COUNTING POINTS ON CURVES USING A MAP TO $\mathbf{P}^1$.

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Abstract. We introduce a new algorithm to compute the zeta function of a curve over a finite field. This method extends Kedlaya's algorithm to a very general class of curves using a map to the projective line. We develop all the necessary bounds and analyse the complexity of the algorithm.

1. Introduction

Let $\mathbf{F}_q$ denote the finite field of characteristic p and cardinality $q = p^n$. Moreover, let $\mathbf{Q}_p$ denote the field of p-adic numbers and $\mathbf{Q}_q$ its unique unramified extension of degree n. As usual, let $\sigma \in \mathrm{Gal}(\mathbf{Q}_q/\mathbf{Q}_p)$ denote the unique element that lifts the pth power Frobenius map on $\mathbf{F}_q$. Finally, let $\mathbf{Z}_q$ denote the ring of integers of $\mathbf{Q}_q$, so that $\mathbf{Z}_q/p\mathbf{Z}_q \cong \mathbf{F}_q$. Suppose that X is a smooth proper algebraic curve of genus g over $\mathbf{F}_q$. Recall that the zeta function of X is defined as

$$Z(X,T) = \exp\Big(\sum_{i=1}^{\infty} |X(\mathbf{F}_{q^i})| \frac{T^i}{i}\Big).$$

It follows from the Weil conjectures that $Z(X,T)$ is of the form

$$\frac{\chi(T)}{(1-T)(1-qT)},$$

with $\chi(T) \in \mathbf{Z}[T]$ a polynomial of degree $2g$, the inverse roots of which have complex absolute value $q^{\frac{1}{2}}$ and are permuted by the map $t \to q/t$. Moreover, by the Lefschetz formula for rigid cohomology, we have that

$$\chi(T) = \det(1 - T\,\mathrm{F}_p^n \,|H^1_{\mathrm{rig}}(X)),$$

where F_p denotes the pth power Frobenius map.

In [Ked01], Kedlaya showed how $Z(X,T)$ can be determined efficiently, in the case when X is a hyperelliptic curve and the characteristic p is odd, by explicitly computing the action of F_p on $H^1_{\mathrm{rig}}(X)$. His algorithm was then extended to characteristic 2 [DV06b] and also to superelliptic curves [GG01], C_{ab} curves [DV06a] and nondegenerate curves [CDV06]. However, for C_{ab} and nondegenerate curves these algorithms have proved a lot less efficient in practice than for hyperelliptic and superelliptic curves. The main reason for this is that the algorithms for $C_{a,b}$ and nondegenerate curves use a more complicated Frobenius lift that does not send x to x^p anymore. Moreover, in the case of nondegenerate curves, the linear algebra that is used to compute in the cohomology is not very efficient and when the curve admits a low degree map to $\mathbf{P}^1$, as is the case for most nondegenerate curves, this is not fully exploited.

The aim of this paper is to propose a new algorithm for computing $Z(X,T)$ that avoids these problems and can be applied to more general curves as well. Our

approach combines Kedlaya's original algorithm and Lauder's fibration method [Lau06]. In the work of Lauder, the Frobenius lift is computed by solving a p-adic differential equation. For curves it turns out to be more efficient to compute the Frobenius lift directly by Hensel lifting as in Kedlaya's algorithm, especially since this allows one to avoid the radix conversions that take up most of the time in the examples of the fibration method computed by Walker in his thesis [Wal10].

Our approach can roughly be described as follows. We start with a finite separable map x from the curve X to the projective line. After removing the ramification locus of x from the curve, we can choose a Frobenius lift that sends x to x^p, which we compute by Hensel lifting as in Kedlaya's algorithm. We then compute in the cohomology as in Lauder's fibration method to find the matrix of Frobenius and the zeta function of X. We show that the resulting algorithm is more general, has slightly better complexity and is more practical than the algorithm from [CDV06].

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2. Lifting the curve and Frobenius

Let $x : X \to \mathbf{P}^1_{\mathbf{F}_q}$ be a finite separable map of degree d_x and $y : X \to \mathbf{P}^1_{\mathbf{F}_q}$ a rational function that generates the function field of X over $\mathbf{F}_q(x)$, such that $Q(x, y) = 0$ where $Q \in \mathbf{F}_q[x, y]$ is irreducible and monic of degree d_x in the variable y. The degree of $\mathcal{Q}$ in the variable x will be denoted by d_y. Let $\mathcal{Q} \in \mathbf{Z}_q[x, y]$ be a lift of Q to characteristic 0 containing the same monomials in its support as Q.

Proposition 2.1. *The ring $\mathcal{A} = \mathbf{Z}_q[x, y]/(\mathcal{Q})$ is a free module of rank d over $\mathbf{Z}_q[x]$ and a basis is given by $[1, y, \ldots, y^{d_x-1}]$.*

Definition 2.2. *We let $\Delta(x) \in \mathbf{Z}_q[x]$ denote the discriminant of $\mathcal{Q}$ with respect to the variable y and $r(x) \in \mathbf{Z}_q[x]$ the squarefree polynomial $r = \Delta/(\gcd(\Delta, \frac{d\Delta}{dx}))$. Note that $\Delta(x) \neq 0 \pmod p$ since $x : X \to \mathbf{P}^1_{\mathbf{F}_q}$ is separable. We denote*

$$\mathcal{S} = \mathbf{Z}_q[x, \frac{1}{r}], \qquad \mathcal{R} = \mathbf{Z}_q[x, \frac{1}{r}, y]/(\mathcal{Q}),$$

and write $\mathcal{V} = \operatorname{Spec} \mathcal{S}$, $\mathcal{U} = \operatorname{Spec} \mathcal{R}$, so that x defines a finite étale morphism from $\mathcal{U}$ to $\mathcal{V}$. Finally, we let $U = \mathcal{U} \otimes_{\mathbf{Z}_q} \mathbf{F}_q$, $V = \mathcal{V} \otimes_{\mathbf{Z}_q} \mathbf{F}_q$ denote the special fibres and $\mathbb{U} = \mathcal{U} \otimes_{\mathbf{Z}_q} \mathbf{Q}_q$, $\mathbb{V} = \mathcal{V} \otimes_{\mathbf{Z}_q} \mathbf{Q}_q$ the generic fibres of $\mathcal{U}$ and $\mathcal{V}$, respectively.

Assumption 1. *We will assume that:*

1. *There exists a smooth proper curve $\mathcal{X}$ over $\mathbf{Z}_q$ and a smooth relative divisor $\mathcal{D}_\mathcal{X}$ on $\mathcal{X}$ such that $\mathcal{U} = \mathcal{X} \setminus \mathcal{D}_\mathcal{X}$.*
2. *There exists a smooth relative divisor $\mathcal{D}_{\mathbf{P}^1}$ on $\mathbf{P}^1_{\mathbf{Z}_q}$ such that $\mathcal{V} = \mathbf{P}^1_{\mathbf{Z}_q} \setminus \mathcal{D}_{\mathbf{P}^1}$.*

Remark 2.3. *A relative divisor $\mathcal{D}$ on a smooth curve over $\mathbf{Z}_q$ is smooth over $\mathbf{Z}_q$ if and only if it is reduced and all of the points in its support are smooth over $\mathbf{Z}_q$, or equivalently if and only if it reduces modulo p to a reduced divisor D. Hence by Assumption 1, all ramification and branch points of the map x restricted to $\mathbb{X}$ are distinct modulo p.*

We write $\mathbb{X} = \mathcal{X} \otimes \mathbf{Q}_q$ for the generic fibre of $\mathcal{X}$. At every point $P \in \mathcal{X} \setminus \mathcal{U}$, we let z_P denote an étale local coordinate on $\mathcal{X}$. By a slight abuse of notation, we write $\operatorname{ord}_P(\cdot)$ for the discrete valuation on $\mathcal{O}_{\mathbb{X},P}$. We let e_P denote the ramification index of the map x. Note that the e_P are the same on X as on $\mathbb{X}$, since they can only increase under reduction modulo p, but add up to d_x in every fibre.

Assumption 2. *We will assume that the zero locus of $\mathcal{Q}(x,y)$ in $\mathbf{A}^2_{\mathbf{Q}_q}$ is smooth.*

Proposition 2.4. *The element*

$$s(x,y) = r(x)/\frac{\partial \mathcal{Q}}{\partial y}$$

of $\mathbf{Q}_q(x,y)$ is contained in $\mathcal{A}$.

Proof. For $k \in \mathbf{N}$, we let W_k denote the free $\mathbf{Z}_q[x]$-module of polynomials in $\mathbf{Z}_q[x,y]$ of degree at most $k-1$ in the variable y. Let Σ be the matrix of the $\mathbf{Z}_q[x]$-module homomorphism:

$$W_{d-1} \oplus W_d \to W_{2d-1}, \qquad (a,b) \mapsto a\mathcal{Q} + b\frac{\partial \mathcal{Q}}{\partial y}, \tag{1}$$

with respect to the bases $[1, y, \ldots, y^{d_x-2}]$, $[1, y, \ldots, y^{d_x-1}]$ and $[1, y, \ldots, y^{2d_x-2}]$. By definition we have $\Delta = \det(\Sigma)$, so that Δ is contained in the image of (1) and $\Delta(x)/\frac{\partial \mathcal{Q}}{\partial y}$ is contained in $\mathcal{A}$. By Assumption 2, the ring $\mathcal{A} \otimes \mathbf{Q}_q$ is the integral closure of $\mathbf{Q}_q[x]$ in $\mathbf{Q}_q(x,y)$. Note that the basis $[1, y, \ldots, y^{d_x-1}]$ of $\mathcal{A} \otimes \mathbf{Q}_q$ is therefore an integral basis for $\mathbf{Q}_q(x,y)$ over $\mathbf{Q}_q[x]$. Since $\mathcal{Q}$ is monic in y, for any irreducible polynomial $\pi \in \mathbf{Q}_q[x]$ the element $\frac{\partial \mathcal{Q}}{\partial y}/\pi$ of $\mathbf{Q}_q(x,y)$ is not integral at the place (π), and hence its inverse $\pi/\frac{\partial \mathcal{Q}}{\partial y}$ is integral (even zero) at (π). Hence s is contained in $\mathcal{A}$. □

Definition 2.5. *We denote the ring of overconvergent functions on $\mathcal{U}$ by*

$$\mathcal{R}^\dagger = \mathbf{Z}_q\langle x, \frac{1}{r}, y\rangle^\dagger/(\mathcal{Q}).$$

Note that $\mathcal{R}^\dagger$ is a free module of rank d_x over $\mathcal{S}^\dagger = \mathbf{Z}_q\langle x, 1/r\rangle^\dagger$ and that a basis is given by $[y^0, \ldots, y^{d_x-1}]$. A Frobenius lift $\mathrm{F}_p : \mathcal{R}^\dagger \to \mathcal{R}^\dagger$ is defined as a σ-semilinear ringhomomorphism that reduces modulo p to the pth power Frobenius map.

Theorem 2.6. *There exists a Frobenius lift $\mathrm{F}_p : \mathcal{R}^\dagger \to \mathcal{R}^\dagger$ for which $\mathrm{F}_p(x) = x^p$.*

Proof. Define sequences $(\alpha_i)_{i\geq 0}$, $(\beta_i)_{i\geq 0}$, with $\alpha_i \in S^\dagger$ and $\beta_i \in \mathcal{R}^\dagger$, by the following recursion:

$$\begin{aligned}
&\alpha_0 = \frac{1}{r^p},\\
&\beta_0 = y^p,\\
&\alpha_{i+1} = \alpha_i(2 - \alpha_i r^\sigma(x^p)) && (\mathrm{mod}\ \ p^{2^{i+1}}),\\
&\beta_{i+1} = \beta_i - \mathcal{Q}^\sigma(x^p, \beta_i)s^\sigma(x^p, \beta_i)\alpha_i && (\mathrm{mod}\ \ p^{2^{i+1}}).
\end{aligned}$$

Then one easily checks that the σ-semilinear ringhomomorphism $\mathrm{F}_p : \mathcal{R}^\dagger \to \mathcal{R}^\dagger$ defined by

$$\mathrm{F}_p(x) = x^p, \qquad \mathrm{F}_p\Big(\frac{1}{r}\Big) = \lim_{i\to\infty} \alpha_i, \qquad \mathrm{F}_p(y) = \lim_{i\to\infty} \beta_i,$$

is a Frobenius lift. □

Proposition 2.7. *Let $G \in M_{d_x \times d_x}(\mathbf{Z}_q[x, 1/r])$ denote the matrix such that*

$$d(y^j) = \sum_{i=0}^{d_x-1} G_{i+1,j+1} y^i dx,$$

for all $0 \leq j \leq d_x - 1$ *as* 1*-forms on* $\mathcal{U}$*. Then we can write* $G = M/r$ *with* $M \in M_{d_x \times d_x}(\mathbf{Z}_q[x])$*.*

Proof. This follows from the formula

$$d(y^j) = -jy^{(j-1)}\Big(\frac{s}{r}\Big)\frac{\partial \mathcal{Q}}{\partial x}dx. \tag{2}$$

□

In the terminology of the fibration method, Gdx is the matrix of the Gauss–Manin connection ∇ on $\mathbf{R}^0 x_*(O_{\mathbb{U}})$ with respect to the basis $[1, y, \ldots, y^{d_x-1}]$. By Proposition 2.7, this matrix has at most a simple pole at all points $\neq \infty$ in the support of $\mathcal{D}_{\mathbf{P}^1}$. At $x = \infty$ we will have to make a change of basis for this to be the case.

Assumption 3. *We will assume that a matrix* $W^\infty \in Gl_{d_x}(\mathbf{Z}_q[x, x^{-1}])$ *is known such that if we denote* $b_j^\infty = \sum_{i=0}^{d_x-1} W^\infty_{i+1,j+1} y^i$ *for all* $0 \leq j \leq d_x - 1$*, then* $[b_0^\infty, \ldots, b_{d_x-1}^\infty]$ *is an integral basis for* $\mathbf{Q}_q(x, y)$ *over* $\mathbf{Q}_q[x^{-1}]$*.*

Proposition 2.8. *Let* $G^\infty \in M_{d_x \times d_x}(\mathbf{Z}_q[x, x^{-1}, 1/r])$ *denote the matrix such that*

$$db_j^\infty = \sum_{i=0}^{d_x-1} G^\infty_{i+1,j+1} b_i^\infty dx,$$

for all $0 \leq j \leq d_x - 1$ *as* 1*-forms on* $\mathcal{U}$*. Then* $G^\infty dx$ *has at most a simple pole at* $x = \infty$*.*

Proof. We denote $t = 1/x$ and let $H \in M_{d_x \times d_x}(\mathbf{Q}_q(t))$ be defined by $H(t)dt = G^\infty(x)dx$. Note that $\mathrm{ord}_P(dt/t) = -1$ at every point $P \in \mathcal{X} \setminus \mathcal{U}$ lying over $t = 0$. At every such P and for all $0 \leq i \leq d_x - 1$ we clearly have $\mathrm{ord}_P(db_i^\infty) \geq 0$, so that $\mathrm{ord}_P(tdb_i^\infty) - \mathrm{ord}_P(dt) \geq 1$. Since $[b_0^\infty, \ldots, b_{d_x-1}^\infty]$ is an integral basis for $\mathbf{Q}_q(x, y)$ over $\mathbf{Q}_q[t]$, we conclude that tH does not have a pole at $t = 0$, so that Hdt has at most a simple pole there. □

Definition 2.9. *Let* $x_0 \neq \infty$ *be a geometric point of* $\mathbf{P}^1(\bar{\mathbf{Q}}_q)$*. The exponents of* Gdx *at* x_0 *are defined as the eigenvalues of the residue matrix* $(x - x_0)G|_{x=x_0}$*. Moreover, the exponents of* $G^\infty dx$ *at* $x = \infty$ *are defined as its exponents at* $t = 0$*, after substituting* $x = 1/t$*.*

Proposition 2.10. *The exponents of* Gdx *at any point* $x_0 \neq \infty$ *and the exponents of* $G^\infty dx$ *at* $x = \infty$ *are elements of* $\mathbf{Q} \cap \mathbf{Z}_p$ *and are contained in the interval* $[0, 1)$*.*

Proof. Let $\lambda \in \bar{\mathbf{Q}}_q$ denote an exponent of Gdx at $x_0 \neq \infty$. Then there exists $f = \sum_{i=0}^{d_x-1} a_i y^i$ with $a_0, \ldots, a_{d_x-1} \in \bar{\mathbf{Q}}_q$ such that

$$df = \big(\frac{\lambda f}{x - x_0} + g\big)dx \tag{3}$$

as 1-forms on $\mathbb{U} \otimes \bar{\mathbf{Q}}_q$, where $g \in \mathcal{O}(\mathbb{U} \otimes \bar{\mathbf{Q}}_q)$ satisfies $\mathrm{ord}_P(g) \geq 0$ at all points $P \in x^{-1}(x_0)$. Note that for at least one $P \in x^{-1}(x_0)$ we have $\mathrm{ord}_P(f) < \mathrm{ord}_P(x - x_0)$, since otherwise $f/(x-x_0)$ would be integral over $\mathbf{Q}_q[x]$, contradicting Assumption 2. For such a P, dividing by f in (3) and taking residues, we obtain

$$\mathrm{ord}_P(f) = \lambda \, \mathrm{ord}_P(x - x_0).$$

Since $0 \leq \mathrm{ord}_P(f) < \mathrm{ord}_P(x - x_0) = e_P$, we see that $\lambda \in \mathbf{Q} \cap [0, 1)$. By Assumption 1, elements of $\mathcal{S}$ have p-adically integral Laurent series expansions at x_0, so that $(x - x_0)G|_{x=x_0} \in M_{d_x \times d_x}(\mathbf{Z}_q)$. Since p-adically integral matrices have p-adically integral eigenvalues, we conclude that $\lambda \in \mathbf{Z}_p$. To obtain the same result for the exponents of $G^\infty dx$ at $x = \infty$, replace x_0 by ∞ and $(x - x_0)$ by $t = 1/x$ in the argument. □

Definition 2.11. *For a geometric point $x_0 \in \mathbf{P}^1(\bar{\mathbf{Q}}_q)$, we let $\mathrm{ord}_{x_0}(\cdot)$ denote the discrete valuation on $\bar{\mathbf{Q}}_q(x)$ corresponding to x_0. We extend these definitions to matrices over $\bar{\mathbf{Q}}_q(x)$ by taking the minimum over their entries.*

Proposition 2.12. *Let $N \in \mathbf{N}$ be a positive integer.*

(1) *The element $\mathrm{F}_p(1/r)$ of $\mathcal{S}^\dagger$ is congruent modulo p^N to*

$$\sum_{i=p}^{pN} \frac{\rho_i(x)}{r^i},$$

where $\rho_i \in \mathbf{Z}_q[x]$ satisfies $\deg(\rho_i) < \deg(r)$ for all $p \leq i \leq pN$.

(2) *For all $0 \leq i \leq d_x - 1$, the element $\mathrm{F}_p(y^i)$ of $\mathcal{R}^\dagger$ is congruent modulo p^N to $\sum_{j=0}^{d-1} \phi_{i,j}(x) y^j$, where*

$$\phi_{i,j} = \sum_{k=0}^{p(N-1)} \frac{\phi_{i,j,k}(x)}{r^k}$$

for all $0 \leq j \leq d_x - 1$ and $\phi_{i,j,k} \in \mathbf{Z}_q[x]$ satisfies

$$\deg(\phi_{i,j,0}) \leq -\mathrm{ord}_\infty(W^\infty) - p\,\mathrm{ord}_\infty((W^\infty)^{-1}),$$
$$\deg(\phi_{i,j,k}) < \deg(r),$$

for all $0 \leq j \leq d_x - 1$ and $1 \leq k \leq p(N-1)$.

(3) *For all $0 \leq i \leq d_x - 1$, the element $\mathrm{F}_p(y^i/r)$ of $\mathcal{R}^\dagger$ is congruent modulo p^N to $\sum_{j=0}^{d_x-1} \psi_{i,j}(x)(y^j/r)$, where*

$$\psi_{i,j} = \sum_{k=0}^{pN-1} \frac{\psi_{i,j,k}(x)}{r^k}$$

for all $0 \leq j \leq d_x - 1$ and $\psi_{i,j,k} \in \mathbf{Z}_q[x]$ satisfies

$$\deg(\psi_{i,j,0}) \leq -\mathrm{ord}_\infty(W^\infty) - p\,\mathrm{ord}_\infty((W^\infty)^{-1}) - (p-1)\deg(r),$$
$$\deg(\psi_{i,j,k}) < \deg(r),$$

for all $0 \leq j \leq d_x - 1$ and $1 \leq k \leq pN - 1$.

Proof.

(1) Since $r^\sigma(x^p) \equiv r^p \pmod{p}$, this follows from

$$\mathrm{F}_p(\frac{1}{r}) = \frac{1}{r^\sigma(x^p)} = \frac{1}{r^p}\Big(1 - \frac{r^p - r^\sigma(x^p)}{r^p}\Big)^{-1} = \frac{1}{r^p}\sum_{i=0}^{\infty}\Big(\frac{r^p - r^\sigma(x^p)}{r^p}\Big)^i.$$

(2) The matrix $\Phi = (\phi_{i,j}) \in M_{d_x \times d_x}(\mathcal{S}^\dagger)$ defines a pth power Frobenius structure on $\mathbf{R}^0 x_*(O_{\mathbb{U}})$. By definition we have $\mathrm{ord}_p(\Phi) \geq 0$ and by Poincaré duality we find that $\mathrm{ord}_p(\Phi^{-1}) \geq 0$ as well. The result now follows from a theorem of Kedlaya and the author [KT12, Corollary 2.6] using Proposition 2.10.

(3) Analogous to (2).

□

3. Computing (in) the cohomology

Definition 3.1. *The rigid cohomology of U in degree 1 can be defined as*

$$H^1_{rig}(U) = \operatorname{coker}(d : \mathcal{R}^\dagger \to \Omega^1(\mathbb{U}) \otimes \mathcal{R}^\dagger).$$

Theorem 3.2.

$$H^1_{rig}(U) \cong H^1_{dR}(\mathbb{U})$$

Proof. This follows as a special case from the comparison theorem between rigid and de Rham cohomology of Baldassarri and Chiarellotto [BC94], since by Assumption 1 $\mathcal{D}_\mathcal{X}$ is smooth over $\mathbf{Z}_q$. □

We can effectively reduce any 1-form to one of low pole order using linear algebra following work of Lauder [Lau06]. The procedure consists of two parts, reducing the pole order at the points not lying over $x = \infty$ and at those lying over $x = \infty$, respectively. From now on we let r' denote the polynomial $\frac{dr}{dx}$. We start with the points not lying over $x = \infty$.

Proposition 3.3. *For all $\ell \in \mathbf{N}$ and every vector $w \in \mathbf{Q}_q[x]^{\oplus d_x}$, there exist vectors $u, v \in \mathbf{Q}_q[x]^{\oplus d_x}$ with $\deg(v) < \deg(r)$, such that*

$$\frac{\sum_{i=0}^{d_x-1} w_i y^i}{r^\ell}\frac{dx}{r} = d\Big(\frac{\sum_{i=0}^{d_x-1} v_i y^i}{r^\ell}\Big) + \frac{\sum_{i=0}^{d_x-1} u_i y^i}{r^{\ell-1}}\frac{dx}{r}.$$

Proof. Note that since r is separable, r' is invertible in the ring $\mathbf{Q}_q[x]/(r)$. One checks that v has to satisfy the $d_x \times d_x$ linear system

$$\Big(\frac{M}{r'} - \ell I\Big)v \equiv \frac{w}{r'} \pmod r$$

over $\mathbf{Q}_q[x]/(r)$. However, since $\ell \geq 1$ is not an exponent of Gdx by Proposition 2.10, we have that $\det(\ell I - M/r')$ is invertible in $\mathbf{Q}_q[x]/(r)$, so that this system has a unique solution v. We now take

$$u = \frac{w - \Big(M - \ell r' I\Big)v}{r} - \frac{dv}{dx}.$$

□

We now move on to the points lying over $x = \infty$.

Proposition 3.4. *For every vector $w \in \mathbf{Q}_q[x, x^{-1}]^{\oplus d_x}$ with*

$$\operatorname{ord}_\infty(w) \leq -\deg(r),$$

there exist vectors $u, v \in \mathbf{Q}_q[x, x^{-1}]^{\oplus d_x}$ with $\operatorname{ord}_\infty(u) > \operatorname{ord}_\infty(w)$ such that

$$(\sum_{i=0}^{d_x-1} w_i b_i^\infty)\frac{dx}{r} = d(\sum_{i=0}^{d_x-1} v_i b_i^\infty) + (\sum_{i=0}^{d_x-1} u_i b_i^\infty)\frac{dx}{r}.$$

Proof. We still denote $t = 1/x$. By Proposition 2.8, we can expand

$$G^\infty dx = \Big(\frac{G^\infty_{-1}}{t} + G^\infty_0 + \dots\Big)dt,$$

where $G^\infty_i \in M_{d_x \times d_x}(\mathbf{Q}_q)$ for all $i \geq -1$. Writing $m = -\operatorname{ord}_\infty(w) - \deg(r) + 1$, we can also expand

$$w\frac{dx}{r} = \sum_{j=-(m+1)}^{\infty} \bar{w}_j t^j dt,$$

where $\bar{w}_j \in \mathbf{Q}_q^{\oplus d_x}$ for all $j \geq -(m+1)$. Note that $m \geq 1$. By Proposition 2.10, we have that $\det(mI - G^\infty_{-1})$ is nonzero, so that the linear system

$$(G^\infty_{-1} - mI)\bar{v} = \bar{w}_{-(m+1)}$$

has a unique solution $\bar{v} \in \mathbf{Q}_q^{\oplus d_x}$. We can now take

$$v = \bar{v}x^m, \qquad\qquad u = w - r(G^\infty v + \frac{dv}{dx}).$$

□

Remark 3.5. *Note that when* $\operatorname{ord}_\infty(w) \leq \operatorname{ord}_0(W^\infty) - \deg(r) + 1$*, we have that* $\operatorname{ord}_0(v) \geq -\operatorname{ord}_0(W^\infty)$*, so that the function* $\sum_{i=0}^{d_x-1} v_i b_i^\infty$ *only has poles at points lying over* $x = \infty$*.*

We now give an explicit description of the cohomology space $H^1_{\mathrm{rig}}(U)$.

Theorem 3.6. *Define the following* $\mathbf{Q}_q$*-vector spaces:*

$$E_0 = \{\Big(\sum_{i=0}^{d_x-1} u_i(x)y^i\Big)\frac{dx}{r} \quad : u \in \mathbf{Q}_q[x]^{\oplus d_x}\},$$

$$E_\infty = \{\Big(\sum_{i=0}^{d_x-1} u_i(x,x^{-1})b_i^\infty\Big)\frac{dx}{r} \quad : u \in \mathbf{Q}_q[x,x^{-1}]^{\oplus d_x}, \operatorname{ord}_\infty(u) > \operatorname{ord}_0(W^\infty) - \deg(r) + 1\},$$

$$B_0 = \{\sum_{i=0}^{d_x-1} v_i(x)y^i \quad : v \in \mathbf{Q}_q[x]^{\oplus d_x}\},$$

$$B_\infty = \{\sum_{i=0}^{d_x-1} v_i(x,x^{-1})b_i^\infty \quad : v \in \mathbf{Q}_q[x,x^{-1}]^{\oplus d_x}, \operatorname{ord}_\infty(v) > \operatorname{ord}_0(W^\infty)\}.$$

Then $E_0 \cap E_\infty$ *and* $d(B_0 \cap B_\infty)$ *are finite dimensional* $\mathbf{Q}_q$*-vector spaces and*

$$H^1_{rig}(U) \cong (E_0 \cap E_\infty)/d(B_0 \cap B_\infty).$$

Proof. First, note that elements of E_0, B_0 have bounded poles everywhere but at the points lying over $x = \infty$ and elements of E_∞, B_∞ everywhere but at the points lying over $x = 0$. So elements of $E_0 \cap E_\infty$ and $d(B_0 \cap B_\infty)$ have bounded poles everywhere on $\mathbb{X}$. Hence these vector spaces are contained in the space of global sections of some line bundle on $\mathbb{X}$ and are therefore finite dimensional.

Next, we show that every class in $H^1_{\mathrm{rig}}(U)$ can be represented by a 1-form in $E_0 \cap E_\infty$. Note that by Theorem 3.2 we can restrict to classes in $H^1_{\mathrm{dR}}(\mathbb{U})$. Now every such class can be represented by a 1-form in E_0 by (repeatedly) applying

Proposition 3.3. Then we change basis by the matrix W^∞ from Assumption 3. Observe that this change of basis might introduce a pole at $x = 0$. Now our cohomology class can be represented by 1-form in $E_0 \cap E_\infty$ by (repeatedly) applying Proposition 3.4 and Remark 3.5.

Finally, we have to prove that if a 1-form $\omega \in E_0 \cap E_\infty$ is exact, then it lies in $d(B_0 \cap B_\infty)$. So let $\omega \in E_0 \cap E_\infty$ denote such an exact 1-form. From Assumption 2 and the definition of $[b_0^\infty, \ldots, b_{d_x-1}^\infty]$, it follows that $\mathrm{ord}_P(\omega) \geq -1$ all points P not lying over $x = \infty$ and $\mathrm{ord}_P(\omega) \geq \mathrm{ord}_0(W^\infty + 1)e_P - 1$ at all points P lying over $x = \infty$. Note that the exterior derivative lowers the order by at most 1. So if $\omega = df$ for some $f \in \mathcal{O}(\mathbb{U})$, then $\mathrm{ord}_P(f) \geq 0$ at all points P not lying over $x = \infty$ and $\mathrm{ord}_P(f) \geq (\mathrm{ord}_0(W^\infty) + 1)e_P$ at all points P lying over $x = \infty$. Using Assumption 2 and the definition of $[b_0^\infty, \ldots, b_{d_x-1}^\infty]$ again, it follows that f is an element of $B_0 \cap B_\infty$. □

Note that by the proof of Theorem 3.6, we can effectively reduce any 1-form to one in $E_0 \cap E_\infty$ with the same cohomology class. However, the reduction procedure will introduce p-adic denominators and therefore suffer from loss of p-adic precision. In the following two propositions we bound these denominators. Our bounds and their proofs generalise the ones from [Ked01].

Proposition 3.7. *Let $\omega \in \Omega^1(\mathcal{U})$ be of the form*

$$\omega = \frac{\sum_{i=0}^{d_x-1} w_i y^i}{r^\ell} \frac{dx}{r},$$

where $\ell \in \mathbf{N}$ and $w \in \mathbf{Z}_q[x]^{\oplus d_x}$ satisfies $\deg(w) < \deg(r)$. We define

$$e = \max\{e_P | P \in \mathcal{X} \setminus \mathcal{U}, x(P) \neq \infty\}.$$

If we represent the class of ω in $H^1_{rig}(U)$ by

$$\Big(\sum_{i=0}^{d_x-1} u_i y^i\Big) \frac{dx}{r},$$

with $u \in \mathbf{Q}_q[x]^{\oplus d_x}$ as in the proof of Theorem 3.6, then

$$p^{\lfloor \log_p(\ell e) \rfloor} u \in \mathbf{Z}_q[x]^{\oplus d_x}.$$

Proof. We have

$$\omega = df + \Big(\sum_{i=0}^{d_x-1} u_i y^i\Big) \frac{dx}{r}$$

with $f = \sum_{j=1}^{\ell} (\sum_{i=0}^{d_x-1} (v_j)_i y^i)/r^j$, where $v_j \in \mathbf{Q}_q[x]^{\oplus d_x}$ satisfies $\deg(f_j) < \deg(r)$ for all $1 \leq j \leq \ell$. Note that it is sufficient to show that $p^{\lfloor \log_p(\ell e) \rfloor} f \in \mathcal{R}$. By Assumption 1, we have that

$$\mathcal{O}(\mathcal{X} - x^{-1}(\infty))/(r)^k \cong \prod_{P \in \mathcal{X} \setminus \mathcal{U}, x(P) \neq \infty} \mathcal{O}_{\mathcal{X},P}/(z_P^{e_P})^k,$$

for all $k \in \mathbf{N}$. Moreover, we have that $\mathcal{O}(\mathbb{X} - x^{-1}(\infty)) \cong \mathcal{A} \otimes \mathbf{Q}_q$ by Assumption 2. To show that $p^{\lfloor \log_p(\ell e) \rfloor} f$ is integral, it is therefore enough to show that for every $P \in \mathcal{X} \setminus \mathcal{U}$ with $x(P) \neq 0$, the Laurent series expansion

$$a_{-\ell e_P} z_P^{-\ell e_P} + \ldots + a_{-e_P-1} z_P^{-e_P-1} + \mathcal{O}(z_P^{-e_P})$$

of $p^{\lfloor \log_p(\ell e)\rfloor} f$ is integral. However, the differential df has a pole of order at most $\ell e_P + 1$ at P, and its Laurent series expansion

$$\Big(b_{-\ell e_P - 1} z_P^{-\ell e_P - 1} + \ldots + b_{-e_P-2} z_P^{-e_P-2} + \mathcal{O}(z_P^{-e_P-1})\Big) dz_P$$

is integral since ω is integral. The worst denominator we get by integrating this series is therefore $p^{\lfloor \log_p(\ell e)\rfloor}$ and the result follows. □

Proposition 3.8. *Let $\omega \in \Omega^1(\mathcal{U})$ be of the form*

$$\omega = (\sum_{i=0}^{d_x - 1} w_i(x, x^{-1}) b_i^{\infty}) \frac{dx}{r},$$

where $w \in \mathbf{Z}_q[x, x^{-1}]^{\oplus d_x}$ satisfies $\mathrm{ord}_\infty(w) \leq \mathrm{ord}_0(W^\infty) - \deg(r) + 1$. We define

$$m = -\mathrm{ord}_\infty(w) - \deg(r) + 1,$$
$$e_\infty = \max\{e_P | P \in \mathcal{X} \setminus \mathcal{U}, x(P) = \infty\}.$$

If we represent the class of ω in $H^1_{rig}(U)$ by

$$\Big(\sum_{i=0}^{d_x - 1} u_i y^i\Big) \frac{dx}{r},$$

with $u \in \mathbf{Q}_q[x, x^{-1}]^{\oplus d_x}$ such that $\mathrm{ord}_\infty(u) > \mathrm{ord}_0(W^\infty) - \deg(r) + 1$ as in the proof of Theorem 3.6, then

$$p^{\lfloor \log_p(m e_\infty)\rfloor} u \in \mathbf{Z}_q[x, x^{-1}]^{\oplus d_x}.$$

Proof. We have

$$\omega = df + (\sum_{i=0}^{d_x - 1} u_i y^i) \frac{dx}{r}$$

with $f = \sum_{j=-\mathrm{ord}_0(W^\infty)}^{m} (\sum_{i=0}^{d_x-1} (v_j)_i y^i) x^j$, where $v_j \in \mathbf{Q}_q^{\oplus d_x}$ for all $-\mathrm{ord}_0(W^\infty) \leq j \leq m$. Note that it is sufficient to show that $p^{\lfloor \log_p(\ell e)\rfloor} f \in \mathcal{R}$. By Assumption 1, we have that

$$\mathcal{O}(\mathcal{X} - x^{-1}(0))/(t)^k \cong \prod_{P \in \mathcal{X} \setminus \mathcal{U}, x(P) = \infty} \mathcal{O}_{\mathcal{X},P}/(z_P^{e_P})^k, \tag{4}$$

for all $k \in \mathbf{N}$. Moreover, by definition $[b_0^\infty, \ldots, b_{d_x-1}^\infty]$ is a basis for $\mathcal{O}(\mathbb{X} - x^{-1}(0))$ over $\mathbf{Q}_q[x^{-1}]$. To show that $p^{\lfloor \log_p(\ell e_\infty)\rfloor} f$ is integral, it is therefore enough to show that for every $P \in \mathcal{X} \setminus \mathcal{U}$ with $x(P) = 0$, the Laurent series expansion

$$a_{-m e_P} z_P^{-m e_P} + \ldots + a_{(\mathrm{ord}_0(W^\infty)+1)e_P - 1} z_P^{(\mathrm{ord}_0(W^\infty)+1)e_P - 1} + \mathcal{O}(z_P^{(\mathrm{ord}_0(W^\infty)+1)e_P})$$

of $p^{\lfloor \log_p(\ell e_\infty)\rfloor} f$ is integral. However, the differential df has a pole of order at most $m e_P + 1$ at P, and its Laurent series expansion

$$\Big(b_{-m e_P - 1} z_P^{-m e_P - 1} + \ldots + b_{(\mathrm{ord}_0(W^\infty)+1)e_P} z_P^{(\mathrm{ord}_0(W^\infty)+1)e_P} + \mathcal{O}(z_P^{(\mathrm{ord}_0(W^\infty)+1)e_P - 1})\Big) dz_P$$

is integral since ω is integral. The worst denominator we get by integrating this series is therefore $p^{\lfloor \log_p(m e_\infty)\rfloor}$ and the result follows. □

Remark 3.9. *Note that Propositions 3.3, 3.4, 3.7 and 3.8 can be used to give an alternative effective proof of Theorem 3.2.*

Recall that in Theorem 3.6 the computation of a basis for $H^1_{\mathrm{rig}}(U)$ was reduced to a (small) finite dimensional linear algebra problem. However, the dimension of $H^1_{\mathrm{rig}}(U)$ is generally about d_x times the dimension of $H^1_{\mathrm{rig}}(X)$, so that we would like to compute a basis for this last space. For this we will need to compute the kernel of a cohomological residue map.

Definition 3.10. *For a 1-form $\omega \in \Omega^1(\mathcal{U})$ and a point $P \in \mathcal{X} \setminus \mathcal{U}$, we let*

$$res_P(\omega) \in \mathcal{O}_{\mathcal{X},P}/(z_P)$$

denote the coefficient a_{-1} in the Laurent series expansion

$$\omega = (a_{-k} z_P^k + \ldots + a_{-1} z_P^{-1} + \cdots) dz_P.$$

Moreover, we denote

$$res = \bigoplus_{P \in \mathcal{X}\setminus\mathcal{U}:\, x(P) \neq \infty} res_P, \qquad res_\infty = \bigoplus_{P \in \mathcal{X}\setminus\mathcal{U}:\, x(P) = \infty} res_P.$$

Theorem 3.11. *We have an exact sequence*

$$0 \longrightarrow H^1_{rig}(X) \longrightarrow H^1_{rig}(U) \xrightarrow{(res \oplus res_\infty)\otimes \mathbf{Q}_q} \bigoplus_{P \in \mathcal{X}\setminus\mathcal{U}} \mathcal{O}_{\mathcal{X},P}/(z_P) \otimes \mathbf{Q}_q.$$

Proof. This is well known. □

The kernels of res and res_∞ can be computed without having to compute Laurent series expansions.

Proposition 3.12. *Let $\omega \in \Omega^1(\mathbb{U})$ be a 1-form of the form*

$$\omega = \Big(\sum_{i=0}^{d_x-1} u_i(x) y^i\Big) \frac{dx}{r},$$

with $u \in \mathbf{Q}_q[x]^{\oplus d_x}$. Then

$$res(\omega) = 0 \quad \Leftrightarrow \quad \frac{\partial Q}{\partial y} \sum_{i=0}^{d_x-1} u_i y^i = 0 \quad in \quad \mathcal{O}(\mathbb{X} - x^{-1}(\infty))/(r).$$

Proof. Let P run over all points in $\mathcal{X} \setminus \mathcal{U}$ such that $x(P) \neq \infty$. One checks that $\mathrm{ord}_P(\frac{dx}{r}) = -1$ and $\mathrm{ord}_P(\omega) \geq -1$. Hence $res_P(\omega) = 0$ if and only if $\mathrm{ord}_P(\sum_{i=0}^{d_x-1} u_i y^i) \geq 1$. However, since $\mathrm{ord}_P(\frac{\partial Q}{\partial y}) = e_P - 1$ by Assumption 2, this is the case if and only if $\mathrm{ord}_P(\frac{\partial Q}{\partial y} \sum_{i=0}^{d_x-1} u_i y^i) \geq e_P$. Finally, $\mathrm{ord}_P(\frac{\partial Q}{\partial y} \sum_{i=0}^{d_x-1} u_i y^i) \geq e_P$ at all P in $\mathcal{X} \setminus \mathcal{U}$ such that $x(P) \neq \infty$ if and only if $\frac{\partial Q}{\partial y} \sum_{i=0}^{d_x-1} u_i y^i$ maps to 0 in $\mathcal{O}(\mathbb{X} - x^{-1}(\infty))/(r)$. □

Proposition 3.13. *Let $\omega \in \Omega^1(\mathbb{U})$ be a 1-form of the form*

$$\omega = \Big(\sum_{i=0}^{d_x-1} u_i(x, x^{-1}) b_i^\infty\Big) \frac{dx}{r},$$

with $u \in \mathbf{Q}_q[x, x^{-1}]^{\oplus d_x}$ satisfying $\mathrm{ord}_\infty(u) > -\deg(r)$ and let e_∞ be defined as in Proposition 3.8. Then

$$res_\infty(\omega) = 0 \quad \Leftrightarrow \quad \Big(x^{1-\deg(r)} \sum_{i=0}^{d_x-1} u_i b_i^\infty\Big)^{e_\infty} = 0 \quad in \quad \mathcal{O}(\mathbb{X} - x^{-1}(0))/(t).$$

Proof. Let P run over all points in $\mathcal{X} \setminus \mathcal{U}$ such that $x(P) = \infty$. One checks that $\mathrm{ord}_P(\frac{dx}{r}) = -1 + (\deg(r) - 1)e_P$ and $\mathrm{ord}_P(\omega) \geq -1$. Hence $res_P(\omega) = 0$ if and only if $\mathrm{ord}_P(x^{1-\deg(r)} \sum_{i=0}^{d_x-1} u_i b_i^\infty) \geq 1$. However, by (4) this is the case at all P in $\mathcal{X} \setminus \mathcal{U}$ such that $x(P) = \infty$ if and only if $(x^{1-\deg(r)} \sum_{i=0}^{d_x-1} u_i b_i^\infty)^{e_\infty}$ maps to 0 in $\mathcal{O}(\mathbb{X} - x^{-1}(0))/(t)$. □

Remark 3.14. *For any $\omega \in \Omega^1(\mathbb{U})$ we can first apply Propositions 3.3 and 3.4 to represent the class of ω in $H^1_{rig}(U)$ by 1-forms to which we can apply Propositions 3.12 and 3.13.*

Remark 3.15. *Let e be defined as in Proposition 3.7. We can replace the statement in Proposition 3.12 by*

$$res(\omega) = 0 \quad \Leftrightarrow \quad \Big(\sum_{i=0}^{d_x-1} u_i y^i\Big)^e = 0 \quad in \quad \mathcal{O}(\mathbb{X} - x^{-1}(\infty))/(r).$$

Note that this is computationally a bit more involved, as we have to compute a power instead of a single product in the ring $\mathcal{O}(\mathbb{X} - x^{-1}(\infty))/(r)$. However, the proof is now the same as for Proposition 3.13 and does not require Assumption 2.

4. The complete algorithm and its complexity

In this section we describe all the steps in the algorithm and determine bounds for the complexity. Recall that X is a curve of genus g over a finite field $\mathbf{F}_q$ with $q = p^n$ and that d_x and d_y denote the degrees of the defining polynomial Q in the variables y and x, respectively. All computations are carried out to p-adic precision N which will be specified later. We use the $\tilde{\mathcal{O}}(-)$ notation that ignores logarithmic factors, i.e. $\tilde{\mathcal{O}}(f)$ denotes the class of functions that lie in $\mathcal{O}(f \log^k(f))$ for some $k \in \mathbf{N}$. For example, two elements of $\mathbf{Z}_q$ can be multiplied in time $\tilde{\mathcal{O}}(\log(p)nN)$. The least exponent for matrix multiplication will be denoted by θ, so that two $k \times k$ matrices can be multiplied in $\mathcal{O}(k^\theta)$ ring operations. It is known that $2 \leq \theta \leq 2.3729$. We start with some bounds that will be useful later on.

Proposition 4.1. *Let Δ, s, r be defined as in Section 2 and e, e_∞ as in Section 3. We then have that:*

$$\deg(\Delta), \deg(r), \deg(s) \leq 2(d_x - 1)d_y \qquad \in \mathcal{O}(d_x d_y), \tag{5a}$$

$$e, e_\infty \leq d_x \qquad \in \mathcal{O}(d_x), \tag{5b}$$

$$g \leq (d_x - 1)(d_y - 1) \in \mathcal{O}(d_x d_y). \tag{5c}$$

Proof. (5a) Note that the matrix Σ from Proposition 2.4 is a $(2d_x - 1) \times (2d_x - 1)$ matrix over $\mathbf{Z}_q[x]$ of degree at most d_y and that the row corresponding to y^{2d_x-2} has degree 0. Since $\Delta = \det(\Sigma)$, this implies that $\deg(\Delta) \leq (2d_x - 2)d_y$. Writing $s = \sum_{i=0}^{d_x-1} s_i(x)y^i$ with $s_i \in \mathbf{Z}_q[x]$, the s_i are entries of $r\Sigma^{-1}$, so that $\deg(s_i) \leq (2d_x - 2)d_y$ for all $0 \leq i \leq d_x - 1$.
(5b) All the ramification indices e_P are at most d_x.
(5c) It is known [BP00] that g is at most the number of interior points of the Newton polygon of Q, which is clearly bounded by $(d_x - 1)(d_y - 1)$. □

Proposition 4.2. *We have that*

$$\operatorname{ord}_\infty(W^\infty) \geq -(d_x-1)d_xd_y \in -\mathcal{O}(d_x^2d_y), \tag{6a}$$

$$\operatorname{ord}_\infty((W^\infty)^{-1}) \geq -(d_x-1)d_y \quad \in -\mathcal{O}(d_xd_y). \tag{6b}$$

Moreover, we may assume that

$$\operatorname{ord}_0(W^\infty) \geq -(d_x-1)d_y \quad \in -\mathcal{O}(d_xd_y). \tag{6c}$$

Proof. We still denote $t = 1/x$. One easily checks that the minimal polynomial $\mathcal{Q}^\infty$ of $y' = y/x^{d_y}$ over $\mathbf{Q}_q[t]$ is monic. Hence the functions $1, y', \ldots, y'^{d_x-1}$ are $\mathbf{Q}_q[t]$-linear combinations of $b_0^\infty, \ldots, b_{d_x-1}^\infty$, so that $\operatorname{ord}_\infty((W^\infty)^{-1}) \geq -(d_x-1)$.

Since the degree of $\mathcal{Q}^\infty$ in the variable t is at most d_xd_y, its discriminant $\Delta^\infty \in \mathbf{Z}_q[t]$ with respect to the variable y' has degree $\leq 2(d_x-1)d_xd_y$ by the argument from Proposition 4.1. Defining the matrix $W^{\infty'} \in Gl_d(\mathbf{Z}_q[x,x^{-1}])$ such that

$$b_j^\infty = \sum_{i=0}^{d_x-1} W_{i+1,j+1}^{\infty'} y'^i$$

for all $0 \leq j \leq d_x - 1$, it follows from basic properties of the discriminant that $\operatorname{ord}_\infty(W^{\infty'}) \geq -\deg(\Delta^\infty)/2$. Clearly $\operatorname{ord}_\infty(W^\infty) \geq \operatorname{ord}_\infty(W^{\infty'})$, so this implies that $\operatorname{ord}_\infty(W^\infty) \geq -(d_x-1)d_xd_y$.

We may assume that $\operatorname{ord}_0(W^{\infty'}) \geq 0$. When this is not the case, we can proceed as in [vH94] to obtain another integral basis such that $\operatorname{ord}_0(W^{\infty'}) \geq 0$. Note that this does not involve computing Puiseux expansions etc. as in [vH94], since we already have the integral basis $[b_0^\infty, \ldots, b_{d_x-1}^\infty]$ at our disposal. Finally, clearly $\operatorname{ord}_0(W^{\infty'}) \geq 0$ implies that $\operatorname{ord}_0(W^\infty) \geq -(d_x-1)d_y$. □

Note that so far we have assumed that W^∞ was given. In general algorithms like the one from [vH94] are available for computing integral bases in function fields. However, for almost all polynomials Q the problem is easier and we can write down $[b_0^\infty, \ldots, b_{d_x-1}^\infty]$ directly.

Proposition 4.3. *Let Γ denote the Newton polygon of Q, i.e. the convex hull of the points $(i,j) \in \mathbf{Z}^2$ such that Q contains the monomial x^iy^j and let Γ_{tr} with vertices $(0,0), (0,d_x)$ and $(a,b) \in \mathbf{Z}^2$ be the smallest triangle that contains Γ. For every integer $-b \leq i \leq d_x - 1 - b$, we let $\epsilon(i)$ denote the largest integer such that $(\epsilon(i), i)$ lies inside the translation $\Gamma_{tr} - (a,b)$ of Γ_{tr}. Now if Q is nondegenerate with respect to Γ, then*

$$\{y^i x^{\epsilon(i)} | -b \leq i \leq d_x - 1 - b\}$$

is an integral basis for $\mathbf{Q}_q(x,y)$ over $\mathbf{Q}_q[x^{-1}]$.

Proof. Note that if Q is nondegenerate with respect to Γ, then so is $\mathcal{Q}$ [CDV06]. Let Σ_Γ denote the projective toric surface over $\mathbf{Q}_q$ associated to Γ. The closure in Σ_Γ of the zero locus of $\mathcal{Q}$ on the dense orbit is the smooth projective model $\mathbb{X}$ of $\mathbf{Q}_q(x,y)$, since $\mathcal{Q}$ is nondegenerate with respect to Γ. The ring of regular functions on the open set $x \neq 0$ of Σ_Γ is given by $\mathbf{Q}_q[\Gamma_{tr} - (a,b)]$ and $\mathbb{X}$ is defined on this open set by $x^{-a}y^{-b}\mathcal{Q}$. Therefore, we have to show that the $y^ix^{\epsilon(i)}$ with $-b \leq i \leq d_x - 1 - b$ form a basis for the $\mathbf{Q}_q[x^{-1}]$-module

$$\mathcal{O}(\mathbb{X} - x^{-1}(0)) = \mathbf{Q}_q[\Gamma_{tr} - (a,b)]/(x^{-a}y^{-b}\mathcal{Q}).$$

However, it is clear that modulo $x^{-a}y^{-b}\mathcal{Q}$ every polynomial supported on the the cone generated by $\Gamma_{tr}-(a,b)$ can be reduced to one supported on $\Gamma_{tr}-(a,b)$ and that any polynomial supported on $\Gamma_{tr}-(a,b)$ can be written as a linear combination over $\mathbf{Q}_q[x^{-1}]$ of the $y^i x^{\epsilon(i)}$ with $-b \leq i \leq d_x - 1 - b$. □

Remark 4.4. *From Proposition 4.3, we can easily determine the matrix W^∞. If $b = 0$, then W^∞ is simply the diagonal matrix $diag(\epsilon(0), \ldots, \epsilon(d_x - 1))$. If $b > 0$, then the origin is the unique lowest point of Γ and modulo $\mathcal{Q}$ we can reduce a negative power of y to a linear combination over $\mathbf{Q}_q[x]$ of larger powers of y. Note that this procedure does not introduce p-adic denominators, since the defining polynomial Q and its lift $\mathcal{Q}$ contain the same monomials. Also observe that we need at most $b \leq d_x - 1$ reduction steps and each time the order at infinity of the coefficients decreases by at most d_y, so that $\mathrm{ord}_\infty(W^\infty) \geq -(d_x - 1)d_y$.*

Assumption 4. *In the complexity analysis we will assume a couple of times that*

$$\mathrm{ord}_\infty(W^\infty) \in -\mathcal{O}(d_x d_y),$$

as is the case when Q is nondegenerate with respect to its Newton polygon by Remark 4.4.

4.1. **Step I: Determine a basis for the cohomology.**

We want to find $\omega_1, \ldots, \omega_\kappa \in (E_0 \cap E_\infty) \cap \Omega^1(\mathcal{U})$ such that:

(1) $[\omega_1, \ldots, \omega_\kappa]$ is a basis for $H^1_{\mathrm{rig}}(U) \cong (E_0 \cap E_\infty)/d(B_0 \cap B_\infty)$,
(2) the class of every element of $(E_0 \cap E_\infty) \cap \Omega^1(\mathcal{U})$ in $H^1_{\mathrm{rig}}(U)$ has p-adically integral coordinates with respect to $[\omega_1, \ldots, \omega_\kappa]$,
(3) $[\omega_1, \ldots, \omega_{2g}]$ is a basis for the kernel of $res \oplus res_\infty$ and hence for the subspace $H^1_{\mathrm{rig}}(X)$ of $H^1_{\mathrm{rig}}(U)$.

This can be done using standard linear algebra over $\mathbf{Z}_q$, i.e. by computing the Smith normal forms (including unimodular transformations) of two matrices. Note for an element

$$\Big(\sum_{i=0}^{d_x-1} u_i(x)y^i\Big)\frac{dx}{r} \in E_0 \cap E_\infty,$$

we have that $\deg(u) \leq \deg(r) - 1 - \mathrm{ord}_0(W^\infty) - \mathrm{ord}_\infty(W^\infty)$. Hence the dimensions of the matrices involved are at most

$$d_x\big(\deg(r) - \mathrm{ord}_0(W^\infty) - \mathrm{ord}_\infty(W^\infty)\big).$$

Therefore, under Assumption 4 we need $\mathcal{O}((d_x^2 d_y)^\theta)$ ring operations in $\mathbf{Z}_q$ by [Sto00, Chapter 7], each of which can be carried out in time $\tilde{\mathcal{O}}(\log(p)nN)$, so that the time complexity of this step is

$$\tilde{\mathcal{O}}(\log(p) d_x^{2\theta} d_y^\theta nN).$$

4.2. **Step II: Compute the map F_p.**

We use Theorem 2.6 to compute approximations:

$$\mathrm{F}_p(\frac{1}{r}) = \alpha_i + \mathcal{O}(p^{2^i}),$$
$$\mathrm{F}_p(y) = \beta_i + \mathcal{O}(p^{2^i}),$$

for $i = 1, \dots, \nu = \lceil \log_2(N) \rceil$. We carry out all computations using r-adic expansions for the elements of $\mathcal{R}$ and $\mathcal{S}$, e.g. we represent α_i, β_i as:

$$\alpha_i = \sum_{j \in J} \frac{\alpha_{i,j}(x)}{r^j}, \qquad \beta_i = \sum_{k=0}^{d_x - 1} \Big(\sum_{j \in J} \frac{\beta_{i,j,k}(x)}{r^j} \Big) y^k,$$

where $J \subset \mathbf{Z}$ is finite and $\alpha_{i,j}, \beta_{i,j,k} \in \mathbf{Z}_q[x]$ satisfy $\deg(\alpha_{i,j}), \deg(\beta_{i,j,k}) < \deg(r)$, for all i, j, k. By Propositions 2.12 and 4.2, we have that

$$|J| \in \mathcal{O}\Big(p\Big(N + d_x^2 d_y / \deg(r) \Big) \Big).$$

Hence, a single ring operation in $\mathcal{R}$ takes time

$$\tilde{\mathcal{O}}(\log(p)|J|nN) \subset \tilde{\mathcal{O}}\Big(p d_x^2 d_y \big(N + d_x \big) nN \Big).$$

Moreover, the image of an element of $\mathbf{Q}_q$ under the map σ can be computed in time $\tilde{\mathcal{O}}(\log^2(p)n + \log(p)nN)$ by [Hub10]. We need $\mathcal{O}(d_x \log(N))$ ring operations in $\mathcal{R}$ and $\mathcal{O}(d_x d_y)$ applications of σ in order to compute (α_ν, β_ν). Therefore, this can be done in time

$$\tilde{\mathcal{O}}\Big(p d_x^3 d_y \big(N + d_x \big) nN \Big).$$

Now for each $\omega_i = (\sum_{k=0}^{d-1} u_k(x) y^k) \frac{dx}{r}$ with $1 \leq i \leq 2g$, we compute

$$\mathrm{F}_p(\omega_i) = \sum_{k=0}^{d_x - 1} p x^{p-1} u_k^\sigma(x^p) \, \mathrm{F}_p\Big(\frac{y^k}{r} \Big) dx = \sum_{k=0}^{d_x - 1} p x^{p-1} u_k^\sigma(x^p) \alpha_\nu \beta_\nu^k dx + \mathcal{O}(p^N). \quad (7)$$

For a single ω_i this takes $\mathcal{O}(d_x)$ ring operations in $\mathcal{R}$ and $\mathcal{O}(d_x \deg(r))$ applications of σ. Hence the complete set of $\mathrm{F}_p(\omega_i)$ can be computed in time

$$\tilde{\mathcal{O}}\Big(g p d_x^3 d_y \big(N + d_x \big) nN \Big) \subset \tilde{\mathcal{O}}\Big(p d_x^4 d_y^2 \big(N + d_x \big) nN \Big),$$

which is also the total time complexity of this step.

4.3. **Step III: Reduce back to the basis.**

We want to find the matrix $\Phi \in M_{2g \times 2g}(\mathbf{Q}_q)$ such that

$$\mathrm{F}_p(\omega_i) = \sum_{j=1}^{2g} \Phi_{j,i} \omega_j$$

in $H^1_{\mathrm{rig}}(U)$. In the previous step, we have obtained an approximation

$$\mathrm{F}_p(\omega_i) = \sum_{j \in J} \Big(\sum_{k=0}^{d_x - 1} \frac{w_{i,j,k}(x)}{r^j} y^k \Big) \frac{dx}{r} + \mathcal{O}(p^N), \quad (8)$$

where $J \subset \mathbf{Z}$ is finite and $w_{i,j,k}(x) \in \mathbf{Z}_q[x]$ satisfies $\deg(w_{i,j,k}(x)) < \deg(r)$ for all i, j, k. We now use Proposition 3.3 and Proposition 3.4 (repeatedly) to reduce this 1-form to an element of $E_0 \cap E_\infty$ as in Theorem 3.6.

To carry out the reduction procedure, it is sufficient to solve a linear system with parameter (ℓ or m) only once in Propositions 3.3 and 3.4. After that, every reduction step corresponds to a multiplication of a vector by a $d_x \times d_x$ matrix (over

$\mathbf{Q}_q[x]/(r)$ or $\mathbf{Q}_q$, respectively). First, the linear systems with parameter can be solved in time

$$\tilde{\mathcal{O}}(\log(p)d_x^{\theta+1}\deg(r)nN) \subset \tilde{\mathcal{O}}(\log(p)d_x^{\theta+2}d_ynN),$$

where one factor d_x is from the degree in the parameter. Then, the number of reduction steps at the points not lying over $x = \infty$ is $\mathcal{O}(pN)$ for each $\mathrm{F}_p(\omega_i)$. Every single finite reduction step takes time $\tilde{\mathcal{O}}(\log(p)d_x^2\deg(r)nN)$, so all $\mathrm{F}_p(\omega_i)$ can be reduced in time

$$\tilde{\mathcal{O}}(g(pN)d_x^2\log(p)\deg(r)nN) \subset \tilde{\mathcal{O}}(pd_x^4d_y^2nN^2).$$

Finally, the number of reduction steps at the points lying over $x = \infty$ is $\mathcal{O}(pd_x^2d_y)$ for each $\mathrm{F}_p(\omega_i)$. Every single infinite reduction takes time $\tilde{\mathcal{O}}(\log(p)d_x^2nN)$, so all $\mathrm{F}_p(\omega_i)$ can be reduced in time

$$\tilde{\mathcal{O}}(g(pd_x^2d_y)\log(p)d_x^2nN) \subset \tilde{\mathcal{O}}(pd_x^5d_y^2nN).$$

After this reduction procedure, we project from $E_0 \cap E_\infty$ onto the basis $[\omega_1, \ldots, \omega_{2g}]$ and read off the entries of Φ. This involves computing $\mathcal{O}(g)$ products of a vector by a matrix of size $\mathcal{O}(d_x^2d_y)$ under Assumption 4. Therefore, it can be done in time

$$\tilde{\mathcal{O}}(\log(p)g(d_x^2d_y)^2nN) \subset \tilde{\mathcal{O}}(\log(p)d_x^5d_y^3nN).$$

Combining all of this, the total time complexity of this step is

$$\tilde{\mathcal{O}}(pd_x^4d_y^2nN^2 + d_x^5d_y^3nN).$$

4.4. **Step IV: Determine $Z(X,T)$.**

It follows from the Lefschetz formula for rigid cohomology that

$$Z(X,T) = \frac{\chi(T)}{(1-T)(1-qT)},$$

where

$$\chi(T) = \det\bigl(1 - \mathrm{F}_p^n\, T | H^1_{\mathrm{rig}}(X)\bigr).$$

Since F_p is not linear but σ-semilinear, the matrix of F_p^n with respect to the basis $[\omega_1, \ldots, \omega_{2g}]$ is given by

$$\Phi^{(n)} = \Phi^{\sigma^{(n-1)}}\Phi^{\sigma^{(n-2)}}\cdots\Phi.$$

Note that $\chi(T)$ is the reverse characteristic polynomial of $\Phi^{(n)}$. It is known (see for example [PT13]) that $\Phi^{(n)}$ can be computed from Φ in time $\tilde{\mathcal{O}}(\log^2(p)g^\theta nN)$ and that $\chi(T)$ can be computed from $\Phi^{(n)}$ in time $\tilde{\mathcal{O}}(\log(p)g^\theta nN)$. Therefore, the total time complexity of this step is

$$\tilde{\mathcal{O}}(\log^2(p)g^\theta nN) \subset \tilde{\mathcal{O}}(\log^2(p)(d_xd_y)^\theta nN).$$

4.5. **The p-adic precision.**

So far we have only obtained an approximation to $\chi(T)$, since we have computed to p-adic precision N. Moreover, because of loss of precision in the computation, in general $\chi(T)$ will not even be correct to precision N. So what precision N is sufficient to determine $\chi(T)$ exactly?

Proposition 4.5. *The least p-adic precision N that is sufficient to determine $\chi(T)$ is contained in $\tilde{\mathcal{O}}(d_x d_y n)$.*

Proof. We assume for simplicity as in [Ked01] that $\mathrm{ord}_p(\Phi) \geq 0$. Right after the proof we will say something more about the general case.

It follows from the Weil conjectures that $\chi(T)$ is determined by the bottom half of its coefficients, all of which are bounded in absolute value by $\binom{2g}{g} q^{\frac{g}{2}}$. Therefore, if $\chi(T)$ is known to p-adic precision at least $\lceil \log_p\big(2\binom{2g}{g} q^{\frac{g}{2}}\big)\rceil$, then it is determined exactly. Since $\mathrm{ord}_p(\Phi) \geq 0$, there will be no loss of precision in computing $\Phi^{(n)}$ and $\chi(T)$, so that it is sufficient to compute Φ to p-adic precision $\lceil \log_p\big(2\binom{2g}{g} q^{\frac{g}{2}}\big)\rceil$.

From Proposition 2.12 and formula (7), it follows that in equation (8) we have

$$\max\{J\} \leq p(N-1) - 1.$$

Therefore, the loss of precision during the reductions at the points not lying over $x = \infty$ is at most

$$\lfloor \log_p(p(N-1)e) \rfloor$$

by Proposition 3.7.

Similarly, the coefficients of $\mathrm{F}_p(y^i/r)$ with respect to the basis $[b_0^\infty, \ldots, b_{d_x-1}^\infty]$ have order at $x = \infty$ at least

$$p\Big(\mathrm{ord}_\infty((W^\infty)^{-1}) + \deg(r)\Big)$$

by the proof of Proposition 2.12. It follows from formula (7) and the definition of E_∞ that the coefficients of $\mathrm{F}_p(\omega_i)$ with respect to the basis $[b_0^\infty, \ldots, b_{d_x-1}^\infty]$, which are elements of $\Omega^1(\mathbb{V})$ now, have order at $x = \infty$ at least

$$p\Big(\mathrm{ord}_\infty((W^\infty)^{-1}) + \deg(r)\Big) - (p-1) + p\Big(\mathrm{ord}_0(W^\infty) - \deg(r) + 2\Big) - 2 \geq$$
$$p\Big(\mathrm{ord}_\infty((W^\infty)^{-1}) + \mathrm{ord}_0(W^\infty)\Big) - 1.$$

Note that the reductions at the points not lying over $x = \infty$ can introduce poles at $x = \infty$, but these can be ignored since they have order at $x = \infty$ at least

$$\mathrm{ord}_\infty((W^\infty)^{-1}) \geq p\Big(\mathrm{ord}_\infty((W^\infty)^{-1}) + \mathrm{ord}_0(W^\infty)\Big) - 1,$$

using that $\mathrm{ord}_\infty((W^\infty)^{-1})$, $\mathrm{ord}_0(W^\infty)$ are both negative. Hence, when applying Proposition 3.8 to the 1-form that remains after the reductions at the points not lying over $x = \infty$, we have that

$$m \leq -p\big(\mathrm{ord}_\infty((W^\infty)^{-1}) + \mathrm{ord}_0(W^\infty)\big).$$

Therefore, the loss of precision during the reductions at the points lying over $x = \infty$ is at most

$$\lfloor \log_p\Big(-p\big(\mathrm{ord}_\infty((W^\infty)^{-1}) + \mathrm{ord}_0(W^\infty)\big)e_\infty\Big) \rfloor.$$

By construction of our basis $[\omega_1, \ldots, \omega_{2g}]$, there will be no further loss of precision computing the matrix Φ. We conclude that it is sufficient for N to satisfy

$$N - \lfloor \log_p(p(N-1)e) \rfloor - \lfloor \log_p\Big(-p\big(\mathrm{ord}_\infty((W^\infty)^{-1}) + \mathrm{ord}_0(W^\infty)\big)e_\infty\Big) \rfloor \geq \lceil \log_p\big(2\binom{2g}{g} q^{\frac{g}{2}}\big) \rceil,$$

from which the result follows using Propositions 4.1 and 4.2 □

Remark 4.6. *We can use Propositions 2.12, 3.7 and 3.8 to obtain a lower bound for* $\mathrm{ord}_p(\Phi)$ *such that taking into account the extra loss of precision* $(n-1)\,\mathrm{ord}_p(\Phi)$ *for computing* $\Phi^{(n)}$ *and* $(2g-1)n\,\mathrm{ord}_p(\Phi)$ *for computing* $\chi(T)$*, we can still take* $N \in \tilde{\mathcal{O}}(d_x d_y n)$*. However, any concrete bound for* N *obtained this way will be bad in practice, while the bound for* N *from the proof of Proposition 4.5 is very good when* $\mathrm{ord}_p(\Phi) \geq 0$*. One can obtain a much sharper bound for* $\mathrm{ord}_p(\Phi)$*, and the loss of precision in computing* $\Phi^{(n)}$ *and* $\chi(T)$*, using the existence of the* F_p*-invariant* $\mathbf{Z}_q$*-lattice coming from the (log)-crystalline cohomology inside the rigid cohomology.*

Theorem 4.7. *The time complexity of the algorithm presented in this section is* $\tilde{\mathcal{O}}(p d_x^6 d_y^4 n^3)$*.*

Proof. We take the sum of the complexities of the different steps using Proposition 4.5, leaving out terms and factors that are absorbed by the $\tilde{\mathcal{O}}$. □

For the analysis of the space complexity, we will not go into the same detail as for the time complexity. However, using Assumption 4 at the same two points as in the analysis of the time complexity, one can prove the following theorem.

Theorem 4.8. *The space complexity of the algorithm presented in this section is* $\tilde{\mathcal{O}}(p d_x^4 d_y^3 n^3)$*.*

Proof. One can show that the space complexity of the algorithm is that of storing a single $\mathrm{F}_p(\omega_i)$, or equivalently an element of $\mathcal{R}$, which is $\tilde{\mathcal{O}}(p d_x^2 d_y\big(N + d_x\big) n N)$. The result now follows using Proposition 4.5. □

Remark 4.9. *When* Q *is nondegenerate with respect to its Newton polygon* Γ*, we have that* $Vol(\Gamma) \in \mathcal{O}(g)$ [CDV06]*, so that* $d_x d_y \in \mathcal{O}(g)$*. Consequently, the time and space complexity of our algorithm are then* $\tilde{\mathcal{O}}(p g^6 n^3)$ *and* $\tilde{\mathcal{O}}(p g^4 n^3)$*, respectively. Note that this improves the complexity estimate from* [CDV06]*. Moreover, if additionally we fix* d_x*, then* d_y *is* $\mathcal{O}(g)$*, so that the time and space complexity of our algorithm are* $\tilde{\mathcal{O}}(p g^4 n^3)$ *and* $\tilde{\mathcal{O}}(p g^3 n^3)$*, respectively. Note that this extends the complexity estimate from* [Ked01] *to* $d_x > 2$*.*

Remark 4.10. *There are some standard ways to improve the algorithm in this section in practice:*

1. *We computed the Frobenius lift by working with p-adic precision* $N_i = 2^i$ *in the* i*th step of the Hensel lift. Setting* $N_\nu = N$ *and* $N_{i-1} = \lceil N_i/2 \rceil$ *for all* $1 \leq i \leq \nu$*, we still obtain the correct Frobenius lift to precision* N*, while having to compute to lower precision in every step.*
2. *The bound* $\log_p\big(2\binom{2g}{g} q^{\frac{g}{2}}\big)$ *for the p-adic precision of* $\chi(T)$ *can be lowered somewhat using the Newton-Girard identities* [Ked08]*.*

These improvements will not affect the complexity of the algorithm, but are important for practical implementations.

4.6. **Our assumptions.**

Without Assumption 1, Theorem 3.2 does not hold and we cannot compute in $H^1_{\mathrm{rig}}(U)$ as in Section 3. Therefore, Assumption 1 is essential and cannot be lifted. It would be interesting to know under what conditions a lift satisfying this assumption can be found. Note that for a curve and a map to $\mathbf{P}^1$ defined over a number field K, the assumption is satisfied at all but finitely many prime ideals of $\mathcal{O}_K$.

Assumption 2 serves to simplify the exposition and can be weakened as follows. Note that the assumption is equivalent to asking that $[y^0, \ldots, y^{d_x-1}]$ is an integral basis for $\mathbf{Q}_q(x,y)$ over $\mathbf{Q}_q[x]$. Let us assume instead that a matrix $W^0 \in Gl_d(\mathbf{Z}_q[x, 1/r])$ is known such that if we denote $b^0_j = \sum_{i=0}^{d_x-1} W^0_{i+1,j+1} y^i$ for all $0 \leq j \leq d-1$, then $[b^0_0, \ldots, b^0_{d_x-1}]$ is an integral basis for $\mathbf{Q}_q(x,y)$ over $\mathbf{Q}_q[x]$. Then everything in this paper still works with some minor changes, as we will now briefly sketch.

We replace $[y^0, \ldots, y^{d_x-1}]$ by $[b^0_0, \ldots, b^0_{d_x-1}]$ in Proposition 2.7, using the same proof as for Proposition 2.8, as well as in Assumption 3. With these changes we have that Proposition 2.10 still holds. Similarly, Proposition 3.3, Theorem 3.6 and Proposition 3.7 continue to hold with $[y^0, \ldots, y^{d_x-1}]$ replaced by $[b^0_0, \ldots, b^0_{d_x-1}]$. In Proposition 2.12, the bounds have to be adapted slightly and will now also involve the matrix W^0. In the algorithm we still compute the Frobenius lift with respect to the basis $[y^0, \ldots, y^{d_x-1}]$, replacing r by Δ in Proposition 2.4 and Theorem 2.6. We compute the kernel of the residue map as in Remark 3.15. To carry out the reductions at the points not lying over $x = \infty$, we first change basis from $[y^0, \ldots, y^{d_x-1}]$ to $[b^0_0, \ldots, b^0_{d_x-1}]$ and to carry out the reductions at the points lying over $x = \infty$, we then change basis from $[b^0_0, \ldots, b^0_{d_x-1}]$ to $[b^\infty_0, \ldots, b^\infty_{d_x-1}]$.

So Assumptions 2 and 3 are in fact very similar: we need an integral basis for $\mathbf{Q}_q(x,y)$ over both $\mathbf{Q}_q[x]$ and $\mathbf{Q}_q[x^{-1}]$. In both cases, if all singularities are nondegenerate with respect to their Newton polygon, we can proceed as in Proposition 4.3 and Remark 4.4. Otherwise, algorithms like the one from [vH94] are available for computing the integral bases.

Assumption 4 is the least important of all the assumptions. We have used it twice in the complexity analysis, to bound the complexity of doing linear algebra in $E_0 \cap E_\infty$. The most natural way of doing linear algebra in $E_0 \cap E_\infty$ is by embedding it in a space of dimension $d_x\big(\deg(r) - \mathrm{ord}_0(W^\infty) - \mathrm{ord}_\infty(W^\infty)\big)$ as in Step I. However, it can be shown using the Riemann-Roch theorem that the dimension of $E_0 \cap E_\infty$ is always $\mathcal{O}(d_x^2 d_y)$ anyway. It therefore seems likely that this assumption can be removed. As it is generically satisfied by Remark 4.4 and in practice the time spent on linear algebra in $E_0 \cap E_\infty$ is always negligble, we have not thought much about removing Assumption 4.

4.7. **Implementation.**

We have (partially) implemented our algorithm in the computer algebra package MAGMA [BCP97]. In examples where we can compare against either [CDV06] or [Wal10], our implementation runs 2 to 3 orders of magnitude faster. The code and some examples can be found at http://perswww.kuleuven.be/jan_tuitman. Note that the implementation is still very much work in progress.

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