Problem 2

C. Kawczynski
Department of Mechanical and Aerospace Engineering
University of California Los Angeles, USA

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1 Derive the vorticity equation with the Lorentz force

1.1 Question

Derive the vorticity equation $\omega = \frac{\partial U}{\partial y} - \frac{\partial V}{\partial x}$ for a 2D MHD flow (in the x-y plane) of electrically conducting fluid in a constant spanwise magnetic field (the field is in z-direction). Based on this equation, conclude what kind of MHD effect will be experienced by the flow.

1.2 Solution

Assumptions

First, not that $\mathbf{j} = \nabla \times \left(\frac{\mathbf{B}}{\mu}\right)$ and $\partial_z() = 0$, therefore $\mathbf{j} = j\hat{\mathbf{e}}_z$. So our assumptions are

1. two-dimensional ($\partial_z() = 0$)
2. 1 component of vorticity $\omega_x = \omega_y = 0, \omega = \omega_z$
3. induced magnetic field is small compared to applied (low magnetic Reynolds number $Re_m << 1$)
4. currents close at infinity ($\mathbf{j} = j\hat{\mathbf{e}}_z$)

Analysis

The vorticity equation is derived by taking the curl of the momentum equation.

$$\epsilon_{lmi}\partial_m \left(\underbrace{\partial_t u_i}_{1} + \underbrace{u_j \partial_j u_i}_{2} = \underbrace{-\frac{1}{\rho}\partial_i p}_{3} + \underbrace{\nu \partial_{jj} u_i}_{4} + \underbrace{\frac{1}{\rho}\epsilon_{ijk} j_j B_k}_{5} \right)$$

First, let $\omega_l = \epsilon_{lmi}\partial_m u_i$, and we get the following:

Unsteady term

$$\begin{aligned}\epsilon_{lmi}\partial_m\partial_t u_i &= \partial_t\epsilon_{lmi}\partial_m u_i \\ &= \partial_t\omega_l\end{aligned} \tag{1}$$

Convection term

$$\begin{aligned}\epsilon_{lmi}\partial_m u_j\partial_j u_i &= \epsilon_{lmi}\partial_m u_j\partial_j u_i \\ &= \epsilon_{lmi}\partial_m\partial_j(u_i u_j) \qquad (\partial_i u_i = 0) \\ &= \epsilon_{lmi}\partial_j\partial_m(u_i u_j) \qquad (\text{swap } \partial \text{ order}) \\ &= \partial_j\partial_m(\epsilon_{lmi}u_i u_j) \\ &= \partial_j(\omega_l u_j) \\ &= u_j\partial_j\omega_l \qquad (\partial_j u_j = 0)\end{aligned} \tag{2}$$

Pressure term

$$\begin{aligned}-\epsilon_{lmi}\partial_m\frac{1}{\rho}\partial_i p &= -\frac{1}{\rho}\epsilon_{lmi}\partial_m\partial_i p \\ &= 0 \qquad (\text{by identity})\end{aligned} \tag{3}$$

Diffusion term

$$\begin{aligned}\epsilon_{lmi}\partial_m\nu\partial_{jj}u_i &= \nu\partial_m\partial_{jj}\epsilon_{lmi}u_i \\ &= \nu\partial_j\partial_{mj}\epsilon_{lmi}u_i \qquad (\text{swap } \partial \text{ order}) \\ &= \nu\partial_j\partial_{jm}\epsilon_{lmi}u_i \qquad (\text{swap index}) \\ &= \nu\partial_j\partial_j\omega_l \\ &= \nu\partial_{jj}\omega_l\end{aligned} \tag{4}$$

The last term (5) is the contribution from the electromagnetic Lorentz force.

1.2.1 Assuming low magnetic Reynolds number

Making use of the vector identity

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \bullet \mathbf{B}) - \mathbf{B}(\nabla \bullet \mathbf{A}) + (\mathbf{B} \bullet \nabla)\mathbf{A} - (\mathbf{A} \bullet \nabla)\mathbf{B}$$

We have

$$\begin{aligned}
\nabla \times \mathbf{j} \times \mathbf{B} &= \underbrace{\mathbf{j}(\nabla \bullet \mathbf{B})}_{=0} - \underbrace{\mathbf{B}(\nabla \bullet \mathbf{j})}_{=0} + (\mathbf{B} \bullet \nabla)\mathbf{j} - (\mathbf{j} \bullet \nabla)\mathbf{B} \qquad \text{(vector identity)} \\
&= (\mathbf{B} \bullet \nabla)\mathbf{j} - (\mathbf{j} \bullet \nabla)\mathbf{B} \\
&= \underbrace{B_x \partial_x \mathbf{j} + B_y \partial_y \mathbf{j}}_{=0,\ \text{(assumption 3)}} + \underbrace{B_z \partial_z \mathbf{j}}_{=0, (\partial_z() = 0)} - \left(\underbrace{j_x \partial_x \mathbf{B} + j_y \partial_y \mathbf{B}}_{=0,\ \text{(assumption 4)}} + \underbrace{j_z \partial_z \mathbf{B}}_{=0,\ (\partial_z() = 0)} \right) \\
&= \mathbf{0}
\end{aligned} \tag{5}$$

1.2.2 Assuming **finite magnetic Reynolds number**

Here, we relax the assumption of low magnetic Reynolds number. Making use of the vector identity

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \bullet \mathbf{B}) - \mathbf{B}(\nabla \bullet \mathbf{A}) + (\mathbf{B} \bullet \nabla)\mathbf{A} - (\mathbf{A} \bullet \nabla)\mathbf{B}$$

We have

$$\begin{aligned}
\nabla \times \mathbf{j} \times \mathbf{B} &= \underbrace{\mathbf{j}(\nabla \bullet \mathbf{B})}_{=0} - \underbrace{\mathbf{B}(\nabla \bullet \mathbf{j})}_{=0} + (\mathbf{B} \bullet \nabla)\mathbf{j} - (\mathbf{j} \bullet \nabla)\mathbf{B} \qquad \text{(vector identity)} \\
&= (\mathbf{B} \bullet \nabla)\mathbf{j} - (\mathbf{j} \bullet \nabla)\mathbf{B} \\
&= B_x \partial_x \mathbf{j} + B_y \partial_y \mathbf{j} + \underbrace{B_z \partial_z \mathbf{j}}_{=0, (\partial_z() = 0)} - \left(\underbrace{j_x \partial_x \mathbf{B} + j_y \partial_y \mathbf{B}}_{=0,\ \text{(assumption 4)}} + \underbrace{j_z \partial_z \mathbf{B}}_{=0,\ (\partial_z() = 0)} \right) \\
&= B_x \partial_x j + B_y \partial_y j
\end{aligned} \tag{6}$$

1.3 Final result

1.3.1 Low magnetic Reynolds number

Putting this all together, we have

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \bullet \nabla)\omega = \nu \nabla^2 \omega$$

Therefore, the vorticity equation is unaffected by the Lorentz force for this 2D flow.

1.3.2 Finite magnetic Reynolds number

Putting this all together, we have

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \bullet \nabla)\omega = \nu \nabla^2 \omega + (\mathbf{B} \bullet \nabla)\mathbf{j}$$