CHAPTER 1

Pebble Interaction Analysis: Theory

This chapter will walk through the analysis first performed by Heinrich Hertz in 1880 of idealized interacting spherical objects to represent our ceramic pebbles. We will begin with mechanical interactions of purely elastic spheres and then use the results to develop a theory for conductive heat transfer between the pebbles. Finally, Hertz's theory will also be used to analyze pebble interaction with flat plattens in our test stand used to measure crush force. A relationship is developed allowing us to translate from experimental results to pebble bed ensembles.

1.1 Hertz theory for normal contact of spheres

[DRAW SOME COORDINATE DIAGRAMS TO SHOW HOW Z R AND X-Y WHATEVER ARE ACTUALLY RELATED AND CAN BE VISUALIZED] We consider two non-conforming solids approaching and then contacting under load. We only wish to analyze the contact of spheres (of differeng radii), so we are able define the surface curvature of the two contacting bodies as

$$z_1 = \frac{1}{2R_1} r^2 \tag{1.1}$$

$$z_2 = \frac{1}{2R_2} r^2 \tag{1.2}$$

respectively. The radius, laying in the $x - y$ plane is related to cartesian coordinates as $r^2 = x^2 + y^2$. Before the surfaces are in contact, each point on the two surfaces are separated by a distance $h(r)$,

$$
\begin{aligned}
h &= z_1 - z_2 \\
h &= \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{r^2}{2}
\end{aligned}
\tag{1.3}
$$

We define the relative radius of curvature as

$$
\frac{1}{R^*} = \frac{1}{R_1} + \frac{1}{R_2}
\tag{1.4}
$$

and then the separation is simply $h = (1/2R^*)r^2$.

We allow the two surfaces to approach, and then, under an external load F, contact. The cross-section of these bodies after contact are shown in Fig. 1.1. If we first imagine that the two surfaces do not interact and their surfaces pass through each other unimpeded, their surfaces would be overalapped to a distance δ. In such a case, we examine two points deep within the bodies, along the axis of contact, calling them T_1 and T_2. These points will have moved δ_1 and δ_2, respectively. The total overlap is obviously related to these displacements by $\delta = \delta_1 + \delta_2$.

However, under actual contact, the two surfaces are going to deform as the load F presses them into contact. So now we consider two points on the surfaces, such as S_1 and S_2. Before contact, these two points are initially are separated by a distance h (from Eq. 1.3), then displace by $\bar{u}_{z1}$ and $\bar{u}_{z2}$ due to contact pressure.

If the points S_1 and S_2 are inside of the contact region under load, these distances are related by

$$
\bar{u}_{z1} + \bar{u}_{z2} + h = \delta
\tag{1.5}
$$

Then using Eq. 1.3, we have an expression for the elastic displacements:

$$
\bar{u}_{z1} + \bar{u}_{z2} = \delta - \frac{1}{2R^*} r^2
\tag{1.6}
$$

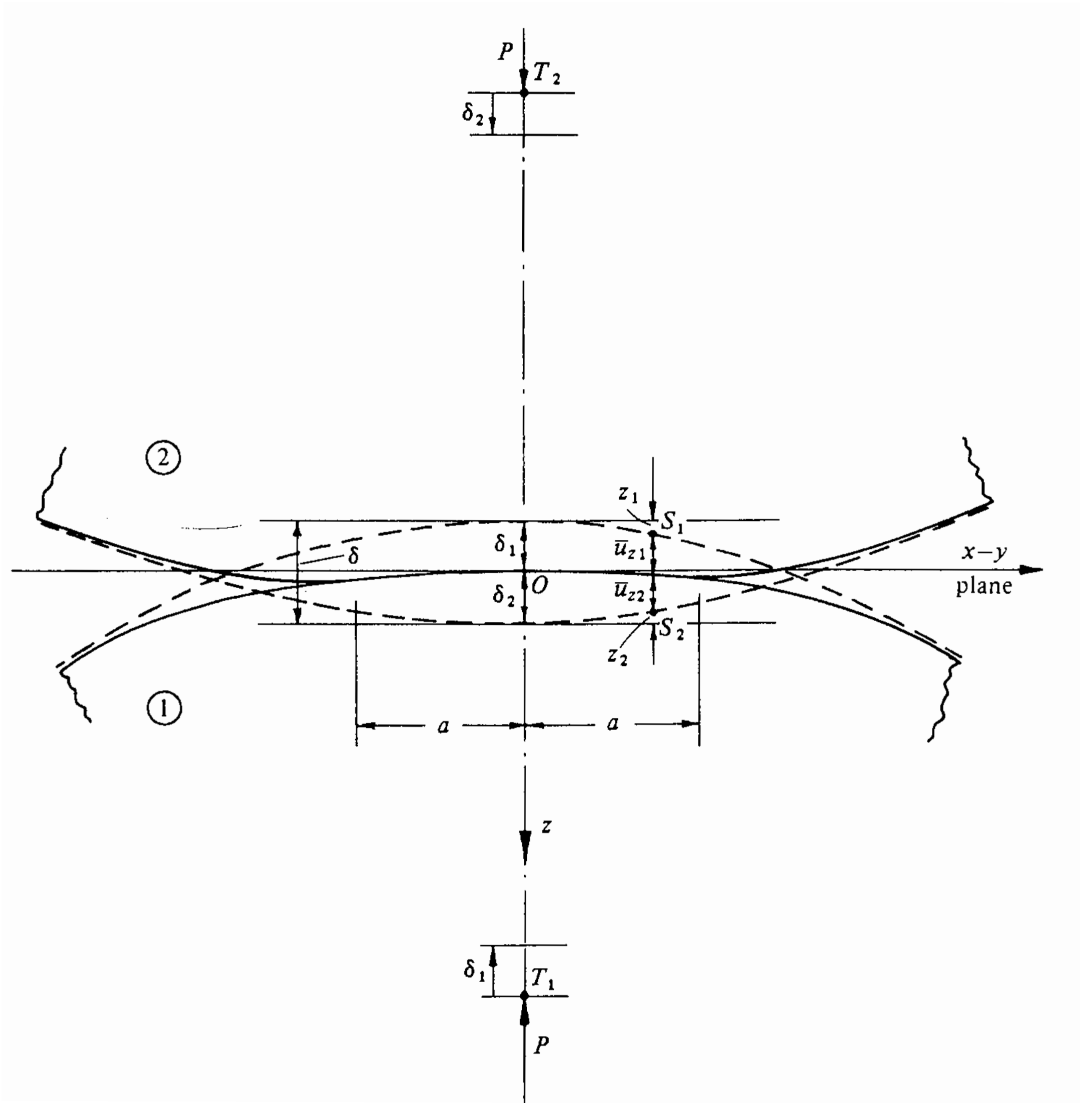


Figure 1.1: default

Alternatively, if after deformation the points S_1 and S_2 are outside of the contact region, this is simply

$$\bar{u}_{z1} + \bar{u}_{z2} > \delta - \frac{1}{2R^*} r^2 \tag{1.7}$$

It now is necessary to find a pressure distribution that satisfies these boundary conditions of displacement. Heinrich Hertz first formulated the expressions of Eqs. **??** in 1882. The power of his solution is borne out by the continuous use of his theory since that time. Hertz simplified the problem by regarding each body as an elastic half-space upon which the load is applied over a small, elliptical region (the contact area). This simplification allows for treatment of the highly concentrated stresses near the region of contact without consideration of either the general response of stresses in the bulk of the body or the manner in which they are supporting the load. This assumption is justifiable if the dimensions of each body as well as the relative radii of curvature are very large compared to the contact area. These assumptions are sufficient to proceed with the analysis, but the curious are pointed to an excellent discussion and background of Hertz's theory as given in KE Johnson's textbook.[?]

For solids of revolution, a distribution of pressure to satisfy the displacements of Eq. **??** is proposed by Hertz as

$$p = p_0 \left[1 - \left(\frac{r}{a} \right)^2 \right]^{1/2} \tag{1.8}$$

where a is the radius of the contact area.

The total load, F is found from the pressure distribution as

$$F = \int_0^a p(r) 2\pi r \, \mathrm{d}r \tag{1.9}$$

$$F = \frac{2}{3} p_0 \pi a^2 \tag{1.10}$$

From the distributed load over the circular region, stresses and deflections are found from superposition of point loads. The pressure is integrated (see Ref.[?]) to find the normal displacement for either solid body as

$$\bar{u}_z = \frac{1-\nu^2}{E} \frac{\pi p_0}{4a} \left(2a^2 - r^2 \right) \tag{1.11}$$

This is applied to both bodies and plugged into Eq. 1.6 to yield

$$\frac{\pi p_0}{4aE^*}\left(2a^2 - r^2\right) = \delta - \left(\frac{1}{2R^*}\right) r^2 \tag{1.12}$$

where we have introduced the now-common term of pair Young's modulus,

$$\frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \tag{1.13}$$

for simplification.

With the solution of Eq. 1.12, if we consider $r = a$ and $\delta(a) = 0$, we find the radius of the contact circle is

$$a = \frac{\pi p_0 R^*}{2E^*} \tag{1.14}$$

and when $r = 0$, we find the overlap as

$$\delta = \frac{\pi a p_0}{2E^*} \tag{1.15}$$

and alternatively we find the pressure as a function of overlap

$$p_0 = \frac{2E^*\delta}{\pi a} \tag{1.16}$$

The radius, overlap, and pressure relations are inserted into Eq. 1.9 to find the force (from now on referred to as the Hertz force) as a function of overlap, relative radius, and pair Young's modulus,

$$F = \frac{4}{3}E^*\sqrt{R^*}\,\delta^{3/2} \tag{1.17}$$

Equation 1.17 defines the normal contact forces between any two contacting, elastic spheres. This extremely important result acts as the basis of all discrete element method codes since the concept was first introduced for granular materials by Cundall & Strack in

1979.[?] To differentiate the force from other terms to be derived later, we specificy it as the normal force between sphere i and sphere j as

$$F_{n,ij} = \frac{4}{3} E^*_{ij} \sqrt{R^*_{ij}}\, \delta^{3/2}_{ij} \tag{1.18}$$

CHAPTER 2

Heat transfer in packed beds

This section covers a discussion of different modes of heat transfer experienced by a pebble in a packed bed. The main modes are conduction to neighbors and convection.

2.1 Single particle modes of heat transfer

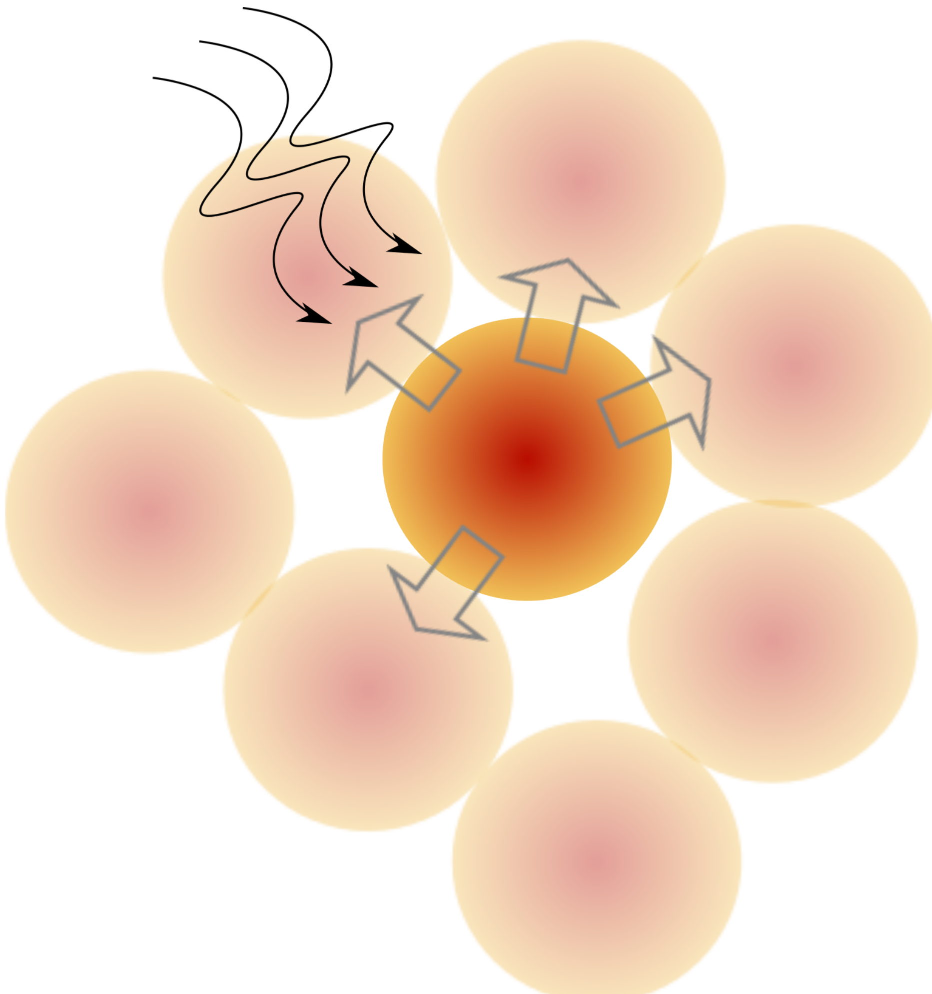
The transient energy balance for an irradiated pebble, shown in Fig. 2.1, in a packed bed with flowing interstitial gas is given by Eq. 2.1,

$$\rho V C \frac{\mathrm{d}T}{\mathrm{d}t} = \dot{Q}_g + \dot{Q}_{\text{conduction}} + \dot{Q}_{\text{convection}} + \dot{Q}_{\text{radiation}} \tag{2.1}$$

We begin with the lumped capacitance assumption that internal temperature gradients inside of the solid particle are negligible thus we can neglect diffusion terms in the solid. The validity of that assumption for the ceramic pebbles in fusion reactors will be discussed in detail in §**??**. The terms on the right-hand-side of Eq. 2.1 are:

1. Conduction through the stagnant fluid between two point- or non-contacted particles.
2. Conduction through the stagnant fluid between two area-contacted particles.
3. Conduction through the contact area between two area-contacted particles.
4. Conduction through the fluid in void space.
5. Radiation between the surfaces of two particles.
6. Radiation between adjacent voids.

Figure 2.1: Each ceramic pebble in a fusion reactor will experience multiple modes of heat transfer.



2.2 Inter-particle heat conduction

Handling the heat transfer between contacting particles has been investigated extensively by researchers in a number of fields.[?,?,?,?,?,?]

In the Hertz analysis we walked through in §1.1, we found the contact radius of two elastic spheres in Eq. 1.14 as a function of the contact pressure. We rewrite the radius in terms of the compression force acting on the bodies,

$$a = \left(\frac{3}{4} \frac{R^*}{E^*} \right)^{1/3} F^{1/3} \tag{2.2}$$

where $\frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}$ and $\frac{1}{R^*} = \frac{1}{R_1} + \frac{1}{R_2}$ as before.

Batchelor and O'Brien[?] made the brilliant observation that the temperature fields in the near-region of contacting spheres are analogous to the velocity potential of the potential flow of a fluid passing from from one reservoir to another through a circular hole in a planar wall. With the analogy, they could make use of the fluid flow solution to write the total flux across the circle of contact,

$$Q_{ij} = H_{ij}(T_i - T_j) \tag{2.3}$$

with the heat conductance,

$$H_{ij} = 2k_s a = 2k_s \left(\frac{3}{4} \frac{R^*}{E^*} \right)^{1/3} F^{1/3} \tag{2.4}$$

governing the time rate of energy transferred per temperature difference between particles, T_i and T_j, respectively. This approach, laid out by Batchelor and O'Brien, is valid when the thermal conductivity ratio of solid and fluid is well above unity and the contact area is small relative to the particle. The condition is expressed as,

$$\frac{k_s}{k_f} \frac{a}{R^*} = \lambda \gg 1 \tag{2.5}$$

The model, being derived from Hertz theory, also carries with it many of the assumptions and limitations inherent with that theory. The assumptions are discussed in detail in §1.1.

Recently, Cheng, et al.[?] proposed a slightly modified variant of the conductance given by Batchelor and O'Brien. In their model, they allow for contacting materials of different thermal conductivity. Therefore they have,

$$H_{ij} = 2k^* a = 2k^* \left(\frac{3}{4} \frac{R^*}{E^*} \right)^{1/3} F^{1/3} \tag{2.6}$$

where $\frac{1}{k^*} = \frac{1}{k_i} + \frac{1}{k_j}$. As well as being a more general, flexible formulation, the models analyzed by Cheng, et al.[?] are in good agreement with experiments and will be used in this study.

2.3 Nusselt number for spheres in packed beds

2.3.1 Convection by interstitial gas

Engineers have paid considerable attention to the calculation of convective heat transfer in packed beds. Correlations for determining the Nusselt number of a sphere in dilute and dense packings over a range of Reynolds and Prandtl number are available. We will cover the detail of many of these correlations in §??. The methods all come down to calculating Nusselt number to find the heat transfer coefficient and then computing the rate of heat transfer from convection as

$$\dot{Q}_{\text{convection}} = -hA(T_s - T_f) \tag{2.7}$$

where T_s is the temperature of the solid with surface area, A, and T_f is the local bulk temperature of the passing fluid. The negative sign is to maintain convention that energy transfer into the solid is positive.

2.3.2 Radiative transfer with neighboring particles

The temperatures expected in the solid breeder are high enough that we can not a priori neglect radiation. The radiation exchange between contacting neighbors in a packed bed becomes extremely complex due to the local and semi-local nature of radiation. A standard approach to treat radiation exchange between surfaces is to consider the view factor between them. In a dense, randomly packed bed of spheres the computation of view factors between pebbles can be done via a method such as that proposed Feng and Han.[?] Ideally, we could show this mode of heat transport is negligible compared to the others already discussed.

In ceramic breeder designs, the tritium breeding volume is rarely more than 2cm wide with pebbles that are, generally, 1mm in diameter. The maximum expected temperature in the breeding zone is about 1000K, roughly at the centerline of the 2cm width. The walls of the coolant must be held below the operable steel temperature of roughly 700K. This works out to a 300K differences spanning 10 pebble diameters. From this we can make a first-order approximation of 30K difference between neighboring pebbles. At the elevated temperatures, an estimate for the radiation exchange between two pebbles (allowing them to act as black bodies for this approximation) is

$$\dot{Q}_{\text{radiation}} = \sigma A \left(T_{\max}^4 - (T_{\max} - 30)^4\right) \approx 0.022\text{W} \tag{2.8}$$

which is the highest amount of radiation exchange we might expect between pebbles.

.1 Sphere with heat generation

To provide homogeneous boundary conditions, the temperature will everywhere be in reference to its difference from the fluid temperature, i.e. $\mathbb{T} = T - T_f$.

Energy equation

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\mathbb{T}) + \frac{g}{k} = \frac{1}{\alpha}\frac{\partial \mathbb{T}}{\partial t} \tag{.9}$$

Subject to

$$\frac{\partial \mathbb{T}}{\partial r}\Big|_b + \frac{h}{k}\mathbb{T}|_b = 0 \tag{.10}$$

and

$$\mathbb{T}|_0 = \text{finite} \tag{.11}$$

with initial condition

$$\mathbb{T}(r,0) = \mathbb{T}_0 \tag{.12}$$

Transform with

$$U(r,t) = r\mathbb{T}(r,t) \tag{.13}$$

Energy equation becomes

$$\frac{\partial^2 U}{\partial r^2} + \frac{gr}{k} = \frac{1}{\alpha}\frac{\partial U}{\partial t} \tag{.14}$$

Subject to

$$\frac{\partial U}{\partial r}\Big|_b + \left(\frac{h}{k} - \frac{1}{b}\right) U|_b = \frac{hb}{k} T_f \tag{.15}$$

and

$$U|_0 = 0 \tag{.16}$$

with initial condition

$$U(r,0) = T_0 r \tag{.17}$$

Break up the problem into simpler problems

1. A set of steady-state problems defined by $U_{ss}(r)$
2. A homogeneous time-dependent problem defined by $U_h(r,t)$

The solution of U_{ss} is found from the solution of

$$\frac{\partial^2 U_{ss}}{\partial r^2} + \frac{gr}{k} = 0 \tag{.18}$$

subject to the boundary conditions of

$$U_{ss}\big|_0 = 0 \tag{.19}$$

$$\frac{\partial U_{ss}}{\partial r}\big|_b + \left(\frac{h}{k} - \frac{1}{b}\right) U_{ss}\big|_b = 0 \tag{.20}$$

We separate and integrate,

$$\frac{\partial U_{ss}}{\partial r} = -\frac{g}{2k} r^2 + C_1 \tag{.21}$$

$$U_{ss} = -\frac{g}{6k} r^3 + C_1 r + C_2 \tag{.22}$$

The first boundary condition of Eq. .19 directly provides $C_2 = 0$. The second boundary condition is solved to give

$$C_1 = \frac{gb^2}{6k}\left(1 + \frac{2}{Bi}\right) \tag{.23}$$

valid for $Bi > 0$. Thus,

$$U_{ss} = \frac{gb^2}{6k}\left(1 + \frac{2}{Bi} - \frac{r^2}{b^2}\right) r \tag{.24}$$

We can now transform back to the temperature version of the equation with the simple reduction in power of r on U, (U = r$\mathbb{T}$),

$$\mathbb{T}_{ss} = \frac{gb^2}{6k}\left(1 + \frac{2}{Bi} - \frac{r^2}{b^2}\right) \tag{.25}$$

Now we non-dimensionalize with

$$\theta = \frac{\mathbb{T}}{gb^2/k} \tag{.26}$$

and

$$r^* = \frac{r}{b} \tag{.27}$$

$$\theta_{ss} = \frac{1}{6}\left(1 + \frac{2}{Bi} - r^{*2}\right) \tag{.28}$$

The next step is to find the homogeneous solution of

$$\frac{\partial^2 U_h}{\partial r^2} = \frac{1}{\alpha}\frac{\partial U_h}{\partial t} \tag{.29}$$

Subject to

$$U_h\big|_0 = 0 \tag{.30}$$

$$\frac{\partial U_h}{\partial r}\big|_b + \left(\frac{h}{k} - \frac{1}{b}\right) U_h\big|_b = 0 \tag{.31}$$

and the initial condition of

$$U_{h,0} = \mathbb{T}_0 r - U_{ss} \tag{.32}$$

$$= \left[\frac{\mathbb{T}_0}{gb^2/k} - \frac{1}{6}\left(1 + \frac{2}{Bi} - \frac{r^2}{b^2}\right)\right] r \tag{.33}$$

We will again transform the spatial variable into a dimensionless form, $r^* = r/b$. The equation to solve is then,

$$\frac{\partial^2 U_h}{\partial r^{*2}} = \frac{b^2}{\alpha}\frac{\partial U_h}{\partial t} \tag{.34}$$

with boundary conditions of

$$U_h\Big|_0 = 0 \tag{.35}$$

$$\frac{\partial U_h}{\partial r^*}\Big|_1 + (Bi-1)U_h\Big|_1 = 0 \tag{.36}$$

and an initial condition of

$$U_h(t=0) = T_0 r^* b - \sum_{j=0}^{N} U_{0j} \tag{.37}$$

$$= \left[\frac{\mathbb{T}_0}{gb^2/k} - \frac{1}{6}\left(1 + \frac{2}{Bi} - r^*\right)\right] r^* b \tag{.38}$$

The solution is assumed of the form

$$U_h = R(r^*)\Gamma(t) \tag{.39}$$

The solution for Γ is given as

$$\Gamma = \exp(-\zeta^2 t/\tau) \tag{.40}$$

where $\tau = b^2/\alpha$. The space-variable function $R(\zeta, r^*)$ satisfies the following eigenvalue problem:

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^{*2}} + \zeta^2 R = 0 \tag{.41}$$

subject to

$$R = 0 \tag{.42}$$

at $r^* = 0$, and

$$\frac{\mathrm{d}R}{\mathrm{d}r^*} + (Bi-1)R = 0 \tag{.43}$$

at $r^* = 1$. This eigenvalue problem is a special case of the Sturm-Liouville problem. The solution for U_h can be constructed from known eigenvalue solutions,

$$U(r,t) = \sum_{n=1}^{\infty} c_n R(\zeta_n, r^*) \exp(-\zeta^2 t/\tau) \tag{.44}$$

Application of the initial condition gives

$$F(r) = \sum_{n=1}^{\infty} c_n R(\zeta_n, r^*) \tag{.45}$$

where $F(r^*)$ is the initial condition we defined above,

$$F(r^*) = \left[\frac{\mathbb{T}_0}{gb^2/k} - \frac{1}{6}\left(1 + \frac{2}{Bi} - r^*\right)\right] \frac{gb^2}{k} r^* b \tag{.46}$$

The coefficients of c_n can be determined by applying the operator $\int_0^1 R(\zeta_n, r^*)\,\mathrm{d}r^*$ and utilizing the orthogonality property of eigenfunctions. The coefficients are found as

$$c_n = \frac{1}{N(\zeta_n)} \int_0^1 R(\zeta_n, r^{*'}) F(r^{*'})\,\mathrm{d}r^{*'} \tag{.47}$$

The norm, N is defined as

$$N(\zeta_n) = \int_0^1 \left[R(\zeta_n, r^*)\right]^2\,\mathrm{d}r^* \tag{.48}$$

With the coefficients defined in Eq. .47, they can be substituted back into Eq. .44,

$$U(r^*,t) = \sum_{n=1}^{\infty} \exp(-\zeta^2 t/\tau) \frac{R(\zeta_n, r^*)}{N(\zeta_n)} \int_0^1 R(\zeta_n, r^{*'}) F(r^{*'})\,\mathrm{d}r^{*'} \tag{.49}$$

The eigenfunctions for Eq. .41 are

$$R(\zeta_n, r^*) = \sin(\zeta_n r^*) \tag{.50}$$

where the eigenvalues are the root of

$$\zeta_n \cot(\zeta_n) = -H \tag{.51}$$

and the normalization integral is given as

$$\frac{1}{N(\zeta_n)} = 2\frac{\zeta_n^2 + H^2}{\zeta_n^2 + H^2 + H} \tag{.52}$$

where $H = (Bi - 1)$

In order to explicitly express the solution, we will first evaluate the integral

$$Z(\zeta_n)\frac{gb^2}{k}b = \int_0^1 R(\zeta_n, r^{*'})F(r^{*'})\,\mathrm{d}r^{*'} \tag{.53}$$

$$= \int_0^1 \sin(\zeta_n r^{*'})\left[\frac{\mathbb{T}_0}{gb^2/k} - \frac{1}{6}\left(1 + \frac{2}{Bi} - r^*\right)\right] r^* b\,\mathrm{d}r^{*'} \tag{.54}$$

$$= \left\{\left[\frac{\mathbb{T}_0}{gb^2/k} - \frac{1}{6}\left(1 + \frac{2}{Bi}\right)\right]\int_0^1 \sin(\zeta_n r^{*'})r^{*'}\,\mathrm{d}r^{*'} + \frac{1}{6}\int_0^1 \sin(\zeta_n r^{*'})r^{*'3}\,\mathrm{d}r^{*'}\right\}\frac{gb^2}{k}b \tag{.55}$$

The two unique integrals are evaluated as

$$C_n = \int_0^1 \sin(\zeta_n r^{*'})r^{*'}\,\mathrm{d}r^{*'} = \frac{\sin\zeta_n - \zeta_n\cos\zeta_n}{\zeta_n^2} \tag{.56}$$

$$K_n = \int_0^1 \sin(\zeta_n r^{*'})r^{*'3}\,\mathrm{d}r^{*'} = \frac{3(\zeta_n^2 - 2)\sin\zeta_n - \zeta_n(\zeta_n^2 - 6)\cos\zeta_n}{\zeta_n^4} \tag{.57}$$

Thus

$$Z(\zeta_n) = \left[\frac{\mathbb{T}_0}{gb^2/k} - \frac{1}{6}\left(1 + \frac{2}{Bi}\right)\right]C_n + \frac{1}{6}K_n \tag{.58}$$

or in terms of our dimensionless temperature,

$$Z(\zeta_n) = \left[\theta_0 - \frac{1}{6}\left(1 + \frac{2}{Bi}\right)\right]C_n + \frac{1}{6}K_n \tag{.59}$$

The solution in terms of the transformed variable, $U(r^*, T)$ is now written as

$$U(r^*,t) = \sum_{n=1}^{\infty} \exp(-\zeta^2 t/\tau)\sin(\zeta_n r^*)\frac{Z(\zeta_n)\frac{gb^2}{k}b}{N(\zeta_n)} \tag{.60}$$

Now we can write the solution for $\mathbb{T}(r^*,t)$ direclty from $U(r^*,t)$,

$$\mathbb{T}(r^*,t) = \sum_{n=1}^{\infty} \exp(-\zeta^2 t/\tau)\frac{\sin(\zeta_n r^*)}{r^*}\frac{Z(\zeta_n)\frac{gb^2}{k}}{N(\zeta_n)} \tag{.61}$$

With one last step, we introduce the dimensionless temperature defined from Eq. .26

$$\theta(r^*,t)_{t.g.} = \sum_{n=1}^{\infty} \exp(-\zeta^2 t/\tau)\frac{\sin(\zeta_n r^*)}{r^*}\frac{Z(\zeta_n)}{N(\zeta_n)} \tag{.62}$$

For comparison, we will non-dimensionalize the heat transfer as:

$$Q^* = \frac{Q}{Q_\infty} \tag{.63}$$

where Q_∞ is the maximum possible amount of energy transfer between the solid and fluid. This value is equal for both lumped capacitance and the exact solution and is:

$$Q_\infty = -\rho_r C_r V(T_f - T_i) \tag{.64}$$

Introducing this non-dimensional heat transfer term and the energy balance is expressed as:

$$Q^*_{t.g.} = \int \frac{-\rho_r C_r \big(T(r,t) - T_0\big)dV}{-\rho_r C_r V(T_\infty - T_0)} \tag{.65}$$

$$Q^*_{t.g.} = \frac{1}{V}\int 1 - \theta_{t.g.}\,\mathrm{d}V \tag{.66}$$

For a circle in spherical coordinates:

$$\mathrm{d}V = r^2 \sin(\phi)\mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta \tag{.67}$$

In non-dimensional terms and assuming only a radial dependence, this becomes:

$$\mathrm{d}V = 4\pi R^3 r^{*2} \mathrm{d}r^* \tag{.68}$$

The exact equation for dimensionless heat transfer for a sphere is then:

$$Q^*_{t.g.} = \frac{4\pi R^3}{V} \int_0^1 \left[1 - \sum_{n=1}^{\infty} \exp(-\zeta^2 t/\tau) \frac{\sin(\zeta_n r^*)}{r^*} \frac{Z(\zeta_n)}{N(\zeta_n)} \right] r^{*2} \, \mathrm{d}r^* \tag{.69}$$

This reduces to:

$$Q^*_{t.g.} = 1 - 3 \sum_{n=1}^{\infty} \exp(-\zeta^2 t/\tau) \frac{Z(\zeta_n)}{N(\zeta_n)} \int_0^1 r^* \sin(\zeta_n r^*) \, \mathrm{d}r^* \tag{.70}$$

And we recognize the integral as one which we solved previously,

$$Q^*_{t.g.} = 1 - 3 \sum_{n=1}^{\infty} \exp(-\zeta^2 t/\tau) \frac{Z(\zeta_n)}{N(\zeta_n)} C_n(\zeta_n) \tag{.71}$$

And we are left with a formula for the total heat removed from the sphere as only a function of the eigenvalues and time.