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#2

Problem #2

The vorticity comes from taking the curl of the MHD fluid momentum equation.

MHD mom: $$\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\bar{u} + \underbrace{\frac{1}{\rho}(\bar{J}\times\bar{B})}_{\text{Lorentz body force from}}$$

This is essentially the Navier-Stokes equation with an extra term for the Lorentz force caused by the induced electric current.

Taking the curl:

$$\nabla\times NS \Rightarrow \underbrace{\nabla\times\frac{\partial\bar{u}}{\partial t}}_{①} + \underbrace{\nabla\times\left[(\bar{u}\cdot\nabla)\bar{u}\right]}_{②} = \underbrace{\nabla\times\left[-\frac{1}{\rho}\nabla p\right]}_{③} + \underbrace{\nabla\times\left[\nu\nabla^2\bar{u}\right]}_{④} + \underbrace{\nabla\times\left[\frac{1}{\rho}(\bar{J}\times\bar{B})\right]}_{⑤}$$

① $= \frac{\partial}{\partial t}(\nabla\times\bar{u})$, which, using the definition $\bar{\omega}\equiv\nabla\times\bar{u}$ gives $\frac{\partial\bar{\omega}}{\partial t}$ ✓

② $(\bar{u}\cdot\nabla)\bar{u} = \nabla\left(\frac{1}{2}\bar{u}\cdot\bar{u}\right) - \bar{u}\times(\nabla\times\bar{u}) = \nabla\left(\frac{1}{2}|\bar{u}|^2\right) - \bar{u}\times\bar{\omega}$

Taking curl ⇒ $\underbrace{\nabla\times\left[\nabla\left(\frac{1}{2}|\bar{u}|^2\right)\right]}_{\text{curl(grad) of scalar fcn}=0} - \nabla\times(\bar{u}\times\bar{\omega}) = -\nabla\times(\bar{u}\times\bar{\omega})$

=0 for uniform density, but ok

③ $\nabla\times\left(-\frac{1}{\rho}\nabla p\right) = -\frac{1}{\rho}\underbrace{(\nabla\times\nabla p)}_{\text{curl(grad)}=0} + \cancel{\nabla\left(-\frac{1}{\rho}\right)}\times\nabla p = \frac{\nabla\rho\times\nabla p}{\rho^2}$

~~④~~ Assume ν = const $\rightarrow$ $= \nu\left[\nabla\times\nabla^2\bar{u}\right]$

Simplifications can be made in general. And assuming low Rem, which was mostly emphasized in class, it can be shown that curl (jxB)=0. Therefore the Lorentz force has no impact on the vorticity of this 2D flow.

~~⑤~~ $\nabla\times\frac{1}{\rho}(\bar{J}\times\bar{B})$

can put this in terms of vorticity

ok

⇒ full vorticity eq: ∴ $$\boxed{\frac{\partial\bar{\omega}}{\partial t} = \nabla\times(\bar{u}\times\bar{\omega}) + \frac{\nabla\rho\times\nabla p}{\rho^2} + \cancel{\nu(\nabla\times\nabla^2\bar{u})} + \cancel{\nabla\times\left(\frac{\bar{J}\times\bar{B}}{\rho}\right)}}$$

MHD effects: Vortexes will form and diffuse throughout the domain. Ignoring viscosity, assuming incompressible, and treating steady state gives:

$$\nabla\times(\bar{u}\times\bar{\omega}) = -\frac{1}{\rho}\nabla\times(\bar{J}\times\bar{B}) \rightarrow \bar{u}\times\bar{\omega} = -\underbrace{\frac{\bar{J}\times\bar{B}}{\rho}}_{\text{Lorentz force}}$$

→ Lorentz body force causes "swirling".