PS4_junhoc

February 5, 2019

0.1 MACS30150 PS4

0.1.1 Dr. Richard Evans

Submitted by Junho Choi

0.1.2 Problem 1

Problem 1-(a) In the below few code chunks, I import the data from *incomes.txt* and first try to provide a few rows of the dataset. Also, it is available to us that the minimum and maximum of the dataset are approximately 49278.80 and 135865.03 dollars, respectively.

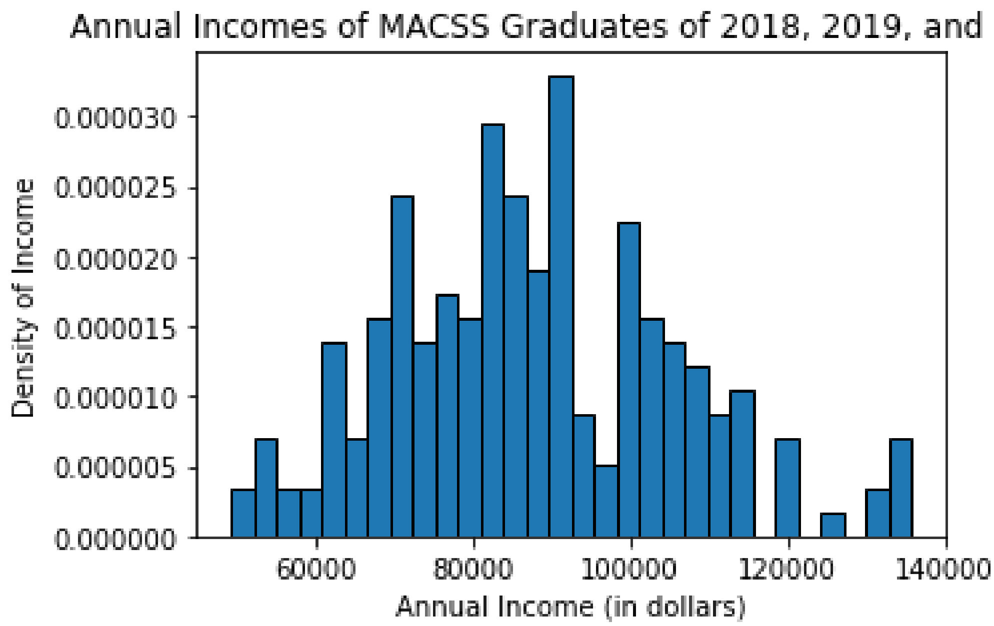
```
In [1]: import numpy as np
        from matplotlib import pyplot as plt
        import scipy.stats as sts
        import pandas as pd
```

```
In [2]: ### Loading in the data
        y_dat = np.loadtxt('incomes.txt')
        print(y_dat[0:5]) ## checking a few elements
        print("Minimum is", '%.2f' % y_dat.min())
        print("Maximum is", '%.2f' % y_dat.max())
```

```
[  51253.49715631  100630.32024137   83009.27613739   82882.10654304
   77338.29483892]
Minimum is 49278.80
Maximum is 135865.03
```

In the below code chunk, I plot of the histogram of annual incomes of MACSS graduates of 2018, 2019, and 2020 as requested.

```
In [3]: num_bins = 30
        count, bins, ignored = plt.hist(y_dat,
                                        num_bins, normed=True,
                                        edgecolor='k')
        plt.title('Annual Incomes of MACSS Graduates of 2018, 2019, and 2020', fontsize=12)
        plt.xlabel(r'Annual Income (in dollars)')
        plt.ylabel(r'Density of Income')
        plt.subplots_adjust(bottom=.25, left=.25)
        plt.show()
```



Problem 1-(b) Firstly, I prepare function *lognormalpdf* for returning the PDF density value given some point x on the distribution, *mu* and *sig* where we assume that $x \sim LN(\mu = mu, \sigma = sig)$.

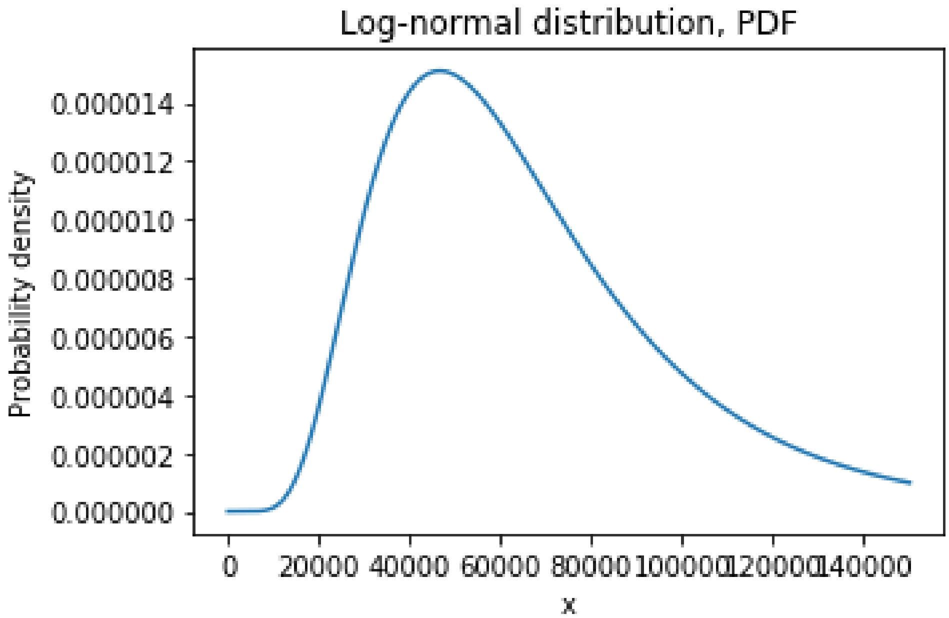
```
In [4]: def lognormalpdf(x, mu, sig):

            err_msg = "standard dev. must be positive"
            assert sig > 0, err_msg

            err_msg2 = "input must be positive"
            if type(x) == np.array or type(x) == np.ndarray:
                assert x.all() >= 0, err_msg2
            elif type(x) == int or type(x) == float:
                assert x >= 0, err_msg2

            fricpart = 1 / (x * sig * ((2 * np.pi)**0.5))
            expopart = (-1) * ((np.log(x)-mu)**2) / (2*(sig**2))
            var = fricpart * (np.e ** expopart)

            return var
```

In the below code chunk, I first create a vector of equally-spaced x values for PDF graph construction -- this vector is called *onebyone* below. Notice that I plot from 0.1 to 150000.1 as 0.0 is actually not included in the support of log-normal distributions. I then plug this vector into the *lognormalpdf* function that I have created above with $mu = 11.0$ and $sig = 0.5$. This yields the vector of densities for log-normal distributions for the aforementioned x values.

```
In [5]: onebyone = np.linspace(0.1, 150000.1, 1500000)
        log_byone = lognormalpdf(onebyone, 11.0, 0.5)
```

Below is the resulting PDF distribution.

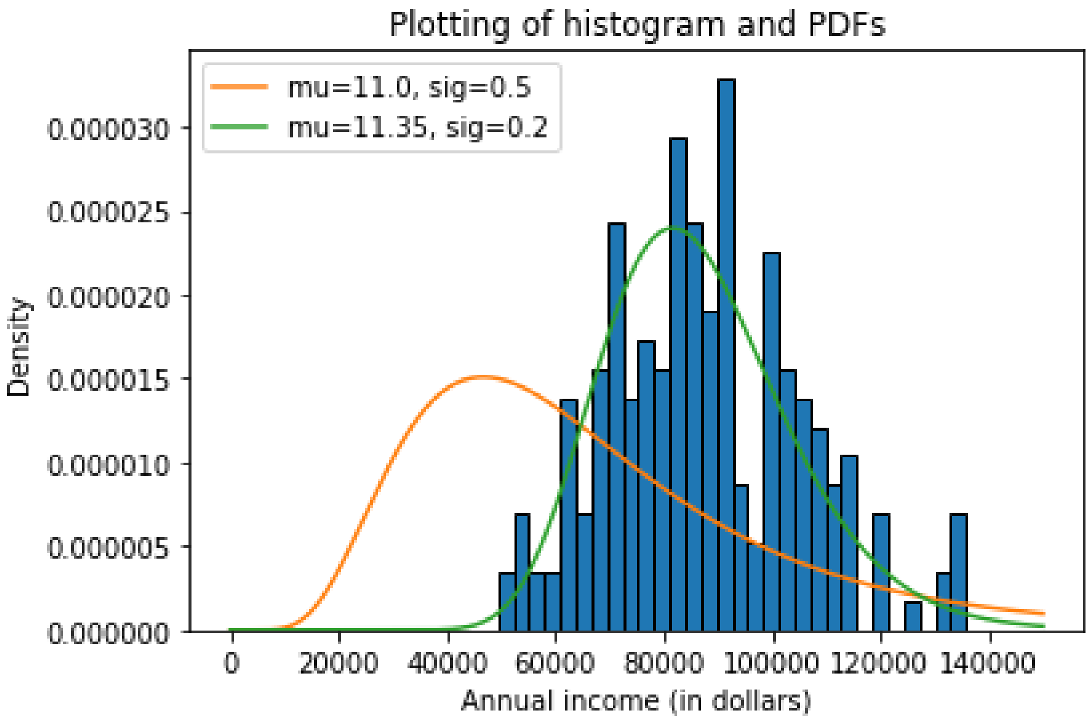
```
In [6]: plt.plot(onebyone,log_byone)
        plt.title("Log-normal distribution, PDF")
        plt.ylabel(r'Probability density')
        plt.xlabel(r'x')
        plt.subplots_adjust(bottom=.25, left=.25)
        plt.show()
```



Notice that the above distribution, when plotted together with the histogram, seems to be a bad fit for representing the data. Therefore, I try out another log-normal distribution with $mu = 11.35$ and $sig = 0.2$ -- this certainly seems to be a better representation than before.

```
In [13]: ## histogram
         num_bins = 30
         count, bins, ignored = plt.hist(y_dat,
                                         num_bins, normed=True,
                                         edgecolor='k')

         log_byone = lognormalpdf(onebyone, 11.0, 0.5)
         plt.plot(onebyone, log_byone, label='mu=11.0, sig=0.5')

         log_byone2 = lognormalpdf(onebyone, 11.35, 0.2)
         plt.plot(onebyone, log_byone2, label='mu=11.35, sig=0.2')
         plt.title("Plotting of histogram and PDFs")
```

```
plt.ylabel(r'Density')
plt.xlabel(r'Annual income (in dollars)')
plt.legend()
plt.show()
```



In the below code chunk, I define the function *loglike_calc* that takes in function *fn*, x values (in numpy vector format)*xvals*, mean variable *mu*, and standard deviation variable *sig*.

```
In [14]: def loglike_calc(fn, xvals, mu, sig):

             msg = "please let the input be in numpy array format"
             assert type(xvals) == np.ndarray or type(xvals) == np.array, msg

             likeli_vec = fn(xvals, mu, sig)
             loglike_vec = np.log(likeli_vec)
             loglike = np.sum(loglike_vec)

             return loglike
```

Given $\mu = 11.0$ and $\sigma = 0.5$, the log likelihood is found to be approximately -2385.857. On the other hand, that with $\mu = 11.35$ and $\sigma = 0.5$ is found to be approximately -2242.253, which certainly seems to perform better than the given set of parameters.

```
In [15]: initial_one = loglike_calc(lognormalpdf, y_dat, 11.0, 0.5)
         better_one = loglike_calc(lognormalpdf, y_dat, 11.35, 0.2)
```

```
print("With mu=11.00 and sigma=0.5, the log likelihood is", '%.3f' % initial_one)
print("With mu=11.35 and sigma=0.2, the log likelihood is", '%.3f' % better_one)
```

```
With mu=11.00 and sigma=0.5, the log likelihood is -2385.857
With mu=11.35 and sigma=0.2, the log likelihood is -2242.253
```

Problem 1-(c) In the below code chunk, I have created the function *crit_lognormal* for returning the negative of log likelihood given some parameters as well as x values. This will serve as the criterion function for determining the values of μ and σ that maximize log likelihood (i.e. that are maximum-likelihood estimates).

```
In [19]: def crit_lognormal(params, xvals):

             mu, sigma = params
             loglikeli = loglike_calc(lognormalpdf, xvals, mu, sigma)
             negloglikeli = -loglikeli

             return negloglikeli
```

Yet since I have not put in the x values that I will be using for this exercise, I have defined a lambda function *crit_lognormal_with_x* in the code chunk below that puts in the provided income data. The maximum-likelihood estimates for μ and σ are found to be $\mu_{MLE} \approx 11.359$ and $\sigma_{MLE} \approx 0.208$.

```
In [20]: import scipy.optimize as opt

         crit_lognormal_with_x = lambda params: crit_lognormal(params, y_dat)

         mu_init = 11.35  # mu_2
         sig_init = 0.2  # sig_2
         params_init = np.array([mu_init, sig_init])
         mle_args = y_dat
         results_uncstr = opt.minimize(crit_lognormal_with_x, params_init)
         mu_MLE, sig_MLE = results_uncstr.x
         print('mu_MLE=', mu_MLE, ' sig_MLE=', sig_MLE)
```

```
mu_MLE= 11.3590230068  sig_MLE= 0.20817732092
```

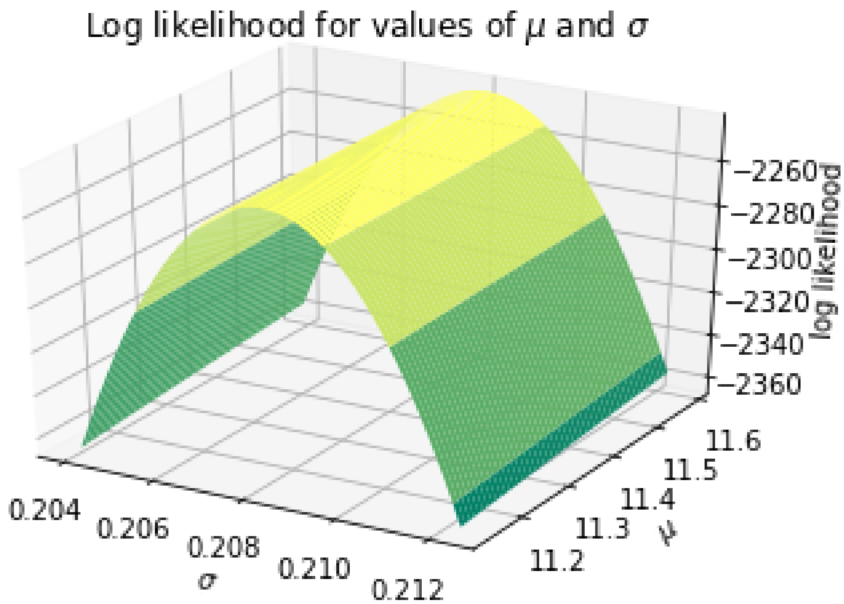
We can certainly check below that log likelihood for MLE approach certainly is higher than the two previous ones that we had examined.

```
In [21]: initial_one = loglike_calc(lognormalpdf, y_dat, 11.0, 0.5)
         better_one = loglike_calc(lognormalpdf, y_dat, 11.35, 0.2)
         MLE_one = loglike_calc(lognormalpdf, y_dat, mu_MLE, sig_MLE)

         print("With mu=11.00 and sigma=0.5, the log likelihood is", '%.3f' % initial_one)
         print("With mu=11.35 and sigma=0.2, the log likelihood is", '%.3f' % better_one)
         print("MLE log likelihood is", '%.3f' % MLE_one)
```

```
With mu=11.00 and sigma=0.5, the log likelihood is -2385.857
With mu=11.35 and sigma=0.2, the log likelihood is -2242.253
MLE log likelihood is -2241.719
```

Just in case, let us show the well-behavedness of the maximum-likelihood estimation for this problem as well. The most ideal shape of the 3-dimensional graph would be that of a paraboloid, but it seems that -- at least in the neighborhood of the maximum likelihood estimates that we had found -- the optimization depends a lot on finding the right value of σ and not so much with that for μ.

```
In [22]: import matplotlib
         from mpl_toolkits.mplot3d import Axes3D
         from matplotlib import cm
         cmap1 = matplotlib.cm.get_cmap('summer')

         # mu_vals = np.linspace(300, 635, 50)
         # sig_vals = np.linspace(30, 210, 50)
         mu_vals = np.linspace(0.98*mu_MLE, 1.02*mu_MLE, 50)
         sig_vals = np.linspace(0.98*sig_MLE, 1.02*sig_MLE, 50)
         lnlik_vals = np.zeros((50, 50))
         for mu_ind in range(50):
             for sig_ind in range(50):
                 lnlik_vals[mu_ind, sig_ind] = \
                     loglike_calc(lognormalpdf, y_dat,
                                  mu_vals[mu_ind], sig_vals[sig_ind])

         mu_mesh, sig_mesh = np.meshgrid(mu_vals, sig_vals)

         fig = plt.figure()
         ax = fig.gca(projection='3d')
         ax.plot_surface(sig_mesh, mu_mesh, lnlik_vals, rstride=8,
                         cstride=1, cmap=cmap1)
         ax.set_title(r'Log likelihood for values of $\mu$ and $\sigma$')
         ax.set_xlabel(r'$\sigma$')
         ax.set_ylabel(r'$\mu$')
         ax.set_zlabel(r'log likelihood')

Out[22]: Text(0.5,0,'log likelihood')
```



The variance-covariance matrix is also presented below.

```
In [23]: vcv_mle = results_uncstr.hess_inv
         stderr_mu_mle = np.sqrt(vcv_mle[0,0])
         stderr_sig_mle = np.sqrt(vcv_mle[1,1])
         print('Variance-Covariance Matrix of MLE:')
         print(vcv_mle)
         print()
         print('Standard error for mu estimate = ', stderr_mu_mle)
         print('Standard error for sigma estimate = ', stderr_sig_mle)
```

```
Variance-Covariance Matrix of MLE:
[[  7.31778409e-04   3.43486900e-05]
 [  3.43486900e-05   1.70453877e-05]]

Standard error for mu estimate =  0.0270514030802
Standard error for sigma estimate =  0.00412860602646
```

Problem 1-(d) The p-value for this likelihood ratio test yields 0, which means that we can reject the null hypothesis that the income data came from a log-normal distribution with $\mu = 11.0$ and $\sigma = 0.5$.

```
In [24]: mu_in_b, sig_in_b = 11.0, 0.5
         log_lik_h0 = loglike_calc(lognormalpdf, y_dat, mu_in_b, sig_in_b)
         print('hypothesis value log likelihood:', log_lik_h0)
         print()
         log_lik_mle = loglike_calc(lognormalpdf, y_dat, mu_MLE, sig_MLE)
```

```
        print('MLE log likelihood:', log_lik_mle)
        print()
        LR_val = 2 * (log_lik_mle - log_lik_h0)
        print('likelihood ratio value:', LR_val)
        print()
        pval_h0 = 1.0 - sts.chi2.cdf(LR_val, 2)
        print('chi squared of H0 with 2 degrees of freedom p-value:', pval_h0)
```

```
hypothesis value log likelihood: -2385.85699781

MLE log likelihood: -2241.71930136

likelihood ratio value: 288.275392902

chi squared of H0 with 2 degrees of freedom p-value: 0.0
```

Problem 1-(e) In the below code chunks, I have presented two ways -- numerical integration via the trapezoid method and directly calculating the CDF of log-normal distribution -- of calculating the percentages for earning more than 100,000 dollars and less than 75,000 dollars. The percentages are found to be approximately 22.99% for the former, and 26.02% for the latter.

```
In [25]: from scipy.special import erf

         def lognormalcdf(x, mu, sig):
             err_msg = "standard dev. must be positive"
             assert sig > 0, err_msg

             err_msg = "input must be positive"
             if type(x) == np.array or type(x) == np.ndarray:
                 assert x.all() > 0, err_msg2
             elif type(x) == int or type(x) == float:
                 assert x > 0, err_msg2

             inerf = (np.log(x) - mu) / (sig * (2 ** 0.5))
             rtnval = 0.5 + 0.5 * erf(inerf)

             return rtnval

In [26]: poor = lognormalcdf(75000, mu_MLE, sig_MLE)
         rich = 1 - lognormalcdf(100000, mu_MLE, sig_MLE)

In [27]: print("Probability of earning less than $75000 is", "%.2f" % (100*poor) + "%")
         print("Probability of earning more than $100000 is", "%.2f" % (100*rich) + "%")
```

```
Probability of earning less than $75000 is 26.02%
Probability of earning more than $100000 is 22.99%
```

```
In [28]: MLE_lognorm = lambda x: lognormalpdf(x, mu_MLE, sig_MLE)

In [29]: def trapintegr(func, a, b, N):
             bin_cuts = np.linspace(a, b, N + 1)
             binsize = (b-a)/N

             bin_fvals = func(bin_cuts)
             bin_vals = (bin_fvals[1:] + bin_fvals[0:N]) / 2 * binsize

             return bin_vals.sum()

In [30]: poor2 = trapintegr(MLE_lognorm, 0.1, 75000, 75000)
         rich2 = 1 - trapintegr(MLE_lognorm, 0.1, 100000, 75000)

In [31]: print("Probability of earning less than $75000 is", "%.2f" % (100*poor2) + "%")
         print("Probability of earning more than $100000 is", "%.2f" % (100*rich2) + "%")

Probability of earning less than $75000 is 26.02%
Probability of earning more than $100000 is 22.99%
```

0.1.3 Problem 2

Problem 2-(a) To start, let us load the data and check out how the data is structured.

```
In [32]: import pandas as pd
         from sklearn.linear_model import LinearRegression as lin

In [33]: ### Loading in the data
         sick_dat = pd.read_csv('sick.txt')
         sick_dat[0:5] ## checking a few rows
```

Out[33]:

	sick	age	children	avgtemp_winter
0	1.67	57.47	3.04	54.10
1	0.71	26.77	1.20	36.54
2	1.39	41.85	2.31	32.38
3	1.37	51.27	2.46	52.94
4	1.45	44.22	2.72	45.90

To provide ourselves with a starting point for finding the maximum-likelihood estimates, let us use OLS to find the estimates for parameters.

```
In [34]: ### trying the OLS first; first prepping the y and X
         y_vals = sick_dat['sick']
         x_mat = sick_dat[['age', 'children', 'avgtemp_winter']]

         y_vals = np.array(y_vals)
         x_mat = np.array(x_mat)
```

It seems that the OLS regression estimates are as follows: $\beta_0 \approx 0.252, \beta_1 \approx 0.013, \beta_2 \approx 0.400$, and $\beta_3 \approx -0.010$.

```
In [35]: reg = lin().fit(x_mat, y_vals)
         base_coef = reg.coef_
         base_intercept = reg.intercept_
         print("Regression coefficients:", reg.coef_)
         print("Regression intercept:", base_intercept)
```

```
Regression coefficients: [ 0.01293366  0.40049939 -0.00999174]
Regression intercept: 0.251641374723
```

Let us now use the below function *eps_creation* to find a vector (numpy array) of error terms (epsilons) and find the standard deviation for it. It seems that the error terms have a standard deviation of approximately 0.003.

```
In [36]: def eps_creation(y, x1, x2, x3, b0, b1, b2, b3):

             epsilons = y - b0 - b1*x1 - b2*x2 - b3*x3

             return epsilons
```

```
In [37]: epsilons = eps_creation(y_vals, x_mat[:, 0], x_mat[:, 1], x_mat[:, 2],
                                 base_intercept, base_coef[0], base_coef[1], base_coef[2])

         print("Standard deviation of the error terms is",'%.5f' % epsilons.std())
```

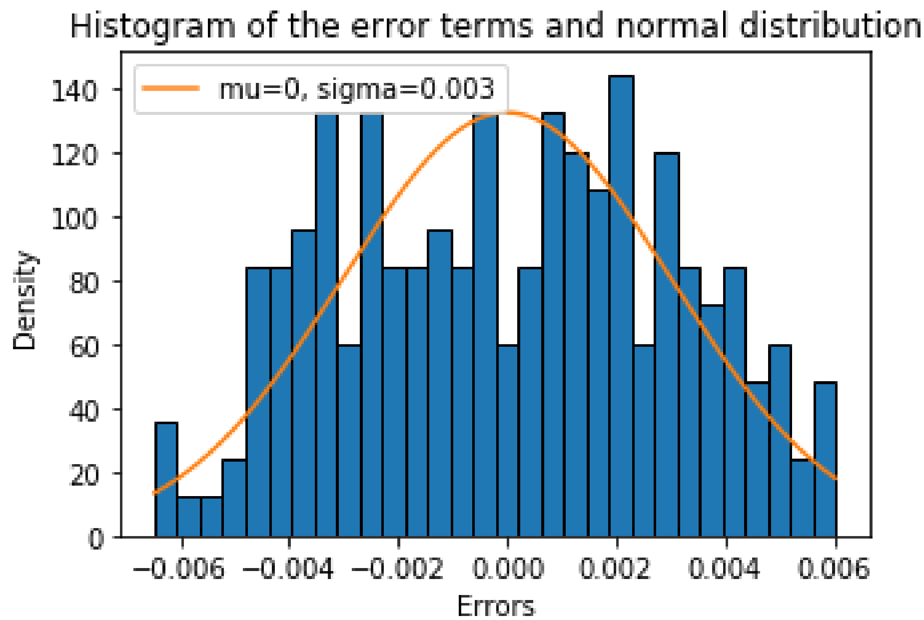
```
Standard deviation of the error terms is 0.00302
```

Now, let us use the below function *normalpdf* to draw the normal distribution which will be presented along with the histogram for the error terms.

```
In [38]: def normalpdf(x, mu, sig):

             sig = abs(sig)

             fracpart = 1 / ((2 * np.pi * (sig ** 2)) ** 0.5)
             expopart = (-1) * ((x - mu)**2)/(2*(sig**2))
             var = fracpart * (np.e ** expopart)

             return var
```

```
In [40]: ## histogram
         num_bins = 30
         count, bins, ignored = plt.hist(epsilons,
                                         num_bins, normed=True,
                                         edgecolor='k')

         ## normal distribution
         lb, ub= np.min(epsilons), np.max(epsilons)
```

```
xspace = np.linspace(lb, ub, 1001)
norm_byone = normalpdf(xspace, 0, epsilons.std())
plt.plot(xspace, norm_byone, label='mu=0, sigma=0.003')
plt.title("Histogram of the error terms and normal distribution")
plt.ylabel(r'Density')
plt.xlabel(r'Errors')
plt.subplots_adjust(bottom=.25, left=.25)
plt.legend()
plt.show()
```



To create the log-likelihood for the normal distribution (as the error terms are assumed to be normally distributed), I use the function *ll_normal*. I admit that this function would not be suitable for those trying to use more or less than 3 regressors; in the future, I will fix this. But for now, this would be sufficient.

```
In [41]: def ll_normal(y, x1, x2, x3, b0, b1, b2, b3, sigma):

             epsilons = eps_creation(y, x1, x2, x3, b0, b1, b2, b3)
             normal_densities = normalpdf(epsilons, 0, sigma)
             ln_densities = np.log(normal_densities)
             log_ll = np.sum(ln_densities)

             return log_ll
```

I also provide the criterion function that returns the negative of the log likelihood given the parameters and arguments.

```
In [42]: def crit_normal(params, *args):
```

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I also provide the criterion function that returns the negative of the log likelihood given the parameters and arguments.

```
In [42]: def crit_normal(params, *args):

             b0, b1, b2, b3, sigma = params
             y, x1, x2, x3 = args

             return -ll_normal(y, x1, x2, x3, b0, b1, b2, b3, sigma)
```

I use the initialization $(0.2, 0.01, 0.4, -0.01, 0.05)$ which are close to the OLS parameters. The below process, for some reason, has produced programming errors; however, the estimated (using MLE approach) coefficients are exactly the same as that of the OLS, which are $\beta_0 \approx 0.252, \beta_1 \approx 0.013, \beta_2 \approx 0.400$, and $\beta_3 \approx -0.010$. The value of the log likelihood is approximately 876.87.

```
In [43]: ## base_coef and base_intercept from OLS estimation
         arguments = (y_vals, x_mat[:, 0], x_mat[:, 1], x_mat[:, 2])
         params_init2 = (0.2, 0.01, 0.4, -0.01, 0.05)

         results_uncstr2 = opt.minimize(crit_normal, params_init2, arguments)
         print(results_uncstr2.x)
         print(-results_uncstr2.fun)
```

```
[ 0.25164623  0.01293336  0.40050207 -0.00999168  0.00301767]
876.8650464975143


C:\ProgramData\Anaconda3\lib\site-packages\ipykernel_launcher.py:5: RuntimeWarning: divide by z
  """
C:\ProgramData\Anaconda3\lib\site-packages\scipy\optimize\optimize.py:643: RuntimeWarning: inva
  grad[k] = (f(*((xk + d,) + args)) - f0) / d[k]
C:\ProgramData\Anaconda3\lib\site-packages\ipykernel_launcher.py:5: RuntimeWarning: divide by z
  """
C:\ProgramData\Anaconda3\lib\site-packages\ipykernel_launcher.py:5: RuntimeWarning: divide by z
  """
C:\ProgramData\Anaconda3\lib\site-packages\scipy\optimize\optimize.py:643: RuntimeWarning: inva
  grad[k] = (f(*((xk + d,) + args)) - f0) / d[k]
C:\ProgramData\Anaconda3\lib\site-packages\ipykernel_launcher.py:5: RuntimeWarning: divide by z
```

```
"""
```

Below code presents the variance-covariance matrix.

```
In [44]: vcv_mle2 = results_uncstr2.hess_inv
         print('Variance-Covariance Matrix of MLE:')
         print(vcv_mle2)
```

```
Variance-Covariance Matrix of MLE:
[[  6.83931654e-07   8.08438267e-09  -1.50571088e-07  -1.65440760e-08
    1.84103852e-09]
 [  8.08438267e-09   3.95820755e-09  -3.52328259e-08  -2.49904455e-09
   -1.36777360e-10]
 [ -1.50571088e-07  -3.52328259e-08   3.58034576e-07   2.23578038e-08
   -6.98637598e-11]
 [ -1.65440760e-08  -2.49904455e-09   2.23578038e-08   1.83866092e-09
    9.79430515e-11]
 [  1.84103852e-09  -1.36777360e-10  -6.98637598e-11   9.79430515e-11
    2.19890756e-08]]
```

Problem 2-(b) Below presents the results of the likelihood ratio test; because the p-value is 0.0, we can reject the null hypothesis that the regressors have no effect on number of sick days. Moreover, such a likelihood is, once again, 0%.

```
In [46]: hyp_b0, hyp_b1, hyp_b2, hyp_b3, hyp_sigma = 1, 0, 0, 0, 0.1
         mle_b0, mle_b1, mle_b2, mle_b3, mle_sigma = results_uncstr2.x

         log_lik_h0 = ll_normal(y_vals, x_mat[:, 0], x_mat[:, 1], x_mat[:, 2],
                                hyp_b0, hyp_b1, hyp_b2, hyp_b3, hyp_sigma)
         print('hypothesis value log likelihood:', log_lik_h0)
         print()
         log_lik_mle = ll_normal(y_vals, x_mat[:, 0], x_mat[:, 1], x_mat[:, 2],
                                 mle_b0, mle_b1, mle_b2, mle_b3, mle_sigma)
         print('MLE log likelihood:', log_lik_mle)
         print()
         LR_val = 2 * (log_lik_mle - log_lik_h0)
         print('likelihood ratio value:', LR_val)
         print()
         pval_h0 = 1.0 - sts.chi2.cdf(LR_val, 5)
         print('chi squared of H0 with 5 degrees of freedom p-value:', pval_h0)
```

```
hypothesis value log likelihood: -2253.70068804

MLE log likelihood: 876.865046498

likelihood ratio value: 6261.13146908
```

```
chi squared of H0 with 5 degrees of freedom p-value: 0.0
```