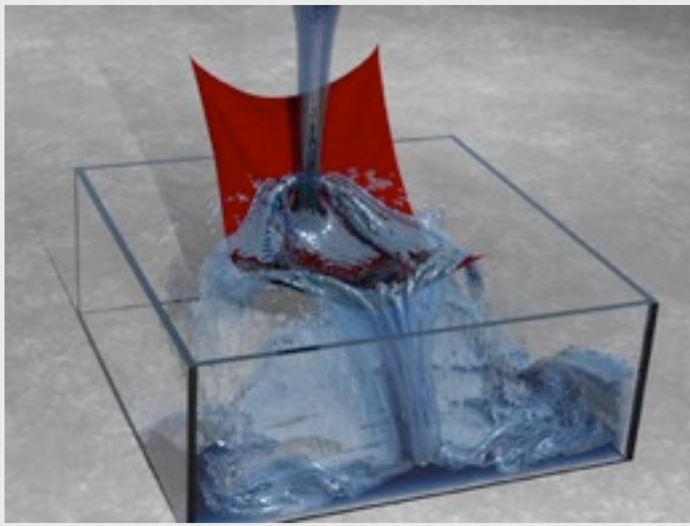
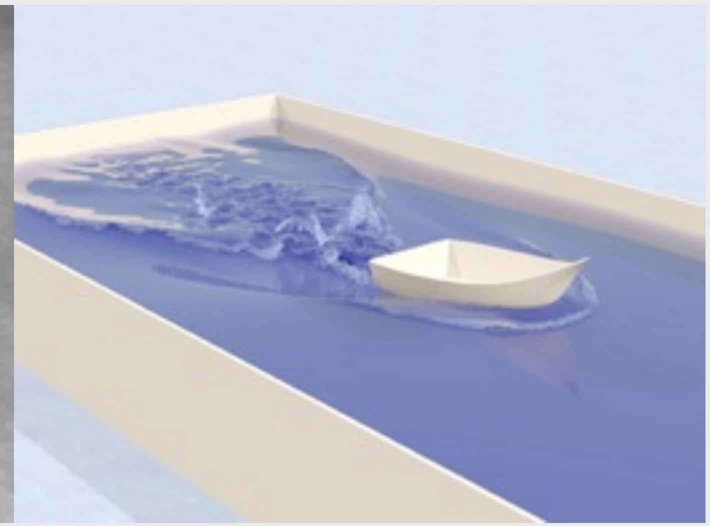
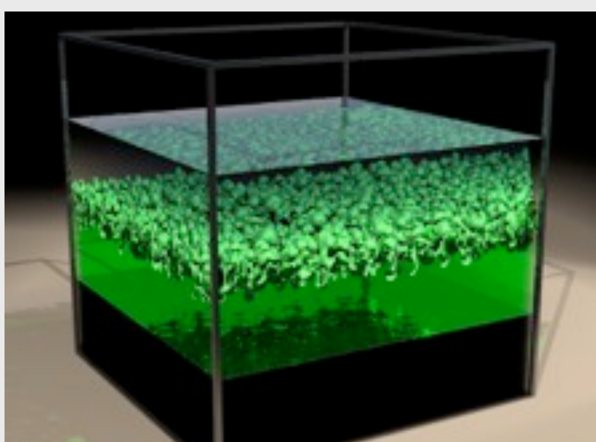
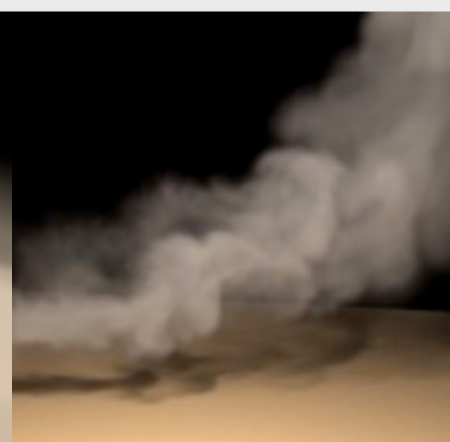
Fluid dynamics


- Math background
- Physics
- Simulation
- Related phenomena
- Frontiers in graphics
- Rigid fluids

Fields

- Domain

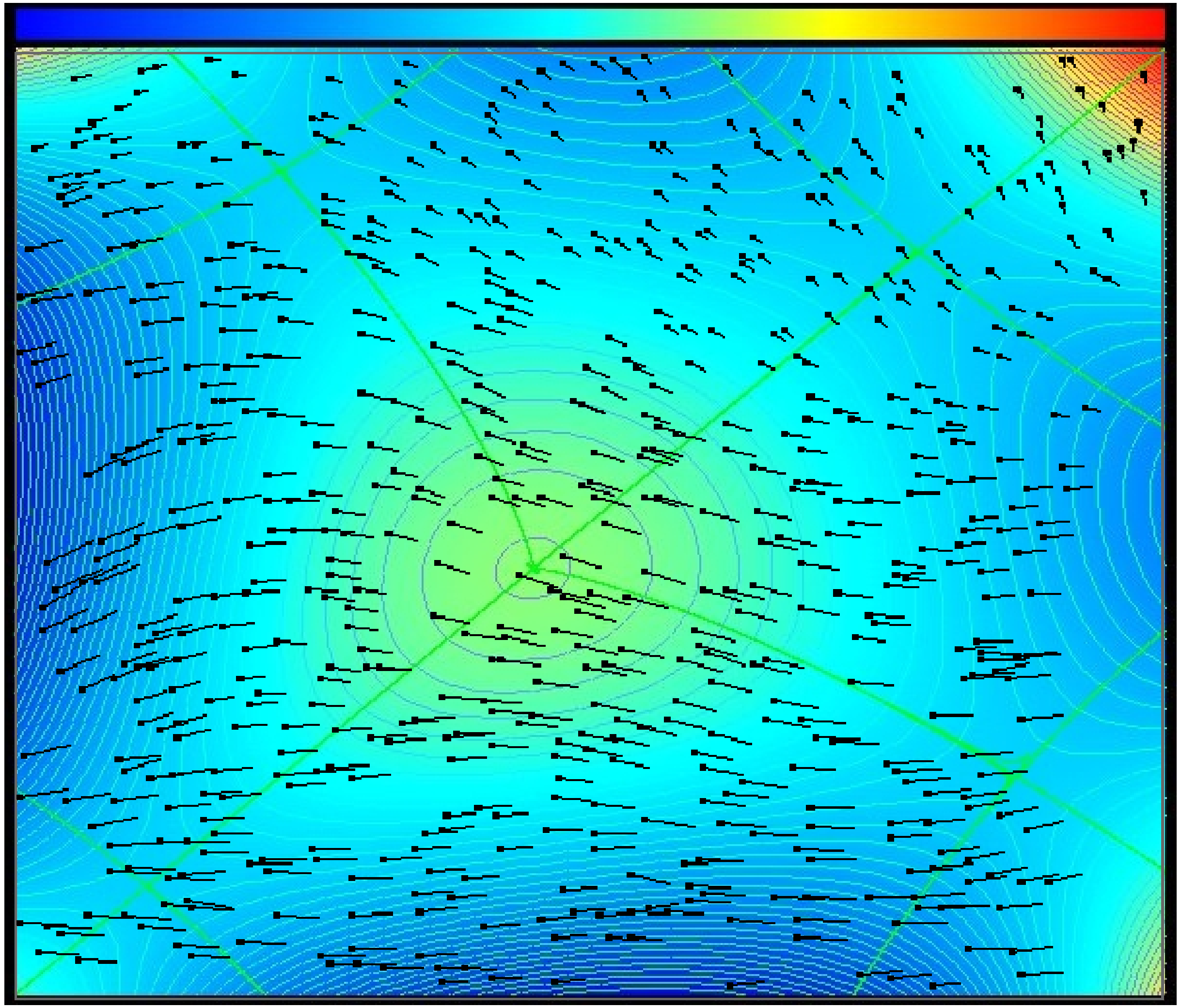
$$\Omega \subseteq \mathbf{R}^2$$

- Scalar field

$$f : \Omega \to \mathbf{R}$$

- Vector field

$$\mathbf{f} : \Omega \to \mathbf{R}^2$$



Types of derivatives

- Derivatives measure how something changes due to its parameters
- Temporal derivatives $\frac{\partial f}{\partial t}$
- Spatial derivatives
    - gradient operator $\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]^T$
    - divergence operator $\nabla \cdot \mathbf{f} = \frac{\partial \mathbf{f}^x}{\partial x} + \frac{\partial \mathbf{f}^y}{\partial y}$
    - Laplacian operator $\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x} + \frac{\partial^2 f}{\partial y}$

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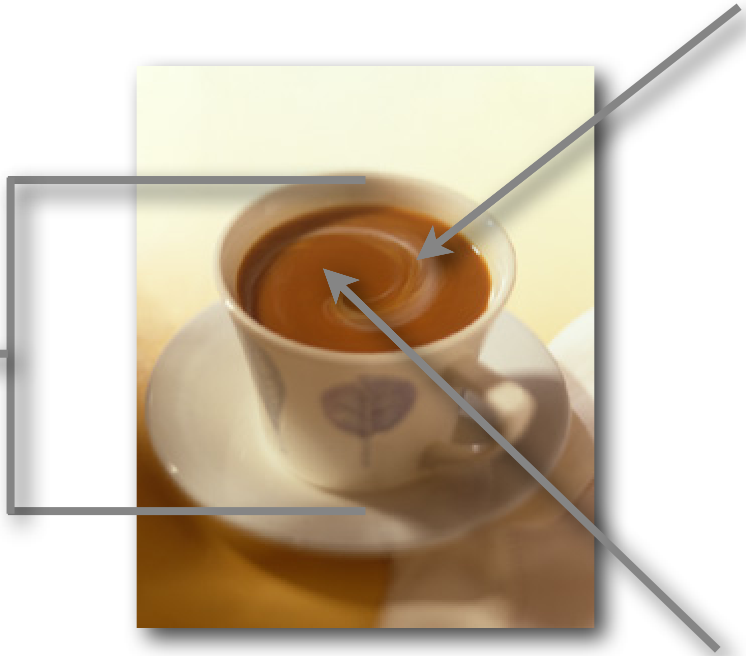
Representation

Density $\rho : \Omega \to [0, 1]$

Domain Ω



Velocity $\mathbf{u} : \Omega \to \mathbf{R}^3$

~~"Coffee cup"~~ equation

Navier-Stokes

Momentum equation

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + s\nabla^2\mathbf{u} + \mathbf{f}$$

Incompressibility condition

$$\nabla \cdot \mathbf{u} = 0$$

$\mathbf{u}$: velocity
p: pressure
s: kinematic viscosity
$\mathbf{f}$: body force
ϱ: fluid density

Momentum equation

- Each particle represents a little blob of fluid with mass m, a volume V, and a velocity $\mathbf{u}$
- The acceleration of the particle

$$\mathbf{a} \equiv \frac{D\mathbf{u}}{Dt}$$

- The Newton's law

$$m\frac{D\mathbf{u}}{Dt} = \mathbf{F}$$

Forces acting on fluids

- Gravity: $m\mathbf{g}$
- Pressure: $-\nabla p$
  - Imbalance of higher pressure
- Viscosity: $\mu\nabla \cdot \nabla\mathbf{u}$
  - Force that makes particle moves at average speed
  - dynamic viscosity coefficient: μ

Momentum equation

The movement of a blob of fluid

$$m\frac{D\mathbf{u}}{Dt} = m\mathbf{g} - V\nabla p + V\mu\nabla \cdot \nabla \mathbf{u}$$

Divide by volume

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{g} - \nabla p + \mu\nabla \cdot \nabla \mathbf{u}$$

Rearrange equation giving Navier-Stoke

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p = \mathbf{g} + s\nabla \cdot \nabla \mathbf{u}$$

Lagrangian v.s. Eulerian

- Lagrangian point of view describes motion as points travel around space over time
- Eulerian point of view describes motion as the change of velocity field in a stationary domain
- Lagrangian approach is conceptually easier, but Eulerian approach makes the spatial gradient easier to compute/approximate

Material derivative

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} + w\frac{\partial q}{\partial z}$$

$$\begin{aligned}\frac{d}{dt}q(t,\mathbf{x}) &= \frac{\partial q}{\partial t} + \nabla q \cdot \frac{d\mathbf{x}}{dt} \\ &= \frac{\partial q}{\partial t} + \nabla q \cdot \mathbf{u} \\ &= \frac{Dq}{Dt}\end{aligned}$$

Advection

- Advection describe how quantity, q, moves with the velocity field $\mathbf{u}$
- Density

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho$$

- Velocity

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}$$

Advection

momentum equation

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p = \mathbf{g} + s\nabla \cdot \nabla \mathbf{u}$$

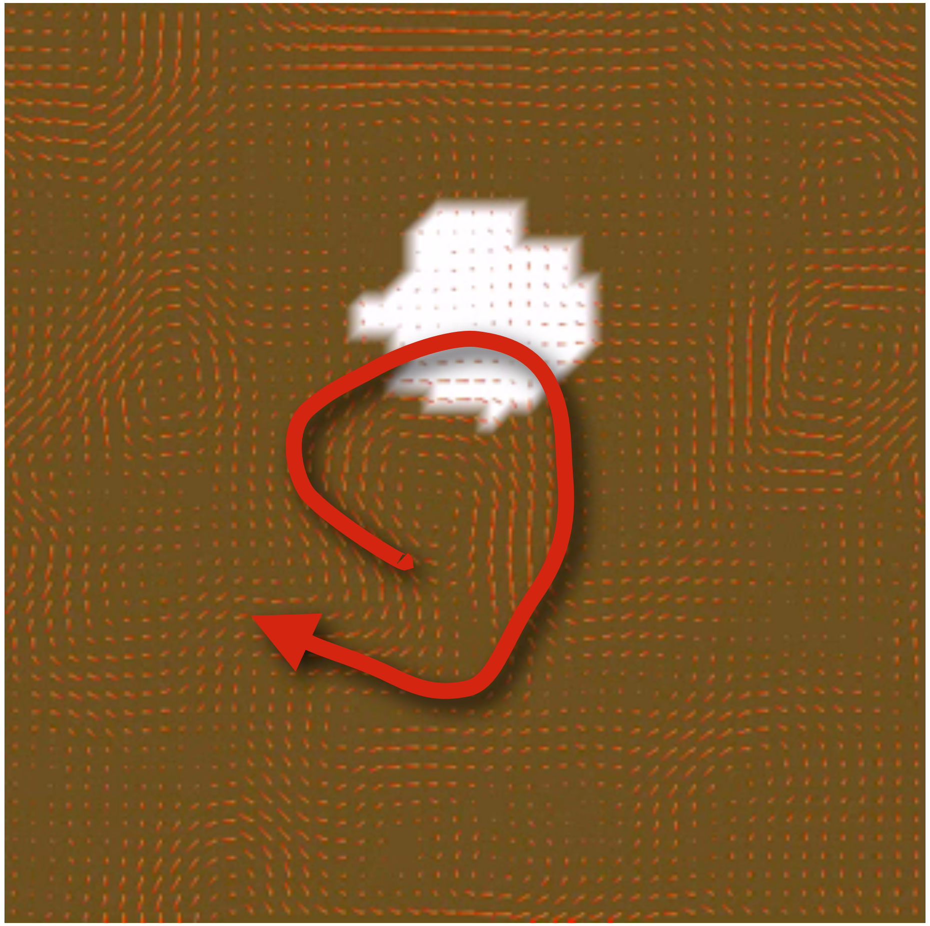
Advection Projection Diffusion Body force

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + s\nabla^2 \mathbf{u} + \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

Density advection

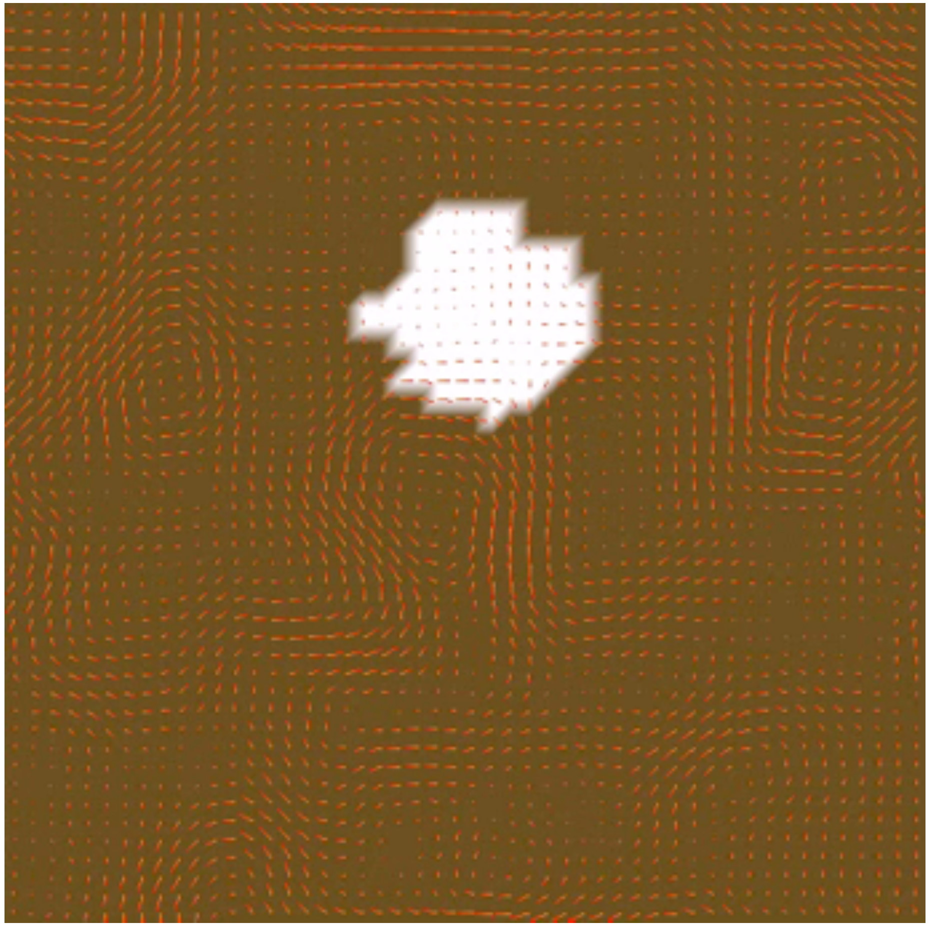
$$\frac{\partial \rho}{\partial t} = -(\mathbf{u} \cdot \nabla)\rho$$



Velocity advection

$$\frac{\partial \mathbf{u}}{\partial t} = \boxed{-(\mathbf{u} \cdot \nabla)\mathbf{u}} - \frac{1}{\rho}\nabla p + s\nabla^2 \mathbf{u} + \mathbf{f}$$

$$\text{s.t. } \nabla \cdot \mathbf{u} = 0$$



Projection

Advection Projection Diffusion Body force

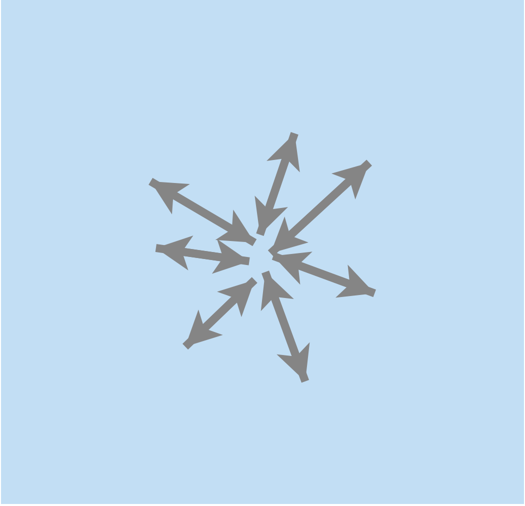
$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} \boxed{- \frac{1}{\rho}\nabla p} + s\nabla^2\mathbf{u} + \mathbf{f}$$

$$\boxed{\nabla \cdot \mathbf{u} = 0}$$

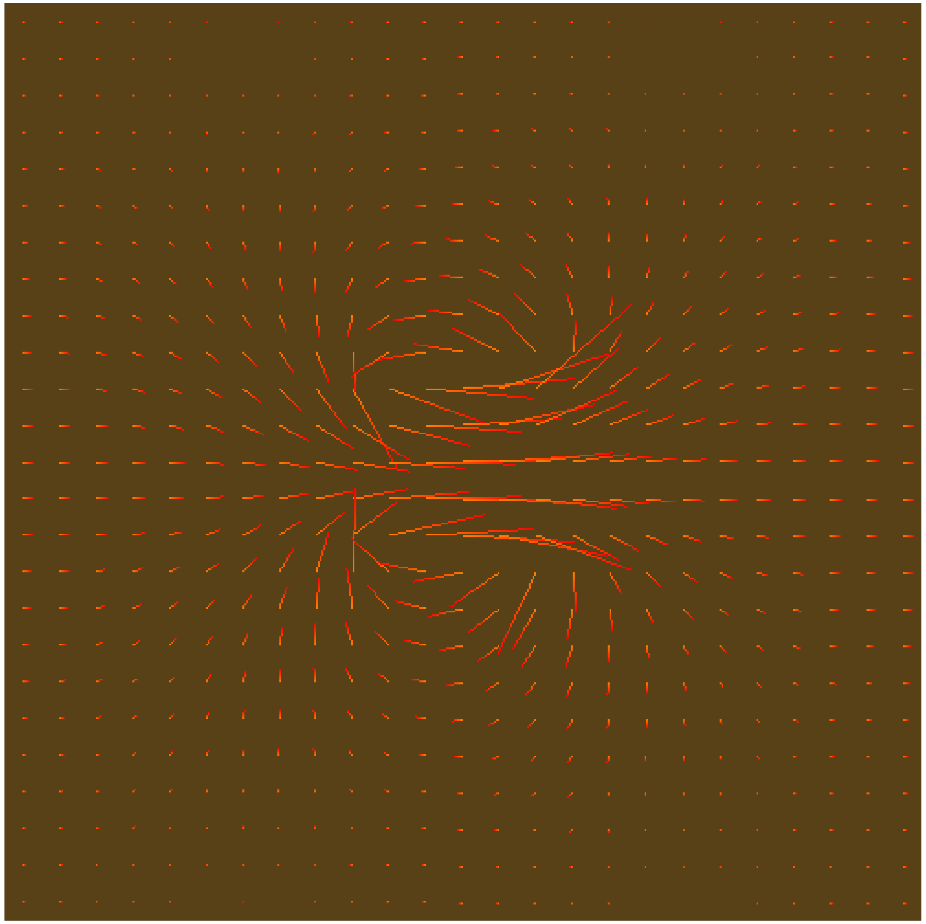
Divergence free

$$\nabla \cdot \mathbf{u} > 0?$$

$$\nabla \cdot \mathbf{u} < 0?$$



Divergence free

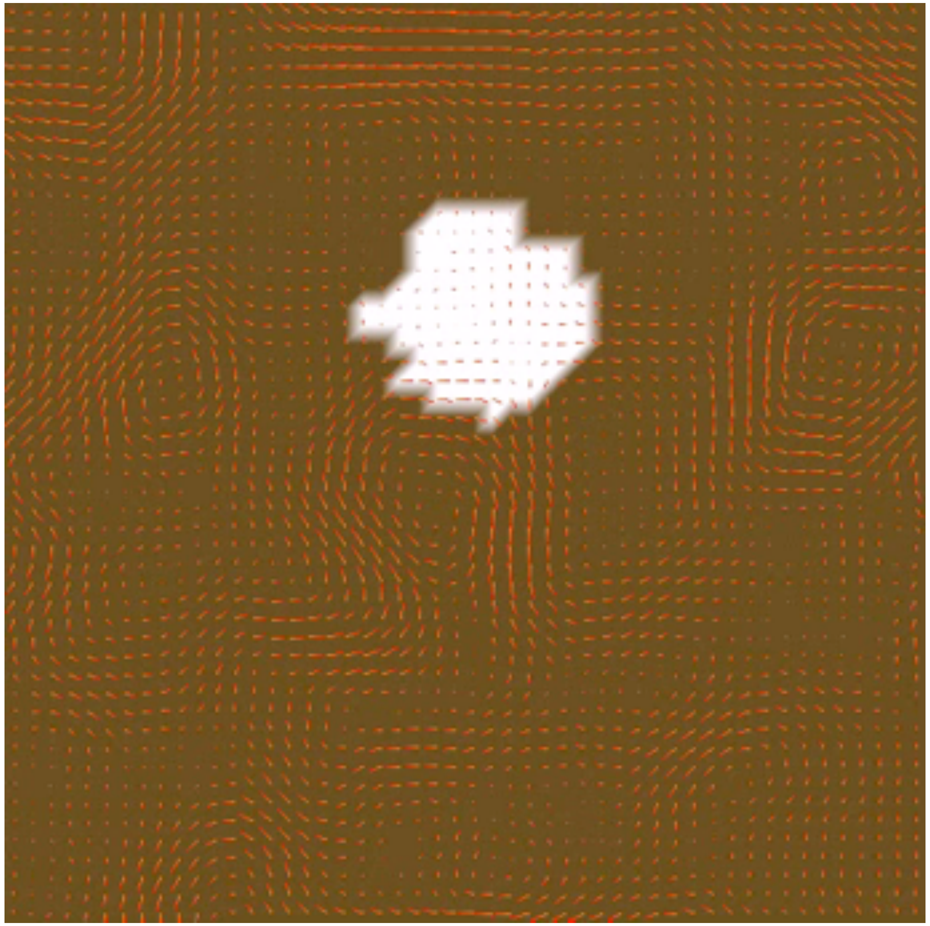


$$\nabla \cdot \mathbf{u} \neq 0$$

Projection

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \boxed{\frac{1}{\rho}\nabla p} + s\nabla^2\mathbf{u} + \mathbf{f}$$

$$\boxed{\text{s.t. } \nabla \cdot \mathbf{u} = 0}$$



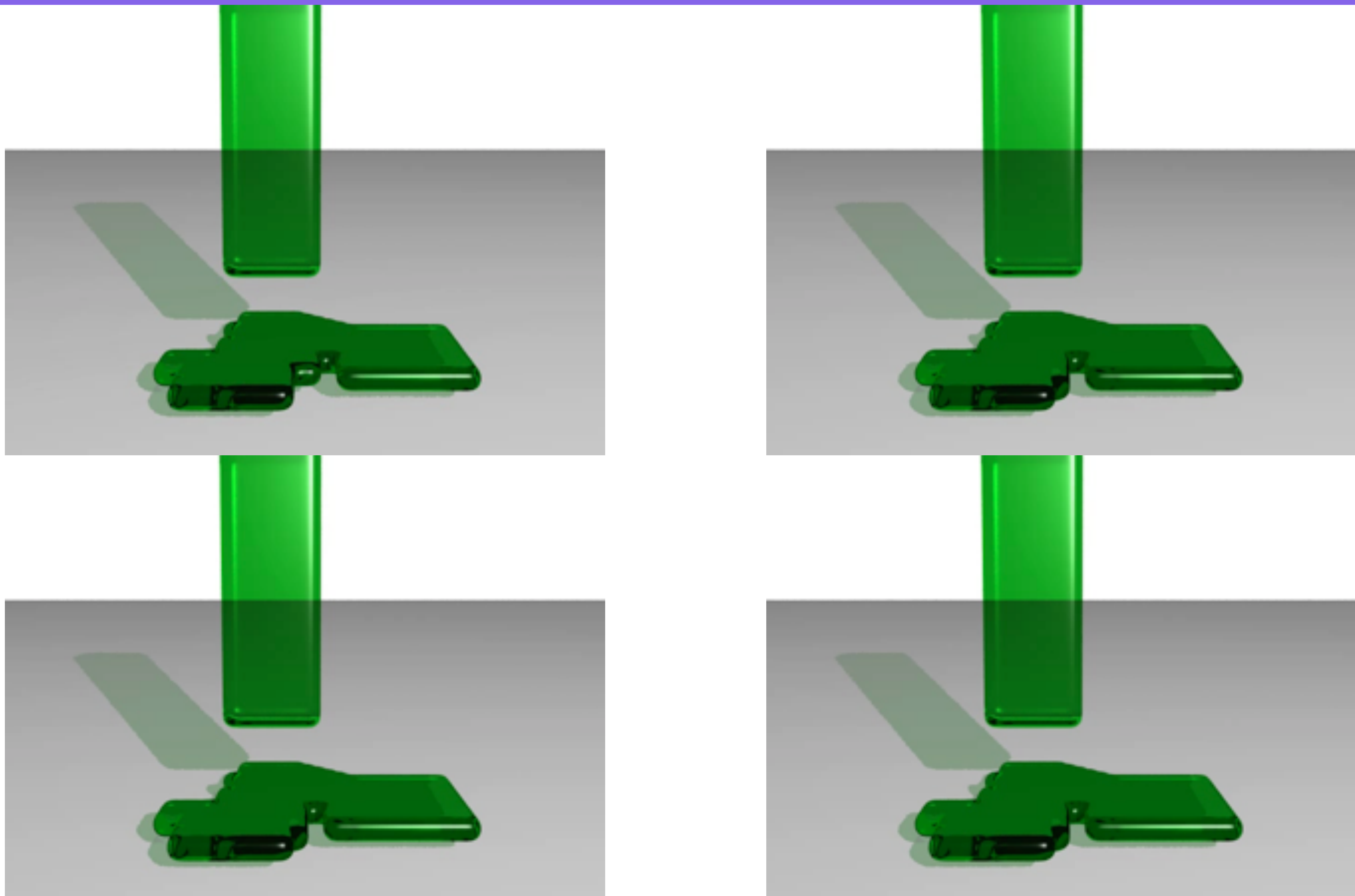
Diffusion

Advection Projection Diffusion Body force

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + s\nabla^2 \mathbf{u} + \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

High viscosity fluids



Dropping viscosity

- Viscosity plays a minor role in most fluids
- Numeric methods introduce errors which can be physically reinterpreted as viscosity
- Fluid with no viscosity is called “inviscid”

Body force

Advection Projection Diffusion Body force

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + s\nabla^2 \mathbf{u} \boxed{+ \mathbf{f}}$$

$$\nabla \cdot \mathbf{u} = 0$$

External forces

- Gravity
- Heat
- Surface tension
- User-specified forces (stirring coffee)

Boundary conditions

- Solid walls
  - The fluid cannot go in and out of it
  - Control the normal velocity
- Free surfaces
  - Air can be represented as a region where the pressure is zero
  - Do not control the velocity at the surface

Solid boundary

- Velocity at the boundary

  $$\mathbf{u} \cdot \hat{\mathbf{n}} = \mathbf{u}_{solid} \cdot \hat{\mathbf{n}}$$

  - No-slip condition: $\mathbf{u} = \mathbf{u}_{solid}$
- Pressure at the boundary
  - Make the fluid incompressible AND enforce the solid boundary condition

Physics recap

- Physical quantity represented as fields
- Navier-Stokes PDE describes the dynamics

- Math background
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- Frontiers in graphics
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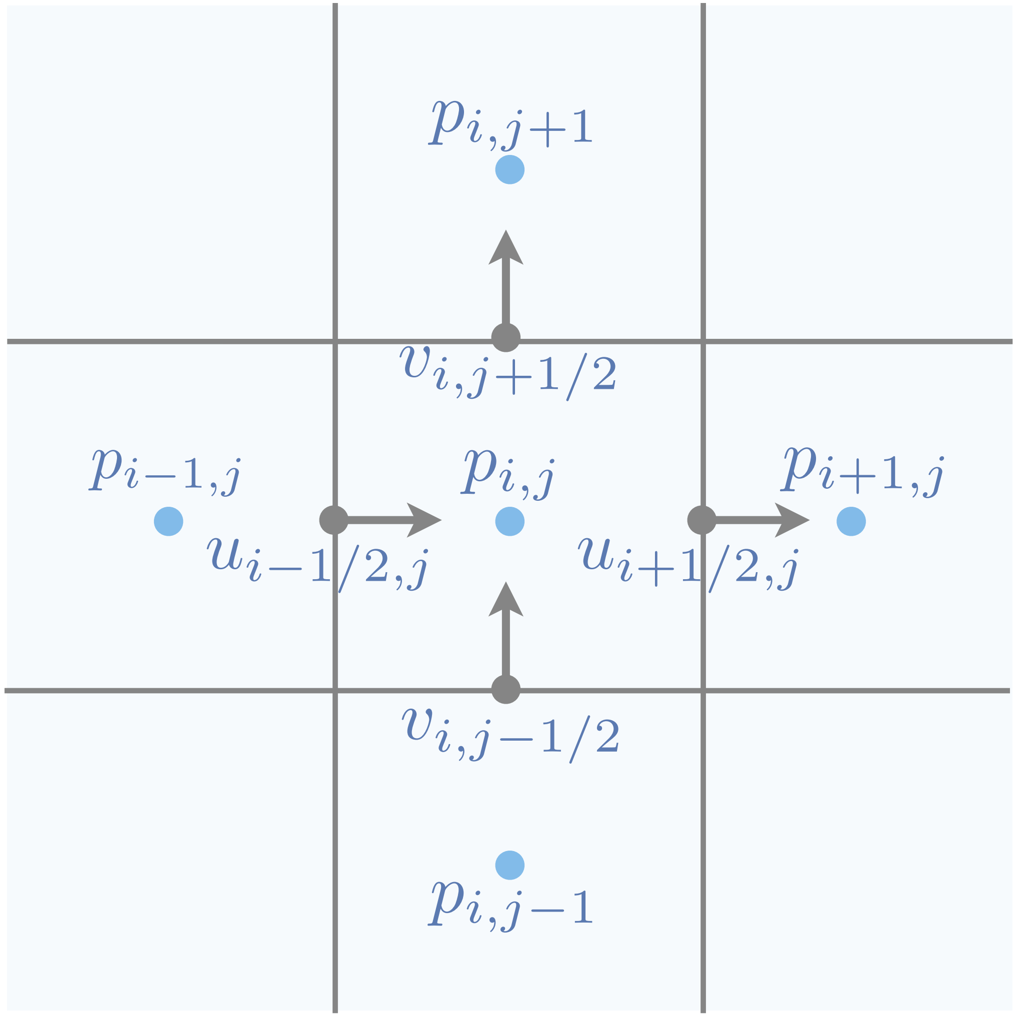
Challenges

- What is the representation for fluids?
  - represent velocity and pressure
- What is the equation of motion for fluids?
  - approximate Navier-Stokes in a discrete domain
  - compute Navier-Stokes efficiently

Grid structure



Explicit integration

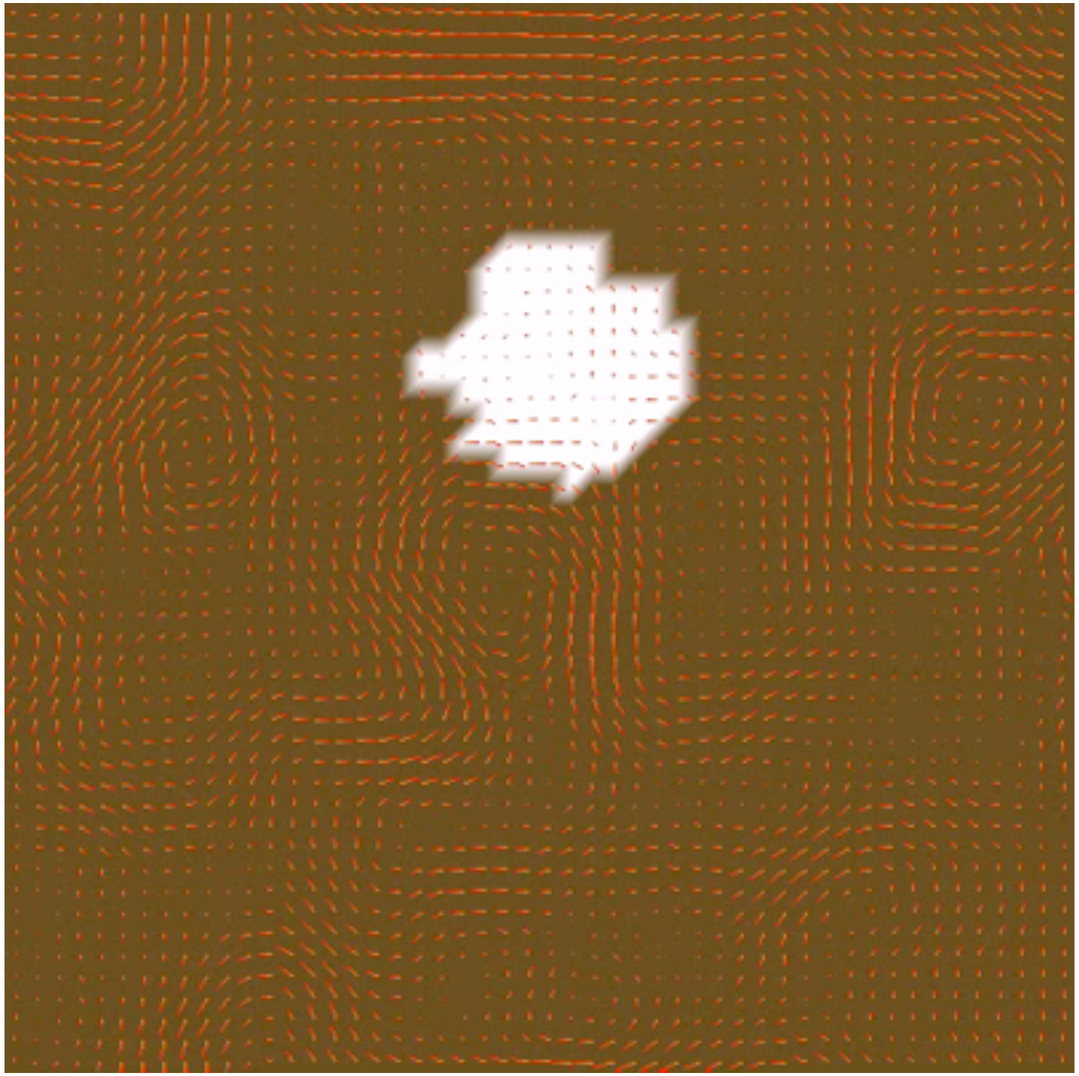
- Solving for the differential equation explicitly, namely

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + s\nabla^2\mathbf{u} + \mathbf{f}$$

$$\mathbf{u}^{t+1} = \mathbf{u}^t + h\dot{\mathbf{u}}(t)$$

- You get this...

Explicit integration



Stable fluids

Invented by Jos Stam

Simple, fast, and unconditionally stable

Splitting methods

- Suppose we had a system

$$\frac{\partial x}{\partial t} = f(x) = g(x) + h(x)$$

- We define simulation function S_f

$$S_f(x, \Delta t) : x(t) \to x(t) + \Delta t f(x)$$

- Then we could define

$$S_f(x, \Delta t) : x(t + \Delta t) = S_g(x, \Delta t) \circ S_h(x, \Delta t)$$

Splitting methods

$$\frac{\partial \mathbf{u}}{\partial t} = -\boxed{(\mathbf{u} \cdot \nabla)\mathbf{u}} - \boxed{\frac{1}{\rho}\nabla p} + \boxed{s\nabla^2 \mathbf{u}} + \boxed{\mathbf{f}}$$

$$\mathbf{w}_0 = \mathbf{u}(\mathbf{x}, t)$$

$$\mathbf{w}_0(\mathbf{x}) \xrightarrow{\text{add force}} \mathbf{w}_1(\mathbf{x}) \xrightarrow{\text{Advect}} \mathbf{w}_2(\mathbf{x}) \xrightarrow{\text{Diffuse}} \mathbf{w}_3(\mathbf{x}) \xrightarrow{\text{Project}} \mathbf{w}_4(\mathbf{x})$$

$$\mathbf{u}(\mathbf{x}, t+\Delta t) = \mathbf{w}_4$$

Splitting methods

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u}\cdot\nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + s\nabla^2\mathbf{u} + \mathbf{f}$$

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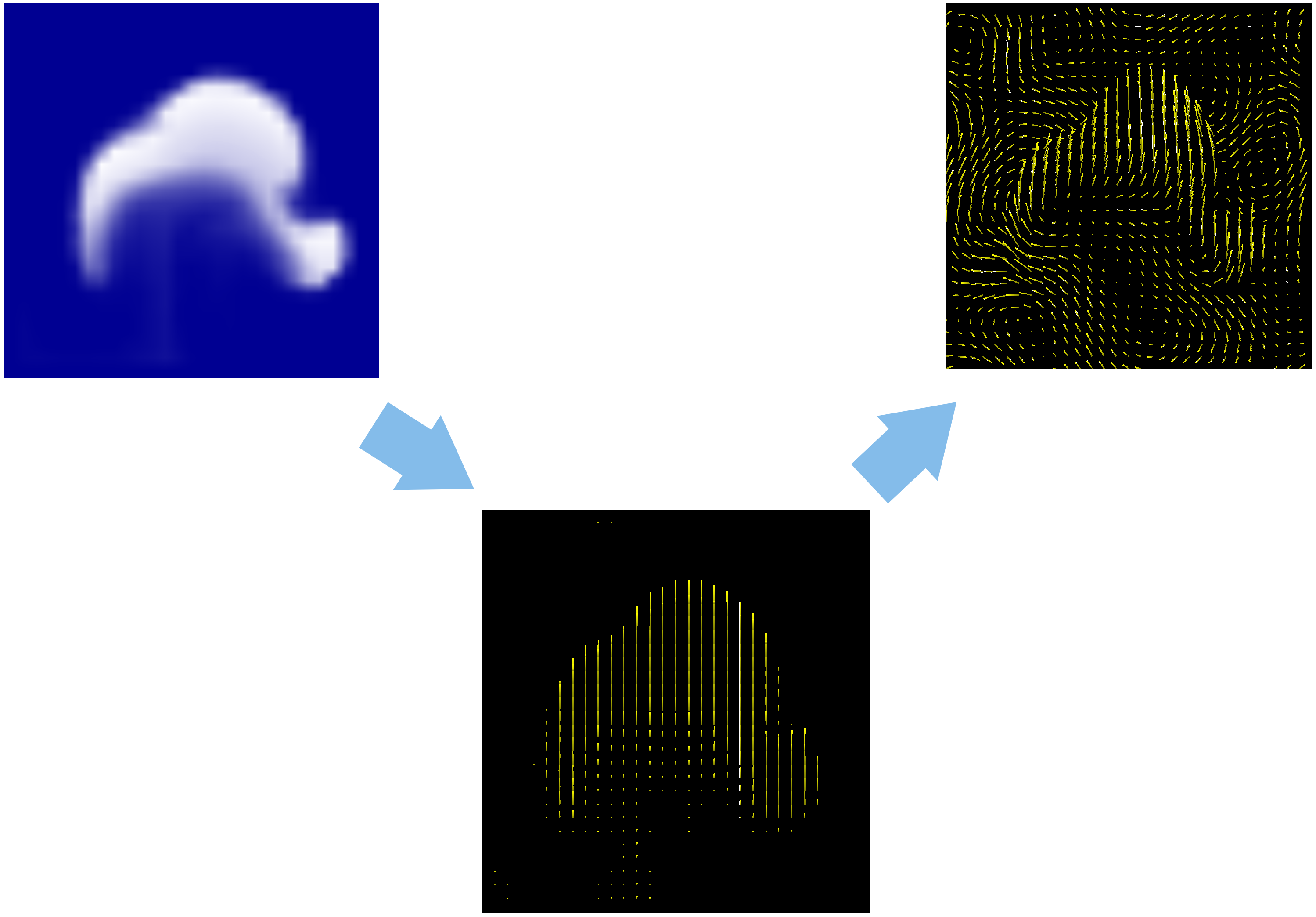
Body forces



Splitting methods

$$\frac{\partial \mathbf{u}}{\partial t} = \boxed{-(\mathbf{u} \cdot \nabla)\mathbf{u}} - \frac{1}{\rho}\nabla p + s\nabla^2\mathbf{u} + \mathbf{f}$$

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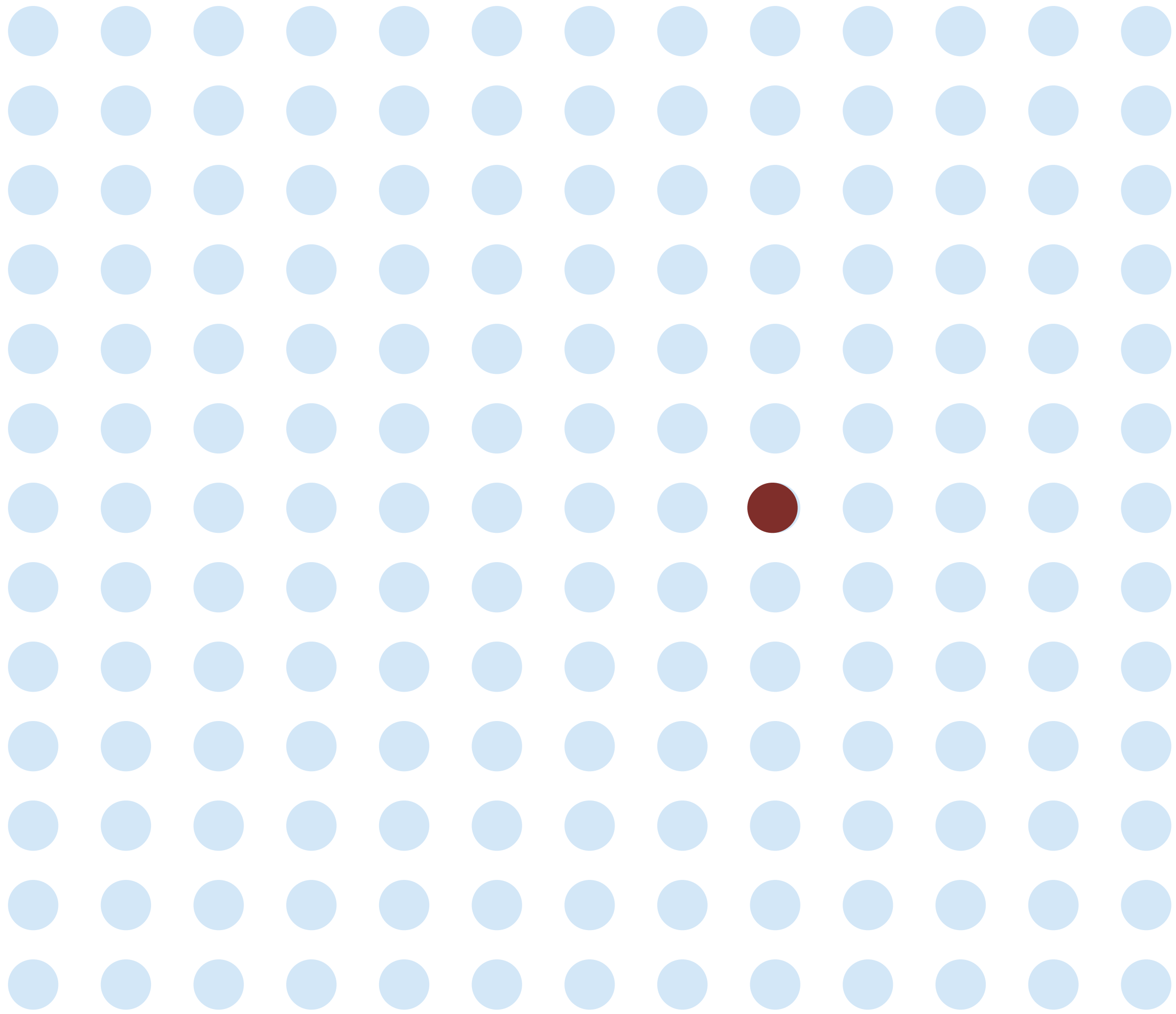
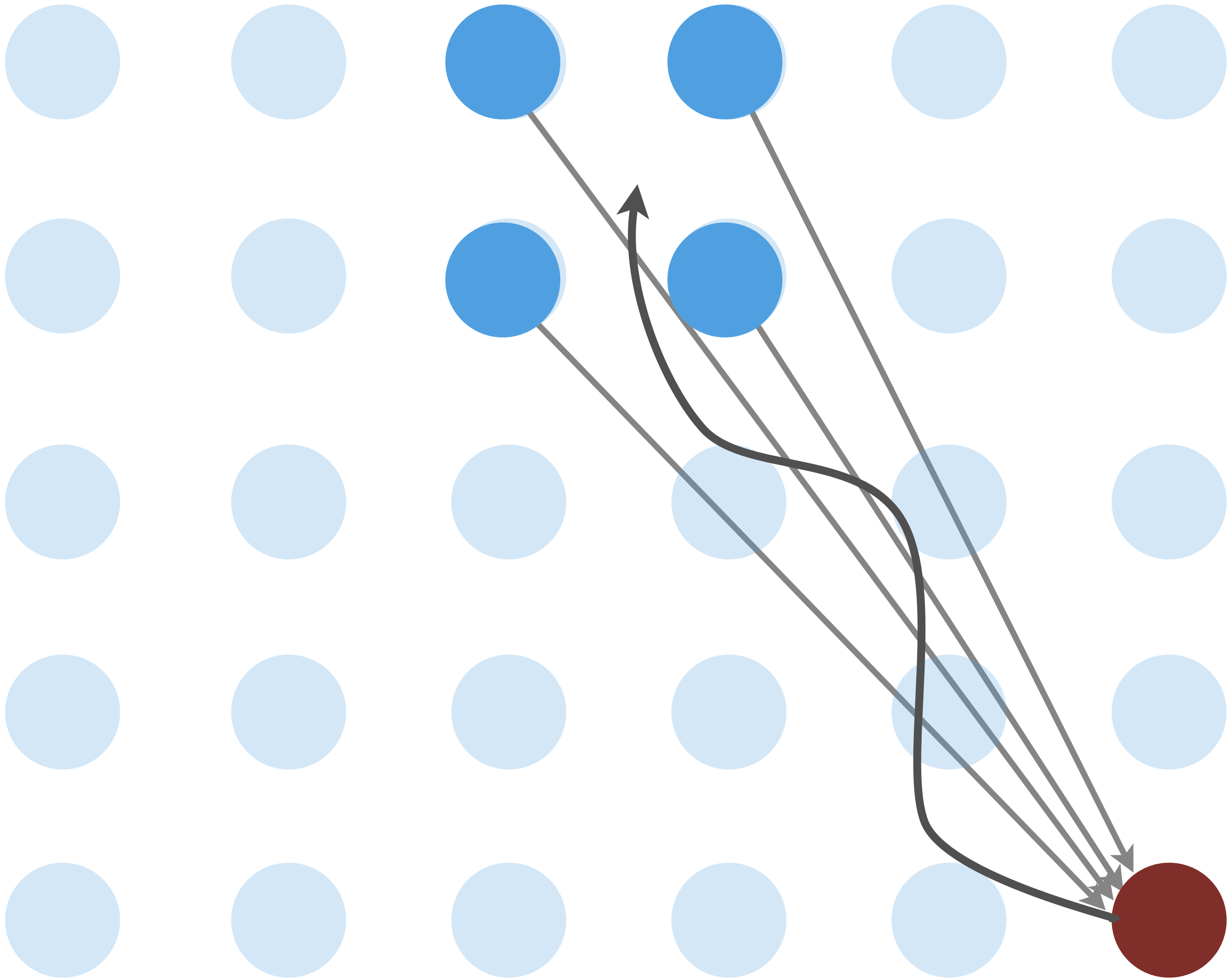
$$\mathbf{u}(\mathbf{x}, t+\Delta t) = \mathbf{w}_4$$

Advection





Numerical dissipation

- Semi-Lagrangian advection tend to smooth out sharp features by averaging the velocity field
- The numerical errors result in a different advection equation solved by semi-Largrangian

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} = u\Delta x\frac{\partial^2 q}{\partial x^2}$$

- Smooth out small vortices in inviscid fluids

Splitting methods

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + \boxed{s\nabla^2 \mathbf{u}} + \mathbf{f}$$

$$\mathbf{w}_0 = \mathbf{u}(\mathbf{x}, t)$$

add force — Advect — Diffuse — Project

$$\mathbf{w}_0(\mathbf{x}) \xrightarrow{\text{add force}} \mathbf{w}_1(\mathbf{x}) \xrightarrow{\text{Advect}} \mathbf{w}_2(\mathbf{x}) \xrightarrow{\text{Diffuse}} \mathbf{w}_3(\mathbf{x}) \xrightarrow{\text{Project}} \mathbf{w}_4(\mathbf{x})$$

$$\mathbf{u}(\mathbf{x}, t+\Delta t) = \mathbf{w}_4$$

Diffusion

- Solve for the effect of viscosity

$$\frac{\partial \mathbf{w}_2}{\partial t} = \nu \nabla^2 \mathbf{w}_2$$

- Use an implicit method for stable result

$$(\mathbf{I} - \nu \Delta t \nabla^2)\mathbf{w}_3(\mathbf{x}) = \mathbf{w}_2(\mathbf{x})$$

Splitting methods

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \boxed{\frac{1}{\rho}\nabla p} + s\nabla^2\mathbf{u} + \mathbf{f}$$

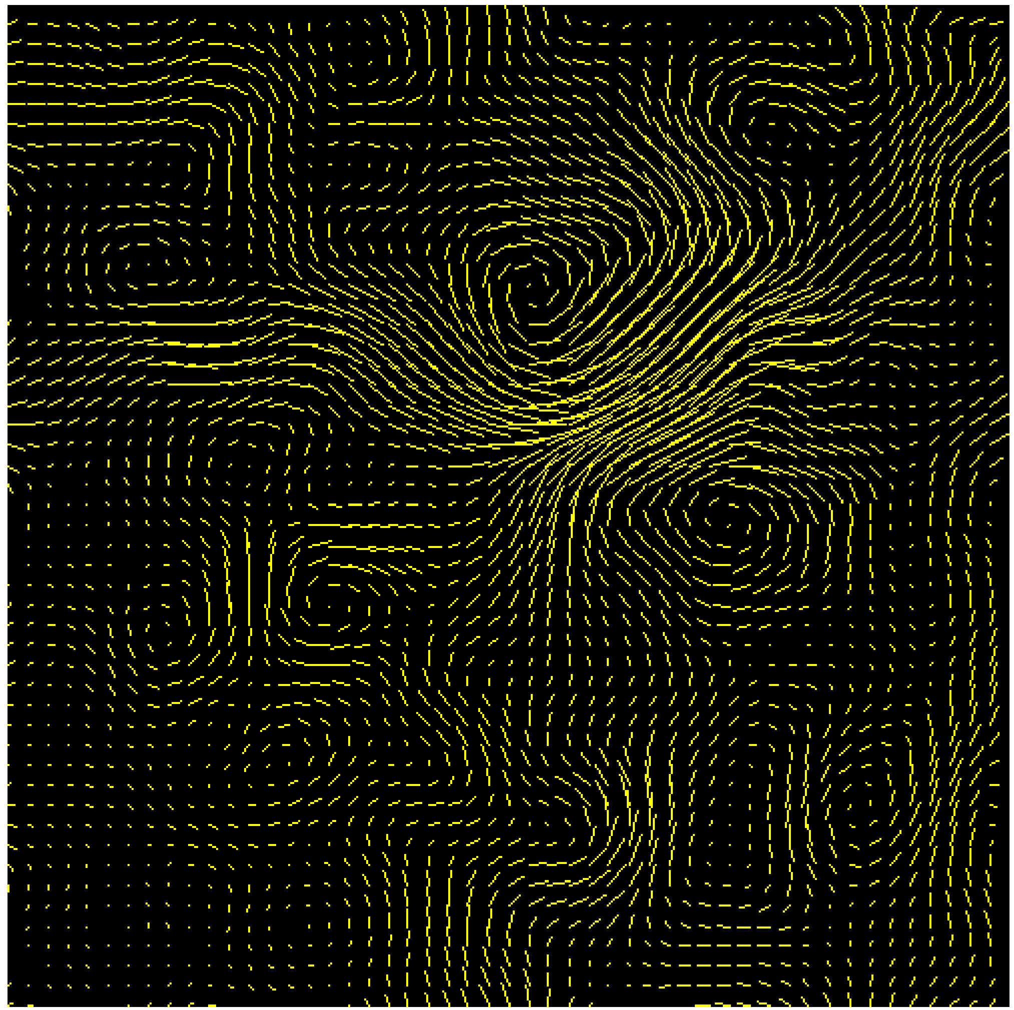
$$\mathbf{w}_0 = \mathbf{u}(\mathbf{x}, t)$$

$$\mathbf{w}_0(\mathbf{x}) \xrightarrow{\text{add force}} \mathbf{w}_1(\mathbf{x}) \xrightarrow{\text{Advect}} \mathbf{w}_2(\mathbf{x}) \xrightarrow{\text{Diffuse}} \boxed{\mathbf{w}_3(\mathbf{x}) \xrightarrow{\text{Project}} \mathbf{w}_4(\mathbf{x})}$$

$$\mathbf{u}(\mathbf{x}, t+\Delta t) = \mathbf{w}_4$$

Projection

P(



)

Projection

- Projection step subtracts off the pressure from the intermediate velocity field $\mathbf{w}_3$

$$\mathbf{w}_4 = \mathbf{w}_3 - \Delta t \frac{1}{\rho} \nabla p$$

- `project`(Δt, $\mathbf{u}$) must satisfies two conditions:
  - divergence free: $\nabla \cdot \mathbf{u}^{t+1} = 0$
  - boundary velocity: $\mathbf{u}^{t+1} \cdot \hat{\mathbf{n}} = \mathbf{u}_{solid} \cdot \hat{\mathbf{n}}$

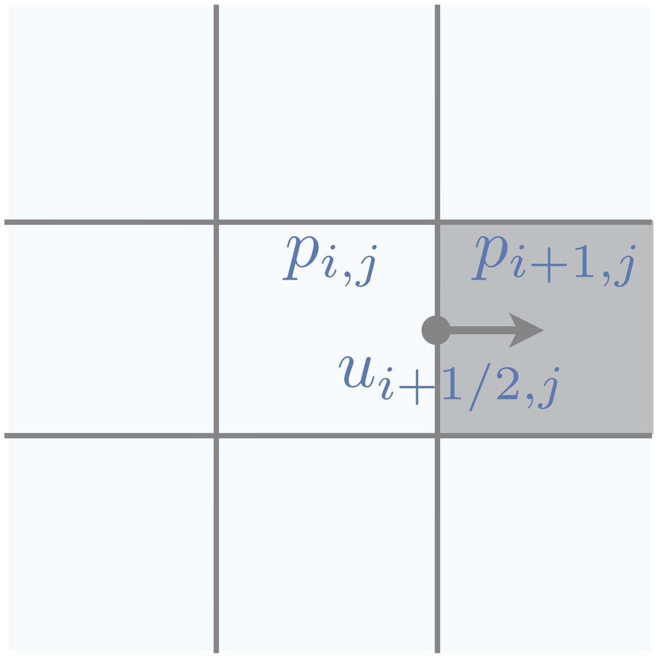
Boundary conditions

- Dirichlet boundary condition for free surfaces
- Neumann boundary condition for solid walls

$$u^{t+1}_{i+1/2,j} = u_{i+1/2,j} - \Delta t \frac{1}{\rho} \frac{p_{i+1,j} - p_{i,j}}{\Delta x}$$

since $\quad u^{t+1}_{i+1/2,j} = u_{solid}$

$$p_{i+1,j} = p_{i,j} + \frac{\rho \Delta x}{\Delta t}(u_{i+1/2,j} - u_{solid})$$

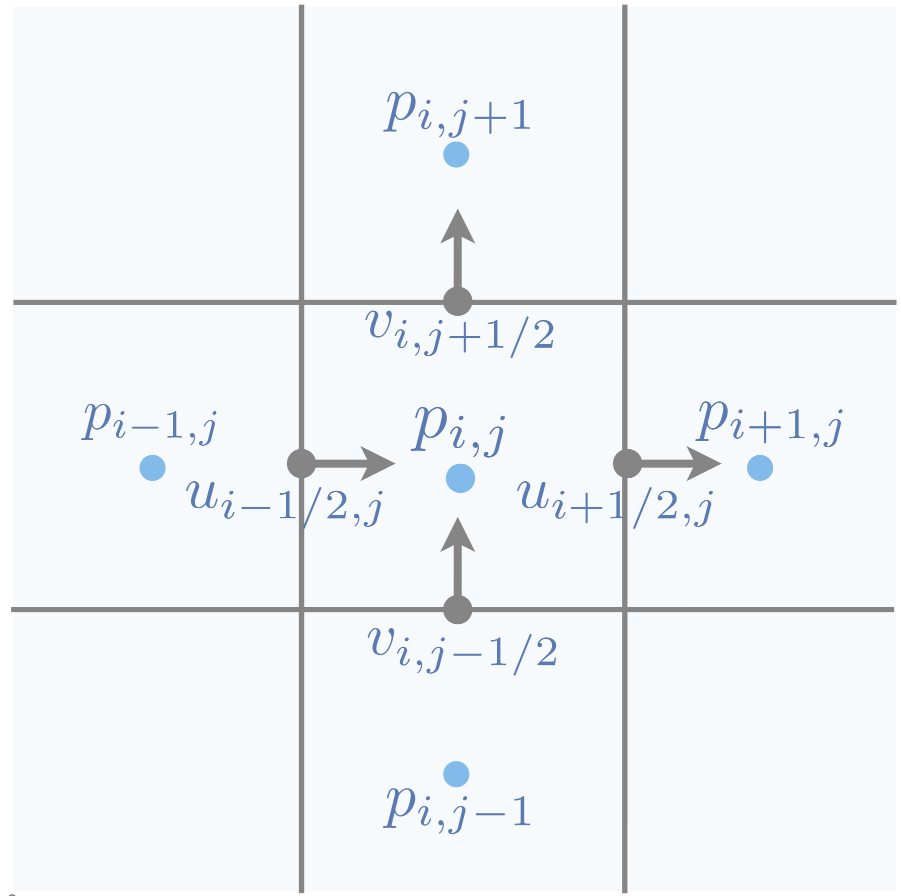


Divergence-free condition

$$\frac{u^{t+1}_{i+1/2,j} - u^{t+1}_{i-1/2,j}}{\Delta x} + \frac{v^{t+1}_{i,j+1/2} - v^{t+1}_{i,j-1/2}}{\Delta x} = 0$$

replace $u^{t+1}_{i+1/2,j}$ (and other terms)

with $u_{i+1/2,j} - \Delta t \frac{1}{\rho} \frac{p_{i+1,j} - p_{i,j}}{\Delta x}$

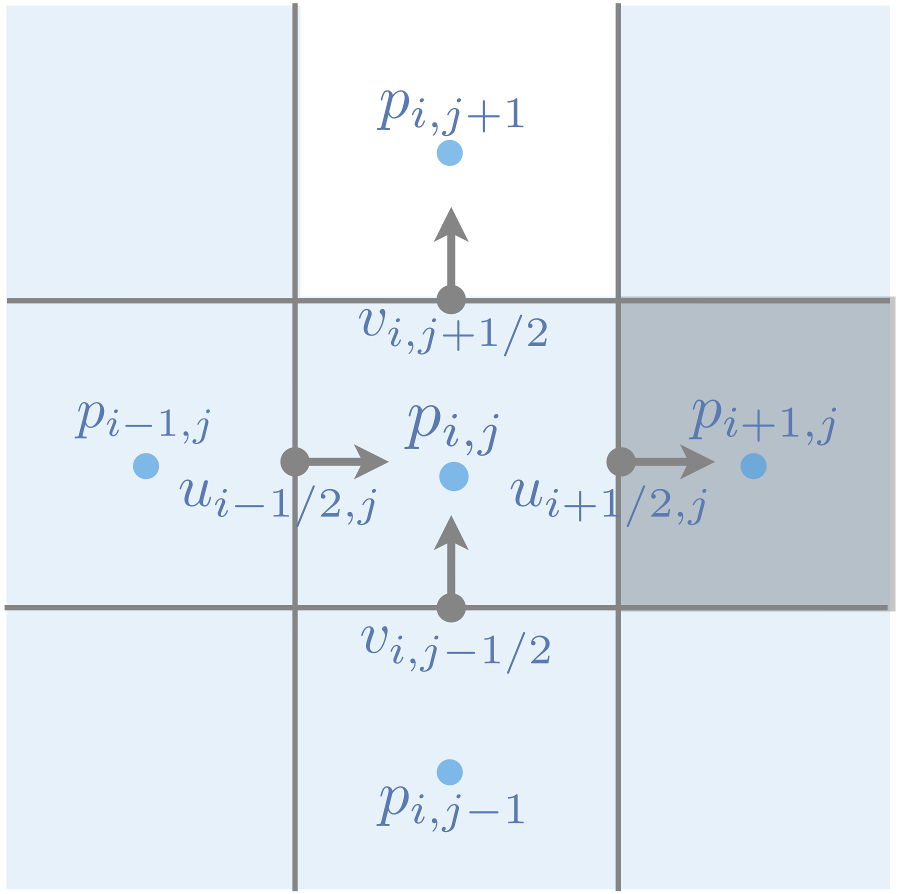


result in a discrete Poisson equation

$$\frac{\Delta t}{\rho}\left(\frac{4p_{i,j} - p_{i+1,j} - p_{i,j+1} - p_{i-1,j} - p_{i,j-1}}{\Delta x^2}\right) = -\left(\frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta x} + \frac{v_{i,j+1/2} - v_{i,j-1/2}}{\Delta x}\right)$$

The pressure equations

- If grid cell (i, j + 1) is in air and (i+1, j) is a solid
  - replace $p_{i,j+1}$ with zero
  - replace $p_{i+1,j}$ with the value from Neumann condition



$$\frac{\Delta t}{\rho}\left(\frac{4p_{i,j} - \overset{p_{i,j} + \frac{\rho \Delta x}{\Delta t}(u_{i+1/2,j} - u_{solid})}{\cancel{p_{i+1,j}}} - \overset{0}{\cancel{p_{i,j+1}}} - p_{i-1,j} - p_{i,j-1}}{\Delta x^2}\right) = -\left(\frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta x} + \frac{v_{i,j+1/2} - v_{i,j-1/2}}{\Delta x}\right)$$

Solve a linear system

- $\mathbf{Ap} = \mathbf{b}$
- Construct matrix **A**:
  - diagonal entry: # of non-solid neighbors
  - off-diagonal: 0 if the neighbor is non-fluid cell and -1 if the neighbor is fluid cell
- **A** is symmetric positive definite
- Use preconditioned conjugate gradient