Chapter 4

2D Kinematics

1

Motion in Two Dimensions

refer to components not vector components

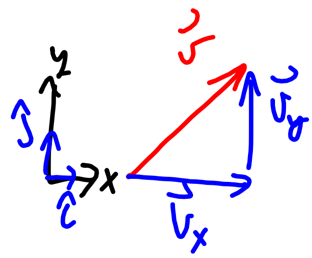
Using + or –signs is not always sufficient to fully describe motion in more than one dimension

- Vectors can be used to more fully describe motion

- Still interested in displacement, velocity, and acceleration

Eg] Velocity $\vec{v}$

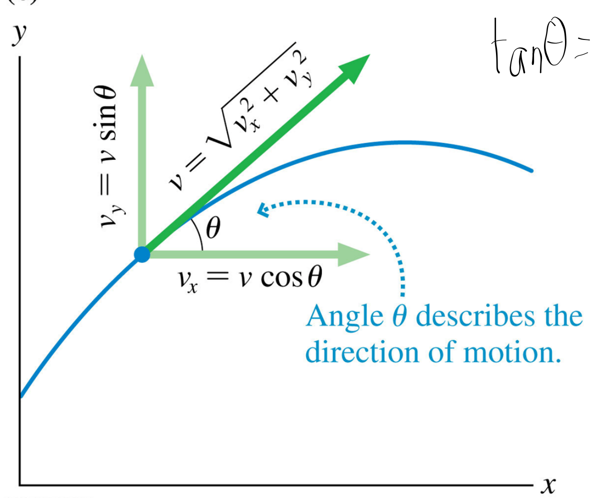
$$\vec{v} = \vec{v}_x + \vec{v}_y$$



$$v_x \hat{i} + v_y \hat{j}$$

Components ← magnitude and sign

(c)

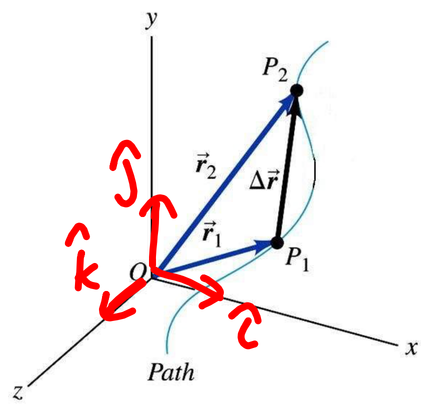


$$\tan\theta = \frac{v_y}{v_x}$$

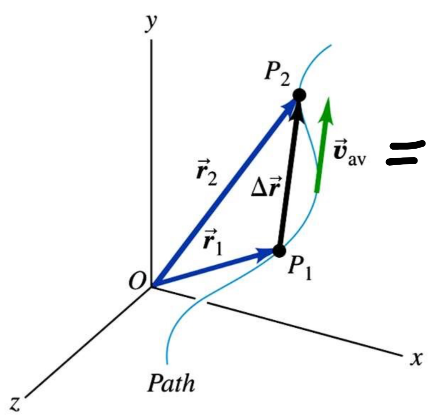
Position and Displacement

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

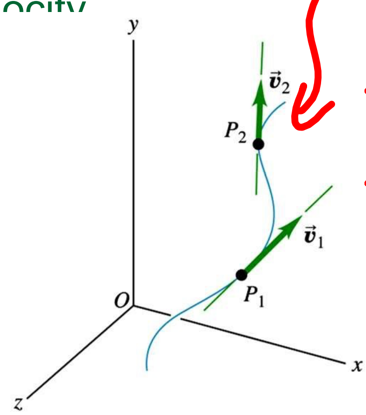
$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$
$$= \underbrace{(\Delta x\,\hat{i} + \Delta y\,\hat{j})}_{2D} + \Delta z\,\hat{k}$$



Average and Instantaneous Velocity



$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t}$$



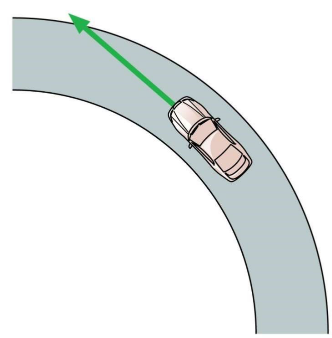
always heading tangent to the path.

Average Acceleration

$$\vec{a}_{avg} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

change in state of motion

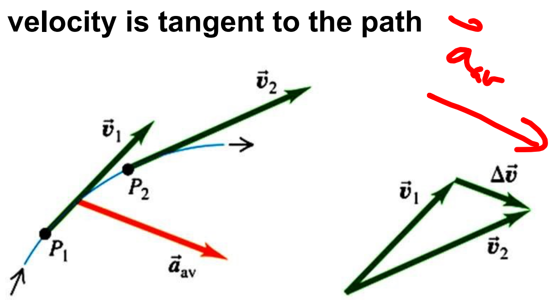
A car is traveling around a curve at a steady 45 mph. Is the car accelerating?



A. Yes

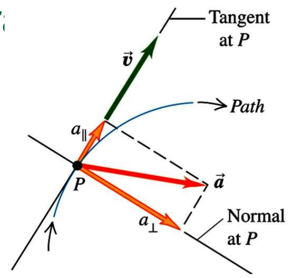
B. No

The velocity is tangent to the path



(not the slope of the tangent)

Producing An Acceler

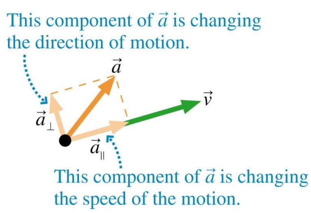


Recall motion in 1D

y

$\uparrow \vec{v}$ $\uparrow \vec{a}$ $\Rightarrow$ speeding up

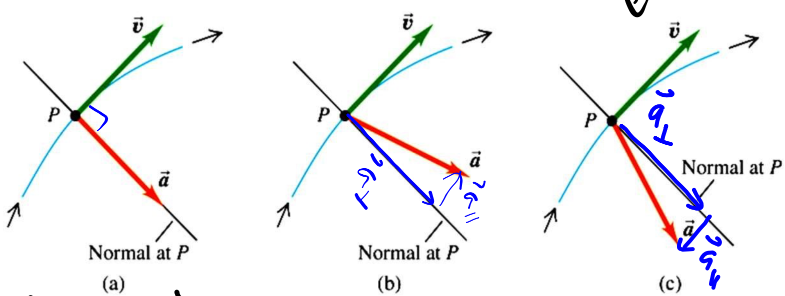
$\uparrow \vec{v}$ $\downarrow \vec{a}$ $\Rightarrow$ slowing down.



Breaking acceleration into "parallel" & "perpendicular" pieces.

$$\vec{a} = \underbrace{\vec{a}_\parallel}_{\text{change in speed}} + \underbrace{\vec{a}_\perp}_{\text{change in direction}}$$

constant speed $\downarrow$ | speeding up $\downarrow$ | slowing down $\downarrow$

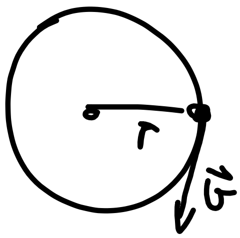


$\vec{a}_\parallel = \vec{0}$

$\vec{a}_\parallel$, same direction as $\vec{v}$

$\vec{a}_\parallel$, opp. direction as $\vec{v}$

Proportional Reasoning aside

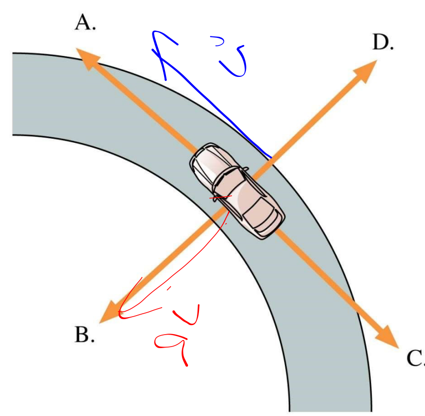


$$a_r \propto \frac{1}{r}$$

$$a_r \propto v^2$$

$$\boxed{a_r = v^2/r}$$

A car is traveling around a curve at a steady 45 mph. Which vector shows the direction of the car's acceleration?



A. B. C. D.

E. The acceleration is zero.

Constant Acceleration in 2D

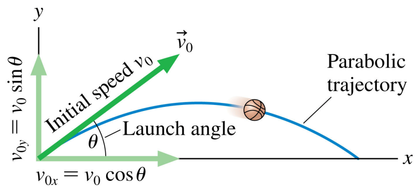
$$x_f = x_i + v_{ix}\,\Delta t + \tfrac{1}{2}a_x(\Delta t)^2 \qquad y_f = y_i + v_{iy}\,\Delta t + \tfrac{1}{2}a_y(\Delta t)^2$$

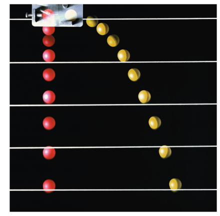
$$v_{fx} = v_{ix} + a_x\,\Delta t \qquad v_{fy} = v_{iy} + a_y\,\Delta t$$

∴ break quantities into x & y components.

- Can be treated as 2 1D problems.
  connected by time

Projectile Motion





$a_x = 0$, $a_y = -g$ } in the absence of air resistance

- y-direction behaves as free-fall
- x-direction behaves as uniform motion.

Kinematic equations:

x] $x = x_0 + v_{0x} t$

y] $y = y_0 + v_{0y}\Delta t - \frac{1}{2} g \Delta t^2$

$v_y = v_{0y} - g\Delta t$

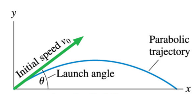
$v_y^2 = v_{0y}^2 - 2g\Delta y$

MODEL 4.1

Projectile motion

For motion under the influence of only gravity.

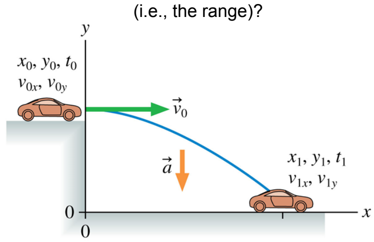
- Model the object as a particle launched with speed v_0 at angle θ:
- Mathematically:
  - **Uniform motion** in the horizontal direction with $v_x = v_0 \cos\theta$.
  - **Constant acceleration** in the vertical direction with $a_y = -g$.
  - Same Δt for both motions.
- Limitations: Model fails if air resistance is significant.



A projectile follows a parabolic trajectory.

Exercise 9

Example: In a stunt, a car is driven off a 10.0m high cliff cliff at a speed of 20.0m/s. How far away does the car land from the base of the cliff (i.e., the range)?



Notice the relevant states have 5 pieces of information

position (x, y);
velocity (v_x, v_y);
time (shared between x and y)

Known			Find
$x_0 = 0$ m	$v_{0y} = 0$ m/s	$t_0 = 0$ s	x_1
$y_0 = 10.0$ m	$v_{0x} = v_0 = 20.0$ m/s		
$a_x = 0$ m/s^2	$a_y = -g$	$y_1 = 0$ m	

horizontal launch!

Sol'n

$x_1 = x_0 + v_{0x}\Delta t$

(1) $x_1 = 20 t_1$

$y_1 = y_0 + v_{0y}\Delta t - \frac{1}{2} g \Delta t^2$

$0 = 10 - \frac{1}{2} g t_1^2 \Rightarrow$ (2) $t_1^2 = 20/g$

$\therefore t_1 = 1.43s$

Sub t_1 into (1):

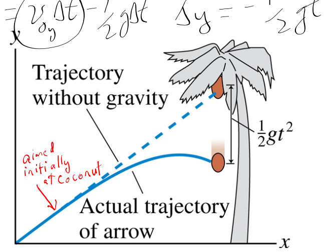
$x_1 = 20(1.43) = 28.6m$

$\therefore$ The range is $28.6m$

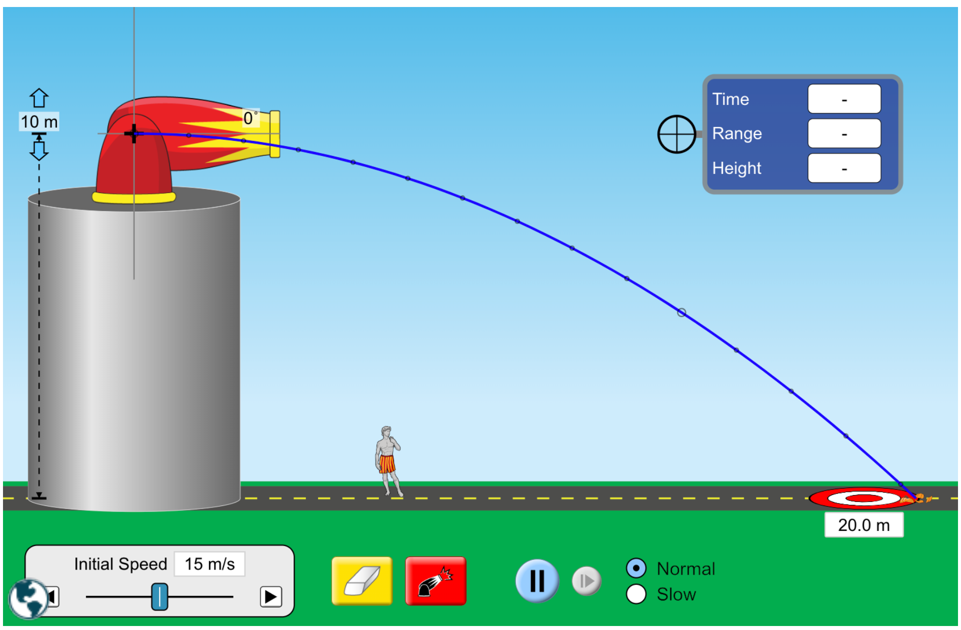
$\Delta y = v_{0y}\Delta t - \frac{1}{2} g \Delta t^2$ $\quad \Delta y = -\frac{1}{2} g t^2$

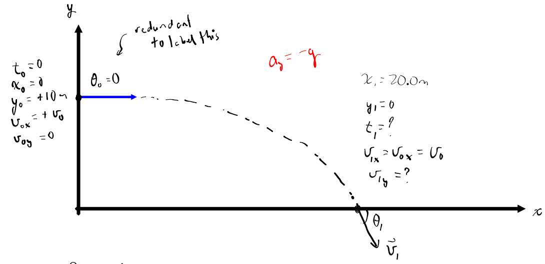
See knight discussion on reasoning about projectile motion.

Does the arrow hit the coconut if it falls at the same time as the arrow is launched?



PhET Activity





a) $v_0 = ?$ initial speed b) $\vec{v}_1 = ?$ find velocity

a) Given:

- The projectile is launched horizontally ($v_{0x} = v_0, v_{0y} = 0$).
- Horizontal distance (range) $\Delta x = 20.0$ m.
- Vertical displacement from launch to landing $\Delta y = -10.0$ m (negative because the displacement is downward).

Find initial speed v_0
Find the time it would take for the object to fall 10.0 m using:

$$\Delta y = v_{0y}\Delta t + \frac{1}{2}a_y(\Delta t)^2 \tag{1}$$

$$\Delta y = -\frac{1}{2}gt_1^2 \tag{2}$$

Given $g = 9.81$ m/s^2 we find:

$$t \approx \sqrt{\frac{20.0 \text{ m}}{9.81 \text{ m/s}^2}} \approx 1.42 \text{ s} \tag{3}$$

the total time the projectile is in the air.
For the horizontal motion:

$$\Delta x = v_{0x}\Delta t \tag{4}$$

Given that the launch is horizontal, $v_{0x} = v_0$, we have:

$$v_0 = \frac{\Delta x}{t_1} \approx \frac{20.0 \text{ m}}{1.42 \text{ s}} \approx 14.1 \text{ m/s} \tag{5}$$

Thus, the required initial speed for the projectile to hit a 20.0 m target from a height of 10.0 m above the ground when launched horizontally is approximately 14.1 m/s.
b) The vertical component of the final velocity:

$$v_y = v_{0y} + a_y\Delta t = -gt_1 = -9.81 \text{ m/s}^2 * 1.42 \text{ s} \approx -13.9 \text{ m/s} \tag{6}$$

Thus final velocity is

$$v_1 = \sqrt{v_{1x}^2 + v_{1y}^2} \approx \sqrt{(14.1 \text{ m/s})^2 + (13.9 \text{ m/s})^2} \tag{7}$$

$$\approx 19.8 \text{ m/s} \tag{8}$$

The direction θ of the final velocity:

$$\tan(\theta) = \frac{v_y}{v_x} = \tan^{-1}\left(\frac{-13.9 \text{ m/s}}{14.1 \text{ m/s}}\right) \tag{9}$$

$$\approx 44.6^\circ \text{ (measured from the horizontal, downward)} \tag{10}$$

Therefore, the magnitude of the final velocity of the projectile just before it hits the ground is approximately 19.82 m/s, and it makes an angle of about 44.7° below the horizontal.

In-class Activity September 28

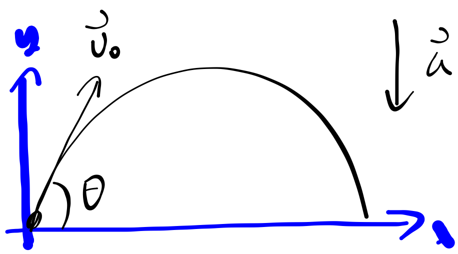
- Derive the relationship between x and y in projectile motion.
- Find an expression for which

$$y(x) \equiv y \text{ as a function of } x$$

$$a_y = -g \;/\; a_x = 0$$

$a_x = 0$

$a_y = -g$



$v_{0x} = v_0 \cos\theta$

$v_{0y} = v_0 \sin\theta$

$x_0 = y_0 = 0$

$t_0 = 0$

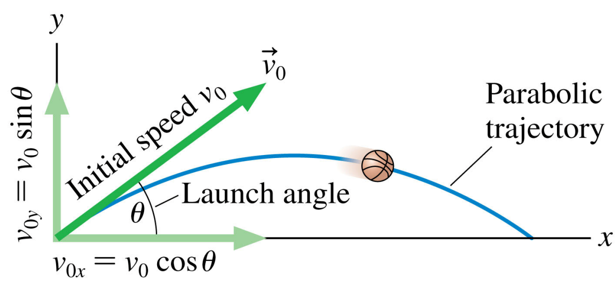
①	②
$x = x_0 + v_{0x} t$	$y = y_0 + v_{0y} t - \frac{1}{2} g t^2$ $v_y = v_{0y} - g t$

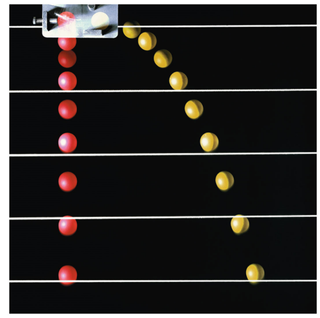
sub ① into ②: $t = x / v_{0x}$

$$y = 0 + v_{0y}\left(\frac{x}{v_{0x}}\right) - \frac{1}{2} g \left(\frac{x}{v_{0x}}\right)^2$$

$$= \tan\theta\, x - \frac{1\, g}{2 v_0 \cos^2\theta} x^2$$

Projectile Motion





$a_x = 0$ } in the absence
$a_y = -g$ } of air resistance

- y-direction behaves as free-fall
- x-direction behaves as uniform motion.

Kinematic equations:

x] $x = x_0 + v_{0x} t$

y] $y = y_0 + v_{0y} \Delta t - \frac{1}{2} g \Delta t^2$

$v_y = v_{0y} - g \Delta t$

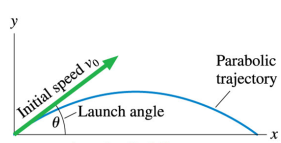
$v_y^2 = v_{0y}^2 - 2g\Delta y$

MODEL 4.1

Projectile motion

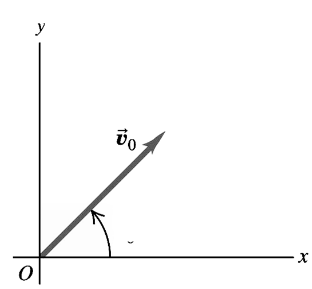
For motion under the influence of only gravity.

- Model the object as a particle launched with speed v_0 at angle θ:
- Mathematically:
  - **Uniform motion** in the horizontal direction with $v_x = v_0 \cos\theta$.
  - **Constant acceleration** in the vertical direction with $a_y = -g$.
  - Same Δt for both motions.
- Limitations: Model fails if air resistance is significant.



A projectile follows a parabolic trajectory.

Exercise 9



Using

$$y = \frac{v_{oy}}{v_{ox}} x - \frac{1}{2}\frac{g}{v_{ox}^2} x^2$$

Find maximum height.

$$y = y_{top}$$

at $x = x_{top}$

1. Take $\frac{dy}{dx}$
2. Set $= 0$

Find max height

3. Show that

$$\frac{d^2y}{dx^2} < 0$$

to confirm it's a max.

Answer:

max. height: $y_{top} = \frac{v_{oy}^2}{2g}$

$$= \frac{v_0^2 \sin^2\theta}{2g}$$

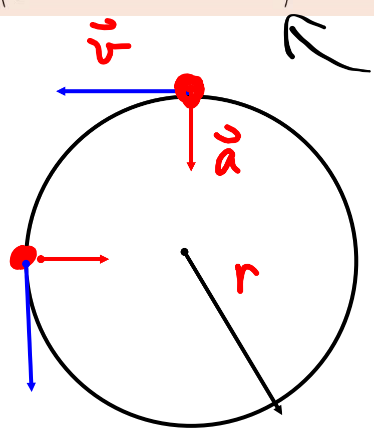
Motion in a circle –in brief

Uniform circular motion (**u.c.m**) occurs when an object moves in a circular path with a constant speed

The change in the velocity is due to the change in direction only.

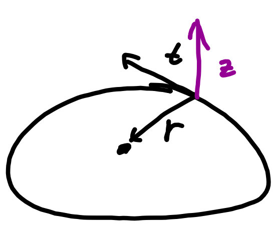
This produces a centripetal ("center-seeking") acceleration.

$$\vec{a} = \left(\frac{v^2}{r}, \text{ toward center of circle}\right) \qquad \text{(centripetal acceleration)}$$



direction will be defined as radial to give (+) or (−) sign. The convention will be (+) toward center of circle.

The directions/coordinates for cir. mot. are usually given as r and t. (radial and tangential)



← z is ⊥ to plane of the circle

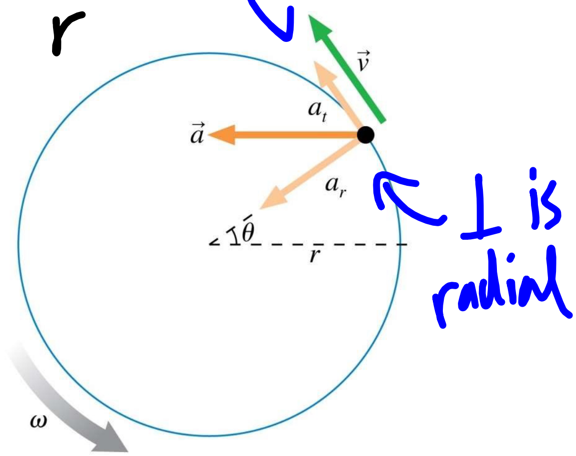
Non-Uniform Circular Motion: Tangential Acceleration

$a_r = \frac{v^2}{r}$

The magnitude of the velocity could also be changing

In this case, there would be a **tangential acceleration:**

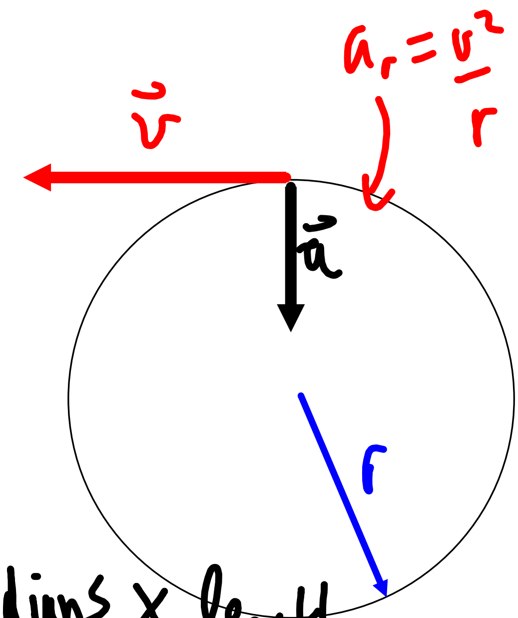
$$\vec{a}_t = \frac{d\vec{v}}{dt}$$



E.g.: a particle moving in a vertical circle like a pendulum

Since $a_r = \frac{v_t^2}{r}$ and v (speed) is changing, then so is a_r. as the particle moves around circle.

Uniform Circular Motion



Consider a particle that moves at constant speed around a circle of radius r

The time interval to complete one revolution is called the period, T

The period T is related to the speed v

radius.x length

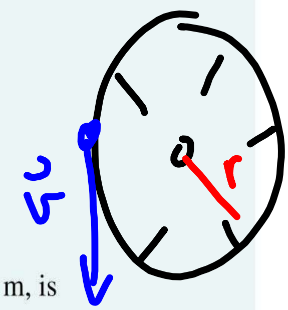
$$v = \frac{1 \text{ circumference}}{1 \text{ period}} = \frac{2\pi r}{T}$$

EXAMPLE 4.9 **A rotating crankshaft**

A 4.0-cm-diameter crankshaft turns at 2400 rpm (revolutions per minute). What is the speed of a point on the surface of the crankshaft?

SOLVE We need to determine the time it takes the crankshaft to make 1 rev. First, we convert 2400 rpm to revolutions per second:

$$\frac{2400 \text{ rev}}{1 \text{ min}} \times \frac{1 \text{ min}}{60 \text{ s}} = 40 \text{ rev/s}$$



If the crankshaft turns 40 times in 1 s, the time for 1 rev is

$$T = \frac{1}{40} \text{ s} = 0.025 \text{ s}$$

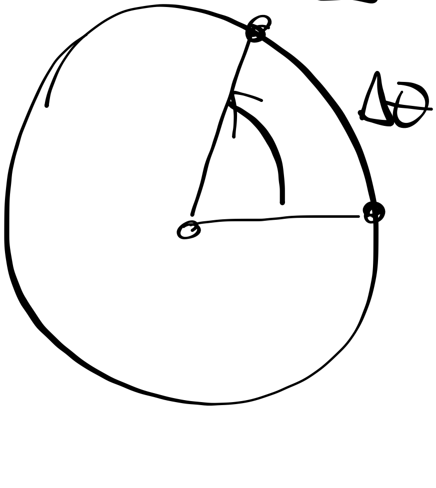
Thus the speed of a point on the surface, where $r = 2.0 \text{ cm} = 0.020 \text{ m}$, is

$$v = \frac{2\pi r}{T} = \frac{2\pi(0.020 \text{ m})}{0.025 \text{ s}} = 5.0 \text{ m/s}$$

$$\omega = \frac{\Delta\theta}{\Delta t}$$

, angular speed.

is the rotation rate.



Angular Velocity in Uniform Circular Motion

- When angular velocity ω is constant, this is u.c.m.
- For one revolution, the angular displacement is $\Delta\theta = 2\pi$ and the time is one period $\Delta t = \Delta T$.

$\Delta\theta = 2\pi$ rad

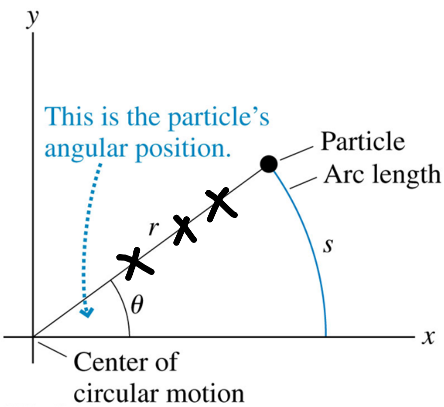
ω

units of radians/sec.

The absolute value of the constant angular velocity is related to the period of the motion by

what is the rate of rotation.

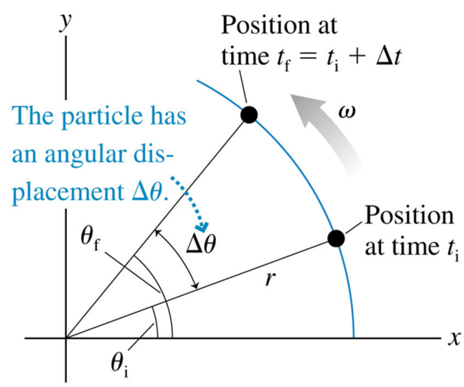
$$\boxed{|\omega| = \frac{2\pi \text{ rad}}{T} \quad \text{or} \quad T = \frac{2\pi \text{ rad}}{|\omega|}}$$



$$s = r\theta \qquad \text{(with } \theta \text{ in rad)}$$

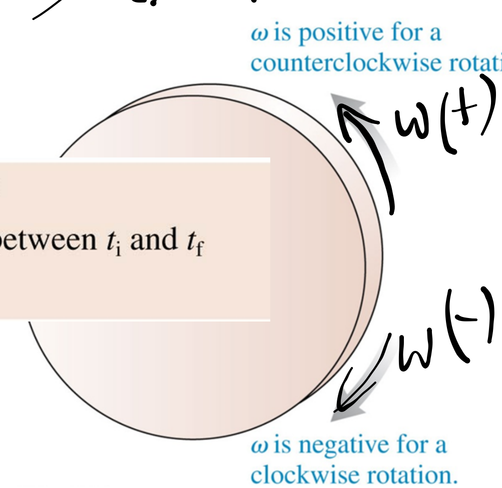
$$\text{average angular velocity} \equiv \frac{\Delta\theta}{\Delta t}$$

$$\omega = \frac{\Delta\theta}{\Delta t}$$



$$\theta_f = \theta_i + \omega\Delta t \Longleftrightarrow x_f = x_i + v\Delta t$$

for uniform motion.



ω = slope of the θ-versus-t graph at time t

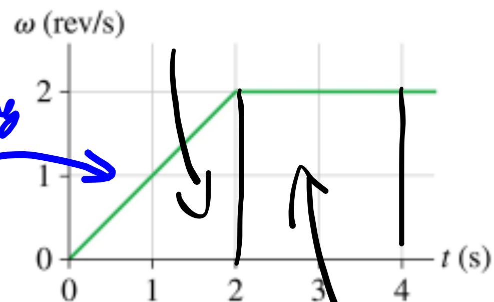
$\theta_f = \theta_i$ + area under the ω-versus-t curve between t_i and t_f

$= \theta_i + \omega\Delta t$

This is the angular velocity graph of a wheel. How many revolutions does the wheel make in the first 4 s?

$A_1 = \frac{1}{2}(2 \times 2) = 2$ rev

the wheel is speeding up.

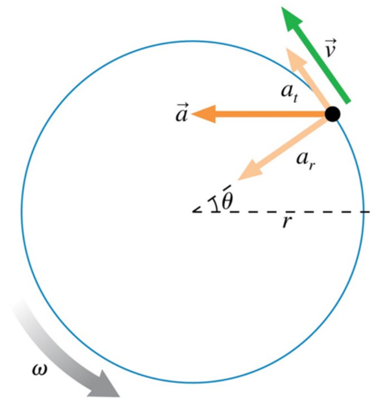


$A_2 = 2 \times 2 = 4$ rev

A. 1
B. 2
C. 4
D. 6
E. 8

Notice the units.

Connection between angular and linear velocity



Tangential Velocity

$s = \theta r$

magnitude

$$v_t = \omega r \qquad \text{(with } \omega \text{ in rad/s)}$$

Units of v_t are m/s; Units of ω are rad/s

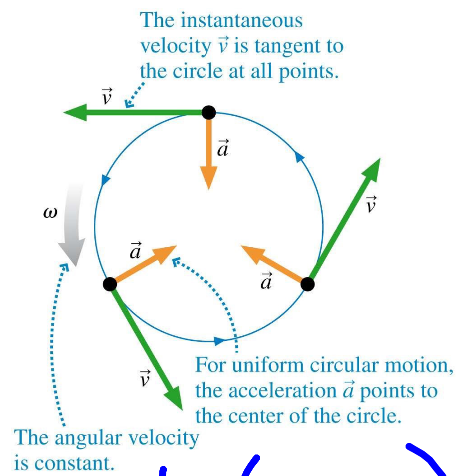
Centripetal Acceleration

$$\vec{a} = \left(\frac{v^2}{r}, \text{ toward center of circle}\right) \qquad \text{(centripetal acceleration)}$$

Can be written in terms of angular velocity as:

magnitude

$$a_r = \omega^2 r$$



For non-uniform circular motion we have a changing speed, v_t so there is a:

Tangential Acceleration

$$a_t = \frac{dv_t}{dt}$$

only if v is changing! ($v = \omega r$)

$a = \frac{dv}{dt}$

Angular Acceleration

$a = r\frac{d\omega}{dt}$

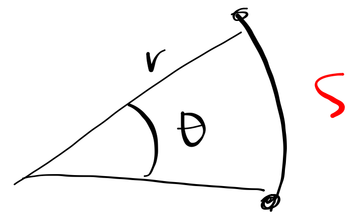
$$\alpha \equiv \frac{d\omega}{dt} \qquad \text{(angular acceleration)}$$

← similar to? linear acceleration

$a_t = \alpha r$

$$s = \theta r$$

$$v_t = \frac{ds}{dt} = \frac{d\theta}{dt} r$$



$$v_t = \omega r$$

$$a_t = \frac{dv_t}{dt} = \frac{d\omega}{dt} r$$

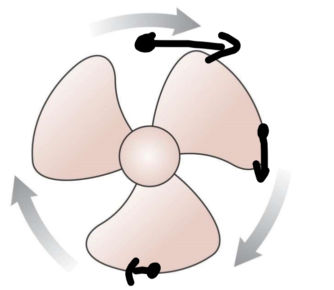
$$a_t = \alpha r$$

θ = angular position

ω = angular velocity

α = angular acceleration

The fan blade is slowing down.
What are the signs of ω and α?



A. ω is positive and α is positive.

B. ω is positive and α is negative.

C. ω is negative and α is positive.

D. ω is negative and α is negative.

E. ω is positive and α is zero.

$v_t = \omega r$

$a_t = \alpha r$

centripetal (radial)

Acceleration in Nonuniform Circular Motion

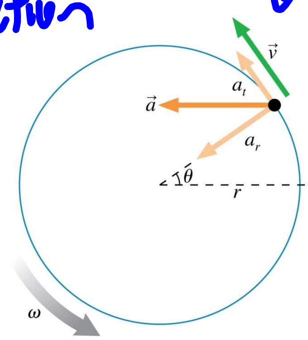
$a_t = 0$ for U.C.M.

$\frac{\Delta v}{\Delta t}$

$a_r = \frac{v^2}{r}$ ← change in direction

$a_t = \frac{dv}{dt}$

$a = \sqrt{a_r^2 + a_t^2}$ ← change in speed.



Kinematics Summarized

1D or 2D components.

Rotational kinematics	Linear kinematics
$\omega_f = \omega_i + \alpha\,\Delta t$	$v_{fs} = v_{is} + a_s\,\Delta t$
$\theta_f = \theta_i + \omega_i\,\Delta t + \frac{1}{2}\alpha(\Delta t)^2$	$s_f = s_i + v_{is}\,\Delta t + \frac{1}{2}a_s(\Delta t)^2$
$\omega_f^2 = \omega_i^2 + 2\alpha\,\Delta\theta$	$v_{fs}^2 = v_{is}^2 + 2a_s\,\Delta s$

$v_t = \omega r$

$a_t = \alpha r$

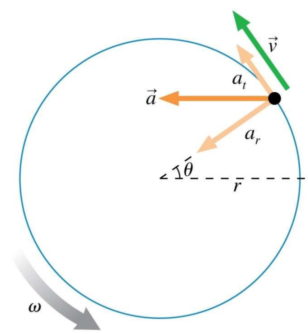
$s = \theta r$

arc length

for uniform linear acceleration.

Nonuniform Circular Motion-Summary

A particle moves along a circle and may be changing speed. The distance traveled along the circle is related to the angle:



$$s = r\theta \qquad \text{(with } \theta \text{ in rad)}$$

The tangential velocity is related to the angular velocity:

$$v_t = \omega r \qquad \text{(with } \omega \text{ in rad/s)}$$

The tangential acceleration is related to the angular acceleration:

$$a_t = \frac{dv_t}{dt} = \frac{d(\omega r)}{dt} = \frac{d\omega}{dt} r = \alpha r$$ = 0 when ω is constant

rad/s^2