Chapter 1

automata abstract

1.1 Finite automata

Definition 1.1 (Finite automation). A finite automaton(an FA) is a 6-tuple (Q,V,T,E,S,F) where

- Q is a finite set of states,
- V is an alphabet,
- $T \in \mathbb{P}(Q \times V \times Q)$ is a transition relation,
- $E \in \mathbb{P}(Q \times Q)$ is an ε-transition relation
- $S \subseteq Q$ is a set of start states, and
- $F \subseteq Q$ is a set of final states.

□

```
class FA: virtual public FAabs {
        // Q is a finite set of states
        StatePool Q;
        // S is a set of start states, F is a
            set of final states
        StateSet S, F;
        // Transitions maps each State to its
            out-transitions.
        TransRel Transitions;
```

```
        // E is the epsilon transition
           relation.
        StateRel E;
}
```

StatePool:All states in an automaton are allocated from a StatePool. StatePool's can be merged together to form a larger one. (Care must be taken to rename any relations or sets (during merging) that depend on the one StatePool.) State is in [0,next)

```
class StatePool {
        int next; // The next one to be
           allocated.
}
```

StateSet:The StateSet is normally associated (implicitly) with a particular StatePool; whenever a StuteSet is to interact with another (from a different StatePool), its States must be renamed (to avoid a name clash). The capacity of a StateSet must be explicitly managed; many set operations are not bounds-checked when assert() is turned off.

```
class StateSet :protected BitVec {
        // How many States can this set
           contain?
        // [O, domain()) can be contained in
           *this.
        inline int domain() const;

        // set How many States can this set
           contain.
        // [O, r) can be contained in *this.
        inline void set_domain(const int r);
}
class BitVec {
```

```
            // uesd max number bits in data,
               denote width(domain),[0,
               bits_in_use) ==> [0,width)
            int bits_in_use;
            // number of words, 1,2,3,...
            int words;
            // save bytes of words,[0,1,2,...
               width(domain)]
            unsigned int *data;
    }
```

transition relation: $T \in Q \to \mathbb{P}(V \times Q), T(p) = \{(a,q) | (p,a,q) \in T\}$, 表示状态 p 的 out-transitions. see Fig 1.1

```
    // V --> Q
    struct TransPair {
    CharRange transition_label;
    State transition_destination;
    }
    class TransImpl { TransPair *data; }
    class Trans:protected TransIml { }

    // map: state(r) --> (T=Trans) out-
       transitions of r
    // SteteTo::data[r] = out-transitions of
       state r
    class TransRel:public StateTo<Trans> {}

    // map: state(r) --> T
    // data[r] = T
    template <class T> class SteteTo {
    T *data; // 动态数组的index(即状态的index)状
       态的out-transitions
    }
```

```
class FA: virtual public FAabs {
TransRel Transitions; // maps each State to its out-transitions.
}
```

ε-relation: $E \in \mathbb{P}(Q \times Q) \Rightarrow E \in Q \to \mathbb{P}(Q), E(p) = \{q | (p,q) \in E\}$, 表示 ε 连接状态 p 和状态 q.

```
// Implement binary relations on States. This is most often used for epsilon transitions.
// map: state(r) ---> {StateSet}
// StateTo::data[r] = {StateSet}, 表示状态r与{StateSet}的二元关系
class StateRel :protected StateTo<StateSet> {
}

class FA: virtual public FAabs {
// E is the epsilon transition relation.
StateRel E;
}
```

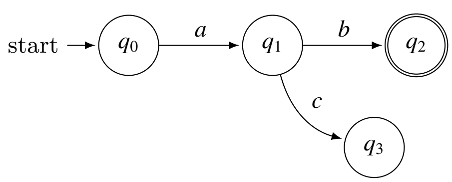


图 1.1: q_1 in-transition: $\{(q_0,a,q_1)\}$; q_1 out-transition: $\{(q_1,b,q_2),(q_1,c,q_3)\}$

[WATSON93a, p6] **the signatures of the transition relations:**

$T \in \mathbb{P}(Q \times V \times Q)$

$T \in V \to \mathbb{P}(Q \times Q)$

$T \in Q \times Q \to \mathbb{P}(V)$

$T \in Q \times V \to \mathbb{P}(Q)$

$T \in Q \to \mathbb{P}(V \times Q)$

for example, the function $T \in Q \to \mathbb{P}(V \times Q)$ is defined as $T(p) = \{(a,q) : (p,a,q) \in T\}$

$T \in \mathbb{P}(Q \times V \times Q), T = \{(p,a,q)\}$
$T \in Q \to \mathbb{P}(V \times Q), T(p) = \{(a,q) : (p,a,q) \in T\}$

$p,q \in Q, a \in V$
$T : Q \times V \to \mathbb{P}(Q)$
$T(p,a) = \{q\}$

According to Convention A.4 (Tuple projection):
$T \in \mathbb{P}(Q \times V \times Q), T = \{(p,a,q)\}$
$\pi_2(T) = \{a|(p,a,q) \in T\}, \bar{\pi}_2(T) = \{(p,q)|(p,a,q) \in T\}$
$T \in Q \to \mathbb{P}(V \times Q), T(p) = \{(a,q) : (p,a,q) \in T\}$
$\pi_2(T(p)) = \{q|(p,a,q) \in T\}, \bar{\pi}_2(T(p)) = \{a|(p,a,q) \in T\}$
$Q_{map} : Q \times V, T(p) = \{(a,q) : (p,a,q) \in T\}$
$Q_{map}(q) = \{a\}$
$Q_{map} : Q \times V, T \in \mathbb{P}(Q \times V \times Q$
$\pi_1(T) = \{p|(p,a,q) \in T\}, \bar{\pi}_1(T) = \{(a,q)|(p,a,q) \in T\}$
$Q_{map} = (\bar{\pi}_1(T))^R = \{(a,q)|(p,a,q) \in T\}^R = \{(q,a)|(p,a,q) \in T\}$

1.2 Properties of finite automata

$$M = (Q,V,T,E,S,F), M_0 = (Q_0,V_0,T_0,E_0,S_0,F_0), M_1 = (Q_1,V_1,T_1,E_1,S_1,F_1)$$

Definition 1.2 (Size of an FA). Define the size of an FA as $|M| = |Q|$

Definition 1.3 (Isomorphism 同构 ($\cong$) of $FA's$). We define isomorphism ($\cong$) as an equivalence relation on FA's. M_0 and M_1 are isomorphic (written $M_0 \cong M_1$) if and only if $V_0 = V_1$ and there exists a bijection 双射 $g \in Q_0 \to Q_1$ such that

- $T_1 = \{(g(p),a,g(q))|(p,a,q) \in T_0\}$

- $E_1 = \{(g(p), g(q)|(p,q) \in E_0\}$
- $S_1 = \{g(s)|s \in S_0\}$ and
- $F_1 = \{g(f)|f \in F_0\}$

(see Fig 1.2). □

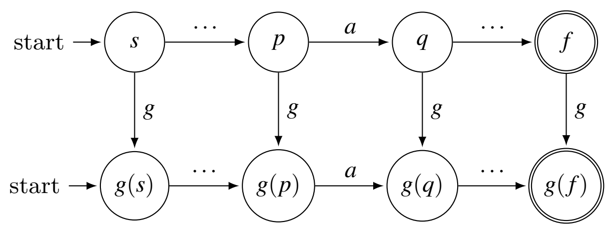


图 1.2: Isomorphism$M_0 \cong M_1$ if and only if $V_0 = V_1$ and there exists a bijection $g \in Q_0 \to Q_1$

Definition 1.4 (Extending the transition relation T). We extend transition relation $T \in V \to \mathbb{P}(Q \times Q)$ to $T^* \in V^* \to \mathbb{P}(Q \times Q)$ as follows:

$T^*(\varepsilon) = E^*$

and (for $a \in V, w \in V^*$)

$T^*(aw) = E^* \circ T(a) \circ T^*(w)$

Operator $\circ$ (composition is defined in Convention 1).

This definition could also have been presented symmetrically. □

Note 1.1. $s_1, s_2, s_3, s_4 \in Q, a \in V, w \in V^*$

$E = T(\varepsilon) = \{(s_1, s_2)\}, T(a) = \{(s_2, s_3)\}, T^*(w) = \{(s_3, s_4)\}$

$$T^*(aw) = E^* \circ T(a) \circ T^*(w)$$
$$= \{(s_1, s_2)\} \circ \{(s_2, s_3)\} \circ \{(s_3, s_4)\} \quad = \{(s_1, s_3)\} \circ \{(s_3, s_4)\} = \{(s_1, s_4)\}$$

Note 1.2. $T \in Q \times V \to \mathbb{P}(Q)$, extend to: $T^* \in Q \times V^* \to \mathbb{P}(Q)$

$\forall q \in Q, w \in V^*, a \in V$,

1. $T^*(q, \varepsilon) = q$
2. $T^*(q, wa) = T(T^*(q, w), a)$

$$\begin{aligned} T^*(q,a) &= T^*(q,\varepsilon a) \\ &= T(T^*(q,\varepsilon),a) \\ &= T(q,a) \end{aligned}$$

两值相同，不用区分这两个符号。

Convention 1 (Relation composition) *Given sets A,B,C (not necessarily different) and two relations, $E \subseteq A \times B$ and $F \subseteq B \times C$, we define relation composition (infix operator 中缀操作符 $\circ$) as:*

$E \circ F = \{(a,c) | (\exists b \in B), (a,b) \in E \wedge (b,c) \in F)\}$ □

Note 1.3. if $\exists b \in B, (a,b) \in E, (b,c) \in F$, then

$E : A \to B \Rightarrow E(a) = b$

$F : B \to C \Rightarrow F(b) = c$

$E \circ F = \{(a,b)\} \circ \{(b,c)\} = \{a,c\}$

$$\begin{aligned} (E \circ F)(a) &= F(E(a)) \\ &= F(b) = c \end{aligned}$$

$$\begin{tikzcd} A \arrow[r, "E"] \arrow[dr, "f"'] & B \arrow[d, "F"] \\ & C \end{tikzcd}$$

图 1.3: $E \circ F = (F \circ E)(a) = F(E(a)) = c = f(a)$

Remark 1.1. We also sometimes use the signature $T^* \in Q \times Q \to \mathbb{P}(V^*)$ □

Note 1.4. $T(p,q) = \{w | p,q \in Q, w \in V^*\}$

Remark 1.2. if $E = \emptyset$ then $E^* = \emptyset^* = I_Q$ where I_Q is the identity relation 单位关系 on the states of M.

Definition 1.5 (The language between states). The language between any two states $q_0, q_1 \in Q$ is $T^*(q_0, q_1)$. □

Definition 1.6 (Left and right languages). The left language of a state (in M) is given by function, $\overleftarrow{L}_M \in Q \to \mathbb{P}(V^*)$,where

$\overleftarrow{L}_M(q) = (\cup s : s \in S : T^*(s,q))$

The right language of a state (in M) is given by function $\overrightarrow{L}_M \in Q \to \mathbb{P}(V^*)$, where

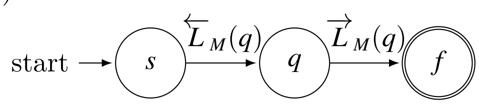
$\overrightarrow{L}_M(q) = (\cup f : f \in F : T^*(q,f))$

The subscript M is usually dropped when no ambiguity can arise.□

Example 1.1. $T^* \in Q \times Q \to \mathbb{P}(V^*), \overleftarrow{L}_M, \overrightarrow{L}_M \in Q \to \mathbb{P}(V^*)$.

$\overleftarrow{L}_M(q)$ ={能引导 M 从开始状态到达 q 状态的字符串集合}, (从 q 往左看)

$\overrightarrow{L}_M(q)$ ={能引导 M 从开始状态到达 q 状态的字符串集合}, (从 q 往右看)



see Fig 1.4.

$$\begin{aligned}
\overleftarrow{L}_M(q_2) &= (s \to q_1 \to q_2) \cup (s \to (q_1 \to q_3)^* \to q_1 \to q_2) \cup (s \to (q_1 \to q_3)^* \to q_3 \to q_2) \\
&= [(s \to q_1 \to q_2) \cup (s \to (q_1 \to q_3)^* \to q_1 \to q_2)] \cup (s \to (q_1 \to q_3)^* \to q_3 \to q_2) \\
&= (s \to (q_1 \to q_3)^* \to q_1 \to q_2) \cup (s \to (q_1 \to q_3)^* \to q_3 \to q_2) \\
&= \{1(10)^*0, 1(10)^*1\} \\
\overrightarrow{L}_M(q_2) &= \{01^*0, 10^*1(001^*0 + (10)^*1)\}
\end{aligned}$$

Definition 1.7 (Language of an *FA*). The language of a finite automaton (with alphabet V) is given by the function $L_{FA} \to \mathbb{P}(V^*)$ defined as:

$L_{FA}(M) = (\cup s, f : s \in S \wedge f \in F : T^*(s,f))$ (所有从开始状态到接受状态的字符串集合) □

Property 1.1 (Language of an FA). From the definition of left and right languages (of a state), we can also write:

$L_{FA}(M) = (\cup f : f \in F : \overleftarrow{L}(f))$ (所有从 s 到 f 的字符串集合，从 f 向左看)

and

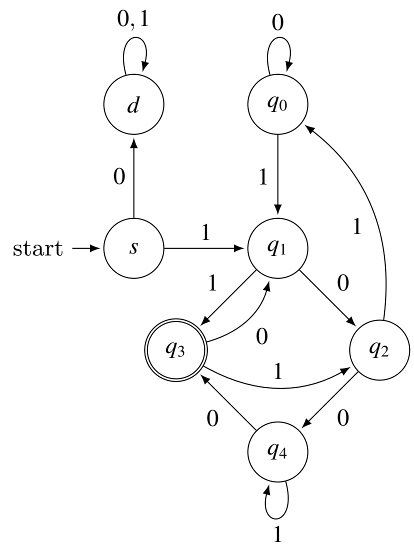


图 1.4: $\{x|x \in \{0,1\}^+$且当把 x 看成二进制数时，x 模 5 与 3 同余，要求当 x 为 0 时，$|x|=1$, 且当 $x \neq 0$ 时，x 的首字符为 1$\}$ 语言对应的 *DFA*

$L_{FA}(M) = (\cup s : s \in S : \overrightarrow{L}(s))$ (所有从 s 到 f 的字符串集合，从 s 向右看) □

Definition 1.8 (ε-$free$ 无 ε 转移). Automaton M is ε-$free$ if and only if $E = \emptyset$. □

Remark 1.3. Even if M is ε-$free$ it is still possible that $\varepsilon \in L_{FA}(M)$: $inthiscaseS \cap F \neq \emptyset$. (开始状态也是接受状态) □

Form [WATSON93a, Convention A.4] (Tuple projection).

Convention 2 (Tuple projection) *For an n-tuple $t = (x_1, x_2, \ldots, x_n)$ we use the notation $\pi_i(t)(1 \leq i \leq n)$ to denote tuple element x_i; we use the notation $\bar{\pi}_i(t)(1 \leq i \leq n)$ to denote the $(n-1)$-tuple $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Both $\pi$and $\bar{\pi}$ extend naturally to sets of tuples.* □

Form [WATSON93a, Definition A.20] (Tuple and relation reversal).

Definition 1.9 (Tuple and relation reversal). For an n-tuple $(x_1, x_2, \ldots, x_n)$ define reversal as (postfix and superscript) function R:

$(x_1, x_2, \ldots, x_n)^R = (x_n, x_n - 1, \ldots, x_2, x_1)$

Given a set A of tuples, we define $A^R = \{x^R : x \in A\}$. □

Definition 1.10 (Reachable states). For M we can define a reachability relation $Reach(M) \subseteq (Q \times Q)$ defined as

$Reach(M) = (\bar{\pi}_2(T) \cup E)^*$ see[1]

Functions π and $\bar{\pi}$ are defined in Convention 2. Similarly the set of start-reachable states is defined to be:

$SReachable(M) = Reach(M)(S)$ see[2]

and the set of final-reachable states is defined to be:

$FReachable(M) = (Reach(M))^R(F)$ see[3]

Reversal of a relation is defined in Definition 1.9. The set of useful states is: $Reachable(M) = SReachable(M) \cap FReachable(M)$ □

Remark 1.4. For *FA* $M = (Q, V, T, E, S, F)$,function *SReachable* satisfies the following interesting property:

$q \in SReachable(M) \equiv \overleftarrow{L}_M(q) \neq \emptyset$

FReachable satisfies a similar property:

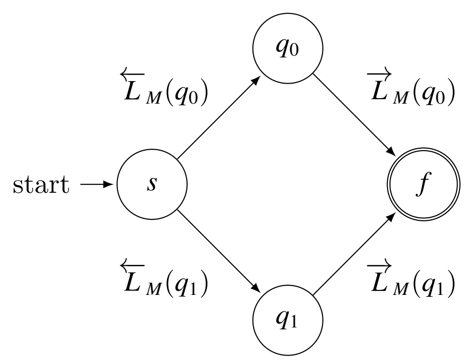
$q \in FReachable(M) \equiv \overrightarrow{L}_M(q) \neq \emptyset$ □

Example 1.2. $T \in \mathbb{P}(Q \times V \times Q), T = \{(p, a, q) | p, q \in Q, a \in V\}$,

$\bar{\pi}_2(T) = \{(p, q) | (p, a, q) \in T\}$

$Q_{map} = (\bar{\pi}_1(T))^R, Q_{map} = \{(a, q) | (p, a, q) \in T\}^R = \{(q, a) | (p, a, q) \in T\}$

□



[1] $\{(p_1, q_1), (p_2, q_2), \ldots\}$

[2] 从 start state 可以到达的状态集合

[3] 可以到达 final state 的状态集合

e.g. $p=\{1,2\}\in Q_1\subseteq\mathbb{P}(Q_0), \overrightarrow{L}_{M_1}(p)=\overrightarrow{L}_{M_0}(1)\cup\overrightarrow{L}_{M_0}(2)$

$$\begin{tikzcd}
Q \arrow[r, "subset"] \arrow[dr, "f"'] & P(Q) \arrow[d, "useful_s"] \\
& D
\end{tikzcd}$$

图 1.5: $subset\circ usefus_s=usefus_s(subset(Q,V,\emptyset,S,F))=(D,V,T',\emptyset,S',F')$

1.3 Σ-algebras and regular expressions

Σ-homomorphism

X 集合中的元素与有序集 S 中的元素一一对应，称 X 是 S-sorted.
$S=\{1,3,7,9\},X=\{d,a,c,f\},s\in S,X_s\in X$
S 是有序的,$S_{s_1}=1,S_{s_2}=3,S_{s_3}=7,S_{s_4}=9$
X 与 S 中的元素一一对应。$X_{s_1}=d,X_{s_2}=a,X_{s_3}=c,X_{s_4}=f$

Σ-homomorphism 同态: $(V,F)\Leftrightarrow(W,G)$，载体 (V,W) 和操作 (F,G) 一一对应。
Σ-homomorphism function: $h\in V\to W$
$L(v)=(h\circ f)(v)=h(f(v))=g(w)=L_{reg}=L_V=L_W$
$L(v)=(g\circ h)(v)=g(h(v))=g(w)=L_{reg}=L_V=L_W$
$\Rightarrow h(f(v))=g(h(v))$

Example 1.3. $\Sigma=(S,\Gamma)$, sort: $expr$, $\Gamma:=\{a,plus\},a$ is a constant. operator $plus: expr\times expr\to expr$.
$\Sigma-$term algebra: $plus[a,a],plus[plus[a,plus[a,a]],a]$
$\Sigma-$algebra X,carrier set: natural number, constant 0. operator $f_{plus}(x,y)=(x\quad max\quad y)+1$

$$\begin{tikzcd}
V \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, "L"] & h(v) \arrow[d, "g"] \\
f(v) \arrow[r, "h"] & g(h(v))
\end{tikzcd}$$

图 1.6: $(h \circ f)(v) = (g \circ h)(v) \Rightarrow h(f(v)) = g(h(v))$

$$\begin{tikzcd}
(e_1, e_2) \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, "L"] & (h(e_1), h(e_2)) \arrow[d, "g"] \\
f(e_1, e_2)\circ \arrow[r, "h"] & g(h(e_1), h(e_2))
\end{tikzcd}$$

图 1.7: $(h \circ f)(e_1, e_2) = (g \circ h)(e_1, e_2) \Rightarrow h(f(e_1, e_2)) = g(h(e_1), h(e_2))$

Σ-homomorphism function("expression tree height"): $h_{expr} : \Sigma-$term algebra $\rightarrow X$

$$\begin{aligned}
(h_{expr} \circ plus)(s) &= (f_{plus} \circ h_{expr})(s) \\
h_{expr}(plus(s)) &= f_{plus}(h_{expr}(s)) \\
left : s \leftarrow e, f \quad &\Rightarrow \quad plus[e, f] \\
right : s \leftarrow e, f \quad &\Rightarrow \quad f_{plus}(h_{expr}(e), h_{expr}(f) \\
h_{expr}(plus(e, f)) &= f_{plus}(h_{expr}(e), h_{expr}(f)) \\
&= (h_{expr}(e) \quad max \quad h_{expr}(f)) + 1) \\
and, & \\
h_{expr}(a) &= 0
\end{aligned}$$

$$\begin{tikzcd}
S \arrow[r, "h_{expr}"] \arrow[d, "plus"'] \arrow[dr, "L"] & X \arrow[d, "f_{plus}"] \\
\circ \arrow[r, "h_{expr}"] & \circ
\end{tikzcd}$$

图 1.8: $(h_{expr} \circ plus)(s) = (f_{plus} \circ h_{expr})(s) \Rightarrow h_{expr}(plus(s)) = f_{plus}(h_{expr}(s))$

$$
\begin{tikzcd}
(e,f) \arrow[r, "h_{expr}"] \arrow[d, "plus"'] \arrow[dr, "L"] & (h_{expr}(e),h_{expr}(f)) \arrow[d, "f_{plus}"] \\
plus[e,f] \arrow[r, "h_{expr}"] & f_{plus}(h_{expr}(e),h_{expr}(f))
\end{tikzcd}
$$

图 1.9: $(h_{expr}\circ plus)(e,f)=(f_{plus}\circ h_{expr})(e,f)\Rightarrow h_{expr}(plus[e,f])=f_{plus}(h_{expr}(e),h_{expr}(f))$

Definition 1.11 (Regular expressions). We define regular expressions (over alphabet V) as the $\Sigma-$term algebra over signature $\Sigma=(S,O)$ where

- S consists of a single sort Reg (for regular expression), and
- O is a set of several constans: $\varepsilon,\emptyset,a_1,a_2,\ldots,a_n;Reg$ (where $V=\{a_1,a_2,\ldots,a_n\}$) and five operators $.:Reg\times Reg\to Reg$ (the dot operator), $\cup:Reg\times Reg\to Reg,*:Reg\to Reg,+:Reg\to Reg,$ and$?:Reg\to Reg$.

$V:=RE$(正则表达式), $W:=FA$(有限自动机)
Σ-homomorphism function: $f\in RE\to FA$
$F:RE_{opt}$ 运算，二元: union(or),concat; 一元: star,plus,question;
常量:epsilon,empty,symbol
$G:FA_{opt}$ 运算, 同上

$$
\begin{tikzcd}
RE \arrow[r, "f"] \arrow[dr, "L_{RE}"'] & FA \arrow[d, "L_{FA}"] \\
& L_{reg}
\end{tikzcd}
\qquad
\begin{tikzcd}
(re_1,re_2) \arrow[r, "f"] \arrow[d, "RE_{opt}"'] \arrow[dr] & (f(re_1),f(re_2)) \arrow[d, "FA_{opt}"] \\
RE_{opt}(re_1,re_2) \arrow[r, "f"] & FA_{opt}(f(re_1),f(re_2))
\end{tikzcd}
$$

```
//Sigma.h
template<class T>
class Reg :public T {
// Helper for constructing the homomorphic
   image of a regular expression.
// T is carrier set: RE,FA,RFA，
// 各自的操作，分别在Sig-RE.cpp,Sig-FA.cpp,
   Sig-RFA.cpp中定义
```

```
    inline void homomorphic_image(const RE& r);
    Reg<T>& epsilon();
    Reg<T>& empty();
    Reg<T>& symbol(const CharRange r);
    Reg<T>& Or(const Reg<T>& r);
    Reg<T>& concat(const Reg<T>& r);
    Reg<T>& star();
    Reg<T>& plus();
    Reg<T>& question();
    }
```

Definition 1.12 (The *nullable* $\Sigma-$algebra). We define the *nullable* $\Sigma-$algebra as follows:

- The carrier set is $\{true, false\}$.
- $a \in V, E_1, E_2 \in RE, \varepsilon \in E_1^*, \varepsilon \in E_1^?, \varepsilon \notin E_1^+$

$$
\begin{aligned}
nullable(\varepsilon) &= true \\
nullable(\emptyset) &= nullable(a) = false \\
nullable(E_1 \vee E_2) &= nullable(E_1 \cup E_2) \\
nullable(E_1 \wedge E_2) &= nullable(E_1 \cdot E_2) \\
nullable(E_1^*) &= true \\
nullable(E_1^+) &= nullable(E_1) \\
nullable(E_1^?) &= true \\
nullable(E_1) &= \begin{cases} true & \varepsilon \in E_1 \\ false & \varepsilon \notin E_1 \end{cases}
\end{aligned}
$$

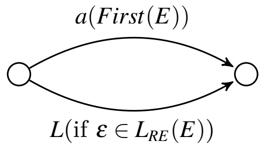


图 1.10: Lookahead function:$look(E,L) = First \cup$ if $(Null(E))$ then L else $\emptyset$ fi

$\varepsilon \in L_{FA}(M) \equiv s \in F$

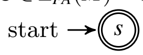


图 1.11: $\varepsilon \in L_{FA}(M) \equiv s \in F$

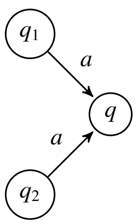


图 1.12: All in-transitions to a state are on the same symbol (in V)

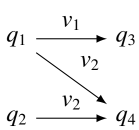


图 1.13: $q_4(in-transition) = \{(q_1,b),(q_2,b)\}, q_1(out-transition) = \{(a,q_3),(b,q_4)\}$

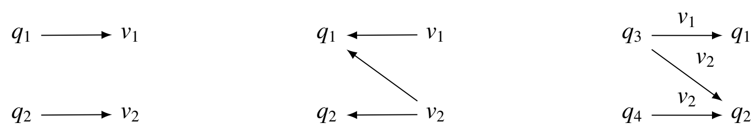
1.4 Constructing ε-lookahead automata

1.5 Towards the Berry-Sethi construction

a	b	c

Definition 1.13 (RFA). A reduced FA (RFA) is a 7-tuple $(Q,V,follow,first,last,null,Q_{map})$ where

- Q is a finite set of states,
- V is an alphabet,



(a) $Q \to V$ 一一对应关系 (b) $V \nrightarrow \mathbb{P}(Q)$ 一对多的关系 (c) 物理含义, 进入状态的字母是唯一的。

$Q_{map}(q_1) = \{v_1\}, Q_{map}^{-1}(v_1) = \{q_1\}, (q_3, v_1, q_1) \in T$

$Q_{map}(q_2) = \{v_2\}, Q_{map}^{-1}(v_2) = \{q_2\}, (q_3, v_2, q_2) \in T, (q_4, v_2, q_2) \in T$

图 1.14: Q_{map}, Q_{map}^{-1}

- $follow \in \mathbb{P}(Q \times Q)$ is a follow relation (relpace the transition relation: $T \in \mathbb{P}(Q \times V \times Q)$) ,
- $first \subseteq Q$ is a set of initial states (replacing T(s) in an LBFA),
- $null \in \{true, false\}$ is a Boolean value (encoding $s \in F$ in an LBFA,$\varepsilon \in L_{FA}(M) \equiv s \in F$), and
- $Q_{map} \in \mathbb{P}(Q \times V), Q_{map}(q) = \{v\}, one \to one$. maps each state to exactly one symbol. i.e. $Q_{map} \in Q \to V$.
  $Q_{map}(q) = \{a | (p, a, q) \in T\}$ 表示 (q,v) 的一一对应关系。物理含义是进入 q 状态的唯一字母 a
  class RFA 中表示 its inverse: $Q_{map}^{-1} : V \nrightarrow \mathbb{P}(Q)$,部分函数$Q_{map}^{-1}(a) = \{q | (p, a, q) \in T\}$

□

```
class RFA : virtual public FAabs{
// Q is a finite set of states
StatePool Q;

// first(subset Q) is a set of initial states
    (replacing T(s) in an LBFA),
// last(subset Q) is a set of final states,
StateSet first , last;

```

```
// Qmap (in P( Q x V)) maps each state to
   exactly one symbol (it is also viewed as
   Qmap in Q —> V,
// and its inverse as Qmap^−1 in V —/—>P(Q)
   [the set of all partial functions from V
   to P(Q)]).
// Trans用struct TransPair 表示:T(a) = { q |
   (p,a,q) in T },
// 因此这里表示Qmap的inverse, V —> P(Q)
Trans Qmap_inverse;

// follow(in P(Q x Q)) is a follow relation(
   replacing the transition relation),
StateRel follow;

// null (in {true,false}) is a Boolean value
   (encoding s in F in an LBFA)
// if epsilon属于LBFA, true; final set中包含s
// {true, flase} ==> {1, 0}
int Nullable;
}
```

```
// V —> Q
struct TransPair {
        CharRange transition_label;
        State transition_destination;
}

class TransImpl { TransPair *data; }
class Trans:protected TransIml { }

}
```

$rfa \circ R(E) = R \circ rfa(E)$

$$\begin{tikzcd}
E \arrow[r, "rfa"] \arrow[d, "R"'] \arrow[dr] & RFA \arrow[d, "R"] \\
\circ \arrow[r, "rfa"'] & \circ
\end{tikzcd}$$

图 1.15: $rfa \circ R(E) = R \circ rfa(E)$

Definition 1.14. (Dual of a function) We assume two sets A and B whose reversal operators are R and R' respectively. Two functions, $f \in A \to B$ and $f_d \in A \to B$ are one another's dual if and only if

$f(a) = (f_d(a^R))^{R'}$

In some cases we relax the equality to isomorphism (when isomorphism is defined on B). □

$f_d \circ R(a) = R' \circ f(a) \Rightarrow f_d(R(a)) = R'(f(a)) \Rightarrow$
$f_d(a^R) = (f(a))^{R'} \Rightarrow f(a) = (f_d(a^R))^{R'}$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "R"'] \arrow[dr] & B \arrow[d, "R'"] \\
\circ \arrow[r, "f_d"'] & \circ
\end{tikzcd}$$

图 1.16: $f(a) = (f_d(a^R))^{R'}$)

1.6 Others

Definition 1.15 (Prefix-closure[Chrison2007]). Let $L \subseteq V^*$,then

$$\overline{L} := \{s \in V^* : (\exists t \in V^*)[st \in L]\}$$

In words, the prefix closure of L is the language denoted by $\overline{L}$ and consisting of all the prefixes in L. In general, $L \subseteq \overline{L}$.

L is said to be prefix-closed if $L = \overline{L}$. Thus language L is prefix-closed if any prefix of any string in L is also an element of L.

$L_1 = \{\varepsilon, a, aa\}, L_1 = \overline{L_1}, L_1$ is prefix-closed.

$L_2 = \{a, b, ab\}, \overline{L_2} = \{\varepsilon, a, b, ab\}, L_2 \subset \overline{L_2}, L_2$ is not prefix closed.

Definition 1.16 (Post-closure[Chrison2007]). Let $L \subseteq V^*$ and $s \in L$. Then the post-language of L after s, denoted by L/s, is the language

$$L/s := \{t \in V^* : st \in L\}$$

By definition, $L/s = \emptyset$ if $s \notin \overline{L}$.

Definition 1.17 (Left derivatives[WATSON93a]). Given language $A \subseteq V^*$ and $w \in V^*$ we define the left derivative of A with respect to w as:

$$w^{-1}A = \{x \in V^* : wx \in A\}$$

A 关于 w 的左导数，就是 A 中：$\{w$ 的后缀组成的字符串集合$\}$。

Sometimes derivatives are written as $D_w A$ or as $\frac{dA}{dw}$. Right derivatives are analogously defined. Derivatives can also be extended to $B^{-1}A$ where B is also a language.

Example 1.4. $A = \{a, aab, baa\}, a^{-1}A = D_a A = \frac{dA}{da} = \{\varepsilon, ab, \emptyset\} = \{\varepsilon, ab\}$ □

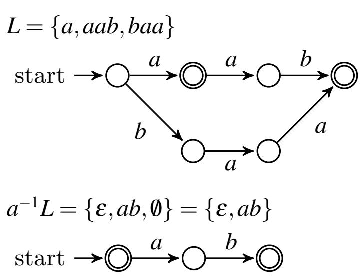
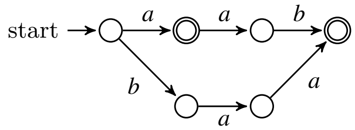


图 1.17: $a^{-1}L$

Example 1.5. $L = \{ba, baa, baab, ca\}, w = \{ba\}$,

$w^{-1}L = \{\varepsilon, a, ab, \emptyset\} = \{\varepsilon, a, ab\}$

$L = \{a, aab, baa\}$



$V^{-1}L = \{\varepsilon, aa, ab\}, V \in \{a, b\}$

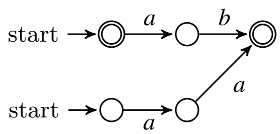


图 1.18: $V^{-1}L$

$$(wa)^{-1}L = (baa)^{-1}L = \{\emptyset, \varepsilon, b, \emptyset\} = \{\varepsilon, b\}$$
$$a^{-1}(w^{-1}L) = a^{-1}\{\varepsilon, a, ab\} = \{\emptyset, \varepsilon, b\} = \{\varepsilon, b\}$$
$$w \in L \equiv \varepsilon \in w^{-1}L, and (wa)^{-1}L = a^{-1}(w^{-1}L)$$ □

Example 1.6. $a^{-1}\{a\} = \{\varepsilon\}; \quad a^{-1}\{b\} = \emptyset, \quad \Leftarrow if(a \neq b)$ □

Example 1.7. $L_0 = \{ab\}, L_1 = \{ac\}, L_0L_1 = \{abac\}$

$$a^{-1}(L_0L_1) = \{bac\}$$
$$a^{-1}(L_0L_1) = (a^{-1}L_0)L_1 \cup \emptyset \quad \Leftarrow (\varepsilon \notin L_0)$$
$$= \{b\}L_1 = \{bac\}$$ □

Example 1.8. $L_0 = \{\varepsilon, ab\}, L_1 = \{ac\}, L_0L_1 = \{ac, abac\}$

$$a^{-1}(L_0L_1) = \{c, bac\}$$
$$a^{-1}(L_0L_1) = (a^{-1}L_0)L_1 \cup a^{-1}L_1 \quad \Leftarrow (\varepsilon \in L_0)$$
$$= \{\emptyset, b\}L_1 \cup \{c\} = \{c, bac\}$$ □

证明. $a^{-1}(L_0L_1)$

$$1. if(\varepsilon \in L_0) \Rightarrow a^{-1}(L_0L_1) = (a^{-1}L_0)L_1 \cup a^{-1}L_1$$
$$L_0 = (L_0 \backslash \{\varepsilon\}) \cup \{\varepsilon\}$$
$$a^{-1}(L_0L_1) = a^{-1}(((L_0 \backslash \{\varepsilon\}) \cup \{\varepsilon\})L_1)$$
$$= a^{-1}(L_0L_1 \cup L_1)$$
$$a^{-1}L_0 = a^{-1}((L_0 \backslash \{\varepsilon\}) \cup \{\varepsilon\})$$
$$= a^{-1}(L_0 \backslash \{\varepsilon\}) \cup a^{-1}\{\varepsilon\}$$
$$= a^{-1}L_0 \cup \emptyset = a^{-1}L_0$$

From [Hopcroft2008, p99]

(1) 如果 L 是一个语言，a 是一个符号，则 L/a(称作 L 和 a 的商) 是所有满足如下条件的串 w 的集合：wa 属于 L。例如，如果 $L = \{a, aab, baa\}$, 则 $L/a = \{\varepsilon, ba\}$, 证明：如果 L 是正则的，那么 L/a 也是。提示：从 L 的 DFA 出发，考虑接受状态的集合。

(2) 如果 L 是一个语言，a 是一个符号，则 $a\backslash L$ 是所有满足如下条件的串 w 的集合: aw 属于 L。例如，如果 $L = \{a, aab, baa\}$, 则 $a\backslash L = \{\varepsilon, ab\}$, 证明：如果 L 是正则的，那么 $a\backslash L$ 也是。提示：记得正则语言在反转运算下是封闭的，又由 (1) 知，正则语言的商运算下是封闭的。

Definition 1.18 (Kleene-closure[Chrison2007]). Let $L \subseteq V^*$, then

$$L^* := \{\varepsilon\} \cup L \cup LL \cup LLL \cup \cdots$$

This is the same operation that we defined above for the set V, except that now it is applied to set L whose elements may be strings of length greater than one. An element of L^* is formed by the concatenation of a finite (but possibly arbitrarily large) number of elements of L; this includes the concatenation of ”zero” elements, that is the empty string ε. Note that $*$ operation is idempotent: $(L^*)^* = L^*$.

$$\begin{aligned} L^* &= \{\varepsilon\} + L^+ \\ &= \{\varepsilon\} \cup (L\backslash\{\varepsilon\})L^* \\ &= \{\varepsilon\} + L + LL + LLL + \cdots \end{aligned}$$

1.7 Linear equation

see [Jean2018, 5.3,p64].

We give an algorithm to covert an automaton to a rational(regular) expression. The algorithm amounts to solving a system of linear equations on languages. We first consider an equation of the form

$$X = KX + L \tag{1.1}$$

Proposition 1.1 (Arden's Lemma). *if K does not contain the empty word, then $X = K^\star L$ is the unique solution of the equation $X = KX + L$.*

where K and L are languages and X is the unknown. When K does not contain the empty word, the equation admits a unique solution.

证明. Replacing X by K^*L in the expression $KX+L$,one gets

$$K(K^*)L+L=K^+L+L=(K^+L+L)=K^*L,$$

and hence $X=K^*L$ is a solution of (1.1). see[1]

To Prove uniqueness, consider two solutions X_1 and X_2 of (1.1). By symmetry,it suffices to show that each word u of X_1 also belongs to X_2. Let us prove this result by induction on the length of u.

If $|u|=0$, u is the empty word[2] and if $u\in X_1=KX_1+L$, then necessarily $u\in L$ since $\varepsilon\notin K$. But in this case, $u\in KX_2+L=X_2$. see[3]

For the induction step, consider a word u of X_1 of length $n+1$. Since $X_1=KX_1+L, u$ belongs either to L or to KX_1. if $u\in L$, then $u\in KX_2+L=X_2$. If $u\in KX_1$ then $u=kx$ for some $k\in K$ and $x\in X_1$. Since k is not the empty word, one has necessarily $|x|\leq n$ and hence by induction $x\in X_2$.[see[4]] It follows that $u\in KX_2$ and finally $u\in X_2$. This conclude the induction and the proof of the proposition. □

From [Wonham2018, p74] The *length* $|s|$ of a string $s\in\Sigma^*$ is defined according to

$$|\varepsilon|=0;|s|=k,\text{if } s=\sigma_1\sigma_2\cdots\sigma_k\in\Sigma^+$$

Thus $|cat(s,t)|=|s|+|t|$.

---

[1]

$$\begin{aligned}K^*&=\{\varepsilon\}+K^+\\&=\{\varepsilon\}+(K\backslash\{\varepsilon\})K^*\\&=\{\varepsilon\}+K+KK+KKK+\cdots\end{aligned}$$

[2] The empty word $=\varepsilon,|\varepsilon|=0$; if a language $M=\{\varepsilon\},|M|=1$, The empty language $M=\emptyset,|M|=0$. 文献 [Jean2018] 用 1 表示 ε, 因为 $\varepsilon K=K\varepsilon=K$, 因此,$\varepsilon$ 是连接运算的单位元，正是 1 表示的用意。0 表示 $\emptyset$, 它是并运算的单位元,$K\cup\emptyset=\emptyset\cup K=K$.

[3] In this case,$|u|=0,X=\{\varepsilon\},|X|=1$. *i.e.* $\varepsilon=K\varepsilon+L,\varepsilon=K+L$

[4] $u=kx,|u|=|kx|=n+1,\varepsilon\notin K,|k|\geq1,|x|\leq n$, 由假设知，$u$ 属于 X_1, 归纳 $|x|=0,|x|=1,\cdots,n,x\in X_2$.

A *language* over Σ is any subset of Σ^*, i.e. an element of the power set $Pwr(\Sigma^*)$; thus the definition includes both the empty language $\emptyset$, and Σ^* itself.

Note the distinction between $\emptyset$ (the language with no strings) and ε (the string with no symbols). For instance the language $\{\varepsilon\}$ is nonempty, but contains only the empty string.

From [Wonham2018, p78]

Proposition 1.2 ([Wonham2018]).

1. *If* $L = M^*N$ *then* $L = ML + N$
2. *If* $\varepsilon \notin M$ *then* $L = ML + N$ *implies* $L = M^*N$ □

Part(2) is Known as Arden's rule. Taken with Part(1) it says that if $\varepsilon \notin M$ then $L = M^*N$ is the unique solution of $L = ML + N$; in particular if $L = ML$ (with $\varepsilon \notin M$) then $L = \emptyset$

Exercise 1.1. Show by counterexample that the restriction $\varepsilon \notin M$ in Arden's rule cannot be dropped.

Solution 1.1. Examples text goes here.

Exercise 1.2. Prove Arden's rule. Hint: If $L = ML + N$ then for every $k \geq 0$

$$L = M^{k+1}L + (M^k + M^{k-1} + \cdots + M + \varepsilon)N$$

Solution 1.2.

Preliminaries :

$$\begin{aligned}
M^* &= M^k + M^{k-1} + \cdots + M^1 + M^0 \qquad (k \geq 0) \\
&= M^k + M^{k-1} + \cdots + M^1 + \varepsilon \\
&= M^+ + \varepsilon \\
&= MM^* + \varepsilon \\
&= (M \backslash \{\varepsilon\}) M^* + \varepsilon \\
M^+ &= M^k + M^{k-1} + \cdots + M^1 \qquad (k > 0) \\
&= M(M^k + M^{k-1} + \cdots + M^1 + M^0) \\
&= MM^* \\
M^0 &= \{\varepsilon\} = 1 \\
M\varepsilon &= \varepsilon M = M \\
\varepsilon + \varepsilon &= \varepsilon \\
M + M &= M
\end{aligned}$$

证明.

$$L = ML + N \Rightarrow$$

$$M^0 L = M^1 L + M^0 N \tag{1.2}$$
$$M^1 L = M^2 L + M^1 N \tag{1.3}$$
$$M^2 L = M^3 L + M^2 N \tag{1.4}$$
$$\tag{1.5}$$

$$\cdots$$

$\Rightarrow$

$(M^0 + M^1 + M^2 + \cdots)L = (M^1 + M^2 + M^3 + \cdots)L + (M^0 + M^1 + M^2 + \cdots)N$

$\Rightarrow$

so,if $L = ML + N$,then for every $k \geq 0$

$L = M^{k+1} L + (M^k + M^{k-1} + \cdots + M + M^0)N$

$\Rightarrow$

$$L = M^{k+1} L + (M^k + M^{k-1} + \cdots + M + \varepsilon)N \tag{1.6}$$

(1) $k=0$

$L = ML + (\varepsilon)N = ML + N$

$\Rightarrow (1-M)L = N$

$(\varepsilon - M)L = N$

由于 $\varepsilon \notin M$, 左端不会消去 $\{\varepsilon\}$. 因此, 只能在 N 中找 L, 仅有唯一解: $L = \{\varepsilon\} = \{\text{empty word}\} \subseteq N$.

From [R.Su and Wonham2004, definition 2.3]

Definition 1.19. Let

$G_A = (X_A, \Sigma, \xi_A, x_{A,0}, X_{A,m})$

$G_A = (X_B, \Sigma, \xi_B, x_{B,0}, X_{B,m})$

G_B is a DES-epimorphic image(满射像) of G_A under DES-epimorphism $\theta : X_A \to X_B$ if

1. $\theta : X_A \to X_B$ is surjective(满射)
2. $\theta(x_{A,0}) = x_{B,0}$ and $\theta(X_{A,m}) = X_{B,m}$
3. $(\forall x \in X_A)(\forall \sigma \in \Sigma)\xi_A(x,\sigma)! \Rightarrow [\xi_B(\theta(x),\sigma)!\&\xi_B(\theta(x),\sigma) = \theta(\xi_A(x,\theta))]$
4. $(\forall x \in X_B)(\forall \sigma \in \Sigma)\xi_B(x,\sigma)! \Rightarrow [(\exists x' \in X_A)\xi_A(x',\sigma)!\&\theta(x') = x]$

In particular,G_B is DES-isomorphic(同构) to G_A if $\theta : X_A \to X_B$ is bijective(双射).

see figure 1.19.

$\theta(x_{A,0}) = x_{B,0}$ and $\theta(X_{A,m}) = X_{B,m}$

$\theta(x_A) = x_B$ and $\theta(x'_A) = x'_B$

$\xi_A(x_A,\sigma) = x'_A$ and $\xi_B(x_B,\sigma) = x'_B \Rightarrow$ definition 1.19 (3,4)

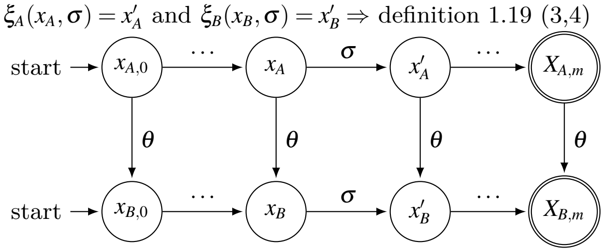


图 1.19: definition 1.19, G_B is a DES-epimorphic image(满射像) of G_A under DES-epimorphism $\theta : X_A \to X_B$

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