Wavelet and filter bank

1 Introduction

1.1 The Fourier Transform

1.1.1 The continuous case

The Fourier transform was introduced for the first time in the 19th century by Joseph Fourier, as part of his work on the conduction of heat [2]. It has become a fundamental tool in mathematics, physics and signal processing.

Let s be a function defined on the space $L^2(\mathbb{R})$ of finite energy functions. The Fourier integral of s, which we note $\widehat{s}$ is defined by:

$$\hat{s}(\omega) = \int_{-\infty}^{+\infty} s(x)e^{-i\omega x}dx \;<\; +\infty \tag{1}$$

where $\omega \in \mathbb{R}$ is a pulsation. If $\hat{s}$ is also integrable, the inverse Fourier transform is then defined by:

$$s(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{s}(\omega)e^{ix\omega}d\omega \tag{2}$$

The Fourier transform can be generalized on R^n. In the 2-D case, the Fourier transform of $s \in L^2(\mathbb{R}^2)$, a two-variable function x_1 and x_2, is:

$$\hat{s}(\omega_1, \omega_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(x_1, x_2)e^{-i(\omega_1 x_1 + \omega_2 x_2)}dx_1 dx_2 \tag{3}$$

1.1.2 The discrete case

The Discrete Fourier Transform (DFT) of s, a function of the space of discrete signals of finite length N, is defined by:

$$\hat{s}[k] = \sum_{n=0}^{N-1} s[n]e^{-i\frac{2\pi kn}{N}} \quad \text{avec} \quad n, k \in Z \tag{4}$$

and its discrete inversion formula is:

$$s[n] = \frac{1}{N}\sum_{k=0}^{N-1} \hat{s}[k]e^{-i\frac{2\pi nk}{N}} \tag{5}$$

The discrete Fourier transform is also generalized for larger dimensions. It is commonly used in dimension 2 for applications in image processing. The DFT of a discrete image s of size $(N_1 \times N_2)$ is written:

$$\hat{s}[k_1, k_2] = \frac{1}{N_1 N_2}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} s[n_1, n_2]e^{-i\frac{2\pi}{N_1 N_2}(k_1 n_1 + k_2 n_2)} \tag{6}$$

The limit of Fourier's analysis is the concealment of temporal information. This is why new representations have appeared in recent years, transformations seeking to represent both temporal and frequential information. This is the case of the wavelets.

1.2 Wavelet transform

1.2.1 The continuous case

Grossmann and Morlet have formalized the wavelet transforms in 1984 [3]. Many scientists have contributed to research on wavelet transforms in the fields of mathematics as well as in the processing of signals and images. We present in this section the notions on the wavelet transforms. For more details, there are some excellent books on this topic such as [7] or [6].

A wavelet is a zero average function ψ of $L^2(\mathbb{R})$:

$$\int_{-\infty}^{+\infty} \psi(t)dt = 0 \tag{7}$$

It is normalized to $\|\psi\| = 1$. A family of time-frequency elements is obtained by dilating by a factor a and translating the wavelet by a factor u:

$$\psi_{a,u}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-u}{a}\right) \tag{8}$$

Factor $\frac{1}{\sqrt{a}}$ normalizes $\psi_{a,u}$ to preserve the energu of the analysing pattern:

$$\|\psi_{a,u}\|_2^2 = \int_{-\infty}^{+\infty} |\psi_{a,u}(t)|^2 \, dt = 1 \qquad \forall a, u \in R \tag{9}$$

Wavelet coefficients $W_s(a, u)$ of $s \in L^2(\mathbb{R})$ at time u ans scale a are obtained by projecting signal s onto the family $\{\psi_{a,u}\}$ of functions obtained by dilatation of a and by translation of u and the mother wavelet ψ.

$$W_s(a, u) = \langle s, \psi_{a,u}\rangle = \int_{-\infty}^{+\infty} s(t)\frac{1}{\sqrt{a}}\psi\left(\frac{t-u}{a}\right) dt \tag{10}$$

To reconstruct the s signal from its wavelet coefficients W_s, wavelet ψ must check the following eligibility condition [1, 3]:

$$C_\psi = \int_{0}^{+\infty} \frac{\left|\hat{\psi}(\nu)\right|^2}{|\nu|} d\nu \; < \; +\infty \tag{11}$$

with $\hat{\psi}$ the Fourier transform of ψ.

The inverse wavelet transform is defined according to normalization coefficient C_ψ :

$$s(t) = \frac{1}{C_\psi} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} W_s(a, u)\frac{1}{\sqrt{a}}\psi\left(\frac{t-u}{a}\right) du\frac{da}{a^2} \tag{12}$$

1.2.2 The discrete case

We now want to compute the wavelet representation on a computer with perfect orthogonality or digital reconstruction properties. We discretize the continuous representation by sampling both the translation parameter u and the scale parameter a [7]. The discrete wavelet transform is defined by:

$$W_s(a, u) = m_0^{-\frac{a}{2}} \int_{-\infty}^{+\infty} s(t)\bar{\psi}\left(m_0^{-a}t - un_0\right) dt \tag{13}$$

with the logarithmically uniform sampling grid:

$$\left\{(m, n) = (m_0^a, un_0m_0^a) \left| m_0 > 1, n_0 > 0, (a, u) \in \mathbb{Z}^2 \right.\right\} \tag{14}$$

Generally, we use a so-called dyadic network which corresponds to the pair of parameters $a = 2^L$ and $u = 1$. This choice of the pair led to propose discrete wavelet decomposition algorithms thanks to interpolation and decimation operators by a factor of 2. Thus these operators constitute the basis of the theory of multiresolution analysis which is origin of the construction of discrete wavelet bases.

$\{2^{-\frac{a}{2}}\psi\left(2^{-a}t-u\right)|(a,u)\in\mathbb{Z}^2\}$ then forms a multiresolution basis of $L^2(\mathbb{R})$. The inverse discrete wavelet transform is written as:

$$x(t)=\sum_{s\in\mathbb{Z}}\sum_{u\in\mathbb{Z}}W_x(s,u)\psi_{s,u}(t) \tag{15}$$

Multiresolution analysis made it possible to construct orthonormal wavelet bases [4, 5, 8]. The principle consists of decomposing the signal to be analyzed into a series of approximation coefficients. The projection of a signal $s\in L^2(\mathbb{R})$ on subspace V_l to step l is an approximation of s at scale 2^{-1}. Space V_l is generated by a scaling function $\phi(t)$ from an integer translation and a dyadic dilation according to $\phi_{l,k}(t)=2^{-\frac{l}{2}}\phi\left(2^{-l}-k\right)$. Approximation coefficients $a_{l,k}$ at the resolution level j are defined by the scalar product of s with $phi_{l,k}$:

$$s_{l,k}=\langle s,\phi_{l,k}\rangle \tag{16}$$

where $l\in\mathbb{Z}$ is the dilatation factor and $k\in\mathbb{Z}$ the translation factor. Scale function ϕ verifies the bi-scale property:

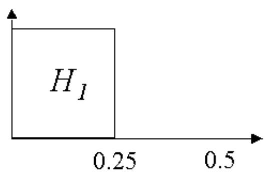
$$\phi(t)=\sqrt{2}\sum_{n}h[n]\phi(2t-n) \tag{17}$$

where $\{h[n],n\in\mathbb{Z}\}$ is impulse response of a low-pass filter.

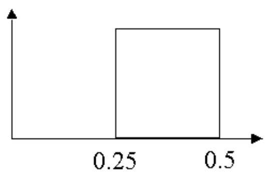
2 Principle of the wavelet decomposition

The Fourier transform has some limitations. The more important is the fact that we loose the notion of chronology in the frequency domain. To obtain a performant strategy of signal analysis, we need a joint temporal/frequential representation. The wavelet transform is element of this representation family. The aim of the wavelet decomposition is the study of the behavior of the different frequency bands. In order to realize this transform, the algorithm proposes to use some filtering operations.

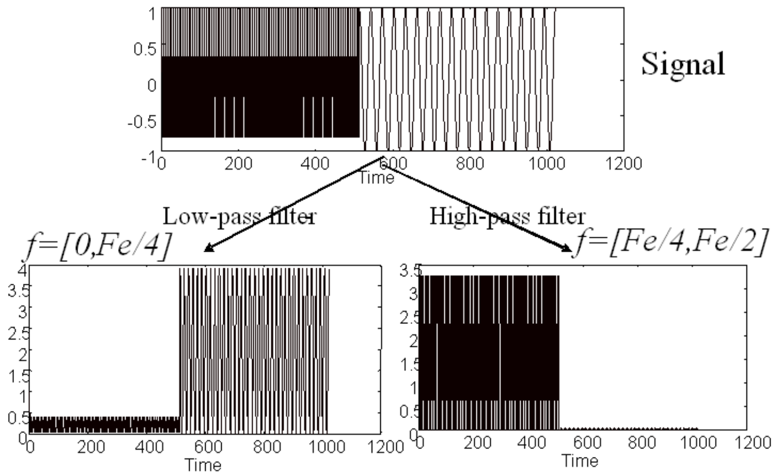
In order to illustrate the principle, we propose the following example: If we apply a lowpass filter on the signal with filter gain such that



The filtered signal indicates the time evolution of the frequency information $\in\left[0,\frac{F_e}{4}\right]$. With the same strategy, if we apply a high-pass filter such that

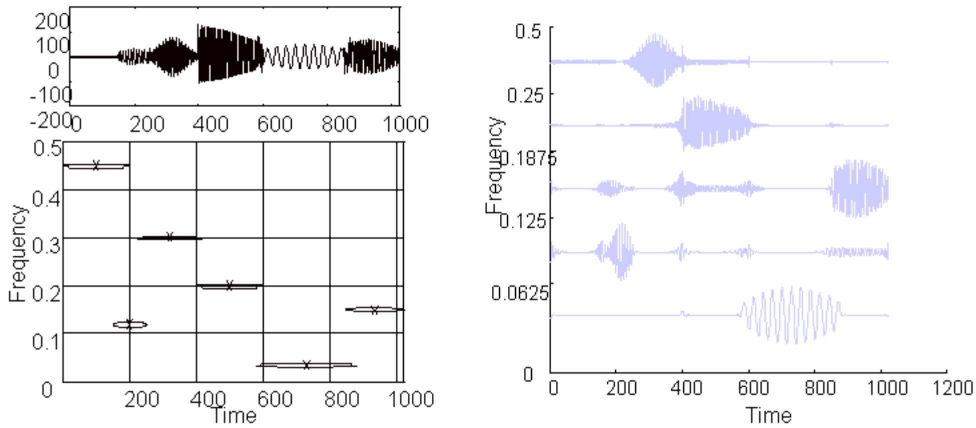


The filtered signal indicates the time evolution of the frequency information $\in\left[\frac{F_e}{4},\frac{F_e}{2}\right]$. We illustrate this process in the following example.



With this two filtering operation, we observe that we have defined a simple temporal/frequency representation.

More generaly, if we can define a set of filters that cut the frequency axis in several segments, we can also study, from all the filtered signals, the time behavior of the different frequency bands. We illustrate this process in the following figure. We have built a test signal with six components. Each component corresponds to a temporal interval and to a frequency. We notice that by applying five filters that cut the frequency axis $[0, 0.5]$, the behavior of the different components of the signal are individualy characterized.



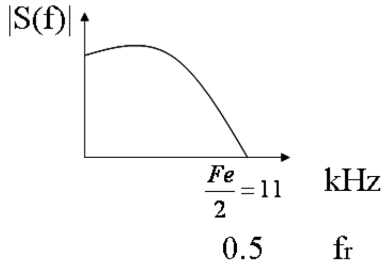
3 Definition of filters bank : decomposition

As illustrated in the precedent example, we want to define a representation that permits to extract the different frequency components of a signal. This representation must be not redundant: if the signal is represented with N samples, the wavelet representation must contains N coefficients. Moreover, this representation must be inversible: a perfect reconstruction of the original signal from the wavelet coefficients. And, finaly we want to define a simple algorithm: all the set of the filters must be defined from a single lowpass filter.

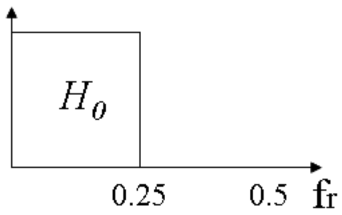
Now, we propose to study the strategy to extract the low frequency components with a compact representation

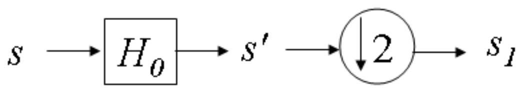
3.1 Lowpass frequency component

We start with a signal that contains N samples that represents the sampling of a continuous signal with a frequency sampling $F_e = 22KHz$.

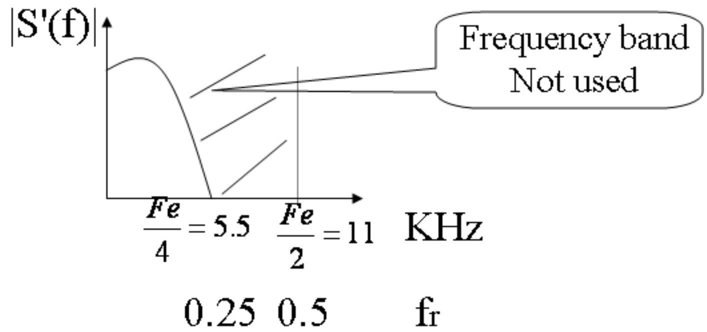


We apply a lowpass filtering in order to select the low frequency components:

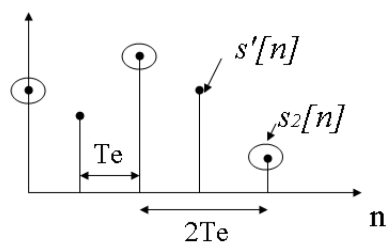




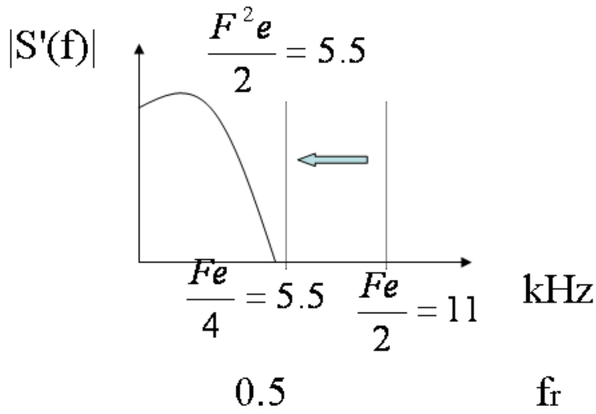
After the filtering operation, we obtain :



We notice that we have a frequency interval that is not used by the signal: the representation contains too many coefficients. We propose then to apply a dowsampling operation: we keep only half of the outputs of the filter (for example the odd-numbered samples are lost).



After the dowsampling, the signal $s_2[n]$ represents the sampling of the continuous signal with a new sampling period $T_e^2 = 2T_e$. In the frequency domain, this operation moves the axis $\frac{F_e}{2}$ ($F_e^2 = \frac{F_e}{2}$).



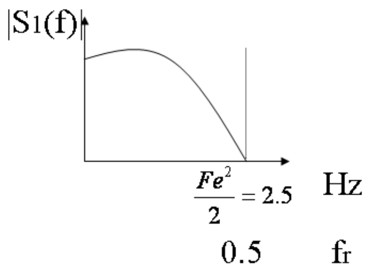
We can see that the filtered information fills the numerical frequency interval $[0, 0.5]$ and the information is not disturbed by the dowsampling operation: the low frequency information is kept with a compact mode.

If we want to select a low frequency component with a smaller frequency interval, we can apply again a lowpass filter that splits the frequency interval into halves.

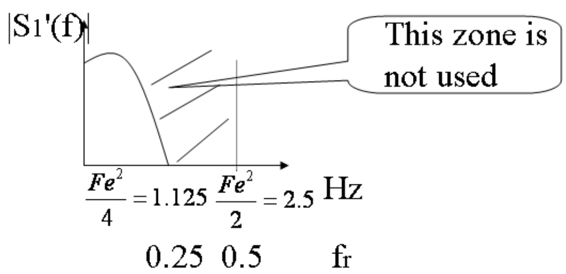
For this we propose to apply again the low-pass filter H_0 to s_1 :

$$s_1 \longrightarrow \boxed{H_0} \longrightarrow s'_1$$

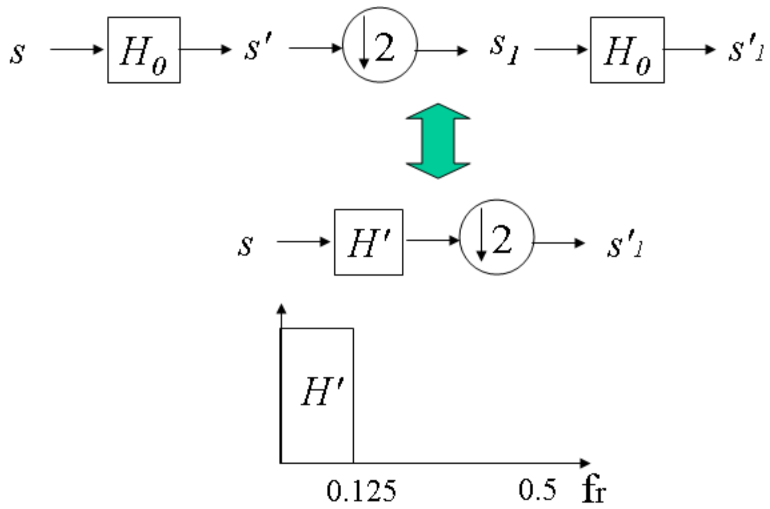
Since the spectrum of s_1 is such that (it fills the numerical frequency interval $[0, 0.5]$):



After the filtering of s_1 by H_0, we obtain in the frequency domain:

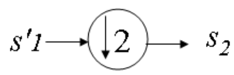


We conclude that the totality of the process is similar to a filtering by a low-pass filter with a gain $H' = 1_{0,0.125}$:



Notice that if we apply the filter H_0 directly on s' and not on s_1 (we remove the dowsampling), then the signal remains unchanged. We deduce that lowpass filter and dowsampling operation permits one to define different filters from a single lowpass filter.

Moreover, we can see that we have a frequency interval that is not used by the signal. We propose again to apply a dowsampling operation: we keep only half of the outputs of the filter.



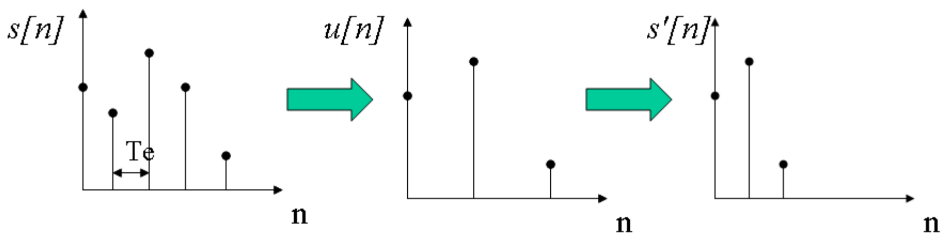
To conclude, we can say that s_2 contains the signal information associated to the frequency band $\left[0, \frac{F_e}{8} = 2.25\right] KHz$ with more compact representation (with $N/4$ samples).

3.2 Definition of the dowsampling

We want to modelise the influence in the frequency domain of the dowsampling in the time domain. Dowsampling is represented by (pronounced down two):

$$s \rightarrow \downarrow 2 \rightarrow s'$$

In order to analyse the dowsampling in the frequency domain, we split the process $\downarrow 2$ into two operations:



Let u be the vector s with its odd-numbered components set to zero:

$$u[n] = \begin{cases} s[n] \text{ if } n \text{ even} \\ 0 \text{ otherwise} \end{cases}$$

$u[n]$ can be defined such that :

$$u[n] = \frac{1}{2}\left(s[n] + (-1)^n s[n]\right) = \frac{1}{2}\left(s[n] + (e^{-j\pi})^n s[n]\right)$$

The second terms includes $(-1)^n$ or $e^{-jn\pi}$ so that addition knocks out odd n. In the frequency domain, we have:

$$\begin{aligned} U(f) &= \sum_n u[n] e^{-2j\pi fn} = \sum_n \frac{1}{2}\left(s[n] + (e^{-j\pi})^n s[n]\right) e^{-2j\pi fn} \\ &= \frac{1}{2}\left[\sum_n s[n] e^{-2j\pi fn} + \sum_n s[n] e^{-2j\pi fn - j\pi n}\right] \end{aligned}$$

We can write $e^{-2j\pi fn - j\pi n} = e^{-2j\pi n(f+\frac{1}{2})}$, then

$$U(f) = \frac{1}{2}\left[S(f) + S(f + \frac{1}{2})\right]$$

The second step is such that :

$$s'[n] = u[2n]$$

because the result only involves even n.

In the frequency domain, we obtain

$$\begin{aligned} S'(f) &= \sum_n u[2n] e^{-2j\pi fn} \text{ with the following variable change } n' = 2n \\ &= \sum_{n'} u[n'] e^{-2j\pi n'(\frac{f}{2})} = U(\frac{f}{2}) \end{aligned}$$

It means that the downsampling of the signal can be defined in the frequency domain:

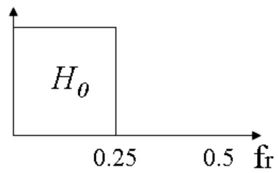
$$s' = [\downarrow 2]\, s \Longrightarrow S'(f) = \frac{1}{2}\left[S(\frac{f}{2}) + S(\frac{f}{2} + \frac{1}{2})\right]$$

3.3 Design of a High-pass filter from a low-pass filter.

Let's

$$H(z) = \sum_k b_k z^{-k} \text{ low-pass filter with } f_c = 0.25$$

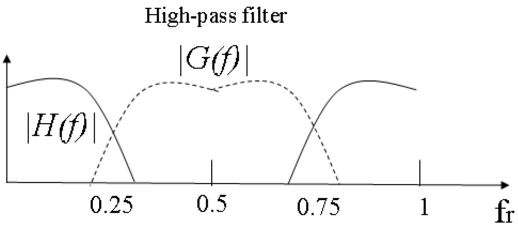
If we are plotting $|H(f)| = \left|\sum_k b_k e^{-2j\pi fk}\right|$, we obtain



We propose to apply the transform $z \to -z$

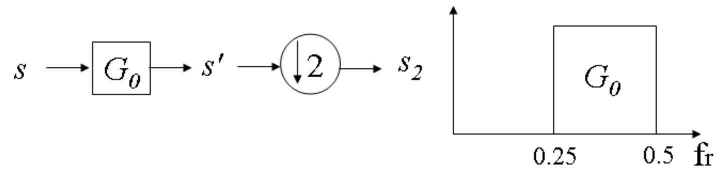
$$\begin{aligned} G(z) &= H(-z) = \sum_k b_k z^{-k} (-1)^k = \sum_k b_k z^{-k} (e^{-\pi j})^k \\ G(f) &= \sum_k b_k e^{-2j\pi fk} (e^{-\pi j})^k = \sum_k b_k \left(e^{-2j\pi(f+\frac{1}{2})}\right)^k \\ &= H(f + \frac{1}{2}) \end{aligned}$$

We deduce that $G(z) = H(-z) \Longrightarrow G(f) = H(f + \frac{1}{2})$. We obtain thus the following representation for $G(f)$

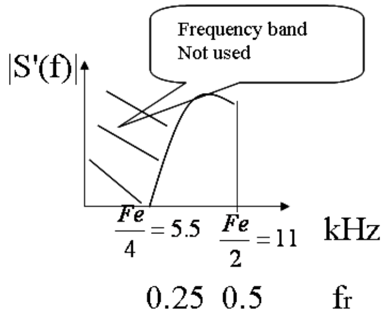


3.4 Study of the High-pass analysis

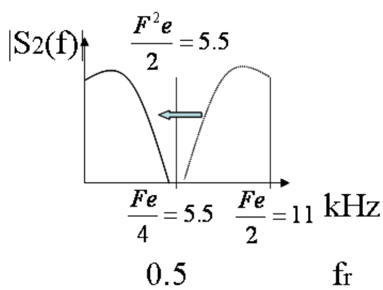
In order to extract the information removed with the low-pass treatment, we propose to apply on the analysed signal an highpass filter:



The frequency representation of the filtered signal s' is such that



We notice that we have a frequency interval that is not used by the signal. As previously, we propose then to apply a dowsampling operation. In the frequency domain, this operation moves the axis $\frac{F_e}{2}$ ($F_e^2 = \frac{F_e}{2}$). Since the spectrum of the filtered signal is out of the interval $[0, 0.5]$, we have an aliasing phenomenon: the frequency information of the filteried signal is moved to the interval $[0, 0.5]$ but is not degraded.



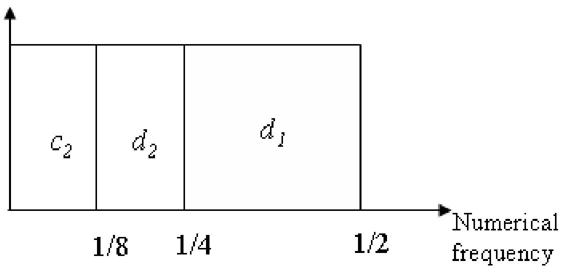
To conclude, we can say that s_2 contains the signal information associated to the frequency band $\left[\frac{F_e}{4}, \frac{F_e}{2}\right]$ with a more compact representation (with $N/2$ samples).

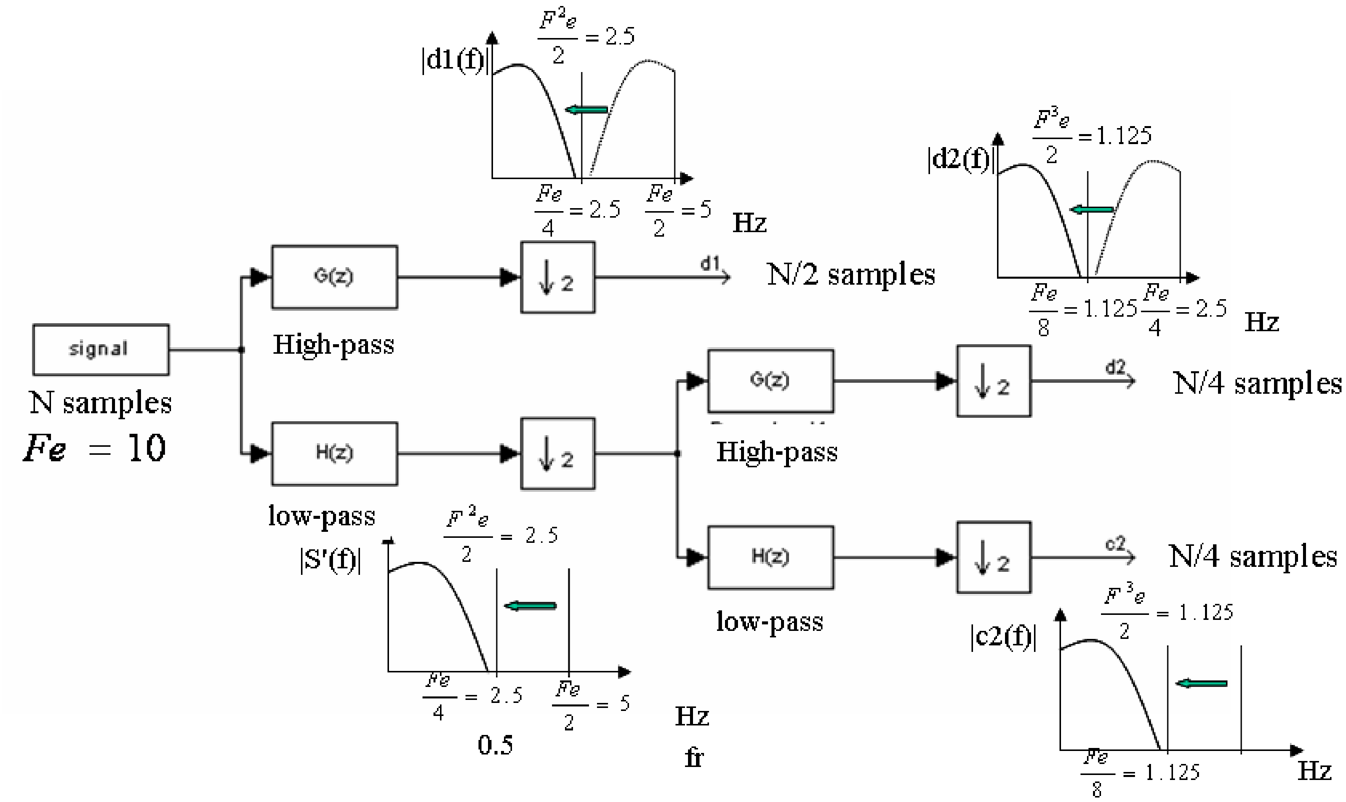
3.5 Wavelet transform and filter bank

A wavelet decomposition is associated to a splitting of the frequency axis such that:

- The low frequency band is splitted into small segments in order to separate all the components of the signal;
- The high frequency band is splitted into large segments. The high frequency band contains less information.

We illustrate an example of the frequency splitting associated with a wavelet transform :





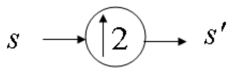
4 The synthesis bank

A well-organized synthesis bank is the inverse of the analysis bank. The analysis bank had two steps: filtering and downsampling. The synthesis bank has also two steps: upsampling and filtering.

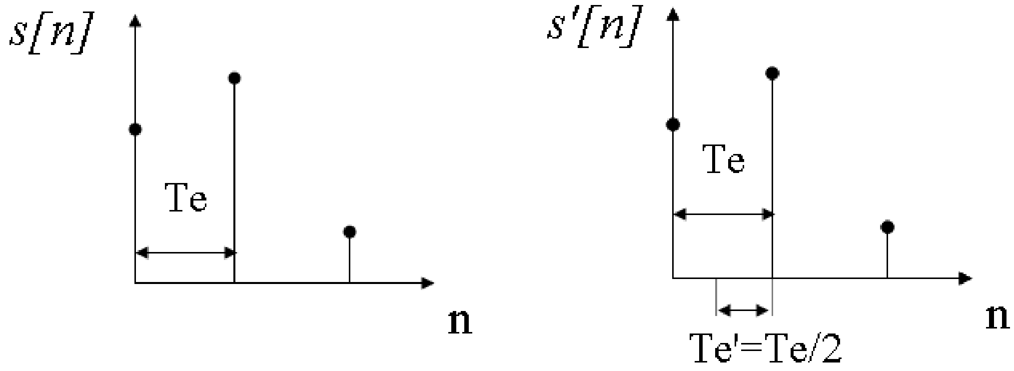
4.1 Principle of the upsampling

The first step is to bring back full-length vectors. For this, we apply the upsampling operation.

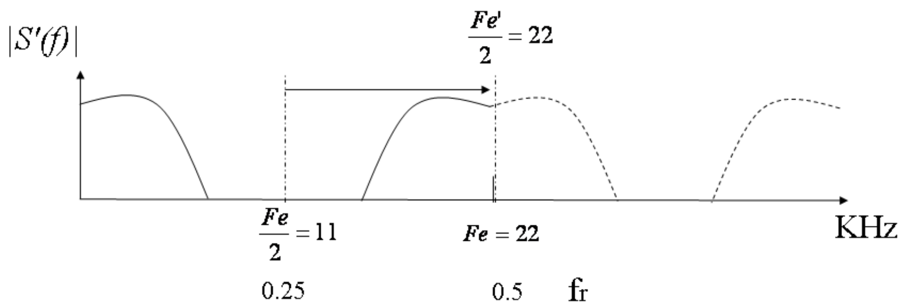
We start with a signal that contains N samples that represent the sampling of a continuous signal. The dowsampling operation is not invertible, but upsampling is as close as we can come.



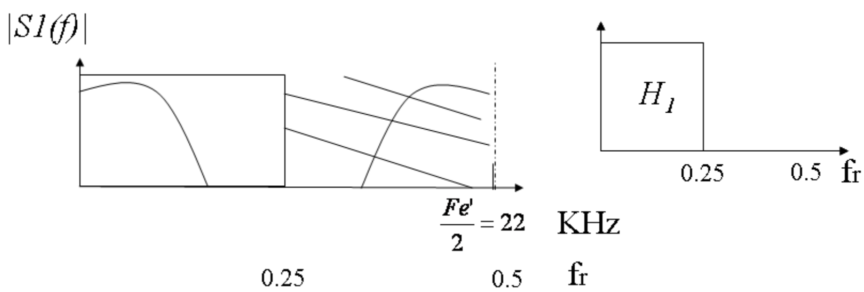
The odd-numbered components are returned as zeros by upsampling. Applied to a half-length signal, upsampling inserts zeros :



After the upsampling, the signal $s'[n]$ represents the sampling of the continuous signal with a new sampling period $T_e^2 = T_e/2$. In the frequency domain, this operation moves the axis $\frac{F_e}{2}$ (($F_e' = 2F_e$). We observe that the "negative" part of the spectrum of s is now element of the band $[0, 0.5]$.



The second step in the synthesis bank, after upsampling, is filtering in order to remove the "negative" part of the spectrum of the original signal. If we apply a low-pass filter such that :



We can see that we obtain the same signal in the frequency domain but the length of the signal is multiplied by two: the signal is interpolated without distortion of the information.

We can apply the same operations in order to interpolate the high information of the wavelet decomposition. But, in this case the used filter must be a high-pass filter.

4.2 Definition of the upsampling

We want to analyse the upsampling in the frequency domain. The signal reached by upsampling have zeros in their odd components:

$$s' = [\uparrow 2]\, s \Longrightarrow \begin{cases} s'[2k] = s[k] \\ s'[2k+1] = 0 \end{cases}$$

The Fourier transform of s' is

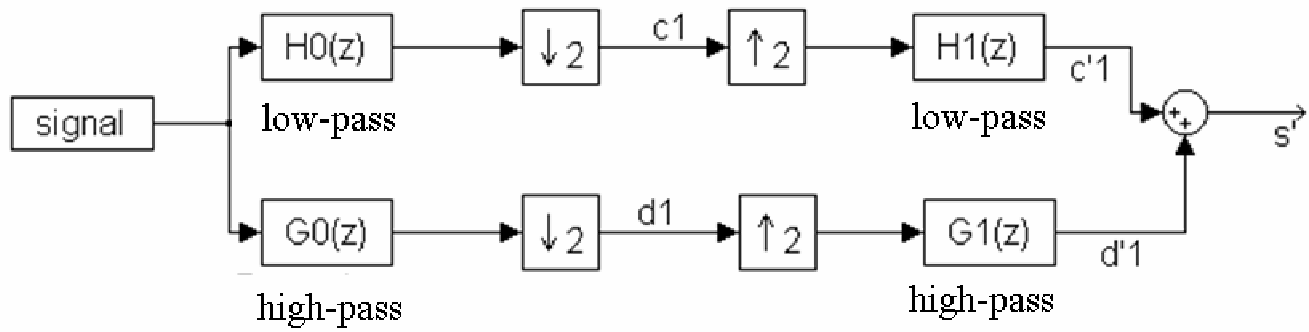
$$S'(f) = \sum_n s'[n] e^{-2j\pi f n}$$

If n is even then $s'[n] = s[n/2]$ otherwise $s'[n] = 0$. The fourier transform can thus be written :

$$\begin{aligned} S'(f) &= \sum_{n \text{ even}} s[n/2] e^{-2j\pi f n} \text{ we set } k = n/2 \\ &= \sum_k s[k] e^{-2j\pi k(2f)} = S(2f) \end{aligned}$$

5 Filter banks with perfect reconstruction

A filter bank is a set of filters, linked by sampling operators. The dowsampling operators are decimators, the upsampling operators are expanders. In a two-channel filter bank, the analysis/synthesis filters are normally lowpass and highpass.



The goal of this section is to discover the conditions for perfect reconstruction : $s = s'$

Study of the lowpass channel

$$\begin{aligned} c_1 &= [\downarrow 2]\,(h_0 * s) \\ C_1(f) &= \frac{1}{2}\left[H_0(\frac{f}{2})S(\frac{f}{2}) + H_0(\frac{f}{2}+\frac{1}{2})S(\frac{f}{2}+\frac{1}{2})\right] \end{aligned}$$

Now upsample and filtering:

$$C_1'(f) = H_1(f)C_1(2f) = \frac{1}{2}H_1(f)\left[H_0(f)S(f) + H_0(f+\frac{1}{2})S(f+\frac{1}{2})\right]$$

Study of the highpass channel The highpass output is the same formula with filters H changed to filters G:

$$D_1'(f) = G_1(f)D_1(2f) = \frac{1}{2}G_1(f)\left[G_0(f)S(f) + G_0(f+\frac{1}{2})S(f+\frac{1}{2})\right]$$

The reconstructed signal

The filter bank combines the two channel to get s'

$$S'(f) = \frac{1}{2}\left[G_0(f)G_1(f) + H_0(f)H_1(f)\right].S(f) + \frac{1}{2}\left[G_0(f+\frac{1}{2})G_1(f) + H_0(f+\frac{1}{2})H_1(f)\right]S(f+\frac{1}{2})$$

We deduce that a 2-channel filter bank gives perfect reconstruction when:

- No distortion :

$$G_0(f)G_1(f) + H_0(f)H_1(f) = 2.\underbrace{e^{-2j\pi nf}}_{delay}$$

- Alias cancellation

$$G_0(f+\frac{1}{2})G_1(f) + H_0(f+\frac{1}{2})H_1(f) = 0$$

6 An example: the Haar basis

6.1 Lowpass filter

In order to define a lowpass operation, we propose to make an average between two neighbour samples:

$$s[n] = e[n] + e[n-1] \rightarrow S(f) = (1 + e^{-2j\pi f})E(f) \rightarrow H_0(f) = 1 + e^{-2j\pi f} \rightarrow H_0(z) = 1 + z^{-1}$$

We propose to normalize H_0 :

$$H_0(f) = \frac{1}{\sqrt{2}}(1 + e^{-2j\pi f})$$

If $|H_0(z=f)|^2$ is plotted for $f = 0...0.5$, we observe that this filter is a lowpass filter with $f_c = 0.25$.

6.2 Highpass filter

We have studied a method that permits one to define a highpass filter from a lowpass filter

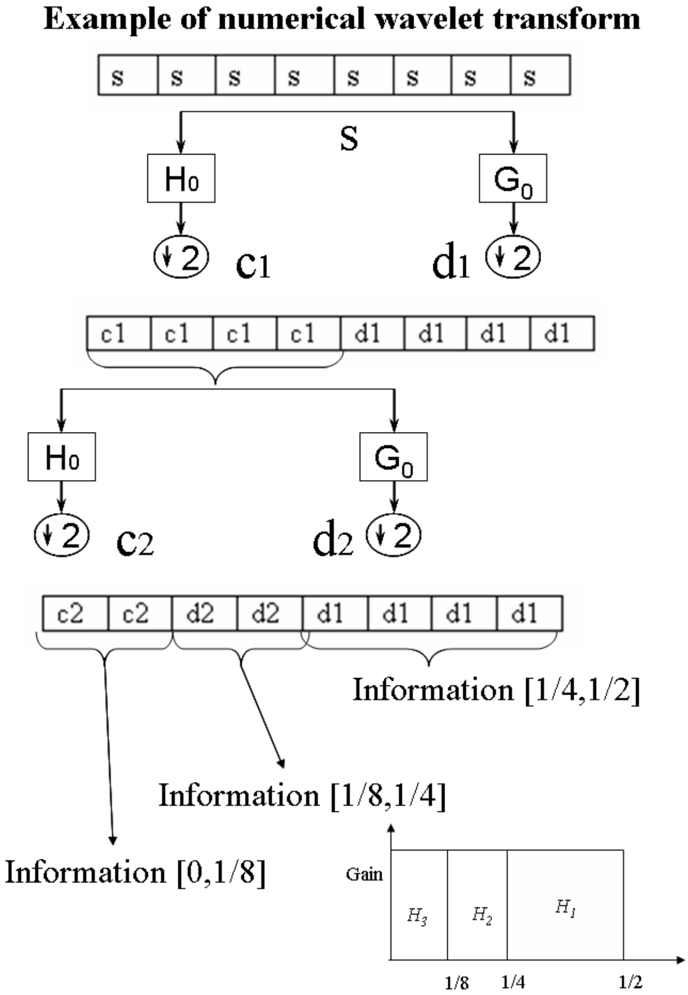
$$G_0(z) = H_0(-z)$$

If we applied this method,

$$G_0(z) = \frac{1}{\sqrt{2}}(1 - z^{-1}) \rightarrow G_0(f) = \frac{1}{\sqrt{2}}(1 - e^{-2j\pi f})$$

We have defined the two filters required for the wavelet decomposition. We can now compute praticaly the wavelet decomposition by applying:

- the two filtering operations
- the dowsampling operations



6.3 Synthesis filters H_1 and G_1

In order to obtain a perfect reconstruction, we must define synthesis filters such that

$$G_0(f+\frac{1}{2})G_1(f)+H_0(f+\frac{1}{2})H_1(f)=0 \text{ Alias cancellation}$$

This condition can be written in the Z-domain

$$H_0(-z)H_1(z)+G_0(-z)G_1(z)=0$$

Moreover H_1 must be a lowpass filter and G_1 must be a highpass filter.

We propose the following construction:

$$\begin{aligned} H_1(z) &= G_0(-z),\ H_1 \text{ is a lowpass filter } f_c=0.25 \\ G_1(z) &= -H_0(-z),\ G_1 \text{ is a highpass filter } f_c=0.25 \end{aligned}$$

If we applied this method, we obtain

$$H_1(z)=\frac{1}{\sqrt{2}}(1+z^{-1}) \text{ et } G_1(z)=\frac{1}{\sqrt{2}}(-1+z^{-1})$$

6.4 Verification

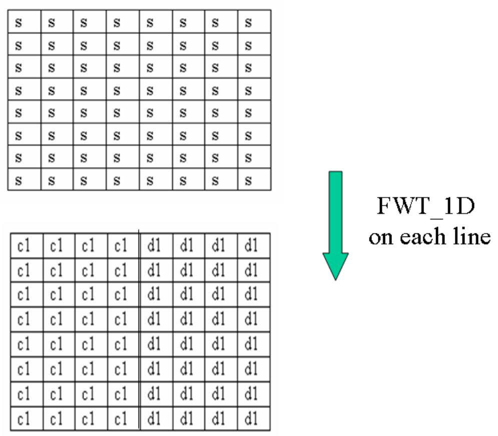
$$\begin{aligned} H_0(-z)H_1(z)+G_0(-z)G_1(z) &= \frac{1}{\sqrt{2}}(1-z^{-1})\frac{1}{\sqrt{2}}(1+z^{-1})+\frac{1}{\sqrt{2}}(1+z^{-1})\frac{1}{\sqrt{2}}(-1+z^{-1}) \\ &= \frac{1-z^{-2}}{2}+\frac{-1+z^{-2}}{2}=0 \end{aligned}$$

$$
\begin{aligned}
H_0(z)H_1(z) + G_0(z)G_1(z) &= \frac{1}{\sqrt{2}}(1+z^{-1})\frac{1}{\sqrt{2}}(1+z^{-1}) + \frac{1}{\sqrt{2}}(1-z^{-1})\frac{1}{\sqrt{2}}(-1+z^{-1}) \\
&= 2z^{-1}
\end{aligned}
$$

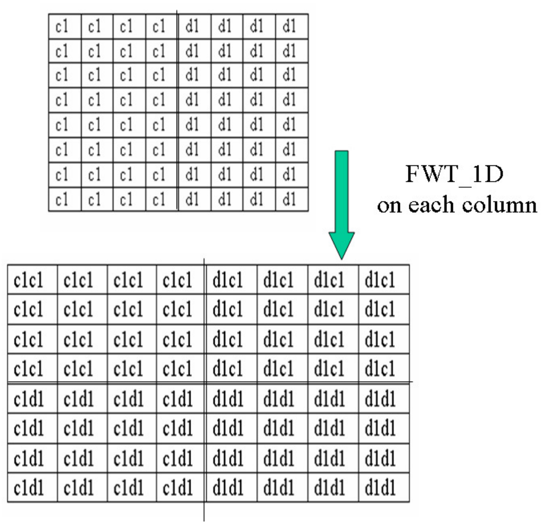
We deduce that $S'(z) = z^{-1}S(z)$ then $s'[n] = s[n-1]$.

7 2D wavelet transform

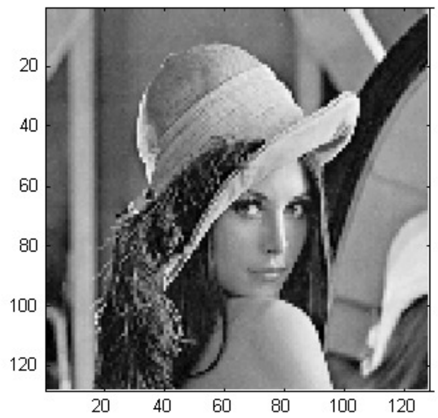
Images are two-dimensional. Processing those images is an extremely important application of subband filtering. For this, we must have two-dimensional filters. Their construction can be easy if we use a separable strategy: products of one-dimensional filters. For this, we apply the 1D wavelet transform on each line of the image

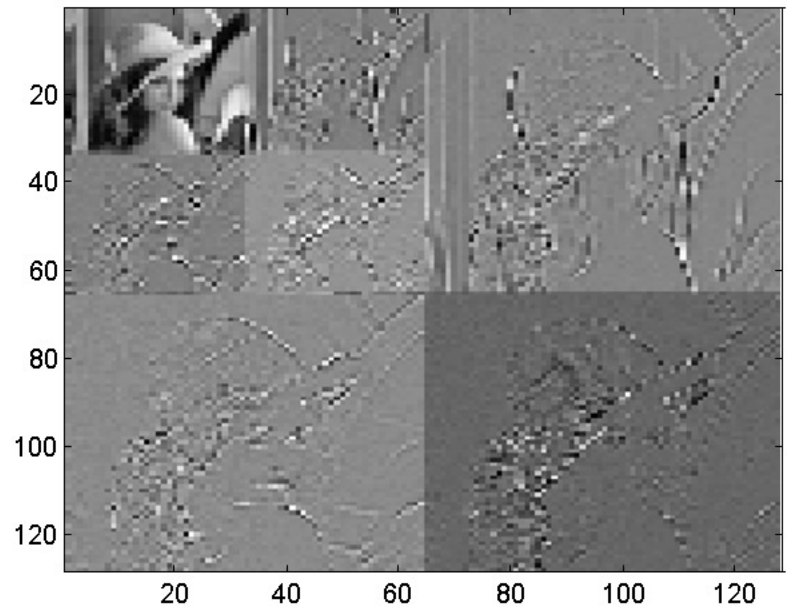


Now we apply the 1D wavelet transform on each column of the precedent matrix



We propose an example of the 2D wavelet transform of the image ”Lenna”





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