信号检测与估值

段江涛
机电工程学院

XIDIAN UNIVERSITY

2019 年 10 月

ch4. 信号波形的检测

ch4-2. 高斯白噪声确知信号波形检测及性能分析

高斯白噪声中确知信号波形的检测

基本要求

- 简单二元信号的波形检测(正交级数展开法和充分统计量)
- 一般二元信号的波形检测(正交级数展开法和充分统计量)
- 检测表达式、检测机结构、检测性能分析

简单二元信号波形的检测—信号模型

在简单二元信号的波形检测中，假设 H_0 下和假设 H_1 的接收信号分别为:

$$H_0: x(t) = n(t), \qquad 0 \le t \le T$$

$$H_1: x(t) = s(t) + n(t), \qquad 0 \le t \le T$$

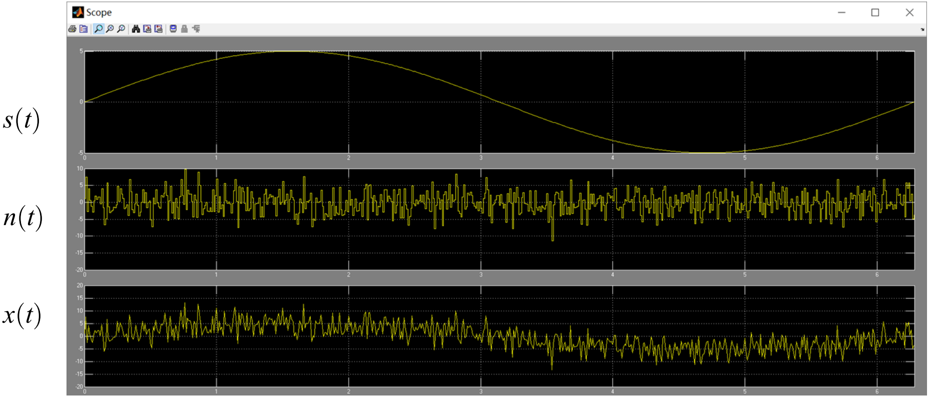
其中, $s(t)$ 是能量为 E_s 的确知信号

$$E_s = \int_0^T s^2(t)dt$$

$n(t)$ 是均值为零, 功率谱密度为 $N_0/2$ 的**高斯白噪声**。

$$\text{信源发送信号}: s(t) = \sin(t), 0 \leq t \leq T$$

$$\text{接收信号}: x(t) = s(t) + n(t), 0 \leq t \leq T$$



简单二元信号波形检测—检测思路

1. 首先,利用随机过程的正交级数展开,将随机过程用一组随机变量来表示;
2. 然后,针对展开得到的随机变量,利用第三章的统计检测方法,构建贝叶斯检测表达式;
3. 最后,令 N 趋向于无穷大,利用展开系数与随机过程之间的表示关系,构建波形信号的检测表达式。

简单二元信号波形检测—检测思路

$$x_k = \int_0^T x(t) f_k(t) dt$$

$$s_k = \int_0^T s(t) f_k(t) dt$$

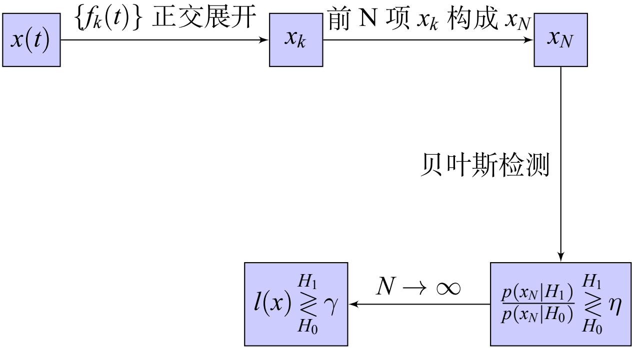
$$n_k = \int_0^T n(t) f_k(t) dt$$

$$x(t) = \lim_{N\to\infty} \sum_{k=1}^{N} x_k f_k(t)$$

$$\lambda(\boldsymbol{x}_N) = \frac{p(\boldsymbol{x}_N|H_1)}{p(\boldsymbol{x}_N|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\ln\lambda(x(t)) \overset{def}{=} \lim_{N\to\infty} [\lambda(\boldsymbol{x}_N)]$$

$$l[x(t)] \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$



判决表达式—步骤 1

$H_0 : x(t) = n(t)$

$H_1 : x(t) = s(t) + n(t)$

步骤 1,选一组完备的正交函数集 $\{f_k(t), k = 1, 2, \cdots\}$, 对接收信号进行正交级数展开,得到一组随机变量 x_k

因为信号 s(t) 是确知信号,n(t) 是均值为 0 的高斯白噪声,所以可以任选正交函数集 $\{f_k(t)\}$

$$x_k = \int_0^T x(t) f_k(t) dt \qquad x(t) = \lim_{N\to\infty} \sum_{k=1}^{N} x_k f_k(t)$$

$$H_0 : x_k = n_k, k = 1, 2, \ldots \qquad n_k = \int_0^T n(t) f_k(t) dt$$

$$H_1 : x_k = s_k + n_k, k = 1, 2, \ldots \qquad s_k = \int_0^T s(t) f_k(t) dt$$

判决表达式—步骤 1

- 信号 s(t) 是确知信号,n(t) 是均值为 0, 功率谱密度为 $P_n(\omega) = N_0/2$ 的高斯白噪声;
- 无论在假设 H_1 下还是在假设 H_2 下 , 接收信号的 $x(t)$ 都是高斯随机过程;
- 展开系数 x_k 是高斯随机过程的积分结果, 因而 x_k 是高斯随机变量;
- 展开系数 x_k 之间是互不相关的, 也是相互统计独立的;
- 高斯随机变量由均值和方差决定。由此求出两个假设下的概率密度函数 $p(x_k|H_j), k = 1, 2, \ldots; j = 0, 1$。

判决表达式—步骤 1: 假设 H_0 下 x_k 的均值和方差

$$H_0 : x_k = n_k, \quad n_k = \int_0^T n(t) f_k(t) dt$$

由于 $n(t)$ 是均值为零的高斯白噪声

$$E[n(t)] = 0, \qquad E[n(t)n(u)] = r_n(t-u) = \frac{N_0}{2}\delta(t-u) = \frac{N_0}{2}, (\delta(t-u) = 1, t = u)$$

$f_k(t)$ 是一组正交函数集 $\Longrightarrow$ $\int_0^T f_j(t) f_k(t) dt = 1, (j = k)$

$$E[x_k|H_0] = E[n_k] = E\left[\int_0^T n(t) f_k(t) dt\right] = \int_0^T E[n(t)] f_k(t) dt = 0$$

$$\begin{aligned} Var[x_k|H_0] &= E[n_k^2] = E\left[\int_0^T n(t) f_k(t) dt \int_0^T n(u) f_k(u) du\right] \\ &= \int_0^T f_k(t) \left\{\int_0^T E[n(t)n(u)] f_k(u) du\right\} dt = \int_0^T f_k(t) \left[\int_0^T \frac{N_0}{2}\delta(t-u) f_k(u) du\right] dt \\ &= \int_0^T f_k(t) \frac{N_0}{2} f_k(t) dt = \frac{N_0}{2} \end{aligned}$$

判决表达式—步骤 1: 假设 H_1 下 x_k 的均值和方差

$$H_1: x_k = s_k + n_k, \quad s_k = \int_0^T s(t) f_k(t) dt, \quad n_k = \int_0^T n(t) f_k(t) dt$$

$$\begin{aligned}
E[x_k|H_1] &= E[s_k + n_k] && \text{by } x_k = s_k + n_k \\
&= E(s_k) + E(n_k) && \text{by } E(n_k) = 0 \\
&= E(s_k) = s_k && \text{(确知信号展开系数为确定量, 其均值就是本身)} \\
Var[x_k|H_1] &= E[(x_k - E[x_k])^2] && \text{by } x_k = s_k + n_k, E[x_k] = s_k \\
&= E[(s_k + n_k - s_k)^2] \\
&= E[n_k^2] = \frac{N_0}{2}
\end{aligned}$$

判决表达式—步骤 1: 假设 H_0, H_1 下 x_k 的概率密度

$$E[x_k|H_0] = 0, \quad Var[x_k|H_1] = \frac{N_0}{2}$$
$$E[x_k|H_1] = s_k, \quad Var[x_k|H_1] = \frac{N_0}{2}$$

$$p(x_k|H_0) = \left(\frac{1}{\pi N_0}\right)^{1/2} \exp\left(-\frac{x_k^2}{N_0}\right), k = 1, 2, \cdots$$
$$p(x_k|H_1) = \left(\frac{1}{\pi N_0}\right)^{1/2} \exp\left(-\frac{(x_k^2 - s_k)^2}{N_0}\right), k = 1, 2, \cdots$$

判决表达式—步骤 2

步骤 2，利用前 N 项展开系数，构建似然比检验。由于信道是加性高斯白噪声，由卡亨南—洛维展开可知，各展开系数是不相关的，因而也是相互独立的。

$$p(\boldsymbol{x}_N|H_0)=\prod_{k=1}^{N}p(x_k|H_0)=\prod_{k=1}^{N}\left(\frac{1}{\pi N_0}\right)^{1/2}\exp\left(-\frac{x_k^2}{N_0}\right)$$

$$p(\boldsymbol{x}_N|H_1)=\prod_{k=1}^{N}p(x_k|H_1)=\prod_{k=1}^{N}\left(\frac{1}{\pi N_0}\right)^{1/2}\exp\left(-\frac{(x_k^2-s_k)^2}{N_0}\right)$$

$$\boldsymbol{x}_N=(x_1,x_2,\cdots)^T$$

判决表达式—步骤 2: 似然比

$$\lambda(\boldsymbol{x}_N) = \frac{p(\boldsymbol{x}_N|H_1)}{p(\boldsymbol{x}_N|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\frac{\prod_{k=1}^{N}\left(\frac{1}{\pi N_0}\right)^{1/2}\exp\left(-\frac{x_k^2}{N_0}\right)}{\prod_{k=1}^{N}\left(\frac{1}{\pi N_0}\right)^{1/2}\exp\left(-\frac{(x_k^2-s_k)^2}{N_0}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

化简,得到

$$\ln\lambda(\boldsymbol{x}_N) = \frac{p(\boldsymbol{x}_N|H_1)}{p(\boldsymbol{x}_N|H_0)} = \frac{2}{N_0}\sum_{k=1}^{N} x_k s_k - \frac{1}{N_0}\sum_{k=1}^{N} s_k^2 \underset{H_0}{\overset{H_1}{\gtrless}} \ln\eta$$

判决表达式—步骤 3

$$\ln\lambda(\boldsymbol{x}_N)=\frac{p(\boldsymbol{x}_N|H_1)}{p(\boldsymbol{x}_N|H_0)}=\frac{2}{N_0}\sum_{k=1}^{N}x_ks_k-\frac{1}{N_0}\sum_{k=1}^{N}s_k^2\underset{H_0}{\overset{H_1}{\gtrless}}\ln\eta$$

步骤 3, 令 $N\to\infty$, 将离散判决式变成连续形式 因为在两个假设下接收信号 $x(t)(0\le t\le T)$ 的展开系数 $x_k(k=1,2,\cdots)$ 是无穷多个,而离散形式判决式只是取前有限 N 项的结果,因此应对上式取 $N\to\infty$ 的极限。

$$\ln\lambda(x(t))\overset{def}{=}\lim_{N\to\infty}[\ln\lambda(\boldsymbol{x}_N)]=\lim_{N\to\infty}\left(\frac{2}{N_0}\sum_{k=1}^{N}x_ks_k-\frac{1}{N_0}\sum_{k=1}^{N}s_k^2\right)$$

$$\ln\lambda(x(t))\underset{H_0}{\overset{H_1}{\gtrless}}\ln\eta$$

$$\ln\lambda(x(t))\overset{def}{=}\lim_{N\to\infty}\left(\frac{2}{N_0}\sum_{k=1}^{N}x_ks_k-\frac{1}{N_0}\sum_{k=1}^{N}s_k^2\right)=\frac{2}{N_0}\lim_{N\to\infty}\sum_{k=1}^{N}x_ks_k-\frac{1}{N_0}\lim_{N\to\infty}\sum_{k=1}^{N}s_k^2\underset{H_0}{\overset{H_1}{\gtrless}}\ln\eta$$

判决表达式—步骤 3: 推导 (1)

$$\ln\lambda(x(t)) \overset{def}{=} \frac{2}{N_0}\lim_{N\to\infty}\sum_{k=1}^{N} x_k s_k - \frac{1}{N_0}\lim_{N\to\infty}\sum_{k=1}^{N} s_k^2 \underset{H_0}{\overset{H_1}{\gtrless}} \ln\eta$$

$$\begin{aligned}
\lim_{N\to\infty}\sum_{k=1}^{N} x_k s_k &= \left[\lim_{N\to\infty}\sum_{k=1}^{N} x_k\right] s_k && \text{by } s_k = \int_0^T s(t) f_k(t)\,dt \\
&= \left[\lim_{N\to\infty}\sum_{k=1}^{N} x_k\right] \int_0^T s(t) f_k(t)\,dt \\
&= \int_0^T s(t)\left[\lim_{N\to\infty}\sum_{k=1}^{N} x_k f_k(t)\right] dt && \text{by } x(t) = \lim_{N\to\infty}\sum_{k=1}^{N} x_k f_k(t) \\
&= \int_0^T s(t) x(t)\,dt
\end{aligned}$$

判决表达式—步骤 3: 推导 (2)

$$\ln\lambda(x(t)) \overset{def}{=} \frac{2}{N_0}\lim_{N\to\infty}\sum_{k=1}^{N} x_k s_k - \frac{1}{N_0}\lim_{N\to\infty}\sum_{k=1}^{N} s_k^2 \underset{H_0}{\overset{H_1}{\gtrless}} \ln\eta$$

$$\begin{aligned}
\lim_{N\to\infty}\sum_{k=1}^{N} s_k^2 &= \left[\lim_{N\to\infty}\sum_{k=1}^{N} s_k\right] s_k & \text{by } s_k = \int_0^T s(t) f_k(t)\,dt \\
&= \left[\lim_{N\to\infty}\sum_{k=1}^{N} s_k\right] \int_0^T s(t) f_k(t)\,dt \\
&= \int_0^T s(t)\left[\lim_{N\to\infty}\sum_{k=1}^{N} s_k f_k(t)\right] dt & \text{by } s(t) = \lim_{N\to\infty}\sum_{k=1}^{N} s_k f_k(t) \\
&= \int_0^T s(t)s(t)\,dt = \int_0^T s^2(t)\,dt & \text{by } E_s = \int_0^T s^2(t)\,dt \\
&= E_s
\end{aligned}$$

判决表达式—步骤 3: 推导 (3)

$$\ln \lambda(x(t)) \overset{def}{=} \frac{2}{N_0} \lim_{N\to\infty} \sum_{k=1}^{N} x_k s_k - \frac{1}{N_0} \lim_{N\to\infty} \sum_{k=1}^{N} s_k^2 \mathop{\gtrless}_{H_0}^{H_1} \ln \eta$$

$$\lim_{N\to\infty} \sum_{k=1}^{N} x_k s_k = \int_0^T s(t)x(t)dt, \qquad \lim_{N\to\infty} \sum_{k=1}^{N} s_k^2 = E_s$$

判决表达式:

$$\ln \lambda(x(t)) \overset{def}{=} \frac{2}{N_0} \int_0^T s(t)x(t)dt - \frac{E_s}{N_0} \mathop{\gtrless}_{H_0}^{H_1} \ln \eta$$

化简为:

$$l[x(t)] \overset{def}{=} \int_0^T s(t)x(t)dt \mathop{\gtrless}_{H_0}^{H_1} \frac{N_0}{2} \ln \eta + \frac{E_s}{2} \overset{def}{=} \gamma$$

简单二元信号波形的检测—检测系统结构

检测系统的结构:最佳检测系统 (又称最佳接收机) 的结构,根据信号最佳检测的判决表达式来设计。

检测统计量 $l[x(t)]$ 是由接收信号 $x(t)$ 与确知信号 $s(t)$ 经相关运算得到的,所以这种结构称为**相关检测系统**,是由互相关器和判决器实现。

白噪声下 t=T 时刻匹配滤波器输出信号与相关器输出信号式相等的。所以也可以用匹配滤波器和判决器来实现。

$$l[x(t)] \overset{def}{=} \int_0^T s(t)x(t)dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln\eta + \frac{E_s}{2} \overset{def}{=} \gamma$$

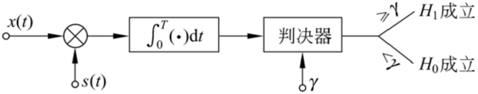


图4.8 相关检测系统结构 (相关接收机)

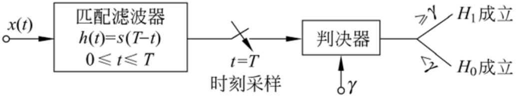
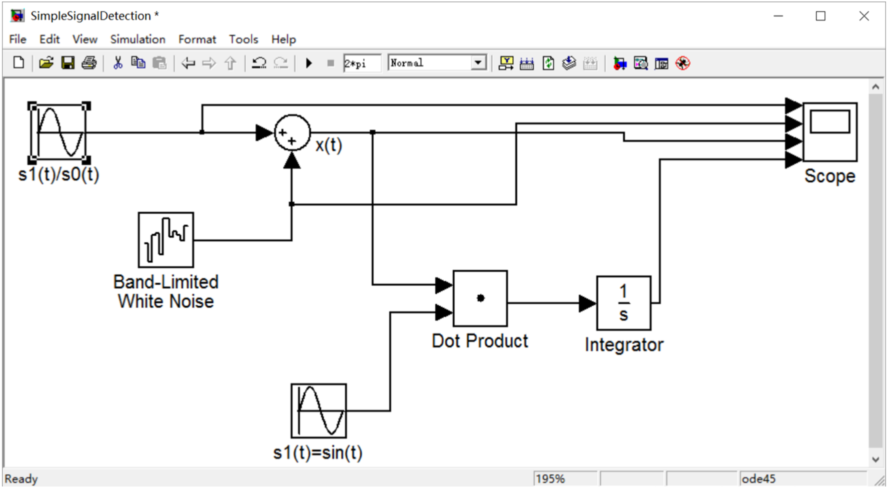


图4.9 匹配滤波器检测系统结构

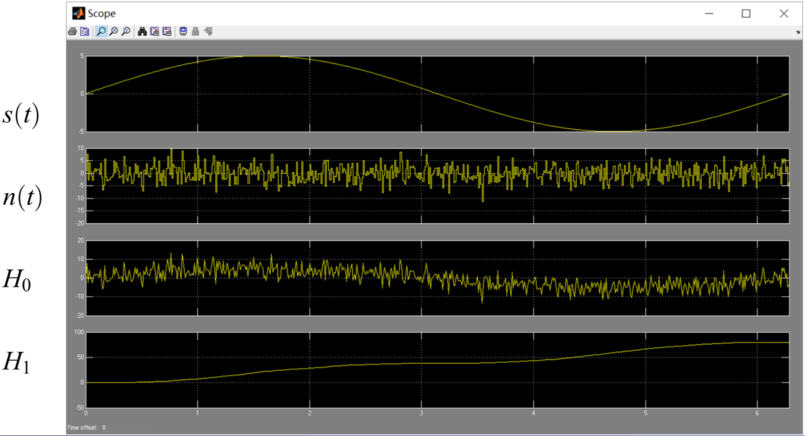
信源发送信号: $s(t) = \sin(t), 0 \leq t \leq T$

接收信号: $x(t) = s(t) + n(t), 0 \leq t \leq T$



$$\text{信源发送信号: } s(t) = 5\sin(t), 0 \leq t \leq T$$

$$\text{接收信号: } x(t) = s(t) + n(t), 0 \leq t \leq T$$



$$\text{信源发送信号}: s(t) = 0, 0 \le t \le T$$

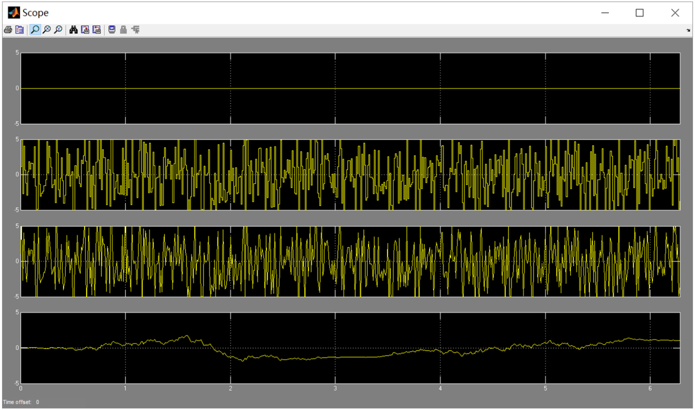
$$\text{接收信号}: x(t) = s(t) + n(t), 0 \le t \le T$$

$s(t)$

$n(t)$

H_0

H_1



简单二元信号波形检测—步骤归纳

1. 首先,利用随机过程的正交级数展开,将随机过程用一组随机变量来表示;
2. 然后,针对展开得到的随机变量,利用第三章的统计检测方法,构建贝叶斯检测表达式;
3. 最后,令 N 趋向于无穷大, 利用展开系数与随机过程之间的表示关系,构建波形信号的检测表达式。

$$l[x(t)] \overset{def}{=} \int_0^T s(t)x(t)dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln\eta + \frac{E_s}{2} \overset{def}{=} \gamma$$

简单二元信号波形检测—步骤归纳

$$x_k = \int_0^T x(t) f_k(t)\,dt$$

$$s_k = \int_0^T s(t) f_k(t)\,dt$$

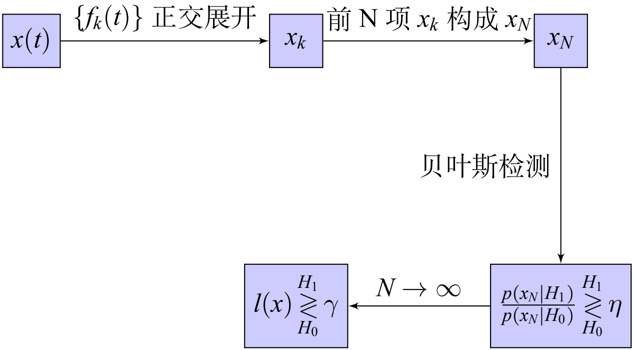
$$n_k = \int_0^T n(t) f_k(t)\,dt$$

$$x(t) = \mathop{\lim}\limits^{N}_{N\to\infty} \sum_{k=1}^{N} x_k f_k(t)$$

$$\lambda(\boldsymbol{x}_N) = \frac{p(\boldsymbol{x}_N|H_1)}{p(\boldsymbol{x}_N|H_0)} \mathop{\gtrless}\limits_{H_0}^{H_1} \eta$$

$$\ln \lambda(x(t)) \overset{def}{=} \lim_{N\to\infty} [\lambda(\boldsymbol{x}_N)]$$

$$l[x(t)] \mathop{\gtrless}\limits_{H_0}^{H_1} \gamma$$



$$l[x(t)] \overset{def}{=} \int_0^T s(t)x(t)\,dt \mathop{\gtrless}\limits_{H_0}^{H_1} \frac{N_0}{2} \ln \eta + \frac{E_s}{2} \overset{def}{=} \gamma$$

简单二元信号波形检测要点

$$H_0 : x(t) = n(t)$$
$$H_1 : x(t) = s(t) + n(t)$$
$$x(t) = \lim_{N\to\infty} \sum_{k=1}^{N} x_k f_k(t)$$
$$x_k = \int_0^T x(t) f_k(t) dt, k = 1, 2, \ldots$$
$$s_k = \int_0^T s(t) f_k(t) dt, k = 1, 2, \ldots$$
$$n_k = \int_0^T n(t) f_k(t) dt, k = 1, 2, \ldots$$
$$H_0 : x_k = n_k, k = 1, 2, \ldots$$
$$H_1 : x_k = s_k + n_k, k = 1, 2, \ldots$$
$$\int_0^T s(t)x(t)dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2} \ln\eta + \frac{E_s}{2}$$

- 信号 s(t) 是确知信号,n(t) 是均值为 0, 功率谱密度为 $P_n(\omega) = N_0/2$ 的高斯白噪声;
- 无论在假设 H_1 下还是在假设 H_2 下,接收信号的 $x(t)$ 都是高斯随机过程;
- 展开系数 x_k 是高斯随机过程的积分结果,因而 x_k 是高斯随机变量;
- 展开系数 x_k 之间是互不相关的,也是相互统计独立的;
- 高斯随机变量由均值和方差决定。由此求出两个假设下的概率密度函数 $p(x_k|H_j), k = 1, 2, \ldots; j = 0, 1$。

x_k, s_k, n_k 之间的关系

x_k, s_k, n_k 之间的关系

$$x(t) = s(t) + n(t); \quad x_k = s_k + n_k$$

随机变量 $x(t)$ 展开系数 x_k= 确知信号 $s(t)$ 展开系数 s_k+ 噪声 $n(t)$ 展开系数 n_k

$$\begin{aligned}
x_k &= \int_0^T f_k(t)x(t)dt \\
&= \int_0^T f_k(t)(s(t) + n(t))dt \\
&= \int_0^T f_k(t)s(t)dt + \int_0^T f_k(t)n(t))dt \\
&= s_k + n_k
\end{aligned}$$

简单二元信号波形检测-检测性能 (1)

$H_0 : x(t) = n(t), \quad H_1 : x(t) = s(t) + n(t)$

判决表达式:

$$l[x(t)] \stackrel{def}{=} \int_0^T x(t)s(t)dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln\eta + \frac{E_s}{2} \stackrel{def}{=} \gamma$$

检验统计量 $l[x(t)]$ 无论在假设 H_0 下，还是在假设 H_1 下，都是由高斯随机过程 $x(t)s(t)(0 \le t \le T)$ 经积分得到的，所以**$l[x(t)]$ 是高斯随机变量**。

1. 求出检验统计量 $l[x(t)]$ 在两个假设下的均值 $E(l|H_j)$ 和方差 $Var(l|H_j), j = 0, 1;$
2. 求各种判决概率 $P(H_i|H_j), i, j = 0, 1;$
   简单二元信号检测与雷达信号检测相对应：$P(H_1|H_0) \stackrel{def}{=} P_F$(称为虚警概率)，$P(H_1|H_1) \stackrel{def}{=} P_D$(称为检测概率)
3. 计算检测性能。

简单二元信号波形检测-检测性能 (2)

1. 定义统计量：

$$l[x(t)] \stackrel{def}{=} \int_0^T x(t)s(t)dt$$

2. 假设 H_0, H_1 下检验统计量 $l[x(t)]$ 的均值和方差分别为

$$E[l|H_0] = E\left[\int_0^T x(t)s(t)dt|H_0\right] = E\left[\int_0^T n(t)s(t)dt\right] = 0$$

$$Var[l|H_0] = E[((l|H_0) - E(l|H_0))^2] = \frac{N_0}{2}E_s$$

$$E[l|H_1] = E\left[\int_0^T x(t)s(t)dt|H_1\right] = E\left[\int_0^T (s(t)+n(t))s(t)dt\right] = E_s$$

$$Var[l|H_1] = E[((l|H_1) - E(l|H_1))^2] = \frac{N_0}{2}E_s$$

3. 假设 H_0, H_1 下服从高斯分布的检验统计量 $l[x(t)]$ 的概率密度函数分别为

$$p(l|H_0) = \left(\frac{1}{\pi N_0 E_s}\right)^{1/2} \exp\left(-\frac{l^2}{N_0 E_s}\right), \quad p(l|H_1) = \left(\frac{1}{\pi N_0 E_s}\right)^{1/2} \exp\left(-\frac{(l-E_s)^2}{N_0 E_s}\right)$$

简单二元信号波形检测-检测性能 (3)

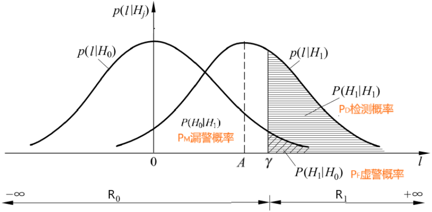
4. 求各种判决概率 $P(H_i|H_j), i,j=0,1$

虚警概率: $P(H_1|H_0) \overset{def}{=} P_F = Q\left[\frac{\ln\eta}{d}+\frac{d}{2}\right]$

检测概率: $P(H_1|H_1) \overset{def}{=} P_D = Q\left[\frac{\ln\eta}{d}-\frac{d}{2}\right]$

$$P(H_0|H_1) = 1-P(H_1|H_1) = 1-Q\left[\frac{\ln\eta}{d}-\frac{d}{2}\right]$$

$$P(H_0|H_0) = 1-P(H_1|H_0) = 1-Q\left[\frac{\ln\eta}{d}+\frac{d}{2}\right]$$



$$d^2 \overset{def}{=} \frac{(E(l|H_1)-E(l|H_0))^2}{Var(l|H_0)} = \frac{2E_s}{N_0}$$ 偏移系数 d^2 表示功率信噪比

结论

对简单二元信号来讲,只要保持确知信号 $s(t)$ 的能量不变,信号波形可以任意设计,检测性能不发生变化。

计算 $E[l|H_0]$ 和 $Var[l|H_0]$

$$l[x(t)] \overset{def}{=} \int_0^T x(t)s(t)dt, \quad E_s = \int_0^T s^2(t)dt, \quad H_0 : x(t) = n(t), \quad E[n(t)] = 0$$

$$E[n(t)n(u)] = r_n(t-u) = \frac{N_0}{2}\delta(t-u) = \frac{N_0}{2}, (t = u, \delta(t-u) = 1)$$

$$E[l|H_0] = E\left[\int_0^T x(t)s(t)dt|H_0\right] = E\left[\int_0^T n(t)s(t)dt\right] = \int_0^T E[n(t)]s(t)dt = 0$$

$$Var[l|H_0] = E[((l|H_0) - E(l|H_0))^2] = E[(l|H_0)^2] = E\left[\left(\int_0^T x(t)s(t)dt\right)^2\right]$$

$$= E\left[\int_0^T n(t)s(t)dt \int_0^T n(t)s(t)dt\right] = E\left[\int_0^T n(t)s(t)dt \int_0^T n(u)s(u)du\right]$$

$$= \int_0^T s(t)\left\{\int_0^T E[n(u)n(t)]s(u)du\right\}dt = \int_0^T s(t)\left[\int_0^T \frac{N_0}{2}\delta(t-u)s(u)du\right]dt$$

$$= \frac{N_0}{2}\int_0^T s(t)\left(\int_0^T s(u)du\right)dt = \frac{N_0}{2}\int_0^T s^2(t)dt = \frac{N_0}{2}E_s$$

计算 $p(l|H_0)$

$$E[l|H_0] = 0, \quad Var[l|H_0] = \frac{N_0}{2}E_s$$

$$\begin{aligned} p(l|H_0) &= \left(\frac{1}{2\pi Var[l|H_0]}\right)^{1/2} \exp\left(-\frac{(l - E[l|H_0])^2}{2Var[l|H_0]}\right) \\ &= \left(\frac{1}{\pi N_0 E_s}\right)^{1/2} \exp\left(-\frac{l^2}{N_0 E_s}\right) \end{aligned}$$

计算 $E[l|H_1]$

$$l[x(t)] \stackrel{def}{=} \int_0^T x(t)s(t)dt, \quad E_s = \int_0^T s^2(t)dt, \quad H_1 : x(t) = s(t) + n(t), \quad E[n(t)] = 0$$

$$\begin{aligned}
E[l|H_1] &= E\left[\int_0^T x(t)s(t)dt|H_1\right] && \text{by } H_1 : x(t) = s(t) + n(t) \\
&= E\left[\int_0^T (s(t) + n(t))s(t)dt\right] \\
&= E\left[\int_0^T s^2(t)dt\right] + \int_0^T E[n(t)]s(t)dt && \text{by } E[n(t)] = 0 \\
&= E\left[\int_0^T s^2(t)dt\right] && \text{by } E_s = \int_0^T s^2(t)dt \\
&= E_s
\end{aligned}$$

计算 $Var[l|H_1]$

$$l[x(t)] \overset{def}{=} \int_0^T x(t)s(t)dt, \quad E_s = \int_0^T s^2(t)dt, \quad H_1 : x(t) = s(t) + n(t), \quad E[n(t)] = 0$$

$$E[n(t)n(u)] = r_n(t-u) = \frac{N_0}{2}\delta(t-u) = \frac{N_0}{2}, (t = u, \delta(t-u) = 1)$$

$$\begin{aligned}
Var[l|H_1] &= E[((l|H_1) - E(l|H_1))^2] = E\left[\left(\int_0^T (s(t)+n(t))s(t)dt - E_s\right)^2\right] \\
&= E\left[\left(\int_0^T (s^2(t)dt + \int_0^T n(t)s(t)dt - E_s\right)^2\right] = E\left[\left(\int_0^T n(t)s(t)dt\right)^2\right] \\
&= E\left[\int_0^T n(t)s(t)dt \int_0^T n(t)s(t)dt\right] = E\left[\int_0^T n(t)s(t)dt \int_0^T n(u)s(u)du\right] \\
&= \int_0^T s(t)\left\{\int_0^T E[n(u)n(t)]s(u)du\right\}dt = \int_0^T s(t)\left[\int_0^T \frac{N_0}{2}\delta(t-u)s(u)du\right]dt \\
&= \frac{N_0}{2}\int_0^T s(t)\left(\int_0^T s(u)du\right)dt = \frac{N_0}{2}\int_0^T s^2(t)dt = \frac{N_0}{2}E_s
\end{aligned}$$

计算 $p(l|H_1)$

$$E[l|H_1] = E_s, \quad Var[l|H_1] = \frac{N_0}{2}E_s$$

$$\begin{aligned} p(l|H_1) &= \left(\frac{1}{2\pi Var[l|H_1]}\right)^{1/2} \exp\left(-\frac{(l - E[l|H_1])^2}{2Var[l|H_1]}\right) \\ &= \left(\frac{1}{\pi N_0 E_s}\right)^{1/2} \exp\left(-\frac{(l - E_s)^2}{N_0 E_s}\right) \end{aligned}$$

计算 $P(H_1|H_0)$

$$p(l|H_0) = \left(\frac{1}{\pi N_0 E_s}\right)^{1/2} \exp\left(-\frac{l^2}{N_0 E_s}\right)$$

$$P(H_1|H_0) \overset{def}{=} P_F = \int_\gamma^\infty p(l|H_0)dl \quad \Longrightarrow \quad Q(x) = \int_x^\infty \left(\frac{1}{2\pi}\right)^{1/2} \exp\left(-\frac{u^2}{2}\right) du$$

$$= \int_\gamma^\infty \left(\frac{1}{\pi N_0 E_s}\right)^{1/2} \exp\left(-\frac{l^2}{N_0 E_s}\right) dl \quad \text{by } l = u\sqrt{N_0 E_s/2}$$

$$= \int_{\frac{\gamma}{\sqrt{N_0 E_s/2}}}^\infty \left(\frac{1}{2\pi}\right)^{1/2} \exp\left(-\frac{u^2}{2}\right) du$$

$$= Q\left[\frac{\gamma}{\sqrt{N_0 E_s/2}}\right] \quad \text{by } \gamma = \frac{N_0}{2}\ln\eta + \frac{E_s}{2}$$

$$= Q\left[\frac{\frac{N_0}{2}\ln\eta + \frac{E_s}{2}}{\sqrt{N_0 E_s/2}}\right] \quad \text{by } d^2 = \frac{2E_s}{N_0}$$

偏移系数 d^2 表示功率信噪比。

$$= Q\left[\frac{\ln\eta}{d} + \frac{d}{2}\right]$$

计算 $P(H_0|H_1)$

$$p(l|H_1) = \left(\frac{1}{\pi N_0 E_s}\right)^{1/2} \exp\left(-\frac{(l-E_s)^2}{N_0 E_s}\right)$$

$$P(H_1|H_1) \overset{def}{=} P_D = \int_{\gamma}^{\infty} p(l|H_1)dl \qquad \Longrightarrow \qquad Q(x) = \int_{x}^{\infty} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left(-\frac{u^2}{2}\right) du$$

$$= \int_{\gamma}^{\infty} \left(\frac{1}{\pi N_0 E_s}\right)^{1/2} \exp\left(-\frac{(l-E_s)^2}{N_0 E_s}\right) dl \quad \text{by } l = u\sqrt{N_0 E_s/2} + E_s$$

$$= \int_{\frac{\gamma - E_s}{\sqrt{N_0 E_s/2}}}^{\infty} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left(-\frac{u^2}{2}\right) du$$

$$= Q\left[\frac{\gamma - E_s}{\sqrt{N_0 E_s/2}}\right] \qquad \text{by } \gamma = \frac{N_0}{2}\ln\eta + \frac{E_s}{2}$$

$$= Q\left[\frac{\frac{N_0}{2}\ln\eta - \frac{E_s}{2}}{\sqrt{N_0 E_s/2}}\right] \qquad \text{by } d^2 = \frac{2E_s}{N_0}$$

偏移系数 d^2 表示功率信噪比。

$$= Q\left[\frac{\ln\eta}{d} - \frac{d}{2}\right]$$

欢迎批评指正！