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The $\omega\psi$-perfection of graphs

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Abstract

In this paper we study a natural generalization for the perfection of graphs to other interesting parameters related with colorations. This generalization was introduced partially by Christen and Selkow in 1979 and Yegnanarayanan in 2001.

Let $a, b \in \{\omega, \chi, \Gamma, \alpha, \psi\}$ where ω is the clique number, χ is the chromatic number, Γ is the Grundy number, α is the achromatic number and ψ is the pseudoachromatic number. A graph G is *ab-perfect* if for every induced subgraph H, $a(H) = b(H)$. In this work we characterize the $\omega\psi$-perfect graphs.

Keywords: Perfect graphs, Grundy, achromatic and pseudoachromatic numbers.

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1 Introduction

Let G be a simple graph. A vertex coloring $\varsigma\colon V(G) \to \{1, \dots, k\}$ is called *complete* if each pair of different colors appears in an edge. The *pseudoachromatic number* $\psi(G)$ is the maximum k for which a complete coloring of G exists.

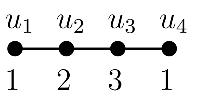
A vertex coloring of G is called *proper* if uv is an edge of G then $\varsigma(u) \neq \varsigma(v)$. The *achromatic number* $\alpha(G)$ is the maximum k for which a proper and complete coloring of G exists.

A vertex coloring of G is called *Grundy* if it is a proper vertex coloring such that, for every two colors $\{i, j\} \in \mathbb{Z}^+$ with $i < j$, every vertex colored j has a neighbor colored i. Consequently, every Grundy coloring is a complete coloring. The *Grundy number* $\Gamma(G)$ is the maximum k for which a Grundy coloring of G exists.

Clearly,

$$\omega(G) \leq \chi(G) \leq \Gamma(G) \leq \alpha(G) \leq \psi(G) \tag{1}$$

where $\omega(G)$ denotes, as usual, the clique number of G and $\chi(G)$ denotes the chromatic number.



ω	χ	Γ	α	ψ
2	2	3	3	3

Fig. 1. A complete and proper coloring of P_4 with 3 colors.

The Grundy number was introduced by Grundy in 1939, the achromatic number by Harary, Hedetniemi and Prins in 1967 and the pseudoachromatic number by Gupta in 1969.

Let $a, b \in \{\omega, \chi, \Gamma, \alpha, \psi\}$. A graph G is called *ab-perfect* if for every induced subgraph H of G, $a(H) = b(H)$. This definition extends the usual notion of *perfect graph* introduced by Berge in 1961; with this notation a perfect graph is called $\omega\chi$-perfect. This reference and the following do not appear in the bibliography due to space restriction. Lóvasz proved in 1972 that G is $\omega\chi$-perfect if and only if G^c is $\omega\chi$-perfect, where G^c denotes the complement of G (that is, if G is a graph of order n, then $V(G^c) = V(G)$ and $E(G^c) = E(K_n) - E(G)$). Chudnovsky, Robertson, Seymour and Thomas proved in 2006 that G is $\omega\chi$-perfect if and only if G and G^c are C_{2k+1}-free for all $k \geq 2$ [4]; and Seinsche proved in 1974 the following theorem which we will use to prove our Theorem 2.1:

[4] A graph G without an induced subgraph H is called *H-free*. A graph H_1-free, H_2-free, $\dots$, H_m-free is called $(H_1, H_2, \dots, H_m)$-free.

Theorem 1.1 *(Seinsche, 1974) If a graph G is P_4-free then G is $\omega\chi$-perfect.*

The concept of the ab-perfect graphs was introduced in [3] and extended in [5], but in this paper only was considered ab-perfect graphs for $a, b \in \{\omega, \chi, \alpha, \psi\}$. The following statements are given in [5]:

Theorem 1.2 *(Yegnanarayanan, 2001) For any finite graph G the following are equivalent:*

$\langle 1 \rangle$ *G is $\omega\psi$-perfect,*
$\langle 2 \rangle$ *G is $\chi\psi$-perfect,*
$\langle 3 \rangle$ *G is $\alpha\psi$-perfect*
$\langle 4 \rangle$ *G is C_4-free.*

Corollary 1.3 *(Yegnanarayanan, 2001) Every $\alpha\psi$-perfect graph is $\chi\alpha$-perfect.*

Corollary 1.4 *(Yegnanarayanan, 2001) Every $\chi\alpha$-perfect graph is $\omega\chi$-perfect.*

Unfortunately, this theorem is false (a counterexample is P_4 because it is C_4-free but not $\omega\psi$-perfect, see Figure 1); i.e., $\langle 4 \rangle$ does not necessarily imply $\langle 1 \rangle$. Consequently the corollaries are not well founded, however, while Corollary 1.3 is false (again the counterexample is P_4, see Figure 1), Corollary 1.4 is true (see Theorem 2.1). Note that if a graph G is $\omega\psi$-perfect then it is immediately $\omega\chi$-perfect (see Equation 1). That is, G is perfect in the usual sense and it is well known that G does not allow odd cycles (except triangles) or their complements and then the condition C_4-free is not sufficient. In Appendix 4 of [2] we exhibit the exact place where the proof of the Theorem 1.2, given by Yegnanarayanan, is wrong.

In this paper we examine this theorem. In the next section we will state some theorems that characterize the $\omega\psi$-graphs. Also, we will show a graph G that has all of these parameters different, and we will exhibit, using a directed transitive graph D, the relationship between the ab-graphs for $a, b \in \{\omega, \chi, \alpha, \psi\}$.

There exists interesting results related with these invariants; the authors of this paper and others have studied some specific $\alpha\psi$-perfect graphs: the line graph of the complete graph K_n for some specific values of n and their relation with the projective planes. For more information on this topic see [1] and the references therein.

As background to this paper, in [4] was proved that for any graph G and for every integer a with $\chi(G) \leq a \leq \alpha(G)$, there is a complete and proper coloring of G with a colors. In [3] the authors proved that for any graph G and for every integer b with $\chi(G) \leq b \leq \Gamma(G)$ there is a Grundy coloring of

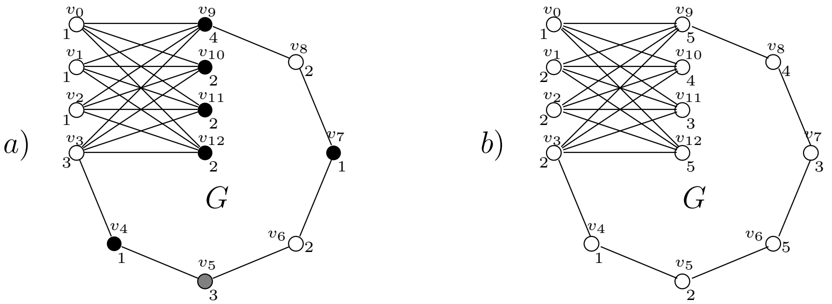


Fig. 2. Different coloration of G.

G with b colors. Also, they give characterizations of ωa-perfect graphs when a is Γ or α (see [3]).

Yegnanarayanan, Balakrishnan and Sampathkunar proved in 2000 that if $2 \leq a \leq b \leq c$ there exists a graph G with chromatic number a, achromatic number b, and pseudoachromatic number c. Finally, Chartrand, Okamoto, Tuza and Zhang proved in 2010 that for integers a, b and c with $2 \leq a \leq b \leq c$ there exists a connected graph G with $\chi(G) = a$, $\Gamma(G) = b$ and $\alpha(G) = c$ if and only if $a = b = c = 2$ or $b \geq 3$.

2 Results

To start this section we show graph G, depicted in Figure 2, with all of these parameters being different. In particular G has $\omega(G) = 2$, $\chi(G) = 3$, $\Gamma(G) = 4$, $\alpha(G) = 5$ and $\psi(G) = 6$. The graph G is constructed as follows: Let e_1 be an edge of $K_{4,4}$ and let e_2 be an edge of C_7, identifying e_1 with e_2 we obtain G. For construction, G does not contain C_3 as an induced subgraph, then $\omega(G) = 2$. Figure 2a) shows a proper vertex coloring of G with three colors (black, white and gray), then $\chi(G) \leq 3$,, and G contains a C_7 then $\chi(G) = 3$. Also to the left we show a Grundy vertex coloring of G with four colors (numbers), then $\Gamma(G) \geq 4$. Figure 2b) shows a complete and proper vertex coloring of G with five colors and $\alpha(G) \geq 5$. The proof of $\Gamma(G) \leq 4$, $\alpha(G) \leq 5$ and $\psi(G) = 6$ are given in Appendix 4 of [2].

In Figure 3 we show a directed transitive graph D where the vertices of D represent the classes of ab-perfect graphs and their label is ab respectively for $a, b \in \{\omega, \chi, \Gamma, \alpha, \psi\}$. If two classes are equal they define the same vertex (see Theorem 2.1). If a class of ab-perfect graphs is contained in a class of cd-perfect graphs then there is an arrow from vertex ab to vertex cd. To prove that D does not contain another arrow we have the following counterexamples: P_4, C_4, C_5 and $P_3 \cup K_2$.

Finally, here are the theorems that we prove in [2]:

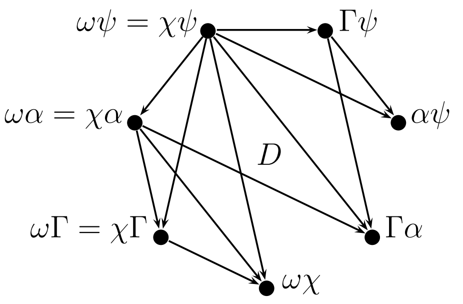


Fig. 3. Relationship between ab-perfect graphs with $a, b \in \{\omega, \chi, \Gamma, \alpha, \psi\}$.

$\alpha\psi \nrightarrow \omega\psi$	$\Gamma\psi \nrightarrow \omega\psi$	$\Gamma\alpha \nrightarrow \omega\psi$	$\omega\chi \nrightarrow \omega\psi$
$\alpha\psi \nrightarrow \omega\alpha$	$\Gamma\psi \nrightarrow \omega\alpha$	$\Gamma\alpha \nrightarrow \omega\alpha$	$\omega\chi \nrightarrow \omega\alpha$
$\alpha\psi \nrightarrow \omega\Gamma$	$\Gamma\psi \nrightarrow \omega\Gamma$	$\Gamma\alpha \nrightarrow \omega\Gamma$	$\omega\chi \nrightarrow \omega\Gamma$

Table 1
P_4 is a counterexample in these cases.

$\omega\alpha \nrightarrow \omega\psi$	$\omega\alpha \nrightarrow \Gamma\psi$	$\omega\alpha \nrightarrow \alpha\psi$	$\omega\chi \nrightarrow \Gamma\psi$	$\Gamma\alpha \nrightarrow \Gamma\psi$
$\omega\Gamma \nrightarrow \omega\psi$	$\omega\Gamma \nrightarrow \Gamma\psi$	$\omega\Gamma \nrightarrow \alpha\psi$	$\omega\chi \nrightarrow \alpha\psi$	$\Gamma\alpha \nrightarrow \alpha\psi$

Table 2
C_4 is a counterexample in these cases.

$\alpha\psi \nrightarrow \omega\chi$	$\Gamma\psi \nrightarrow \omega\chi$	$\Gamma\alpha \nrightarrow \omega\chi$

Table 3
C_5 is a counterexample in these cases.

$\alpha\psi \nrightarrow \Gamma\psi$	$\alpha\psi \nrightarrow \Gamma\alpha$	$\omega\Gamma \nrightarrow \Gamma\alpha$	$\omega\chi \nrightarrow \Gamma\alpha$	$\omega\Gamma \nrightarrow \omega\alpha$

Table 4
$P_3 \cup K_2$ is a counterexample in this cases.

Theorem 2.1 *Let G be a graph and let $a \in \{\Gamma, \alpha, \psi\}$. G is χa-perfect if and only if G is ωa-perfect.*

Theorem 2.2 *G is a connected graph of order n (C_4, P_4)-free if and only if there exists a set of connected graphs $\{G_1, \ldots, G_k\}$ for some $k \in \mathbb{N}$ also (C_4, P_4)-free and $m \in \mathbb{Z}^+$ such that $G = K_m \oplus \bigcup_{i=1}^{k} G_i$ and $G - K_m$ is disconnected* *(if $k \geq 2$)*[5] .

In the following two theorems we prove some equivalences between graphs. Note that, in the hypothesis of Theorem 2.3, G is a connected graph, while in Theorem 2.4, G is any graph (not necessarily connected).

Theorem 2.3 *For any connected graph G the following are equivalent:*

$\langle 1\rangle$ *G is $\omega\psi$-perfect,*
$\langle 2\rangle$ *G is $\chi\psi$-perfect,*
$\langle 3\rangle$ *G is $(C_4, P_4, P_3 \cup K_2, 3K_2)$-free, and*
$\langle 4\rangle$ *$G = K_{n_1} \oplus (K_{n_2} \cup K_{n_3} \cup n_4K_1)$ or $G = K_{n_1} \oplus (G' \cup n_2K_1)$ for $n_1 \in \mathbb{Z}^+$ and $n_2, n_3, n_4 \in \mathbb{N}$ where G' is a non complete connected graph and $(C_4, P_4, P_3 \cup K_2, 3K_2)$-free.*

Theorem 2.4 *For any graph G the following are equivalent:*

$\langle 1\rangle$ *G is $\omega\psi$-perfect,*
$\langle 2\rangle$ *G is $\chi\psi$-perfect,*
$\langle 3\rangle$ *G is $(C_4, P_4, P_3 \cup K_2, 3K_2)$-free, and*
$\langle 4\rangle$ *$G = K_{n_1} \cup K_{n_2} \cup n_3K_1$ or $G = G' \cup n_2K_1$ for $n_1 \in \mathbb{Z}^+$ and $n_2, n_3 \in \mathbb{N}$, where G' is a non complete connected graph and $(C_4, P_4, P_3 \cup K_2, 3K_2)$-free.*

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[5] For convenience, in this paper $0 \in \mathbb{N}$ and $\oplus$ denotes the *joint* of graphs.