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Minimization of lattice multiset finite automata

Yongbing Wang[a,b,*] and Yongming Li[b]
[a]*College of Mathematics and Computation, Anqing Normal University, Anqing, China*
[b]*College of Computer Science, Shaanxi Normal University, Xi'an, China*

In this paper, we study the minimization of lattice-valued multiset finite automata with membership values in a distributive lattice. First, we establish the equivalence of (nondeterministic) lattice multiset finite automata (LMA) and deterministic lattice multiset finite automata (DLMA). Furthermore, we present some operations on lattice-valued regular multiset languages, and prove that the family of lattice-valued regular multiset languages is closed under this operations. We also introduce and study the minimal DLMAs and present an effective algorithm to obtain a minimal DLMA for a given LMA. Finally, we give a decomposition of lattice-valued regular multiset language by some simple lattice-valued regular multiset languages accepted by some special minimal DLMAs.

Keywords: Lattice-valued multiset finite automata, lattice-valued regular multiset languages, deterministic lattice-valued multiset finite automata, minimization, lattice-valued unitary language

1. Introduction

As a modeling tool, finite automata play an important role in various areas of Theoretical Computer Sciences, as in digital image compression or date base theory. The study of fuzzy finite automata was introduced by Santos [26, 27], Wee [31], Lee and Zadeh [12] in the late 1960s, which are studied as an extended model that reduces the gap between precision and vagueness of computer languages. For a comprehensive overview of the area, the reader is refereed to [9, 10, 14–16, 20, 23, 24] that highlights the recent development and ongoing research. In recent years, novel applications of fuzzy finite automata have emerged from numerous sciences, like learning systems, computing with words, fuzzy discrete event systems, pattern recognition, and database theory [13, 20, 23, 32].

The multiset (set with multiplicities associated with its element, in the form of natural numbers) is a notion which has appeared frequently in various areas of mathematics, computer science, biology and biochemistry (cf., [1, 5, 21, 25, 29], sometimes called a bag. Multiset languages and their characterizations have important applications such as in Concurrency Theory Membrane Computing, which can be characterized by grammars as well as finite automata. For example, multiset automata and multiset grammars were discussed in [2]. Mealy multiset automata with output were presented in [3]. Furthermore, multiset pushdown automata and the algebra of multiset languages were considered in [19]. Multiset automata and multiset grammars is covered in the literature, cf. [5, 6, 18, 19].

*Corresponding author. Yongbing Wang. E-mail: wangyongb@snnu.edu.cn, liyongm@snnu.edu.cn.

Recently, Wang, Yin and Gu [30] introduced the concepts of fuzzy multiset finite automata and fuzzy multiset grammars, as the generalizations of multiset grammars and multiset finite automata, respectively. Also, in [30], the relationship between fuzzy multiset regular grammars and fuzzy multiset finite automata is discussed. Tiwari [28] showed that a deterministic fuzzy multiset finite automata is equally powerful as a fuzzy multiset finite automaton in the sense of acceptance of a fuzzy multiset language. Furthermore, minimization of deterministic fuzzy multiset automata for a given fuzzy multiset language was discussed in [28]. In this study, we shall use distributive lattices as the basic structure of membership values. This generalizes those dissertations carried out in the unit interval [0, 1]. We first introduce lattice-valued multiset finite automata, and prove that a (nondeterministic) lattice-valued multiset finite automata (LMA) into deterministic LMA (DLMA). Further, we show that DLMA can be minimized, and the state-sets of the minimal DLMA is finite, then we present an effective algorithm to realize the minimization of DLMA. The construction of the minimal DLMA exhibits close links with the lattice-valued multiset languages being accepted. Using the structure of the minimal DLMA, we study L-unitary language.

The paper is arranged in the following ways. In Section 2, we introduce the basic notions of LMA, DLMA and prove that a DMLA cloud be constructed from LMA. Furthermore, some fundamental properties of LMA are presented. In Section 3, we study the minimization algorithm of DLMA. In Section 4, we study the decomposition of lattice-valued regular multiset language by some simple lattice-valued regular multiset languages accepted by some special minimal DLMAs. Finally, conclusions are covered in Section 5.

2. Preliminaries

In this section, We recall the notions of multisets and multiset finite automata which are required in the subsequent sections. Interested readers are referred to [1, 2, 30] for more information.

2.1. Multesets

Definition 1. ([1, 30]) If Σ is a finite alphabet, then $\alpha : \Sigma \to N$ is a *multiset* over Σ, where N denotes the set of natural numbers including 0. The α norm of Σ is defined by $\mid \alpha \mid = \sum_{a\in\Sigma} \alpha(a)$.

Let $\Sigma^{\oplus}$ denote the set of all multiset over Σ. The multiset $0_\Sigma \in \Sigma^{\oplus}$ is defined $0_\Sigma(a) = 0$ for any $a \in \Sigma$. And for each $b \in \Sigma$, we shall denote by $\langle b\rangle$, a singleton multiset, and is defined as follows:

$$\langle b\rangle(a) = \begin{cases} 1, & \text{if } b = a, \\ 0, & \text{if } b \neq a \end{cases}$$

for any $a \in \Sigma$. For example, let $\Sigma = \{a, b, c\}$, the multiset $\alpha = \{1/a, 2/b, 0/c\}$ may be written $\alpha = \langle a\rangle \oplus \langle b\rangle \oplus \langle b\rangle$.

For two multisets $\alpha, \beta \in \Sigma^{\oplus}$, the operations inclusion $\subseteq$, addition $\oplus$ and difference $\ominus$ are defined as follows, respectively,

(i) $\alpha \subseteq \beta \Leftrightarrow \alpha(\alpha) \leq \beta(\alpha)$;
(ii) $(\alpha \oplus \beta)(a) = \alpha(a) + \beta(a)$;
(iii) $(\alpha \ominus \beta)(a) = \max\{0, \alpha(a) - \beta(a)\}$

for any $a \in \Sigma$. Furthermore, $\alpha \subset \beta$ if $\alpha \subseteq \beta$ and $\alpha \neq \beta$.

Clearly, $\Sigma^{\oplus}$ is a commutative monoid with the element 0_Σ as the identity with respect to $\oplus$.

Now, we review the following definitions of multiset finite automata and multiset languages from [2].

Definition 2. [2] A *multiset finite automaton* (*MA*, for short) is a quintuple $\mathcal{W} = (Q, \Sigma, \delta, q_0, F)$ with a finite set of states Q, a finite alphabet Σ, a finite set of instructions $\delta \subseteq Q \times \Sigma^{\oplus} \times Q$, an initial state q_0 and a set of final states $F \subseteq Q$.

A *configuration* is a pair $(p, \alpha) \in (Q, \Sigma^{\oplus})$. A step $(p, \alpha) \to (q, \beta)$ leads from one configuration to another if there exists a multiset $\gamma \in \Sigma^{\oplus}$ with $\gamma \subseteq \alpha$, $(p, \gamma, q) \in \delta$, and $\beta = \alpha \ominus \gamma$. $\to^*$ denotes the reflexive and transitive closure of $\to$.

A multiset $\alpha \in \Sigma^{\oplus}$ is accepted by $\mathcal{W}$ if there is a successful computation with label α, that is, if α is reduced in finitely many steps to 0_Σ, after which $\mathcal{W}$ is in a final state.

Definition 3. [2] The *multiset language* accepted by $\mathcal{W}$ is $\mathcal{L}(\mathcal{W}) = \{\alpha \in \Sigma^{\oplus} \mid \exists q \in F : (p, \alpha) \to^* (q, 0_\Sigma)\}$.

A multiset language $\mathcal{L}$ is *regular* if there is an MA $\mathcal{W}$ with $\mathcal{L} = \mathcal{L}(\mathcal{W})$. For two MAs $\mathcal{W}_1$ and $\mathcal{W}_2$, we say that they are equivalent if they accept the same regular multiset language, that is, $\mathcal{L}(\mathcal{W}_1) = \mathcal{L}(\mathcal{W}_2)$.

Note that the commutativity of the operation $\oplus$ for multisets, the choice between applicable instructions (p, α, q) and (p, β, q), possibly reading different parts of the input is non-deterministic.

An MA $\mathcal{W} = (Q, \Sigma, \delta, q_0, Q_F)$ is said to be *deterministic multiset finite automata* (*DMA*) if $|\delta(p, \alpha)| \leq 1$ for any $p \in Q$ and $\alpha \in \Sigma^{\oplus}$.

MAs and DMAs have the same computation power, accept the same set of multiset languages.

Lemma 1. If $\mathcal{L}_1, \mathcal{L}_2$ are regular, then so are $\mathcal{L}_1 \cup \mathcal{L}_2$, $\mathcal{L}_1 \cap \mathcal{L}_2$, $\Sigma^{\oplus} - \mathcal{L}_1$.

Proof. The proof is similar to that given for finite automata in [4].

2.2. Lattice multiset finite automata and their languages

In the sequel L always denotes a distributive lattice with the least element 0 and the largest element 1. We use $\vee$ (or $\bigvee$) to represent the finite supremum operation, and $\wedge$ (or $\bigwedge$) to represent the finite infimum operation. Then the following distributive laws hold for any $a, a_1, a_2, \cdots, a_n \in L$, where n is a positive integer, $a \wedge (\bigvee_{i=1}^{n} a_i) = \bigvee_{i=1}^{n}(a \wedge a_i)$, $a \vee (\bigwedge_{i=1}^{n} a_i) = \bigwedge_{i=1}^{n}(a \vee a_i)$.

Let X be a non-empty set. An L-fuzzy subset of X is just a mapping from X to L, which is also called fuzzy subset of X in this paper. The formal definition of lattice multiset finite automata is given as follows, which is a fuzzy finite multiset automata as defined in [28, 30] with membership values restricted in the unit interval [0, 1].

Definition 4. A *lattice multiset finite automata* (or *LMA*, for short) is a five tuple $\mathcal{M} = (Q, \Sigma, \delta, I, F)$, where

(i) Q and Σ are nonempty finite sets called the state-set and input-set, respectively;

(ii) $\delta : Q \times \Sigma^{\oplus} \times Q \to L$ is an L-fuzzy subset of $Q \times \Sigma^{\oplus} \times Q$, called fuzzy transition function, and let $\Delta_{\mathcal{M}} = \{(q, \alpha, p) \mid \delta(q, \alpha, p) > 0, q, p \in Q, \alpha \in \Sigma^{\oplus}\}$, where the elements of $\Delta_{\mathcal{M}}$ are finite;

(iii) $I : Q \to L$ is a map called the fuzzy set of initial states; and

(iv) $F : Q \to L$ is a map called the fuzzy set of final states.

A configuration of LMA $\mathcal{M}$ is an element of $Q \times \Sigma^{\oplus}$. The configuration (p, α) is meant to represent the situation that $\mathcal{M}$ is in state p, α is the remaining multiset on the input tape. The transition in an LMA are described with the help of configurations. The LMA is nondeterministic, so there may be several transitions that are possible in a given configuration. Thus, the transition from configuration (p, α) leads to configuration (q, β) with membership value $l \in L$ if there exists a multiset $\gamma \in \Sigma^{\oplus}$ with $\gamma \subseteq \alpha$, $\delta(p, \gamma, q) = l$ and $\beta = \alpha \ominus \gamma$ and is denoted by $(p, \alpha) \xrightarrow{l} (q, \beta)$.

Furthermore, $\overset{k'}{\rightarrow^*}$ means the reflexive and transitive closure of $\overset{k'}{\rightarrow}$, that is, for $(p, \alpha),(q, \beta) \in (Q, \Sigma^{\oplus})$, $(p, \alpha) \overset{l'}{\rightarrow^*} (q, \beta)$ if for some $n \geq 0$, there exists $(n+1)$ configurations $(p, \alpha), (q_1, \alpha_1), \cdots, (q_{n-1}, \alpha_{n-1}), (q, \beta)$, such that $(p, \alpha) \xrightarrow{l_1} (q_1, \alpha_1) \xrightarrow{l_2} \cdots \xrightarrow{l_{n-1}} (q_{n-1}, \alpha_{n-1}) \xrightarrow{l_n} (q, \beta)$, where $l = l_1 \wedge l_2 \wedge \cdots \wedge l_{n-1} \wedge l_n$. Naturally the degree of the configuration (p, α) deriving the configuration (q, β) is defined as

$\mu_{\mathcal{M}}\big((p, \alpha) \to^* (q, \beta)\big) = \bigvee \big\{\mu_{\mathcal{M}}\big((p, \alpha) \to^* (r, \alpha \ominus \gamma)\big) \wedge \mu_{\mathcal{M}}\big((r, \alpha \ominus \gamma) \to^* (q, \beta)\big) : r \in Q, \gamma \in \Sigma^{\oplus}, \text{ and } \gamma \subseteq \alpha\big\}$,

and

$$\mu_{\mathcal{M}}\big((p, \alpha) \to^* (q, \alpha)\big) = \begin{cases} 1, & \text{if } p = q, \\ 0, & \text{if } p \neq q. \end{cases}$$

Example 1. Consider an LMA $\mathcal{M} = (Q, \Sigma, \delta, q_0, I, F)$, where $Q = \{q_0, q_1, q_2, q_3\}$, $\Sigma = \{a, b\}$, and the fuzzy transition function is given as under:
$\delta(q_0, \langle a\rangle \oplus \langle a\rangle, q_1) = 0.4$, $\delta(q_0, \langle a\rangle, q_2) = 0.5$, $\delta(q_0, \langle b\rangle \oplus \langle b\rangle \oplus \langle b\rangle, q_2) = 0.5$, $\delta(q_0, \langle b\rangle \oplus \langle b\rangle, q_3) = 0.8$, $\delta(q_1, \langle a\rangle, q_0) = 0.9$, $\delta(q_1, \langle a\rangle \oplus \langle a\rangle \oplus \langle b\rangle, q_3) = 0.7$, $\delta(q_1, \langle b\rangle \oplus \langle b\rangle, q_2) = 0.6$, $\delta(q_2, \langle a\rangle, q_3) = 0.3$, $\delta(q_2, \langle b\rangle, q_1) = 0.6$.

If $\alpha = \langle a\rangle \oplus \langle a\rangle \oplus \langle a\rangle \oplus \langle b\rangle \oplus \langle b\rangle$, and $\beta = 0_\Sigma$, Then the transition steps, $\mu_{\mathcal{M}}\big((q_0, \alpha) \to^* (q_3, \beta)\big)$ are as follows:

(1) $(q_0, \langle a\rangle \oplus \langle a\rangle \oplus \langle a\rangle \oplus \langle b\rangle \oplus \langle b\rangle) \xrightarrow{0.4} (q_1, \langle a\rangle \oplus \langle b\rangle \oplus \langle b\rangle) \xrightarrow{0.6} (q_2, \langle a\rangle) \xrightarrow{0.3} (q_3, 0_\Sigma)$;

(2) $(q_0, \langle a\rangle \oplus \langle a\rangle \oplus \langle a\rangle \oplus \langle b\rangle \oplus \langle b\rangle) \xrightarrow{0.4} (q_1, \langle a\rangle \oplus \langle b\rangle \oplus \langle b\rangle) \xrightarrow{0.9} (q_0, \langle b\rangle \oplus \langle b\rangle) \xrightarrow{0.8} (q_3, 0_\Sigma)$;

(3) $(q_0, \langle a\rangle \oplus \langle a\rangle \oplus \langle a\rangle \oplus \langle b\rangle \oplus \langle b\rangle) \xrightarrow{0.5} (q_2, \langle a\rangle \oplus \langle a\rangle \oplus \langle b\rangle \oplus \langle b\rangle) \xrightarrow{0.6} (q_1, \langle a\rangle \oplus \langle a\rangle \oplus \langle b\rangle) \xrightarrow{0.7} (q_3, 0_\Sigma)$.

Thus
$\mu_{\mathcal{M}}\big((q_0, \alpha) \to^* (q_3, \beta)\big) = \vee\{0.4 \wedge 0.6 \wedge 0.3, 0.4 \wedge 0.9 \wedge 0.8, 0.5 \wedge 0.6 \wedge 0.7\} = 0.5$.

Any L-fuzzy subset in $\Sigma^{\oplus}$ is called an *L-multiset language* on Σ. An L-multiset language accepted by an LMA $\mathcal{M} = (Q, \Sigma, \delta, I, F)$, denoted as $f_{\mathcal{M}} : \Sigma^{\oplus} \to L$, is expressed in the form:

$$f_{\mathcal{M}}(\alpha) = \bigvee_{p,q\in Q} \{I(p) \wedge \mu_{\mathcal{M}}((p,\alpha) \to^* (q, 0_\Sigma)) \wedge F(q)\}$$

for any $\alpha \in \Sigma^{\oplus}$. An L-multiset language which is accepted by an LMA is called an *L-regular multiset language*.

Indeed, the above equality means that the membership degree of the multiset α to the L-multiset language $f_{\mathcal{M}}$ is equal to the degree to which $\mathcal{M}$ accepts the multiset α. Meanwhile, we need to realize about computation of the lattice multiset finite automata $\mathcal{M}$ does not depend on some strict order of symbols in the "input multiset".

For two LMAs, $\mathcal{M}_1$ and $\mathcal{M}_2$, we say that they are equivalent if they accept the same L-regular multiset language, that is, $f_{\mathcal{M}_1} = f_{\mathcal{M}_2}$.

As to classical multiset automata theory, we have the notions of deterministic multiset finite automata and nondeterministic multiset finite automata. The notion of an LMA is a generalization of the notion of nondeterministic automata: instead of sets of initial and final states we have fuzzy sets of initial and final states; instead of a (bivalent) transition relation we have fuzzy transition relation.

The LMA is nondeterministic in nature: there may be nonzero truth degrees that the automata can go to more than one state (given a state and "input multiset"). In the following we are going to present a deterministic counterpart of the notion of an LMA.

Definition 5. A *deterministic lattice multiset finite automata* (or *DLMA*, for short) is a five tuple $\mathcal{M} = (Q, \Sigma, \delta, q_0, F)$, such that $\delta : Q \times \Sigma^{\oplus} \to Q$, $q_0 \in Q$ is the initial state, we denote $\nabla_{\mathcal{M}} = \{(q,\alpha)|$there exist a unique $p \in Q$ such that $\delta(q,\alpha) = p, q \in Q, \alpha \in \Sigma^{\oplus}\}$, the set $\nabla_{\mathcal{M}}$ is finite; and $F : Q \to L$.

A configuration of a DLMA is the same case of a multiset auotmata.

The L-multiset language $f_{\mathcal{M}} : \Sigma^{\oplus} \to L$ accepted by a DLMA is defined as, for any $\alpha \in \Sigma^{\oplus}$, $f_{\mathcal{M}}(\alpha) = F(\delta(q_0, \alpha))$. And, the L-multiset language f_q accepted by $\mathcal{M}$ in state q is a mapping $f_q : \Sigma^{\oplus} \to L$ such that $f_q(\alpha) = F(\delta(q, \alpha))$ for any $\alpha \in \Sigma^{\oplus}$.

Lemma 2. [7] If L is a distributive lattice, and A is a subset of L, then the sublattice L_A of L generated by A is given as the set $L_A = \{\bigvee Z : Z$ is a finite subset of $B\}$, where $B = \{\bigwedge T : T$ is a finite subset of $A\}$. Furthermore, if A is a finite subset of L, then L_A is also finite.

In the following theorem we present a method to transform an LMA into an equivalent DLMA.

Let $\mathcal{M} = (Q, \Sigma, \delta, I, F)$ be an LMA, where $Q = \{q_1, q_2, \cdots, q_n\}$. Construct the DLMA $\mathcal{N} = (P, \Sigma, \eta, p_0, E)$ as follows:

(1) Let $A = \{\delta(q_i, \alpha, q_j) : (q_i, \alpha, q_j) \in \Delta_{\mathcal{M}}, q_i, q_j \in Q$, and $\alpha \in \Sigma^{\oplus}\} \cup \{I(q_i) : q_i \in Q\}$. According to Lemma 2, we generate a finite sublattice L_A of L.

(2) Let $P = \{(p_1, p_2, \cdots, p_n) : p_i \in L_A\}$, we define $\eta : P \times \Sigma^{\oplus} \to P$ by

$\eta((p_1, p_2, \cdots, p_n), \alpha) = (r_1^\alpha, r_2^\alpha, \cdots, r_n^\alpha)$,

where $r_i^\alpha = \bigvee_{j=1}^n \{p_j \wedge \mu_{\mathcal{M}}((q_j, \alpha) \to^* (q_i, 0_\Sigma))\}$. $p_0 = (I(q_1), I(q_2), \cdots, I(q_n))$, $E : P \to L$ is defined by $E(p_1, p_2, \cdots, p_n) = \bigvee_{j=1}^n \{p_i \wedge F(q_i)\}$.

Theorem 1. If DLMA is constructed from LMA by the above construction, then $f_{\mathcal{N}} = f_{\mathcal{M}}$, i.e., $f_{\mathcal{N}}(\alpha) = f_{\mathcal{M}}(\alpha)$ for any $\alpha \in \Sigma^{\oplus}$.

Proof. It is easy to see that $\mathcal{N}$ is a well defined DLMA. We now proof that $f_{\mathcal{N}} = f_{\mathcal{M}}$. For any $\alpha \in \Sigma^{\oplus}$,

$$\begin{aligned}
f_{\mathcal{N}}(\alpha) &= E(\eta(p_0, \alpha)) \\
&= E((r_1^\alpha, r_2^\alpha, \cdots, r_n^\alpha)) \\
&= \bigvee_{i=1}^n \{r_i^\alpha \wedge F(q_i)\} \\
&= \bigvee_{i=1}^n \Big\{ \bigvee_{j=1}^n \big[p_j \wedge \mu_{\mathcal{M}}((q_j, \alpha) \to^* (q_i, 0_\Sigma))\big] \wedge F(q_i) \Big\} \\
&= \bigvee_{q', q'' \in Q} \{I(q') \wedge \mu_{\mathcal{M}}((q', \alpha) \to^* (q'', 0_\Sigma)) \wedge F(q'')\} \\
&= f_{\mathcal{M}}(\alpha).
\end{aligned}$$

Example 2. Let $\mathcal{M} = (Q, \Sigma, \delta, I, F)$ be an LMA, where $Q = \{q_1, q_2\}$, $\Sigma = \{a, b\}$, $L = [0, 1]$, $I(q_1) = 0.6$, $I(q_2) = 0$, $F(q_1) = 0$, $F(q_2) = 1$, and the fuzzy transition function is given as under:

$\delta(q_1, \langle a\rangle, q_1) = 1$, $\delta(q_1, \langle a\rangle, q_2) = 0.6$,
$\delta(q_1, \langle b\rangle, q_1) = 1$, $\delta(q_1, \langle b\rangle, q_2) = 1$,
$\delta(q_1, \langle a\rangle \oplus \langle b\rangle, q_2) = 0.6$, $\delta(q_2, \langle a\rangle, q_1) = 0.6$,
$\delta(q_2, \langle b\rangle, q_2) = 0.6$, $\delta(q_2, \langle a\rangle \oplus \langle b\rangle, q_2) = 1$,
$\delta(q_2, \langle a\rangle \oplus \langle b\rangle, q_1) = 0.6$.

According to Theorem 1, the equivalent DLMA $\mathcal{N} = (P, \Sigma, \eta, p_0, E)$ as follows: $P = \{p_1, p_2, \cdots, p_n\}$, where $p_1 = \{0, 0\}$, $p_2 = \{0, 0.6\}$, $p_3 = \{0, 1\}$, $p_4 = \{0.6, 0\}$, $p_5 = \{0.6, 0.6\}$, $p_6 = \{0.6, 1\}$, $p_7 = \{1, 0\}$, $p_8 = \{1, 0.6\}$, $p_9 = \{1, 1\}$, and $p_0 = p_4$; $E(p_1) = 0$, $E(p_2) = 0.6$, $E(p_3) = 1$, $E(p_4) = 0.6$, $E(p_5) = 0.6$,

$E(p_6) = 1$, $E(p_7) = 0$, $E(p_8) = 0.6$, $E(p_9) = 1$. The transition function $\eta : P \times \Sigma^{\oplus} \to P$ is given as $\delta(p_1, \langle a\rangle) = p_1$, $\delta(p_1, \langle b\rangle) = p_1$, $\delta(p_1, \langle a\rangle \oplus \langle b\rangle) = p_1$, $\delta(p_2, \langle a\rangle) = p_4$, $\delta(p_2, \langle b\rangle) = p_2$, $\delta(p_2, \langle a\rangle \oplus \langle b\rangle) = p_5$, $\delta(p_3, \langle a\rangle) = p_8$, $\delta(p_3, \langle b\rangle) = p_2$, $\delta(p_2, \langle a\rangle \oplus \langle b\rangle) = p_6$, $\delta(p_4, \langle a\rangle) = p_5$, $\delta(p_4, \langle b\rangle) = p_5$, $\delta(p_4, \langle a\rangle \oplus \langle b\rangle) = p_2$, $\delta(p_5, \langle a\rangle) = p_4$, $\delta(p_5, \langle b\rangle) = p_5$, $\delta(p_5, \langle a\rangle \oplus \langle b\rangle) = p_5$, $\delta(p_6, \langle a\rangle) = p_5$, $\delta(p_6, \langle b\rangle) = p_5$, $\delta(p_6, \langle a\rangle \oplus \langle b\rangle) = p_6$, $\delta(p_7, \langle a\rangle) = p_8$, $\delta(p_7, \langle b\rangle) = p_9$, $\delta(p_7, \langle a\rangle \oplus \langle b\rangle) = p_1$, $\delta(p_8, \langle a\rangle) = p_8$, $\delta(p_8, \langle b\rangle) = p_9$, $\delta(p_8, \langle a\rangle \oplus \langle b\rangle) = p_5$, $\delta(p_9, \langle a\rangle) = p_8$, $\delta(p_9, \langle b\rangle) = p_9$, $\delta(p_9, \langle a\rangle \oplus \langle b\rangle) = p_6$.

In the following, we use Theorem 1 to prove some results concerning L-regular multiset languages.

For any L-multiset language $f : \Sigma^{\oplus} \to L$, and $\lambda \in L$, we define the λ-*cut* of f, f_λ, in the usual manner, that is, $f_\lambda = \{\alpha \in \Sigma^{\oplus} : f(\alpha) \geq \lambda\}$. The *image* of f, in notation $\mathrm{Im}(f)$, is a subset of L given by $\mathrm{Im}(f) = \{f(\alpha) : \alpha \in \Sigma^{\oplus}\}$, define a subset f of $\Sigma^{\oplus}$ as $f^{[\lambda]} = \{\alpha \in \Sigma^{\oplus} : f(\alpha) = \lambda\}$. The *support* of f, *supp*(f), that is, *supp*$(f) = \{\alpha \in \Sigma^{\oplus} : f(\alpha) > 0\}$.

Theorem 2. For any L-multiset language $f : \Sigma^{\oplus} \to L$, the following conditions are equivalent:

(i) f is an L-regular multiset language.

(ii) The image set of f, $\mathrm{Im}(f)$, is finite, and for each $\lambda \in L$, f_λ is a crisp regular multiset language.

(iii) The image set of f, $\mathrm{Im}(f)$, is finite, and for each $\lambda \in L$, $f^{[\lambda]}$ is a crisp regular multiset language.

Proof. (i) $\Rightarrow$ (ii) Suppose f is L-regular multiset language, then there is a DLMA $\mathcal{M} = (Q, \Sigma, \delta, q_0, F)$ such that the multiset language accepted by $\mathcal{M}$ is f, i.e., $f_{\mathcal{M}} = f$. This means that for any $\alpha \in \Sigma^{\oplus}$, $f(\alpha) = F(\delta(q_0, \alpha))$, then $\mathrm{Im}(f) \subseteq \mathrm{Im}(F) = \{F(q) : q \in Q\}$ is a finite subset of L. For any $\lambda \in L$, define a crisp multiset finite automata $\mathcal{M}_\lambda = (Q, \Sigma, \delta, q_0, F_\lambda)$, then $\alpha \in L(\mathcal{M}_\lambda)$ if and only if $\delta(q_0, \alpha) \in F_\lambda$, that is, $F(\delta(q_0, \alpha)) \geq \lambda$ if and only if $f(\alpha) \geq \lambda$. Hence, $L(\mathcal{M}_\lambda) = f_\lambda$, f_λ is a crisp regular multiset language.

(ii) $\Rightarrow$ (iii) For any $\lambda \in \mathrm{Im}(f)$, let $C_\lambda = \{\lambda' \in \mathrm{Im}(f) : \lambda' \geq \lambda$ and $\lambda' \neq \lambda\}$. Notice that $f^{[\lambda]} = f_\lambda - \bigcup_{\lambda' \in C_\lambda} f_{\lambda'}$ and the family of regular multiset languages is closed under finite unions and complement with respect to Lemma 1, which implies that $f^{[\lambda]}$ is a regular multiset language.

(iii) $\Rightarrow$ (i) For any $\lambda \in L$, since $f^{[\lambda]}$ is a regular multiset language, there exists a crisp deterministic multiset finite automata $\mathcal{W}_\lambda = (Q_\lambda, \Sigma, \delta_\lambda, s_\lambda, Q_{F_\lambda})$ such that $f^{[\lambda]} = \mathcal{L}(\mathcal{W})$. Without loss of generality, we assume that if $\lambda_1 \neq \lambda_2$, $Q_{\lambda_1} \neq Q_{\lambda_2}$. We construct an LMA $\mathcal{M} = (Q, \Sigma, \delta, q_0, I, F)$ as follows: $Q = \bigcup_{\lambda \in \mathrm{Im}(f)} (Q_\lambda - s_\lambda) \cup \{q_0\}$, $q_0 \notin \bigcup_{\lambda \in \mathrm{Im}(f)} Q_\lambda$, $F : Q \to L$ is define as

$$F(q) = \begin{cases} \lambda, & \text{if } q \in Q_{F_\lambda}, \\ \bigvee\{\lambda : \lambda \in \mathrm{Im}(f)\}, & \text{if } q = q_0, \end{cases}$$

where $\delta : Q \times \Sigma^{\oplus} \times Q \to L$ is a mapping such that

$$\mu_{\mathcal{M}}\big((p, \alpha) \to^* (q, 0_\Sigma)\big) = \begin{cases} 1, & p \in Q_\lambda - \{s_\lambda\} \text{ and } q = \delta_\lambda(p, \alpha) \text{ or} \\ & p = q_0 \text{ and there exists } \lambda \in \mathrm{Im}(f) \\ & \text{such that } q = \delta_\lambda(s_\lambda, \alpha), \\ 0, & \text{otherwise.} \end{cases}$$

It is not difficult to show that $f_{\mathcal{M}} = f$, that is, f is an L-regular multiset language.

Theorem 3. For any L-multiset language $f : \Sigma^{\oplus} \to L$, the following conditions are equivalent:

(i) f is an L-regular multiset language.

(ii) There exists an LMA $\mathcal{N}$ with transition function $\delta : Q \times \Sigma^{\oplus} \to Q$ ans crisp final states $F \subseteq Q$, and fuzzy initial state $I : Q \to L$.

Proof. (ii) $\Rightarrow$ (i) Obviously.

(i) $\Rightarrow$ (ii) Suppose f is an L-regular multiset language, then there exists a DLMA $\mathcal{M} = (Q, \Sigma, \delta, q_0, F)$ such that $f_{\mathcal{M}} = f$. Let $L_1 = \mathrm{Im}(F)$ and $L_2 = \{\bigvee T : T$ is a finite subset of $L_1\}$, then L_2 is finite. Let $P = L_2^Q$, the set of all mapping form Q to L_2. Clearly, P is also a finite set. The transition function $\eta : P \times \Sigma^{\oplus} \to P$ is define in the form $\eta(y, \alpha)(p) = \bigvee\{y(q) : \delta(q, \alpha) = p\}$. Furthermore, $\rho_0 : P \to L_2$ is defined as $\rho_0(y) = y(q_0)$, and $E = \{F\}$. Then $\mathcal{N} = (P, \Sigma, \eta, \rho_0, E)$ is a DLMA with crisp transition function η and crisp final states $E \subseteq P$. Now we show that $f_{\mathcal{N}} = f$. For any $\alpha \in \Sigma^{\oplus}$,

$$\begin{aligned} f_{\mathcal{N}}(\alpha) &= \bigvee\{\rho_0(y) : y \in P \text{ and } \eta(y, \alpha) \in E\} \\ &= \bigvee\{y(q_0) : y \in P \text{ and } \eta(y, \alpha) = E\} \\ &= \bigvee\Big\{y(q_0) : y \in P \text{ and } \vee\{y(q) : \delta(q, \alpha) = p\} = F(p)\Big\} \\ &= F(\delta(q_0, \alpha)) \\ &= f_{\mathcal{M}}(\alpha) = f(\alpha). \end{aligned}$$

2.3. Operations on the family L-multiset languages

Similarly to [14] where some closure properties of fuzzy regular languages were studied, we will have a look to these properties of L-regular multiset languages.

Definition 6. Given L-multiset languages f, g on Σ, the *union*, *addition*, *Kleene closure operation* are given as follows:

Let Σ be an alphabet and, $g, h : \Sigma^{\oplus} \to L$. We define the union $g \cup h$, the addition $g \oplus h$, and the Kleene closure f by

(i) $(g \cup h)(\gamma) = \max\{g(\gamma), h(\gamma)\}$ for all $\gamma \in \Sigma^{\oplus}$.

(ii) $(g \oplus h)(\gamma) = \max_{\alpha\oplus\beta=\gamma}\{g(\gamma) \wedge h(\gamma)\}$ for all $\gamma \in \Sigma^{\oplus}$.

(iii) $f^{\oplus} = f^0 \cup f \cup f^2 \cup \cdots \cup f^n \cup \cdots$, where f^0 denotes that the multiset 0_Σ, that is

$$f^0(\gamma) = \begin{cases} 1, & \text{if } \gamma = 0_\Sigma, \\ 0, & \text{if } \gamma \neq 0_\Sigma \end{cases}$$

and f^n is inductively defined as $f^n = f^{n-1} \oplus f$ for $n \geq 2$.

Theorem 4. The family of L-regular multiset languages on Σ is closed under the operations of union, addition and Kleene closure.

Proof. The family of L-regular multiset languages is closed under the union and addition is direct from Theorem 2.

Consider the Kleene closure operation. Suppose that $\mathcal{M} = (Q, \Sigma, \delta, I, F)$ is an LMA. Construct an LMA $\mathcal{M}' = (P, \Sigma, \delta', I', F')$ as, $P = Q \cup \{s_0\}$, $I' : P \to L$ and $F' : P \to L$ are defined as follows:

$$I'(q') = \begin{cases} I(q'), & \text{if } q' \in Q, \\ 1, & \text{if } q' = s_0; \end{cases}$$

$$F'(q') = \begin{cases} F(q'), & \text{if } q' \in Q, \\ 1, & \text{if } q' = s_0, \end{cases}$$

where $\delta' : P \times \Sigma^{\oplus} \times P \to L$ is a mapping such that

$$\mu_{\mathcal{M}'}\big((q_1', 0_\Sigma) \to^* (q_2', 0_\Sigma)\big) = \begin{cases} 1, & \text{if } q_1' = q_1', \\ 0, & \text{if } q_1' \neq q_1' \end{cases}$$

for any $\alpha \in \Sigma^{\oplus}(\alpha \neq 0_\Sigma)$,

$$\mu_{\mathcal{M}'}\big((q_1', \alpha) \to^* (q_2', 0_\Sigma)\big) = \begin{cases} \mu_\sharp\big((q_1', \alpha) \to^* (q_2', 0_\Sigma)\big), & \text{if } q_1', q_2' \in Q, \\ \bigvee_{q\in Q}\{I(q) \wedge \mu_\sharp\big((q, \alpha) \to^* (q_2', 0_\Sigma)\big)\}, & \text{if } q_1' = s_0, \\ & q_2' \in Q, \\ 0, & \text{if } q_2' = s_0, \end{cases}$$

where

$$\begin{aligned} &\mu_\sharp\big((q_1', \alpha) \to^* (q_2', 0_\Sigma)\big) \\ &= \bigvee_{k\geq 0} \bigvee_{\alpha_1\oplus\alpha_2\oplus\cdots\oplus\alpha_n=\alpha} \bigvee_{q_1, q_2, \cdots, q_{2(k-1)}\in Q} \big\{\mu_{\mathcal{M}}\big((q_1', \alpha_1) \\ &\to^* (q_1, 0_\Sigma)\big) \wedge F(q_1) \wedge I(q_2) \wedge \cdots \\ &\wedge \mu_{\mathcal{M}}\big((q_{2(k-1)}, \alpha_k) \to^* (q_2', 0_\Sigma)\big)\big\}. \end{aligned}$$

Therefore, for any $\alpha \in \Sigma^{\oplus}$, we have

(1) If $\alpha = 0_\Sigma$, $f_{\mathcal{M}'}(0_\Sigma) = \bigvee_{q'\in Q'}\{I'(q') \wedge F'(q')\} = 1 = f^*_{\mathcal{M}}(0_\Sigma)$.

(2) If $\alpha \neq 0_\Sigma$, then
$f_{\mathcal{M}'}(\alpha) = \bigvee_{q_1', q_2'\in Q'}\{I'(q_1') \wedge \mu_{\mathcal{M}'}\big((q_1', \alpha) \to^* (q_2', 0_\Sigma)\big) \wedge F'(q_2')\}$
$= \bigvee_{k\geq 1} \bigvee_{\alpha_1\oplus\alpha_2\oplus\cdots\oplus\alpha_n=\alpha} \bigvee_{q_1', q_2', \cdots, q_{2(k-1)'}\in Q} \big\{I(q') \wedge \mu_{\mathcal{M}}\big((q', \alpha_1) \to^* (q_1', 0_\Sigma)\big) \wedge F(q_1') \wedge I(q_2') \wedge \cdots \wedge \mu_{\mathcal{M}}\big((q'_{2(k-1)}, \alpha_k) \to^* (q_2', 0_\Sigma)\big) \wedge F(q_2')\big\}$
$= \bigvee_{k\geq 1} \bigvee_{\alpha_1\oplus\alpha_2\oplus\cdots\oplus\alpha_n=\alpha} \big\{f_{\mathcal{M}}(\alpha_1) \wedge f_{\mathcal{M}}(\alpha_2) \wedge \cdots \wedge f_{\mathcal{M}}(\alpha_k)\big\} = f^*_{\mathcal{M}}(\alpha)$.

Therefore, $f_{\mathcal{M}'} = f^*_{\mathcal{M}}$ is an L-regular multiset language.

3. Minimization of DLMA

Multiset finite automata are used to design complex system. Finding a minimum representation of multiset finite automata is a critical issue arising in such design. In this part, we shall introduce and study the concept of minimization of a DLMA. To make the meaning of minimization of a DLMA clear, we first introduce some necessary notions, which are slight modifications of related notions in [28].

Definition 7. Let $\mathcal{M} = (Q, \Sigma, \delta, q_0, F)$ be a DLMA. The *reachability* mapping $r : \Sigma^{\oplus} \to Q$ such that

(1) $r(0_\Sigma) = q_0$;

(2) $r(\alpha \oplus \langle a\rangle) = \delta(r(\alpha), \langle a\rangle)$ for any $\alpha, \langle a\rangle \in \Sigma^{\oplus}$.

$\mathcal{M}$ is called *reachable* if r is onto. And the following facts can be easily verified for any $\alpha, \beta \in \Sigma^{\oplus}$,

(1) $r(\alpha) = \delta(q_0, \alpha)$;

(2) $r(\alpha \oplus \beta) = \delta(r(\alpha), \beta)$.

Let $Q^r = \{\delta(q_0, \alpha) : \alpha \in \Sigma^{\oplus}\}$. Clearly, $Q = Q^r$, then $\mathcal{M}$ is reachable. The elements of Q^r are called *reachable states*, and the elements of $Q - Q^r$ are called *inreachable elements*.

Definition 8. Given a DLMA $\mathcal{M} = (Q, \Sigma, \delta, q_0, F)$, the *coreachability* mapping $\sigma : Q \to \mathcal{F}(\Sigma^{\oplus})$ such that $\sigma(q) = f_q$, where $\mathcal{F}(\Sigma^{\oplus})$ denotes the set of L-multiset languages.

$\mathcal{M}$ is called *coreachable* if σ is one-to-one.

Definition 9. Given a DLMA $\mathcal{M} = (Q, \Sigma, \delta, q_0, F)$ is *minimal* if $\mathcal{M}$ is both reachable and coreachable.

Next, we give the following concepts of homomorphisms necessary to compare two or more DLMAs.

Given two DLMAs $\mathcal{M} = (Q, \Sigma, \delta, q_0, F)$ and $\mathcal{N} = (P, \Sigma, \eta, p_0, E)$, a *homomorphism* from $\mathcal{M}$ to $\mathcal{N}$, denoted $\varphi : \mathcal{M} \to \mathcal{N}$, is a mapping $\varphi : Q \to P$ such that

(1) $\varphi(q_0) = p_0$;

(2) $\varphi(\delta(q, \alpha)) = \eta(\varphi(q), \alpha)$;

(3) $E(\varphi(q)) \le F(q)$

for any $q \in Q$, $\alpha \in \Sigma^{\oplus}$. Furthermore, if $E(\varphi(p)) = \bigvee\{F(q) : \varphi(q) = p\}$, then φ is called a *strong homomorphism*. A strong homomorphism $\varphi : \mathcal{M} \to \mathcal{N}$ is called an *isomorphism* if φ is one-to-one and onto. By applying this notions we get the following result.

Theorem 5. If $\varphi : \mathcal{M} \to \mathcal{N}$ is a homomorphism between two DLMAs, then $f_{\mathcal{N}} \le f_{\mathcal{M}}$ as two L-multiset languages on $\Sigma^{\oplus}$. Furthermore, if φ is strong, then $f_{\mathcal{N}} = f_{\mathcal{M}}$.

Proof. For any $\alpha \in \Sigma^{\oplus}$, if φ is a homomorphism, then

$$\begin{aligned} f_{\mathcal{N}}(\alpha) &= E(\eta(p_0, \alpha)) \\ &= E(\eta(\varphi(q_0), \alpha)) \\ &= E(\varphi(\delta(q_0, \alpha))) \\ &\ge F(\delta(q_0, \alpha)) = f_{\mathcal{M}}(\alpha). \end{aligned}$$

Thus, $f_{\mathcal{N}} \le f_{\mathcal{M}}$. In particular, if φ is a strong homomorphism, then

$$\begin{aligned} f_{\mathcal{N}}(\alpha) &= E(\eta(p_0, \alpha)) \\ &= E(\eta(\varphi(q_0), \alpha)) \\ &= E(\varphi(\delta(q_0, \alpha))) \\ &= \bigvee\{F(q) : \varphi(q) = \varphi(\delta(q_0, \alpha))\} \\ &\ge F(\delta(q_0, \alpha)) = f_{\mathcal{M}}(\alpha), \end{aligned}$$

that is, $f_{\mathcal{N}} \ge f_{\mathcal{M}}$. Therefore, $f_{\mathcal{N}} = f_{\mathcal{M}}$.

The following theorem presents a characterization of L-multiset language that can be accepted by a DLMA.

Theorem 6. For any L-multiset language $f : \Sigma^{\oplus} \to L$, the following conditions are equivalent:

(i) f is an L-regular multiset language.

(ii) There exists a congruence $\equiv$ defined by

$\alpha_1 \equiv \alpha_2$ if and only if for any $\beta \in \Sigma^{\oplus}$, $\alpha_1 \oplus \beta \equiv \alpha_2 \oplus \beta$

has a finite index; And, if $\alpha_1 \equiv \alpha_2$, then $f(\alpha_1) = f(\alpha_2)$.

(iii) The equivalent relation $\equiv_f$ on $\Sigma^{\oplus}$ defined as, for $\alpha_1, \alpha_2 \in \Sigma^{\oplus}$, $\alpha_1 \equiv_f \alpha_2$ if and only if $f(\alpha_1 \oplus \beta) = f(\alpha_2 \oplus \beta)$ for any $\beta \in \Sigma^{\oplus}$, has finite index.

Proof. (i) $\Rightarrow$ (ii) For an L-regular multiset language f, let $\mathcal{M} = (Q, \Sigma, \delta, q_0, F)$ be a DLMA accepting f. Define a relation $\equiv$ on $\Sigma^{\oplus}$ in the form: for any $\alpha_1, \alpha_2 \in \Sigma^{\oplus}$, $\alpha_1 \equiv \alpha_2$ if and only if $\delta(q_0, \alpha_1) = \delta(q_0, \alpha_2)$. Clearly, $\equiv$ is an equivalence relation on $\Sigma^{\oplus}$. Hence, if $\alpha_1 \equiv \alpha_2$, then $\delta(q_0, \alpha_1) = \delta(q_0, \alpha_2)$, and for any $\beta \in \Sigma^{\oplus}$, $\delta(q_0, \alpha_1 \oplus \beta) = \delta(\delta(q_0, \alpha_1), \beta) = \delta(\delta(q_0, \alpha_2), \beta) = \delta(q_0, \alpha_2 \oplus \beta)$. This implies $\alpha_1 \oplus \beta \equiv \alpha_2 \oplus \beta$. Let $[\alpha]_{\equiv}$ be the equivalence class of $\equiv$, that is, for each $q \in Q$, $[\alpha]_{\equiv} = \{\alpha \in \Sigma^{\oplus} | \delta(q_0, \alpha) = q\}$. Since Q is finite implies $\equiv$ with finitely many equivalence classes. And, if $\alpha_1 \equiv \alpha_2$, then $\delta(q_0, \alpha_1) = \delta(q_0, \alpha_2)$, that is, $f(\alpha_1) = f(\delta(q_0, \alpha_1)) = f(\delta(q_0, \alpha_2)) = f(\alpha_2)$.

(ii) $\Rightarrow$ (iii) For $\alpha_1, \alpha_2 \in \Sigma^{\oplus}$, if $\alpha_1 \equiv \alpha_2$, then for any $\beta \in \Sigma^{\oplus}$, $\alpha_1 \oplus \beta \equiv \alpha_2 \oplus \beta$, and so $f(\alpha_1 \oplus \beta) = f(\alpha_1 \oplus \beta)$. Thus, $\alpha_1 \equiv_f \alpha_2$, which implies that $\equiv$ refines $\equiv_f$. Note that $\equiv$ has finite index, so is $\equiv_f$.

(iii) $\Rightarrow$ (i) Let $Q_{\equiv_f} = \Sigma^*/_{\equiv_f} = \{[\alpha]_{\equiv} | \alpha \in \Sigma^{\oplus}\}$, i.e., $Q_{\equiv_f}$ is finite. Then the formal construction of DLMA $\mathcal{M}_{\equiv_f} = (Q_{\equiv_f}, \Sigma, \delta_{\equiv_f}, [0_\Sigma]_{\equiv_f}, F_{\equiv_f})$ is as follows: $\delta_{\equiv_f} : Q_{\equiv_f} \times \Sigma^{\oplus} \to Q_{\equiv_f}$ is a mapping such that $\delta_{\equiv_f}([\alpha]_{\equiv_f}, \beta) = [\alpha \oplus \beta]_{\equiv_f}$, and $F_{\equiv_f} : Q \to L$ is defined by $F_{\equiv_f}([\alpha]_{\equiv}) = f(\alpha)$ for any $\alpha \in \Sigma^{\oplus}$. Thus, for any $\alpha \in \Sigma^{\oplus}$, $f_{\equiv_f}(\alpha) = F_{\equiv_f}(\delta_{\equiv_f}([0_\Sigma]_{\equiv_f}, \alpha) = F_{\equiv_f}([\alpha]_{\equiv_f}) = f(\alpha)$, that is, $f_{\equiv_f} = f$, and f is an L-regular multiset language.

The DLMA constructed in the above theorem will be denoted by $\mathcal{M}_{\equiv_f}$ in the following.

Theorem 7. For any L-regular multiset language f, $\mathcal{M}_{\equiv_f}$ is a strong homomorphic image of any reachable DLMA accepting f.

Proof. Note that $\mathcal{M}_{\equiv_f}$ constructed in Theorem 6 is reachable. Let $\mathcal{N} = (P, \Sigma, \eta, p_0, E)$ be any reachable DLMA accepting f. Since $\mathcal{N}$ is a reachable DLMA, that is, for any $p \in P$, there exists $\alpha_p \in \Sigma^{\oplus}$ such that $\eta(p_0, \alpha_p) = p$. Define a mapping $\varphi : P \to Q_{\equiv_f}$ as $\varphi(p) = [\alpha_p]_{\equiv_f}$. Clearly, φ is well defined. In fact, if $\eta(p_0, \alpha_p) = \eta(p_0, \alpha)$ for another $\alpha \in \Sigma^{\oplus}$, then $\alpha_p \equiv_{\equiv_f} \alpha$, where $\equiv_{\equiv_f}$ is defined in Theorem 6, and by Theorem 6, we have $\alpha_p \equiv_f \alpha$, that is, $[\alpha_p]_{\equiv_f} = [\alpha]_{\equiv_f}$. Hence, φ is well defined. For any $\alpha \in \Sigma^{\oplus}$, since $\varphi(\eta(p_0, \alpha)) = [\alpha]_{\equiv_f} \in Q_{\equiv_f}$, φ is a surjective. Finally, we know that $\eta(p_0, 0_\Sigma) = p_0$, $\varphi(p_0) = [0_\Sigma]_{\equiv_f}$. Furthermore, for any $\beta \in \Sigma^{\oplus}$,

$$\begin{aligned}\varphi(\eta(p, \beta)) &= \varphi(\eta(p_0, \alpha_p), \beta)\\ &= \varphi(\eta(p_0, \alpha_p \oplus \beta))\\ &= [\alpha_p \oplus \beta]_{\equiv_f}\\ &= \delta_{\equiv_f}([\alpha_p]_{\equiv_f}, \beta)\\ &= \eta(\varphi(p), \beta).\end{aligned}$$

Similarly we have $E(p) = F_{\equiv_f}(\varphi(p))$. Therefore, φ is a strong, surjective homomorphism, and $\mathcal{M}_{\equiv_f}$ is a homomorphism image of $\mathcal{N}$.

Theorem 8. For any L-regular multiset language f, there exists a minimal reachable DLMA accepting f, which is just the one $\mathcal{M}_{\equiv_f}$ constructed in Theorem 6.

Proof. The proof is similar to the proof of Proposition 5.5 in [28] and, hence, is omitted.

Remark. In fact, the construction of $\mathcal{M}_{\equiv_f}$ is not valid. Since the construction of $\mathcal{M}_{\equiv_f}$ depends on the L-regular multiset language f and the equivalence relation $\equiv_f$ defined in Theorem 6. As the following theorem states, there is an efficient algorithm to minimize a given DLMA $\mathcal{N} = (P, \Sigma, \eta, p_0, E)$.

Consider a DLMA $\mathcal{N} = (P, \Sigma, \eta, p_0, E)$, define the following equivalence relation $\equiv_s$ on P as follows: $q \equiv_s p$ if and only if $(\forall \alpha \in \Sigma^{\oplus})\ E(\eta(q, \alpha)) = E(\eta(p, \alpha))$. We say that q is *distinguishable* from p if there exists an $\alpha \in \Sigma^{\oplus}$ such that $E(\delta(q, \alpha)) \neq E(\delta(p, \alpha))$.

The proof of the following theorem is the same as in the crisp case so it will be omitted.

Theorem 9. Let $\mathcal{N} = (P, \Sigma, \eta, p_0, E)$ be a DLMA. and let a sequence $\{R_i\}_{i \in \mathbb{N}}$ of equivalences on P be defined as follows:

(1) $R_0 = \{(q, p) \mid E(q) = E(p)\}$;

(2) $R_{i+1} = \{(q, p) \in R_i \mid$ for any $(q, \alpha), (p, \alpha) \in \nabla_{\mathcal{N}}$, $\big(\eta(q, \alpha), \eta(p, \alpha)\big) \in R_i\}$, $(i = 0, 1, 2, \cdots)$,

then $R_0 \supseteq R_1 \supseteq \cdots \supseteq R_i \supseteq R_{i+1}$, and there exists $k \in \mathbb{N}$ such that $R_k = R_{k+1}$, that is, $R_k = \equiv_s$.

Let $\mathcal{N} = (P, \Sigma, \eta, p_0, E)$ be a DLMA, we write $\nabla_{\mathcal{N}}^{\Sigma^{\oplus}} = \{\alpha \in \Sigma^{\oplus} \mid$ there exists $p \in Q$ such that $(p, \alpha) \in \nabla_{\mathcal{N}}\}$. Clearly, $\nabla_{\mathcal{N}}^{\Sigma^{\oplus}}$ is a finite set. The following algorithm presents an efficient method to minimize a given DLMA.

Algorithm 1. The algorithm of deterministic lattice multiset finite automata

Input: Given a DLMA $\mathcal{N} = (P, \Sigma, \eta, p_0, E)$, where $P = \{p_0, p_1, \cdots, p_{n-1}\}$

Output: The minimization of DLMA $\mathcal{M} = (Q, \Sigma, \delta, q_0, F)$

Step 1. For $p_i, p_j \in P$, where $i < j$, if $E(p_i) \neq E(p_j)$, mark (p_i, p_j) as p_i being distinguishable form p_j.

Step 2. For each pair (p_i, p_j), where $i < j$, and if (p_i, p_j) is unmarked, do

If there exists a multiset $\alpha \in \nabla_{\mathcal{N}}^{\Sigma^{\oplus}}$, such that $(p_i, \alpha), (p_j, \alpha) \in \nabla_{\mathcal{N}}$, and $\big(\eta(p_i, \alpha), \eta(p_j, \alpha)\big)$ is marked, then

Step 2.1. Mark(p_i, p_j).

Step 2.2. Recursively check all unmarked pairs on the list (p_i, p_j) with respect to $\nabla_{\mathcal{N}}^{\Sigma^{\oplus}}$.

else

Step 2.3. For each α in $\nabla_{\mathcal{N}}^{\Sigma^{\oplus}}$, do

Put the pair (p_i, p_j) on the list for $(\eta(p_i, \alpha), \eta(p_j, \alpha))$, $\alpha \in \Sigma^{\oplus}$ unless $\eta(p_i, \alpha) = \eta(p_j, \alpha)$.

Step 3. The formal construction of equivalence classes of $\mathcal{N}$ is as follows:

For $i = 0$ to $n - 1$ do

For $j = i + 1$ to n do

If (p_i, p_j) is unmarked, p_j is in $[p_i]_{\equiv_s}$.

Step 4. Define minimal $\mathcal{M} = (Q, \Sigma, \delta, q_0, F)$ as follows:

$Q = \{[p_i]_{\equiv_s}, p_i \in P\}$, $\delta([p_i]_{\equiv_s}, \alpha) = [\eta(p_i), \alpha]_{\equiv_s}$, $q_0 = [p_0]_{\equiv_s}$, and $F([p_i]_{\equiv_s}) = E(p_i)$.

Step 5. Return $\mathcal{M}$

Theorem 10. The DLMA constructed by means of Algorithm 1, with inreachable states removed, is minimal. The algorithm of complexity of the minimal DLMA is $O(kn^2)$, where $|\nabla_{\mathcal{N}}^{\Sigma^{\oplus}}| = k$ and $|P| = n$.

Proof. By Definition 9, it suffice to show that the DLMA $\mathcal{M} = (Q, \Sigma, \delta, q_0, F)$ constructed in of Algorithm 1 is coreachable. Let $\mathcal{M}$ accept L-regular multiset language f, and let $[p_i]_{\equiv_s}, [p_j]_{\equiv_s} \in Q$ such that $\sigma([p_i]_{\equiv_s}) = \sigma([p_j]_{\equiv_s})$, where the coreachability mapping σ is defined in Definition 8. Hence, $f_{[p_i]_{\equiv_s}} = f_{[p_j]_{\equiv_s}}$. Therefore, for any $\alpha \in \Sigma^{\oplus}$,

$$f_{[p_i]_{\equiv_s}}(\alpha) = f_{[p_j]_{\equiv_s}}(\alpha)$$
$$\Rightarrow F(\delta([p_i]_{\equiv_s}, \alpha)) = F(\delta([p_j]_{\equiv_s}, \alpha))$$
$$\Rightarrow F([\eta(p_i, \alpha)]) = F([\eta(p_j, \alpha)])$$
$$\Rightarrow p_i \equiv_s p_i$$
$$\Rightarrow [p_i]_{\equiv_s} = [p_j]_{\equiv_s}.$$

Hence, $\sigma : Q \to \mathcal{F}(\Sigma^{\oplus})$ is one-one. Therefore, $\mathcal{M}$ is minimal.

The algorithm of complexity can be verified directly.

Example 3. Let $L = \{0, l_1, l_2, 1\}$ such that $l_1 \vee l_2 = 1$ and $l_1 \wedge l_2 = 0$. Clearly, L is a distributive lattice. Consider a DLMA $\mathcal{N} = (P, \Sigma, \eta, p_0, E)$, where $P = \{p_0, p_1, \cdots, p_7\}$, $\Sigma = \{a, b, c\}$, and $E(p_1) = E(p_6) = l_1$, $E(p_5) = l_2$, $E(p_i) = 0$ $(i \neq 0, 1, 2)$, the fuzzy transition function is given as under:

$\eta(p_0, \langle a\rangle) = p_1$, $\eta(p_0, \langle a\rangle \oplus \langle b\rangle) = p_2$,
$\eta(p_1, \langle a\rangle \oplus \langle b\rangle \oplus \langle c\rangle) = p_5$, $\eta(p_1, \langle a\rangle \oplus \langle c\rangle) = p_5$,
$\eta(p_2, \langle a\rangle) = p_0$, $\eta(p_2, \langle b\rangle) = p_6$,
$\eta(p_3, \langle a\rangle \oplus \langle c\rangle) = p_2$, $\eta(p_5, \langle a\rangle) = p_2$,
$\eta(p_4, \langle a\rangle \oplus \langle b\rangle \oplus \langle c\rangle) = p_7$, $\eta(p_4, \langle a\rangle \oplus \langle b\rangle) = p_2$,
$\eta(p_6, \langle a\rangle) = p_6$, $\eta(p_6, \langle b\rangle \oplus \langle c\rangle) = p_0$,
$\eta(p_7, \langle a\rangle \oplus \langle c\rangle) = p_6$, $\eta(p_7, \langle b\rangle \oplus \langle a\rangle \oplus \langle c\rangle) = p_5$.

By Algorithm 1, we have $p_0 \equiv_s p_4$ and $p_1 \equiv_s p_7$. Since p_3 is inreachable, then p_3 cloud be removed. Thus, we get the minimal DLMA $\mathcal{M} = (Q, \Sigma, \delta, q_0, F)$ as follows: $Q = \{q_0, q_1, q_2, q_3, q_4\}$, $q_0 = \{p_0, p_4\}, q_1 = \{p_1, p_7\}, q_2 = \{p_2\}, q_3 = \{p_4\}$, $q_4 = \{p_5\}$, and $F(q_1) = l_1$, $F(q_3) = l_2$. The transition function δ is defined as follows:

$\delta(q_0, \langle a\rangle) = q_1$, $\delta(q_0, \langle a\rangle \oplus \langle b\rangle) = q_2$,
$\delta(q_1, \langle a\rangle \oplus \langle c\rangle) = q_4$, $\delta(q_1, \langle a\rangle \oplus \langle b\rangle \oplus \langle c\rangle) = q_3$,
$\delta(q_2, \langle a\rangle) = q_0$, $\delta(q_2, \langle b\rangle) = q_4$,
$\delta(q_3, \langle a\rangle) = q_2$, $\delta(q_4, \langle a\rangle) = q_4$,
$\delta(q_4, \langle b\rangle \oplus \langle c\rangle) = q_0$.

4. Finite decomposition of L-regular multiset languages

In this section, we show that an L-regular multiset language admits a finite decomposition into some disjoint joins of some simple L-regular multiset languages over an alphabet Σ with respect to the construction of the corresponding minimal DLMA.

For an L-regular multiset language f, if the fuzzy final state of its minimal DLMA $\mathcal{M} = (Q, \Sigma, \delta, q_0, F)$ has a single support, then we call f an *L-unitary language*. That is, there exist $q \in Q$ and $l \in L - \{0\}$ such that $F(q) = l$ and $F(p) = 0$ if $p \neq q$. We also denote F as q_l, meaning that F has singleton support and the height l.

Theorem 8. For an L-multiset language $f : \Sigma^{\oplus} \to L$, the following conditions are equivalent:

(i) f is an L-unitary language.

(ii) For any $\alpha_1, \alpha_2 \in supp(f)$ and $\alpha \in \Sigma^{\oplus}$, $f(\alpha_1 \oplus \alpha) = f(\alpha_2 \oplus \alpha)$.

(iii) f can be accepted by a DLMA $\mathcal{N} = (P, \Sigma, \eta, p_0, p_l)$ with a singleton support of fuzzy final state such that p is reachable.

Proof. (ii) $\Rightarrow$ (i) Consider the L-regular multiset language f, there exists a minimal reachable DLMA accepting f, which is just the one $\mathcal{M}_{\equiv_f} = (Q_{\equiv_f}, \Sigma, \delta_{\equiv_f}, [0_\Sigma]_{\equiv_f}, F_{\equiv_f})$ constructed in Theorem 6, where the fuzzy final state $F_{\equiv_f} : Q_{\equiv_f} \to L$ is defined by $F_{\equiv_f}([\alpha]_{\equiv_f}) = f(\alpha)$ for any $[\alpha]_{\equiv_f} \in Q_{\equiv_f}$, $\alpha \in \Sigma^{\oplus}$. If $F_{\equiv_f}([\alpha_1]_{\equiv_f}) > 0$, and $F_{\equiv_f}([\alpha_1]_{\equiv_f}) > 0$, that is, $f(\alpha_1) > 0$, and $f(\alpha_2) > 0$, i.e., $\alpha_1, \alpha_2 \in supp(f)$. From (ii), for any $\alpha \in \Sigma^{\oplus}$, we have $f(\alpha_1 \oplus \alpha) = f(\alpha_2 \oplus \alpha)$, i.e., $[\alpha_1 \oplus \alpha]_{\equiv_f} = [\alpha_2 \oplus \alpha]_{\equiv_f}$. By Theorem 6, $[\alpha_1]_{\equiv_f} = [\alpha_2]_{\equiv_f}$. Hence, F has a singleton support.

(i) $\Rightarrow$ (iii) Obviously.

(iii) $\Rightarrow$ (ii) Suppose that $f = f_{\mathcal{N}}$ for some DLMA $\mathcal{N} = (P, \Sigma, \eta, p_0, p_l)$ with a singleton support of fuzzy final state such that p is reachable. Thus, we let $\alpha \in supp(f)$, then $\eta(p_0, \alpha) = p$. Furthermore, $f_{\mathcal{N}}(\alpha \oplus \alpha') = f(\alpha \oplus \alpha') = p_l(\eta(p_0, \alpha \oplus \alpha') = p_l(\eta(p, \alpha'))$, which implies that the value of $f(\alpha \oplus \alpha')$ is independent of the choice of $\alpha \in supp(f)$. Hence, for any $\alpha_1, \alpha_2 \in supp(f)$, $f(\alpha_1 \oplus \alpha) = f(\alpha_2 \oplus \alpha)$.

Theorem 12. For an L-regular multiset language f on Σ and $f \neq \emptyset$. Then $f = \bigvee_{i=1}^{n} f_i$, where f_i is an L-unitary language and $f_i \wedge f_j = \emptyset$ whenever $i \neq j$.

Proof. Let $\mathcal{M}_{\equiv_f} = (Q_{\equiv_f}, \Sigma, \delta_{\equiv_f}, [0_\Sigma]_{\equiv_f}, F_{\equiv_f})$ be the minimal DLMA accepting f. Clearly, Im(f) is finite. Let $supp(f) = \{[\alpha_1]_{\equiv_f}, [\alpha_2]_{\equiv_f}, \cdots, [\alpha_n]_{\equiv_f}\}$. For convenience, we let $F^j_{\equiv_f} = [\alpha_j]^{l_j}_{\equiv_f}$, $l_j = F_{\equiv_f}([\alpha_j]_{\equiv_f}) = f(\alpha_j)$, $[\alpha_j]^{l_j}_{\equiv_f} \in supp(f)$. Thus, consider a DLMA $\mathcal{M}^j_{\equiv_f} = (Q_{\equiv_f}, \Sigma, \delta_{\equiv_f}, [0_\Sigma]_{\equiv_f}, F^j_{\equiv_f})$ with $f_j = f_{\mathcal{M}^j_{\equiv_f}}$. It is easily shown that $f = \bigvee_{i=1}^n f_i$ and $f_i \wedge f_j = \emptyset$ whenever $i \neq j$. Since $\mathcal{M}_{\equiv_f}$ is reachable, i.e., $[\alpha_j]_{\equiv_f}$ is a reachable state, which implies that f_j is an L-unitary language for $j = 1, 2, \cdots, n$.

5. Conclusions

In this paper, we have studied the properties of finite multiset automata whose membership values are in a distributive lattice, and its relationships to the decomposition of L-multiset languages. We have demonstrated that the nondeterministic LMA are equivalent to the DLMA in the sense of maintaining the same ability of accepting L-multiset languages. We have presented an effective algorithm to obtain a minimal DLMA for a given LMA. Some special L-multiset languages related to the structures of the minimal DLMA were also introduced. There is still much work to be done, such as fuzzy mealy multiset automata, fuzzy moore multiset automata.

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