Chap 7 Planarity

Yih-Lang Li (李毅郎)
Computer Science Department
National Yang-Ming Chiao-Tung University, Taiwan

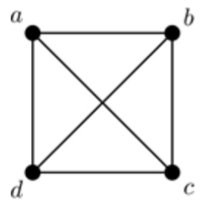
The sources of most figure images are from the textbook

Outline

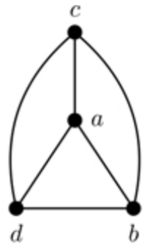
- Kuratowski's Theorem
- Graph Coloring Revisited
- Edge Crossing
- Thickness

Planarity

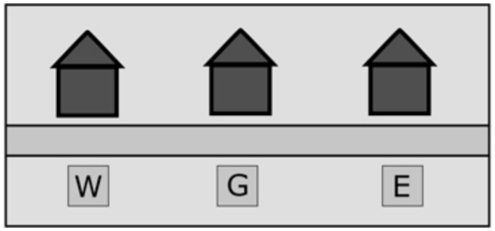
- **Definition 7.1** A graph G is planar if and only if the vertices can be arranged on the page so that edges do not cross (or touch) at any point other than at a vertex.



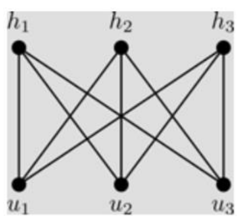
K_4

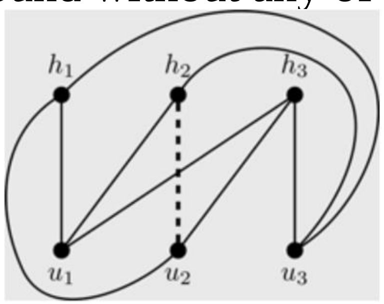


planar drawing of K_4



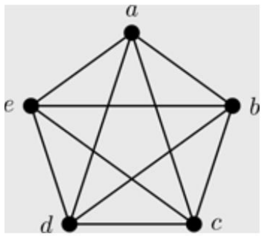
- **Example 7.1** Three houses are set to be built along a new city block; across the street lie access points to the three main utilities each house needs (water, electricity, and gas). Is it possible to run the lines and pipes underground without any of them crossing?

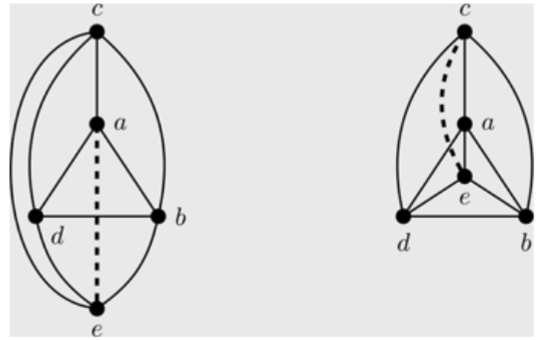




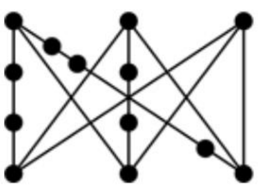
Kuratowski's Theorem

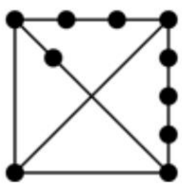
- **Example 7.2** Determine if K_5 is planar. If so, give a planar drawing; if not, explain why not.

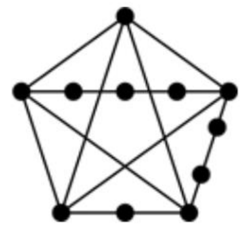


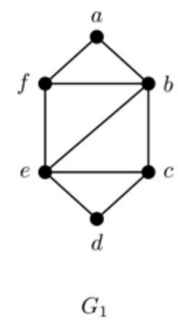


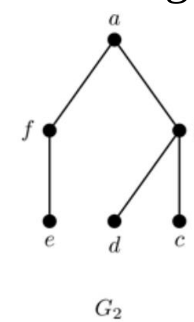
- **Definition 7.2** A subdivision of an edge *xy* consists of inserting vertices so that the edge *xy* is replaced by a path from *x* to *y*. The subdivision of a graph *G* is obtained by subdividing edges in *G*.











- **Theorem 7.3** (*Kuratowski's Theorem*) A graph *G* is planar if and only if it does not contain a subdivision of $K_{3,3}$ or K_5.

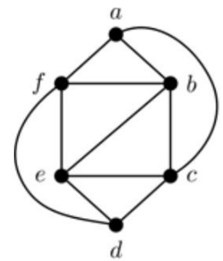
- **Definition 7.4** Given a planar drawing of a graph *G*, a *region* is a portion of the plane completely bounded by the edges of the graph.

Kuratowski's Theorem

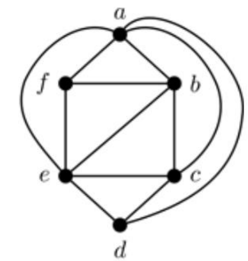
- **Theorem 7.5** (*Euler's Formula*) If G is a connected planar graph with n vertices, m edges, and r regions then $n-m+r=2$.
  - ✓ Argue by induction on m, the number of edges in the graph. If $m=1 \rightarrow G$ is either a tree with one edge and so $n=2$ and $r=1$, or G is a graph with a loop, and so $n=1$ and $r=2$. Euler's Formula holds.
  - ✓ Suppose Euler's Formula holds for all graphs with $m \geq 1$ edges and consider a graph G with $m+1$ edges, n vertices, and r regions.
  - ✓ First, if $G'=G-e$ is ***not connected*** for any edge e in $G \rightarrow e$ must be a bridge of G and G must be a tree $\rightarrow n=m+1$ and $r=1 \rightarrow n-m+r=m+1-m+1=2$.
  - ✓ Next, if $G'=G-e$ is ***connected*** for some edge e of $G \rightarrow e$ must be a part of some cycle in $G \rightarrow$ Two regions R_1 and R_2 in G abutting each other on e merges as a region in G'.
  - ✓ Thus G' has n vertices, $r-1$ regions, and $m-1$ edges. By the induction hypothesis applied to G', we know $n-(m-1)+(r-1)=2$, which simplifies to $n-m+r=2$.
  - ✓ Thus by induction we know Euler's Formula holds for all connected planar graphs.

Kuratowski's Theorem

- How about adding a new edge into a planar graph?



$G_1 + \{ac, df\}$

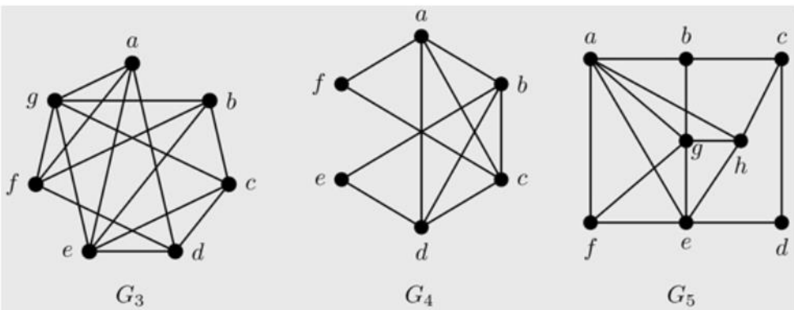


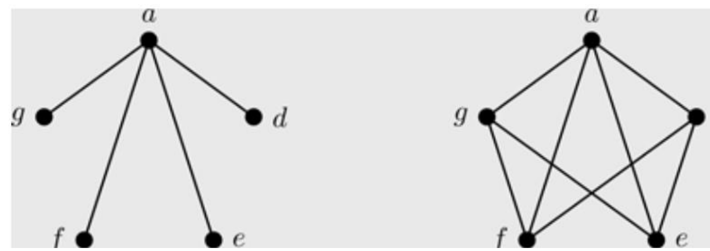
$G_1 + \{ac, ae, ad\}$

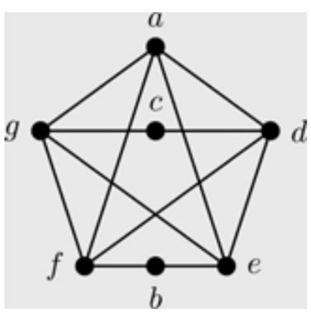
- **Definition 7.6** A graph G is *maximally planar* if $G+e$ is nonplanar for any edge $e=xy$ for any two nonadjacent vertices $x, y \in V(G)$.
- **Theorem 7.7** If G is a maximally planar simple graph with $n \geq 3$ vertices and m edges, then $m=3n-6$.
  - ✓ Assume G is maximally planar. Then every region must be bounded by a triangle, as otherwise we could add a chord to a region bounded by a longer cycle.
  - ✓ Since every edge separates two regions, and every region is bounded by three edges, we know $r=\frac{2m}{3}$.
  - ✓ Thus by Euler's Formula, we have $n-m+\frac{2m}{3}=2$, and so $3n-3m+2m=6$, giving $m=3n-6$.
- **Theorem 7.8** If $G=(V, E)$ is a simple planar graph with m edges and $n \geq 3$ vertices, then $m \leq 3n-6$.

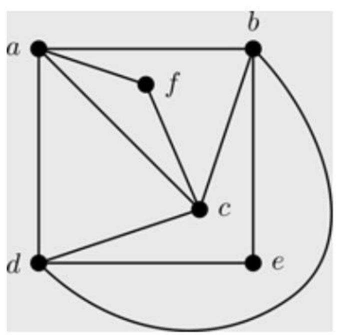
Kuratowski's Theorem

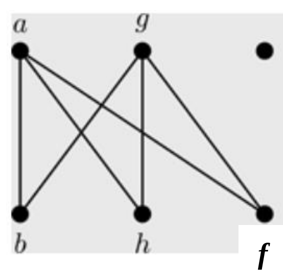
- **Theorem 7.9** If $G=(V, E)$ is a simple planar graph with m edges and $n\geq3$ and no cycles of length 3, then $m\leq 2n-4$.
- **Example 7.3** Determine which of the following graphs are planar. If planar, give a drawing with no edge crossings. If nonplanar, find a $K_{3,3}$ or K_5 subdivision.

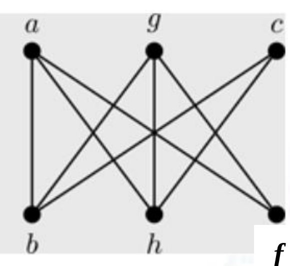


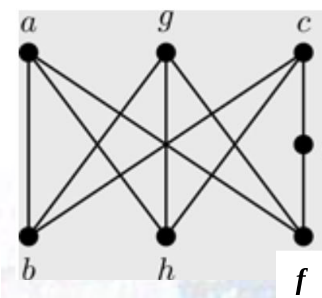






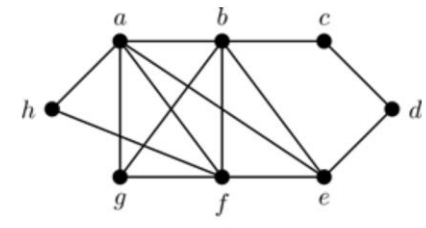




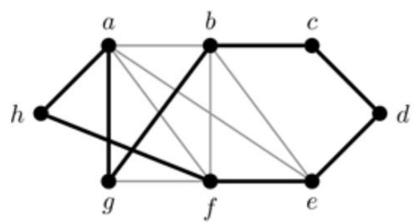


Cycle-Chord Method

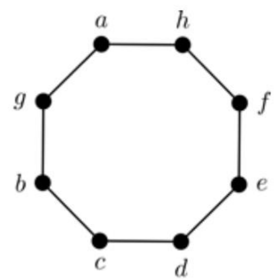
- How to identify a planar drawing?
  - ✓ Step 1 – find a spanning cycle
  - ✓ Step 2 – put as many edges in the interior of the cycle as possible
  - ✓ Step 3 – notice the vertices with unprocessed incident edges that need to be placed in the exterior of the cycle. It would be better that the unprocessed edges are the incident edges of the same vertex



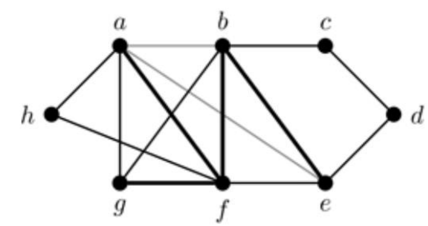
G_6



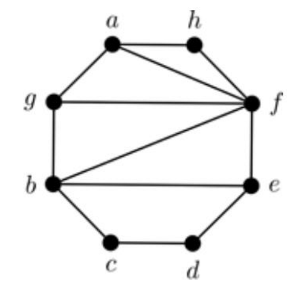
highlighted portion of G_6



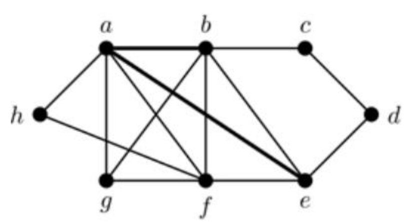
planar drawing



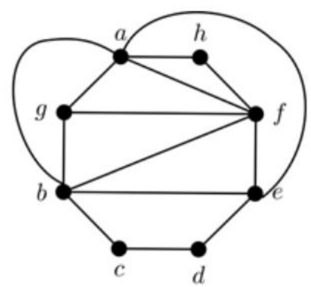
highlighted portion of G_6



planar drawing



highlighted portion of G_6



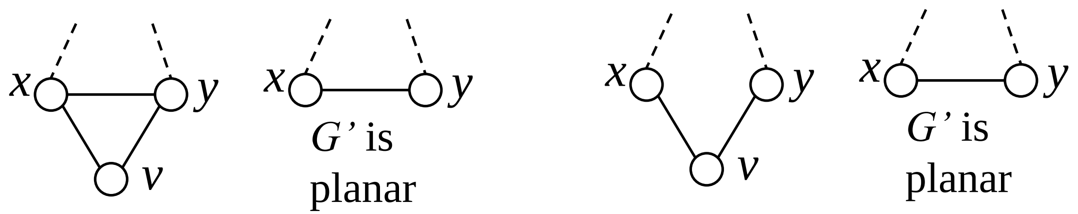
planar drawing

Proof of Kuratowski's Theorem

- **Lemma 7.10** Let H be a subgraph of G. Then G is nonplanar if H is nonplanar.
- **Lemma 7.11** Let G' be a subdivision of G. Then G is planar if and only if G' is planar.
- **Lemma 7.12** A graph G with at least 3 vertices is 2-connected if and only if for every pair of vertices x and y there exists a cycle through x and y.
- **Theorem 7.3** (*Kuratowski's Theorem*) A graph G is nonplanar if and only if it contains a subdivision of $K_{3,3}$ or K_5.
  - ✓ First suppose G contains a subdivision of $K_{3,3}$ or $K_5 \rightarrow G$ is also nonplanar.
  - ✓ Conversely, suppose for a contradiction that there exists a nonplanar graph G that does not contain a subdivision of $K_{3,3}$ or K_5. Choose G to be a minimal such graph; that is, any graph with fewer vertices or edges that does not contain a subdivision of $K_{3,3}$ or K_5 must be planar.
  - ✓ Note that since G is nonplanar then it must contain a nonplanar block B. If B is a proper subgraph of G, then G would not be minimal. Thus we know that G must itself be 2-connected.

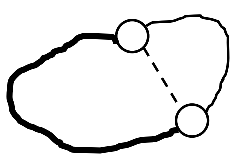
Proof of Kuratowski's Theorem

- *Claim* 1*: Every vertex of G have degree at least* 3.
  - Suppose for a contradiction that there exists some vertex v that does not have degree 3. We know $deg(v)\geq 2$ since G is 2-connected, and so $deg(v)=2$.
  - Let x and y be the two distinct neighbors of v. If x and y are adjacent, then $G'=G-v$ must be planar by the minimality of G.
  - Thus x and y cannot be adjacent. Define $G'=G-v+xy$;



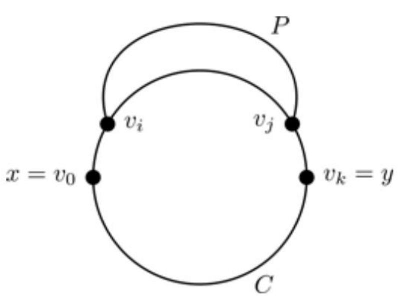
Subdivision to *xy*

- *Claim* 2*: There exists an edge e=xy such that G'=G−e is* 2*-connected.*
  - Since every vertex of G has degree at least 3, we know G cannot simply be a cycle. G is 2-connected $\rightarrow$ G has an ear decomposition by Theorem 4.24.
  - Then last path added to the decomposition must be a singular edge, as otherwise any internal vertex of the path would have degree 2 in G. Thus the graph obtained by removing this edge will remain 2-connected as it still has an ear decomposition.



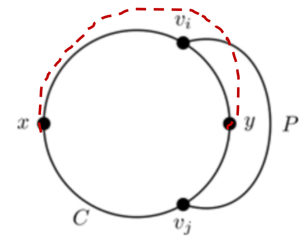
Proof of Kuratowski's Theorem

- Let x, y be chosen so that $G' = G - e\ (xy)$ is 2-connected. Then by the minimality of $G \to G'$ is planar.
- By Corollary 4.20, there exists a cycle C in G' that contains both x and y. Consider the planar drawing of G' with C chosen so that: it contains x and y, the number of regions inside C is maximal among all planar drawings of G'.
- Let $C = v_0 v_1 \cdots v_k v_{k+1} \cdots v_1 v_0$, where $v_0 = x$ and $v_k = y$. Since x and y are not adjacent in G', we know $k \geq 2$.
- *Claim 3: There is no path P connecting two vertices in $\{v_0, v_1, \ldots v_k\}$ or in $\{v_k, v_{k+1}, \ldots v_1, v_0\}$ that lies in the exterior of C.*
  - Suppose such a path exists, say between vertices v_i and v_j. Then construct cycle C' by taking cycle C from v_0 up to v_i then the path P to v_j followed by the cycle C from v_j to v_k and then continuing back to v_0.

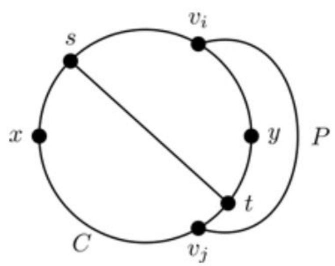


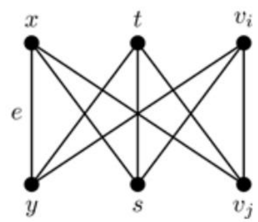
Proof of Kuratowski's Theorem

- Since G is nonplanar, we know that we cannot simply add the edge xy to the exterior of C in the planar drawing of G'. Thus there must be a path along the exterior of C that connects a vertex from the set $\{v_1,\ldots v_{k-1}\}$ to a vertex from the set $\{v_{k+1},\ldots v_l\}$, say P is from v_i to v_j with $1 \le i \le k-1$ and $k+1 \le j \le l$.



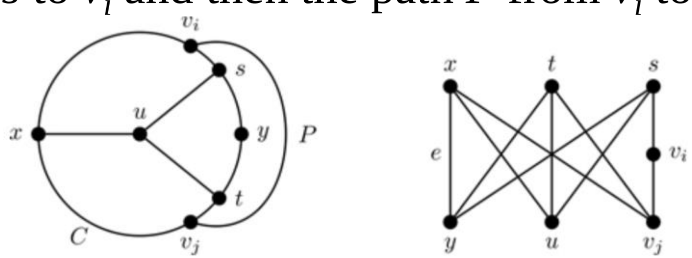
- Note that no vertex of P can be adjacent to any other vertex of C.
- Since P is placed along the exterior of C, there must be some reason why it cannot be placed in the interior of C. The four following cases are exhaustive.
- *Case 1*: There is a path P' between a vertex s from $\{v_0,v_1,\ldots,v_{i-1}\}$ and a vertex t from $\{v_k,\ldots v_{j-1}\}$. Adding the edge e back in produces a $K_{3,3}$ subdivision using portions of the cycle C and paths P and P' as shown below.

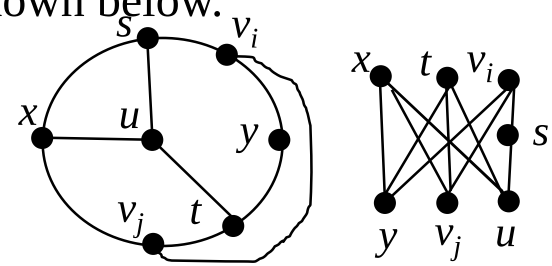




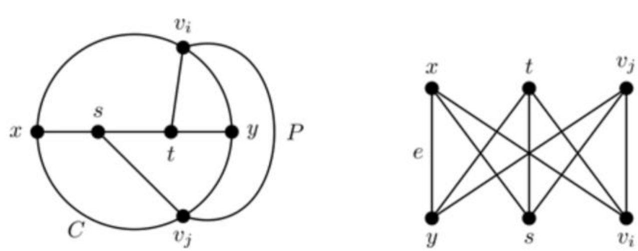
Proof of Kuratowski's Theorem

- *Case* 2: There is vertex u with three disjoint paths to vertices of C, one of which is from $A=\{x=v_0,v_i,y=v_k,v_j\}$, and the other two vertices s and t lie along C between the other three vertices from A. One such option is shown below. Adding the edge e back in produces a $K_{3,3}$ subdivision, where the path from s to v_j uses the portion of the cycle C from s to v_i and then the path P from v_i to v_j, as shown below.



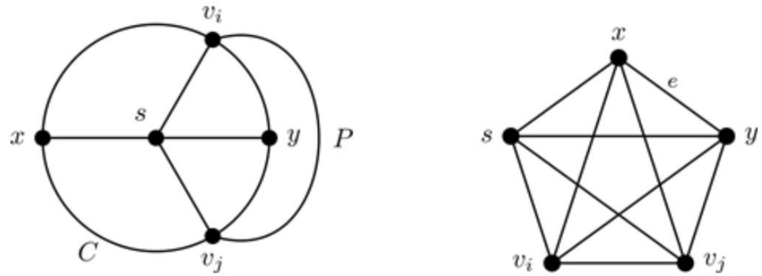


- *Case* 3: There is a path from x to y on the interior of C with two distinct vertices s and t from which there exist disjoint paths to v_i and v_j. Adding the edge e back in produces a $K_{3,3}$ subdivision as shown below.



Proof of Kuratowski's Theorem

- *Case* 4: There is a path from x to y on the interior of C with a vertex s from which there exist disjoint paths to v_i and v_j. Adding the edge e back in produces a K_5 subdivision as shown below.



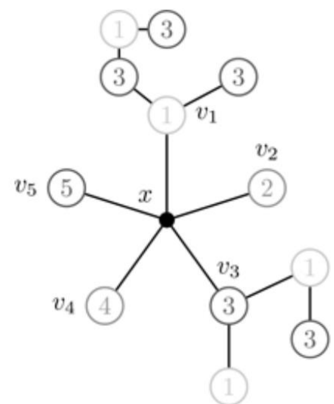
- In each of the cases above we obtain a subgraph of G that is a subdivision of $K_{3,3}$ or K_5. Thus we have shown that every nonplanar graph must contain a subdivision of $K_{3,3}$ or K_5.

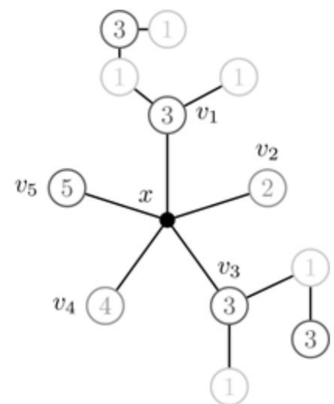
Graph Coloring Revisited

- **Lemma 7.13** All simple planar graphs have a vertex of degree at most 5.
- **Theorem 7.14** Every planar graph can be colored using at most 6 colors.
  - Argue by induction on n, the number of vertices in G. If G has one vertex, then only 1 color is needed, and so 6 colors suffice.
  - Now suppose $n \geq 2$ and every planar graph G' with less than n vertices can be properly colored with at most 6 colors. Let G be a planar graph with n vertices.
  - By ***Lemma*** 7.13, G must contain a vertex x of degree at most 5. Let $G' = G - x$. Then G' can be colored with at most 6 colors by the induction hypothesis.
  - But since $deg(x) \leq 5$, at least one of these 6 colors not used by any neighbor of x.
  - Thus x can be colored with one of the 6 colors not used on its neighbors.
- **Definition 7.15** Let G be a graph in which every vertex has been colored. Then $G_{i,j}$ is the graph induced by colors i and j and a Kempe $i-j$ chain is any component of graph $G_{i,j}$.

Graph Coloring Revisited

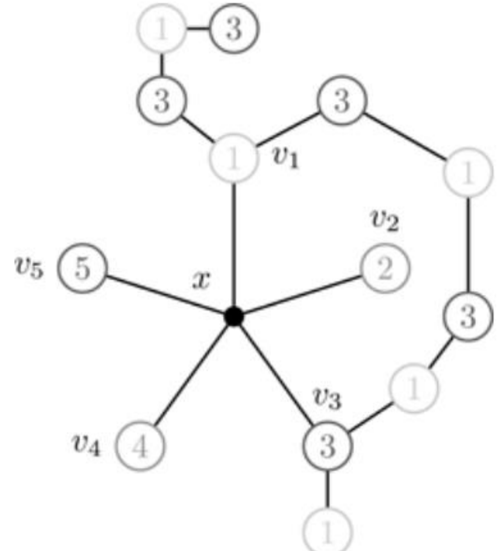
- **Theorem 7.16** Every planar graph can be colored using at most 5 colors.
  - ✓ Argue by induction on n, the number of vertices in G. If $n < 6$, $\chi(G) \leq 5$.
  - ✓ Assume $n \geq 6$ and $\forall$ planar graph G, $|V(G)| < n$, $\chi(G) \leq 5$. By Lemma 7.13, $\exists x$ in G, $deg(x) \leq 5$. Define $G' = G - x$. Then by the induction hypothesis, we can color G' with at most 5 colors.
  - ✓ If the neighbors of x use at most 4 colors $\rightarrow \chi(G) \leq 5$.
  - ✓ Otherwise $deg(x) = 5$ and each of the neighbors of x have been given a unique color.
  - ✓ Let $v_1, \ldots v_5$ be the neighbors of x. Without loss of generality, we may assume that the vertices are arranged in a cyclic nature around x so that v_i has color i.
  - ✓ Consider the graph $G_{1,3}$ that only contains v_1 and v_3.
  - ✓ We will consider whether these vertices are in the same component of $G_{1,3}$.
  - ✓ In particular if the Kempe 1−3 chain K containing v_1 also contains v_3.





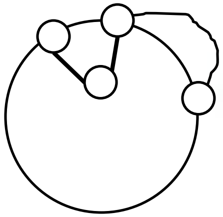
Graph Coloring Revisited

- Case 1: K does not contain v_3. Then we can swap the colors along $K \rightarrow$ no neighbor of x has color 1, making it possible to give x color 1.
- Case 2: K contains v_3. Then there must exist a path P in K from v_1 to v_3 that alternates between a vertex of color 1 and a vertex of color 3 $\rightarrow$ a cycle C in which v_2 and v_4 are on separate faces.
- Then in $G_{2,4}$ there cannot be a Kempe 2−4 chain that contains both v_2 and v_4 as any such path would cross C at either an edge, making G nonplanar, or a vertex, creating a vertex with more than one color.
- Thus we can swap the colors on the Kempe 2−4 chain containing v_2, making v_2 have color 4 and allowing x to be given color 2.



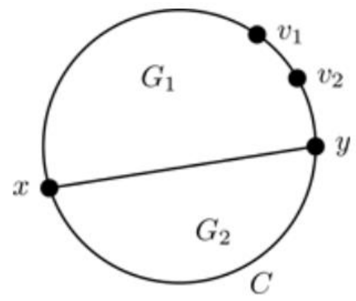
Graph Coloring Revisited

- **Theorem 7.17** Every planar graph is 5-choosable.
  - First note that adding edges to a graph cannot reduce the choosability of a graph, so we will only consider planar graphs in which the exterior boundary is a cycle and all interior regions are triangulated.
  - So suppose the boundary cycle is $C=v_1 v_2 \cdots v_k v_1$. Further, suppose v_1 has been given color 1, v_2 has been given color 2, $\forall v$ in C-$\{v_1, v_2\}$, $|L(v)|\geq 3$, and $\forall v$ in G-C, $|L(v)|=5$.
  - We will prove that the coloring of v_1 and v_2 can be extended to a coloring of the remaining vertices of G by inducting on $|G|$, the number of vertices in G.
  - Suppose $|G|=3$. Since v_3 has a list of size 3, we know there must be a color other than 1 or 2 available for v_3, so G is list-colorable.
  - Now suppose $|G|\geq 4$ and assume our initial assumptions above hold for all graphs with fewer vertices.
  - Two cases for the cycle C based on the existence of a chord for C. In each of these cases we will find a subgraph of G that satisfies the induction hypothesis and explain how to extend its coloring to the entire graph G.



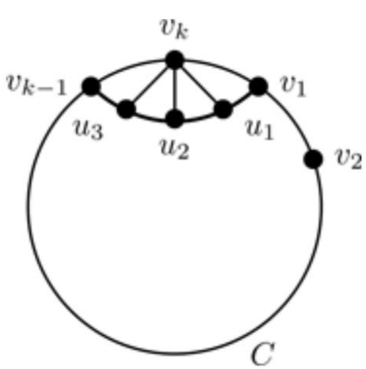
Graph Coloring Revisited

- ***Case* 1**: C has a chord $xy \rightarrow$ two cycles C_1 and C_2 which are contained in the cycle C together with the edge xy.
- Let edge v_1v_2 be a part of C_1, since at most one of v_1 and v_2 can equal x or y. Let G_i be the graph induced by C_i and all vertices in its interior.
- Note that G_1 has fewer vertices than G and with colors 1 and 2 for v_1 and v_2, respectively $\rightarrow$ the induction hypothesis to G_1 to obtain a proper list coloring of G_1.
- This fixes x and y to each have specific color, and since $|V(G_2)|<|V(G)|$, applying the induction hypothesis to G_2 with x and y playing the role of v_1 and v_2, we get a proper list coloring of G_2.
- Combining these two colorings produces a proper list coloring of G.



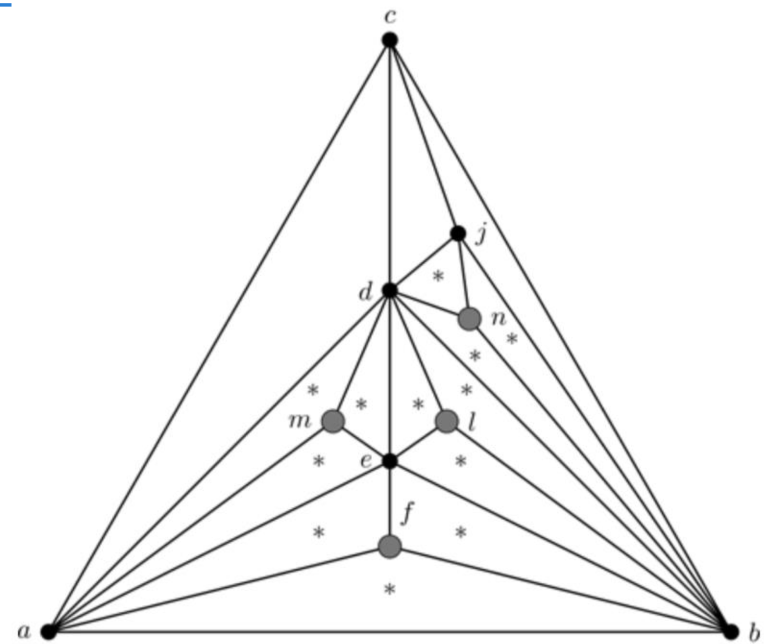
Graph Coloring Revisited

- ***Case* 2**: C does not have any chords. Let $v_1 u_1, u_2, \ldots, u_m, v_{k-1}$ be the neighbors of v_k in their cyclic order around v_k.
- Then by how C was defined, each of the u_i lie in the interior of $C \to$ a path from v_1 to $v_{k\text{-}1}$ using the u_i vertices, that is $P=v_1\, u_1 \cdots u_m\, v_{k\text{-}1}$. Let C' be the cycle formed by removing v_k from C and adding in the path P.
- Since $|L(v_k)|=3$, we know at least two of these c_i, c_j are not 1, namely the color of v_1.
- Remove c_i and c_j from the list of each u_i vertex. Then v_1 and v_2 are on C', with colors 1 and 2, respectively, and all other vertices of C' have lists of size at least 3.
- Then by the induction hypothesis we can properly list color $G-\{v_k\}$. Since c_i and c_j are not used for v_1 and any u_i vertex, at most one of these two colors can be used on v_{k-1}, leaving the other available for v_k. Thus we have obtained a proper list coloring for G.

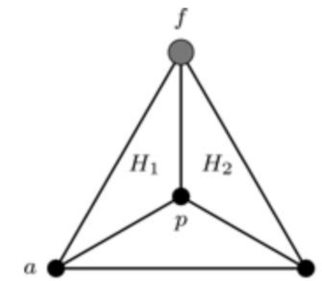


Graph Coloring Revisited

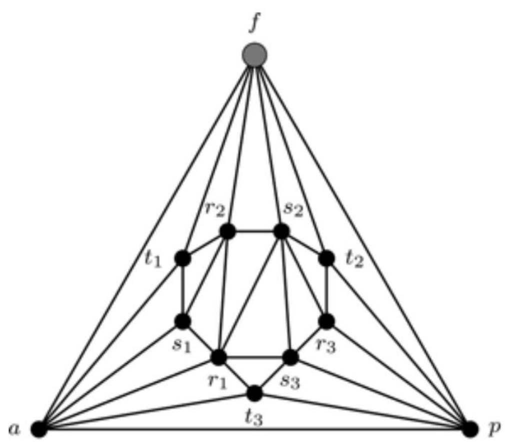
- There exists a planar graph that is not 4-choosable?



G_v

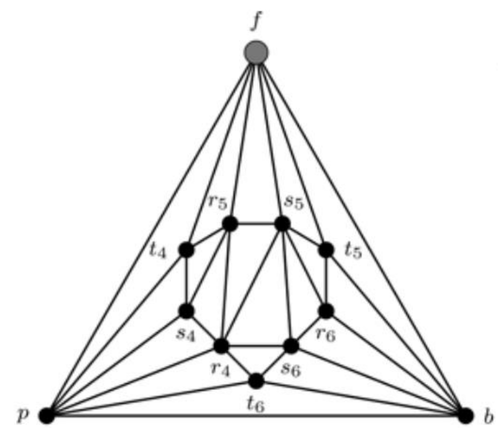


H



H_1

Vertex	List
r_1	$\{2,3,5,6\}$
r_2	$\{1,3,5,6\}$
r_3	$\{4,3,5,6\}$
s_1	$\{2,3,5,6\}$
s_2	$\{1,3,5,6\}$
s_3	$\{4,3,5,6\}$
t_1	$\{1,3,5,6\}$
t_2	$\{1,3,4,5\}$
t_3	$\{2,3,4,5\}$

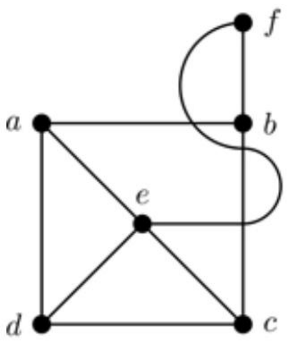


H_2

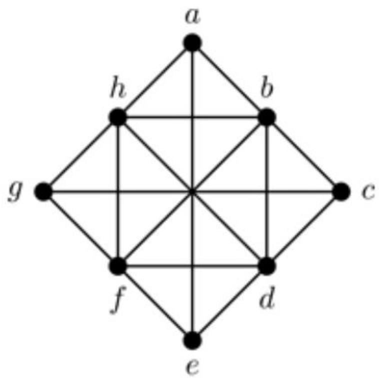
Vertex	List
r_4	$\{3,4,5,6\}$
r_5	$\{1,3,5,6\}$
r_6	$\{2,3,5,6\}$
s_4	$\{3,4,5,6\}$
s_5	$\{1,3,5,6\}$
s_6	$\{2,3,5,6\}$
t_4	$\{1,3,4,5\}$
t_5	$\{1,2,3,5\}$
t_6	$\{2,3,4,5\}$

7.3 Edge- Crossing

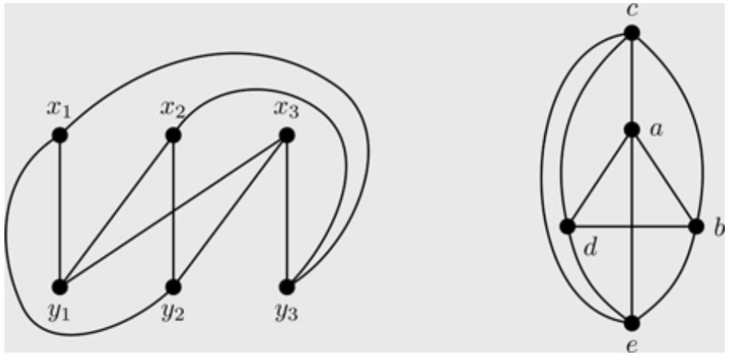
- **Definition 7.18** For any simple graph G the crossing number of G, denoted $cr(G)$, is the minimum number of edge crossings in any drawing of G satisfying the conditions below:
  - ✓ no edge crosses another more than once, and
  - ✓ at most two edges cross at a given point.



G_7



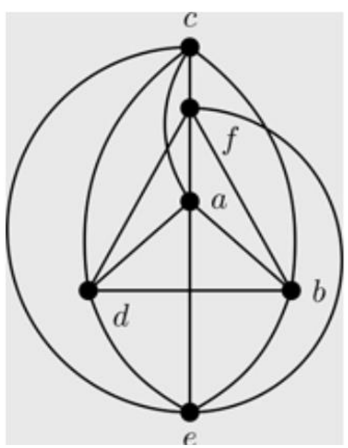
G_8

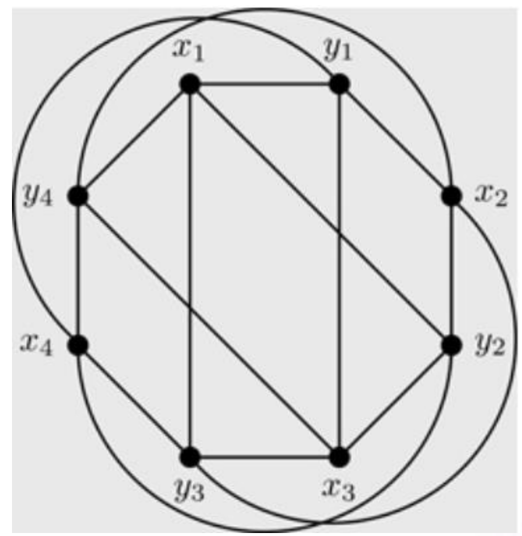


- **Example 7.4** Determine the crossing numbers for K_5 and $K_{3,3}$.
- **Theorem 7.19** Let G be a simple graph with m edges and n vertices. Then $cr(G) \geq m-3n+6$. Moreover, if G is bipartite then $cr(G) \geq m-2n+4$.
- How to compute the number of crossings?
  - ✓ Similar to compute $\chi(G)$. Compute a lower bound and then find the one.

Edge Crossing

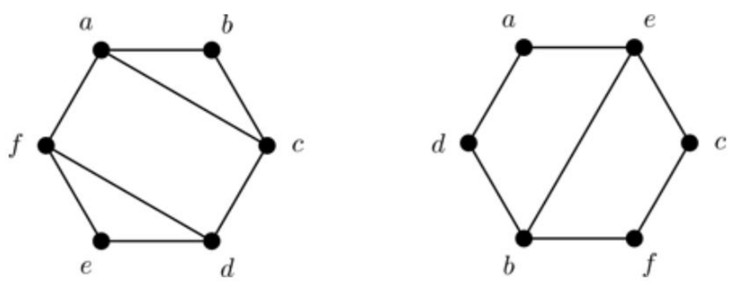
- **Example 7.5** Find the crossing number for K_6 and $K_{4,4}$.
  - ✓ $cr(K_6) \geq 15-3*6+6=3$. Below is a drawing of K_6 with 3 edge crossings, and so we know $cr(K_6)=3$.
  - ✓ $cr(K_{4,4}) \geq 16-2*8+4=4$. Since the drawing below of $K_{4,4}$ has 4 edge crossings, we know $cr(K_{4,4})=4$.





Thickness

- **Definition 7.22** Let $T=\{H_1,H_2,\ldots,H_t\}$ be a set of spanning subgraphs of G so that each H_i is planar and every edge of G appears in exactly one graph from T. The thickness of a graph G, denoted $\theta(G)$, is the minimum size of T among all possible such collections.



- **Corollary 7.23** Let G be a connected simple graph with n vertices and m edges. Then $\theta(G)\geq\left\lceil\frac{m}{3n-6}\right\rceil$
- **Corollary 7.24** Let G be a connected simple bipartite graph with n vertices and m edges. Then $\theta(G)\geq\left\lceil\frac{m}{2n-4}\right\rceil$
- **Theorem 7.25** $\theta(K_n)=\begin{cases}\left\lfloor\frac{n+7}{6}\right\rfloor, n\neq 9{,}10\\ 3, n=9{,}10\end{cases}$