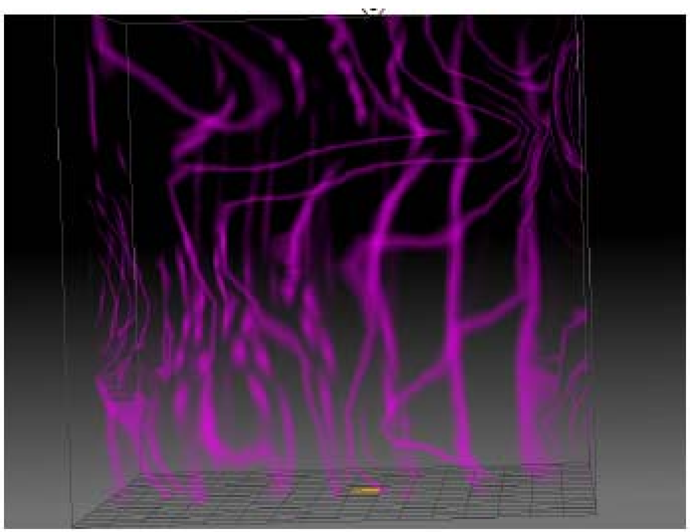
Stable Fluid



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Goal

- Simulate behavior of gas (or liquid)
- Stable method

- Graphics applications
- Sacrifice accuracy for speed and control

Navier-Stokes Equations

- Conservation of momentum

$$
\begin{aligned}
\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial uw}{\partial z} &= -\frac{\partial p}{\partial x}+g_x+\nu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}) \\
\frac{\partial v}{\partial t}+\frac{\partial vu}{\partial x}+\frac{\partial v^2}{\partial y}+\frac{\partial vw}{\partial z} &= -\frac{\partial p}{\partial y}+g_y+\nu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}) \\
\frac{\partial w}{\partial t}+\frac{\partial wu}{\partial x}+\frac{\partial wv}{\partial y}+\frac{\partial w^2}{\partial z} &= -\frac{\partial p}{\partial z}+g_z+\nu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}),
\end{aligned}
$$

- Conservation of mass

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

u, v, w: velocities in x, y, z directions
p: local pressure
g: gravity (local force in general)
ν: viscosity

Compact Form

Notation:

$$\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$$

$$\nabla^2 = \nabla \cdot \nabla$$

u: velocity
p: pressure
f: force
ν: viscosity
ρ: density

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \frac{\partial \mathbf{u}}{\partial t} &= -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \end{aligned}$$

Solving ODE/PDE

- Consider $du/dt = -\lambda u$, $u(0) = 1$, $\lambda > 0$
- Exact solution $u = e^{-\lambda t}$
- Explicit (Euler method):

$$(u_{i+1} - u_i)/\Delta t = -\lambda u_i, \quad u_i \approx u(i\Delta t)$$
$$\boxed{u_{i+1} = (1 - \lambda\Delta t)\, u_i}, \quad u_0 = 1$$
$$u_i = (1 - \lambda\Delta t)^i$$

- Implicit (Backward Euler method):

$$(u_{i+1} - u_i)/\Delta t = -\lambda u_{i+1}$$
$$\boxed{u_{i+1} = (1 + \lambda\Delta t)^{-1}\, u_i}, \quad u_0 = 1$$
$$u_i = (1 + \lambda\Delta t)^{-i}$$

In general: explicit method is cheap, implicit method is expensive

Stability of a Numerical Method

Stability: Is the computed solution bounded?
(assuming that the true solution is bounded)

Explicit: $u_i = (1 - \lambda\Delta t)^i$
Stable when $|1 - \lambda\Delta t| \leq 1$
$\Delta t \leq 2/\lambda$

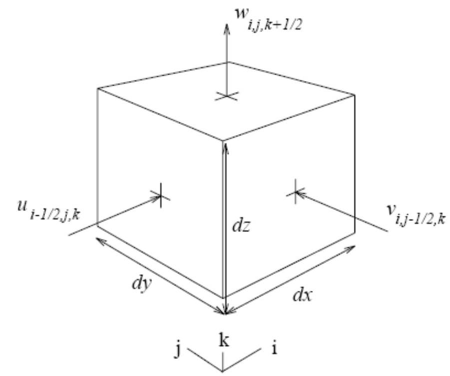
Implicit: $u_i = (1 + \lambda\Delta t)^{-i}$
Stable when $|1 + \lambda\Delta t| \geq 1$
Unconditionally stable!

small time step when λ is large!

Stability is not about the accuracy of the approximation

Stability is necessary for a numerical method to be useful

Previous Talk



$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

u: velocity
p: pressure
f: force
ν: viscosity
ρ: density

- Advance u using explicit method

$$(u_{i+1} - u_i)/\Delta t = -(u_i \cdot \nabla)\, u_i - 1/\rho\, \nabla p_i + \nu\, \nabla^2 u_i + f_i$$

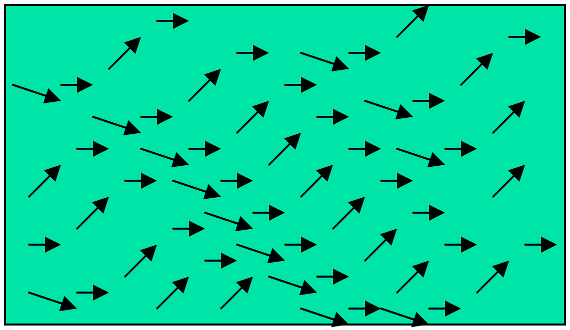
$$u_{i+1} = u_i + \Delta t\, [\, -(u_i \cdot \nabla)\, u_i - 1/\rho\, \nabla p_i + \nu\, \nabla^2 u_i + f_i\,]$$

- “Project” u to ensure conservation of mass and to compute new value for p

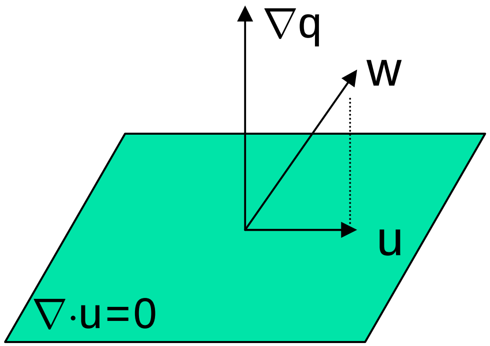
Stable when $\Delta t \approx \Delta x$

Helmholtz-Hodge Decomposition

- Vector field w: $R^d \rightarrow R^d$, scalar field q: $R^d \rightarrow R$, where d = 2, 3



- Any vector field w can be written uniquely as $w = u + \nabla q$, where $\nabla \cdot u = 0$, and q is a scalar field



Helmholtz-Hodge Decomposition

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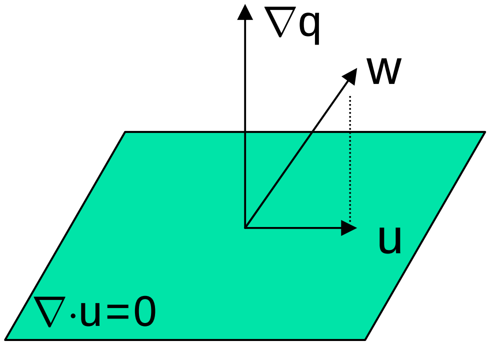


- Proof: Given w, the Poison equation $\nabla^2 q = \nabla \cdot w$ has a unique solution q

- Corollary: There is a projection operator P that projects any vector field into the space of "divergence free" vector fields

  if $w = u + \nabla q$ and $\nabla \cdot u = 0$, then $u = P(w)$

Removing Pressure Term

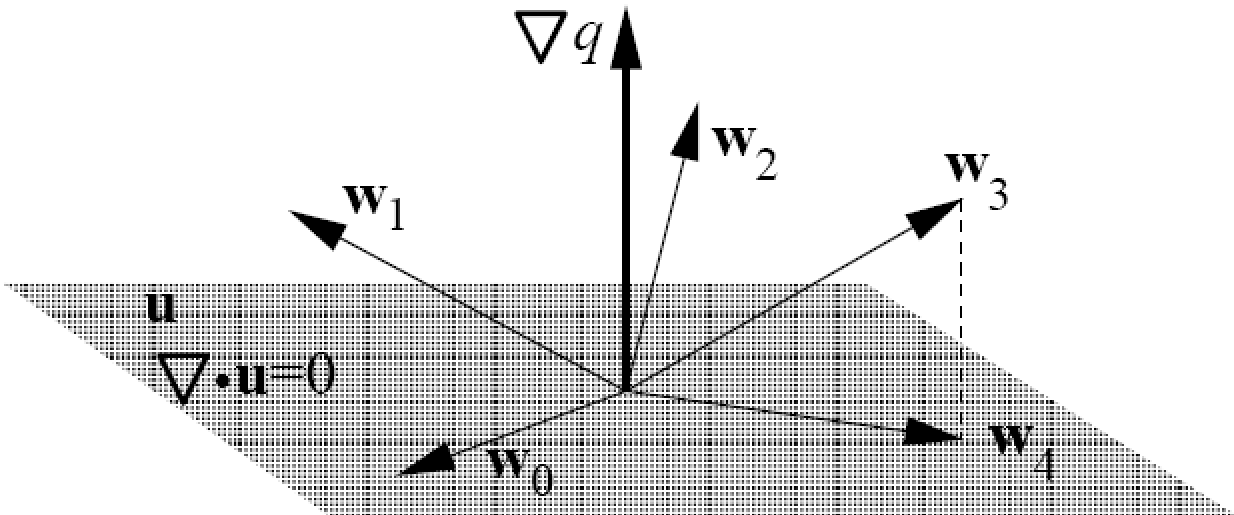
$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \frac{\partial \mathbf{u}}{\partial t} &= -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \end{aligned}$$

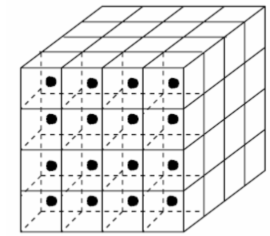
$$\Downarrow$$

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}\left(-(\mathbf{u} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \mathbf{u} + \mathbf{f}\right)$$

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}\left(-(\mathbf{u}\cdot\nabla)\mathbf{u} + \nu\nabla^2\mathbf{u} + \mathbf{f}\right)$$

Strategy





$$\mathbf{w}_0(\mathbf{x}) \overbrace{\longrightarrow}^{\text{add force}} \mathbf{w}_1(\mathbf{x}) \overbrace{\longrightarrow}^{\text{advect}} \mathbf{w}_2(\mathbf{x}) \overbrace{\longrightarrow}^{\text{diffuse}} \mathbf{w}_3(\mathbf{x}) \overbrace{\longrightarrow}^{\text{project}} \mathbf{w}_4(\mathbf{x}).$$

$$\frac{\partial u}{\partial t} = f \qquad \frac{\partial u}{\partial t} = -u\cdot\nabla u \qquad \frac{\partial u}{\partial t} = \nu\nabla^2 u \qquad w_4 = P(w_3)$$

$u(0) = w_0$
$w_1 = u(\Delta t)$

$u(0) = w_1$
$w_2 = u(\Delta t)$

$u(0) = w_2$
$w_3 = u(\Delta t)$

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}\left(-(\mathbf{u}\cdot\nabla)\mathbf{u} + \nu\nabla^2\mathbf{u} + \boxed{\mathbf{f}}\right)$$

Step 1: Add Force

$$\frac{\partial u}{\partial t} = f$$

$u(x,0) = w_0(x)$

$w_1(x) = u(x, \Delta t)$

$$\mathbf{w}_1(\mathbf{x}) = \mathbf{w}_0(\mathbf{x}) + \Delta t\ \mathbf{f}(\mathbf{x}, t)$$

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}\left(\boxed{-(\mathbf{u}\cdot\nabla)\mathbf{u}} + \nu\nabla^2\mathbf{u} + \mathbf{f}\right)$$

Step 2: Advection

$$\frac{\partial u}{\partial t} = -u \cdot \nabla u$$

$u(0) = w_1$
$w_2 = u(\Delta t)$

- "Propagate" disturbance
- Linearized, solving $\partial u/\partial t = - w_1 \cdot \nabla u$ instead

Method of Characteristics

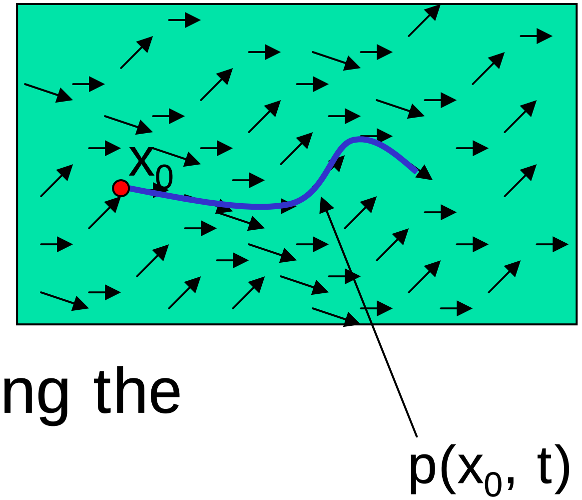
- Problem: solving $\partial a(x,t)/\partial t = -\, v(x) \cdot \nabla a(x,t)$, where $a(x,0) = a_0(x)$

function of t, "characteristic curve at x_0"

- Approach: Fix x_0, and let $p(x_0, t)$ be such that $p(x_0, 0) = x_0$ and that $dp(x_0, t)/dt = v(x_0)$.

- Let $b(t) = a(p(x_0, t), t)$, then by chain rule $db/dt = \nabla a \cdot dp/dt + \partial a/\partial t = 0$

  i.e. $b(t)$ = constant, $a(.,t)$ is a constant along the characteristic curves

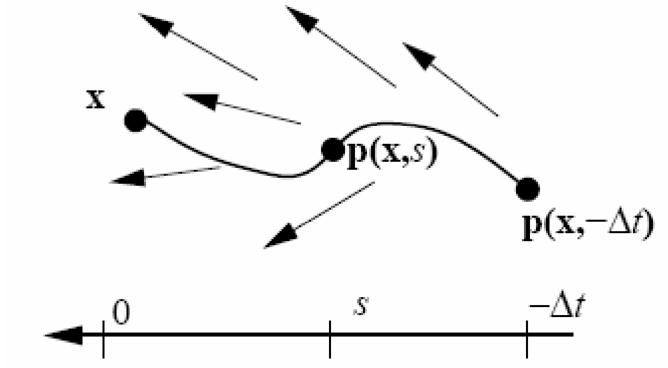


$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}\left(\boxed{-(\mathbf{u}\cdot\nabla)\mathbf{u}} + \nu\nabla^2\mathbf{u} + \mathbf{f}\right)$$

Step 2: Advection

$$\frac{\partial u}{\partial t} = -w_1 \cdot \nabla u$$

$u(0) = w_1$
$w_2 = u(\Delta t)$



$$\mathbf{w}_2(\mathbf{x}) = \mathbf{w}_1(\mathbf{p}(\mathbf{x}, -\Delta t))$$

p(x, –Δt) = x – Δt v(x) if t is small
integrate v(x) if Δt is large

max $w_2(x) \leq$ max $w_1(x)$

explicit method requires small time step $\Delta t \approx \Delta x$ here

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}\left(-(\mathbf{u}\cdot\nabla)\mathbf{u} + \boxed{\nu\nabla^2\mathbf{u}} + \mathbf{f}\right)$$

Step 3: Diffusion

$$\frac{\partial u}{\partial t} = \nu\nabla^2 u$$

$u(0) = w_2$

$w_3 = u(\Delta t)$

previous, explicit method: $w_3(x) = (I + \nu\Delta t\nabla^2)w_2(x)$

$$\left(\mathbf{I} - \nu\Delta t\nabla^2\right)\mathbf{w}_3(\mathbf{x}) = \mathbf{w}_2(\mathbf{x})$$

Sparse matrix

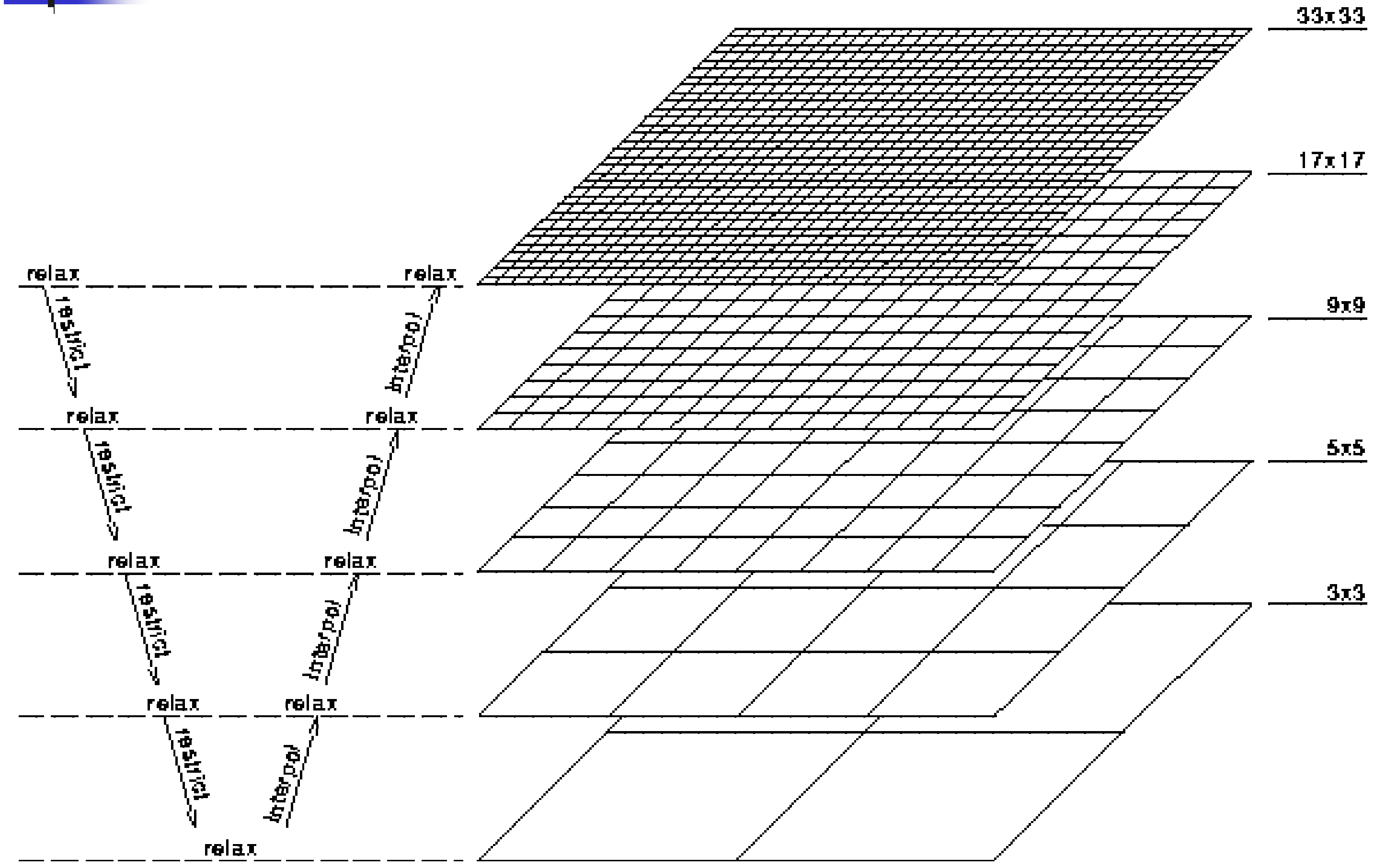
Solved using multi-grid

$\nabla^2 u = \partial^2 u/\partial x^2 + \partial^2 v/\partial x^2 + \partial^2 w/\partial z^2$

$\partial^2 u_{i,j,k}/\partial x^2 = 1/(\Delta x)^2\,(u_{i,j,k+1} - 2\,u_{i,j,k} + u_{i,j,k-1})$

Multi-grid method

$$\left(\mathbf{I} - \nu \Delta t \nabla^2\right) \mathbf{w}_3(\mathbf{x}) = \mathbf{w}_2(\mathbf{x})$$



approximate linear time algorithm for sparse matrix system

$$\frac{\partial \mathbf{u}}{\partial t} = \boxed{\mathbf{P}}\left(-(\mathbf{u}\cdot\nabla)\mathbf{u} + \nu\nabla^2\mathbf{u} + \mathbf{f}\right)$$

Step 4: Projection

Find w_4 such that $\nabla \cdot w_4 = 0$ and $w_3 = w_4 + \nabla q$

$$\nabla^2 q = \nabla\cdot\mathbf{w}_3 \qquad \mathbf{w}_4 = \mathbf{w}_3 - \nabla q.$$

Sparse matrix
Solved using multi-grid
Can be solved in linear time

Periodic Boundary Condition

FourierStep($\mathbf{w}_0$,$\mathbf{w}_4$,Δt):

add force: $\mathbf{w}_1 = \mathbf{w}_0 + \Delta t\ \mathbf{f}$

advect: $\mathbf{w}_2(\mathbf{x}) = \mathbf{w}_1(\mathbf{p}(\mathbf{x}, -\Delta t))$

transform: $\hat{\mathbf{w}}_2 = \mathrm{FFT}\{\mathbf{w}_2\}$

diffuse: $\hat{\mathbf{w}}_3(\mathbf{k}) = \hat{\mathbf{w}}_2(\mathbf{k})/(1 + \nu \Delta t k^2)$

project: $\hat{\mathbf{w}}_4 = \hat{\mathbf{P}}\hat{\mathbf{w}}_3$

transform: $\mathbf{w}_4 = \mathrm{FFT}^{-1}\{\hat{\mathbf{w}}_4\}$

k: wave number

Substances in the Fluid

- Simulate dust density, smoke/water droplets, fluid temperature, texture coordinate

- Propagate scalar quantity a using

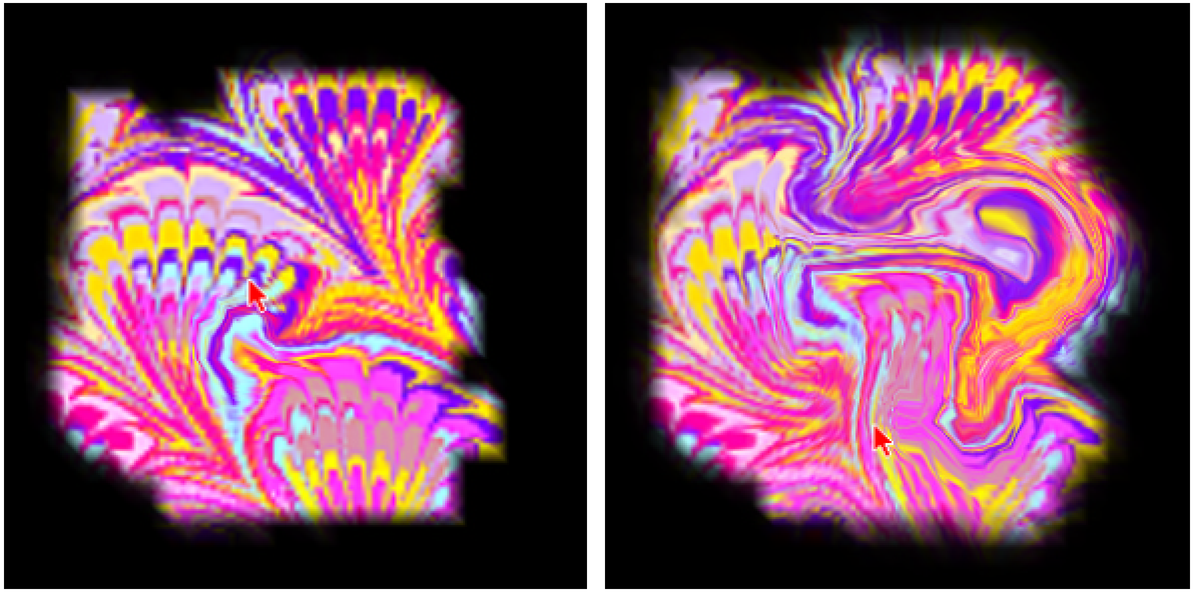
$$\frac{\partial a}{\partial t} = -\mathbf{u} \cdot \nabla a + \kappa_a \nabla^2 - \alpha_a a + S_a,$$

fluid velocity

κ_a: diffusion constant
α_a: dissipation term
S_a: source

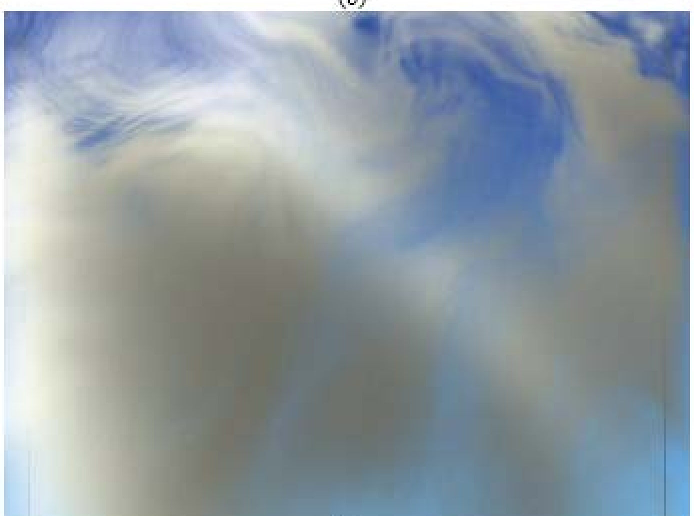
Results

- 16^3 – 30^3 grids
- Texture map is used for rendering
- Fast enough for interactive control of fluid



(b)

(c)



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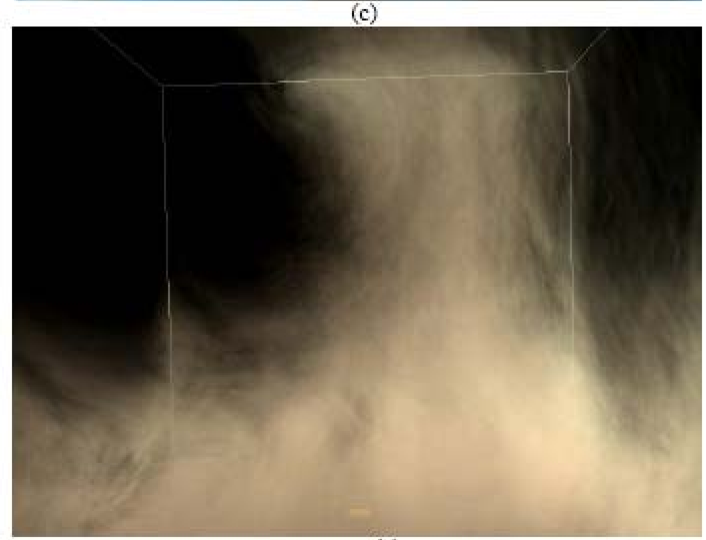

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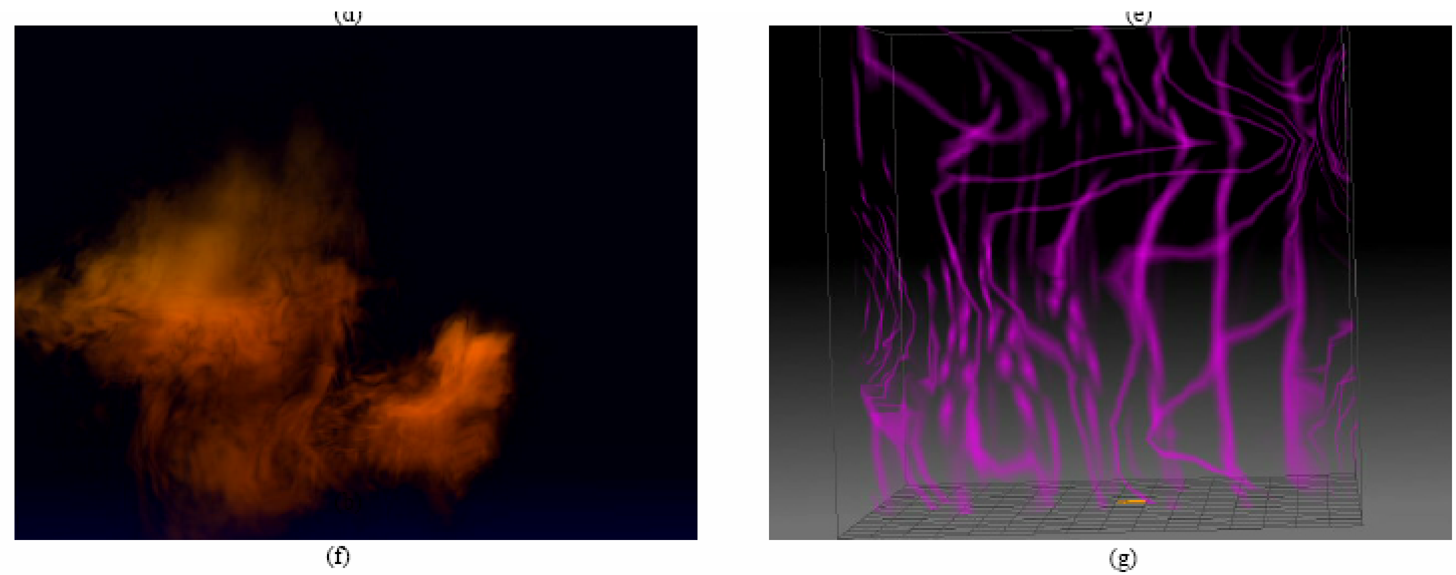
(f)
(g)

Summary

- Unconditional stable algorithm to solve Navier Stokes Equations
- Allowing fast simulation of fluid