信号检测与估值

段江涛
机电工程学院

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ch3. 信号检测与估计理论的基础知识

ch3-3. 贝叶斯准则例题 4

贝叶斯准则例题 4

考虑以下信号检测问题：

$$H_0 : x_k = 1 + n_k, \qquad k = 1, 2, \ldots, N$$
$$H_1 : x_k = -1 + n_k, \qquad k = 1, 2, \ldots, N$$

其中 n_k 是均值为零，方差为 $\sigma_n^2 = 1/2$ 的高斯随机变量，且不同采样时刻的加性噪声之间是相互统计独立的。
若两种假设先验等概的，且代价因子为 $c_{00} = 1, c_{10} = 4, c_{11} = 2, c_{01} = 3$。
请给出上述问题的贝叶斯检测准则和平均代价 C。

贝叶斯准则例题 4: 解

步骤 1: 计算两个似然函数, 构建似然比 $\lambda(\boldsymbol{x})=\frac{p(\boldsymbol{x}|H_1)}{p(\boldsymbol{x}|H_0)}$

步骤 2: 根据两个假设的先验概率和代价因子, 计算判决门限 $\eta \stackrel{def}{=} \frac{P(H_0)(c_{10}-c_{00})}{P(H_1)(c_{01}-c_{11})}$

步骤 3: 形成贝叶斯检测基本表达式 $\lambda(\boldsymbol{x})=\frac{p(\boldsymbol{x}|H_1)}{p(\boldsymbol{x}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$

步骤 4: 化简, 形成贝叶斯检测判决表达式 $l(\boldsymbol{x}) \stackrel{def}{=} \frac{1}{N}\sum_{k=1}^{N} x_k \underset{H_0}{\overset{H_1}{\gtrless}} \stackrel{def}{=} \gamma$

步骤 5: 性能分析

计算判决概率: $P(H_1|H_0), P(H_0|H_0), P(H_0|H_1), P(H_1|H_1)$, 其中,

$$\text{信噪比: } d^2=\frac{4N}{\sigma_n^2}, \quad Q(x)=\int_x^{\infty}\left(\frac{1}{2\pi}\right)^{1/2}\exp\left(-\frac{u^2}{2}\right)du$$

计算平均代价:

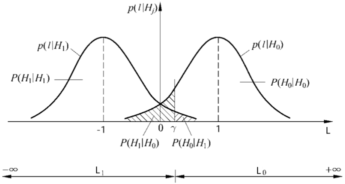
$$C=P(H_0)(c_{00}P(H_0|H_0)+c_{10}P(H_1|H_0))+P(H_1)(c_{01}P(H_0|H_1)+c_{11}P(H_1|H_1))$$

贝叶斯准则例题 4: 判决概率分布分析

$$H_0 : x_k = 1 + n_k, \quad H_1 : x_k = -1 + n_k$$

$$k = 1, 2, \ldots, N, \quad \boldsymbol{x} = (x_1, x_2, \ldots, x_N)^T$$

$$P(H_i|H_j) = \int_{R_i} p(\boldsymbol{x}|H_j) d\boldsymbol{x}$$



$$P(H_0|H_0) = \int_{L_0} p(\boldsymbol{x}|H_0) d\boldsymbol{x}, \qquad P(H_1|H_0) = \int_{L_1} p(\boldsymbol{x}|H_0) d\boldsymbol{x}$$

$$P(H_0|H_1) = \int_{L_0} p(\boldsymbol{x}|H_1) d\boldsymbol{x}, \qquad P(H_1|H_1) = \int_{L_1} p(\boldsymbol{x}|H_1) d\boldsymbol{x}$$

$$\boldsymbol{L} = L_0 \cup L_1, \quad L_0 \cap L_1 = \emptyset, \quad \int_{\boldsymbol{L}} p(\boldsymbol{x}|H_j) d\boldsymbol{x} = 1$$

$$P(H_0|H_0) + P(H_1|H_0) = \int_{L_0} p(\boldsymbol{x}|H_0) d\boldsymbol{x} + \int_{L_1} p(\boldsymbol{x}|H_0) d\boldsymbol{x} = \int_{\boldsymbol{L}} p(\boldsymbol{x}|H_0) d\boldsymbol{x} = 1$$

$$P(H_0|H_1) + P(H_1|H_1) = \int_{L_0} p(\boldsymbol{x}|H_1) d\boldsymbol{x} + \int_{L_1} p(\boldsymbol{x}|H_1) d\boldsymbol{x} = \int_{\boldsymbol{L}} p(\boldsymbol{x}|H_1) d\boldsymbol{x} = 1$$

贝叶斯准则例题 4: 解

解: N 次独立采样, 样本为 $x_k(k=1,2,\cdots,N)$:

$$H_0: x_k = 1 + n_k \qquad k=1,2,\cdots,N$$

$$H_1: x_k = -1 + n_k \qquad k=1,2,\cdots,N$$

步骤 1: 计算两个似然函数, 构建似然比

由于 n 是高斯分布随机变量, 因此在 H_0 假设下, 第 k 次采样值 x_k 服从高斯分布, 且均值为 1, 方差为 σ_n^2; 在 H_1 假设下, 第 k 次采样值 x_k 服从均值为 -1, 方差为 σ_n^2 的高斯分布。

$$p(x_k|H_0) = \left(\frac{1}{2\pi\sigma_n^2}\right)^{1/2} \exp\left(-\frac{(x_k-1)^2}{2\sigma_n^2}\right) \Longrightarrow p(\boldsymbol{x}|H_0) = \prod_{k=1}^{N}\left(\frac{1}{2\pi\sigma_n^2}\right)^{1/2} \exp\left(-\frac{(x_k-1)^2}{2\sigma_n^2}\right)$$

$$p(x_k|H_1) = \left(\frac{1}{2\pi\sigma_n^2}\right)^{1/2} \exp\left(-\frac{(x_k+1)^2}{2\sigma_n^2}\right) \Longrightarrow p(\boldsymbol{x}|H_1) = \prod_{k=1}^{N}\left(\frac{1}{2\pi\sigma_n^2}\right)^{1/2} \exp\left(-\frac{(x_k+1)^2}{2\sigma_n^2}\right)$$

$$\lambda(\boldsymbol{x}) = \frac{p(\boldsymbol{x}|H_1)}{p(\boldsymbol{x}|H_0)} = \exp\left(\frac{\sum\limits_{k=1}^{N}\left((x_k-1)^2-(x_k+1)^2\right)}{2\sigma_n^2}\right)$$

贝叶斯准则例题 4: 解续 (1)

步骤 2: 根据两个假设的先验概率和代价因子, 计算判决门限

$$\eta \overset{def}{=} \frac{P(H_0)(c_{10}-c_{00})}{P(H_1)(c_{01}-c_{11})} = \frac{4-1}{3-2} = 3$$

步骤 3: 形成贝叶斯检测基本表达式

$$\lambda(\boldsymbol{x}) = \frac{p(\boldsymbol{x}|H_1)}{p(\boldsymbol{x}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\exp\left(\frac{\sum_{k=1}^{N}\left((x_k-1)^2-(x_k+1)^2\right)}{2\sigma_n^2}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

步骤 4: 化简, 形成贝叶斯检测判决表达式

$$-4\sum_{k=1}^{N} x_k \underset{H_0}{\overset{H_1}{\gtrless}} 2\sigma_n^2 \ln\eta \implies \sum_{k=1}^{N} x_k \underset{H_1}{\overset{H_0}{\gtrless}} -\frac{\sigma_n^2 \ln\eta}{2}$$

$$l(\boldsymbol{x}) \overset{def}{=} \frac{1}{N}\sum_{k=1}^{N} x_k \underset{H_1}{\overset{H_0}{\gtrless}} -\frac{\sigma_n^2 \ln\eta}{2N} = -\frac{\ln 3}{4N} \overset{def}{=} \gamma$$

贝叶斯准则例题 4: 解续 (2)

经过上述化简, 信号检测的判决式由似然比检验的形式, 简化为检验统计量 $l(x)$ 与检测门限 γ 相比较的形式, 形成贝叶斯检测判决表达式:

$$l(\boldsymbol{x}) \overset{def}{=} \frac{1}{N}\sum_{k=1}^{N} x_k \underset{H_1}{\overset{H_0}{\gtrless}} -\frac{\sigma_n^2 \ln \eta}{2N} = -\frac{\ln 3}{4N} \overset{def}{=} \gamma$$

检验统计量 $l(x) \overset{def}{=} \frac{1}{N}\sum\limits_{k=1}^{N} x_k$ 是观测信号 $x_k(k = 1, 2, \ldots, N)$ 的求和取平均值的结果, 即它是 $x_k(k = 1, 2, \ldots, N)$ 的函数, 是一个随机变量。

因为高斯随机变量的线性组合还是高斯随机变量, 所以两种假设下的观测量 $(l|H_0), (l|H_1)$ 也是服从高斯分布的随机变量。

贝叶斯准则例题 4: 性能分析—观测量 $(l|H_0)$

$$H_0: x_k = 1 + n_k \qquad H_1: x_k = -1 + n_k$$

$$l(\boldsymbol{x}) \overset{def}{=} \frac{1}{N}\sum_{k=1}^{N} x_k \underset{H_1}{\overset{H_0}{\gtrless}} -\frac{\sigma_n^2 \ln\eta}{2N} = -\frac{\ln 3}{4N} \overset{def}{=} \gamma$$

统计量: $l(\boldsymbol{x}) \overset{def}{=} \frac{1}{N}\sum_{k=1}^{N} x_k$

假设 H_0 条件下, 统计量 $l(x)$ 为高斯分布, 均值和方差分别为

$$E[l|H_0] = E\left[\frac{1}{N}\sum_{k=1}^{N}(x_k|H_0)\right] = E\left[\frac{1}{N}\sum_{k=1}^{N}(1+n_k)\right] = \frac{1}{N}\sum_{k=1}^{N}E[1+n_k] = 1$$

$$Var[l|H_0] = E\left[(l|H_0 - E(l|H_0))^2\right] = E\left[\left(\frac{1}{N}\sum_{k=1}^{N}(1+n_k) - 1\right)^2\right]$$

$$= \frac{1}{N^2}\sum_{k=1}^{N}E[n_k^2] = \frac{1}{N^2}\sum_{k=1}^{N}\sigma_n^2 = \frac{\sigma_n^2}{N}$$

因此, $(l|H_0) \sim \mathcal{N}(1, \frac{\sigma_n^2}{N})$

$$p(l|H_0) = \left(\frac{1}{2\pi Var[l|H_0]}\right)^{1/2} \exp\left(-\frac{(l - E[l|H_0])^2}{2Var[l|H_0]}\right) = \left(\frac{N}{2\pi\sigma_n^2}\right)^{1/2} \exp\left(-\frac{N(l-1)^2}{2\sigma_n^2}\right)$$

贝叶斯准则例题 4: 性能分析—观测量 $(l|H_0)$

$$H_0 : x_k = 1 + n_k \qquad l(\boldsymbol{x}) \overset{def}{=} \frac{1}{N}\sum_{k=1}^{N} x_k \underset{H_1}{\overset{H_0}{\gtrless}} -\frac{\sigma_n^2 \ln \eta}{2N} = -\frac{\ln 3}{4N} \overset{def}{=} \gamma \qquad \textbf{统计量: } l(\boldsymbol{x}) \overset{def}{=} \frac{1}{N}\sum_{k=1}^{N} x_k$$

$$H_1 : x_k = -1 + n_k$$

$$p(l|H_0) = \left(\frac{N}{2\pi\sigma_n^2}\right)^{1/2} \exp\left(-\frac{N(l-1)^2}{2\sigma_n^2}\right)$$

$$P(H_1|H_0) = 1 - \int_{\gamma}^{\infty} p(l|H_0)dl \implies Q(x) = \int_{x}^{\infty} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left(-\frac{u^2}{2}\right) du$$

$$= 1 - \int_{\gamma}^{\infty} \left(\frac{N}{2\pi\sigma_n^2}\right)^{1/2} \exp\left(-\frac{N(l-1)^2}{2\sigma_n^2}\right) dl \qquad \text{by } l = \frac{\sigma_n u}{\sqrt{N}} + 1$$

$$= 1 - \int_{\frac{\sqrt{N}(\gamma-1)}{\sigma_n}}^{\infty} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left(-\frac{u^2}{2}\right) du$$

$$= 1 - Q\left(\frac{\sqrt{N}(\gamma-1)}{\sigma_n}\right) = 1 - Q\left(\frac{\sqrt{N}\left(-\frac{\sigma_n^2 \ln \eta}{2N} - 1\right)}{\sigma_n}\right)$$

贝叶斯准则例题 4: 性能分析—观测量 $(l|H_0)$

$$H_0 : x_k = 1 + n_k$$
$$H_1 : x_k = -1 + n_k$$

$$l(\boldsymbol{x}) \overset{def}{=} \frac{1}{N}\sum_{k=1}^{N} x_k \underset{H_1}{\overset{H_0}{\gtrless}} -\frac{\sigma_n^2 \ln \eta}{2N} = -\frac{\ln 3}{4N} \overset{def}{=} \gamma$$

统计量: $l(\boldsymbol{x}) \overset{def}{=} \frac{1}{N}\sum_{k=1}^{N} x_k$

$$\begin{aligned}
P(H_1|H_0) &= 1 - Q\left(\frac{\sqrt{N}\left(-\frac{\sigma_n^2 \ln \eta}{2N} - 1\right)}{\sigma_n}\right) \\
&= 1 - Q\left(-\frac{\sigma_n \ln \eta}{2\sqrt{N}} - \frac{\sqrt{N}}{\sigma_n}\right) \\
&= 1 - Q\left(-\frac{\ln \eta}{d} - \frac{d}{2}\right)
\end{aligned}$$

by $d^2 = \dfrac{4N}{\sigma_n^2}$

$$P(H_0|H_0) = 1 - P(H_1|H_0)$$

贝叶斯准则例题 4: 性能分析—观测量 $(l|H_1)$

$$H_0: x_k = 1 + n_k \qquad H_1: x_k = -1 + n_k$$

$$l(\boldsymbol{x}) \overset{def}{=} \frac{1}{N}\sum_{k=1}^{N} x_k \underset{H_1}{\overset{H_0}{\gtrless}} -\frac{\sigma_n^2 \ln\eta}{2N} = -\frac{\ln 3}{4N} \overset{def}{=} \gamma$$

统计量: $l(\boldsymbol{x}) \overset{def}{=} \frac{1}{N}\sum_{k=1}^{N} x_k$

假设 H_1 条件下, 统计量 $l(x)$ 为高斯分布, 均值和方差分别为

$$E[l|H_1] = E\left[\frac{1}{N}\sum_{k=1}^{N}(x_k|H_1)\right] = E\left[\frac{1}{N}\sum_{k=1}^{N}(-1+n_k)\right] = -1 + \frac{1}{N}\sum_{k=1}^{N}E[n_k] = -1$$

$$Var[l|H_1] = E\left[(l|H_1 - E(l|H_1))^2\right] = E\left[\left(\frac{1}{N}\sum_{k=1}^{N}(-1+n_k)+1\right)^2\right]$$

$$= \frac{1}{N^2}\sum_{k=1}^{N}E[n_k^2] = \frac{1}{N^2}\sum_{k=1}^{N}\sigma_n^2 = \frac{\sigma_n^2}{N}$$

因此, $(l|H_1) \sim \mathcal{N}(-1, \frac{\sigma_n^2}{N})$

$$p(l|H_1) = \left(\frac{1}{2\pi Var[l|H_1]}\right)^{1/2}\exp\left(-\frac{(l-E[l|H_1])^2}{2Var[l|H_1]}\right) = \left(\frac{N}{2\pi\sigma_n^2}\right)^{1/2}\exp\left(-\frac{N(l+1)^2}{2\sigma_n^2}\right)$$

贝叶斯准则例题 4: 性能分析—观测量 $(l|H_1)$

$$H_0: x_k = 1 + n_k \qquad H_1: x_k = -1 + n_k$$

$$l(\boldsymbol{x}) \overset{def}{=} \frac{1}{N}\sum_{k=1}^{N} x_k \underset{H_1}{\overset{H_0}{\gtrless}} -\frac{\sigma_n^2 \ln\eta}{2N} = -\frac{\ln 3}{4N} \overset{def}{=} \gamma$$

统计量: $l(\boldsymbol{x}) \overset{def}{=} \frac{1}{N}\sum_{k=1}^{N} x_k$

$$p(l|H_1) = \left(\frac{N}{2\pi\sigma_n^2}\right)^{1/2} \exp\left(-\frac{N(l+1)^2}{2\sigma_n^2}\right)$$

$$P(H_0|H_1) = \int_{\gamma}^{\infty} p(l|H_1)dl \implies Q(x) = \int_x^{\infty}\left(\frac{1}{2\pi}\right)^{1/2}\exp\left(-\frac{u^2}{2}\right)du$$

$$= \int_{\gamma}^{\infty}\left(\frac{N}{2\pi\sigma_n^2}\right)^{1/2}\exp\left(-\frac{N(l+1)^2}{2\sigma_n^2}\right)dl \qquad \text{by } l = \frac{\sigma_n u}{\sqrt{N}} - 1$$

$$= \int_{\frac{\sqrt{N}(\gamma+1)}{\sigma_n}}^{\infty}\left(\frac{1}{2\pi}\right)^{1/2}\exp\left(-\frac{u^2}{2}\right)du$$

$$= Q\left(\frac{\sqrt{N}(\gamma+1)}{\sigma_n}\right) = Q\left(\frac{\sqrt{N}\left(-\frac{\sigma_n^2\ln\eta}{2N}+1\right)}{\sigma_n}\right)$$

贝叶斯准则例题 4: 性能分析—观测量 $(l|H_1)$

$$H_0: x_k = 1 + n_k \qquad H_1: x_k = -1 + n_k$$

$$l(\boldsymbol{x}) \overset{def}{=} \frac{1}{N}\sum_{k=1}^{N} x_k \underset{H_1}{\overset{H_0}{\gtrless}} -\frac{\sigma_n^2 \ln\eta}{2N} = -\frac{\ln 3}{4N} \overset{def}{=} \gamma$$

统计量: $l(\boldsymbol{x}) \overset{def}{=} \frac{1}{N}\sum_{k=1}^{N} x_k$

$$\begin{aligned}
P(H_0|H_1) &= Q\left(\frac{\sqrt{N}\left(-\frac{\sigma_n^2 \ln\eta}{2N} + 1\right)}{\sigma_n}\right) \\
&= Q\left(-\frac{\sigma_n \ln\eta}{2\sqrt{N}} + \frac{\sqrt{N}}{\sigma_n}\right) \\
&= Q\left(-\frac{\ln\eta}{d} + \frac{d}{2}\right)
\end{aligned}$$

by $d^2 = \frac{4N}{\sigma_n^2}$

$$P(H_1|H_1) = 1 - P(H_0|H_1)$$

贝叶斯准则例题 4: 平均代价

$$H_0: x_k = 1 + n_k \qquad H_1: x_k = -1 + n_k \qquad l(\boldsymbol{x}) \overset{def}{=} \frac{1}{N}\sum_{k=1}^{N} x_k \underset{H_1}{\overset{H_0}{\gtrless}} -\frac{\sigma_n^2 \ln \eta}{2N} = -\frac{\ln 3}{4N} \overset{def}{=} \gamma$$

统计量: $l(\boldsymbol{x}) \overset{def}{=} \frac{1}{N}\sum_{k=1}^{N} x_k$

判决概率: (式中, 信噪比 $d^2 = \frac{4N}{\sigma_n^2}$)

$$P(H_1|H_0) = 1 - Q\left(-\frac{\ln \eta}{d} - \frac{d}{2}\right), \qquad P(H_0|H_0) = 1 - P(H_1|H_0)$$

$$P(H_0|H_1) = Q\left(-\frac{\ln \eta}{d} + \frac{d}{2}\right), \qquad P(H_1|H_1) = 1 - P(H_0|H_1)$$

两种假设先验等概 $\Longrightarrow P(H_0) = P(H_1) = \frac{1}{2}$

代价因子为 $c_{00} = 1, c_{10} = 4, c_{11} = 2, c_{01} = 3$

平均代价:

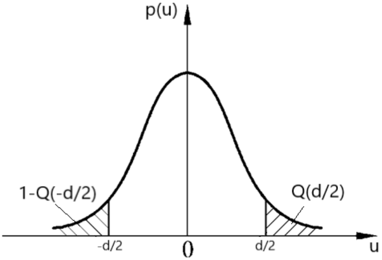
$$C = P(H_0)(c_{00}P(H_0|H_0) + c_{10}P(H_1|H_0)) + P(H_1)(c_{01}P(H_0|H_1) + c_{11}P(H_1|H_1))$$

四个判决概率与信噪比 d 有关,只需要设计信噪比,可得到所需性能。

标准高斯分布的右尾积分

$$Q(x) = \int_x^{\infty} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left(-\frac{u^2}{2}\right) du$$

$$Q\left(\frac{d}{2}\right) = 1 - Q\left(-\frac{d}{2}\right)$$



因此,

$$P(H_1|H_0) = 1 - Q\left(-\frac{\ln\eta}{d} - \frac{d}{2}\right) = Q\left(\frac{\ln\eta}{d} + \frac{d}{2}\right)$$

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