# Administrative Matters

- Course: Elementary Graph Theory
- Time/Location: M56-EC022
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- Prerequisites: 離散數學、計算機概論與程式設計
- Textbook: “Graph theory: An introduction to proofs, algorithms, and applications”, by Karin R. Saoub, CRC press, 1st Ed., 2021
- Reference: “Graph Theory and Its Applications” by Jonathan L. Gross and Jay Yellen, Second Edition. Publisher: Chapman & Hall/CRC

# Administrative Matters

- Course goal:介紹圖形理論中常見的問題、定理以及應用。課程著重在常見問題與其相關基礎定理的介紹，在各章節中也會介紹章節主題在各領域的應用範例。此外也會透過程式作業，讓同學練習圖形理論相關的程式設計，使同學具有圖形理論的基礎知識與實作能力。
- Course contents:
  - Graph Models, Terminology, and Proofs
    - Tournaments, Introduction to Graph Models and Terminology, Matrix Representation, Proof Techniques, Degree Sequence, Tournaments Revisited
  - Graph Routes
    - Eulerian Circuits, Hamiltonian Cycles, Shortest Paths
  - Trees
    - Spanning Trees, Tree Properties, Rooted Trees, Additional Applications
  - Connectivity and Flow
    - Connectivity Measures, Connectivity and Paths, Network Flow
  - Matching
    - Matching in Bipartite Graphs, Matching in General Graphs, Stable Matching
  - Graph Coloring
    - Four Color Theorem, Vertex Coloring, Edge Coloring
  - Planarity
    - Kuratowski's Theorem

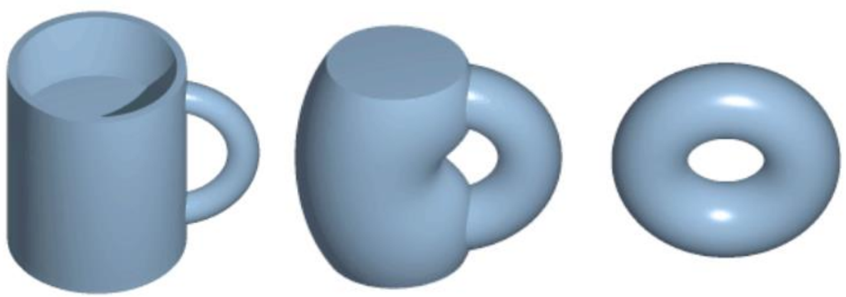
# Administrative Matters

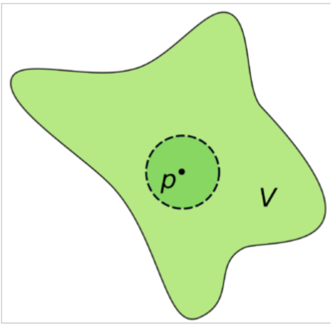
- Grading
  - Program assignments: 50%,
  - Two tests: 50%, (The better one : 30%, the worse one 20%)
  - Class participation: bonus

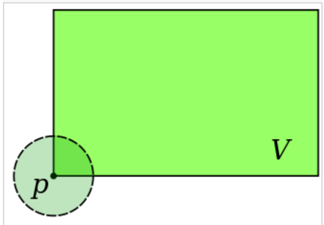
- Course Web Site: eCampus
- Academic Honesty**:** Avoiding *cheating* at all cost.

# Preliminary

- Geometria (Euclidean geometry, Solid geometry, non-Euclidean geometry , Riemannian geometry, Algebraic geometry, Differential geometry, Topology) & Algebra
- Manifold: Each point of an *n*-dimensional manifold has a neighborhood that is homeomorphic (topological isomorphical) to the Euclidean space of dimension *n*.
- 1-D manifold: line and circle. 2-D manifold: surface including plane, sphere and torus.
- Graph theory can be regarded as 1-D topology.





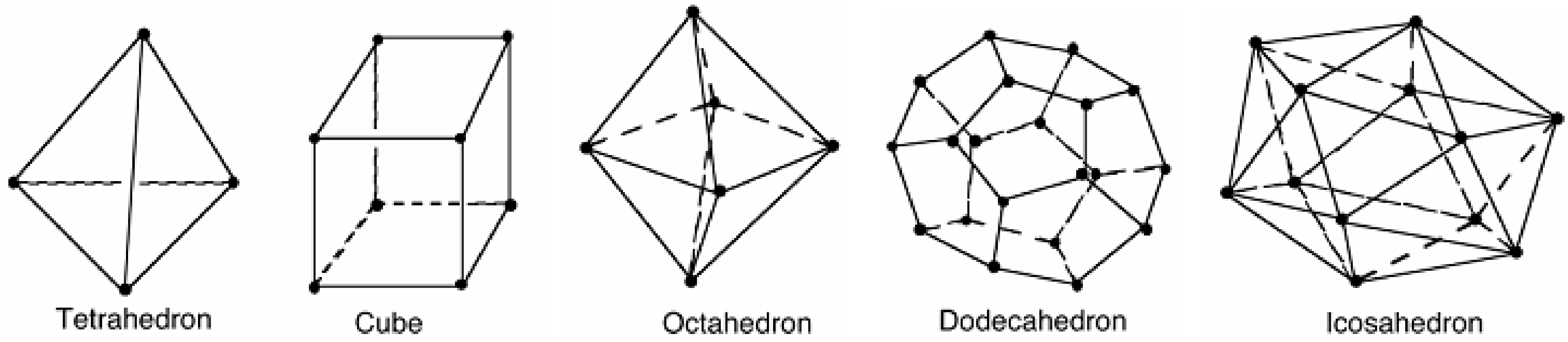

柏拉圖的形上學將世界切割為兩個不同的區塊：「形式的」智慧世界、以及我們所感覺到的世界。我們所感覺到的世界是從有智慧的形式或理想裡所複製的，但這些複製版本並不完美。那些真正的形式是完美的而且無法改變的，而且只有使用智力加以理解才能實現之，這也表示了人的智力並不包含知覺能力或想像力。

source: wiki

# Preliminary

# Regular Graphs

- **DEFINITION**: A regular graph is a graph whose vertices have common degree *k* and is denoted ***k-regular*** graph.




(fire) (earth) (air) (Ether) (water)

- ***Platonic graph***: Only five types of this kind of graphs (in "Timaeus dialogue" & "The elements" ) and each of them consists of regular polygons
  - ✓ ***Tetrahedron***: 4 regular triangles; ***cube***: 6 squares; ***Octahedron***: 8 regular triangles; ***Dodecahedron***: 12 regular pentagons; ***Icosahedron***: 20 regular triangles
  - ✓ **Property 1**: The number of polygons adjacent to each vertex is the same.
  - ✓ **Property 2**: There exists a sphere such that all vertices are on its surface.
  - ✓ **Property 3**: The sum of vertex number and face number equals to the edge number plus 2.

# Preliminary

- **The problem of Seven Bridges of Königsberg** solved by Leonhard Euler in 1735

Figure source: wiki

國立交通大學
National Chiao Tung University

EDA Lab.
Electrical Design Automation Lab.

# Chap 1. Graph Models, Terminology, and Proofs

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**The sources of most figure images are from the textbook**

# Outline

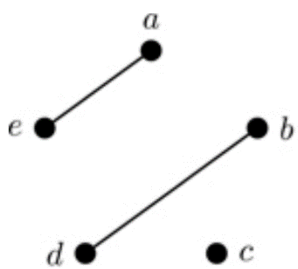
- Tournaments
- Introduction to Graph Models and Terminology
- Matrix Representation
- Proof Techniques
- Degree Sequence
- Tournaments Revisited

# 1.1 Tournaments

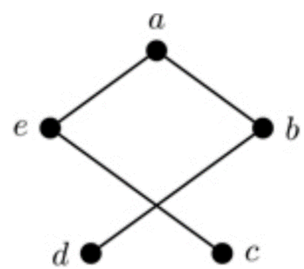
- **SCENARIO**: Five teams (Aardvarks, Bears, Cougars, Ducks, and Eagles) in the Roanoke Soccer League. Game rule: each team plays every other team exactly once and no ties are allowed. The tournament director must determine how many games are needed, how to schedule the games, and how to determine a winner once the tournament is completed.

Vertex: team

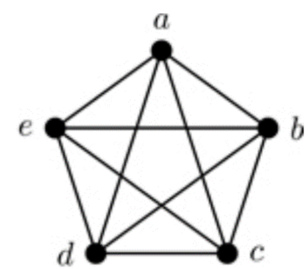
Edge: game




$G_1$

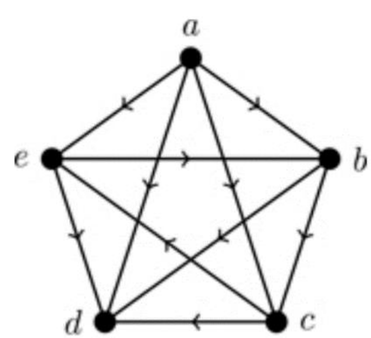



$G_2$




$G_3$

Directed edge or arc: game win

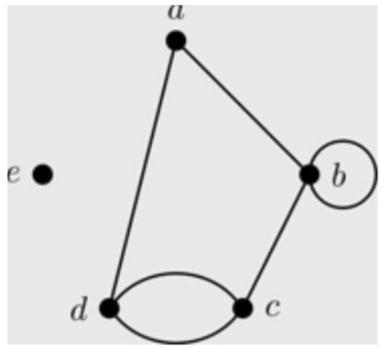



$G_4$

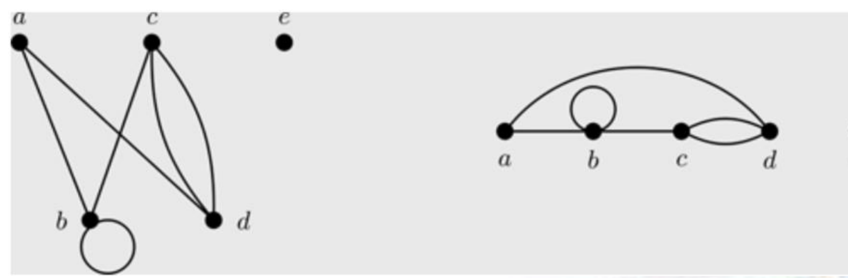
Which team is the tournament winner?

# 1.2 Introduction to Graph Models and Terminology

- **Definition 1.1** A graph $G$ consists of two sets: $V(G)$, called the vertex set, and $E(G)$, called the edge set. An edge, denoted $xy$, is an unordered pair of vertices. We will often use $G$ or $G=(V,E)$ as short-hand.




- **Definition 1.2** The number of vertices in a graph $G$ is denoted $|V(G)|$, or more simply $|G|$. The number of edges is denoted $|E(G)|$ or $\|G\|$.
- What is the best configuration for a graph?

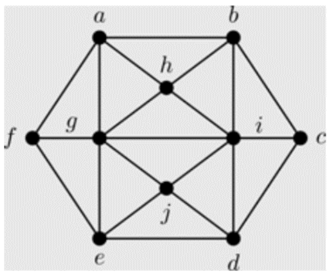


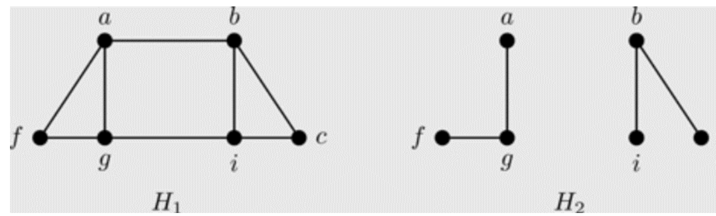
# Formal Specification of Graphs and Digraphs

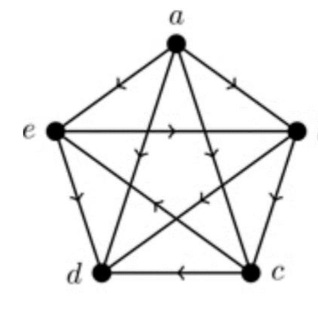
- **Definition 1.3** Let $G$ be a graph.
  - ✓ If $xy$ is an edge, then $x$ and $y$ are the endpoints for that edge. We say $x$ is ***incident*** to edge $e$ if $x$ is an endpoint of $e$.
  - ✓ If two vertices are incident to the same edge, we say the vertices are ***adjacent***, denoted ***x~y***. Similarly, if two edges share an endpoint, we say they are adjacent. If two vertices are adjacent, we say they are ***neighbors*** and the set of all neighbors of a vertex $x$ is denoted ***N*(*x*)**.
  - ✓ If two vertices (or edges) are not adjacent then we call them ***independent***.
  - ✓ If a vertex is not incident to any edge, we call it an ***isolated vertex***.
  - ✓ If both endpoints of an edge are the same vertex, then we say the edge is a ***loop***.
  - ✓ If there is more than one edge with the same endpoints, we call these ***multi-edges***.
  - ✓ If a graph has no multi-edges or loops, we call it ***simple***.
  - ✓ The degree of a vertex $v$, denoted ***deg*(*v*)**, is the number of edges incident to $v$, with a loop adding two to the degree. If the degree is even, the vertex is called ***even***; if the degree is odd, then the vertex is ***odd***.
  - ✓ If all vertices in a graph $G$ have the same degree $k$, then $G$ is called a ***k-regular*** graph. When $k$=3, we call the graph ***cubic***.

# Formal Specification of Graphs and Digraphs

- **Definition 1.4** A *subgraph H* of a graph $G$ is a graph where $H$ contains some of the edges and vertices of $G$; that is, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
- **Example 1.3**

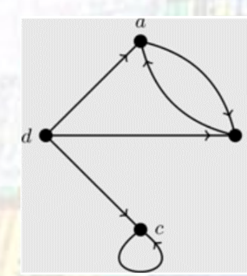


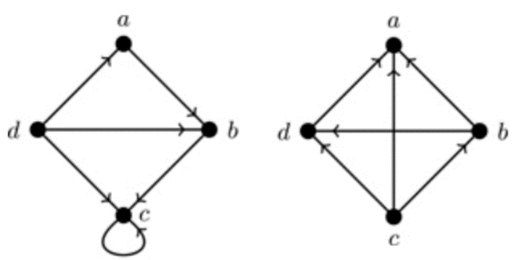

$G_4$

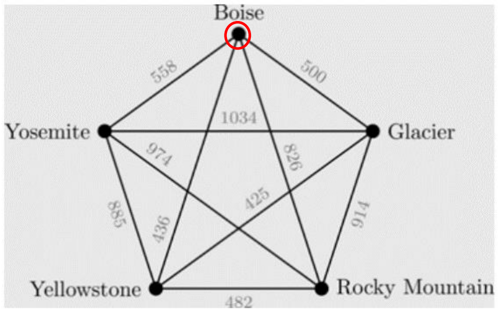
- **Definition 1.5** Given a graph $G=(V,E)$, an *induced subgraph* is a subgraph $G[V']$ where $V' \subseteq V$ and every available edge from $G$ between the vertices in $V'$ is included. We say $H$ is a *spanning subgraph* if it contains all the vertices but not necessarily all the edges of $G$; that is, $V(H)=V(G)$ and $E(H) \subseteq E(G)$.
- **Example 1.4** Find a spanning subgraph of the graph $G$ from Example 1.3 above.
- **Example 1.5** Consider the soccer tournament represented by a graph containing more information such as win/lose information for each game.
- **Definition 1.6** A *directed graph*, or *digraph*, is a graph $G=(V,A)$ that consists of a vertex set $V(G)$ and an arc set $A(G)$. An *arc* is an ordered pair of vertices.
- **Example 1.6**



# Digraph and Weighted Graph

- **Definition 1.7** Let $G=(V,A)$ be a digraph. Given an arc $xy$, the head is the starting vertex $x$ and the tail is the ending vertex $y$.
  - ✓ Given a vertex $x$, the in-degree of $x$ is the number of arcs for which $x$ is a tail, denoted $deg^-(x)$. The out-degree of $x$ is the number of arcs for which $x$ is the head, denoted $deg^+(x)$.
  - ✓ The underlying graph for a digraph is the graph $G'=(V,E)$ which is formed by removing the direction from each arc to form an edge.
- *Does the degree information determine a unique underlying graph structure?*

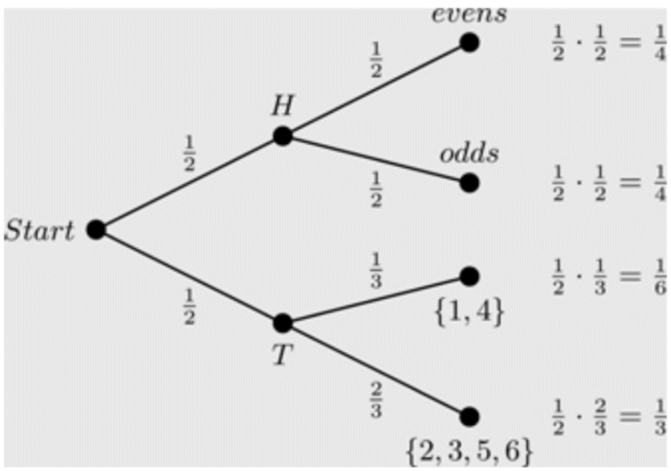



- ***Weighted graphs*** are the graphs with a specific quantity on each edge or arc.
- **Example 1.7** Sam wants to visit 4 national parks over the summer. To save money, he needs to minimize his driving distance.

# Weighted Graphs

- **Definition 1.8** A *weighted graph* $G=(V,E,w)$ is a graph where each of the edges has a real number associated with it. This number is referred to as the weight and denoted $w(xy)$ for the edge $xy$ (weights also can be assigned to vertices).
- Example 1.8 Adam comes to you with a new game. He flips a coin and you roll a die. If he gets heads and you roll an even number, you win \$2; if he gets heads and you roll an odd number, you pay him \$3. If he gets tails and you roll either 1 or 4, you win \$5; if he gets tails and any of 2, 3, 5, or 6 is rolled, you pay him \$2. What is the probability you win \$5? What is the probability you win any amount of money?




you win \$2

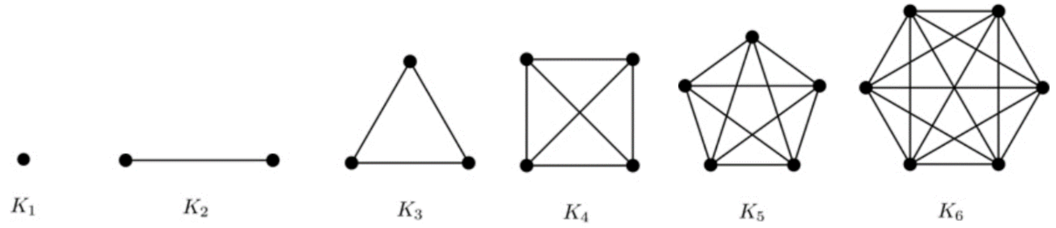
you lose \$3

you win \$5

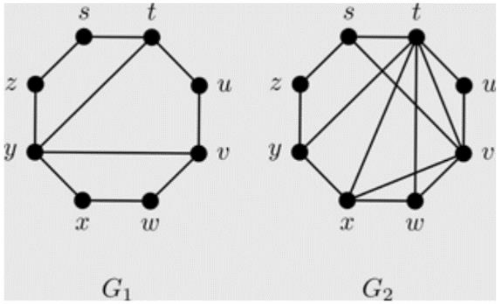
you lose \$2

# Complete Graphs

- **Definition 1.9** A simple graph $G$ is *complete* if every pair of distinct vertices is adjacent. The complete graph on $n$ vertices is denoted $K_n$.

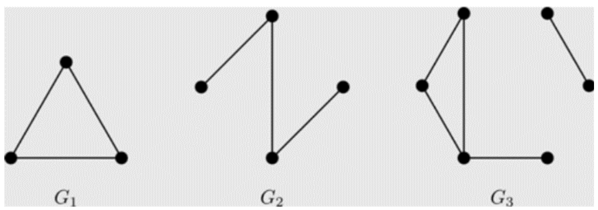


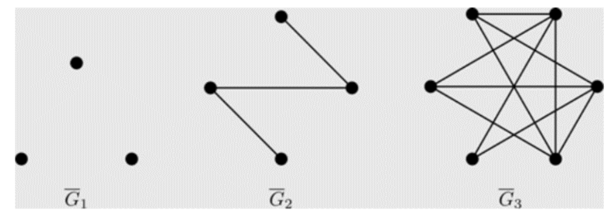

- Properties of $K_n$: (1) Each vertex in $K_n$ has degree $n-1$. (2) $K_n$ has $n(n-1)/2$ edges. (3) $K_n$ contains the most edges out of all simple graphs on $n$ vertices.
- **Definition 1.10** The *clique-size* of a graph, $\omega(G)$, is the largest integer $n$ such that $K_n$ is a subgraph of $G$ but $K_{n+1}$ is not.
- Example 1.9 Find $\omega(G)$ for each of the graphs shown below.



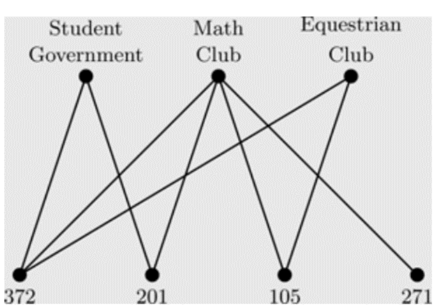
# Graph Complement and Bipartite Graph

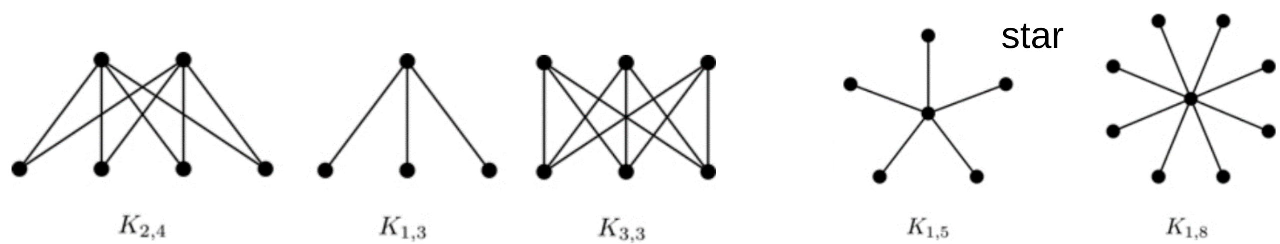
- **Definition 1.11** Given a simple graph $G=(V,E)$, define the *complement* of $G$ as the graph $\bar{G} = (V, \bar{E})$, where an edge $xy \in \bar{E}$ if and only if $xy \notin E$.
- Example 1.10 Find the complements of each graph shown below.



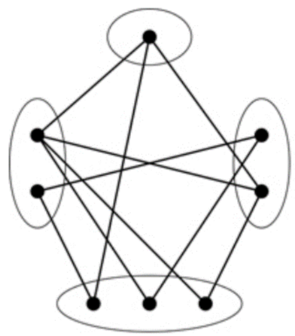

- **Definition 1.12** A graph $G$ is *bipartite* if the vertices can be partitioned into two sets $X$ and $Y$ so that every edge has one endpoint in $X$ and the other in $Y$.
- Example 1.11 Three student organizations (Student Government, Math Club, and the Equestrian Club) are holding meetings on Thursday afternoon.
- **Definition 1.13** $K_{m,n}$ is the complete bipartite graph where $|X|=m$ and $|Y|=n$ and every vertex in $X$ is adjacent to every vertex in $Y$.



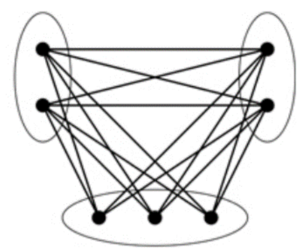



# Bipartite Graph and Graph Combination

- **Definition 1.14** A graph $G$ is *k-partite* if the vertices can be partitioned into $k$ sets $X_1,X_2,\ldots X_k$ so that every edge has one endpoint in $X_i$ and the other in $X_j$ where $i \neq j$.



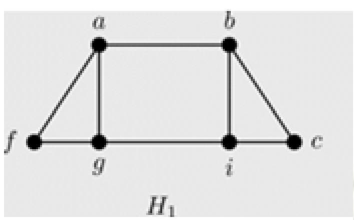
4-partite

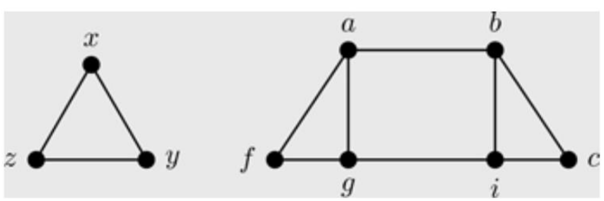


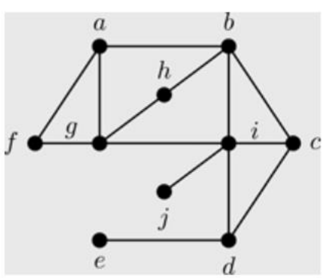
$K_{2,2,3}$

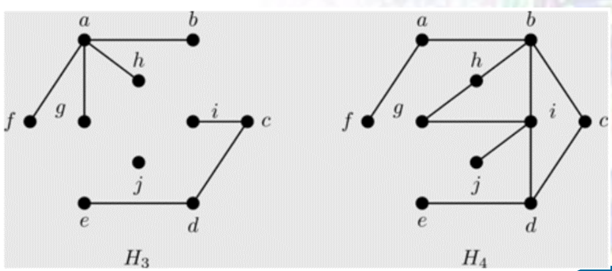
- **Definition 1.15** Given two graphs $G$ and $H$ the *union* $G \cup H$ is the graph with vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$. If the vertex-sets are disjoint (that is $V(G) \cap V(H) = \emptyset$) then we call the disjoint union the *sum*, denoted $G+H$.

- Example 1.12 Find the sum $K_3+H_1$ and the union $H_1 \cup H_4$



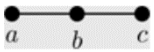



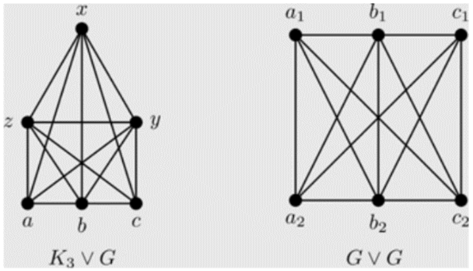



# Graph Combination

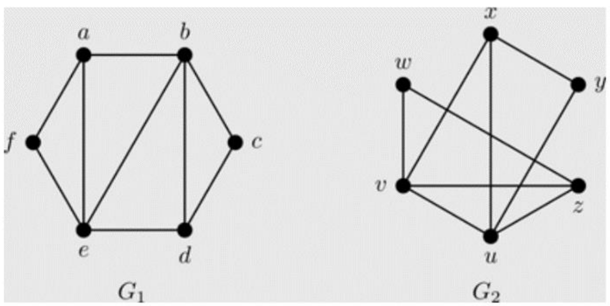
- **Definition 1.16** The *join* of two graphs $G$ and $H$, denoted $G \vee H$, is the sum $G+H$ together with all edges of the form $xy$ where $x \in V(G)$ and $y \in V(H)$.
- Example 1.13 Find the join of $K_3$ and the graph $G$ below consisting of three vertices and two edges, as well as the join $G \vee G$.



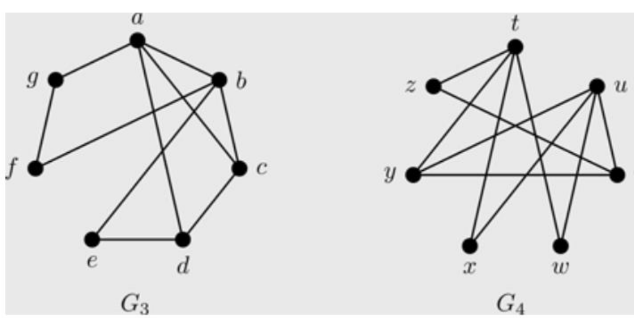
# 1.3 Isomorphism

- **Definition 1.17** Two graphs $G_1$ and $G_2$ are *isomorphic*, denoted $G_1 \cong G_2$, if there exists a bijection $f{:}V(G_1)\to V(G_2)$ so that $xy \in E(G_1)$ if and only if $f(x)f(y)\in E(G_2)$.
- It should be easy to name a few things that are quick to check
  - ✓ number of vertices, number of edges, vertex degrees
- Example 1.14 Determine if the following pair of graphs are isomorphic.




| $V(G_1) \longleftrightarrow V(G_2)$ | Edges | |
|---|---|---|
| $a \longleftrightarrow x$ | $ab \longleftrightarrow xv$ | ✓ |
| $b \longleftrightarrow v$ | $ae \longleftrightarrow xu$ | ✓ |
| $c \longleftrightarrow w$ | $af \longleftrightarrow xy$ | ✓ |
| $d \longleftrightarrow z$ | $bc \longleftrightarrow vw$ | ✓ |
| $e \longleftrightarrow u$ | $bd \longleftrightarrow vz$ | ✓ |
| $f \longleftrightarrow y$ | $be \longleftrightarrow vu$ | ✓ |
| | $cd \longleftrightarrow wz$ | ✓ |
| | $de \longleftrightarrow zu$ | ✓ |
| | $ef \longleftrightarrow uy$ | ✓ |

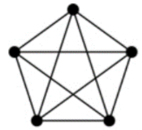
- Example 1.15 Determine if the following pair of graphs are isomorphic.



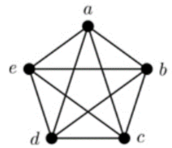
# Isomorphism

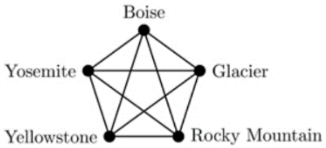
- **Theorem 1.18** Assume $G_1$ and $G_2$ are isomorphic graphs. Then $G_1$ and $G_2$ must satisfy any of the properties listed below; that is, if $G_1$ is connected, has $n$ vertices, has $m$ edges, has $p$ vertices of degree $k_1$, has a cycle of length $k_2$ (see Section 2.1.2), has an eulerian circuit (see Section 2.1.3), has a hamiltonian cycle (see Section 2.2), then so too must $G_2$ (where $n$, $m$, $p$, $k_1$ and $k_2$ are non-negative integers).

- Isomorphism class



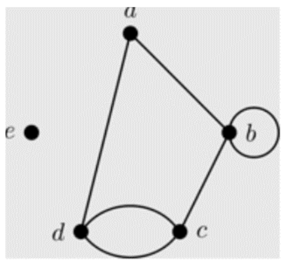
Representatively unlabeled graph



# 1.4 Matrix Representation

- **Definition 1.19** The *adjacency matrix* $A(G)$ of the graph $G$ is the $n \times n$ matrix where vertex $v_i$ is represented by row $i$ and column $i$ and the entry $a_{ij}$ denotes the number of edges between $v_i$ and $v_j$.
- Example 1.16 Find the adjacency matrix for the graph $G_4$ from Example 1.2.

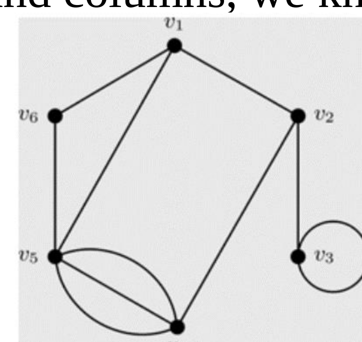



$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

- Example 1.17 Draw the graph whose adjacency matrix is shown below.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Solution: Since the matrix has 6 rows and columns, we know that



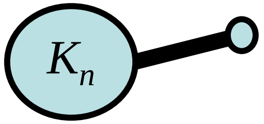
# 1.5 Proof Techniques – Direct and Indirect Proofs

- **Proposition 1.20** The sum of two odd integers is even.
  - ✓ Proof: **Assume** $x$ and $y$ are odd integers. **Then** there exist integers $n$ and $m$ such that $x=2n+1$ and $y=2m+1$. **Thus** $x+y=(2n+1)+(2m+1)=2(n+m+1)=2k$, where $k$ is the integer given by $n+m+1$. **Therefore** $x+y$ is even.
- A proper mathematical proof should be *self-contained*, *concise*, and *complete*.
- **Theorem 1.21** (Handshaking Lemma) Let $G=(V,E)$ be a graph and $|E|$ denote the number of edges in $G$. Then the sum of the degrees of the vertices equals twice the number of edges; that is if $V=\{v_1,v_2,\ldots,v_n\}$, then $\sum_{i=1}^{n}\deg(v_i) = \deg(v_1) + \deg(v_2) + \cdots + \deg(v_n) = 2|E|$
- **Corollary 1.22** Every graph has an even number of vertices of odd degree.

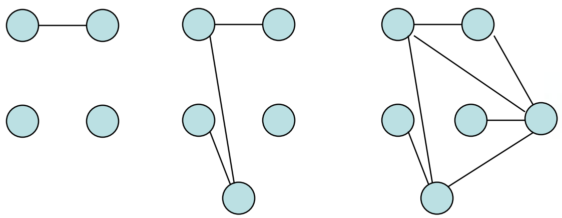
- **Indirect proof**: *contradiction* and *contraposition*
- **Proposition 1.23** For any integer $n$, if $n^2$ is odd then $n$ is odd.
  - ✓ By contradiction
  - ✓ By contraposition
- **Proposition 1.24** For every simple graph $G$ on at least 2 vertices, there exist two vertices of the same degree.

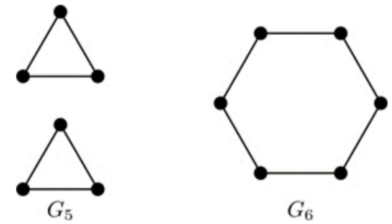
# 1.6 Degree Sequence

- **Mathematical induction**
- **Proposition 1.25** The complete graph $K_n$ has $n(n-1)/2$ edges.



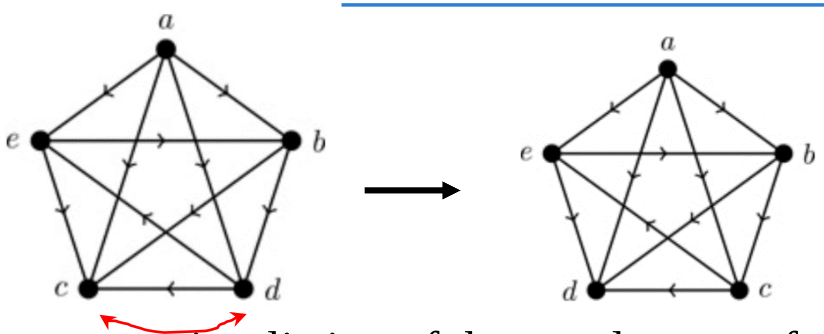

  - ✓ $n+n(n-1)/2 = n(n+1)/2$
- **Definition 1.26** The *degree sequence* of a graph is a listing of the degrees of the vertices. It is customary to write these in decreasing order. If a sequence is a degree sequence of a simple graph then we call it *graphical*.
- Example 1.18 Explain why neither 4, 4, 2, 1, 0 nor 4, 4, 3, 1, 0 can be graphical.
- **Theorem 1.27** (*Havel-Hakimi Theorem*) An non-increasing sequence $S$:$s_1, s_2, \ldots, s_n$ (for $n \geq 2$) of nonnegative integers is graphical if and only if the sequence $S'$:$s_2-1, s_3-1, \ldots, s_{s_1+1}-1, s_{s_1+2}, \ldots, s_n$ is graphical.
- Example 1.19 Determine if either of S: 4, 4, 2, 1, 1, 0 or T: 4, 3, 3, 2, 1, 1 is graphical.
  - ✓ 4,4,2,1,1,0→3,1,0,0,0→0,-1,-1,0
  - ✓ 4,3,3,2,1,1→2,2,1,1,0→1,1,0,0
- Same degree sequence → isomorphism?



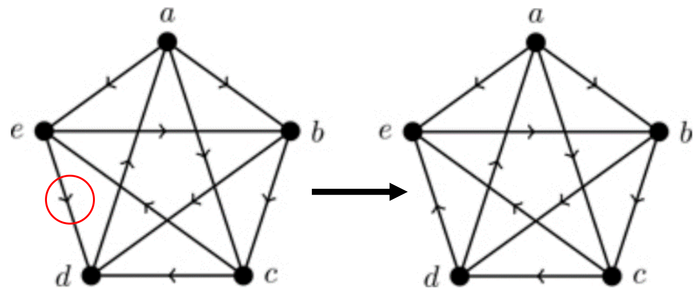


# Tournaments Revisited – Score Sequence

| Team | Teams they Beat |
|---|---|
| Aardvarks | Bears, Cougars, Ducks, Eagles |
| Bears | Cougars, Ducks |
| Cougars | Ducks, Eagles |
| Ducks | |
| Eagles | Bears, Ducks |




- **Definition 1.28** The *score sequence* of a tournament is a listing of the out-degrees of the vertices. It is customary to write these in increasing order.




- Properties of Score Sequences: The score sequence of any tournament $T_n$ must satisfy the following:
  - ✓ $s_1, s_2, \ldots, s_n$ is a sequence of integers satisfying $0 \le s_k \le n-1$ for all $k=1,2,\ldots,n$.
  - ✓ at most one $s_k$ equals 0
  - ✓ at most one $s_k$ equals $n-1$
  - ✓ $s_1+s_2+\cdots+s_n=n(n-1)/2$

# Score Sequence

- **Theorem 1.29** An increasing sequence $S$: $s_1, s_2, \ldots, s_n$ (for $n \geq 2$) of nonnegative integers is a score sequence if and only if $s_1 + s_2 + \cdots + s_k \geq \frac{k(k-1)}{2}$ for each $k$ between 1 and $n$ with equality holding at $k=n$.

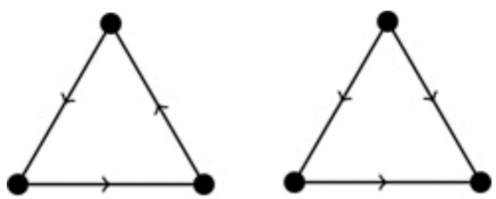
- Example 1.20 Determine if the sequence 1, 2, 2, 3, 3, 4 is the score sequence of a tournament.

| $k$ | $s_1 + \cdots + s_k$ | $\frac{k(k-1)}{2}$ |
|---|---|---|
| 1 | $1$ | 0 |
| 2 | $1+2=3$ | 1 |
| 3 | $1+2+2=5$ | 3 |
| 4 | $1+2+2+3=8$ | 6 |
| 5 | $1+2+2+3+3=11$ | 10 |
| 6 | $1+2+2+3+3+4=15$ | 15 |

- **Theorem 1.30** An increasing sequence $S$: $s_1, s_2, \ldots, s_n$ (for $n \geq 2$) of nonnegative integers is a score sequence of a tournament if and only if the sequence $S_1$: $s_1, s_2, \ldots, s_{s_n}, s_{s_n+1} - 1, \ldots, s_{n-1} - 1$ is a score sequence.

- 1,2,2,3,3,4 $\rightarrow$ 1,2,2,2,3

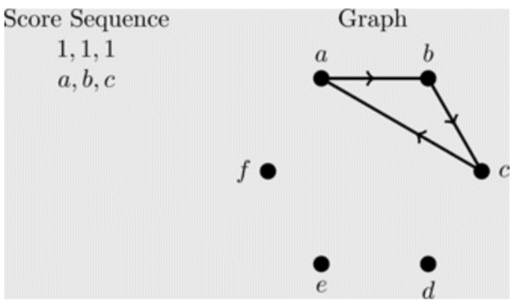
  $\rightarrow$ 1,1,2,2 $\rightarrow$ 1,1,1

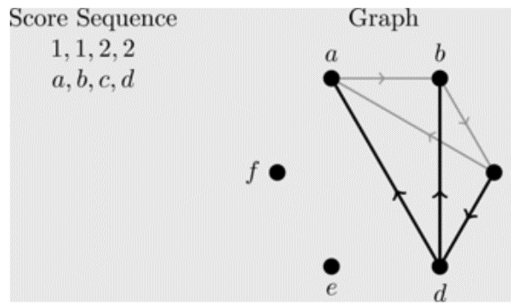


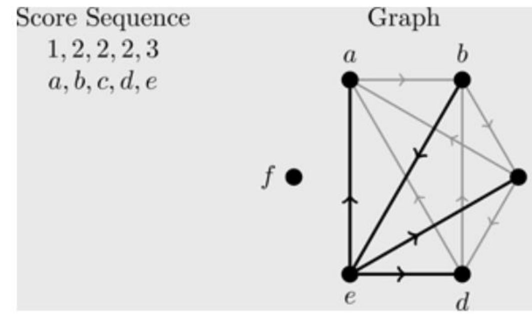
1, 1, 1     0, 1, 2

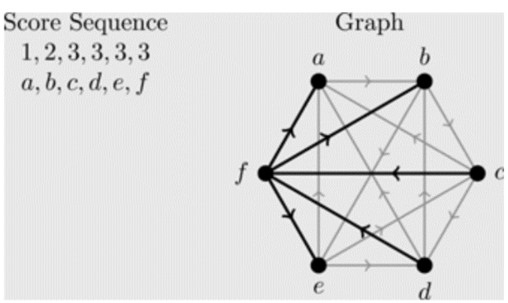
# Score Sequence

- **Example 1.22** Using the results from Example 1.21, draw a tournament with score sequence 1, 2, 3, 3, 3, 3.
  - ✓ 1,2,3,3,3,3 → 1,2,2,2,3 →1,1,2,2 →1,1,1

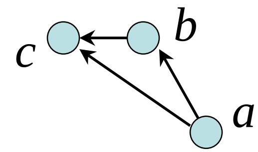


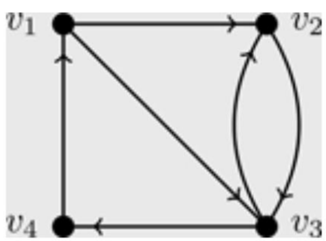

- **Definition 1.31** A tournament $T_n$ is *transitive* if it does not contain any directed cycles. It has score sequence $0,1,2,\ldots,n-1$.



# Matrix Representation

- **Definition 1.32** The adjacency matrix $A(G)$ of the digraph $G$ is the $n{\times}n$ matrix where vertex $v_i$ is represented by row $i$ and column $i$ and the entry $a_{ij}$ denotes the number of arcs from $v_i$ to $v_j$.
- Example 1.23 Find the adjacency matrix for the digraph given below.

✓

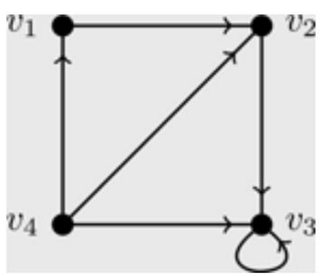



$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

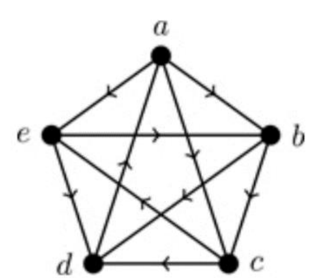
- Example 1.24 Draw the digraph with the adjacency matrix given below.

✓

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$




- Another tournament information

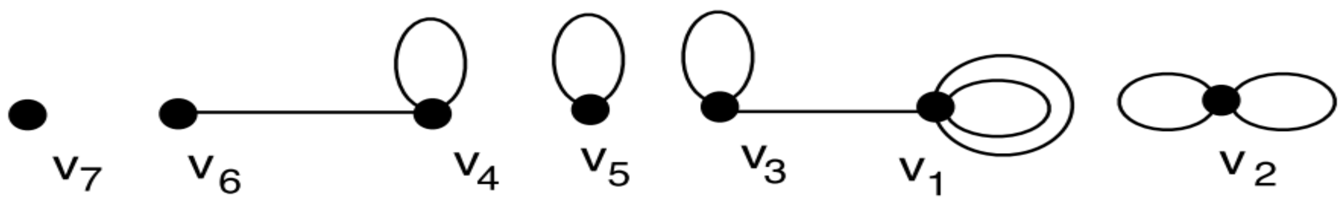



$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

# Supplement – Graphic Degree Sequence

- **Question:** For a non-increasing nonnegative sequence of integers, is this sequence the degree sequence of some graph if the sum of all integers is even?
- **Example:** To construct a graph whose degree sequence is < 5, 4, 3, 3, 2, 1, 0 >

$v_1\ v_2\ \ v_3\ v_4\ \ v_5\ \ v_6\ v_7$




- **Theorem S1.1.** *Suppose that* $<d_1, d_2, \ldots, d_n>$ *is a sequence of nonnegative integers whose sum is even. Then there exists a graph with vertices* $v_1, v_2, \ldots, v_n$ *such that* $deg(v_i)=d_i$*, for* $i = 1, \ldots, n$.
- **DEFINITION:** A sequence $<d_1, d_2, \ldots, d_n>$ is said to be graphic if there is a permutation of it that is the degree sequence of some simple graph. Such a simple graph is said to ***realize*** the sequence.

# Supplement – Graphic Degree Sequence

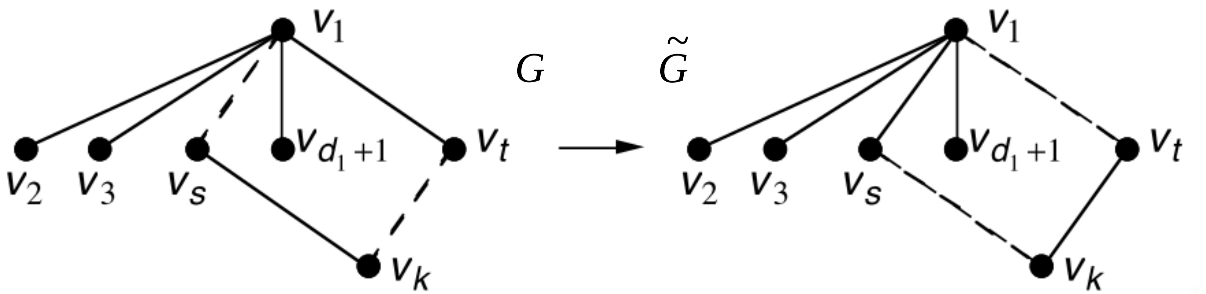
- **Theorem S1.2.** *Let <$d_1, d_2, \ldots, d_n$> be a graphic sequence, with $d_1 \geq d_2 \geq \ldots \geq d_n$. Then there is a simple graph with vertex-set $\{v_1, \ldots, v_n\}$ satisfying $deg(v_i)=d_i$ for $i = 1, 2, \ldots, n$,* ***such that $v_1$ is adjacent to vertices $v_2, \ldots, v_{d_1+1}$.***
  - ✓ **Proof:**

$$r = |N_G(v_1) \cap \{v_2, \ldots, v_{d_1+1}\}| \text{ Select } G \text{ such that } r \text{ is maximum.}$$

*Assume there is a vertex $v_s$, $2 \leq s \leq d_1+1 < t$, $deg(v_1) \geq deg(v_s) \geq deg(v_{d_1+1}) \geq deg(v_t)$,*

*In case $deg(v_s) > deg(v_t)$, $\exists v_k$. $G \to \tilde{G}$, $|N_{\tilde{G}}(v_1) \cap \{v_2, \ldots, v_{d_1+1}\}| = r+1 \to$ contradiction to the assumption.*

*In case $deg(v_s) = deg(v_t)$, swap $v_t$ with $v_s$.*



# Supplement – Graphic Degree Sequence

- **Theorem 1.27** (*Havel-Hakimi Theorem*) An non-increasing sequence $S{:}s_1, s_2, \ldots, s_n$ (for $n \geq 2$) of nonnegative integers is graphical if and only if the sequence $S'{:}s_2-1, s_3-1, \ldots, s_{s_1+1}-1, s_{s_1+2}, \ldots, s_n$ is graphical.

ALGORITHM: RECURSIVE GRAPHICSEQUENCE($\langle d_1, d_2, \ldots, d_n \rangle$)
*Input*: a non-increasing sequence $\langle d_1, d_2, \ldots, d_n \rangle$.
*Output*: TRUE if the sequence is graphic; FALSE if it is not.

```
If d_1 = 0 && d_n = 0
    Return TRUE
Else
    If d_n < 0
        Return FALSE
    Else
        Let ⟨a_1, a_2, ..., a_{n-1}⟩ be a non-incr permutation
          of ⟨d_2 − 1, ..., d_{d_1+1} − 1, d_{d_1+2}, ..., d_n⟩.
        Return GraphicSequence(⟨a_1, a_2, ..., a_{n-1}⟩)
```