Chapter 4: Euclid's Algorithm

Exercise 4.4 Prove that, for any odd square number x, there is an even square number y such that $x + y$ is a square number.

Proof. Since x is square and odd, there must be an $n \in \mathbb{N}$ such that $x = (2n+1)^2$. Let y be some even square number. Thus, there must be an $m \in \mathbb{N}$ such that $y = (2m)^2$. It follows that

$$\begin{aligned} x + y &= (2n+1)^2 + (2m)^2 \\ &= 4(n^2 + m^2) + 4n + 1 \end{aligned}$$

We must define m as a function on n in such a way that this number conforms a square. In order to do this, let's see what happens for some small cases:

- If $n = 1$, then $x = 9$. If we set $m = 2$, $y = 16$ and $x + y = 25$, which is a square.
- If $n = 2$, then $x = 25$. Taking $m = 6$, $y = 144$ and $x + y = 169$, which is a square (since $13^2 = 169$).
- If $n = 3$, then $x = 49$. Now, m can be 12, and then $y = 576$ and $x + y = 625 = 25^2$.

A careful analysis of these cases reveals a pattern: $m = n^2 + n$. Substituting this in the equation shown before,

$$\begin{aligned} x + y &= 4(n^2 + m^2) + 4n + 1 \\ &= 4(n^2 + (n^2 + n)^2) + 4n + 1 \\ &= 4n^2 + 4(n^2 + n)^2 + 4n + 1 \\ &= 4(n^2 + n)^2 + 4(n^2 + n) + 1 \\ &= (2(n^2 + n) + 1)^2 \end{aligned}$$

$\square$