STAT8008: Time Series & Multi-Variate Analysis

Department of Mathematics
Cork Institute of Technology

Lecture 04: Exponential Smoothing

Outline

Lecture objectives

- Introduce exponential smoothing as a time series method.
- Discuss the different series patterns that can be fit using exponential smoothing.
- Learn how to apply exponential smoothing to time series data.

Introduction

- Exponential smoothing is a time series technique that can be a relatively **quick way of developing forecasts**.
- The technique is a **pure time series method**; this means that the technique is suitable when data have only been collected for the series that you wish to forecast but you have no independent variables.
- Exponential smoothing can therefore be applied in instances when there are not enough variables measured to construct good causal time series models, or when the quality of the data is such that causal time series models give poor forecasts.

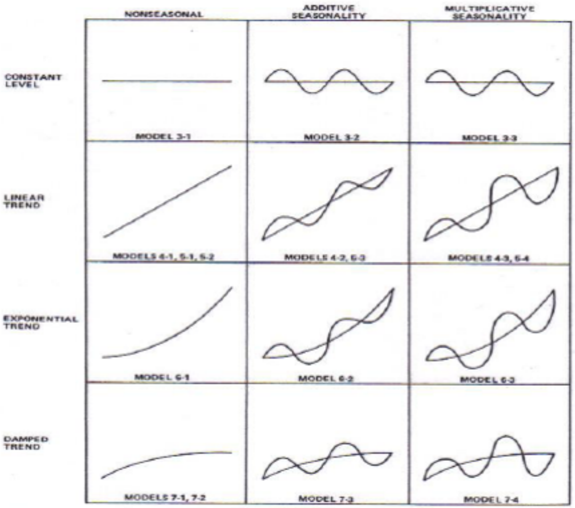
Introduction

- Exponential smoothing takes the approach that **recent observations should have relatively more weight** in forecasting than distant (in time) observations.
- Smoothing implies predicting an observation by a weighted combination of previous values, which is an **extension of prior moving average** procedure.
- Exponential smoothing implies that **weights decrease exponentially** as observations get older.

Types of Exponential Smoothing

1. Simple Exponential Smoothing
2. Browns Exponential Smoothing
3. Holts Exponential Smoothing
4. Damped Exponential Smoothing
5. Simple Seasonal (Additive)
6. Winters Additive
7. Winters Multiplicative

Types of Exponential Smoothing



Simple Exponential Smoothing

- If the sequence chart shows that there is **no general increase or decrease** in the series values and **no repeating seasonal pattern**, then the simple exponential smoothing model should be specified.
- Simple exponential smoothing (no trend, no seasonality) can be described in two algebraically equivalent ways. One common formula, known as the recurrence form, is as follows:

$$S_{(t)} = \alpha y_{(t)} + (1 - \alpha) S_{(t-1)}$$

- The predicted value at time m steps ahead is:

$$y_{(t+m)} = S_{(t)}$$

Simple Exponential Smoothing

- Where $y_{(t)}$ is the observed value of the time series in period t, $S_{(t-1)}$ is the smoothed level of the series at time $t-1$.
- α (alpha) is the smoothing parameter for the level of the series.
- $S_{(t)}$ is the smoothed level of the series at time t, computed after $y_{(t)}$ is observed.
- The formula states that the current smoothed value is obtained by combining information from two sources: the current point and the history embodied in the series.
- α is a weight ranging between 0 and 1. The closer α is to 1, the more exponential smoothing weights the most recent observation and the less it weights the historical pattern of the series.
- The smoothed value for the current case becomes the forecast value at all future time points.

Simple Exponential Smoothing

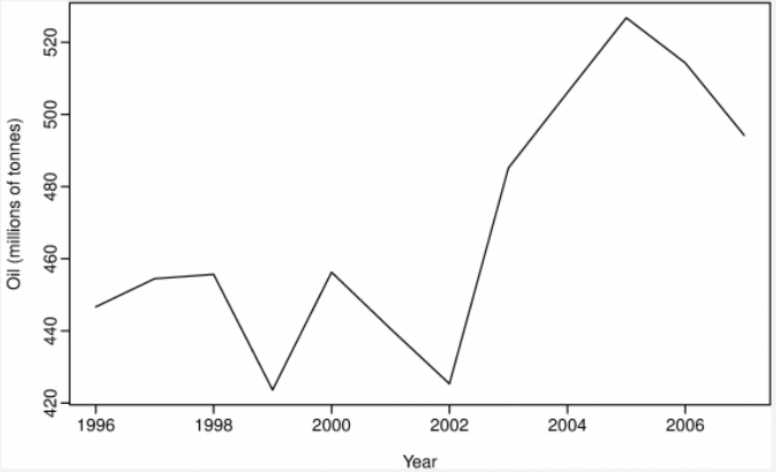


Figure 7.1: Oil production in Saudi Arabia from 1996 to 2007.

Brown's Exponential Smoothing

- A linear trend model assumes a constant increase or decrease in the series over time.
- Exponential smoothing generalises this to allow for a **slowly changing slope.** In other words, it allows a locally (but not necessarily globally) constant linear trend model.
- Browns model uses the **same weight coefficient** (α) when updating both the smoothed level and trend effects:

$$\begin{aligned} S_{(t)} &= S_{(t-1)} + T_{(t-1)} + \alpha\epsilon_{(t)}, \\ T_{(t)} &= T_{(t-1)} + \alpha^2\epsilon_{(t)}. \end{aligned}$$

- Predicted value:

$$y_{(t+m)} = S_{(t)} + mT_{(t)}.$$

Brown's Exponential Smoothing

- The smoothed level of the series at time t is given by the sum of:
  1. the smoothed level of the series at time $(t-1)$,
  2. the smoothed trend of the series at time $(t-1)$, and
  3. a fraction (determined by the value of α (alpha)) of the one-step-ahead forecast error.
- The smoothed trend at the end of time t is given by the sum of:
  1. the smoothed trend at the end of time $(t-1)$ and
  2. a fraction (determined by α^2) of the one-step-ahead forecast error.
- Finally, the forecast for m periods ahead from the time t is given by the linear equation shown.

Brown's Exponential Smoothing

- The second equation allows the estimate of the trend to change. An alpha value of 0 implies that no update of the trend and level components are necessary, and a constant slope and level will be fit.
- An alpha value near 1 suggests the level shifts and the trend involves a linear slope that changes over time.
- Since the slope is a function of α^2, a given error will influence the slope (by α^2) less than the level (by α).

Holt's Exponential Smoothing

- Holts model (discussed below) is more general in that it assigns **different weight coefficients** to trend and level effects.
- Holts exponential smoothing model with linear trend can be algebraically represented as follows:

$$\begin{aligned} S_{(t)} &= S_{(t-1)} + T_{(t-1)} + \alpha\epsilon_{(t)}, \\ T_{(t)} &= T_{(t-1)} + \alpha\gamma\epsilon_{(t)}. \end{aligned}$$

- Predicted value:

$$y_{(t+m)} = S_{(t)} + mT_{(t)}$$

Holt's Exponential Smoothing

- The smoothed level of the series at time t is given by the sum of:
  1. the smoothed level of the series at time $(t-1)$,
  2. the smoothed trend of the series at time $(t-1)$, and
  3. fraction (determined by the value of α (alpha)) of the one-step-ahead forecast error.
- The smoothed trend at the end of time t is given by the sum of:
  1. the smoothed trend at the end of time $(t-1)$ and
  2. a fraction (determined by the product of α (alpha) and γ (gamma)) of the one-step-ahead forecast error.
- Finally, the forecast for m periods ahead from the time t is given by the linear equation shown.

Holt's Exponential Smoothing

- The second equation allows the estimate of the trend to change. A gamma value of 0 implies that no update of the trend component is necessary and a constant slope will be fit.
- A gamma value near 1 suggests the trend involves a linear slope that changes over time.
- Since the weight coefficient for the trend is separate from that for the level, Holts model is more general than Browns

Damped Exponential Smoothing

- If there are signs that the series increases or decreases, but at a **decreasing rate** as the time series proceeds, then the damped exponential smoothing model should be specified.
- Remember: a linear trend model assumes a constant increase or decrease in the series over time, while Browns and Holts models permit the trend to change slowly over time. Damped exponential smoothing applies when there is a **linear trend that is dying out over time**.
- Algebraically represented as follows:

$$\begin{aligned} S_{(t)} &= S_{(t-1)} + \phi T_{(t-1)} + \alpha\epsilon_{(t)}, \\ T_{(t)} &= \phi T_{(t-1)} + \alpha\gamma\epsilon_{(t)}. \end{aligned}$$

- Predicted value:

$$y_{(t+m)} = S_{(t)} + \sum_{j=1}^{m} \phi^j T_{(t)}.$$

Damped Exponential Smoothing

- The smoothed level of the series at time t is given by the sum of:
    1. the smoothed level of the series at time $(t-1)$,
    2. the damped trend (damped by the value of ϕ (phi)) of the series at time $(t-1)$,
    3. a fraction (determined by the value of α (alpha)) of the one-step-ahead forecast error.
- The smoothed trend at the end of time t is given be the sum of:
    1. the damped trend (damped by the value of ϕ (phi)) at the end of time $(t-1)$,
    2. a fraction (determined by the product of α (alpha) and γ (gamma)) of the one-step-ahead forecast error.
- Finally, the forecast for m periods ahead from the time t is given by the equation shown in which the trend effect decreases at each time point by a factor of phi.

Damped Exponential Smoothing

- The second equation thus allows the estimate of the trend to change and explicitly takes damping into account.
- A gamma value of zero implies that no update of the trend component is necessary, and the slope will change only due to damping.
- A gamma value near one suggests the trend involves a linear slope that changes over time.
- The closer phi is to one, the more gradual is the effect on damping (a phi value of one indicates no damping occurs).
- Note that the coefficient for damping is separate from the weight coefficient for trend.

Simple Seasonal (Additive)

- Additive seasonality describes a series with a **repeating, seasonal pattern that maintains the same magnitude** if the series level increases or decreases.
- Algebraically represented as follows:

$$\begin{aligned} S_{(t)} &= S_{(t-1)} + \alpha \epsilon_{(t)}, \\ I_{(t)} &= I_{(t-p)} + \delta(1-\alpha)\epsilon_{(t)}. \end{aligned}$$

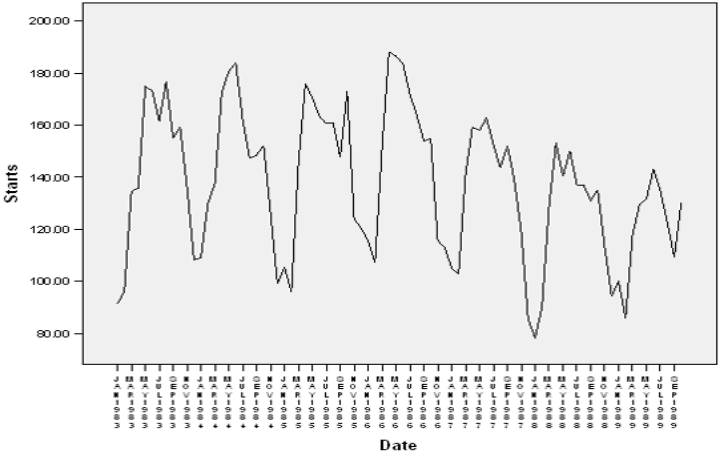
- Predicted value:

$$y_{(t+m)} = S_{(t)} + I_{(t-p+m)}$$

Simple Seasonal (Additive)

- α and γ represent the alpha and delta weight coefficients.
- $S_{(t)}$ is the smoothed level of the series computed after $y_{(t)}$ is observed.
- $I_{(t)}$ is the smoothed seasonal factor at the end of period t.
- m is the number of periods in the forecast lead time.
- $\epsilon_{(t)}$ is the one-step-ahead forecast error from the previous period.
- The predicted $y_{(t+m)}$ is the forecast for m periods ahead of the period t.

Simple Seasonal (Additive)



Winters Additive (Linear Trend and Additive Seasonality)

- If there are signs that the **series increases (decreases) over time** and there is a repeating **seasonal pattern that maintains the same magnitude** when the series level increase (decreases), then the Winter's Additive model should be used.
- Algebraically represented as follows:

$$\begin{aligned} S_{(t)} &= S_{(t-1)} + T_{(t-1)} + \alpha\epsilon_{(t)} \\ T_{(t)} &= T_{(t-1)} + \alpha\gamma\epsilon_{(t)} \\ I_{(t)} &= I_{(t-p)} + \delta(1-\alpha)\epsilon_{(t)} \end{aligned}$$

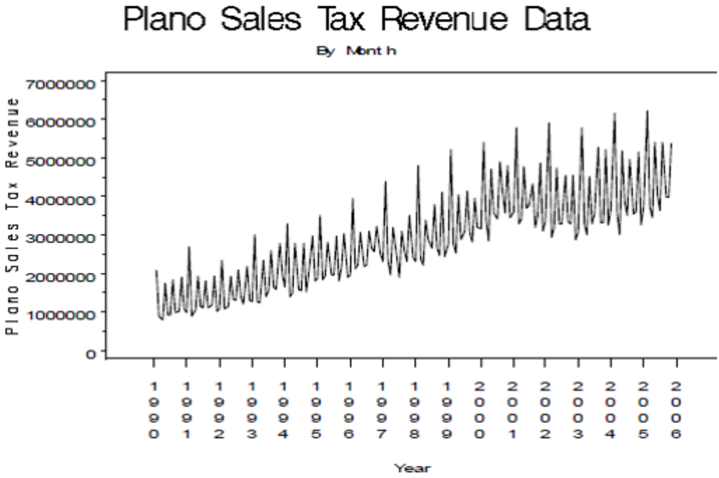
- Predicted value:

$$y_{(t+m)} = S_{(t)} + mT_{(t)} + I_{(t-p+m)}.$$

Winters Additive (Linear Trend and Additive Seasonality)

- α, γ and δ represent the alpha, gamma, and delta weight coefficients.
- $S_{(t)}$ is the smoothed level of the series computed after $y_{(t)}$ is observed.
- $T_{(t)}$ is the smoothed trend at the end of period t.
- $I_{(t)}$ is the smoothed seasonal factor at the end of period t.
- m is the number of periods in the forecast lead time.
- p is the number of time periods in one season.
- $\epsilon_{(t)}$ is the one-step-ahead error from the previous period.
- The predicted $y_{(t+m)}$ is the forecast for m periods ahead of the current period t.

Winters Additive (Linear Trend and Additive Seasonality)



Winters Multiplicative (Linear Trend and Multiplicative Seasonality)

- This method is used if there are signs that the series **increases (decreases) over time** and there is a repeating **seasonal pattern that increases (decreases) in magnitude**.
- Winters Multiplicative model accommodates **linear trend** and **multiplicative seasonality**; the seasonal pattern becomes more (less) pronounced when the series values increase (decrease).
- Algebraically represented as follows:

$$\begin{aligned} S_{(t)} &= S_{(t-1)} + T_{(t-1)} + \frac{\alpha\epsilon_{(t)}}{I_{(t-p)}} \\ T_{(t)} &= T_{(t-1)} + \frac{\alpha\gamma\epsilon_{(t)}}{I_{(t-p)}} \\ I_{(t)} &= I_{(t-p)} + \frac{\delta(1-\alpha)\epsilon_{(t)}}{S_{(t)}} \end{aligned}$$

- Predicted value:

$$y_{(t+m)} = \left(S_{(t)} + mT_{(t)}\right) I_{(t-p+m)}$$

Winters Multiplicative (Linear Trend and Multiplicative Seasonality)

- α, γ and δ represent the alpha, gamma, and delta weight coefficients.
- $S_{(t)}$ is the smoothed level of the series computed after $y_{(t)}$ is observed.
- $T_{(t)}$ is the smoothed trend at the end of period t.
- $I_{(t)}$ is the smoothed seasonal factor at the end of period t.
- m is the number of periods in the forecast lead time.
- p is the number of time periods in one season.
- $\epsilon_{(t)}$ is the one-step-ahead error from the previous period.
- The predicted $y_{(t+m)}$ is the forecast for m periods ahead of the current period t.

Recommended approach

The general approach when developing an exponential smoothing model is as follows:

- **Define the time series dates** using the define date procedure in SPSS
- Run a **sequence chart** for the variable which you wish to forecast, so that general trend and seasonal patterns can be identified
- Specify the appropriate **trend and seasonal patterns** for the exponential smoothing model or use the *Expert Modeler*
- Estimate the **model parameters** for the exponential smoothing model
- Test model **performance**
- Refine the model if possible
- **Forecast**

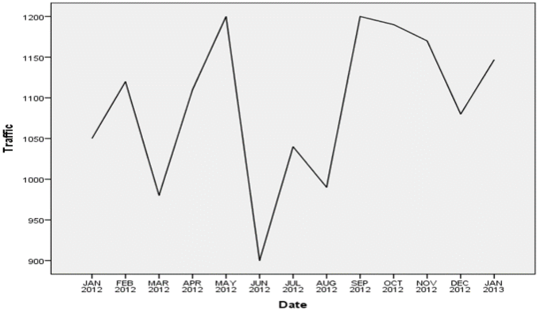
Example

Suppose we have set of data on the average monthly network traffic on a software module during a year. The data contains the month and the average level of traffic as shown:

Month (M)	Traffic (T)
1	1050
2	1120
3	980
4	1110
5	1200
6	900
7	1040
8	990
9	1200
10	1190
11	1170
12	1080

Example

Represent this data on a time series graph.



Use simple exponential smoothing to smooth this data.

Notes:

- Value for α???
- Value for $S_{(0)}$???

Value for α?

- If data show large randomness, use small α - i.e.

$$0.01 < \alpha < 0.3$$

- If data show pattern, use large α - i.e.

$$\alpha \to 1$$

  Suggest trend or seasonality

- Choose α which minimise MSE, MAPE... over a test set.
- In this example, use $\alpha = 0.6$

Initial condition?

Several alternatives for $S_{(0)}$:

1. $S_{(0)} = y_{(1)}$ or
2. $\phantom{S_{(0)}} =$ mean of all observations, or
3. $\phantom{S_{(0)}} =$ mean of the first 4, 5 or 6 observations, or
4. $\phantom{S_{(0)}} =$ mean of half of the data

Note: The weight attached to $S_{(0)}$ is $(1-\alpha)^t$, which is usually small. So the choice of $S_{(0)}$ becomes less important after processing many observations or large α is used.
In this example, use $S_{(0)} = 1000$.

Example: Solution

Month (M)	Traffic (T)	$\alpha y_{(t)} + (1-\alpha)S_{(t-1)}$	ES $S_{(t)}$
1	1050	— n/a —	1000
2	1120		
3	980		
4	1110		
5	1200		
6	900		
7	1040		
8	990		
9	1200		
10	1190		
11	1170		
12	1080		
13	1147		

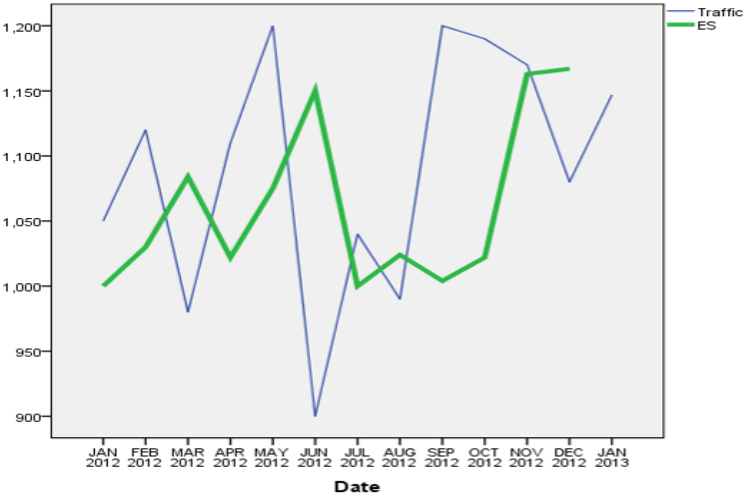
Example: Solution

Month (M)	Traffic (T)	ES
1	1050	1000
2	1120	1030
3	980	1084
4	1110	1022
5	1200	1075
6	900	1150
7	1040	1000
8	990	1024
9	1200	1004
10	1190	1022
11	1170	1163
12	1080	1167
13	1147	

Example: Solution

Month (M)	Traffic (T)	ES
1	1050	1000
2	1120	1030
3	980	1084
4	1110	1022
5	1200	1075
6	900	1150
7	1040	1000
8	990	1024
9	1200	1004
10	1190	1022
11	1170	1163
12	1080	1167
13	1147	1115

Example: Solution



Exercise 1

The table below shows passenger movements by air for a small airport. Figures are in thousands.

	Quarter			
Year	I	II	III	IV
2008	62	98	144	86
2009	51	84	128	68
2010	41	67	104	52

1. Plot the data on a time series graph.
2. Smooth the data using simple exponential smoothing with $\alpha = 0.6$ and $S_{(0)} = 97.5$.