the two halves each extend into the overlap though, but since they don't agree, one of them must sit on top of the other in that overlapping area. It is like when two area rugs overlap: one sits on top of the other.

As a final point, note that this fails to be monopresheaf, because there are two sections in $F(O)$ that restrict to $\perp$. The gluings are trivial here, but nonetheless, gluing is not unique precisely over $\perp$. Hence, this is a presheaf, but not a monopresheaf. This is a limit case where presheaves allow multiple assemblies of the same parts, but monopresheaves do not.

The previous two examples were spatial. But parts come in non-spatial guises too, and sheaf theory can model them just as well.

Example 4. *Suppose we say that human society (under some description) consists of the mesh of a specified set of relationships between the people that participate in that society.*

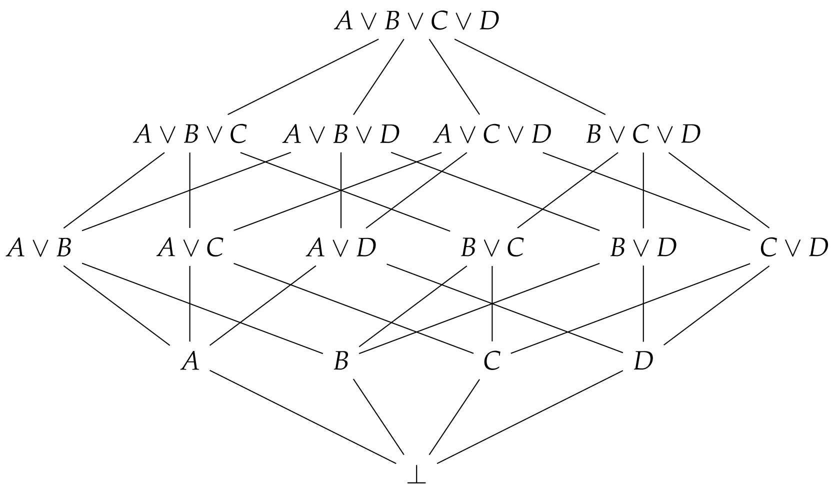
Let P be the population in question (a finite set of individual people), and let the regions of our locale be subsets of such individuals. Then the ambient locale is given by the presentation:

- $\mathbb{L} = \langle G, R \rangle = \langle P, \varnothing \rangle$

For concreteness, suppose:

- $P = \{A, B, C, D\}$, *with A short for Alice, B for Bob, C for Carol, and D for Denny.*

Then the Hasse diagram is isomorphic to the powerset of P:



Let us next define a presheaf F that models the mesh of a selected set of relationships over P. To do that, let us first specify a set R that picks out the (binary, symmetric) relationships of interest:

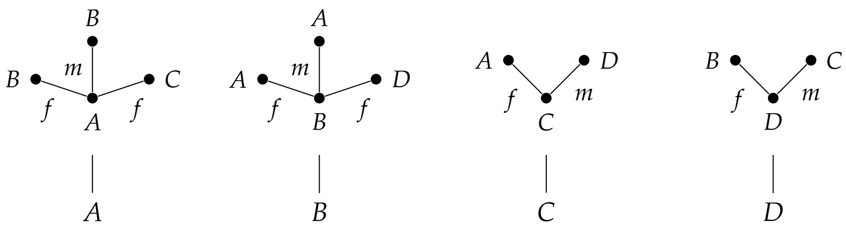
- $R = \{f, m, \ldots\}$, *with f short for being friends, m for being married, etc.*

For convenience, if $U, V \in P$, $r \in R$, *and U and V stand in relationship r, we will write* $r(U, V)$.

For the generators, let us fix a choice of local data by assigning to each person the relations they stand in, e.g.:

- $F(A) = \{\langle\{f(A, B), f(A, C), m(A, B)\}\rangle\}$
- $F(B) = \{\langle\{f(B, A), m(B, A), f(B, D)\}\rangle\}$
- $F(C) = \{\langle\{f(C, A), m(C, D)\}\rangle\}$
- $F(D) = \{\langle\{f(D, B), m(D, C)\}\rangle\}$

To visualize this data, we can picture each fiber as a mini-graph:



For example, in the fiber over A:

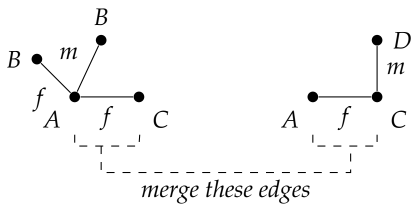
- *The f-labeled edge from A to B represents* $f(A, B)$*: A and B are friends.*
- *The m-labeled edge from A to B represents* $m(A, B)$*: A and B are married.*
- *The f-labeled edge from A to C represents* $f(A, C)$*: A and C are friends.*

In the bottom fiber, over the region of no people to carry relationships, assign the special symbol 0*:*

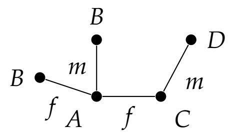
- $F(\perp) = \{0\}$

Next, let us extend the above data to binary joins by merging mini-graphs along shared edges, wherever the components share exactly the same edges. To see how this works, consider (for example) the mini-graphs over A and C:

Can these be merged? The answer is yes, because they share exactly one edge, namely the one labeled f. If you rotate the graphs sideways a bit, you can see how they can be merged along $f(A, C)$:



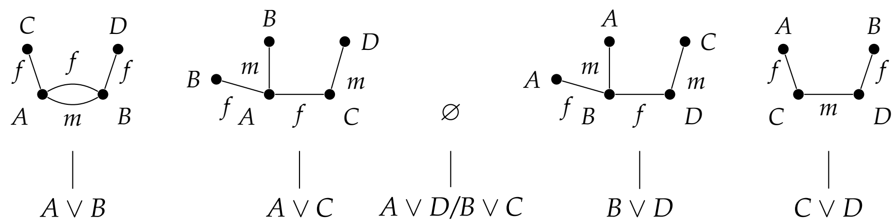
Merging along $f(A, C)$ yields the following graph:



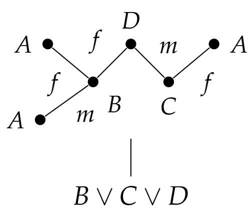
By merging all binary joins in this fashion, we get:

- $F(A \vee B) = \{\langle \{f(A, B), m(A, B), f(A, C)\}, \{f(B, A), m(B, A), f(B, D)\}\rangle\}$
- $F(A \vee C) = \{\langle \{f(B, A), m(B, A), f(B, D)\}\rangle, \{f(C, A), m(C, D)\}\}$
- $F(A \vee D) = \varnothing$
- $F(B \vee C) = \varnothing$
- $F(B \vee D) = \{\langle \{f(B, A), m(B, A), f(B, D)\}, \{f(D, B), m(D, C)\}\rangle\}$
- $F(C \vee D) = \{\langle \{f(C, A), m(C, D)\}, \{f(D, B), m(D, C)\}\rangle\}$

As pictures:



Having merged the graphs from joins of two regions, we must next merge the graphs of joins from three regions. For instance, take $B \vee C \vee D$. We can merge the graphs of $B \vee C$ trivially (because they share no edges), we can merge the graphs of $C \vee D$ along their shared f-edge, and we can merge the graphs of $B \vee D$ along their shared f-edge. That yields:



By merging the graphs of all joins of three regions in this fashion, we get:

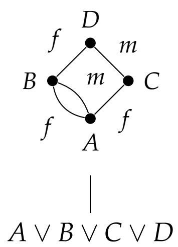
- $F(A \vee B \vee C) = \left\{ \begin{array}{ll} \langle & \{f(A, B), f(A, C), m(A, B)\}, \\ & \{f(B, A), m(B, A), f(B, D)\}, \\ & \{f(C, A), m(C, D)\} \quad \rangle \end{array} \right\}$
- $F(A \vee B \vee D) = \left\{ \begin{array}{ll} \langle & \{f(A, B), f(A, C), m(A, B)\}, \\ & \{f(B, A), m(B, A), f(B, D)\}, \\ & \{f(D, B), m(D, C)\} \quad \rangle \end{array} \right\}$

- $$F(A \vee C \vee D) = \left\{ \begin{array}{ll} \langle & \{f(A,B), f(A,C), m(A,B)\}, \\ & \{f(C,A), m(C,D)\}, \\ & \{f(D,B), m(D,C)\} \quad \rangle \end{array} \right\}$$
- $$F(B \vee C \vee D) = \left\{ \begin{array}{ll} \langle & \{f(B,A), m(B,A), f(B,D)\}, \\ & \{f(C,A), m(C,D)\}, \\ & \{f(D,B), m(D,C)\} \quad \rangle \end{array} \right\}$$

At the top-most join of the locale, if we merge the graphs of all four regions, we get:

- $$F(A \vee B \vee C \vee D) = \left\{ \begin{array}{ll} \langle & \{f(A,B), f(A,C), m(A,B)\}, \\ & \{f(B,A), m(B,A), f(B,D)\}, \\ & \{f(C,A), m(C,D)\}, \\ & \{f(D,B), m(D,C)\} \quad \rangle \end{array} \right\}$$

As a picture:



In effect, the fiber over each region U is a consistent assignment of relationships involving exactly the people in U, obtained by restriction from the full society. A restriction from $F(U)$ to $F(V)$ in effect forget any edges and vertices not involving people in V. It truly restricts the relationships mesh from those people in U to only those in V.

This is in fact a sheaf. The sheaf condition requires that the relationship meshes over two opens agrees precisely on their overlap: when you restrict the meshes down to their overlap, you get the same sub-mesh.

The result is a sheaf: a fused mesh of relationships over the population, which is glued together from smaller meshes over smaller subsets of the population.

- *Each fiber is a part of the whole (human society), and its data encodes the internal (relational) structure of that part.*
- *Mereological overlap is then modeled by shared relationships: two parts overlap if their relational graphs intersect coherently.*
- *Regions that are not covered by a mesh (as in $F(A \vee D) = \varnothing$ and $F(B \vee C) = \varnothing$) reflect mereological separation: the regions in question are simple not related, so there is nothing to glue.*

For another example, consider processes. A process (or more generally any sequence of events, states, etc.) can be seen as a part-whole complex too.

Example 5. *Imagine a scenario where something can do one of two things repeatedly: at each step, it can do one thing ("option a") or another thing ("option b"), and then repeat the choice again.*

To model this, fix a finite alphabet $\Sigma = \{a, b\}$, with "a" for "option a" and "b" for "option b." Then let Σ^* be the set of all finite sequences (words) over Σ, with ϵ denoting the empty sequence. For instance, the sequence aab represents the sequence of length 3 that picks "option a" first, then "option a" again, and then finally "option b."

Let us say that $\Sigma^{\leqslant n}$ is the set of all finite sequences less than length n, and let us say that $\Sigma^{=n}$ is the set of finite sequences of exactly length n. Hence:

- $\Sigma^{=0} = \{\epsilon\}$.

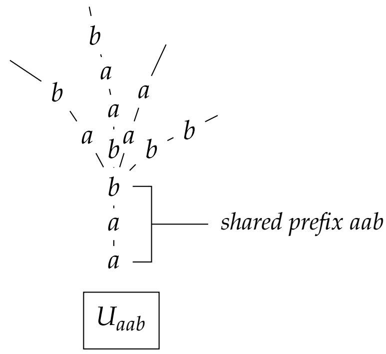
- $\Sigma^{=1} = \Sigma^{\leqslant 1} = \{\epsilon, a, b\}$.
- $\Sigma^{\leqslant 2} = \{\epsilon, a, b, aa, bb, ab, ba\}$.
- $\Sigma^{=2} = \{aa, bb, ab, ba\}$.
- *Etc.*

Given sequences $w, v \in \Sigma^{\leqslant n}$ with $length(w) \leqslant length(v)$, let us write $w \subseteq v$ to denote that w is a prefix of v, as in $aab \subseteq aabc$.

Next, define a topology over $\Sigma^{\leqslant n}$ by setting the open sets to be sequences that share a prefix:

- $U_w = \{v \in \Sigma^{\leqslant n} \mid w \subseteq v\}$.

So U_w consists of all sequences that continue w. For instance, if $w = aab$, then we might picture U_w as a kind of bouquet or bundle of sequences that are all bound at their shared stem (aab) but then branch out in different directions:



We can form a locale from this topology. Let $\mathbb{L}$ be the locale given by the presentation $\langle G, R\rangle$, where:

- $G = \{U_w \mid w \in \Sigma^n\}$, *i.e., each open is a generator.*
- $R = \{U_w \preccurlyeq U_v \mid v \subseteq w\}$, *i.e., bouquets with longer prefixes are lower.*

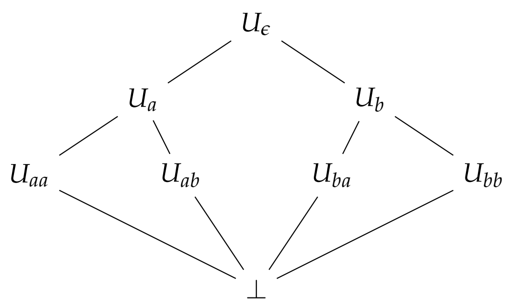
For example, given $\Sigma^{\leqslant 2}$, we have the following generators:

- $G = \{U_\epsilon, U_a, U_b, U_{aa}, U_{bb}, U_{ab}, U_{ba}\}$.

Here are some of the relations:

- $U_{aa} \preccurlyeq U_a$ *and* $U_{ab} \preccurlyeq U_a$, *since "a" is a prefix of aa and ab.*
- $U_{bb} \preccurlyeq U_b$ *and* $U_{ba} \preccurlyeq U_b$, *since "b" is a prefix of bb and ba.*
- *Every generator is lower than U_ϵ, since ϵ (the empty sequence) is a prefix of every sequence.*

The Hasse diagram looks like this:



Think of moving upwards in this locale as forgetting information about (or alternatively, as committing less to) the history of the sequence. For example, think of U_{ab} as a region where we know that "a" happened first and then "b" happened, but think of U_a as a region where we know only that "a" happened first and we don't know what happened after that. The top element is U_ϵ, which means we don't know anything about the sequence of actions. The $\perp$ element indicates not that we know nothing, but that there is no sequence at all.

Notice that implication moves upwards: U_{ab} implies U_a because if I know (at U_{ab}) that "a" happened first and then "b" happened, then I certainly know that "a" happened first.

This particular locale is interesting because it models the "process space" of any 2-stage sequence that can make one of two choices at each stage. Let us now assign some actual processes to this ambient space, using a presheaf.

Imagine a machine m that can run multiple concurrent processes, all of whom share the same memory. For simplicity, let us suppose that the machine has two registers ($R = \{r_1, r_2\}$), each of which can hold one bit (1 or 0). So, at any point in time the machine's memory state $S : \{0,1\} \times \{0,1\}$ can be one of the following:

- $S = \{\langle 0,0\rangle, \langle 1,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle\}$, *with initial state* $s_0 = \langle 0,0\rangle$.

We can think of the concurrent processes of interest as a selection of programs that we want to run on the machine all at the same time. In terms of behavior, let us say that each program-run reads a word from its input stream, one character at a time, and in response to each character, it takes one of the following actions A: it writes a value (1 or 0) to one of the registers, it writes (possibly distinct) values to both registers, or it does nothing and leaves the registers as they are:

- $A = \{\{r1 \mapsto v\}, \{r2 \mapsto v\}, \{r1 \mapsto v, r2 \mapsto w\}, \varnothing\}$, *where* $v, w \in \{0,1\}$.

We can define a process (program trace) as a map from n-length words to n-length sequences of write actions, where we require that such maps agree on prefixes (since a process responding to ab and aa would do the same thing on the first a). This way, a program trace records for each input stream the sequence of write actions that result. For concreteness, here are two such traces:

- $f : \Sigma^{=2} \to A \times A$
  - $f(aa) = \langle\{r1 \mapsto 1\}, \{r1 \mapsto 0\}\}\rangle$
  - $f(ab) = \langle\{r1 \mapsto 1\}, \{r2 \mapsto 0\}\}\rangle$
  - $f(bb) = \langle\{\{r1 \mapsto 0\}, \{r2 \mapsto 1\}\}\rangle$
  - $f(ba) = \langle\{\{r1 \mapsto 0\}, \{r1 \mapsto 1\}\}\rangle$
- $g : \Sigma^{=2} \to A \times A$
  - $g(aa) = \langle\{r2 \mapsto 0\}, \{r2 \mapsto 0\}\}\rangle$
  - $g(ab) = \langle\{r2 \mapsto 0\}, \varnothing\}\rangle$
  - $g(bb) = \langle\{\{r1 \mapsto 1\}, \varnothing\}\rangle$
  - $g(ba) = \langle\{\{r1 \mapsto 1\}, \varnothing\}\rangle$

Let us say that we now want to run f and g concurrently on the machine. For each word $w \in \Sigma^{=2}$, let us write $\mathcal{R}(w)$ for

$$(f(w), g(w)),$$

i.e., the joint run of f and g on w. For instance:

$$\mathcal{R}(aa) = (f(aa), g(aa)) = (\langle\{r1 \mapsto 1\}, \{r1 \mapsto 0\}\}\rangle, \langle\{r2 \mapsto 0\}, \{r2 \mapsto 0\}\}\rangle).$$

Next, let us say that two joint runs $\mathcal{R}(w)$ and $\mathcal{R}(v)$ are prefix-compatible iff:

- *For every stage k such that w and v share their first k letters, the combined writes of f and g at stage k do not assign different values to the same register.*

For instance, $\mathcal{R}(aa)$ and $\mathcal{R}(ab)$ are prefix-compatible because at stage 1 (at letter a), their writes do not conflict:

- *$f(aa)$ and $f(ab)$ write 1 to r1, while $g(aa)$ and $g(ab)$ write 0 to r2, and this is no conflict because they write to different registers.*

By contrast, $\mathcal{R}(bb)$ and $\mathcal{R}(ba)$ are not prefix-compatible because at stage 1 (at letter b) their writes do conflict:

- *$f(bb)$ and $f(ba)$ write 0 to r1, while $g(bb)$ and $g(ba)$ write 1 to r1, so they attempt to write conflicting values to the same register.*

For a region U_w with $|w| \leq 2$, let $\mathrm{ext}(U_w)$ be the set of length-2 words that extend w. For instance, $\mathrm{ext}(U_a) = \{aa, ab\}$, since aa and ab extend a, and $\mathrm{ext}(U_{ab}) = \{ab\}$, since ab is already a fully extended length-2 word.

Now let us define a presheaf F as follows. For each region U_w, let us provide an execution model that describes how f and g execute w and then continue on. More specifically, we need to assign prefix-compatible program runs for the extensions of w. The idea is that $F(U_w)$ will present a coherent description of how the machine will behave, as f and g jointly execute w and then continue on to their continuations.

At the most specified regions (aa, ab, bb, and ba), everything is fully specified, so the joint runs are fully determined already:

- $F(U_{aa}) = \{aa \mapsto \mathcal{R}(aa)\}$
- $F(U_{ab}) = \{ab \mapsto \mathcal{R}(ab)\}$
- $F(U_{bb}) = \{bb \mapsto \mathcal{R}(bb)\}$
- $F(U_{ba}) = \{ba \mapsto \mathcal{R}(ba)\}$

The more complicated case involves a less specified region, e.g. U_a. At this region, we know that the first step a happened, but we don't yet know whether the second step will be a or b. So, from the point of view of U_a, both continuations are possible.

Thus, what we need for $F(U_a)$ is a coherent description of how the machine will behave no matter which continuation actually occurs next. We must therefore:

- *Pick a joint run of aa.*
- *Pick a joint run of ab.*
- *But these two choices must be prefix-compatible, since they are supposed to represent the same joint run that has two different futures.*

Are there any joint runs of f and g on aa and ab that are prefix-compatible at a? Yes:

- $aa \mapsto \mathcal{R}(aa)$
- $ab \mapsto \mathcal{R}(ab)$

For as we saw earlier, $\mathcal{R}(aa)$ and $\mathcal{R}(ab)$ are prefix-compatible. Hence, there is a coherent execution model for f and g at U_a:

- $F(U_a) = \{aa \mapsto \mathcal{R}(aa), ab \mapsto \mathcal{R}(ab)\}$

Now consider U_b. The continuations of b are bb and ba. Are there any joint runs of f and g on bb and ba that are prefix-compatible? Here, the answer is no, for as we saw above, $\mathcal{R}(bb)$ and $\mathcal{R}(ba)$ are not prefix-compatible. Hence, there is no execution model for f and g at U_b:

- $F(U_b) = \varnothing$

What about U_ϵ? This would have to be an execution model of a joint run that is prefix compatible with the execution models at both U_a and U_b. Since there is no possible execution model at U_b, no execution model is possible for U_ϵ:

- $F(U_\epsilon) = \varnothing$

Formally, we can summarize the above description of F as follows. Let $\mathcal{R} = \{\mathcal{R}(v) \mid v \in \Sigma^{=2}\}$. Then:

$$F(U_w) = \left\{ s : \mathsf{ext}(w) \to \mathcal{R} \;\middle|\; \begin{array}{l} \textit{(i) } s(v) = \mathcal{R}(v) \textit{ for all } v \in \mathsf{ext}(w), \\ \textit{(ii) } \{s(v) \mid v \in \mathsf{ext}(w)\} \textit{ is pairwise prefix-compatible} \end{array} \right\}$$

For the bottom fiber, where no processes occur, assign the special symbol zero:

- $F(\bot) = \{0\}$.

To notate the restriction of a function $f : B \to D$ *to a smaller domain* $C \subseteq B$*, write* $f|_C$*. Then, whenever* $U_v \preccurlyeq U_w$*, the restriction maps are straightforward restrictions:*

$$\rho^{U_w}_{U_v}(s) = s|_{\mathsf{ext}(U_v)}\,.$$

Hence, for example:

- $\rho^{U_a}_{U_{ab}}(\{aa \mapsto \mathcal{R}(aa), ab \mapsto \mathcal{R}(ab)\}) = \{ab \mapsto \mathcal{R}(ab)\}$.
- $\rho^{U_a}_{U_{aa}}(\{aa \mapsto \mathcal{R}(aa), ab \mapsto \mathcal{R}(ab)\}) = \{aa \mapsto \mathcal{R}(aa)\}$.

The maximal fusions in this presheaf occur over U_a*: the execution model in* U_a*'s fiber is glued from those of* U_{aa} *and* U_{ab}*, exactly as one would expect, since* f *and* g *can run concurrently without conflict at* a*. By contrast, there is no gluing over* U_b*, since* f *and* g *cannot run concurrently without conflict at* b*.*

This is not a sheaf, because it is missing gluings (e.g., over U_b*). However, it is a monopresheaf, because when gluings exist (over* U_a*), they are unique.*

This relevance of this example is t hat it illustrates how sheaf theory can model processes, concurrency, and resource conflicts. Here the processes are programs running on a simple machine, but they could just as easily be biological processes competing for resources, etc.

Whatever the concrete details may be, this example captures how local behaviors integrate and extend over larger regions of the process space. One might naively think that the "parts" of such systems are the processes. But there is a different way to slice it: if you want to talk about the integrity of the "whole" of a concurrent system, you need to talk about how that involves coherent, integrated behavior that is functionally united locally across the various "regions" and "stages" of the system's evolution.

Sheaf theory works in continuous environments too.

Example 6. *Suppose we want to model the inhabitants of an apartment building over time. Let* $T = (0, 10) \subseteq \mathbb{R}$ *be an open interval representing a period of time (a span of 10 years, say).*

Let T *have the standard Euclidean topology, and let* $\mathbb{L}$ *be the locale of opens of the topology. This is the ambient locale we want to work with.*

Next, let P *be the set of people who at some point or other lived in the building:*

$$P = \{A, B, C, \ldots\}, \textit{ with } A \textit{ short for Alice, } B \textit{ for Bob, } C \textit{ for Carol, and so on.}$$

Let I *assign to each person the set of intervals during which they lived in the building. For instance:*

- $I(A) = \{(1, 3), (4, 7)\}$
- $I(B) = \{(4, 9)\}$
- $I(C) = \{(6, 10)\}$

Let F *be a presheaf given as follows. For each time span* $U \in \mathbb{L}$*:*

$$F(U) = \{p \in P \mid U \subseteq V, \textit{ for some } V \in I(p)\}$$

In other words, $F(U)$ is the set of people who live in the building for the entire duration of U.

For $V \preccurlyeq U$, the restriction maps are just inclusion:

$$\rho_V^U(J) = J,$$

since if the set of people J lived in the building throughout the span U, then they most certainly lived there during the smaller interval V.

This is a sheaf, since any compatible selection of patch candidates glues uniquely. For any cover $\{U_i\}_{i \in I}$ of U:

$$F(U) = \bigcap_i F(U_i).$$

A fusion in this sheaf is a glued section, and its parts are the patch candidates it is glued from. For example, take $U = (6, 7)$, with cover $\{U_1 = (6, 6.5), U_2 = (6.4, 7)\}$. Any selection of compatible patch candidates from this cover glue uniquely.

This is a particularly simple sheaf, but the sheaf's strong gluing conditions tell us why the parts glue together so straightforwardly here. Gluing is thoroughly integrated throughout the structure.

Example 7. *For another continuous example, let's model a lump of clay through time.*

Let $T = (0, 10)$ be a span of time, and let $\mathbb{L}$ be the locale of the opens of T with its standard Euclidean topology again.

For simplicity, let us assume that 3-dimensional space is just $\mathbb{R}^3$. Then fix a function

$$\phi : T \to \wp(\mathbb{R}^3)$$

that, for each $t \in T$,

$$\phi(t) \subseteq \mathbb{R}^3$$

is the open region of space occupied by the clay at time t.

If we think of $\phi(t)$ as a snapshot of the clay at t, then at some ts, $\phi(t)$ might be shaped like a statue, at other times like a lump, at still other times like two disconnected lumps, and so on.

With that background fixed, let us now define a presheaf F that assigns material parts to each interval of T. In particular, for each interval $U \in \mathbb{L}$:

$$F(U) = \{S \subseteq \mathbb{R}^3 \mid S \text{ is open and } \forall t \in U, S \subseteq \phi(t)\}.$$

In other words, F(U) is the set of the clay's material parts that persist through the duration of U.

Restriction is inclusion again, since persistence through a larger time span U implies persistence through a smaller span $V \preccurlyeq U$:

$$\rho_V^U(S) = S.$$

This is a sheaf, since compatible local section glue by intersection. The fibers of this sheaf are typically infinite: each fiber includes the whole lump of clay, all of is sub-regions, and these regions are themselves spatially continuous.

Given a time span U, if

$$\bigcap_{t \in U} \phi(t)$$

is connected, then there will be a maximal connected section $S \in F(U)$ such that all other sections in $F(U)$ sit inside it. This models a single lump of clay through time.

By contrast, if

$$\bigcap_{t \in U} \phi(t)$$

has two or more connected components, then there will not be a maximal connected section $S \in F(U)$ that all other sections of $F(U)$ fit inside. This models a fragmented lump of clay through time.

What exactly does the sheaf give us here that ϕ doesn't? What ϕ gives us is a time-indexed history of spatial occupation. It is like a movie: each frame is just a picture. It says nothing about what the material parts are, or which parts persist through time. That information has to be imposed on top of ϕ.

F provides that extra information explicitly. F provides precisely what the material parts are that persist through time, along with a principled notion of fusion. In short, ϕ gives us what points in space are occupied at each instant, whereas F gives us which parts persist over regions of time, and how they compose.

Adopting weaker gluing conditions allows one to model more exotic part-whole complexes that fail extensionality.

Example 8. *Take the example of the lump of clay from Example 7. Examples like this can be enriched to model further features.*

For instance, suppose we model Socrates in the fashion of Example 7. But then we might want to enrich the example and say that the fibers carry not just persistent material parts, but rather that they carry persistent material parts tagged with an accidental mode (e.g., a posture, like seated or standing).

Let $\mathsf{Posture} = \{seated, standing, \ldots,\} \cup \{unspecified\}$*. Then for every $U \in \mathbb{L}$, we can say that $F(U)$ is a set of pairs $\langle S, \pi \rangle$ such that $\forall t \in U$, $S \subseteq \phi(t)$, and* $\pi \in \mathsf{Posture}$*. Intuitively, these pairs are the material parts in standing-Socrates, seated-Socrates, and Socrates-simpliciter that persist in the given mode throughout U. Such a presheaf would then have multiple, distinct persistent material fusions (e.g., Socrates-simpliciter and seated-Socrates) that simultaneously inhabit the same regions of space and differ only by their accidental mode.*

Note that in the sheaf-theoretic setting, we can model part-whole complexes where gluing need not be unique (via presheaves). In other words, if we choose to give up the stricter gluing conditions of monopresheaves and sheaves, we can then use presheaves to model situations where multiple fusions are glued from the same parts. For instance, we can model seated-/standing-Socrates, Athena and Lumpl, matter-and-substantial-form, a soul-and-its-powers, etc.

Sheaf-theoretic mereology is rare in the literature, but there are a few notable nearby cases. We conclude this section by distinguishing two kindred approaches from one genuine instance of the framework developed here.

Example 9. *Thomas Mormann's work on trope sheaves [1] provides an explicit philosophical application of sheaf theory to the construction of complex entities. Although Mormann does not frame this work in mereological terms, and does not define parts and fusions via restriction and gluing of sections, the motivations and mathematical setting are clearly allied with our approach.*

In later work on what he calls "structural mereology" ([2], [3], [4]), Mormann frequently employs presheaf- and sheaf-like constructions (often without using that terminology). In these cases, however, parts and wholes are defined via the category-theoretic notion of a subobject, and fusions are given by unions (joins) of subobjects. Since mereological structure then lives in the lattice of an object's subobjects rather than in the fibers of the presheaf, this yields a distinct mereological theory.

Thus, while Mormann's structural mereology is a close conceptual relative of sheaf-theoretic mereology, it is not an instance of the gluing-based framework developed here.

Example 10. *A genuine instance of our framework appears in the theory of "behavioral mereology" developed by Fong, Myers, and Spivak ([5]; cf. [6]). On this approach, a system is defined to be a sheaf of behaviors. Global sections represent complete system behaviors (wholes), while restrictions of sections to smaller regions represent parts of the system.*

In this setting, composition of parts is given by the sheaf-theoretic gluing operation, exactly as in the framework presented here. Parts are not subobjects but fiber-level data: they are local behaviors that arise as restrictions of global ones. So this is an exact instance of our framework.

Interestingly, Fong, Myers, and Spivak impose an additional constraint, requiring restriction maps to be surjective. The effect is that a part of a system consists precisely of those behaviors observable when attention is restricted to a smaller region. In this sense, a part is legitimate only insofar as it is genuinely realized by the whole system.

This yields a distinctly Aristotelian conception of parthood (cf. [7]). Famously, Aquinas ([8]) held that a hand is a part of a body only insofar as it remains functionally integrated with it; a severed hand is no longer a hand. Interpreting a human organism as a system, the hand is exactly that portion of the organism's behavior observable at the hand. Once detached, it is no longer a surjective restriction of the whole system's behavior.

This example illustrates the flexibility of the sheaf-theoretic setting. By imposing additional constraints — such as surjectivity of restrictions — one can recover substantively different metaphysical views of parthood within a single mathematical framework.

References

1. Mormann, T. Trope Sheaves: A Topological Ontology of Tropes. *Journal of Philosophical Logic* **1995**, *3*, 129–150.
2. Mormann, T. Updating Classical Mereology. In Proceedings of the Proceedings of the XIII International Conference on Logic, Methodology, and Philosophy of Science, Beijing 2007 International Conference on Logic, Methodology and Philosophy of Science (Beijing 2007); Glymour, C.; Wang, W.; Westerstahl, D., Eds., London, 2009; pp. 326–343.
3. Mormann, T. Structural Universals as Structural Parts: Toward a General Theory of Parthood and Composition. *Axiomathes* **2010**, *20*, 209–227.
4. Mormann, T. On the Mereological Structure of Complex States of Affairs. *Synthese* **2012**, *187*, 403–418.
5. Fong, B.; Myers, D.J.; Spivak, D. Behavioral Mereology: A Modal Logic for Passing Constraints. In Proceedings of the Proceedings of the 3rd Annual International Applied Category Theory Conference 2020 (ACT2020); Spivak, D.; Vicary, J., Eds., 2020.
6. Schultz, P.; Spivak, D. *Temporal Type Theory: A Topos-Theoretic Approach to Systems and Behavior*; Birkhäuser, 2019.
7. Inman, R. *Substance and the Fundamentality of the Familiar: A Neo-Aristotelian Mereology*; Routledge: London, 2018.
8. Svoboda, D. Thomas Aquinas on Whole and Part. *The Thomist* **2012**, *76*, 273–304.

Todo list