信号检测与估值

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ch4. 信号波形的检测

ch4-5. 一般二元信号波形的检测—充分统计量的方法

1. 一般二元信号波形的检测—充分统计量的方法
2. 二元信号波形的检测归纳

一般二元信号波形的检测—充分统计量的方法

- **条件:** 功率谱密度为 $P_n(\omega) = N_0/2$ 的高斯白噪声背景中一般二元信号波形检测
- **正交级数展开法:** 信道噪声是白噪声,正交函数集可任意选取。
- **充分统计量法:** 选取特定的正交函数集,使得有关发送信号的信息只包含在有限的展开系数中。

一般二元信号波形的检测—充分统计量的方法

信号模型

$$H_0 : x(t) = s_0(t) + n(t), 0 \le t \le T$$

$$H_1 : x(t) = s_1(t) + n(t), 0 \le t \le T$$

$n(t)$ 为零均值高斯白噪声

正交函数集 $\{f_k(t)\}$ 的构造问题

波形相关系数 ρ:

$$\rho = \frac{1}{\sqrt{E_0 E_1}} \int_0^T s_0(t) s_1(t) dt, \quad (|\rho| \le 1)$$

$\rho = 0$ 时, 信号 $s_0(t)$ 与 $s_1(t)$ 正交。

$\rho \ne 0$ 时, 信号 $s_0(t)$ 与 $s_1(t)$ 不正交。

一般二元信号波形的检测—充分统计量的方法

$H_0 : x(t) = s_0(t) + n(t), 0 \leq t \leq T$

$H_1 : x(t) = s_1(t) + n(t), 0 \leq t \leq T$

(1) 选择一组完备正交函数集,构造两个坐标函数:

第一个坐标函数满足:

$$f_1(t) = \frac{1}{\sqrt{E_1}} s_1(t), \quad E_1 = \int_0^T s_1^2(t)dt$$

$f_1(t)$ 为确知信号 $s_1(t)$ 的**归一化函数**。

其余坐标函数 $f_k(t), k \geq 2$ 是与 $f_1(t)$ 正交, 且两两正交的任意归一化函数, 即

$$f_j(t) \text{和} f_k(t) \text{是正交的}, k \geq 1, j \geq 1, k \neq j$$

格拉姆—施密特正交化法构造 $f_2(t)$

格拉姆—施密特 (Gram—Schmidt) 正交化法构造第二个坐标函数:

利用 $s_0(t)$ 构造与 $f_1(t)$ 正交的信号 $g_2(t)$。

$$
\begin{aligned}
g_2(t) &= s_0(t) - s_{01} f_1(t) \\
&= s_0(t) - \left[\int_0^T s_0(t) f_1(t) dt\right] f_1(t) \\
&= s_0(t) - \left[\int_0^T s_0(t) \frac{1}{\sqrt{E_1}} s_1(t) dt\right] \frac{1}{\sqrt{E_1}} s_1(t) \quad \text{by } f_1(t) = \frac{1}{\sqrt{E_1}} s_1(t) \\
&= s_0(t) - \rho \sqrt{\frac{E_0}{E_1}} s_1(t) \quad \text{by } \rho = \frac{1}{\sqrt{E_0 E_1}} \int_0^T s_0(t) s_1(t) dt
\end{aligned}
$$

格拉姆—施密特法构造 $f_2(t)$

$$g_2(t)=s_0(t)-\rho\sqrt{\frac{E_0}{E_1}}s_1(t),\ \int_0^T s_0^2(t)dt=E_0,\ \int_0^T s_1^2(t)dt=E_1,\ \int_0^T s_0(t)s_1(t)dt=\rho\sqrt{E_0E_1}$$

归一化 $g_2(t)$, 得到**第二个坐标函数:**

$$f_2(t)=\frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)dt}}=\frac{s_0(t)-\rho\sqrt{\frac{E_0}{E_1}}s_1(t)}{\sqrt{\int_0^T\left(s_0(t)-\rho\sqrt{\frac{E_0}{E_1}}s_1(t)\right)^2dt}}$$

$$=\frac{s_0(t)-\rho\sqrt{\frac{E_0}{E_1}}s_1(t)}{\sqrt{\int_0^T\left(s_0^2(t)-2\rho\sqrt{\frac{E_0}{E_1}}s_0(t)s_1(t)+\rho^2\frac{E_0}{E_1}s_1^2(t)\right)dt}}$$

$$=\frac{s_0(t)-\rho\sqrt{\frac{E_0}{E_1}}s_1(t)}{\sqrt{E_0-2\rho\sqrt{\frac{E_0}{E_1}}\rho\sqrt{E_0E_1}+\rho^2\frac{E_0}{E_1}E_1}}=\frac{1}{\sqrt{(1-\rho^2)E_0}}\left(s_0(t)-\rho\sqrt{\frac{E_0}{E_1}}s_1(t)\right)$$

证明: $f_1(t)$ 和 $f_2(t)$ 是正交函数集的前两个坐标函数 (1)

$$f_1(t)=\frac{1}{\sqrt{E_1}}s_1(t),\quad f_2(t)=\frac{1}{\sqrt{(1-\rho^2)E_0}}\left(s_0(t)-\rho\sqrt{\frac{E_0}{E_1}}s_1(t)\right)$$

$$\int_0^T s_0^2(t)dt=E_0,\int_0^T s_1^2(t)dt=E_1,\int_0^T s_0(t)s_1(t)dt=\rho\sqrt{E_0E_1}$$

证明: $f_1(t)$ 和 $f_2(t)$ 满足正交集坐标函数的定义。

(1) 先证明 $f_1(t)$,$f_2(t)$ 是**归一化函数**。因为

$$\int_0^T f_1^2(t)dt=\frac{1}{E_1}\int_0^T s_1^2(t)dt=1$$

$$\begin{aligned}\int_0^T f_2^2(t)dt&=\frac{1}{(1-\rho^2)E_0}\int_0^T\left(s_0(t)-\rho\sqrt{\frac{E_0}{E_1}}s_1(t)\right)^2dt\\&=\frac{1}{(1-\rho^2)E_0}\int_0^T\left(s_0^2(t)-2\rho\sqrt{\frac{E_0}{E_1}}s_0(t)s_1(t)+\rho^2\frac{E_0}{E_1}s_1^2(t)\right)dt\\&=\frac{1}{(1-\rho^2)E_0}\left(E_0-2\rho\sqrt{\frac{E_0}{E_1}}\rho\sqrt{E_0E_1}+\rho^2\frac{E_0}{E_1}E_1\right)=1\end{aligned}$$

证明: $f_1(t)$ 和 $f_2(t)$ 是正交函数集的前两个坐标函数 (2)

$$f_1(t)=\frac{1}{\sqrt{E_1}}s_1(t),\quad f_2(t)=\frac{1}{\sqrt{(1-\rho^2)E_0}}\left(s_0(t)-\rho\sqrt{\frac{E_0}{E_1}}s_1(t)\right)$$

$$\int_0^T s_0^2(t)dt=E_0,\int_0^T s_1^2(t)dt=E_1,\int_0^T s_0(t)s_1(t)dt=\rho\sqrt{E_0E_1}$$

(2) 再证明 $f_1(t),f_2(t)$ 是相互正交的两个函数。因为

$$\begin{aligned}\int_0^T f_1(t)f_2(t)dt&=\int_0^T\frac{1}{\sqrt{E_1}}s_1(t)\frac{1}{\sqrt{(1-\rho^2)E_0}}\left(s_0(t)-\rho\sqrt{\frac{E_0}{E_1}}s_1(t)\right)dt\\&=\frac{1}{\sqrt{(1-\rho)^2E_0E_1}}\left(\int_0^T s_0(t)s_1(t)dt-\rho\sqrt{\frac{E_0}{E_1}}\int_0^T s_1^2(t)dt\right)\\&=\frac{1}{\sqrt{(1-\rho)^2E_0E_1}}\left(\rho\sqrt{E_0E_1}-\rho\sqrt{\frac{E_0}{E_1}}E_1\right)=0\end{aligned}$$

所以,$f_1(t),f_2(t)$ 是相互正交的两个函数。

综上 (1), (2), $f_1(t),f_2(t)$ 是归一化函数, 且满足正交性, 是正交函数集的前两个坐标函数。

$\square$

充分统计量的方法, 选择一组完备正交函数集

$$f_1(t) = \frac{1}{\sqrt{E_1}} s_1(t), \quad 0 \le t \le T$$

$$f_2(t) = \frac{1}{\sqrt{(1-\rho^2)E_0}} \left(s_0(t) - \rho\sqrt{\frac{E_0}{E_1}} s_1(t) \right), \quad 0 \le t \le T$$

其余坐标函数 $f_k(t), k \ge 3$ 是与 $f_1(t)$ 和 $f_2(t)$ 正交, 且两两相互正交的任意归一化函数, 即 $f_j(t)$ 和 $f_k(t)$ 是正交的, $k \ge 1, j \ge 1, k \ne j$

$$\int_0^T f_j(t) f_k(t) dt = 0, \quad k \ge 1, j \ge 1, k \ne j$$

对接收信号进行正交展开 (假设 $H_0 : x_1$)

假设 $H_0 : x(t) = s_0(t) + n(t)$ 下, 展开系数 x_1

$$
\begin{aligned}
x_1 &= \int_0^T x(t) f_1(t) dt = \int_0^T [s_0(t) + n(t)] f_1(t) dt = \int_0^T s_0(t) f_1(t) dt + \int_0^T n(t) f_1(t) dt \\
&= \int_0^T s_0(t) [\frac{1}{\sqrt{E_1}} s_1(t)] dt + n_1 = \frac{1}{\sqrt{E_1}} \int_0^T s_0(t) s_1(t) dt + n_1 \\
&= \rho \sqrt{E_0} + n_1
\end{aligned}
$$

$$
f_1(t) = \frac{1}{\sqrt{E_1}} s_1(t), \quad \int_0^T n(t) f_1(t) dt = n_1
$$

$$
\rho = \frac{1}{\sqrt{E_0 E_1}} \int_0^T s_0(t) s_1(t) dt \implies \int_0^T s_0(t) s_1(t) dt = \rho \sqrt{E_0 E_1}
$$

对接收信号进行正交展开 (假设 $H_0 : x_2$)

假设 $H_0 : x(t) = s_0(t) + n(t)$ 下, 展开系数 x_2

$$
\begin{aligned}
x_2 &= \int_0^T x(t) f_2(t) dt = \int_0^T [s_0(t) + n(t)] f_2(t) dt = \int_0^T s_0(t) f_2(t) dt + \int_0^T n(t) f_2(t) dt \\
&= \int_0^T s_0(t) \frac{1}{\sqrt{(1-\rho^2)E_0}} \left(s_0(t) - \rho \sqrt{\frac{E_0}{E_1}} s_1(t) \right) dt + n_2 \\
&= \frac{1}{\sqrt{(1-\rho^2)E_0}} \left[\int_0^T s_0^2(t) dt - \rho \sqrt{\frac{E_0}{E_1}} \int_0^T s_0(t) s_1(t) dt \right] + n_2 \\
&= \frac{1}{\sqrt{(1-\rho^2)E_0}} \left[E_0 - \rho \sqrt{\frac{E_0}{E_1}} \rho \sqrt{E_0 E_1} \right] + n_2 = \sqrt{(1-\rho^2)E_0} + n_2
\end{aligned}
$$

$$
f_2(t) = \frac{1}{\sqrt{(1-\rho^2)E_0}} \left(s_0(t) - \rho \sqrt{\frac{E_0}{E_1}} s_1(t) \right), \quad E_0 = \int_0^T s_0^2(t) dt, \quad \int_0^T n(t) f_2(t) dt = n_2
$$

$$
\rho = \frac{1}{\sqrt{E_0 E_1}} \int_0^T s_0(t) s_1(t) dt \implies \int_0^T s_0(t) s_1(t) dt = \rho \sqrt{E_0 E_1}
$$

对接收信号进行正交展开 (假设 $H_0 : x_k$)

假设 $H_0 : x(t) = s_0(t) + n(t)$ 下, 展开系数 x_k

$$
\begin{aligned}
x_k &= \int_0^T x(t) f_k(t) dt = \int_0^T [s_0(t) + n(t)] f_k(t) dt = \int_0^T s_0(t) f_k(t) dt + \int_0^T n(t) f_k(t) dt \\
&= 0 + \int_0^T n(t) f_k(t) dt = n_k \quad k \geq 3
\end{aligned}
$$

$$
f_2(t) = \frac{1}{\sqrt{(1-\rho^2)E_0}} \left(s_0(t) - \rho \sqrt{\frac{E_0}{E_1}} s_1(t) \right), s_1(t) = \sqrt{E_1} f_1(t)
$$

$$
\Longrightarrow s_0(t) = \left(\sqrt{(1-\rho^2)E_0} \right) f_2(t) + \left(\rho \sqrt{E_0} \right) f_1(t)
$$

$$
\int_0^T f_j(t) f_k(t) dt = 0, \quad k \geq 1, j \geq 1, k \neq j
$$

对接收信号进行正交展开 (假设 H_1)

假设 $H_1 : x(t) = s_1(t) + n(t)$ 下，展开系数

$$
\begin{aligned}
x_1 &= \int_0^T x(t)f_1(t)dt = \int_0^T [s_1(t) + n(t)]f_1(t)dt = \int_0^T s_1(t)f_1(t)dt + \int_0^T n(t)f_1(t)dt \\
&= \int_0^T s_1(t)[\frac{1}{\sqrt{E_1}}s_1(t)]dt + n_1 = \frac{1}{\sqrt{E_1}}\int_0^T s_1^2(t)dt + n_1 \\
&= \sqrt{E_1} + n_1 \quad (\text{by } f_1(t) = \frac{1}{\sqrt{E_1}}s_1(t), E_1 = \int_0^T s_1^2(t)dt) \\
x_2 &= \int_0^T x(t)f_2(t)dt = \int_0^T [s_1(t) + n(t)]f_2(t)dt = \int_0^T s_1(t)f_2(t)dt + \int_0^T n(t)f_2(t)dt \\
&= \int_0^T [\sqrt{E_1}f_1(t)]f_2(t)dt + n_2 = 0 + n_2 = n_2 \\
x_k &= \int_0^T x(t)f_k(t)dt = \int_0^T [s_1(t) + n(t)]f_k(t)dt = \int_0^T [\sqrt{E_1}f_1(t) + n(t)]f_k(t)dt \\
&= \int_0^T n(t)f_k(t)dt = n_k \quad k \geq 3 \quad (by \quad s_1(t) = \sqrt{E_1}f_1(t), \int_0^T f_1(t)f_k(t)dt = 0, k \geq 3)
\end{aligned}
$$

充分量统计法

(2) 利用构造的正交函数集 $f_1(t), f_2(t)$ 和 $\{f_k(t)|k \geq 3\}$ 对接收信号进行正交展开

两个假设下展开系数 x_1, x_2

$$x_1|H_0 = \int_0^T x(t) f_1(t) dt = \int_0^T [s_0(t) + n(t)] f_1(t) dt = \rho\sqrt{E_0} + n_1$$

$$x_2|H_0 = \int_0^T x(t) f_2(t) dt = \int_0^T [s_0(t) + n(t)] f_2(t) dt = \sqrt{(1-\rho^2)E_0} + n_2$$

$$x_1|H_1 = \int_0^T x(t) f_1(t) dt = \int_0^T [s_1(t) + n(t)] f_1(t) dt = \sqrt{E_1} + n_1$$

$$x_2|H_1 = \int_0^T x(t) f_2(t) dt = \int_0^T [s_1(t) + n(t)] f_2(t) dt = n_2$$

$$H_0, H_1 : x_k = \int_0^T x(t) f_k(t) dt = n_k \quad (k \geq 3) \implies \text{不含确知信号 } s_0(t), s_1(t) \text{ 信息}$$

$\boldsymbol{x} = (x_1, x_2)^T$ **是充分统计量。且 x_1 和 x_2 为高斯随机变量，相互统计独立。**

充分量统计法: x_1, x_2 的均值和方差

$$E[x_1|H_0] = E\left[\rho\sqrt{E_0} + n_1\right] = \rho\sqrt{E_0}$$

$$E[x_2|H_0] = E\left[\sqrt{(1-\rho^2)E_0} + n_2\right] = \sqrt{(1-\rho^2)E_0}$$

$$Var[x_1|H_0] = Var[x_2|H_0] = E[n_1^2] = E[n_2^2] = \frac{N_0}{2}$$

$$E[x_1|H_1] = E\left[\sqrt{E_1} + n_1\right] = \sqrt{E_1}$$

$$E[x_2|H_1] = E[n_2] = 0$$

$$Var[x_1|H_1] = Var[x_2|H_1] = E[n_1^2] = E[n_2^2] = \frac{N_0}{2}$$

充分量统计法—构建似然比

(3) 利用得到的展开系数,构建似然比表达式

$$\boldsymbol{x} = (x_1, x_2)^T$$

$$\lambda(\boldsymbol{x}) = \frac{p(\boldsymbol{x}|H_1)}{p(\boldsymbol{x}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\lambda(\boldsymbol{x}) = \frac{p(x_1, x_2|H_1)}{p(x_1, x_2|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\frac{\frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{\left(x_1 - \sqrt{E_1}\right)^2}{N_0}\right) \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{x_2^2}{N_0}\right)}{\frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{\left(x_1 - \rho\sqrt{E_0}\right)^2}{N_0}\right) \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{\left(x_2 - \sqrt{(1-\rho^2)E_0}\right)^2}{N_0}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

充分量统计法—构建似然比

(3) 利用得到的展开系数,构建似然比表达式

取对数化简

$$\exp\left(\frac{1}{N_0}\left[\left(x_1-\rho\sqrt{E_0}\right)^2+\left(x_2-\sqrt{(1-\rho^2)E_0}\right)^2\right]-\frac{1}{N_0}\left[\left(x_1-\sqrt{E_1}\right)^2+x_2^2\right]\right)\underset{H_0}{\overset{H_1}{\gtrless}}\eta$$

$$\frac{1}{N_0}\left[2\sqrt{E_1}x_1-2\rho\sqrt{E_0}x_1-2\sqrt{(1-\rho^2)E_0}x_2-E_1+E_0\right]\underset{H_0}{\overset{H_1}{\gtrless}}\ln\eta$$

$$\left(\sqrt{E_1}-\rho\sqrt{E_0}\right)x_1-\left(\sqrt{(1-\rho^2)E_0}\right)x_2\underset{H_0}{\overset{H_1}{\gtrless}}\frac{N_0}{2}\ln\eta+\frac{1}{2}(E_1-E_0)$$

检验统计量:

$$\begin{aligned}l[x(t)]&=\left(\sqrt{E_1}-\rho\sqrt{E_0}\right)x_1-\left(\sqrt{(1-\rho^2)E_0}\right)x_2\\&=\left(\sqrt{E_1}-\rho\sqrt{E_0}\right)\int_0^T x(t)f_1(t)dt-\left(\sqrt{(1-\rho^2)E_0}\right)\int_0^T x(t)f_2(t)dt\end{aligned}$$

充分量统计法—构建似然比

(3) 利用得到的展开系数，构建似然比表达式

检验统计量:

$$
\begin{aligned}
l[x(t)] &= \left(\sqrt{E_1} - \rho\sqrt{E_0}\right)\int_0^T x(t)f_1(t)dt - \left(\sqrt{(1-\rho^2)E_0}\right)\int_0^T x(t)f_2(t)dt \\
&= \left(\sqrt{E_1} - \rho\sqrt{E_0}\right)\int_0^T x(t)\frac{1}{\sqrt{E_1}}s_1(t)dt \\
&- \left(\sqrt{(1-\rho^2)E_0}\right)\int_0^T x(t)\frac{1}{\sqrt{(1-\rho^2)E_0}}\left(s_0(t) - \rho\sqrt{\frac{E_0}{E_1}}s_1(t)\right)dt \\
&= \left(1 - \rho\sqrt{\frac{E_0}{E_1}}\right)\int_0^T x(t)s_1(t)dt - \int_0^T x(t)\left(s_0(t) - \rho\sqrt{\frac{E_0}{E_1}}s_1(t)\right)dt \\
&= \int_0^T x(t)s_1(t)dt - \int_0^T x(t)s_0(t)dt
\end{aligned}
$$

充分量统计法—判决表达式

$$l[x(t)] \overset{def}{=} \int_0^T x(t)s_1(t)dt - \int_0^T x(t)s_0(t)dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln\eta + \frac{1}{2}(E_1 - E_0) \overset{def}{=} \gamma$$

结论

由任意正交函数集对 $x(t)$ 进行正交级数展开法与由充分统计量法导出的判决表达式是完全一样的,因而也具有相同的检测系统结构和相同的检测性能。

一般二元信号波形的检测例题 1

考虑发送信号周期为 $T = 2\pi/\omega_0$ 的二元移频键控系统。在假设 H_0 和 H_1 下的发送信号分别为:

$$H_0 : x(t) = a\sin\omega_0 t + n(t), \quad 0 \le t \le T$$
$$H_1 : x(t) = a\sin 2\omega_0 t + n(t), \quad 0 \le t \le T$$

其中, 信号的振幅 a 和频率 ω_0 已知, 并假定两个假设先验等概。信号在传输中叠加了均值为零, 功率谱密度为 $N_0/2$ 的高斯白噪声 $n(t)$。

现采用最小平均错误概率准则, 设计信号检测系统, 并计算平均错误概率 P_e。

一般二元信号波形的检测例题 1: 解

解: 根据题设, 得到两个确知信号的能量分别为

$$E_0 = \int_0^T s_0^2(t)dt = \int_0^T (a\sin\omega_0 t)^2 dt = \frac{a^2T}{2}$$

$$E_1 = \int_0^T s_1^2(t)dt = \int_0^T (a\sin 2\omega_0 t)^2 dt = \frac{a^2T}{2}$$

$$E_s \overset{def}{=} E_0 = E_1 = \frac{a^2T}{2}$$

由于两个假设的先验概率等概, 因此在最小平均错误概率准则下, 判决门限 $\eta = 1$, 利用一般二元信号检测波形判决表达式, 得

$$l[x(t)] \overset{def}{=} \int_0^T x(t)s_1(t)dt - \int_0^T x(t)s_0(t)dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln\eta + \frac{1}{2}(E_1 - E_0) = 0$$

$$\int_0^T x(t)a\sin 2\omega_0 t dt - \int_0^T x(t)a\sin\omega_0 t dt \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

一般二元信号波形的检测例题 1: 解 (续 1)

为求平均错误概率, 首先需要计算偏移系数 d^2

$$d^2 \stackrel{def}{=} \frac{(E[l|H_1] - E[l|H_0])^2}{Var[l|H_0]}$$

$$l[x(t)] = \int_0^T x(t)a\sin 2\omega_0 t dt - \int_0^T x(t)a\sin\omega_0 t dt$$

$$\begin{aligned}
E[l|H_0] &= E\left[\int_0^T x(t)a\sin 2\omega_0 t dt - \int_0^T x(t)a\sin\omega_0 t dt|H_0\right] \quad \text{by } x(t) = a\sin\omega_0 t + n(t) \\
&= E\left[\int_0^T [a\sin\omega_0 t + n(t)]a\sin 2\omega_0 t dt - \int_0^T [a\sin\omega_0 t + n(t)]a\sin\omega_0 t dt\right] \\
&= E\left[a^2\int_0^T \sin\omega_0 t\sin 2\omega_0 t dt\right] + \int_0^T E[n(t)]a\sin 2\omega_0 t dt \\
&\quad - E\left[\int_0^T (a\sin\omega_0 t)^2 dt\right] - \int_0^T E[n(t)]a\sin\omega_0 t dt \quad \text{by } E[n(t)] = 0 \\
&= 0 + 0 - E\left[\int_0^T (a\sin\omega_0 t)^2 dt\right] - 0 = -E_0 = -E_s
\end{aligned}$$

一般二元信号波形的检测例题 1: 解 (续 2)

$$H_0 : x(t) = a\sin\omega_0 t + n(t), \quad E[l|H_0] = -E_s$$

$$Var[l|H_0] = E\left[((l|H_0) - E[l|H_0])^2\right]$$

$$= E\left[\left(\int_0^T x(t)a\sin 2\omega_0 tdt - \int_0^T x(t)a\sin\omega_0 tdt|H_0 + E_s\right)^2\right]$$

$$= E\left[\left(\int_0^T [a\sin\omega_0 t + n(t)]a\sin 2\omega_0 tdt - \int_0^T [a\sin\omega_0 t + n(t)]a\sin\omega_0 tdt + E_s\right)^2\right]$$

$$= E\left[\left(0 + \int_0^T n(t)a\sin 2\omega_0 tdt - E_0 - \int_0^T n(t)a\sin\omega_0 tdt + E_s\right)^2\right]$$

$$= E\left[\left(\int_0^T n(t)a\sin 2\omega_0 tdt - \int_0^T n(t)a\sin\omega_0 tdt\right)^2\right]$$

$$= E\left[\left(\int_0^T n(t)a\sin 2\omega_0 tdt\right)^2\right] + E\left[\left(\int_0^T n(t)a\sin\omega_0 tdt\right)^2\right]$$

$$- 2E\left[\int_0^T n(t)a\sin 2\omega_0 tdt\int_0^T n(u)a\sin\omega_0 udu\right]$$

一般二元信号波形的检测例题 1: 解 (续 3)

$$E\left[\left(\int_0^T n(t)a\sin 2\omega_0 t dt\right)^2\right] = E\left[\int_0^T n(t)a\sin 2\omega_0 t dt \int_0^T n(u)a\sin 2\omega_0 u du\right]$$

$$= \int_0^T a\sin 2\omega_0 t\left[\int_0^T E[n(t)n(u)]a\sin 2\omega_0 u du\right] dt \quad \text{by } E[n(t)n(u)] = \frac{N_0}{2}\delta(t-u)$$

$$= \int_0^T a\sin 2\omega_0 t\left[\int_0^T \frac{N_0}{2}\delta(t-u)a\sin 2\omega_0 u du\right] dt \quad \text{by } \delta \text{ 的筛选性}$$

$$= \int_0^T a\sin 2\omega_0 t\left[\frac{N_0}{2}a\sin 2\omega_0 t\right] dt = \frac{N_0}{2}\int_0^T (a\sin 2\omega_0 t)^2 dt = \frac{N_0 E_1}{2}$$

类似地有， $$E\left[\left(\int_0^T n(t)a\sin \omega_0 t dt\right)^2\right] = \frac{N_0 E_0}{2}$$

$$E\left[\int_0^T n(t)a\sin 2\omega_0 t dt \int_0^T n(u)a\sin \omega_0 u du\right]$$

$$= \int_0^T a\sin 2\omega_0 t\left[\int_0^T E[n(t)n(u)]a\sin \omega_0 u du\right] dt$$

$$= \frac{a^2 N_0}{2}\int_0^T \sin 2\omega_0 t \sin \omega_0 t dt = 0$$

一般二元信号波形的检测例题 1: 解 (续 4)

$$
\begin{aligned}
Var[l|H_0] &= E\left[\left(\int_0^T n(t)a\sin 2\omega_0 t dt\right)^2\right] + E\left[\left(\int_0^T n(t)a\sin\omega_0 t dt\right)^2\right] \\
&\quad - 2E\left[\int_0^T n(t)a\sin 2\omega_0 t dt \int_0^T n(u)a\sin 2\omega_0 u du\right] \\
&= \frac{N_0E_1}{2} + \frac{N_0E_0}{2} - 0 \\
&= N_0E_s
\end{aligned}
$$

一般二元信号波形的检测例题 1: 解 (续 5)

$$E[l|H_1] = E\left[\int_0^T x(t) a \sin 2\omega_0 t dt - \int_0^T x(t) a \sin \omega_0 t dt | H_1\right] \quad \text{by } x(t) = a \sin 2\omega_0 t + n(t)$$

$$= E\left[\int_0^T [a \sin 2\omega_0 t + n(t)] a \sin 2\omega_0 t dt - \int_0^T [a \sin 2\omega_0 t + n(t)] a \sin \omega_0 t dt\right]$$

$$= E\left[\int_0^T (a \sin 2\omega_0 t)^2 dt\right] + \int_0^T E[n(t)] a \sin 2\omega_0 t dt$$

$$- E\left[a^2 \int_0^T \sin 2\omega_0 t \sin \omega_0 t dt\right] - \int_0^T E[n(t)] a \sin \omega_0 t dt \quad \text{by } E[n(t)] = 0$$

$$= E\left[\int_0^T (a \sin 2\omega_0 t)^2 dt\right] + 0 - 0 - 0 = E_1 = E_s$$

$$Var[l|H_1] = Var[l|H_0] = N_0 E_s$$

一般二元信号波形的检测例题 1: 解 (续 6)

$$d^2 \overset{def}{=} \frac{(E[l|H_1] - E[l|H_0])^2}{Var[l|H_0]} = \frac{(E_s - (-E_s))^2}{N_0 E_s} = \frac{4E_s}{N_0}$$

判决概率为

$$P(H_1|H_0) \overset{def}{=} P_F = \int_\gamma^\infty p(l|H_0)dl = Q\left(\frac{\ln\eta}{d} + \frac{d}{2}\right) = Q\left(\frac{d}{2}\right) = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

$$P(H_1|H_1) \overset{def}{=} P_D = \int_\gamma^\infty p(l|H_1)dl = Q\left(\frac{\ln\eta}{d} - \frac{d}{2}\right) = Q\left(-\sqrt{\frac{E_s}{N_0}}\right)$$

$$P(H_0|H_1) = 1 - P(H_1|H_1) = 1 - Q\left(-\sqrt{\frac{E_s}{N_0}}\right) = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

$$\begin{aligned} P_e &= P(H_0)P(H_1|H_0) + P(H_1)P(H_0|H_1) \\ &= \frac{1}{2}P(H_1|H_0) + \frac{1}{2}P(H_0|H_1) \\ &= Q\left(\sqrt{\frac{E_s}{N_0}}\right) \end{aligned}$$

一般二元信号波形的检测例题 2

设连续相位移频键控通信系统, 在假设 H_0 和 H_1 下的发送信号分别为:

$$H_0 : x(t) = a \sin \omega_0 t + n(t), \quad 0 \le t \le T$$

$$H_1 : x(t) = a \sin \omega_1 t + n(t), \quad 0 \le t \le T$$

其中, 信号的振幅 a 和频率 ω_0, ω_1 已知, 并假定两个假设先验等概。信号在传输中叠加了均值为零, 功率谱密度为 $N_0/2$ 的高斯白噪声 $n(t)$。

问使最小平均错误概率 P_e 最小的两个信号的差频 $\omega_d = \omega_1 - \omega_0$ 为多少?

一般二元信号波形的检测例题 2: 解

解: 根据题设, 得到两个确知信号的能量分别为

$$E_0 = \int_0^T s_0^2(t)dt = \int_0^T (a\sin\omega_0 t)^2 dt = \frac{a^2 T}{2}$$

$$E_1 = \int_0^T s_1^2(t)dt = \int_0^T (a\sin\omega_1 t)^2 dt = \frac{a^2 T}{2}$$

$$E_s \overset{def}{=} E_0 = E_1 = \frac{a^2 T}{2}$$

由于两个假设的先验概率等概, 因此在最小平均错误概率准则下, 判决门限 $\eta = 1$, 利用一般二元信号检测波形判决表达式, 得

$$l[x(t)] \overset{def}{=} \int_0^T x(t)s_1(t)dt - \int_0^T x(t)s_0(t)dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln\eta + \frac{1}{2}(E_1 - E_0) = 0$$

$$\int_0^T x(t)a\sin\omega_1 t dt - \int_0^T x(t)a\sin\omega_0 t dt \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

一般二元信号波形的检测例题 2: 解 (续 1)

为求平均错误概率, 首先需要计算偏移系数 d^2

$$d^2 \overset{def}{=} \frac{(E[l|H_1]-E[l|H_0])^2}{Var[l|H_0]}$$

$$l[x(t)] = \int_0^T x(t)s_1(t)dt - \int_0^T x(t)s_0(t)dt$$

$$\begin{aligned}
E[l|H_0] &= E\left[\int_0^T x(t)s_1(t)dt - \int_0^T x(t)s_0(t)|H_0\right] \quad \text{by } x(t)=s_0(t)+n(t) \\
&= E\left[\int_0^T [s_0(t)+n(t)]s_1(t)dt - \int_0^T [s_0(t)+n(t)]s_0(t)dt\right] \\
&= E\left[\int_0^T s_0(t)s_1(t)dt\right] + \int_0^T E[n(t)]s_1(t)dt - E\left[\int_0^T s_0^2(t)dt\right] - \int_0^T E[n(t)]s_0(t)dt \quad \text{by } E[n(t)]=0 \\
&= E\left[\int_0^T s_0(t)s_1(t)dt\right] - E\left[\int_0^T s_0^2(t)dt\right] \quad \text{by } E_0 = \int_0^T s_0^2(t)dt \\
&= \rho\sqrt{E_0E_1} - E_0 = \rho E_s - E_s \quad \text{by } E_0=E_1=E_s
\end{aligned}$$

相关系数 $\rho = \dfrac{1}{\sqrt{E_0E_1}}\displaystyle\int_0^T s_0(t)s_1(t)dt = \dfrac{1}{E_s}\int_0^T s_0(t)s_1(t)dt, \quad (|\rho| \le 1)$

一般二元信号波形的检测例题 2: 解 (续 2)

$$H_0: x(t) = a\sin\omega_0 t + n(t), \quad E[l|H_0] = \rho E_s - E_s$$

$$\rho = \frac{1}{\sqrt{E_0E_1}}\int_0^T s_0(t)s_1(t)dt = \frac{1}{E_s}\int_0^T s_0(t)s_1(t)dt, \quad (|\rho| \le 1)$$

$$Var[l|H_0] = E\left[((l|H_0) - E[l|H_0])^2\right]$$

$$= E\left[\left(\int_0^T x(t)s_1(t)dt - \int_0^T x(t)s_0(t)dt|H_0 - \rho E_s + E_s\right)^2\right]$$

$$= E\left[\left(\int_0^T [s_0(t)+n(t)]s_1(t)dt - \int_0^T [s_0(t)+n(t)]s_0(t)dt - \rho E_s + E_s\right)^2\right]$$

$$= E\left[\left(\int_0^T s_0(t)s_1(t)dt + \int_0^T n(t)s_1(t)dt - \int_0^T s_0^2(t)dt - \int_0^T n(t)s_0(t)dt - \rho E_s + E_s\right)^2\right]$$

$$= E\left[\left(\rho E_s + \int_0^T n(t)s_1(t)dt - E_0 - \int_0^T n(t)s_0(t)dt - \rho E_s + E_s\right)^2\right] = E\left[\left(\int_0^T n(t)s_1(t)dt - \int_0^T n(t)s_0(t)dt\right)^2\right]$$

$$= E\left[\left(\int_0^T n(t)s_1(t)dt\right)^2\right] + E\left[\left(\int_0^T n(t)s_0(t)dt\right)^2\right] - 2E\left[\int_0^T n(t)s_1(t)dt\int_0^T n(u)s_0(t)du\right]$$

一般二元信号波形的检测例题 2: 解 (续 3)

$$E\left[\left(\int_0^T n(t)s_1(t)dt\right)^2\right] = E\left[\int_0^T n(t)s_1(t)dt \int_0^T n(u)s_1(u)du\right]$$

$$= \int_0^T s_1(t)\left[\int_0^T E[n(t)n(u)]s_1(u)du\right]dt \quad \text{by } E[n(t)n(u)] = \frac{N_0}{2}\delta(t-u)$$

$$= \int_0^T s_1(t)\left[\int_0^T \frac{N_0}{2}\delta(t-u)s_1(u)du\right]dt \quad \text{by } \delta \text{ 的筛选性}$$

$$= \int_0^T s_1(t)\left[\frac{N_0}{2}s_1(t)\right]dt = \frac{N_0}{2}\int_0^T s_1^2(t)dt = \frac{N_0E_1}{2}$$

类似地有， $$E\left[\left(\int_0^T n(t)s_0(t)dt\right)^2\right] = \frac{N_0E_0}{2}$$

$$E\left[\int_0^T n(t)s_1(t)dt \int_0^T n(u)s_0(u)du\right] = \int_0^T s_1(t)\left[\int_0^T E[n(t)n(u)]s_0(u)du\right]dt$$

$$= \frac{N_0}{2}\int_0^T s_1(t)s_0(t)dt = \frac{N_0\rho E_s}{2}$$

一般二元信号波形的检测例题 2: 解 (续 4)

$$Var[l|H_0] = E\left[\left(\int_0^T n(t)s_1(t)dt\right)^2\right] + E\left[\left(\int_0^T n(t)s_0(t)dt\right)^2\right] - 2E\left[\int_0^T n(t)s_1(t)dt \int_0^T n(u)s_0(t)du\right]$$

$$= \frac{N_0E_1}{2} + \frac{N_0E_0}{2} - N_0\rho E_s = \frac{N_0}{2}(E_s + E_s - 2\rho E_s) = N_0E_s(1-\rho)$$

$$E[l|H_1] = E\left[\int_0^T x(t)s_1(t)dt - \int_0^T x(t)s_0(t)|H_1\right] \quad \text{by } x(t) = s_1(t) + n(t)$$

$$= E\left[\int_0^T [s_1(t) + n(t)]s_1(t)dt - \int_0^T [s_1(t) + n(t)]s_0(t)dt\right]$$

$$= E\left[\int_0^T s_1^2(t)dt\right] + \int_0^T E[n(t)]s_1(t)dt - E\left[\int_0^T s_1(t)s_0(t)dt\right] - \int_0^T E[n(t)]s_0(t)dt$$

$$= E\left[\int_0^T s_1^2(t)dt\right] - E\left[\int_0^T s_1(t)s_0(t)dt\right] \quad \text{by } E_1 = \int_0^T s_1^2(t)dt$$

$$= E_1 - \rho\sqrt{E_0E_1} = E_s - \rho E_s \quad \text{by } E_0 = E_1 = E_s$$

相关系数 $\rho = \frac{1}{\sqrt{E_0E_1}}\int_0^T s_0(t)s_1(t)dt = \frac{1}{E_s}\int_0^T s_0(t)s_1(t)dt, \quad (|\rho| \le 1)$

一般二元信号波形的检测例题 2: 解 (续 5)

$$d^2 \overset{def}{=} \frac{(E[l|H_1] - E[l|H_0])^2}{Var[l|H_0]} = \frac{(E_s - \rho E_s - (\rho E_s - E_s))^2}{N_0 E_s (1-\rho)} = \frac{4E_s}{N_0}(1-\rho)$$

判决概率为

$$P(H_1|H_0) \overset{def}{=} P_F = \int_\gamma^\infty p(l|H_0)dl = Q\left[\frac{\ln\eta}{d} + \frac{d}{2}\right] = Q\left[\frac{d}{2}\right] = Q\left[\sqrt{\frac{E_s}{N_0}(1-\rho)}\right]$$

$$P(H_1|H_1) \overset{def}{=} P_D = \int_\gamma^\infty p(l|H_1)dl = Q\left[\frac{\ln\eta}{d} - \frac{d}{2}\right] = Q\left[-\sqrt{\frac{E_s}{N_0}(1-\rho)}\right]$$

$$P(H_0|H_1) = 1 - P(H_1|H_1) = 1 - Q\left[-\sqrt{\frac{E_s}{N_0}(1-\rho)}\right] = Q\left[\sqrt{\frac{E_s}{N_0}(1-\rho)}\right]$$

$$\begin{aligned} P_e &= P(H_0)P(H_1|H_0) + P(H_1)P(H_0|H_1) \\ &= \frac{1}{2}P(H_1|H_0) + \frac{1}{2}P(H_0|H_1) \\ &= Q\left[\sqrt{\frac{E_s}{N_0}(1-\rho)}\right] \end{aligned}$$

一般二元信号波形的检测例题 2: 解 (续 6)

$$P_e = Q\left[\sqrt{\frac{E_s}{N_0}(1-\rho)}\right]$$

由上式可知, 为使平均错误概率最小, 需使 $\frac{E_s}{N_0}(1-\rho)$ 取得最小值, 又由于

$$\begin{aligned}
\rho &= \frac{1}{\sqrt{E_0E_1}}\int_0^T s_0(t)s_1(t)dt = \frac{1}{E_s}\int_0^T s_0(t)s_1(t)dt \\
&= \frac{a^2}{E_s}\int_0^T \sin\omega_1 t\sin\omega_0 t dt \quad \text{by } E_s = \frac{a^2T}{2} \\
&= \frac{2}{T}\int_0^T \sin\omega_1 t\sin\omega_0 t dt \\
&= \frac{1}{T}\int_0^T \cos(\omega_1-\omega_0)tdt - \frac{1}{T}\int_0^T \cos(\omega_1+\omega_0)tdt \\
&= \frac{1}{T}\cdot\frac{1}{\omega_1-\omega_0}\sin(\omega_1-\omega_0)t\Big|_0^T = \frac{\sin\omega_d T}{\omega_d T}
\end{aligned}$$

一般二元信号波形的检测例题 2: 解 (续 7)

$$\rho = \frac{\sin \omega_d T}{\omega_d T}$$

由上式可知, 相关系数与频率差有关, 令

$$\frac{\partial \rho}{\partial \omega_d} = \frac{1}{T} \cdot \frac{(\cos \omega_d T)\omega_d T - \sin \omega_d T}{\omega_d^2} = 0$$

得到满足方程 $\tan \omega_d T = \omega_d T$

使平均错误概率最小的两个信号的频率差

$$\omega_d T = 1.41\pi$$

对应的相关系数为

$$\rho_{min} = -0.21$$

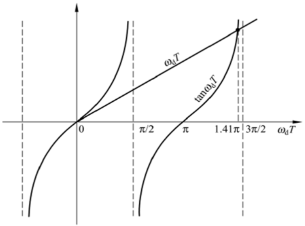

图4.14超越方程$\tan \omega_d T = \omega_d T$的解

一般二元信号波形的检测例题 2: 解 (续 8)

为了实现简单, 通常采用正交信号, 此时相关系数为 0.

为了获得与相关系数为 -0.21 时相同的错误概率, 需要改变发送信号的能量, 即

$$\left.\sqrt{\frac{E_s}{N_0}(1-\rho)}\right|_{\rho=-0.21} = \sqrt{\frac{E_s'}{N_0}}$$

$$E_s' = 1.21E_s$$

二元信号波形的检测归纳 (1)

1. **基本检测方法 (正交级数展开法)**
   - 首先,利用随机过程的正交级数展开,将随机过程用一组随机变量来表示;
   - 然后,针对展开得到的随机变量,取前 N 个展开系数,利用第三章的统计检测方法,构建贝叶斯检测表达式 (白高斯噪声条件下,展开系数是不相关的,也是独立的);
   - 最后,令 N 趋向于无穷大,求极限,得到波形信号的检测表达式。
2. **充份量统计法**
   - 利用信源发送信号,构建一组特殊的正交函数集,可以利用有限个展开系数构建波形检测表达式。

二元信号波形的检测归纳 (2)

检测性能与偏移系数有关

$$d^2 \overset{def}{=} \frac{\left(E[l|H_1] - E[l|H_0]\right)^2}{Var[l|H_0]}$$

简单二元信号

$$d^2 = \frac{2E_s}{N_0}$$

一般二元信号

$$d^2 = \frac{2}{N_0}\left(E_1 + E_0 - 2\rho\sqrt{E_1 E_0}\right)$$

二元信号波形的检测归纳 (3)

- 对简单二元信号,只要保持信号 $s(t)$ 的能量不变,信号波形可以**任意**设计,检测性能不发生变化。
- 在高斯白噪声条件下,对于确知一般二元信号的波形检测,当两个信号设计成**互反信号**时,可在信号能量给定的约束下获得最好的检测性能。

二元信号波形的检测归纳 (4)

一般二元信号波形检测的判决表达式为

$$l[x(t)] \overset{def}{=} \left(\sqrt{E_1} - \rho\sqrt{E_0}\right) x_1 - \left(\sqrt{(1-\rho^2)E_0}\right) x_2 \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2} \ln\eta + \frac{1}{2}(E_1 - E_0)$$

当判决表达式去等号时,

$$\left(\sqrt{E_1} - \rho\sqrt{E_0}\right) x_1 - \left(\sqrt{(1-\rho^2)E_0}\right) x_2 \overset{def}{=} \gamma$$

由该方程确定的直线,是判定假设 H_0 成立还是判决假设 H_1 成立的分界线。

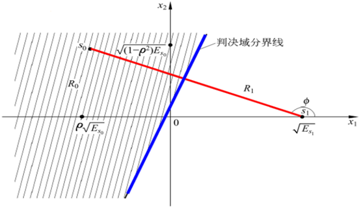

图4.15判决域划分示意图

二元信号波形的检测归纳 (5)

$$x_2 = \frac{\sqrt{E_1} - \rho\sqrt{E_0}}{\sqrt{(1-\rho^2)E_0}} x_1 - \frac{\gamma}{\sqrt{(1-\rho^2)E_0}}$$

直线的斜率为:

$$k_x = \frac{\sqrt{E_1} - \rho\sqrt{E_0}}{\sqrt{(1-\rho^2)E_0}}$$

信号 $s_0(t)$ 与 $s_1(t)$ 差向量为:

$$\begin{aligned} s_0(t) - s_1(t) &= \rho\sqrt{E_0} f_1(t) + \sqrt{(1-\rho^2)E_0} f_2(t) - \sqrt{E_1} f_1(t) \\ &= \left(\rho\sqrt{E_0} - \sqrt{E_1}\right) f_1(t) + \sqrt{(1-\rho^2)E_0} f_2(t) \end{aligned}$$

差向量曲线的斜率为:

$$k_s = \tan\phi = \frac{\sqrt{(1-\rho^2)E_0}}{\rho\sqrt{E_0} - \sqrt{E_1}}$$

$$k_x k_s = -1$$

判决区域分界线是垂直于信号间连线的一条直线。

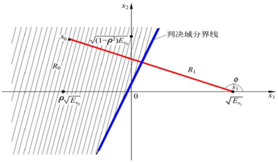

图4.15判决域划分示意图

二元信号波形的检测归纳 (6)

如果二元信号假设的先验概率相等,并采用最小平均错误概率准则,则判决域的分界线应满足如下方程:

$$\exp\left(\frac{1}{N_0}\left[\left(x_1-\rho\sqrt{E_0}\right)^2+\left(x_2-\sqrt{(1-\rho^2)E_0}\right)^2\right]-\frac{1}{N_0}\left[\left(x_1-\sqrt{E_1}\right)^2+x_2^2\right]\right)\underset{H_0}{\overset{H_1}{\gtrless}}\eta=1$$

$$\left(x_1-\rho\sqrt{E_0}\right)^2+\left(x_2-\sqrt{(1-\rho^2)E_0}\right)^2=\left(x_1-\sqrt{E_1}\right)^2+x_2^2$$

即判决域的分界线是信号 $s_0(t)$ 与信号 $s_1(t)$ 连线的垂直平分线。

如果进一步假设两个信号能量相等,则有:

$$\sqrt{1-\rho}x_1=\sqrt{1+\rho}x_2$$

判决域的分界线是信号 $s_0(t)$ 与信号 $s_1(t)$ 连线的垂直平分线,并通过原点。

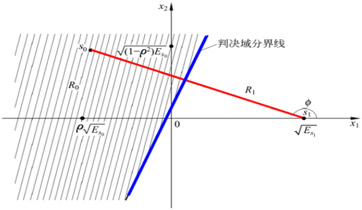

图4.15判决域划分示意图

欢迎批评指正！