Problem 3

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1 Circular pipe with insulating walls

1.1 Question

Consider a fully developed MHD flow in a non-conducting circular pipe with radius R in the presence of a uniform magnetic field in the z-direction ($\mathbf{B} = B\hat{\mathbf{e}}_z$) as shown in the figure.

For such a configuration evaluate the following:

a) The distribution of electric potential along the wall ($r = R$) of the pipe for a given axisymmetric velocity profile $\mathbf{u}(r) = 2u_{ave}\left(1 - \frac{r^2}{R^2}\right)\hat{\mathbf{e}}_x$ (here u_{ave} is the average fluid velocity) by solving a 2D Poisson equation for the electric potential in the y-z plane with the assumption that the velocity profile is not affected by the magnetic field. [HINT: use the method of separation of variables.]

b) Potential difference between points A and B for the magnetic field strength B of 1 Tesla, average velocity u_{ave} of 10 cm/sec and pipe radius R of 10 cm.

1.2 Solution

Assuming axi-symmetric flow and

$$\mathbf{B} = B\hat{\mathbf{e}}_z, \qquad \mathbf{u} = \mathbf{u}(r)\hat{\mathbf{e}}_x. \tag{1}$$

The governing equation for ϕ is

$$\nabla^2\phi = \mathbf{B}\bullet\omega \tag{2}$$

with

$$\frac{\partial\phi}{\partial r} = 0, \qquad \text{for} \qquad r = R. \tag{3}$$

Evaluating this we have

$$\nabla^2 \phi = -B \frac{\partial u}{\partial y} = -B \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} = -B \frac{\partial u}{\partial r} \cos(\theta) \tag{4}$$

Using the vector identity for the Laplacian operator in cylindrical coordinates, we have

$$\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \tag{5}$$

Equating the last two equations, we have

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = -B \frac{\partial u}{\partial r} \cos(\theta) \tag{6}$$

Use the ansatz $\phi(r, \theta) = \cos(\theta) f(r)$ to get

$$\frac{d}{dr} \left(r^2 \frac{df}{dr} - rf \right) = -r^2 B u' \tag{7}$$

Apply the BC $\frac{df}{dr} = 0$ at $r = R$ and integrating the above equation from 0 to R to get

$$\left[r^2 \frac{df}{dr} - rf \right]_0^R = -B \int_0^R r^2 u' dr = \left[-Br^2 u \right]_0^R + 2B \int_0^R r - u dr \tag{8}$$

Second term on RHS of above equation is the flow rate and the first term goes to zero.

$$Rf(R) = \frac{2B\Phi}{2\pi} \tag{9}$$

Where

$$\Phi = \text{flow rate} = \iint_S \mathbf{u} \bullet ds = 2\pi \int_0^R u(r) r dr, \qquad \text{(axisymmetric)} \tag{10}$$

Therefore we have

$$\boxed{f(R) = \frac{B\Phi}{\pi R}} \tag{11}$$

Finally

$$\boxed{\phi(R) = \frac{B\Phi}{\pi R} \cos(\theta)} \tag{12}$$

Voltage difference at points A and B are

$$\cos(\theta) = -1 \text{ and } +1 \tag{13}$$

So we have

$$|\phi_A - \phi_B| = \phi(R, \pi) - \phi(R, 0) = -2f(R) \tag{14}$$

Therefore

$$\boxed{|\phi_A - \phi_B| = \frac{2B\Phi}{\pi R}} \tag{15}$$

This shows that the solution is independent of the velocity profile provided it is axisymmetric.