# Chapter 1

# automata abstract

## 1.1 Finite automata

**Definition 1.1 (Finite automation).** A finite automaton(an FA) is a 6-tuple $(Q,V,T,E,S,F)$ where

- $Q$ is a finite set of states,
- $V$ is an alphabet,
- $T \in \mathbb{P}(Q \times V \times Q)$ is a transition relation,
- $E \in \mathbb{P}(Q \times Q)$ is an $\varepsilon$-transition relation
- $S \subseteq Q$ is a set of start states, and
- $F \subseteq Q$ is a set of final states.

□

```
class FA: virtual public FAabs {
        // Q is a finite set of states
        StatePool Q;
        // S is a set of start states, F is a
            set of final states
        StateSet S, F;
        // Transitions maps each State to its
            out-transitions.
        TransRel Transitions;
```

```
        // E is the epsilon transition
           relation.
        StateRel E;
}
```

**StatePool**:All states in an automaton are allocated from a StatePool. StatePool's can be merged together to form a larger one. (Care must be taken to rename any relations or sets (during merging) that depend on the one StatePool.) State is in [0,next)

```
class StatePool {
        int next; // The next one to be
           allocated.
}
```

**StateSet**:The StateSet is normally associated (implicitly) with a particular StatePool; whenever a StuteSet is to interact with another (from a different StatePool), its States must be renamed (to avoid a name clash). The capacity of a StateSet must be explicitly managed; many set operations are not bounds-checked when assert() is turned off.

```
class StateSet :protected BitVec {
        // How many States can this set
           contain?
        // [O, domain()) can be contained in
           *this.
        inline int domain() const;

        // set How many States can this set
           contain.
        // [O, r) can be contained in *this.
        inline void set_domain(const int r);
}
class BitVec {
```

```
            // uesd max number bits in data,
               denote width(domain),[0,
               bits_in_use) ==> [0,width)
            int bits_in_use;
            // number of words, 1,2,3,...
            int words;
            // save bytes of words,[0,1,2,...
               width(domain)]
            unsigned int *data;
    }
```

**transition relation:** $T \in Q \rightarrow \mathbb{P}(V \times Q), T(p) = \{(a,q)|(p,a,q) \in T\}$, 表示状态 $p$ 的 out-transitions. see Fig 1.1

```
    // V ---> Q
    struct TransPair {
    CharRange transition_label;
    State transition_destination;
    }
    class TransImpl { TransPair *data; }
    class Trans:protected TransIml { }

    // map: state(r) ---> (T=Trans) out-
        transitions of r
    // SteteTo::data[r] = out-transitions of
        state r
    class TransRel:public StateTo<Trans> {}

    // map: state(r) ---> T
    // data[r] = T
    template <class T> class SteteTo {
    T *data; // 动态数组的index(即状态的index)状
        态的out-transitions
    }
```

```
class FA:virtual public FAabs {
TransRel Transitions;// maps each State to
    its out-transitions.
}
```

**ε-relation**: $E \in \mathbb{P}(Q \times Q) \Rightarrow E \in Q \to \mathbb{P}(Q), E(p) = \{q|(p,q) \in E\}$, 表示 $\varepsilon$ 连接状态 $p$ 和状态 $q$.

```
// Implement binary relations on States. This
     is most often used for epsilon
    transitions.
// map: state(r) --> {StateSet}
// StateTo::data[r] = {StateSet}, 表示状态r与
    {StateSet}的二元关系
class StateRel :protected StateTo<StateSet> {
}

class FA:virtual public FAabs {
// E is the epsilon transition relation.
StateRel E;
}
```

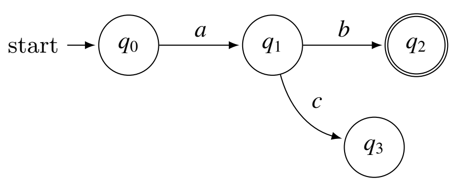



图 1.1: $q_1$ in-transition: $\{(q_0,a,q_1)\}$; $q_1$ out-transition: $\{(q_1,b,q_2),(q_1,c,q_3)\}$

[WATSON93a, p6] **the signatures of the transition relations:**

$T \in \mathbb{P}(Q \times V \times Q)$

$T \in V \to \mathbb{P}(Q \times Q)$

$T \in Q \times Q \to \mathbb{P}(V)$

$T \in Q \times V \to \mathbb{P}(Q)$
$T \in Q \to \mathbb{P}(V \times Q)$

for example, the function $T \in Q \to \mathbb{P}(V \times Q)$ is defined as $T(p) = \{(a,q) : (p,a,q) \in T\}$

$T \in \mathbb{P}(Q \times V \times Q), T = \{(p,a,q)\}$
$T \in Q \to \mathbb{P}(V \times Q), T(p) = \{(a,q) : (p,a,q) \in T\}$

$p,q \in Q, a \in V$
$T : Q \times V \to \mathbb{P}(Q)$
$T(p,a) = \{q\}$

According to Convention A.4 (Tuple projection):
$T \in \mathbb{P}(Q \times V \times Q), T = \{(p,a,q)\}$
$\pi_2(T) = \{a|(p,a,q) \in T\}, \bar{\pi}_2(T) = \{(p,q)|(p,a,q) \in T\}$
$T \in Q \to \mathbb{P}(V \times Q), T(p) = \{(a,q) : (p,a,q) \in T\}$
$\pi_2(T(p)) = \{q|(p,a,q) \in T\}, \bar{\pi}_2(T(p)) = \{a|(p,a,q) \in T\}$
$Q_{map} : Q \times V, T(p) = \{(a,q) : (p,a,q) \in T\}$
$Q_{map}(q) = \{a\}$
$Q_{map} : Q \times V, T \in \mathbb{P}(Q \times V \times Q$
$\pi_1(T) = \{p|(p,a,q) \in T\}, \bar{\pi}_1(T) = \{(a,q)|(p,a,q) \in T\}$
$Q_{map} = (\bar{\pi}_1(T))^R = \{(a,q)|(p,a,q) \in T\}^R = \{(q,a)|(p,a,q) \in T\}$

## 1.2 Properties of finite automata

$M = (Q,V,T,E,S,F), M_0 = (Q_0,V_0,T_0,E_0,S_0,F_0), M_1 = (Q_1,V_1,T_1,E_1,S_1,F_1)$

**Definition 1.2 (Size of an *FA*).** Define the size of an *FA* as $|M| = |Q|$

**Definition 1.3 (Isomorphism 同构 ($\cong$) of *FA′s*).** We define isomorphism ($\cong$) as an equivalence relation on FA's. $M_0$ and $M_1$ are isomorphic (written $M_0 \cong M_1$) if and only if $V_0 = V_1$ and there exists a bijection 双射 $g \in Q_0 \to Q_1$ such that

- $T_1 = \{(g(p), a, g(q)) | (p, a, q) \in T_0\}$
- $E_1 = \{(g(p), g(q)) | (p, q) \in E_0\}$
- $S_1 = \{g(s) | s \in S_0\}$ and
- $F_1 = \{g(f) | f \in F_0\}$

(see Fig 1.2). □

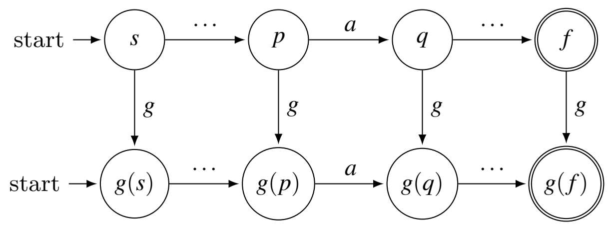


图 1.2: Isomorphism$M_0 \cong M_1$ if and only if $V_0 = V_1$ and there exists a bijection $g \in Q_0 \to Q_1$

**Definition 1.4 (Extending the transition relation $T$).** We extend transition relation $T \in V \to \mathbb{P}(Q \times Q)$ to $T^* \in V^* \to \mathbb{P}(Q \times Q)$ as follows:

$T^*(\varepsilon) = E^*$

and (for $a \in V, w \in V^*$)

$T^*(aw) = E^* \circ T(a) \circ T^*(w)$

Operator $\circ$ (composition is defined in Convention 1).

This definition could also have been presented symmetrically. □

*Note 1.1.* $s_1, s_2, s_3, s_4 \in Q, a \in V, w \in V^*$

$E = T(\varepsilon) = \{(s_1, s_2)\}, T(a) = \{(s_2, s_3)\}, T^*(w) = \{(s_3, s_4)\}$

$$T^*(aw) = E^* \circ T(a) \circ T^*(w)$$
$$= \{(s_1, s_2)\} \circ \{(s_2, s_3)\} \circ \{(s_3, s_4)\} \quad = \{(s_1, s_3)\} \circ \{(s_3, s_4)\} = \{(s_1, s_4)\}$$

*Note 1.2.* $T \in Q \times V \to \mathbb{P}(Q)$, extend to: $T^* \in Q \times V^* \to \mathbb{P}(Q)$

$\forall q \in Q, w \in V^*, a \in V$,

1. $T^*(q, \varepsilon) = q$
2. $T^*(q, wa) = T(T^*(q, w), a)$

$$\begin{aligned} T^*(q,a) &= T^*(q,\varepsilon a) \\ &= T(T^*(q,\varepsilon),a) \\ &= T(q,a) \end{aligned}$$

两值相同，不用区分这两个符号。

**Convention 1 (Relation composition)** *Given sets $A,B,C$ (not necessarily different) and two relations, $E \subseteq A \times B$ and $F \subseteq B \times C$, we define relation composition (infix operator 中缀操作符 $\circ$) as:*

$E \circ F = \{(a,c) | (\exists b \in B), (a,b) \in E \wedge (b,c) \in F)\}$ □

*Note 1.3.* if $\exists b \in B, (a,b) \in E, (b,c) \in F$, then

$E : A \to B \Rightarrow E(a) = b$

$F : B \to C \Rightarrow F(b) = c$

$E \circ F = \{(a,b)\} \circ \{(b,c)\} = \{a,c\}$

$$\begin{aligned} (E \circ F)(a) &= F(E(a)) \\ &= F(b) = c \end{aligned}$$

$$\begin{tikzcd} A \arrow[r, "E"] \arrow[dr, "f"'] & B \arrow[d, "F"] \\ & C \end{tikzcd}$$

图 1.3: $E \circ F = (F \circ E)(a) = F(E(a)) = c = f(a)$

*Remark 1.1.* We also sometimes use the signature $T^* \in Q \times Q \to \mathbb{P}(V^*)$ □

*Note 1.4.* $T(p,q) = \{w | p,q \in Q, w \in V^*\}$

*Remark 1.2.* if $E = \emptyset$ then $E^* = \emptyset^* = I_Q$ where $I_Q$ is the identity relation 单位关系 on the states of $M$.

**Definition 1.5 (The language between states).** The language between any two states $q_0, q_1 \in Q$ is $T^*(q_0, q_1)$. □

**Definition 1.6 (Left and right languages).** The left language of a state (in $M$) is given by function, $\overleftarrow{L}_M \in Q \to \mathbb{P}(V^*)$,where

$\overleftarrow{L}_M(q) = (\cup s : s \in S : T^*(s,q))$

The right language of a state (in $M$) is given by function $\overrightarrow{L}_M \in Q \to \mathbb{P}(V^*)$, where

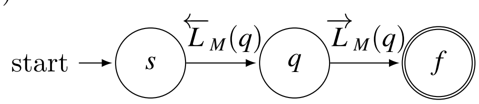
$\overrightarrow{L}_M(q) = (\cup f : f \in F : T^*(q,f))$

The subscript $M$ is usually dropped when no ambiguity can arise.□

*Example 1.1.* $T^* \in Q \times Q \to \mathbb{P}(V^*), \overleftarrow{L}_M, \overrightarrow{L}_M \in Q \to \mathbb{P}(V^*)$.

$\overleftarrow{L}_M(q)$ ={能引导 M 从开始状态到达 q 状态的字符串集合}, (从 $q$ 往左看)

$\overrightarrow{L}_M(q)$ ={能引导 M 从开始状态到达 q 状态的字符串集合}, (从 $q$ 往右看)




see Fig 1.4.

$$\begin{aligned}
\overleftarrow{L}_M(q_2) &= (s \to q_1 \to q_2) \cup (s \to (q_1 \to q_3)^* \to q_1 \to q_2) \cup (s \to (q_1 \to q_3)^* \to q_3 \to q_2) \\
&= [(s \to q_1 \to q_2) \cup (s \to (q_1 \to q_3)^* \to q_1 \to q_2)] \cup (s \to (q_1 \to q_3)^* \to q_3 \to q_2) \\
&= (s \to (q_1 \to q_3)^* \to q_1 \to q_2) \cup (s \to (q_1 \to q_3)^* \to q_3 \to q_2) \\
&= \{1(10)^*0, 1(10)^*1\} \\
\overrightarrow{L}_M(q_2) &= \{01^*0, 10^*1(001^*0 + (10)^*1)\}
\end{aligned}$$

**Definition 1.7 (Language of an $FA$).** The language of a finite automaton (with alphabet $V$) is given by the function $L_{FA} \to \mathbb{P}(V^*)$ defined as:

$L_{FA}(M) = (\cup s, f : s \in S \wedge f \in F : T^*(s,f))$ (所有从开始状态到接受状态的字符串集合) □

*Property 1.1 (Language of an FA).* From the definition of left and right languages (of a state), we can also write:

$L_{FA}(M) = (\cup f : f \in F : \overleftarrow{L}(f))$ (所有从 s 到 f 的字符串集合，从 f 向左看)

and

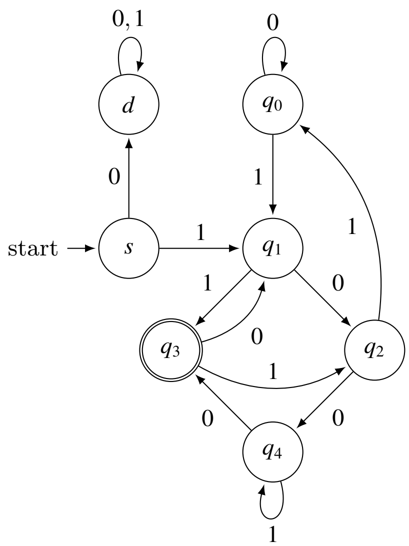



图 1.4: $\{x|x \in \{0,1\}^{+}$且当把 $x$ 看成二进制数时，$x$ 模 5 与 3 同余，要求当 $x$ 为 0 时，$|x| = 1$, 且当 $x \neq 0$ 时，$x$ 的首字符为 1$\}$ 语言对应的 *DFA*

$L_{FA}(M) = (\cup s : s \in S : \overrightarrow{L}(s))$ (所有从 s 到 f 的字符串集合，从 s 向右看) □

**Definition 1.8 ($\varepsilon$-$free$ 无 $\varepsilon$ 转移).** Automaton $M$ is $\varepsilon$-$free$ if and only if $E = \emptyset$. □

*Remark 1.3.* Even if $M$ is $\varepsilon$-$free$ it is still possible that $\varepsilon \in L_{FA}(M) : in this case S \cap F \neq \emptyset$. (开始状态也是接受状态) □

Form [WATSON93a, Convention A.4] (Tuple projection).

**Convention 2 (Tuple projection)** *For an* $n$*-tuple* $t = (x_1, x_2, \ldots, x_n)$ *we use the notation* $\pi_i(t)(1 \le i \le n)$ *to denote tuple element* $x_i$*; we use the notation* $\bar{\pi}_i(t)(1 \le i \le n)$ *to denote the* $(n-1)$*-tuple* $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$*. Both* $\pi$ *and* $\bar{\pi}$ *extend naturally to sets of tuples.* □

Form [WATSON93a, Definition A.20] (Tuple and relation reversal).

**Definition 1.9 (Tuple and relation reversal).** For an $n$-tuple $(x_1, x_2, \ldots, x_n)$ define reversal as (postfix and superscript) function R:

$$(x_1, x_2, \ldots, x_n)^R = (x_n, x_n - 1, \ldots, x_2, x_1)$$

Given a set $A$ of tuples, we define $A^R = \{x^R : x \in A\}$. □

**Definition 1.10 (Reachable states).** For $M$ we can define a reachability relation $Reach(M) \subseteq (Q \times Q)$ defined as

$$Reach(M) = (\bar{\pi}_2(T) \cup E)^* \text{ see}^1$$

Functions $\pi$ and $\bar{\pi}$ are defined in Convention 2. Similarly the set of start-reachable states is defined to be:

$$SReachable(M) = Reach(M)(S) \text{ see}^2$$

and the set of final-reachable states is defined to be:

$$FReachable(M) = (Reach(M))^R(F) \text{ see}^3$$

Reversal of a relation is defined in Definition 1.9. The set of useful states is: $Reachable(M) = SReachable(M) \cap FReachable(M)$ □

*Remark 1.4.* For $FA$ $M = (Q, V, T, E, S, F)$,function $SReachable$ satisfies the following interesting property:

$$q \in SReachable(M) \equiv \overleftarrow{L}_M(q) \neq \emptyset$$

$FReachable$ satisfies a similar property:

$$q \in FReachable(M) \equiv \overrightarrow{L}_M(q) \neq \emptyset$$ □

*Example 1.2.* $T \in \mathbb{P}(Q \times V \times Q), T = \{(p,a,q) | p,q \in Q, a \in V\}$,

$$\bar{\pi}_2(T) = \{(p,q) | (p,a,q) \in T\}$$

$$Q_{map} = (\bar{\pi}_1(T))^R, Q_{map} = \{(a,q) | (p,a,q) \in T\}^R = \{(q,a) | (p,a,q) \in T\}$$

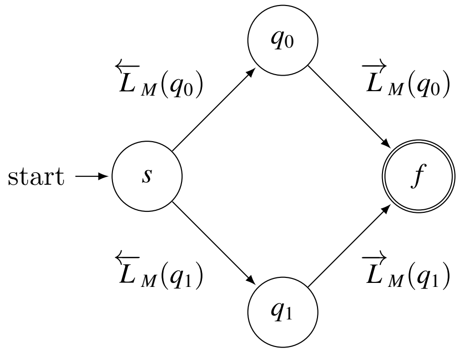
□

---

[1] $\{(p_1, q_1), (p_2, q_2), \ldots\}$

[2] 从 start state 可以到达的状态集合

[3] 可以到达 final state 的状态集合




e.g. $p = \{1,2\} \in Q_1 \subseteq \mathbb{P}(Q_0), \overrightarrow{L}_{M_1}(p) = \overrightarrow{L}_{M_0}(1) \cup \overrightarrow{L}_{M_0}(2)$

$$\begin{tikzcd} Q \arrow[r, "subset"] \arrow[dr, "f"'] & P(Q) \arrow[d, "useful_s"] \\ & D \end{tikzcd}$$

图 1.5: $subset \circ usefus_s = usefus_s(subset(Q,V,\emptyset,S,F)) = (D,V,T',\emptyset,S',F')$

## 1.3 $\Sigma$-algebras and regular expressions

$\Sigma$-homomorphism

$X$ 集合中的元素与有序集 $S$ 中的元素一一对应，称 $X$ 是 $S$-sorted.
$S = \{1,3,7,9\}, X = \{d,a,c,f\}, s \in S, X_s \in X$
$S$ 是有序的,$S_{s_1} = 1, S_{s_2} = 3, S_{s_3} = 7, S_{s_4} = 9$
$X$ 与 $S$ 中的元素一一对应。$X_{s_1} = d, X_{s_2} = a, X_{s_3} = c, X_{s_4} = f$

$\Sigma$-homomorphism 同态: $(V,F) \Leftrightarrow (W,G)$，载体 $(V,W)$ 和操作 $(F,G)$ 一一对应。
$\Sigma$-homomorphism function: $h \in V \to W$

$L(v) = (h \circ f)(v) = h(f(v)) = g(w) = L_{reg} = L_V = L_W$
$L(v) = (g \circ h)(v) = g(h(v)) = g(w) = L_{reg} = L_V = L_W$
$\Rightarrow h(f(v)) = g(h(v))$

$$
\begin{tikzcd}
V \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, "L"] & h(v) \arrow[d, "g"] \\
f(v) \arrow[r, "h"] & g(h(v))
\end{tikzcd}
$$

图 1.6: $(h \circ f)(v) = (g \circ h)(v) \Rightarrow h(f(v)) = g(h(v))$

$$
\begin{tikzcd}
(e_1, e_2) \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, "L"] & (h(e_1), h(e_2)) \arrow[d, "g"] \\
f(e_1, e_2)\circ \arrow[r, "h"] & g(h(e_1), h(e_2))
\end{tikzcd}
$$

图 1.7: $(h \circ f)(e_1, e_2) = (g \circ h)(e_1, e_2) \Rightarrow h(f(e_1, e_2)) = g(h(e_1), h(e_2))$

*Example 1.3.* $\Sigma = (S, \Gamma)$, sort: $expr$, $\Gamma := \{a, plus\}, a$ is a constant. operator $plus : expr \times expr \to expr$.
$\Sigma-$term algebra: $plus[a, a], plus[plus[a, plus[a, a]], a]$
$\Sigma-$algebra $X$,carrier set: natural number, constant 0. operator $f_{plus}(x, y) = (x \quad max \quad y) + 1$
$\Sigma$-homomorphism function(”expression tree height”): $h_{expr} : \Sigma-$term algebra $\to X$

$$
\begin{aligned}
(h_{expr} \circ plus)(s) &= (f_{plus} \circ h_{expr})(s) \\
h_{expr}(plus(s)) &= f_{plus}(h_{expr}(s)) \\
left : s \leftarrow e, f \quad &\Rightarrow \quad plus[e, f] \\
right : s \leftarrow e, f \quad &\Rightarrow \quad f_{plus}(h_{expr}(e), h_{expr}(f) \\
h_{expr}(plus(e, f)) &= f_{plus}(h_{expr}(e), h_{expr}(f)) \\
&= (h_{expr}(e) \quad max \quad h_{expr}(f)) + 1) \\
and, & \\
h_{expr}(a) &= 0
\end{aligned}
$$

$$
\begin{tikzcd}
S \arrow[r, "h_{expr}"] \arrow[d, "plus"'] \arrow[dr, "L"] & X \arrow[d, "f_{plus}"] \\
\circ \arrow[r, "h_{expr}"] & \circ
\end{tikzcd}
$$

图 1.8: $(h_{expr} \circ plus)(s) = (f_{plus} \circ h_{expr})(s) \Rightarrow h_{expr}(plus(s)) = f_{plus}(h_{expr}(s))$

$$
\begin{tikzcd}
(e, f) \arrow[r, "h_{expr}"] \arrow[d, "plus"'] \arrow[dr, "L"] & (h_{expr}(e), h_{expr}(f)) \arrow[d, "f_{plus}"] \\
plus[e, f] \arrow[r, "h_{expr}"] & f_{plus}(h_{expr}(e), h_{expr}(f))
\end{tikzcd}
$$

图 1.9: $(h_{expr} \circ plus)(e, f) = (f_{plus} \circ h_{expr})(e, f) \Rightarrow h_{expr}(plus[e, f]) = f_{plus}(h_{expr}(e), h_{expr}(f))$

**Definition 1.11 (Regular expressions).** We define regular expressions (over alphabet $V$) as the $\Sigma-$term algebra over signature $\Sigma = (S, O)$ where

- $S$ consists of a single sort Reg (for regular expression), and
- $O$ is a set of several constans: $\varepsilon, \emptyset, a_1, a_2, \ldots, a_n; Reg$ (where $V = \{a_1, a_2, \ldots, a_n\}$) and five operators $. : Reg \times Reg \rightarrow Reg$ (the dot operator), $\cup : Reg \times Reg \rightarrow Reg, * : Reg \rightarrow Reg, + : Reg \rightarrow Reg,$ and$? : Reg \rightarrow Reg$.

$V := RE$(正则表达式), $W := FA$(有限自动机)

$\Sigma$-homomorphism function: $f \in RE \to FA$

$F : RE_{opt}$ 运算，二元: union(or),concat; 一元: star,plus,question;

常量:epsilon,empty,symbol

$G : FA_{opt}$ 运算, 同上

$$\begin{tikzcd} RE \arrow[r, "f"] \arrow[dr, "L_{RE}"'] & FA \arrow[d, "L_{FA}"] \\ & L_{reg} \end{tikzcd} \qquad \begin{tikzcd} (re_1, re_2) \arrow[r, "f"] \arrow[d, "RE_{opt}"'] \arrow[dr] & (f(re_1), f(re_2)) \arrow[d, "FA_{opt}"] \\ RE_{opt}(re_1, re_2) \arrow[r, "f"] & FA_{opt}(f(re_1), f(re_2)) \end{tikzcd}$$

```
//Sigma.h
template<class T>
class Reg :public T {
// Helper for constructing the homomorphic
   image of a regular expression.
// T is carrier set: RE,FA,RFA,
// 各自的操作，分别在Sig-RE.cpp,Sig-FA.cpp,
   Sig-RFA.cpp中定义
inline void homomorphic_image(const RE& r);
Reg<T>& epsilon();
Reg<T>& empty();
Reg<T>& symbol(const CharRange r);
Reg<T>& Or(const Reg<T>& r);
Reg<T>& concat(const Reg<T>& r);
Reg<T>& star();
Reg<T>& plus();
Reg<T>& question();
}
```

**Definition 1.12 (The *nullable* $\Sigma-$algebra).** We define the *nullable* $\Sigma-$algebra as follows:

- The carrier set is $\{true, false\}$.

- $a \in V, E_1, E_2 \in RE, \varepsilon \in E_1^*, \varepsilon \in E_1^?, \varepsilon \notin E_1^+$

$$\begin{aligned}
nullable(\varepsilon) &= true \\
nullable(\emptyset) &= nullable(a) = false \\
nullable(E_1 \vee E_2) &= nullable(E_1 \cup E_2) \\
nullable(E_1 \wedge E_2) &= nullable(E_1 \cdot E_2) \\
nullable(E_1^*) &= true \\
nullable(E_1^+) &= nullable(E_1) \\
nullable(E_1^?) &= true \\
nullable(E_1) &= \begin{cases} true & \varepsilon \in E_1 \\ false & \varepsilon \notin E_1 \end{cases}
\end{aligned}$$

## 1.4 Constructing $\varepsilon$-lookahead automata

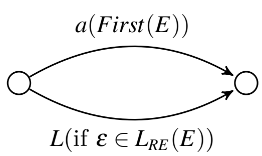



图 1.10: Lookahead

function:$look(E,L) = First \cup$ if $(Null(E))$ then $L$ else $\emptyset$ fi

## 1.5 Towards the Berry-Sethi construction

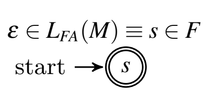



图 1.11: $\varepsilon \in L_{FA}(M) \equiv s \in F$

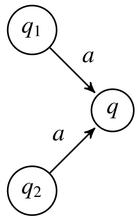



图 1.12: All in-transitions to a state are on the same symbol (in V)

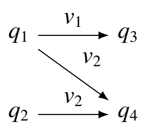
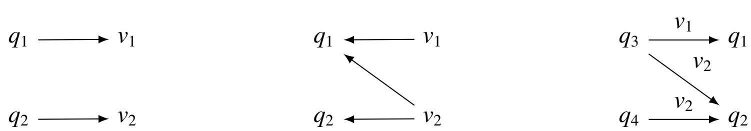



图 1.13: $q_4(in-transition) = \{(q_1,b),(q_2,b)\}, q_1(out-transition) = \{(a,q_3),(b,q_4)\}$

| a | b | c |
|---|---|---|
| | | |




(a) $Q \to V$ 一一对应关系 (b) $V \nrightarrow \mathbb{P}(Q)$ 一对多的关系 (c) 物理含义, 进入状态的字母是唯一的。

$Q_{map}(q_1) = \{v_1\}, Q^{-1}_{map}(v_1) = \{q_1\}, (q_3,v_1,q_1) \in T$

$Q_{map}(q_2) = \{v_2\}, Q^{-1}_{map}(v_2) = \{q_2\}, (q_3,v_2,q_2) \in T, (q_4,v_2,q_2) \in T$

图 1.14: $Q_{map}, Q^{-1}_{map}$

**Definition 1.13 (RFA).** A reduced FA (RFA) is a 7-tuple $(Q,V,follow,first,last,null,Q_{map})$ where

- $Q$ is a finite set of states,
- $V$ is an alphabet,

- $follow \in \mathbb{P}(Q \times Q)$ is a follow relation (relpace the transition relation: $T \in \mathbb{P}(Q \times V \times Q)$) ,
- $first \subseteq Q$ is a set of initial states (replacing T(s) in an LBFA),
- $null \in \{true, false\}$ is a Boolean value (encoding $s \in F$ in an LBFA,$\varepsilon \in L_{FA}(M) \equiv s \in F$), and
- $Q_{map} \in \mathbb{P}(Q \times V), Q_{map}(q) = \{v\}, one \to one$. maps each state to exactly one symbol. i.e. $Q_{map} \in Q \to V$.
  $Q_{map}(q) = \{a | (p,a,q) \in T\}$ 表示 (q,v) 的一一对应关系。物理含义是进入 $q$ 状态的唯一字母 $a$
  class RFA 中表示 its inverse: $Q_{map}^{-1} : V \nrightarrow \mathbb{P}(Q)$,部分函数$Q_{map}^{-1}(a) = \{q | (p,a,q) \in T\}$

□

```
class RFA : virtual public FAabs{
// Q is a finite set of states
StatePool Q;

// first(subset Q) is a set of initial states
   (replacing T(s) in an LBFA),
// last(subset Q) is a set of final states,
StateSet first , last;

// Qmap (in P( Q x V)) maps each state to
   exactly one symbol (it is also viewed as
   Qmap in Q --> V,
// and its inverse as Qmap^-1 in V --/-->P(Q)
   [the set of all partial functions from V
   to P(Q)]).
// Trans用struct TransPair 表示:T(a) = { q |
   (p,a,q) in T },
// 因此这里表示Qmap的inverse, V --> P(Q)
Trans Qmap_inverse;
```

```
// follow(in P(Q x Q)) is a follow relation(
    replacing the transition relation),
StateRel follow;

// null (in {true,false}) is a Boolean value
    (encoding s in F in an LBFA)
// if epsilon属于LBFA, true; final set中包含s
// {true, flase} ==> {1, 0}
int Nullable;
}
```

```
// V —> Q
struct TransPair {
        CharRange transition_label;
        State transition_destination;
}

class TransImpl { TransPair *data; }
class Trans:protected TransIml { }

}
```

$rfa \circ R(E) = R \circ rfa(E)$

$$\begin{tikzcd} E \arrow[r, "rfa"] \arrow[d, "R"'] \arrow[dr] & RFA \arrow[d, "R"] \\ \circ \arrow[r, "rfa"'] & \circ \end{tikzcd}$$

图 1.15: $rfa \circ R(E) = R \circ rfa(E)$

**Definition 1.14.** (Dual of a function) We assume two sets $A$ and $B$ whose reversal operators are $R$ and $R'$ respectively. Two functions, $f \in A \to B$ and $f_d \in A \to B$ are one another's dual if and only if

$f(a) = (f_d(a^R))^{R'}$

In some cases we relax the equality to isomorphism (when isomorphism is defined on $B$). □

$f_d \circ R(a) = R' \circ f(a) \Rightarrow f_d(R(a)) = R'(f(a)) \Rightarrow$
$f_d(a^R) = (f(a))^{R'} \Rightarrow f(a) = (f_d(a^R))^{R'}$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "R"'] \arrow[dr] & B \arrow[d, "R'"] \\
\circ \arrow[r, "f_d"'] & \circ
\end{tikzcd}$$

图 1.16: $f(a) = (f_d(a^R))^{R'})$

## 1.6 The Berry-Sethi construction

$C_{\cdot,RFA}(rfa(\$), rfa(E)) = C_{\cdot,RFA}(C_{\$,RFA}, rfa(E)) = L_{RE}(\$E) = \{\$\}L_{RE}(E)$
*covert* 简单剔除 $E$ 的第一个字符。$L_{FA} \circ convert \circ rfa(E) = V^{-1}L_{RE}(E)$
$convert(C_{\cdot,RFA}(C_{\$,RFA}, rfa(E)) = rfa(E)$

$$\begin{tikzcd}
RE \arrow[r, "rfa"] \arrow[dr, "L_{RE}"'] & RFA \arrow[d, "L_{FA}"] \\
 & L_{reg}
\end{tikzcd}
\qquad
\begin{tikzcd}
(\$, E) \arrow[r, "rfa"] \arrow[d, "RE\cdot"'] \arrow[dr] & (rfa(\$), rfa(E)) \arrow[d, "FA\cdot"] \\
\$ \cdot E \arrow[r, "rfa"] & rfa(\$) \cdot rfa(E)
\end{tikzcd}$$

图 1.17: $convert(C_{\cdot,RFA}(C_{\$,RFA}, rfa(E)) = rfa(E)$

Algorithm 2.45(impements $useful_s \circ subset$):
initial: $D = \emptyset, U = S$

$d := \bigcup_{q \in u} T(q,a)$

$$\{q_1, q_2\} \xrightarrow{a} \{T(q_1,a), T(q_2,a)\}$$

using Algorithm 2.45 for decode(RFA→ LBFA)

$d := \bigcup\{q | q \in first \wedge Q_{map}(q) = a\}$

$$\{s\} \xrightarrow{a} \{d\}$$

note:

$d := \bigcup_{p \in u}\{q | (p,q) \in follow \wedge Q_{map}(q) = a\}$

$u = \{p_1, p_2\}, d = \{q_1, q_2\},$

$follow(p_1) = q_1, follow(p_2) = q_2, Q_{map}(q_1) = Q_{map}(q_2) = a$

$$\{p_1, p_2\} \xrightarrow{a} \{q_1, q_2\}$$

## 1.7 The McNaughton-Yamada-Clushkov construction

$RE \rightarrow [DFA]_{\cong}$

$MYG(E) = useful_s \circ subset \circ decode \circ rfa(E)$

$$\begin{tikzcd}
RE \arrow[r, "rfa"] & {[RFA]_{\cong}} \arrow[r, "decode"] & {[NFA]_{\cong}} \arrow[d, "subset"] \\
{[DFA]_{\cong}} & {[DFA]_{\cong}} \arrow[l, "Complete"'] & P(Q) \arrow[l, "useful_s"']
\end{tikzcd}$$

图 1.18: $MYG(E) = useful_s \circ subset \circ decode \circ rfa(E)$

Dual construction: $R \circ f \circ R$

$$RE \xrightarrow{R} RE \xrightarrow{f} FA \xrightarrow{R} FA$$

$f : FA$ construction

$$\begin{tikzcd} RE \arrow[r, "f"] \arrow[dr, "L_{RE}"'] & FA \arrow[d, "L_{FA}"] \\ & L_{reg} \end{tikzcd}$$

图 1.19: Dual construction: $R \circ f \circ R$

## 1.8 The dual of the Berry-Sethi construction

$R \circ R$ is the identity $\Rightarrow R \circ R(A) = A$

## 1.9 Algorithm 4.52 (Aho-Sethi-Ullman)

note:

$d := \bigcup_{q \in u} \{follow(q) | Q_{map}(q) = a\}$

$$q \xrightarrow{a} follow(q)$$

note:

$T_0(b) = (p, p'), T_1(b) = (q, q')$

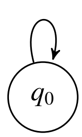
$\pi_2(T_0(b)) = p', \pi_2(T_1(b)) = q'$

$T_0(s_0, a) = p, T_1(s_1, a) = q$

$Q' = \{q_0\} \cup (\bigcup_{b \in V)} \{\pi_2(T_0(b) \times \pi_2(T_1(b))\}$

$$M_0 : s_0 \xrightarrow{a} p \xrightarrow{b} p'$$

$$M_1 : s_1 \xrightarrow{a} q \xrightarrow{b} q'$$




notes:

$[V^*]_{R_L} = V^*/R_L$ 表示右不变的等价关系，每个等价关系对应一个状态。

$[\varepsilon]_{R_L}$ 表示 $\varepsilon$ 所在的等价类对应的状态, 就是开始状态

$T_0(b) = (p, p'), T_1(b) = (q, q')$

$\pi_2(T_0(b)) = p', \pi_2(T_1(b)) = q'$

$T_0(s_0, a) = p, T_1(s_1, a) = q$

$Q' = \{q_0\} \cup \bigcup_{b \in V} (\pi_2(T_0(b)) \times \pi_2(T_1(b)))$

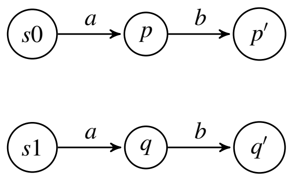



图 1.20: Intersection of LBFA's

$Q' = \bigcup_{b \in V} (Q^{-1}_{map_0}(b) \times Q^{-1}_{map_1}(b))$

$follow' = \{(p, q), (p', q')\}$

$first' = \{(s_0, s_1)\}$

$last' = \{(last_0 \times last_1) \cap Q'\}$

$null' = null_0 \wedge null_1$

$Q'^{-1}_{map}(a) = \{(Q'^{-1}_{map_0}(a) \times Q'^{-1}_{map_1}(a))\}$

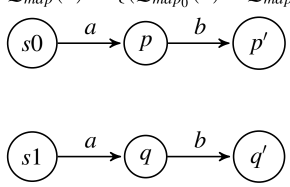



图 1.21: Intersection of RFA's

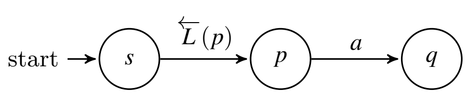



图 1.22: text111

$T([w]_E, a) = \{[wa]_E\}$

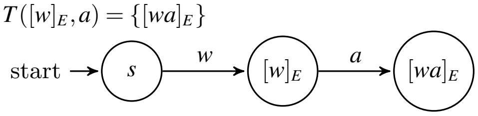



图 1.23: $T([w]_E, a) = \{[wa]_E\}$

$T([w]_{R_L}, a) = \{[wa]_{R_L}\}, (wa)^{-1}L = a^{-1}(w^{-1}L), \varepsilon^{-1}L = L$

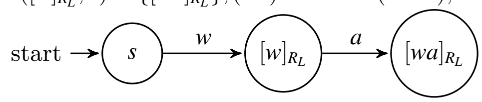



图 1.24: $T([w]_{R_L}, a) = \{[wa]_{R_L}\}$

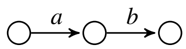



图 1.25: $a, b \in V, (a, b) \in Follow(E)$

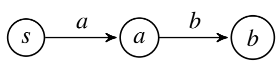



图 1.26: $BSenc(E)$

$\circ \xrightarrow{a} \circ q_0$

图 1.27: text

$[V^*]_E = V^*/R_E$ 表示右不变的等价关系，每个等价关系对应一个状态。
$[\varepsilon]_E$ 表示 $\varepsilon$ 所在的等价类对应的状态, 就是开始状态

$$\begin{aligned}
First((a \cup \varepsilon)b^*) &= (Defn. \quad First(E \cdot F), \varepsilon \in (a \cup \varepsilon), Null(E) = true) \\
&= (\Rightarrow First(E) \cup First(F)) \\
&= First(a \cup \varepsilon) \cup First(b^*) \\
&= (Defn. \quad First(E \cup F) = First(E) \cup First(F), First(E^*) = First(E)) \\
&= (First(a) \cup First(\varepsilon)) \cup First(b) \\
&= \{a \cup \emptyset\} \cup \{b\} = \{a, b\}
\end{aligned}$$

$$\begin{aligned}
First((a\cup\varepsilon)b^*) &= First(ab^*\cup\varepsilon b^*)\\
&= (Defn. \quad Fitst(E\cup F) = First(E)\cup First(F))\\
&= First(ab^*)\cup First(b^*)\\
&= (Defn. \quad First(E\cdot F), \varepsilon\notin\{a\}, Null(E) = false \Rightarrow First(ab^*) = First(a)\cup\emptyset = \{a\})\\
&= First(a)\cup First(b) = \{a,b\}
\end{aligned}$$

$$\begin{aligned}
Last((a\cup\varepsilon)b^*) &= (Defn. \quad Last(E\cdot F), \varepsilon\in(a\cup\varepsilon), Null(E) = true)\\
&= (\Rightarrow Last(E)\cup Last(F))\\
&= Last(a\cup\varepsilon)\cup Last(b^*)\\
&= (Defn. \quad Last(E\cup F) = Last(E)\cup Last(F), Last(E^*) = Last(E))\\
&= (Last(a)\cup Last(\varepsilon))\cup Last(b)\\
&= \{a\cup\emptyset\}\cup\{b\} = \{a,b\}
\end{aligned}$$

$$\begin{aligned}
Null((a\cup\varepsilon)b^*) &= (Defn. \quad Null(E\cdot F) = Null(E\wedge F)), \varepsilon\in L_{RE}\equiv Null(E))\\
&= Null(a\cup\varepsilon)\wedge Null(b^*)\\
&= (Defn. \quad Null(E\cup F) = Null(E\vee F), Null(E^*) = true)\\
&= (Null(a)\vee Null(\varepsilon))\wedge true\\
&= true\wedge true = true
\end{aligned}$$

$$\begin{aligned}
Last(a\cup\varepsilon) &= Last(a)\cup Last(\varepsilon) = \{a\}\cup\emptyset = \{a\}\\
First(b^*) &= First(b) = \{b\}\\
Follow(a\cup\varepsilon) &= Follow(a)\cup Follow(\varepsilon) = \emptyset\cup\emptyset = \emptyset\\
Follow(b^*) &= Follow(b)\cup(Last(b)\times First(b))) = \emptyset\cup\{(b,b)\} = \{(b,b)\}
\end{aligned}$$

$$\begin{aligned} Follow((a\cup\varepsilon)b^*) &= Follow(a\cup\varepsilon)\cup Follow(b^*)\cup(Last(a\cup\varepsilon)\times First(b^*)) \\ &= \emptyset\cup\{(b,b)\}\cup\{(\{a\}\times\{b\})\} \\ &= \{(b,b)\}\cup\{(a,b)\} \\ &= \{(a,b),(b,b)\} \end{aligned}$$

$L_0 = L, L_1 = w^{-1}L, L_2 = a^{-1}(w^{-1}L) = (aw)^{-1}L$

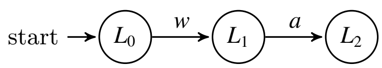



图 1.28: Construction 5.19(MNmin)

$E_0 = E, E_1 = [v^{-1}E]_\sim, E_2 = \{a^{-1}[v^{-1}E]_\sim\} = \{[va]_\sim^{-1}E\}$

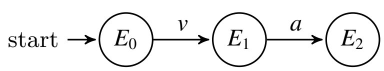



图 1.29: Construction 5.34 (Brzozowski)

$(\forall u, a, u\in V^*\wedge a\in V), (\exists v\in V^*, [u]_E\cdot a\subseteq [v]_E)$

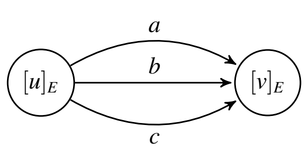



图 1.30: Definition 5.2 (Right invariance of an equivalence relation)

$u, w\in V^*, (\forall u, w, (\exists v\in V^*, [u]_E\cdot\{w\}\subseteq [v]_E))$

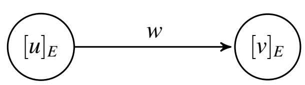



图 1.31: Right invariance of an equivalence relation [u],[v]

$M=(Q,V,E,s,F),M$ 所确定的 $V^*$ 上的关系 $R_M$ 定义为: 对于 $\forall x,y\in V^*$,
$xR_My\Leftrightarrow T^*(s,x)=T^*(s,y)$
$\Rightarrow$
$xR_My\Leftrightarrow \exists q\in Q,x,y\in \overleftarrow{L}(q)$
按照这个定义所得的关系 $R_M$，实际上是 $V^*$ 上的等价关系，利用这个关系，可以将 $V^*$ 划分成不多于 $|Q|$ 个等价类。

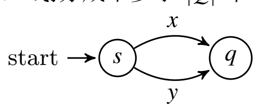



图 1.32: Equivalence classes of an equivalence relation

$\forall x,y\in V^*$, 如果 $xR_Ly$, 则在 $x$ 和 $y$ 后无论接 $V^*$ 中的任何字符串 $z$,$xz$ 和 $yz$ 要么都属于 $L$, 要么都不属于 $L$。
$xR_Ly\Leftrightarrow(\forall z\in V^*,xz\in L\Leftrightarrow yz\in L)$
$q$ 是自动机的一个状态，从开始状态到达该状态的字符串 $(\overleftarrow{L}(q))$ 是一个等价关系, 用 $[q]_E$ 表示。

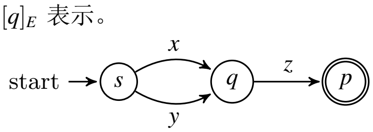



图 1.33: Equivalence classes of an equivalence relation

Def:
$M=(Q,V,T,E,S,F),T^*\in(Q\times Q)\to\mathbb{P}(V^*)$
$\overleftarrow{L}_M(q),\overrightarrow{L}_M(q)\in Q\to\mathbb{P}(V^*)$
$\overleftarrow{L}_M(q)=\{x|x\in V^*,T^*(s,q)=x,s\in S\}$
$\overrightarrow{L}_M(q)=\{x|x\in V^*,T^*(q,f)=x,f\in F\}$
$L_{FA}(Q,V,T,E,S,F)=\bigcup_{f\in F}(\overleftarrow{L}(f))$

## 1.10 Others

**Definition 1.15 (Prefix-closure[Chrison2007]).** Let $L\subseteq V^*$,then

$$\overline{L}:=\{s\in V^*:(\exists t\in V^*)[st\in L]\}$$

In words, the prefix closure of L is the language denoted by $\overline{L}$ and consisting of all the prefixes in L. In general, $L \subseteq \overline{L}$.

L is said to be prefix-closed if $L = \overline{L}$. Thus language L is prefix-closed if any prefix of any string in L is also an element of L.

$L_1 = \{\varepsilon, a, aa\}, L_1 = \overline{L_1}, L_1$ is prefix-closed.

$L_2 = \{a, b, ab\}, \overline{L_2} = \{\varepsilon, a, b, ab\}, L_2 \subset \overline{L_2}, L_2$ is not prefix closed.

**Definition 1.16 (Post-closure[Chrison2007]).** Let $L \subseteq V^*$ and $s \in L$. Then the post-language of L after s, denoted by L/s, is the language

$$L/s := \{t \in V^* : st \in L\}$$

By definition, $L/s = \emptyset$ if $s \notin \overline{L}$.

**Definition 1.17 (Left derivatives[WATSON93a]).** Given language $A \subseteq V^*$ and $w \in V^*$ we define the left derivative of A with respect to $w$ as:

$$w^{-1}A = \{x \in V^* : wx \in A\}$$

$A$ 关于 $w$ 的左导数，就是 $A$ 中：$\{w$ 的后缀组成的字符串集合$\}$。

Sometimes derivatives are written as $D_w A$ or as $\frac{dA}{dw}$. Right derivatives are analogously defined. Derivatives can also be extended to $B^{-1}A$ where B is also a language.

*Example 1.4.* $A = \{a, aab, baa\}, a^{-1}A = D_a A = \frac{dA}{da} = \{\varepsilon, ab, \emptyset\} = \{\varepsilon, ab\}$ □

*Example 1.5.* $L = \{ba, baa, baab, ca\}, w = \{ba\}$,

$w^{-1}L = \{\varepsilon, a, ab, \emptyset\} = \{\varepsilon, a, ab\}$

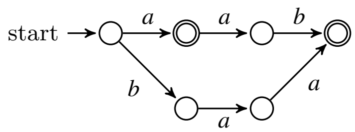
$(wa)^{-1}L = (baa)^{-1}L = \{\emptyset, \varepsilon, b, \emptyset\} = \{\varepsilon, b\}$

$a^{-1}(w^{-1}L) = a^{-1}\{\varepsilon, a, ab\} = \{\emptyset, \varepsilon, b\} = \{\varepsilon, b\}$

$w \in L \equiv \varepsilon \in w^{-1}L, and (wa)^{-1}L = a^{-1}(w^{-1}L)$ □

*Example 1.6.* $a^{-1}\{a\} = \{\varepsilon\}; \quad a^{-1}\{b\} = \emptyset, \quad \Leftarrow if(a \neq b)$ □

$L = \{a, aab, baa\}$




$a^{-1}L = \{\varepsilon, ab, \emptyset\} = \{\varepsilon, ab\}$

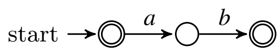
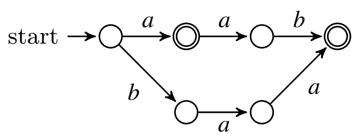



图 1.34: $a^{-1}L$

$L = \{a, aab, baa\}$




$V^{-1}L = \{\varepsilon, aa, ab\}, V \in \{a, b\}$

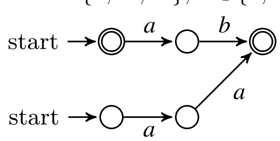



图 1.35: $V^{-1}L$

*Example 1.7.* $L_0 = \{ab\}, L_1 = \{ac\}, L_0L_1 = \{abac\}$

$a^{-1}(L_0L_1) = \{bac\}$

$a^{-1}(L_0L_1) = (a^{-1}L_0)L_1 \cup \emptyset \quad \Leftarrow (\varepsilon \notin L_0)$

$= \{b\}L_1 = \{bac\}$ □

*Example 1.8.* $L_0 = \{\varepsilon, ab\}, L_1 = \{ac\}, L_0L_1 = \{ac, abac\}$

$a^{-1}(L_0L_1) = \{c, bac\}$

$a^{-1}(L_0L_1) = (a^{-1}L_0)L_1 \cup a^{-1}L_1 \quad \Leftarrow (\varepsilon \in L_0)$

$= \{\emptyset, b\}L_1 \cup \{c\} = \{c, bac\}$ □

证明. $a^{-1}(L_0L_1)$

$1.if(\varepsilon \in L_0) \Rightarrow a^{-1}(L_0L_1) = (a^{-1}L_0)L_1 \cup a^{-1}L_1$

$L_0 = (L_0 \backslash \{\varepsilon\}) \cup \{\varepsilon\}$

$a^{-1}(L_0L_1) = a^{-1}(((L_0 \backslash \{\varepsilon\}) \cup \{\varepsilon\})L_1)$

$= a^{-1}(L_0L_1 \cup L_1)$

$$a^{-1}L_0 = a^{-1}((L_0 \setminus \{\varepsilon\}) \cup \{\varepsilon\})$$
$$= a^{-1}(L_0 \setminus \{\varepsilon\}) \cup a^{-1}\{\varepsilon\}$$
$$= a^{-1}L_0 \cup \emptyset = a^{-1}L_0$$

From [Hopcroft2008, p99]

(1) 如果 $L$ 是一个语言，$a$ 是一个符号，则 $L/a$(称作 $L$ 和 $a$ 的商) 是所有满足如下条件的串 $w$ 的集合：$wa$ 属于 $L$。例如，如果 $L=\{a,aab,baa\}$, 则 $L/a=\{\varepsilon,ba\}$, 证明：如果 $L$ 是正则的，那么 $L/a$ 也是。提示：从 $L$ 的 $DFA$ 出发，考虑接受状态的集合。

(2) 如果 $L$ 是一个语言，$a$ 是一个符号，则 $a\backslash L$ 是所有满足如下条件的串 $w$ 的集合: $aw$ 属于 $L$。例如，如果 $L=\{a,aab,baa\}$, 则 $a\backslash L=\{\varepsilon,ab\}$, 证明：如果 $L$ 是正则的，那么 $a\backslash L$ 也是。提示：记得正则语言在反转运算下是封闭的，又由 (1) 知，正则语言的商运算下是封闭的。

**Definition 1.18 (Kleene-closure[Chrison2007]).** Let $L \subseteq V^*$, then

$$L^* := \{\varepsilon\} \cup L \cup LL \cup LLL \cup \cdots$$

This is the same operation that we defined above for the set V, except that now it is applied to set L whose elements may be strings of length greater than one. An element of $L^*$ is formed by the concatenation of a finite (but possibly arbitrarily large) number of elements of L; this includes the concatenation of ”zero” elements, that is the empty string $\varepsilon$. Note that $*$ operation is idempotent: $(L^*)^* = L^*$.

$$\begin{aligned} L^* &= \{\varepsilon\} + L^+ \\ &= \{\varepsilon\} \cup (L\backslash\{\varepsilon\})L^* \\ &= \{\varepsilon\} + L + LL + LLL + \cdots \end{aligned}$$

## 1.11 Linear equation

see [Jean2018, 5.3,p64].

We give an algorithm to covert an automaton to a rational(regular) expression. The algorithm amounts to solving a system of linear equations on languages. We first consider an equation of the form

$$X = KX + L \tag{1.1}$$

**Proposition 1.1 (Arden’s Lemma).** *if $K$ does not contain the empty word, then $X = K^\star L$ is the unique solution of the equation $X = KX + L$.*

where $K$ and $L$ are languages and $X$ is the unknown. When $K$ does not contain the empty word, the equation admits a unique solution.

证明. Replacing $X$ by $K^*L$ in the expression $KX+L$,one gets

$$K(K^*)L+L=K^+L+L=(K^+L+L)=K^*L,$$

and hence $X=K^*L$ is a solution of (1.1). see[1]

To Prove uniqueness, consider two solutions $X_1$ and $X_2$ of (1.1). By symmetry,it suffices to show that each word $u$ of $X_1$ also belongs to $X_2$. Let us prove this result by induction on the length of $u$.

If $|u|=0$, $u$ is the empty word[2] and if $u\in X_1=KX_1+L$, then necessarily $u\in L$ since $\varepsilon\notin K$. But in this case, $u\in KX_2+L=X_2$. see[3]

For the induction step, consider a word $u$ of $X_1$ of length $n+1$. Since $X_1=KX_1+L, u$ belongs either to $L$ or to $KX_1$. if $u\in L$, then $u\in KX_2+L=X_2$. If $u\in KX_1$ then $u=kx$ for some $k\in K$ and $x\in X_1$. Since $k$ is not the empty word, one has necessarily $|x|\leq n$ and hence by induction $x\in X_2$.[see[4]] It follows that $u\in KX_2$ and finally $u\in X_2$. This conclude the induction and the proof of the proposition. □

From [Wonham2018, p74] The *length* $|s|$ of a string $s\in\Sigma^*$ is defined according to

$$|\varepsilon|=0; |s|=k, \text{if } s=\sigma_1\sigma_2\cdots\sigma_k\in\Sigma^+$$

Thus $|cat(s,t)|=|s|+|t|$.

[1]

$$\begin{aligned}K^*&=\{\varepsilon\}+K^+\\&=\{\varepsilon\}+(K\backslash\{\varepsilon\})K^*\\&=\{\varepsilon\}+K+KK+KKK+\cdots\end{aligned}$$

[2] The empty word $=\varepsilon, |\varepsilon|=0$; if a language $M=\{\varepsilon\}, |M|=1$, The empty language $M=\emptyset, |M|=0$. 文献 [Jean2018] 用 1 表示 $\varepsilon$, 因为 $\varepsilon K=K\varepsilon=K$, 因此,$\varepsilon$ 是连接运算的单位元，正是 1 表示的用意。0 表示 $\emptyset$, 它是并运算的单位元,$K\cup\emptyset=\emptyset\cup K=K$.

[3] In this case,$|u|=0, X=\{\varepsilon\}, |X|=1$. *i.e.* $\varepsilon=K\varepsilon+L, \varepsilon=K+L$

[4] $u=kx, |u|=|kx|=n+1, \varepsilon\notin K, |k|\geq 1, |x|\leq n$, 由假设知，$u$ 属于 $X_1$, 归纳 $|x|=0, |x|=1, \cdots, n, x\in X_2$.

A *language* over $\Sigma$ is any subset of $\Sigma^*$, i.e. an element of the power set $Pwr(\Sigma^*)$; thus the definition includes both the empty language $\emptyset$, and $\Sigma^*$ itself.

Note the distinction between $\emptyset$ (the language with no strings) and $\varepsilon$ (the string with no symbols). For instance the language $\{\varepsilon\}$ is nonempty, but contains only the empty string.

From [Wonham2018, p78]

**Proposition 1.2 ([Wonham2018]).**

*1. If $L = M^*N$ then $L = ML + N$*
*2. If $\varepsilon \notin M$ then $L = ML + N$ implies $L = M^*N$* □

Part(2) is Known as Arden's rule. Taken with Part(1) it says that if $\varepsilon \notin M$ then $L = M^*N$ is the unique solution of $L = ML + N$; in particular if $L = ML$ (with $\varepsilon \notin M$) then $L = \emptyset$

**Exercise 1.1.** Show by counterexample that the restriction $\varepsilon \notin M$ in Arden's rule cannot be dropped.

**Solution 1.1.** Examples text goes here.

**Exercise 1.2.** Prove Arden's rule. Hint: If $L = ML + N$ then for every $k \geq 0$

$$L = M^{k+1}L + (M^k + M^{k-1} + \cdots + M + \varepsilon)N$$

**Solution 1.2.**

*Preliminaries* :

$$
\begin{aligned}
M^* &= M^k + M^{k-1} + \cdots + M^1 + M^0 \qquad (k \geq 0) \\
&= M^k + M^{k-1} + \cdots + M^1 + \varepsilon \\
&= M^+ + \varepsilon \\
&= MM^* + \varepsilon \\
&= (M \backslash \{\varepsilon\})M^* + \varepsilon \\
M^+ &= M^k + M^{k-1} + \cdots + M^1 \qquad (k > 0) \\
&= M(M^k + M^{k-1} + \cdots + M^1 + M^0) \\
&= MM^* \\
M^0 &= \{\varepsilon\} = 1 \\
M\varepsilon &= \varepsilon M = M \\
\varepsilon + \varepsilon &= \varepsilon \\
M + M &= M
\end{aligned}
$$

证明.

$$L = ML + N \Rightarrow$$

$$M^0L = M^1L + M^0N \tag{1.2}$$

$$M^1L = M^2L + M^1N \tag{1.3}$$

$$M^2L = M^3L + M^2N \tag{1.4}$$

$$\tag{1.5}$$

$$\cdots$$

$\Rightarrow$

$(M^0 + M^1 + M^2 + \cdots)L = (M^1 + M^2 + M^3 + \cdots)L + (M^0 + M^1 + M^2 + \cdots)N$

$\Rightarrow$

so,if $L = ML + N$,then for every $k \geq 0$
$L = M^{k+1}L + (M^k + M^{k-1} + \cdots + M + M^0)N$

$\Rightarrow$

$$L = M^{k+1}L + (M^k + M^{k-1} + \cdots + M + \varepsilon)N \tag{1.6}$$

(1) $k=0$

$L = ML + (\varepsilon)N = ML + N$

$\Rightarrow (1-M)L = N$

$(\varepsilon - M)L = N$

由于 $\varepsilon \notin M$, 左端不会消去 $\{\varepsilon\}$. 因此, 只能在 $N$ 中找 $L$, 仅有唯一解: $L = \{\varepsilon\} = \{\text{empty word}\} \subseteq N$.

From [R.Su and Wonham2004, definition 2.3]

**Definition 1.19.** Let

$G_A = (X_A, \Sigma, \xi_A, x_{A,0}, X_{A,m})$

$G_A = (X_B, \Sigma, \xi_B, x_{B,0}, X_{B,m})$

$G_B$ is a DES-epimorphic image(满射像) of $G_A$ under DES-epimorphism $\theta : X_A \to X_B$ if

1. $\theta : X_A \to X_B$ is surjective(满射)
2. $\theta(x_{A,0}) = x_{B,0}$ and $\theta(X_{A,m}) = X_{B,m}$
3. $(\forall x \in X_A)(\forall \sigma \in \Sigma)\xi_A(x,\sigma)! \Rightarrow [\xi_B(\theta(x),\sigma)!\&\xi_B(\theta(x),\sigma) = \theta(\xi_A(x,\theta))]$
4. $(\forall x \in X_B)(\forall \sigma \in \Sigma)\xi_B(x,\sigma)! \Rightarrow [(\exists x' \in X_A)\xi_A(x',\sigma)!\&\theta(x') = x]$

In particular,$G_B$ is DES-isomorphic(同构) to $G_A$ if $\theta : X_A \to X_B$ is bijective(双射).

see figure 1.36.

$\theta(x_{A,0}) = x_{B,0}$ and $\theta(X_{A,m}) = X_{B,m}$

$\theta(x_A) = x_B$ and $\theta(x'_A) = x'_B$

$\xi_A(x_A,\sigma) = x'_A$ and $\xi_B(x_B,\sigma) = x'_B \Rightarrow$ definition 1.19 (3,4)

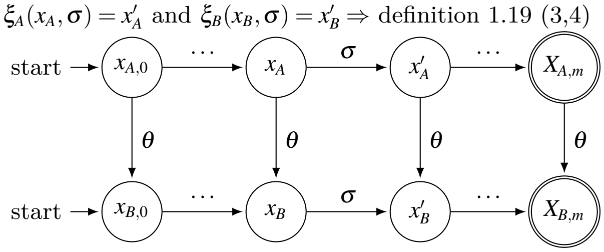



图 1.36: definition 1.19, $G_B$ is a DES-epimorphic image(满射像) of $G_A$ under DES-epimorphism $\theta : X_A \to X_B$

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