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Problem 2

Start with Navier-Stokes equation for MHD

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{u} + \frac{1}{\rho}(\vec{j}\times\vec{B})$$

Take curl ($\nabla\times$) of equation to get vorticity

$$\underset{①}{\nabla\times\frac{\partial\vec{u}}{\partial t}} + \underset{②}{\nabla\times(\vec{u}\cdot\nabla)\vec{u}} = \underset{③}{\nabla\times\left(-\frac{1}{\rho}\nabla p\right)} + \underset{④}{\nabla\times(\nu\nabla^2\vec{u})} + \underset{⑤}{\nabla\times\left(\frac{1}{\rho}\vec{j}\times\vec{B}\right)}$$

1. $\nabla\times\frac{\partial\vec{u}}{\partial t} = \frac{\partial}{\partial t}(\nabla\times\vec{u}) = \frac{\partial\vec{\omega}}{\partial t}$

2. $\nabla\times\left[(\vec{u}\cdot\nabla)\vec{u}\right] = \nabla\times\left[\frac{1}{2}\nabla|\vec{u}|^2 - \vec{u}\times\vec{\omega}\right]$

   Use identity $\nabla\times\nabla\phi = 0$, ϕ = scalar field

   $$\Rightarrow \nabla\times\left[\tfrac{1}{2}\nabla|\vec{u}|^2\right] = 0$$

3. Similarly, $\nabla\times\left(-\frac{1}{\rho}\nabla p\right) = -\frac{1}{\rho}\nabla\times\nabla p = 0$ if $\nabla\frac{1}{\rho} = 0$

4. $\nabla\times(\nu\nabla^2\vec{u}) = \nu\nabla^2(\nabla\times\vec{u}) = \nu\nabla^2\vec{\omega}$

5. $\nabla\times\left(\frac{1}{\rho}\vec{j}\times\vec{B}\right) = \frac{1}{\rho}\nabla\times(\vec{j}\times\vec{B})$

$$\frac{\partial\vec{\omega}}{\partial t} = \nabla\times(\vec{u}\times\vec{\omega}) + \nu\nabla^2\vec{\omega} + \frac{1}{\rho}\nabla\times(\vec{j}\times\vec{B})$$

But $\vec{j} = \sigma\vec{u}\times\vec{B}$ when $\cancel{\vec{E} = 0}$

Not a valid assumption. d/dz() = 0 and using Ampere's law j = j_z only. Therefore E_z is significant since (uxB)_z is negligible (for low Rem)

$$\boxed{\frac{\partial\vec{\omega}}{\partial t} = \nabla\times(\vec{u}\times\vec{\omega}) + \nu\nabla^2\vec{\omega} + \frac{\sigma}{\rho}\nabla\times(\vec{u}\times\vec{B}\times\vec{B})}$$

If $\vec{u} = u\hat{e}_x + v\hat{e}_y$ and $\vec{B} = B\hat{e}_z$, $\vec{\omega} = \omega\hat{e}_z$

$$\vec{u}\times\vec{\omega} = \begin{bmatrix} e_x & e_y & e_z \\ u & v & 0 \\ 0 & 0 & \omega \end{bmatrix} = v\omega\hat{e}_x - u\omega\hat{e}_y$$

$$\vec{U} \times \vec{\omega} = \begin{bmatrix} e_x & e_y & e_z \\ U & V & 0 \\ 0 & 0 & \omega \end{bmatrix} = V\omega\,\hat{e}_x - U\omega\,\hat{e}_y$$

$$\nabla \times (\vec{U} \times \vec{\omega}) = \begin{bmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V\omega & -U\omega & 0 \end{bmatrix} = \frac{\partial}{\partial z}(U\omega)\hat{e}_x + \frac{\partial}{\partial z}(V\omega)\hat{e}_y - \left[\frac{\partial}{\partial x}(U\omega) + \frac{\partial}{\partial y}(V\omega)\right]\hat{e}_z$$

$$\vec{U} \times \vec{B} = VB\,\hat{e}_x - UB\,\hat{e}_y$$

$$\vec{U} \times \vec{B} \times \vec{B} = -UB^2\,\hat{e}_x - VB^2\,\hat{e}_y$$

$$\nabla \times (\vec{U} \times \vec{B} \times \vec{B}) = \begin{bmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -UB^2 & -VB^2 & 0 \end{bmatrix} = \frac{\partial}{\partial z}(VB^2)\hat{e}_x - \frac{\partial}{\partial z}(UB^2)\hat{e}_y - \left[\frac{\partial}{\partial x}(VB^2) - \frac{\partial}{\partial y}(UB^2)\right]\hat{e}_z$$

$\Rightarrow$ in the z-direction,

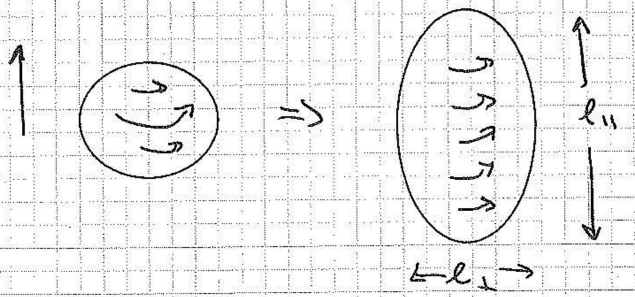
$$\frac{\partial \omega}{\partial t} = -\left[\frac{\partial}{\partial x}(U\omega) + \frac{\partial}{\partial y}(V\omega)\right] + \nu\nabla^2\omega - \frac{\sigma}{\rho}\left[\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right]B^2$$

But $\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$, $\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = -\omega$

$$\boxed{\frac{\partial \omega}{\partial t} + \underbrace{U\frac{\partial \omega}{\partial x} + V\frac{\partial \omega}{\partial y}}_{\sim \vec{U}\cdot\nabla\vec{\omega}} = \nu\nabla^2\omega + \cancel{\frac{\sigma}{\rho}\omega B^2}}$$

This is wrong likely because of the incorrect assumption of E = 0.

$$\underbrace{\frac{D\omega}{Dt} = \nu\nabla^2\omega}_{\text{2D hydrodynamic vorticity equation}} + \underbrace{\frac{\sigma}{\rho}\omega B^2}_{\text{vortex damping by magnetic field}}$$



$$\tau_j = \frac{\sigma}{\rho}B^2$$

There is no effect of jxB on the vorticity equation for this 2D problem assuming low Rem.