NUMERICAL DATA FOR THE PAPER 'EXPLICIT CHABAUTY–KIM FOR THE SPLIT CARTAN MODULAR CURVE OF LEVEL 13'

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This document contains numerical data related to the computation of the rational points on the split Cartan modular curve $X := X_{\mathrm{s}}(13)$. All of the data below can be recomputed using the code in https://github.com/jtuitman/Cartan13. Our working model is $Q(X,Y,Z) = 0$, where

$$\begin{aligned}Q(X,Y,Z) = {}& Y^4 + 5X^4 - 6X^2Y^2 + 6X^3Z + 26X^2YZ + 10XY^2Z - 10Y^3Z - 32X^2Z^2 - 40XYZ^2 \\ & + 24Y^2Z^2 + 32XZ^3 - 16YZ^3\end{aligned}$$

The known rational points are

P_0	P_1	P_2	P_3	P_4	P_5	P_6
$(1:1:1)$	$(1:1:2)$	$(0:0:1)$	$(-3:3:2)$	$(1:1:0)$	$(0:2:1)$	$(-1:1:0)$

We show that these are precisely the common zeroes of two quadratic Chabauty functions at $p = 17$.

First note that we have

$$\begin{aligned}X(\mathbf{F}_{17}) = \{&(1:1:0), (1:-1:0)\,(0:0:1), (0:-1:1), (0:9:1), (0:2:1), (9:13:1), (9:9:1), \\ &(10:4:1), (10:3:1), (13:9:1), (5:15:1), (15:14:1), (8:0:1), (8:3:1), (7:10:1), \\ &(7:6:1), (4:9:1), (4:1:1), (1:1:1), \}\end{aligned}$$

1. Data for Section 6.4: Zeroes on $]\mathcal{U}_1[$

Define $\mathcal{U}_1 := Y_{\mathbf{F}_p} \cap \{q^0_y \neq 0\}$, where $Y : q^0 = 0$ is the affine chart $Z \neq 0$, so $q^0(x,y) = Q(x,y,1)$. This contains the residue disks of all points in $X(\mathbf{F}_{17})$ except for the points $(1:\pm 1:0)$ and $(1:1:1)$. We use the basis of differentials

$$\boldsymbol{\omega} := \begin{pmatrix} 1 \\ x \\ y \\ -\frac{160}{3}x^4 + \frac{736}{3}x^3 - \frac{16}{3}x^2y + \frac{436}{3}x^2 - \frac{440}{3}xy + \frac{68}{3}y^2 \\ -\frac{80}{3}x^3 + 44x^2 - \frac{40}{3}xy + \frac{68}{3}y^2 - 32 \\ -16x^2y + 28x^2 + 72xy - 4y^2 - \frac{160}{3}x + \frac{272}{3} \end{pmatrix} \frac{dx}{q^0_y}.$$

the base point $P_2 = (0,0)$ (which is a Teichmüller point) and the p-adic Tate classes encoded with respect to $\boldsymbol{\omega}$ by the matrices

$$Z_1 = \begin{pmatrix} 0 & -976 & -1104 & 10 & -6 & 18 \\ 976 & 0 & -816 & -3 & 1 & 3 \\ 1104 & 816 & 0 & -3 & 3 & -11 \\ -10 & 3 & 3 & 0 & 0 & 0 \\ 6 & -1 & -3 & 0 & 0 & 0 \\ -18 & -3 & 11 & 0 & 0 & 0 \end{pmatrix}, \qquad Z_2 = \begin{pmatrix} 0 & 112 & -656 & -6 & 6 & 6 \\ -112 & 0 & -2576 & 15 & 9 & 27 \\ 656 & 2576 & 0 & 3 & 3 & -3 \\ 6 & -15 & -3 & 0 & 0 & 0 \\ -6 & -9 & -3 & 0 & 0 & 0 \\ -6 & -27 & 3 & 0 & 0 & 0 \end{pmatrix}$$

Using the basis $\boldsymbol{\omega}$, we find an equivariant splitting of the Hodge filtration

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 224/3 & -880/3 & 0 \\ -880/3 & -1696/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We find the Hodge filtration of the connections $\mathcal{A}_{Z_i}$:

$$\begin{aligned} \eta_{Z_1} &= -(44x^2 + \tfrac{148}{3}xy + 8y^2)\tfrac{dx}{q_y^0} & \quad \eta_{Z_2} &= (-40x^2 + 148xy + 36y^2)\tfrac{dx}{q_y^0} \\ \boldsymbol{\beta}_{\mathrm{Fil},Z_1} &= (0, \tfrac{1}{2}, \tfrac{1}{2})^{\intercal} & \quad \boldsymbol{\beta}_{\mathrm{Fil},Z_2} &= (0, -\tfrac{1}{2}, -\tfrac{5}{2})^{\intercal} \\ \gamma_{\mathrm{Fil},Z_1} &= \tfrac{5}{6}y + \tfrac{3}{2}x & \quad \gamma_{\mathrm{Fil},Z_2} &= -\tfrac{5}{6}y - \tfrac{15}{2}x. \end{aligned}$$

The Frobenius structure is too long to be written down here. The matrices

$$T_i(x) := \begin{pmatrix} \theta_{Z_i}(x) & \Psi_1(Z_i\ , x) & \Psi_2(Z_i\ , x) & \Psi_3(Z_i\ , x) \\ \theta_{Z_1}(P_1) & \Psi_1(Z_1, P_1) & \Psi_2(Z_1, P_1) & \Psi_3(Z_1, P_1) \\ \theta_{Z_1}(P_3) & \Psi_1(Z_1, P_3) & \Psi_2(Z_1, P_3) & \Psi_3(Z_1, P_3) \\ \theta_{Z_1}(P_5) & \Psi_1(Z_1, P_5) & \Psi_2(Z_1, P_5) & \Psi_3(Z_1, P_5) \end{pmatrix}$$

are of the form

$$T_i(x) = \begin{pmatrix} \theta_{Z_i}(x) & \Psi_1(Z_i\ , x) & \Psi_2(Z_i\ , x) & \Psi_3(Z_i\ , x) \\ 16 \cdot 17 + 6 \cdot 17^2 + 10 \cdot 17^3 + 16 \cdot 17^4 & 352 & 818 & 294 \\ 4 \cdot 17 + 5 \cdot 17^2 + 2 \cdot 17^3 + 7 \cdot 17^4 & 162 & 406 & 150 \\ 17 + 4 \cdot 17^2 + 10 \cdot 17^3 + 3 \cdot 17^4 & -36 & -62 & -18 \end{pmatrix},$$

where $\Psi_j(Z, z) := \psi_j(E_1(z) \otimes_{K_p} E_{2,Z}(z))$.

The following table contains the zeroes of the functions $\det(T_1(x))$ and $\det(T_2(x))$ on $]\mathcal{U}_1[$, computed to precision $O(17^5)$; all of them are simple. In those disks which contained rational points, we used local parameters at these points, which means that a zero at 0 in the table below corresponds to the known rational point. The common zeroes are printed in bold.

Disk	$\det(T_1(x)) = 0$	$\det(T_2(x)) = 0$
$](0,0)[$	$\mathbf{0}$	$\mathbf{0}$
	$11 + 16 \cdot 17 + 6 \cdot 17^2 + 16 \cdot 17^3 + 9 \cdot 17^4$	$15 + 7 \cdot 17 + 12 \cdot 17^2 + 14 \cdot 17^3 + 15 \cdot 17^4$
$](0,-1)[$		
$](0,9)[$	$12 + 9 \cdot 17 + 3 \cdot 17^2 + 6 \cdot 17^3 + 3 \cdot 17^4$	$10 + 9 \cdot 17 + 5 \cdot 17^2 + 7 \cdot 17^3 + 8 \cdot 17^4$
	$16 + 10 \cdot 17 + 4 \cdot 17^2 + 11 \cdot 17^3 + 15 \cdot 17^4$	$14 + 3 \cdot 17 + 8 \cdot 17^2 + 7 \cdot 17^3 + 4 \cdot 17^4$
$](0,2)[$	$\mathbf{0}$	$\mathbf{0}$
	$13 + 12 \cdot 17 + 7 \cdot 17^2 + 7 \cdot 17^3 + 10 \cdot 17^4$	$2 + 7 \cdot 17 + 9 \cdot 17^2 + 5 \cdot 17^3 + 4 \cdot 17^4$
$](9,13)[$	$14 + 14 \cdot 17 + 11 \cdot 17^2 + 15 \cdot 17^3 + 17^4$	$8 + 4 \cdot 17 + 9 \cdot 17^2$
		$9 + 9 \cdot 17 + 4 \cdot 17^2 + 6 \cdot 17^3 + 9 \cdot 17^4$
$](9,9)[$	$\mathbf{0}$	$\mathbf{0}$
	$4 + 3 \cdot 17 + 8 \cdot 17^2 + 14 \cdot 17^3 + 15 \cdot 17^4$	
$](10,4)[$	$6 + 4 \cdot 17 + 2 \cdot 17^3 + 4 \cdot 17^4$	
	$11 + 2 \cdot 17 + 8 \cdot 17^2 + 4 \cdot 17^3 + 8 \cdot 17^4$	
$](10,3)[$		
$](13,9)[$	$7 + 17 + 5 \cdot 17^2 + 9 \cdot 17^4$	
	$15 + 6 \cdot 17^2 + 8 \cdot 17^4$	
$](5,15)[$	$7 + 15 \cdot 17 + 17^2 + 9 \cdot 17^3 + 3 \cdot 17^4$	$6 + 13 \cdot 17 + 14 \cdot 17^2 + 7 \cdot 17^3 + 10 \cdot 17^4$
		$16 + 13 \cdot 17 + 12 \cdot 17^2 + 14 \cdot 17^3 + 4 \cdot 17^4$
$](15,14)[$		
$](8,0)[$		
$](8,3)[$		$4 + 13 \cdot 17 + 13 \cdot 17^2 + 5 \cdot 17^3 + 12 \cdot 17^4$
		$15 + 8 \cdot 17 + 4 \cdot 17^2 + 2 \cdot 17^3$
$](7,10)[$	$\mathbf{0}$	$\mathbf{0}$
		$14 + 2 \cdot 17 + 3 \cdot 17^2 + 8 \cdot 17^3 + 6 \cdot 17^4$
	$14 + 2 \cdot 17 + 9 \cdot 17^2 + 13 \cdot 17^3$	
$](7,6)[$	$4 + 15 \cdot 17 + 7 \cdot 17^2 + 16 \cdot 17^3 + 4 \cdot 17^4$	$1 + 9 \cdot 17 + 4 \cdot 17^2 + 3 \cdot 17^3 + 7 \cdot 17^4$
	$14 + 8 \cdot 17 + 14 \cdot 17^2 + 12 \cdot 17^3 + 17^4$	
$](4,9)[$		
$](4,1)[$		

This recovers the rational points P_1, P_2, P_3, P_5 and shows that there are no other rational points in $]\mathcal{U}_1[$.

2. Data for Section 6.5: Zeroes on $]\mathcal{U}_2[-]\mathcal{U}_1[$

Define the affine chart $Y' : q^\infty = 0$ by $X \neq 0$, with coordinates $u := Z/X$ and $v := Y/X$, so $q^\infty(u,v) = Q(1,v,u)$. Let $\mathcal{U}_2 := Y'_{\mathbf{F}_p} \cap \{q_v^\infty \neq 0\}$, where $q_v^\infty = \partial q^\infty / \partial v$. We use the basis of differentials

$$\boldsymbol{\omega}' := \begin{pmatrix} -u \\ -1 \\ -v \\ \frac{768}{5}u^2v - \frac{448}{5}uv^2 - \frac{1536}{5}u^2 + 96uv + 16v^2 + \frac{2272}{15}u - \frac{1648}{15}v + \frac{1712}{15} \\ \frac{128}{7}u^2v^2 - \frac{5056}{35}u^2v + \frac{576}{35}uv^2 + \frac{7552}{35}u^2 - \frac{816}{7}uv + \frac{136}{7}v^2 + \frac{10736}{105}u - \frac{1072}{15}v - \frac{184}{105} \\ -\frac{448}{5}u^2v + \frac{288}{5}uv^2 + \frac{896}{5}u^2 - 80uv - 8v^2 - \frac{2272}{15}u + \frac{96}{5}v - \frac{1432}{15} \end{pmatrix} \frac{du}{q_v^\infty}$$

and the base point $P_6 = (0,-1)$. Note that the matrices Z_1 and Z_2 are the same as above.

We find the Hodge filtration of the connections $\mathcal{A}_{Z_i}$:

$$\begin{aligned} \eta_{\infty,Z_1} &= \left(\tfrac{4812}{35}uv - \tfrac{270}{7}v^2\right)\tfrac{du}{q_v^\infty} & \eta_{\infty Z_2} &= -\left(\tfrac{2412}{35}uv + \tfrac{138}{35}v^2\right)\tfrac{du}{q_v^\infty} \\ \boldsymbol{\beta}_{\mathrm{Fil},\infty,Z_1} &= (\tfrac{3}{8}, 0, \tfrac{3}{7})^\intercal & \boldsymbol{\beta}_{\mathrm{Fil},\infty,Z_2} &= (\tfrac{15}{56}, 0, \tfrac{12}{7})^\intercal \\ \gamma_{\mathrm{Fil},\infty,Z_1} &= \tfrac{6}{35}y + \tfrac{6}{35} & \gamma_{\mathrm{Fil},\infty,Z_2} &= -\tfrac{78}{35}y - \tfrac{78}{35}. \end{aligned}$$

The matrices

$$T_i(x) := \begin{pmatrix} \theta_{Z_i}(x) & \Psi_1(Z_i\ ,x) & \Psi_2(Z_i\ ,x) & \Psi_3(Z_i\ ,x) \\ \theta_{Z_1}(P_1) & \Psi_1(Z_1,P_1) & \Psi_2(Z_1,P_1) & \Psi_3(Z_1,P_1) \\ \theta_{Z_1}(P_3) & \Psi_1(Z_1,P_3) & \Psi_2(Z_1,P_3) & \Psi_3(Z_1,P_3) \\ \theta_{Z_1}(P_5) & \Psi_1(Z_1,P_5) & \Psi_2(Z_1,P_5) & \Psi_3(Z_1,P_5) \end{pmatrix}$$

are of the form

$$T_i(x) = \begin{pmatrix} \theta_{Z_i}(x) & \Psi_1(Z_i\ ,x) & \Psi_2(Z_i\ ,x) & \Psi_3(Z_i\ ,x) \\ 2\cdot 17+11\cdot 17^2+9\cdot 17^3+12\cdot 17^4 & \frac{116}{13} & \frac{340}{13} & \frac{100}{13} \\ 7\cdot 17+9\cdot 17^2+17^3+3\cdot 17^4 & \frac{134}{169} & \frac{1994}{169} & \frac{718}{169} \\ 2\cdot 17+11\cdot 17^2+7\cdot 17^3+5\cdot 17^4 & -\frac{1012}{169} & -\frac{2084}{169} & -\frac{746}{169} \end{pmatrix},$$

where $\Psi_j(Z,z) := \psi_j(E_1(z) \otimes_{K_p} E_{2,Z}(z))$.

Because we've already dealt with $]\mathcal{U}_1[$, it suffices to check the residue disks in $]\mathcal{U}_2[$ not contained in $]\mathcal{U}_1[$, namely the disks of $(1:\pm 1:0)$ (on the projective model $Q=0$). The following table contains the zeroes of the functions $\det(T_1'(u))$ and $\det(T_2'(u))$ on these disks, computed to precision $O(17^5)$; all of them are simple. The common zeroes are printed in bold.

Disk	$\det T_1'(u) = 0$	$\det T_2'(u) = 0$
$](0,-1)[$	$\mathbf{0}$	$\mathbf{0}$
	$2+6\cdot 17+6\cdot 17^2+6\cdot 17^3+5\cdot 17^4$	$5+17+2\cdot 17^2+9\cdot 17^3+12\cdot 17^4$
$](0,1)[$	$\mathbf{0}$	$\mathbf{0}$
	$14+12\cdot 17+11\cdot 17^2+6\cdot 17^3+14\cdot 17^4$	$4+2\cdot 17+8\cdot 17^2+10\cdot 17^3+4\cdot 17^4$

This recovers the points P_6 and P_4 and shows that there are no other rational points in the residue disks we considered.

3. Data for Section 6.6

The Coleman integrals $\int_b^{P_0} \boldsymbol{\omega}$ are

$$\begin{gathered} 3\cdot 17+10\cdot 17^2+6\cdot 17^3+3\cdot 17^4, 8\cdot 17+2\cdot 17^2+15\cdot 17^3+10\cdot 17^4, \\ 6\cdot 17+3\cdot 17^2+11\cdot 17^3+15\cdot 17^4, 10+11\cdot 17+15\cdot 17^2+17^3+11\cdot 17^4, \\ 3+7\cdot 17+4\cdot 17^2+3\cdot 17^3+12\cdot 17^4, 12+15\cdot 17+13\cdot 17^2+13\cdot 17^3+16\cdot 17^4 \end{gathered}$$

As described in Section 5.5, this suffices to write down quadratic Chabauty functions on the disk $]P_0]$, where the Frobenius lift we used is not defined. The quadratic Chabauty matrices are as in Section 1 of this document, except that the first rows are computed with respect to the Frobenius structure of

$A_{Z_i}(P_0, x)$ as described in Section 5.5 and Section 6.6 The only zeros of the resulting functions are at the rational point P_0.

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