

## Chapter 4: Euclid's Algorithm

**Exercise 4.4** Prove that, for any odd square number  $x$ , there is an even square number  $y$  such that  $x + y$  is a square number.

*Proof.* Since  $x$  is square and odd, there must be an  $n \in \mathbb{N}$  such that  $x = (2n + 1)^2$ . Let  $y$  be some even square number. Thus, there must be an  $m \in \mathbb{N}$  such that  $y = (2m)^2$ . It follows that

$$\begin{aligned}x + y &= (2n + 1)^2 + (2m)^2 \\ &= 4(n^2 + m^2) + 4n + 1\end{aligned}$$

We must define  $m$  as a function of  $n$  in such a way that this number conforms a square. In order to do this, let's see what happens for some small cases:

- If  $n = 1$ , then  $x = 9$ . If we set  $m = 2$ ,  $y = 16$  and  $x + y = 25$ , which is a square.
- If  $n = 2$ , then  $x = 25$ . Taking  $m = 6$ ,  $y = 144$  and  $x + y = 169$ , which is a square (since  $13^2 = 169$ ).
- If  $n = 3$ , then  $x = 49$ . Now,  $m$  can be 12, and then  $y = 576$  and  $x + y = 625 = 25^2$ .

A careful analysis of these cases reveals a pattern:  $m = n^2 + n$ . Substituting this in the equation shown before,

$$\begin{aligned}x + y &= 4(n^2 + m^2) + 4n + 1 \\ &= 4(n^2 + (n^2 + n)^2) + 4n + 1 \\ &= 4n^2 + 4(n^2 + n)^2 + 4n + 1 \\ &= 4(n^2 + n)^2 + 4(n^2 + n) + 1 \\ &= (2(n^2 + n) + 1)^2\end{aligned}$$

□

**Exercise 4.5** Prove that, if  $x$  and  $y$  are both sums of two squares, then so is their product  $xy$ .

*Proof.* Being  $x$  and  $y$  both sums of two squares, we can write them like so:

$$\begin{aligned}x &= x_1^2 + x_2^2 \\ y &= y_1^2 + y_2^2\end{aligned}$$

This implies that

$$\begin{aligned}xy &= (x_1^2 + x_2^2)(y_1^2 + y_2^2) \\ &= x_1^2 y_1^2 + x_1^2 y_2^2 + x_2^2 y_1^2 + x_2^2 y_2^2\end{aligned}$$

After several failed attempts at completing the squares (i.e., adding and subtracting the same thing), the following Python script was used to gain some insight into the underlying pattern of  $xy$ :

```
def find_squares(x, y):
    x1, x2 = x
    y1, y2 = y
    n = (x1**2 + x2**2)*(y1**2 + y2**2)
    return [(i,j) for i in xrange(n)
            for j in xrange(i,n)
            if n == i**2 + j**2]
```

For example,

- `find_squares((2,3), (5,7))` → [(1, 31), (11, 29)].
- `find_squares((1,2), (3,4))` → [(2, 11), (5, 10)].

Playing with this script and guessing how to combine the elements in the input tuples in order to generate an output tuple  $(z_1, z_2)$ , the following pattern emerged:

$$\begin{aligned} z_1 &= x_2 y_1 + x_1 y_2 \\ z_2 &= x_2 y_2 - x_1 y_1 \end{aligned}$$

Indeed,

$$\begin{aligned} z_1^2 + z_2^2 &= (x_2 y_1 + x_1 y_2)^2 + (x_2 y_2 - x_1 y_1)^2 \\ &= ((x_2 y_1)^2 + (x_1 y_2)^2 + 2x_2 y_1 x_1 y_2) + ((x_2 y_2)^2 + (x_1 y_1)^2 - 2x_2 y_2 x_1 y_1) \\ &= x_2^2 y_1^2 + x_1^2 y_2^2 + x_2^2 y_2^2 + x_1^2 y_1^2 \\ &= x_1^2 y_1^2 + x_1^2 y_2^2 + x_2^2 y_1^2 + x_2^2 y_2^2 \\ &= xy \end{aligned}$$

□