

# Chapter 1

## automata abstract

### 1.1 Finite automata

**Definition 1.1 (Finite automaton).** A finite automaton (an FA) is a 6-tuple  $(Q, V, T, E, S, F)$  where

- $Q$  is a finite set of states,
- $V$  is an alphabet,
- $T \in \mathbb{P}(Q \times V \times Q)$  is a transition relation,
- $E \in \mathbb{P}(Q \times Q)$  is an  $\varepsilon$ -transition relation
- $S \subseteq Q$  is a set of start states, and
- $F \subseteq Q$  is a set of final states.

□

```
class FA: virtual public FAabs {
    // Q is a finite set of states
    StatePool Q;
    // S is a set of start states, F is a
    // set of final states
    StateSet S, F;
    // Transitions maps each State to its
    // out-transitions.
    TransRel Transitions;
```

```

        // E is the epsilon transition
        relation.
        StateRel E;
    }

```

**StatePool:** All states in an automaton are allocated from a StatePool. StatePool's can be merged together to form a larger one. (Care must be taken to rename any relations or sets (during merging) that depend on the one StatePool.) State is in  $[0, \text{next})$

```

class StatePool {
    int next; // The next one to be
              allocated.
}

```

**StateSet:** The StateSet is normally associated (implicitly) with a particular StatePool; whenever a StateSet is to interact with another (from a different StatePool), its States must be renamed (to avoid a name clash). The capacity of a StateSet must be explicitly managed; many set operations are not bounds-checked when `assert()` is turned off.

```

class StateSet :protected BitVec {
    // How many States can this set
    contain?
    // [O, domain()) can be contained in
    *this.
    inline int domain() const;

    // set How many States can this set
    contain.
    // [O, r) can be contained in *this.
    inline void set__domain(const int r);
}

class BitVec {

```

```

        // used max number bits in data,
        denote width(domain), [0,
        bits_in_use) ==> [0, width)
    int bits_in_use;
    // number of words, 1, 2, 3, ...
    int words;
    // save bytes of words, [0, 1, 2, ...
    width(domain)]
    unsigned int *data;
}

```

**transition relation:**  $T \in Q \rightarrow \mathbb{P}(V \times Q)$ ,  $T(p) = \{(a, q) | (p, a, q) \in T\}$ ,  
表示状态  $p$  的 out-transitions. see Fig 1.1

```

// V -> Q
struct TransPair {
    CharRange transition_label;
    State transition_destination;
}
class TransImpl { TransPair *data; }
class Trans: protected TransImpl { }

// map: state(r) -> (T=Trans) out-
// transitions of r
// StateTo::data[r] = out-transitions of
// state r
class TransRel: public StateTo<Trans> {}

// map: state(r) -> T
// data[r] = T
template <class T> class StateTo {
T *data; // 动态数组的index(即状态的index)状
// 态的out-transitions
}

```

```

class FA: virtual public FAabs {
    TransRel Transitions; // maps each State to
                           its out-transitions.
}

```

**$\varepsilon$ -relation:**  $E \in \mathbb{P}(Q \times Q) \Rightarrow E \in Q \rightarrow \mathbb{P}(Q), E(p) = \{q \mid (p, q) \in E\}$ , 表示  $\varepsilon$  连接状态  $p$  和状态  $q$ .

```

// Implement binary relations on States. This
// is most often used for epsilon
// transitions.
// map: state(r) -> {StateSet}
// StateTo::data[r] = {StateSet}, 表示状态r与
// {StateSet}的二元关系
class StateRel : protected StateTo<StateSet> {
}

class FA: virtual public FAabs {
    // E is the epsilon transition relation.
    StateRel E;
}

```

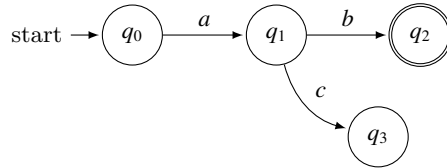


图 1.1:  $q_1$  in-transition:  $\{(q_0, a, q_1)\}$ ;  $q_1$  out-transition:  $\{(q_1, b, q_2), (q_1, c, q_3)\}$

[WATSON93a, p6] **the signatures of the transition relations:**

$$T \in \mathbb{P}(Q \times V \times Q)$$

$$T \in V \rightarrow \mathbb{P}(Q \times Q)$$

$$T \in Q \times Q \rightarrow \mathbb{P}(V)$$

$$T \in \mathcal{Q} \times V \rightarrow \mathbb{P}(\mathcal{Q})$$

$$T \in \mathcal{Q} \rightarrow \mathbb{P}(V \times \mathcal{Q})$$

for example, the function  $T \in \mathcal{Q} \rightarrow \mathbb{P}(V \times \mathcal{Q})$  is defined as  $T(p) = \{(a, q) : (p, a, q) \in T\}$

$$T \in \mathbb{P}(\mathcal{Q} \times V \times \mathcal{Q}), T = \{(p, a, q)\}$$

$$T \in \mathcal{Q} \rightarrow \mathbb{P}(V \times \mathcal{Q}), T(p) = \{(a, q) : (p, a, q) \in T\}$$

$$p, q \in \mathcal{Q}, a \in V$$

$$T : \mathcal{Q} \times V \rightarrow \mathbb{P}(\mathcal{Q})$$

$$T(p, a) = \{q\}$$

According to Convention A.4 (Tuple projection):

$$T \in \mathbb{P}(\mathcal{Q} \times V \times \mathcal{Q}), T = \{(p, a, q)\}$$

$$\pi_2(T) = \{a \mid (p, a, q) \in T\}, \tilde{\pi}_2(T) = \{(p, q) \mid (p, a, q) \in T\}$$

$$T \in \mathcal{Q} \rightarrow \mathbb{P}(V \times \mathcal{Q}), T(p) = \{(a, q) : (p, a, q) \in T\}$$

$$\pi_2(T(p)) = \{q \mid (p, a, q) \in T\}, \tilde{\pi}_2(T(p)) = \{a \mid (p, a, q) \in T\}$$

$$\mathcal{Q}_{map} : \mathcal{Q} \times V, T(p) = \{(a, q) : (p, a, q) \in T\}$$

$$\mathcal{Q}_{map}(q) = \{a\}$$

$$\mathcal{Q}_{map} : \mathcal{Q} \times V, T \in \mathbb{P}(\mathcal{Q} \times V \times \mathcal{Q})$$

$$\pi_1(T) = \{p \mid (p, a, q) \in T\}, \tilde{\pi}_1(T) = \{(a, q) \mid (p, a, q) \in T\}$$

$$\mathcal{Q}_{map} = (\tilde{\pi}_1(T))^R = \{(a, q) \mid (p, a, q) \in T\}^R = \{(q, a) \mid (p, a, q) \in T\}$$

## 1.2 Properties of finite automata

$$M = (\mathcal{Q}, V, T, E, S, F), M_0 = (\mathcal{Q}_0, V_0, T_0, E_0, S_0, F_0), M_1 = (\mathcal{Q}_1, V_1, T_1, E_1, S_1, F_1)$$

**Definition 1.2 (Size of an FA).** Define the size of an FA as  $|M| = |\mathcal{Q}|$

**Definition 1.3 (Isomorphism 同构 ( $\cong$ ) of FA's).** We define isomorphism ( $\cong$ ) as an equivalence relation on FA's.  $M_0$  and  $M_1$  are isomorphic (written  $M_0 \cong M_1$ ) if and only if  $V_0 = V_1$  and there exists a bijection 双射  $g \in \mathcal{Q}_0 \rightarrow \mathcal{Q}_1$  such that

- $T_1 = \{(g(p), a, g(q)) | (p, a, q) \in T_0\}$
- $E_1 = \{(g(p), g(q)) | (p, q) \in E_0\}$
- $S_1 = \{g(s) | s \in S_0\}$  and
- $F_1 = \{g(f) | f \in F_0\}$

(see Fig 1.2). □

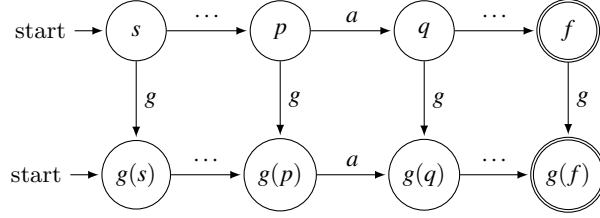


图 1.2: Isomorphism  $M_0 \cong M_1$  if and only if  $V_0 = V_1$  and there exists a bijection  $g \in Q_0 \rightarrow Q_1$

**Definition 1.4 (Extending the transition relation  $T$ ).** We extend transition relation  $T \in V \rightarrow \mathbb{P}(Q \times Q)$  to  $T^* \in V^* \rightarrow \mathbb{P}(Q \times Q)$  as follows:

$$T^*(\varepsilon) = E^*$$

and (for  $a \in V, w \in V^*$ )

$$T^*(aw) = E^* \circ T(a) \circ T^*(w)$$

Operator  $\circ$  (composition is defined in Convention 1).

This definition could also have been presented symmetrically. □

*Note 1.1.*  $s_1, s_2, s_3, s_4 \in Q, a \in V, w \in V^*$

$$E = T(\varepsilon) = \{(s_1, s_2)\}, T(a) = \{(s_2, s_3)\}, T^*(w) = \{(s_3, s_4)\}$$

$$\begin{aligned} T^*(aw) &= E^* \circ T(a) \circ T^*(w) \\ &= \{(s_1, s_2)\} \circ \{(s_2, s_3)\} \circ \{(s_3, s_4)\} = \{(s_1, s_3)\} \circ \{(s_3, s_4)\} = \{(s_1, s_4)\} \end{aligned}$$

*Note 1.2.*  $T \in Q \times V \rightarrow \mathbb{P}(Q)$ , extend to:  $T^* \in Q \times V^* \rightarrow \mathbb{P}(Q)$

$$\forall q \in Q, w \in V^*, a \in V,$$

1.  $T^*(q, \varepsilon) = q$
2.  $T^*(q, wa) = T(T^*(q, w), a)$

$$\begin{aligned}
T^*(q, a) &= T^*(q, \varepsilon a) \\
&= T(T^*(q, \varepsilon), a) \\
&= T(q, a)
\end{aligned}$$

两值相同，不用区分这两个符号。

**Convention 1 (Relation composition)** Given sets  $A, B, C$  (not necessarily different) and two relations,  $E \subseteq A \times B$  and  $F \subseteq B \times C$ , we define relation composition (infix operator 中缀操作符  $\circ$ ) as:

$$E \circ F = \{(a, c) | (\exists b \in B), (a, b) \in E \wedge (b, c) \in F\} \quad \square$$

Note 1.3. if  $\exists b \in B, (a, b) \in E, (b, c) \in F$ , then

$$E : A \rightarrow B \Rightarrow E(a) = b$$

$$F : B \rightarrow C \Rightarrow F(b) = c$$

$$E \circ F = \{(a, b)\} \circ \{(b, c)\} = \{a, c\}$$

$$\begin{aligned}
(E \circ F)(a) &= F(E(a)) \\
&= F(b) = c
\end{aligned}$$

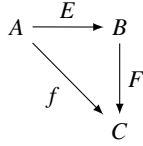


图 1.3:  $E \circ F = (F \circ E)(a) = F(E(a)) = c = f(a)$

Remark 1.1. We also sometimes use the signature  $T^* \in Q \times Q \rightarrow \mathbb{P}(V^*)$   $\square$

Note 1.4.  $T(p, q) = \{w | p, q \in Q, w \in V^*\}$

Remark 1.2. if  $E = \emptyset$  then  $E^* = \emptyset^* = I_Q$  where  $I_Q$  is the identity relation 单位关系 on the states of  $M$ .

**Definition 1.5 (The language between states).** The language between any two states  $q_0, q_1 \in Q$  is  $T^*(q_0, q_1)$ .  $\square$

**Definition 1.6 (Left and right languages).** The left language of a state (in  $M$ ) is given by function,  $\overleftarrow{L}_M \in Q \rightarrow \mathbb{P}(V^*)$ , where

$$\overleftarrow{L}_M(q) = (\cup s : s \in S : T^*(s, q))$$

The right language of a state (in  $M$ ) is given by function  $\overrightarrow{L}_M \in Q \rightarrow \mathbb{P}(V^*)$ , where

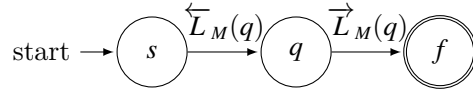
$$\overrightarrow{L}_M(q) = (\cup f : f \in F : T^*(q, f))$$

The subscript  $M$  is usually dropped when no ambiguity can arise.  $\square$

*Example 1.1.*  $T^* \in Q \times Q \rightarrow \mathbb{P}(V^*)$ ,  $\overleftarrow{L}_M, \overrightarrow{L}_M \in Q \rightarrow \mathbb{P}(V^*)$ .

$\overleftarrow{L}_M(q) = \{\text{能引导 } M \text{ 从开始状态到达 } q \text{ 状态的字符串集合}\}$ , (从  $q$  往左看)

$\overrightarrow{L}_M(q) = \{\text{能引导 } M \text{ 从开始状态到达 } q \text{ 状态的字符串集合}\}$ , (从  $q$  往右看)



see Fig 1.4.

$$\begin{aligned} \overleftarrow{L}_M(q_2) &= (s \rightarrow q_1 \rightarrow q_2) \cup (s \rightarrow (q_1 \rightarrow q_3)^* \rightarrow q_1 \rightarrow q_2) \cup (s \rightarrow (q_1 \rightarrow q_3)^* \rightarrow q_3 \rightarrow q_2) \\ &= [(s \rightarrow q_1 \rightarrow q_2) \cup (s \rightarrow (q_1 \rightarrow q_3)^* \rightarrow q_1 \rightarrow q_2)] \cup (s \rightarrow (q_1 \rightarrow q_3)^* \rightarrow q_3 \rightarrow q_2) \\ &= (s \rightarrow (q_1 \rightarrow q_3)^* \rightarrow q_1 \rightarrow q_2) \cup (s \rightarrow (q_1 \rightarrow q_3)^* \rightarrow q_3 \rightarrow q_2) \\ &= \{1(10)^*0, 1(10)^*1\} \\ \overrightarrow{L}_M(q_2) &= \{01^*0, 10^*1(001^*0 + (10)^*1)\} \end{aligned}$$

**Definition 1.7 (Language of an FA).** The language of a finite automaton (with alphabet  $V$ ) is given by the function  $L_{FA} \rightarrow \mathbb{P}(V^*)$  defined as:

$L_{FA}(M) = (\cup s, f : s \in S \wedge f \in F : T^*(s, f))$  (所有从开始状态到接受状态的字符串集合)  $\square$

*Property 1.1 (Language of an FA).* From the definition of left and right languages (of a state), we can also write:

$L_{FA}(M) = (\cup f : f \in F : \overleftarrow{L}(f))$  (所有从  $s$  到  $f$  的字符串集合, 从  $f$  向左看)

and



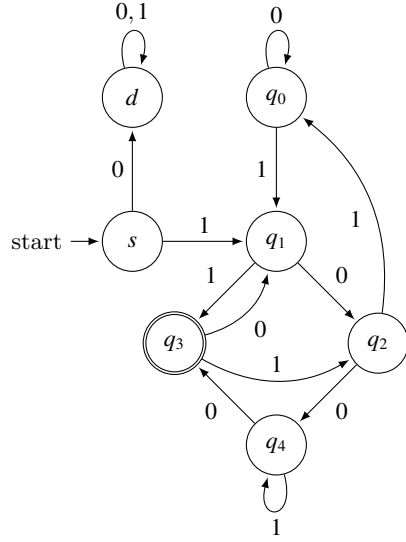


图 1.4:  $\{x|x \in \{0,1\}^+ \text{ 且当把 } x \text{ 看成二进制数时, } x \text{ 模 } 5 \text{ 与 } 3 \text{ 同余,}$   
 要求当  $x$  为 0 时,  $|x| = 1$ , 且当  $x \neq 0$  时,  $x$  的首字符为 1} 语言对应的  
**DFA**

$L_{FA}(M) = (\cup s : s \in S : \vec{L}(s))$  (所有从  $s$  到  $f$  的字符串集合, 从  $s$  向右看)  $\square$

**Definition 1.8 ( $\epsilon$ -free 无  $\epsilon$  转移).** Automaton  $M$  is  $\epsilon$ -free if and only if  $E = \emptyset$ .  $\square$

*Remark 1.3.* Even if  $M$  is  $\epsilon$ -free it is still possible that  $\epsilon \in L_{FA}(M)$ : in this case  $S \cap F \neq \emptyset$ . (开始状态也是接受状态)  $\square$

Form [WATSON93a, Convention A.4] (Tuple projection).

**Convention 2 (Tuple projection)** For an  $n$ -tuple  $t = (x_1, x_2, \dots, x_n)$  we use the notation  $\pi_i(t)$  ( $1 \leq i \leq n$ ) to denote tuple element  $x_i$ ; we use the notation  $\bar{\pi}_i(t)$  ( $1 \leq i \leq n$ ) to denote the  $(n-1)$ -tuple  $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ . Both  $\pi$  and  $\bar{\pi}$  extend naturally to sets of tuples.  $\square$

Form [WATSON93a, Definition A.20] (Tuple and relation reversal).

**Definition 1.9 (Tuple and relation reversal).** For an  $n$ -tuple  $(x_1, x_2, \dots, x_n)$  define reversal as (postfix and superscript) function  $R$ :

$$(x_1, x_2, \dots, x_n)^R = (x_n, x_n - 1, \dots, x_2, x_1)$$

Given a set  $A$  of tuples, we define  $A^R = \{x^R : x \in A\}$ .  $\square$

**Definition 1.10 (Reachable states).** For  $M$  we can define a reachability relation  $Reach(M) \subseteq (Q \times Q)$  defined as

$$Reach(M) = (\bar{\pi}_2(T) \cup E)^* \text{ see}^1$$

Functions  $\pi$  and  $\bar{\pi}$  are defined in Convention 2. Similarly the set of start-reachable states is defined to be:

$$SReachable(M) = Reach(M)(S) \text{ see}^2$$

and the set of final-reachable states is defined to be:

$$FReachable(M) = (Reach(M))^R(F) \text{ see}^3$$

Reversal of a relation is defined in Definition 1.9. The set of useful states is:  $Reachable(M) = SReachable(M) \cap FReachable(M)$   $\square$

*Remark 1.4.* For FA  $M = (Q, V, T, E, S, F)$ , function  $SReachable$  satisfies the following interesting property:

$$q \in SReachable(M) \equiv \overleftarrow{L}_M(q) \neq \emptyset$$

$FReachable$  satisfies a similar property:

$$q \in FReachable(M) \equiv \overrightarrow{L}_M(q) \neq \emptyset \quad \square$$

*Example 1.2.*  $T \in \mathbb{P}(Q \times V \times Q)$ ,  $T = \{(p, a, q) | p, q \in Q, a \in V\}$ ,

$$\bar{\pi}_2(T) = \{(p, q) | (p, a, q) \in T\}$$

$$Q_{map} = (\bar{\pi}_1(T))^R, Q_{map} = \{(a, q) | (p, a, q) \in T\}^R = \{(q, a) | (p, a, q) \in T\}$$

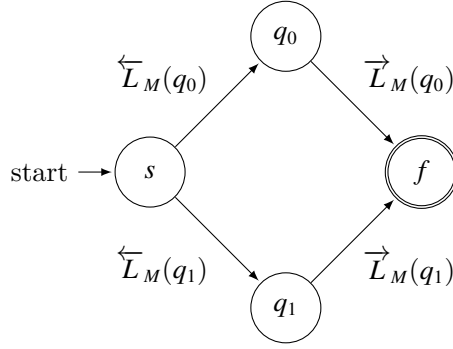
$\square$

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<sup>1</sup>  $\{(p_1, q_1), (p_2, q_2), \dots\}$

<sup>2</sup> 从 start state 可以到达的状态集合

<sup>3</sup> 可以到达 final state 的状态集合



e.g.  $p = \{1, 2\} \in Q_1 \subseteq \mathbb{P}(Q_0)$ ,  $\vec{L}_{M_1}(p) = \vec{L}_{M_0}(1) \cup \vec{L}_{M_0}(2)$

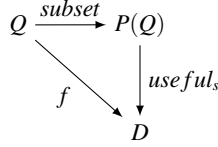


图 1.5:  $\text{subset} \circ \text{usefus}_s = \text{usefus}_s(\text{subset}(Q, V, \emptyset, S, F)) = (D, V, T', \emptyset, S', F')$

### 1.3 $\Sigma$ -algebras and regular expressions

#### $\Sigma$ -homomorphism

$X$  集合中的元素与有序集  $S$  中的元素一一对应, 称  $X$  是  $S$ -sorted.

$S = \{1, 3, 7, 9\}, X = \{d, a, c, f\}, s \in S, X_s \in X$

$S$  是有序的,  $S_{s_1} = 1, S_{s_2} = 3, S_{s_3} = 7, S_{s_4} = 9$

$X$  与  $S$  中的元素一一对应。  $X_{s_1} = d, X_{s_2} = a, X_{s_3} = c, X_{s_4} = f$

$\Sigma$ -homomorphism 同态:  $(V, F) \Leftrightarrow (W, G)$ , 载体  $(V, W)$  和操作  $(F, G)$  一一对应。

$\Sigma$ -homomorphism function:  $h \in V \rightarrow W$

$$\begin{aligned}
L(v) &= (h \circ f)(v) = h(f(v)) = g(w) = L_{reg} = L_V = L_W \\
L(v) &= (g \circ h)(v) = g(h(v)) = g(w) = L_{reg} = L_V = L_W \\
&\Rightarrow h(f(v)) = g(h(v))
\end{aligned}$$

$$\begin{array}{ccc}
V & \xrightarrow{h} & h(v) \\
f \downarrow & \searrow L & \downarrow g \\
f(v) & \xrightarrow{h} & g(h(v))
\end{array}$$

图 1.6:  $(h \circ f)(v) = (g \circ h)(v) \Rightarrow h(f(v)) = g(h(v))$

$$\begin{array}{ccc}
(e_1, e_2) & \xrightarrow{h} & (h(e_1), h(e_2)) \\
f \downarrow & \searrow L & \downarrow g \\
f(e_1, e_2) \circ & \xrightarrow{h} & g(h(e_1), h(e_2))
\end{array}$$

图 1.7:  $(h \circ f)(e_1, e_2) = (g \circ h)(e_1, e_2) \Rightarrow h(f(e_1, e_2)) = g(h(e_1), h(e_2))$

*Example 1.3.*  $\Sigma = (S, \Gamma)$ , sort:  $expr$ ,  $\Gamma := \{a, plus\}$ ,  $a$  is a constant. operator  $plus : expr \times expr \rightarrow expr$ .

$\Sigma$ -term algebra:  $plus[a, a], plus[plus[a, plus[a, a]], a]$

$\Sigma$ -algebra  $X$ , carrier set: natural number, constant 0. operator  $f_{plus}(x, y) = (x \max y) + 1$

$\Sigma$ -homomorphism function("expression tree height"):  $h_{expr} : \Sigma\text{-term algebra} \rightarrow X$

$$\begin{aligned}
(h_{\text{expr}} \circ \text{plus})(s) &= (f_{\text{plus}} \circ h_{\text{expr}})(s) \\
h_{\text{expr}}(\text{plus}(s)) &= f_{\text{plus}}(h_{\text{expr}}(s)) \\
\text{left} : s \leftarrow e, f &\Rightarrow \text{plus}[e, f] \\
\text{right} : s \leftarrow e, f &\Rightarrow f_{\text{plus}}(h_{\text{expr}}(e), h_{\text{expr}}(f)) \\
h_{\text{expr}}(\text{plus}(e, f)) &= f_{\text{plus}}(h_{\text{expr}}(e), h_{\text{expr}}(f)) \\
&= (h_{\text{expr}}(e) \max h_{\text{expr}}(f)) + 1 \\
&\text{and,} \\
h_{\text{expr}}(a) &= 0
\end{aligned}$$

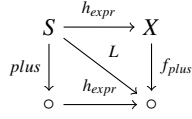


图 1.8:  $(h_{\text{expr}} \circ \text{plus})(s) = (f_{\text{plus}} \circ h_{\text{expr}})(s) \Rightarrow h_{\text{expr}}(\text{plus}(s)) = f_{\text{plus}}(h_{\text{expr}}(s))$

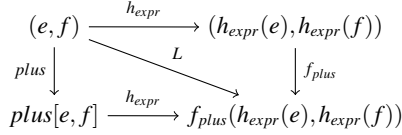


图 1.9:  $(h_{\text{expr}} \circ \text{plus})(e, f) = (f_{\text{plus}} \circ h_{\text{expr}})(e, f) \Rightarrow h_{\text{expr}}(\text{plus}[e, f]) = f_{\text{plus}}(h_{\text{expr}}(e), h_{\text{expr}}(f))$

**Definition 1.11 (Regular expressions).** We define regular expressions (over alphabet  $V$ ) as the  $\Sigma$ -term algebra over signature  $\Sigma = (S, O)$  where

- $S$  consists of a single sort  $\text{Reg}$  (for regular expression), and
- $O$  is a set of several constans:  $\varepsilon, \emptyset, a_1, a_2, \dots, a_n; \text{Reg}$  (where  $V = \{a_1, a_2, \dots, a_n\}$ ) and five operators  $\cdot : \text{Reg} \times \text{Reg} \rightarrow \text{Reg}$  (the dot operator),  $\cup : \text{Reg} \times \text{Reg} \rightarrow \text{Reg}$ ,  $*$  :  $\text{Reg} \rightarrow \text{Reg}$ ,  $+$  :  $\text{Reg} \rightarrow \text{Reg}$ , and  $?$  :  $\text{Reg} \rightarrow \text{Reg}$ .

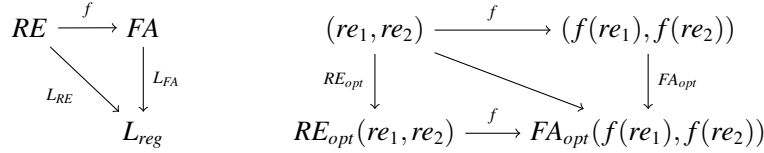
$V := RE$ (正则表达式),  $W := FA$ (有限自动机)

$\Sigma$ -homomorphism function:  $f \in RE \rightarrow FA$

$F : RE_{opt}$  运算, 二元: union(or),concat; 一元: star,plus,question;

常量:epsilon,empty,symbol

$G : FA_{opt}$  运算, 同上



```

//Sigma.h
template<class T>
class Reg : public T {
// Helper for constructing the homomorphic
// image of a regular expression.
// T is carrier set: RE,FA,RFA,
// 各自的操作, 分别在 Sig-RE.cpp, Sig-FA.cpp,
// Sig-RFA.cpp中定义
inline void homomorphic_image(const RE& r);
Reg<T>& epsilon();
Reg<T>& empty();
Reg<T>& symbol(const CharRange r);
Reg<T>& Or(const Reg<T>& r);
Reg<T>& concat(const Reg<T>& r);
Reg<T>& star();
Reg<T>& plus();
Reg<T>& question();
}

```

**Definition 1.12 (The nullable  $\Sigma$ -algebra).** We define the *nullable*  $\Sigma$ -algebra as follows:

- The carrier set is  $\{true, false\}$ .

- $a \in V, E_1, E_2 \in RE, \varepsilon \in E_1^*, \varepsilon \in E_1^?, \varepsilon \notin E_1^+$

$$nullable(\varepsilon) = true$$

$$nullable(\emptyset) = nullable(a) = false$$

$$nullable(E_1 \vee E_2) = nullable(E_1 \cup E_2)$$

$$nullable(E_1 \wedge E_2) = nullable(E_1 \cdot E_2)$$

$$nullable(E_1^*) = true$$

$$nullable(E_1^+) = nullable(E_1)$$

$$nullable(E_1^?) = true$$

$$nullable(E_1) = \begin{cases} true & \varepsilon \in E_1 \\ false & \varepsilon \notin E_1 \end{cases}$$

#### 1.4 Constructing $\varepsilon$ -lookahead automata

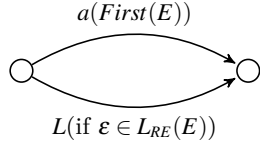


图 1.10: Lookahead

function:  $look(E, L) = First \cup \text{if } (Null(E)) \text{ then } L \text{ else } \emptyset$  fi

#### 1.5 Towards the Berry-Sethi construction

$$\varepsilon \in L_{FA}(M) \equiv s \in F$$

start  $\rightarrow$

图 1.11:  $\varepsilon \in L_{FA}(M) \equiv s \in F$

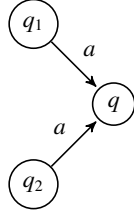
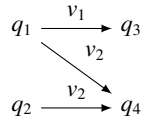
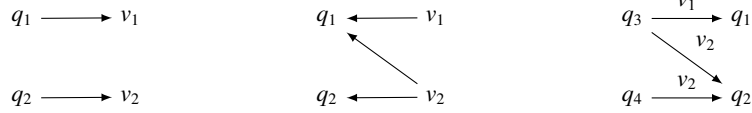
图 1.12: All in-transitions to a state are on the same symbol (in  $V$ )

图 1.13:

$$q_4(\text{in-transition}) = \{(q_1, b), (q_2, b)\}, q_1(\text{out-transition}) = \{(a, q_3), (b, q_4)\}$$

a	b	c



$$Q_{map}(q_1) = \{v_1\}, Q_{map}^{-1}(v_1) = \{q_1\}, (q_3, v_1, q_1) \in T$$

$$Q_{map}(q_2) = \{v_2\}, Q_{map}^{-1}(v_2) = \{q_2\}, (q_3, v_2, q_2) \in T, (q_4, v_2, q_2) \in T$$

图 1.14:  $Q_{map}, Q_{map}^{-1}$ 

**Definition 1.13 (RFA).** A reduced FA (RFA) is a 7-tuple  $(Q, V, follow, first, last, null, Q_{map})$  where

- $Q$  is a finite set of states,
- $V$  is an alphabet,



- $follow \in \mathbb{P}(Q \times Q)$  is a follow relation (replace the transition relation:  $T \in \mathbb{P}(Q \times V \times Q)$ ),
- $first \subseteq Q$  is a set of initial states (replacing  $T(s)$  in an LBFA),
- $null \in \{true, false\}$  is a Boolean value (encoding  $s \in F$  in an LBFA,  $\varepsilon \in L_{FA}(M) \equiv s \in F$ ), and
- $Q_{map} \in \mathbb{P}(Q \times V)$ ,  $Q_{map}(q) = \{v\}$ ,  $one \rightarrow one$ . maps each state to exactly one symbol. i.e.  $Q_{map} \in Q \rightarrow V$ .

$Q_{map}(q) = \{a | (p, a, q) \in T\}$  表示  $(q, v)$  的一一对应关系。物理含义是进入  $q$  状态的唯一字母  $a$

class RFA 中表示 its inverse:  $Q_{map}^{-1} : V \rightarrow \mathbb{P}(Q)$ , 部分函数  $Q_{map}^{-1}(a) = \{q | (p, a, q) \in T\}$

□

```
class RFA : virtual public FAabs{
// Q is a finite set of states
StatePool Q;

// first(subset Q) is a set of initial states
// (replacing T(s) in an LBFA),
// last(subset Q) is a set of final states,
StateSet first, last;

// Qmap (in P( Q x V)) maps each state to
// exactly one symbol (it is also viewed as
// Qmap in Q → V,
// and its inverse as Qmap-1 in V → P(Q)
// [the set of all partial functions from V
// to P(Q)]).
// Trans用struct TransPair 表示:T(a) = { q |
// (p,a,q) in T },
// 因此这里表示Qmap的inverse, V → P(Q)
Trans Qmap_inverse;
```

```

// follow(in P(Q x Q)) is a follow relation(
    replacing the transition relation),
StateRel follow;

// null (in {true,false}) is a Boolean value
    (encoding s in F in an LBFA)
// if epsilon属于LBFA, true; final set中包含s
// {true, false} ==> {1, 0}
int Nullable;
}

```

```

// V -> Q
struct TransPair {
    CharRange transition_label;
    State transition_destination;
}

class TransImpl { TransPair *data; }
class Trans:protected TransImpl { }

}

```

$$rfa \circ R(E) = R \circ rfa(E)$$

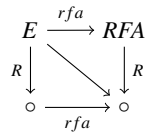


图 1.15:  $rfa \circ R(E) = R \circ rfa(E)$

**Definition 1.14.** (Dual of a function) We assume two sets  $A$  and  $B$  whose reversal operators are  $R$  and  $R'$  respectively. Two functions,  $f \in A \rightarrow B$  and  $f_d \in A \rightarrow B$  are one another's dual if and only if

$$f(a) = (f_d(a^R))^{R'}$$

In some cases we relax the equality to isomorphism (when isomorphism is defined on  $B$ ).  $\square$

$$\begin{aligned} f_d \circ R(a) &= R' \circ f(a) \Rightarrow f_d(R(a)) = R'(f(a)) \Rightarrow \\ f_d(a^R) &= (f(a))^{R'} \Rightarrow f(a) = (f_d(a^R))^{R'} \end{aligned}$$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ R \downarrow & \searrow & \downarrow R' \\ \circ & \xrightarrow{f_d} & \circ \end{array}$$

图 1.16:  $f(a) = (f_d(a^R))^{R'}$

## 1.6 The Berry-Sethi construction

$C_{\cdot, RFA}(rfa(\$), rfa(E)) = C_{\cdot, RFA}(C_{\$, RFA}, rfa(E)) = L_{RE}(\$E) = \{\$ \} L_{RE}(E)$   
 $convert$  简单剔除  $E$  的第一个字符。  $L_{FA} \circ convert \circ rfa(E) = V^{-1} L_{RE}(E)$   
 $convert(C_{\cdot, RFA}(C_{\$, RFA}, rfa(E))) = rfa(E)$

$$\begin{array}{ccc} RE & \xrightarrow{rfa} & RFA \\ & \searrow L_{RE} & \downarrow L_{FA} \\ & & L_{reg} \end{array} \quad \begin{array}{ccc} (\$, E) & \xrightarrow{rfa} & (rfa(\$), rfa(E)) \\ RE \cdot \downarrow & \searrow rfa & \downarrow FA \cdot \\ \$ \cdot E & \xrightarrow{\quad} & rfa(\$) \cdot rfa(E) \end{array}$$

图 1.17:  $convert(C_{\cdot, RFA}(C_{\$, RFA}, rfa(E))) = rfa(E)$

Algorithm 2.45(implements  $useful_s \circ subset$ ):  
 initial:  $D = \emptyset, U = S$

$$d := \bigcup_{q \in u} T(q, a)$$

$$\{q_1, q_2\} \xrightarrow{a} \{T(q_1, a), T(q_2, a)\}$$

using Algorithm 2.45 for  $\text{decode}(\text{RFA} \rightarrow \text{LBFA})$

$$d := \bigcup \{q \mid q \in \text{first} \wedge Q_{\text{map}}(q) = a\}$$

$$\{s\} \xrightarrow{a} \{d\}$$

note:

$$d := \bigcup_{p \in u} \{q \mid (p, q) \in \text{follow} \wedge Q_{\text{map}}(q) = a\}$$

$$u = \{p_1, p_2\}, d = \{q_1, q_2\},$$

$$\text{follow}(p_1) = q_1, \text{follow}(p_2) = q_2, Q_{\text{map}}(q_1) = Q_{\text{map}}(q_2) = a$$

$$\{p_1, p_2\} \xrightarrow{a} \{q_1, q_2\}$$

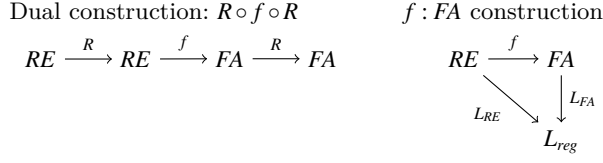
## 1.7 The McNaughton-Yamada-Clushkov construction

$$RE \rightarrow [DFA]_{\cong}$$

$$MYG(E) = \text{useful}_s \circ \text{subset} \circ \text{decode} \circ rfa(E)$$

$$\begin{array}{ccccc}
RE & \xrightarrow{rfa} & [RFA]_{\cong} & \xrightarrow{\text{decode}} & [NFA]_{\cong} \\
& & & & \downarrow \text{subset} \\
[DFA]_{\cong} & \xleftarrow{\text{Complete}} & [DFA]_{\cong} & \xleftarrow{\text{useful}_s} & P(Q)
\end{array}$$

图 1.18:  $MYG(E) = \text{useful}_s \circ \text{subset} \circ \text{decode} \circ rfa(E)$

图 1.19: Dual construction:  $R \circ f \circ R$ 

## 1.8 The dual of the Berry-Sethi construction

$R \circ R$  is the identity  $\Rightarrow R \circ R(A) = A$

## 1.9 Algorithm 4.52 (Aho-Sethi-Ullman)

note:

$$d := \bigcup_{q \in u} \{follow(q) \mid Q_{map}(q) = a\}$$

$$q \xrightarrow{a} follow(q)$$

note:

$$T_0(b) = (p, p'), T_1(b) = (q, q')$$

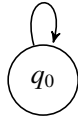
$$\pi_2(T_0(b)) = p', \pi_2(T_1(b)) = q'$$

$$T_0(s_0, a) = p, T_1(s_1, a) = q$$

$$Q' = \{q_0\} \cup (\bigcup_{b \in V} \{\pi_2(T_0(b)) \times \pi_2(T_1(b))\})$$

$$M_0 : s_0 \xrightarrow{a} p \xrightarrow{b} p'$$

$$M_1 : s_1 \xrightarrow{a} q \xrightarrow{b} q'$$



notes:

$[V^*]_{R_L} = V^*/R_L$  表示右不变的等价关系，每个等价关系对应一个状态。

$[\epsilon]_{R_L}$  表示  $\epsilon$  所在的等价类对应的状态，就是开始状态

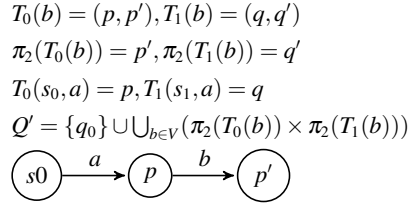


图 1.20: Intersection of LBFA's

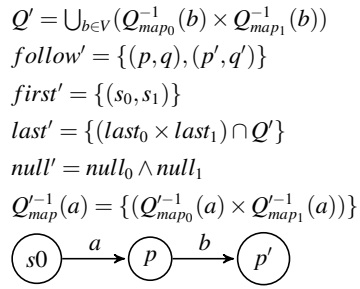


图 1.21: Intersection of RFA's

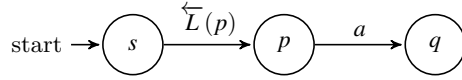
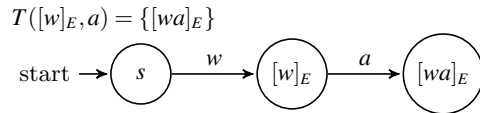


图 1.22: text111

图 1.23:  $T([w]_E, a) = \{[wa]_E\}$

$$T([w]_{R_L}, a) = \{[wa]_{R_L}\}, (wa)^{-1}L = a^{-1}(w^{-1}L), \epsilon^{-1}L = L$$

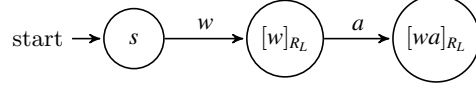
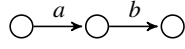
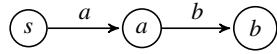
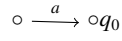
图 1.24:  $T([w]_{R_L}, a) = \{[wa]_{R_L}\}$ 图 1.25:  $a, b \in V, (a, b) \in \text{Follow}(E)$ 图 1.26:  $\text{BSenc}(E)$ 

图 1.27: text

$[V^*]_E = V^*/R_E$  表示右不变的等价关系，每个等价关系对应一个状态。  
 $[\epsilon]_E$  表示  $\epsilon$  所在的等价类对应的状态，就是开始状态

$$\begin{aligned}
 \text{First}((a \cup \epsilon)b^*) &= (\text{Defn. } \text{First}(E \cdot F), \epsilon \in (a \cup \epsilon), \text{Null}(E) = \text{true}) \\
 &= (\Rightarrow \text{First}(E) \cup \text{First}(F)) \\
 &= \text{First}(a \cup \epsilon) \cup \text{First}(b^*) \\
 &= (\text{Defn. } \text{First}(E \cup F) = \text{First}(E) \cup \text{First}(F), \text{First}(E^*) = \text{First}(E)) \\
 &= (\text{First}(a) \cup \text{First}(\epsilon)) \cup \text{First}(b) \\
 &= \{a \cup \emptyset\} \cup \{b\} = \{a, b\}
 \end{aligned}$$

$$\begin{aligned}
First((a \cup \varepsilon)b^*) &= First(ab^* \cup \varepsilon b^*) \\
&= (Defn. \quad First(E \cup F) = First(E) \cup First(F)) \\
&= First(ab^*) \cup First(b^*) \\
&= (Defn. \quad First(E \cdot F), \varepsilon \notin \{a\}, Null(E) = false \Rightarrow First(ab^*) = First(a) \cup \emptyset = \{a\}) \\
&= First(a) \cup First(b) = \{a, b\}
\end{aligned}$$

$$\begin{aligned}
Last((a \cup \varepsilon)b^*) &= (Defn. \quad Last(E \cdot F), \varepsilon \in (a \cup \varepsilon), Null(E) = true) \\
&= (\Rightarrow Last(E) \cup Last(F)) \\
&= Last(a \cup \varepsilon) \cup Last(b^*) \\
&= (Defn. \quad Last(E \cup F) = Last(E) \cup Last(F), Last(E^*) = Last(E)) \\
&= (Last(a) \cup Last(\varepsilon)) \cup Last(b) \\
&= \{a \cup \emptyset\} \cup \{b\} = \{a, b\}
\end{aligned}$$

$$\begin{aligned}
Null((a \cup \varepsilon)b^*) &= (Defn. \quad Null(E \cdot F) = Null(E \wedge F), \varepsilon \in L_{RE} \equiv Null(E)) \\
&= Null(a \cup \varepsilon) \wedge Null(b^*) \\
&= (Defn. \quad Null(E \cup F) = Null(E \vee F), Null(E^*) = true) \\
&= (Null(a) \vee Null(\varepsilon)) \wedge true \\
&= true \wedge true = true
\end{aligned}$$

$$Last(a \cup \varepsilon) = Last(a) \cup Last(\varepsilon) = \{a\} \cup \emptyset = \{a\}$$

$$First(b^*) = First(b) = \{b\}$$

$$Follow(a \cup \varepsilon) = Follow(a) \cup Follow(\varepsilon) = \emptyset \cup \emptyset = \emptyset$$

$$Follow(b^*) = Follow(b) \cup (Last(b) \times First(b)) = \emptyset \cup \{(b, b)\} = \{(b, b)\}$$



$$\begin{aligned}
Follow((a \cup \varepsilon)b^*) &= Follow(a \cup \varepsilon) \cup Follow(b^*) \cup (Last(a \cup \varepsilon) \times First(b^*)) \\
&= \emptyset \cup \{(b, b)\} \cup \{(\{a\} \times \{b\})\} \\
&= \{(b, b)\} \cup \{(a, b)\} \\
&= \{(a, b), (b, b)\}
\end{aligned}$$

$$L_0 = L, L_1 = w^{-1}L, L_2 = a^{-1}(w^{-1}L) = (aw)^{-1}L$$

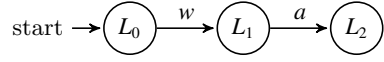


图 1.28: Construction 5.19(MNmin)

$$E_0 = E, E_1 = [v^{-1}E]_{\sim}, E_2 = \{a^{-1}[v^{-1}E]_{\sim}\} = \{[va]_{\sim}^{-1}E\}$$

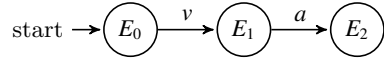


图 1.29: Construction 5.34 (Brzozowski)

$$(\forall u, a, u \in V^* \wedge a \in V), (\exists v \in V^*, [u]_E \cdot a \subseteq [v]_E)$$

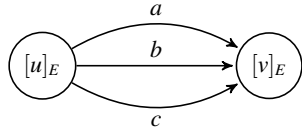
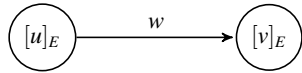


图 1.30: Definition 5.2 (Right invariance of an equivalence relation)

$$u, w \in V^*, (\forall u, w, (\exists v \in V^*, [u]_E \cdot \{w\} \subseteq [v]_E))$$

图 1.31: Right invariance of an equivalence relation  $[u], [v]$

$M = (Q, V, E, s, F), M$  所确定的  $V^*$  上的关系  $R_M$  定义为: 对于  $\forall x, y \in V^*$ ,

$$xR_M y \Leftrightarrow T^*(s, x) = T^*(s, y)$$

$\Rightarrow$

$$xR_M y \Leftrightarrow \exists q \in Q, x, y \in \overleftarrow{L}(q)$$

按照这个定义所得的关系  $R_M$ , 实际上是  $V^*$  上的等价关系, 利用这个关系, 可以将  $V^*$  划分成不多于  $|Q|$  个等价类。

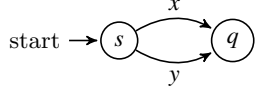


图 1.32: Equivalence classes of an equivalence relation

$\forall x, y \in V^*$ , 如果  $xR_L y$ , 则在  $x$  和  $y$  后无论接  $V^*$  中的任何字符串  $z$ ,  $xz$  和  $yz$  要么都属于  $L$ , 要么都不属于  $L$ 。

$$xR_L y \Leftrightarrow (\forall z \in V^*, xz \in L \Leftrightarrow yz \in L)$$

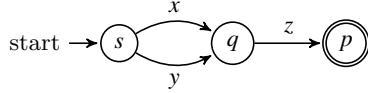


图 1.33: Equivalence classes of an equivalence relation

Def:

$$M = (Q, V, T, E, S, F), T^* \in (Q \times Q) \rightarrow \mathbb{P}(V^*)$$

$$\overleftarrow{L}_M(q), \overrightarrow{L}_M(q) \in Q \rightarrow \mathbb{P}(V^*)$$

$$\overleftarrow{L}_M(q) = \{x | x \in V^*, T^*(s, q) = x, s \in S\}$$

$$\overrightarrow{L}_M(q) = \{x | x \in V^*, T^*(q, f) = x, f \in F\}$$

$$L_{FA}(Q, V, T, E, S, F) = \bigcup_{f \in F} (\overleftarrow{L}(f))$$

## 1.10 Others

**Definition 1.15 (Prefix-closure[Chrison2007]).** Let  $L \subseteq V^*$ , then

$$\overline{L} := \{s \in V^* : (\exists t \in V^*)[st \in L]\}$$

In words, the prefix closure of  $L$  is the language denoted by  $\overline{L}$  and consisting of all the prefixes in  $L$ . In general,  $L \subseteq \overline{L}$ .

$L$  is said to be prefix-closed if  $L = \bar{L}$ . Thus language  $L$  is prefix-closed if any prefix of any string in  $L$  is also an element of  $L$ .

$L_1 = \{\varepsilon, a, aa\}, \bar{L}_1 = \bar{L}_1, L_1$  is prefix-closed.

$L_2 = \{a, b, ab\}, \bar{L}_2 = \{\varepsilon, a, b, ab\}, L_2 \subset \bar{L}_2, L_2$  is not prefix closed.

**Definition 1.16 (Post-closure[Chrison2007]).** Let  $L \subseteq V^*$  and  $s \in L$ . Then the post-language of  $L$  after  $s$ , denoted by  $L/s$ , is the language

$$L/s := \{t \in V^* : st \in L\}$$

By definition,  $L/s = \emptyset$  if  $s \notin \bar{L}$ .

**Definition 1.17 (Left derivatives[WATSON93a]).** Given language  $A \subseteq V^*$  and  $w \in V^*$  we define the left derivative of  $A$  with respect to  $w$  as:

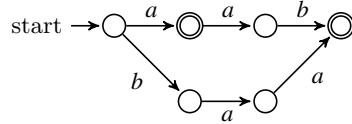
$$w^{-1}A = \{x \in V^* : wx \in A\}$$

$A$  关于  $w$  的左导数，就是  $A$  中：  $\{w$  的后缀组成的字符串集合}。

Sometimes derivatives are written as  $D_w A$  or as  $\frac{dA}{dw}$ . Right derivatives are analogously defined. Derivatives can also be extended to  $B^{-1}A$  where  $B$  is also a language.

*Example 1.4.*  $A = \{a, aab, baa\}, a^{-1}A = D_a A = \frac{dA}{da} = \{\varepsilon, ab, \emptyset\} = \{\varepsilon, ab\}$   $\square$

$L = \{a, aab, baa\}$



$a^{-1}L = \{\varepsilon, ab, \emptyset\} = \{\varepsilon, ab\}$

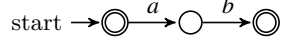
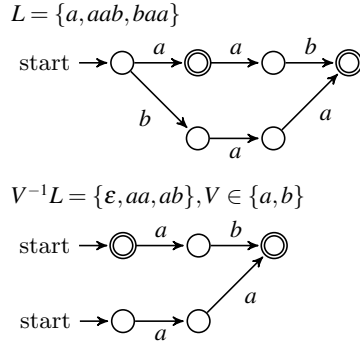


图 1.34:  $a^{-1}L$

图 1.35:  $V^{-1}L$ 

*Example 1.5.*  $L = \{ba, baa, baab, ca\}, w = \{ba\},$

$$w^{-1}L = \{\varepsilon, a, ab, \emptyset\} = \{\varepsilon, a, ab\}$$

$$(wa)^{-1}L = (baa)^{-1}L = \{\emptyset, \varepsilon, b, \emptyset\} = \{\varepsilon, b\}$$

$$a^{-1}(w^{-1}L) = a^{-1}\{\varepsilon, a, ab\} = \{\emptyset, \varepsilon, b\} = \{\varepsilon, b\}$$

$$w \in L \equiv \varepsilon \in w^{-1}L, \text{ and } (wa)^{-1}L = a^{-1}(w^{-1}L)$$

□

*Example 1.6.*  $a^{-1}\{a\} = \{\varepsilon\}; \quad a^{-1}\{b\} = \emptyset, \quad \Leftarrow \text{if } (a \neq b)$

□

*Example 1.7.*  $L_0 = \{ab\}, L_1 = \{ac\}, L_0L_1 = \{abac\}$

$$a^{-1}(L_0L_1) = \{bac\}$$

$$a^{-1}(L_0L_1) = (a^{-1}L_0)L_1 \cup \emptyset \quad \Leftarrow (\varepsilon \notin L_0)$$

$$= \{b\}L_1 = \{bac\}$$

□

*Example 1.8.*  $L_0 = \{\varepsilon, ab\}, L_1 = \{ac\}, L_0L_1 = \{ac, abac\}$

$$a^{-1}(L_0L_1) = \{c, bac\}$$

$$a^{-1}(L_0L_1) = (a^{-1}L_0)L_1 \cup a^{-1}L_1 \quad \Leftarrow (\varepsilon \in L_0)$$

$$= \{\emptyset, b\}L_1 \cup \{c\} = \{c, bac\}$$

□

证明.  $a^{-1}(L_0L_1)$

$$1. \text{if } (\varepsilon \in L_0) \Rightarrow a^{-1}(L_0L_1) = (a^{-1}L_0)L_1 \cup a^{-1}L_1$$

$$L_0 = (L_0 \setminus \{\varepsilon\}) \cup \{\varepsilon\}$$

$$a^{-1}(L_0L_1) = a^{-1}(((L_0 \setminus \{\varepsilon\}) \cup \{\varepsilon\})L_1)$$

$$= a^{-1}(L_0L_1 \cup L_1)$$

$$a^{-1}L_0 = a^{-1}((L_0 \setminus \{\varepsilon\}) \cup \{\varepsilon\})$$

$$\begin{aligned} &= a^{-1}(L_0 \setminus \{\varepsilon\}) \cup a^{-1}\{\varepsilon\} \\ &= a^{-1}L_0 \cup \emptyset = a^{-1}L_0 \end{aligned}$$

From [Hopcroft2008, p99]

(1) 如果  $L$  是一个语言,  $a$  是一个符号, 则  $L/a$  (称作  $L$  和  $a$  的商) 是所有满足如下条件的串  $w$  的集合:  $wa$  属于  $L$ 。例如, 如果  $L = \{a, aab, baa\}$ , 则  $L/a = \{\varepsilon, ba\}$ , 证明: 如果  $L$  是正则的, 那么  $L/a$  也是。提示: 从  $L$  的  $DFA$  出发, 考虑接受状态的集合。

(2) 如果  $L$  是一个语言,  $a$  是一个符号, 则  $a \backslash L$  是所有满足如下条件的串  $w$  的集合:  $aw$  属于  $L$ 。例如, 如果  $L = \{a, aab, baa\}$ , 则  $a \backslash L = \{\varepsilon, ab\}$ , 证明: 如果  $L$  是正则的, 那么  $a \backslash L$  也是。提示: 记得正则语言在反转运算下是封闭的, 又由 (1) 知, 正则语言的商运算下是封闭的。

**Definition 1.18 (Kleene-closure[Chrison2007]).** Let  $L \subseteq V^*$ , then

$$L^* := \{\varepsilon\} \cup L \cup LL \cup LLL \cup \dots$$

This is the same operation that we defined above for the set  $V$ , except that now it is applied to set  $L$  whose elements may be strings of length greater than one. An element of  $L^*$  is formed by the concatenation of a finite (but possibly arbitrarily large) number of elements of  $L$ ; this includes the concatenation of "zero" elements, that is the empty string  $\varepsilon$ . Note that  $*$  operation is idempotent:  $(L^*)^* = L^*$ .

$$\begin{aligned} L^* &= \{\varepsilon\} + L^+ \\ &= \{\varepsilon\} \cup (L \setminus \{\varepsilon\})L^* \\ &= \{\varepsilon\} + L + LL + LLL + \dots \end{aligned}$$

### 1.11 Linear equation

see [Jean2018, 5.3,p64].

We give an algorithm to covert an automaton to a rational(regular) expression. The algorithm amounts to solving a system of linear equations on languages. We first consider an equation of the form

$$X = KX + L \tag{1.1}$$

**Proposition 1.1 (Arden's Lemma).** *if  $K$  does not contain the empty word, then  $X = K^*L$  is the unique solution of the equation  $X = KX + L$ .*

where  $K$  and  $L$  are languages and  $X$  is the unknown. When  $K$  does not contain the empty word, the equation admits a unique solution.

证明. Replacing  $X$  by  $K^*L$  in the expression  $KX + L$ , one gets

$$K(K^*)L + L = K^+L + L = (K^+L + L) = K^*L,$$

and hence  $X = K^*L$  is a solution of (1.1). see<sup>1</sup>

To Prove uniqueness, consider two solutions  $X_1$  and  $X_2$  of (1.1). By symmetry, it suffices to show that each word  $u$  of  $X_1$  also belongs to  $X_2$ . Let us prove this result by induction on the length of  $u$ .

If  $|u| = 0$ ,  $u$  is the empty word<sup>2</sup> and if  $u \in X_1 = KX_1 + L$ , then necessarily  $u \in L$  since  $\varepsilon \notin K$ . But in this case,  $u \in KX_2 + L = X_2$ . see<sup>3</sup>

For the induction step, consider a word  $u$  of  $X_1$  of length  $n + 1$ . Since  $X_1 = KX_1 + L$ ,  $u$  belongs either to  $L$  or to  $KX_1$ . if  $u \in L$ , then  $u \in KX_2 + L = X_2$ . If  $u \in KX_1$  then  $u = kx$  for some  $k \in K$  and  $x \in X_1$ . Since  $k$  is not the empty word, one has necessarily  $|x| \leq n$  and hence by induction  $x \in X_2$ . [see<sup>4</sup>] It follows that  $u \in KX_2$  and finally  $u \in X_2$ . This conclude the induction and the proof of the proposition.  $\square$

From [Wonham2018, p74] The *length*  $|s|$  of a string  $s \in \Sigma^*$  is defined according to

$$|\varepsilon| = 0; |s| = k, \text{ if } s = \sigma_1 \sigma_2 \cdots \sigma_k \in \Sigma^+$$

Thus  $|cat(s, t)| = |s| + |t|$ .

---

1

$$\begin{aligned} K^* &= \{\varepsilon\} + K^+ \\ &= \{\varepsilon\} + (K \setminus \{\varepsilon\})K^* \\ &= \{\varepsilon\} + K + KK + KKK + \cdots \end{aligned}$$

<sup>2</sup> The empty word  $= \varepsilon$ ,  $|\varepsilon| = 0$ ; if a language  $M = \{\varepsilon\}$ ,  $|M| = 1$ , The empty language  $M = \emptyset$ ,  $|M| = 0$ . 文献 [Jean2018] 用 1 表示  $\varepsilon$ , 因为  $\varepsilon K = K\varepsilon = K$ , 因此,  $\varepsilon$  是连接运算的单位元, 正是 1 表示的用意。0 表示  $\emptyset$ , 它是并运算的单位元,  $K \cup \emptyset = \emptyset \cup K = K$ .

<sup>3</sup> In this case,  $|u| = 0$ ,  $X = \{\varepsilon\}$ ,  $|X| = 1$ . i.e.  $\varepsilon = K\varepsilon + L$ ,  $\varepsilon = K + L$

<sup>4</sup>  $u = kx$ ,  $|u| = |kx| = n + 1$ ,  $\varepsilon \notin K$ ,  $|k| \geq 1$ ,  $|x| \leq n$ , 由假设知,  $u$  属于  $X_1$ , 归纳  $|x| = 0$ ,  $|x| = 1, \dots, n$ ,  $x \in X_2$ .

A *language* over  $\Sigma$  is any subset of  $\Sigma^*$ , i.e. an element of the power set  $Pwr(\Sigma^*)$ ; thus the definition includes both the empty language  $\emptyset$ , and  $\Sigma^*$  itself.

Note the distinction between  $\emptyset$  (the language with no strings) and  $\varepsilon$  (the string with no symbols). For instance the language  $\{\varepsilon\}$  is nonempty, but contains only the empty string.

From [Wonham2018, p78]

**Proposition 1.2** ([Wonham2018]).

1. If  $L = M^*N$  then  $L = ML + N$
2. If  $\varepsilon \notin M$  then  $L = ML + N$  implies  $L = M^*N$  □

Part(2) is Known as Arden's rule. Taken with Part(1) it says that if  $\varepsilon \notin M$  then  $L = M^*N$  is the unique solution of  $L = ML + N$ ; in particular if  $L = ML$  (with  $\varepsilon \notin M$ ) then  $L = \emptyset$

**Exercise 1.1.** Show by counterexample that the restriction  $\varepsilon \notin M$  in Arden's rule cannot be dropped.

**Solution 1.1.** Examples text goes here.

**Exercise 1.2.** Prove Arden's rule. Hint: If  $L = ML + N$  then for every  $k \geq 0$

$$L = M^{k+1}L + (M^k + M^{k-1} + \cdots + M + \varepsilon)N$$

**Solution 1.2.**



*Preliminaries :*

$$M^* = M^k + M^{k-1} + \cdots + M^1 + M^0 \quad (k \geq 0)$$

$$= M^k + M^{k-1} + \cdots + M^1 + \varepsilon$$

$$= M^+ + \varepsilon$$

$$= MM^* + \varepsilon$$

$$= (M \setminus \{\varepsilon\})M^* + \varepsilon$$

$$M^+ = M^k + M^{k-1} + \cdots + M^1 \quad (k > 0)$$

$$= M(M^k + M^{k-1} + \cdots + M^1 + M^0)$$

$$= MM^*$$

$$M^0 = \{\varepsilon\} = 1$$

$$M\varepsilon = \varepsilon M = M$$

$$\varepsilon + \varepsilon = \varepsilon$$

$$M + M = M$$

证明.

$$L = ML + N \Rightarrow$$

$$M^0 L = M^1 L + M^0 N \quad (1.2)$$

$$M^1 L = M^2 L + M^1 N \quad (1.3)$$

$$M^2 L = M^3 L + M^2 N \quad (1.4)$$

$$(1.5)$$

...

$\Rightarrow$

$$(M^0 + M^1 + M^2 + \cdots)L = (M^1 + M^2 + M^3 + \cdots)L + (M^0 + M^1 + M^2 + \cdots)N$$

$\Rightarrow$

so, if  $L = ML + N$ , then for every  $k \geq 0$

$$L = M^{k+1}L + (M^k + M^{k-1} + \cdots + M + M^0)N$$

$\Rightarrow$

$$L = M^{k+1}L + (M^k + M^{k-1} + \cdots + M + \varepsilon)N \quad (1.6)$$

$$(1) \ k = 0$$

$$L = ML + (\varepsilon)N = ML + N$$

$$\Rightarrow (1 - M)L = N$$

$$(\varepsilon - M)L = N$$

由于  $\varepsilon \notin M$ , 左端不会消去  $\{\varepsilon\}$ . 因此, 只能在  $N$  中找  $L$ , 仅有唯一解:

$$L = \{\varepsilon\} = \{\text{empty word}\} \subseteq N.$$

From [R.Su and Wonham2004, definition 2.3]

**Definition 1.19.** Let

$$G_A = (X_A, \Sigma, \xi_A, x_{A,0}, X_{A,m})$$

$$G_B = (X_B, \Sigma, \xi_B, x_{B,0}, X_{B,m})$$

$G_B$  is a DES-epimorphic image(满射像) of  $G_A$  under DES-epimorphism

$\theta : X_A \rightarrow X_B$  if

1.  $\theta : X_A \rightarrow X_B$  is surjective(满射)
2.  $\theta(x_{A,0}) = x_{B,0}$  and  $\theta(X_{A,m}) = X_{B,m}$
3.  $(\forall x \in X_A)(\forall \sigma \in \Sigma) \xi_A(x, \sigma)! \Rightarrow [\xi_B(\theta(x), \sigma)! \& \xi_B(\theta(x), \sigma) = \theta(\xi_A(x, \sigma))]$
4.  $(\forall x \in X_B)(\forall \sigma \in \Sigma) \xi_B(x, \sigma)! \Rightarrow [(\exists x' \in X_A) \xi_A(x', \sigma)! \& \theta(x') = x]$

In particular,  $G_B$  is DES-isomorphic(同构) to  $G_A$  if  $\theta : X_A \rightarrow X_B$  is bijective(双射).

see figure 1.36.

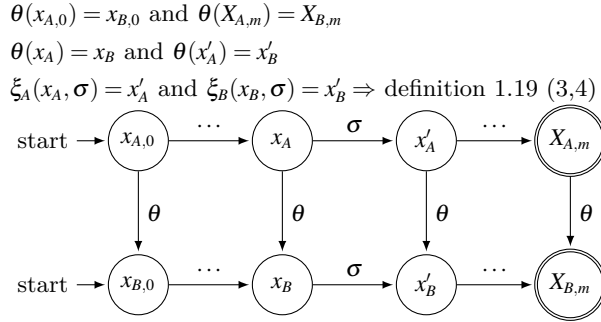


图 1.36: definition 1.19,  $G_B$  is a DES-epimorphic image(满射像) of  $G_A$   
under DES-epimorphism  $\theta : X_A \rightarrow X_B$

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