Erratum to: Faithful Squashed Entanglement

Fernando Brandão, Matthias Christandl and Jon Yard

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Let

$$E_R(\rho_{A:B}) = \min_{\sigma_{AB} \in \mathsf{SEP}(A:B)} D(\rho \| \sigma)$$

be the relative entropy of entanglement and let

$$E_R^\infty(\rho_{A:B}) = \lim_{n \to \infty} \frac{1}{n} E_R(\rho_{A:B}^{\otimes n})$$

be its regularization. Let $D_{\mathsf{M}}(\rho_{A:B})$ be the optimal type-II error exponent for distinguishing the product states $\rho_{AB}^{\otimes n}$ (the null hypothesis) from the set $\mathsf{SEP}(A^n : B^n)$ of separable states (the alternative hypothesis) with arbitrarily small type-I error using measurements of type M, in the limit of large n. Essentially by definition,

$$D_{\mathsf{M}}(\rho_{A:B}) = \liminf_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \min_{\sigma_n \in \mathsf{SEP}(A^n:B^n)} D_{\mathsf{M}_n}^{\varepsilon}(\rho^{\otimes n} \| \sigma_n)$$

(see Appendix).

The proof of the main result (Theorem) in [1], [2] combines

Lemma 1.

$$I(A;B|E) \geq E_R^\infty(\rho_{A:BE}) - E_R^\infty(\rho_{A:E}) \tag{1}$$

and

Lemma 2.

$$E_R^\infty(\rho_{A:BE}) - E_R^\infty(\rho_{A:E}) \geq D_{\overrightarrow{\mathsf{LOCC}}}(\rho_{A:B}) \tag{2}$$

to conclude that

$$I(A;B|E) \geq D_{\overrightarrow{\mathsf{LOCC}}}(\rho_{A:B}). \tag{3}$$

The proof of Lemma 2 uses the statement $E_R^\infty(\rho_{A:B}) = D_{\mathsf{ALL}}(\rho_{A:B})$ from [3]. It was recently noticed [4] that the proof of this statement is incomplete. This leaves (3), and thus Theorem in [1], [2], without a complete proof. In the following we establish a proof of (3) by proving a stronger version

Lemma 1'.

$$I(A;B|E) \geq D_{\mathsf{ALL}}(\rho_{A:BE}) - D_{\mathsf{ALL}}(\rho_{A:E}). \tag{4}$$

of Lemma 1 and weakening Lemma 2 to the following

Lemma 2'.

$$D_{\mathsf{ALL}}(\rho_{A:BE}) - D_{\mathsf{ALL}}(\rho_{A:E}) \geq D_{\overrightarrow{\mathsf{LOCC}}}(\rho_{A:B}), \tag{5}$$

which was proved in [1], where it was combined with the problematic statement from [3] to prove Lemma 2.

To prove Lemma 1', we first let $D_{\max}(\rho\|\sigma) = \min\{\lambda : \rho \leq 2^{\lambda}\sigma\}$ be the max-relative entropy. Its smoothed version

$$D^{\varepsilon}_{\max}(\rho\|\sigma) \quad = \quad \inf\{D_{\max}(\rho'\|\sigma) : \tfrac{1}{2}\|\rho' - \rho\|_1 \leq \varepsilon\}$$

satisfies the following non-lockability bound for all $0 < \varepsilon < 1$:

$$\min_{\sigma \in \mathsf{SEP}(A:BE)} D^{\varepsilon}_{\max}(\rho_{ABE}\|\sigma_{ABE}) \leq \min_{\sigma \in \mathsf{SEP}(A:E)} D^{\varepsilon}_{\max}(\rho_{AE}\|\sigma_{AE}) + 2\log|B|. \tag{6}$$

Proof. Let σ_{AE} and ρ'_{AE} achieve $D^{\varepsilon}_{\max}(\rho_{AB}\|\sigma_{AE})$ so that $\rho'_{AE} \leq 2^{D^{\varepsilon}_{\max}(\rho_{AB}\|\sigma_{AE})}\sigma_{AE}$. Let ρ'_{ABE} be an extension of ρ'_{AE} satisfying $\frac{1}{2}\|\rho'_{ABE} - \rho_{ABE}\|_1 \leq \varepsilon$, which exists by Uhlmann's Theorem. Because

$$\rho'_{ABE} \leq |B|^2 \rho'_{AE} \otimes \tau_B,$$

we get

$$\rho'_{ABE} \leq 2^{2\log|B|}\rho'_{AE} \otimes \tau_B \leq 2^{2\log|B| + D^{\varepsilon}_{\max}(\rho_{AB}\|\sigma_{AE})}\sigma_{AE} \otimes \tau_B,$$

and so $\min_{\sigma \in \mathsf{SEP}} D'_{\max}(\rho_{ABE}\|\sigma) \leq D^{\varepsilon}_{\max}(\rho_{AB}\|\sigma_{AE}) + 2\log|B|$ as required. □

Furthermore it is easily shown to satisfy the following continuity property for all $0 < \varepsilon + \delta < 1$:

$$D^{\varepsilon}_{\max}(\rho\|\sigma) \geq D^{\varepsilon+\delta}_{\max}(\rho'\|\sigma) \text{ for } \tfrac{1}{2}\|\rho - \rho'\|_1 \leq \delta. \tag{7}$$

Next, we use the existence of asymptotically optimal state redistribution protocols transferring B from someone holding E to someone holding a purifier E' of ρ_{ABE}. Specifically, we use the existence [5]–[7] of sequences D_n, F_n and G_n of finite-dimensional Hilbert spaces and real numbers $\delta_n > 0$ satisfying $\frac{1}{n}\log|G_n| \to \frac{1}{2}I(A;B|E)$ and $\delta_n \to 0$, together with a sequence of encoding operations

$$\Lambda_n : B^n E^n D_n \to E^n F_n G_n$$

possessing asymptotic inverses $\Lambda'_n : E^n F_n G_n \to B^n E^n D_n$ such that the states

$$\phi_{A^n E^n F_n G_n} = (\mathrm{id}_{A^n} \otimes \Lambda_n)(\rho^{\otimes n}_{ABE} \otimes \tau_{D_n}) \tag{8}$$

satisfy

$$\frac{1}{2}\left\|\phi_{A^n E^n F_n} - \rho^{\otimes n}_{AE} \otimes \tau_{F_n}\right\|_1 \leq \delta_n \tag{9}$$

and

$$\frac{1}{2}\left\|\Lambda_n'(\phi_{A^nB^nE^nF_nG_n}) - \rho_{ABE}^{\otimes n}\otimes\tau_{D_n}\right\|_1 \leq \delta_n \tag{10}$$

for each n. The first condition implies that F_n is maximally entangled with a subsystem of $B^nG_nD_nE'^n$, so a decoding isometry exists by Uhlmann's Theorem. The second condition guarantees asymptotic reversibility, implying that the encoder Λ_n can be taken as the partial isometry obtained by inverting an Uhlmann decoder so as to maximally entangle D_n with a subsystem of $B^nE^nG_nF_n$

Proof of Lemma 1'. We first observe (see Appendix) that

$$D_{\mathsf{ALL}}(\rho_{A:B}) = \liminf_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\min_{\sigma_n\in\mathsf{SEP}(A:B)} D_H^{\varepsilon}(\rho_{AB}^{\otimes n}\|\sigma_n).$$

Now, given a protocol for quantum state redistribution as above, we find

$$\begin{aligned}
\min_{\sigma_n\in\mathsf{SEP}(A^n:B^nE^n)} D_H^{\varepsilon}(\rho_{ABE}^{\otimes n}\|\sigma_n) &\leq \min_{\sigma_n\in\mathsf{SEP}(A^n:B^nE^n)} D_{\max}^{1-2\varepsilon}(\rho_{ABE}^{\otimes n}\|\sigma_n) - \log(\varepsilon)\\
&\leq \min_{\sigma\in\mathsf{SEP}(A^n:B^nE^n)} D_{\max}^{1-\nu_n}(\phi_{A^nB^nE^nF_nG_n}\|\sigma_{A^nB^nE^nF_nG_n}) - \log(\varepsilon)\\
&\leq \min_{\sigma\in\mathsf{SEP}(A^n:E^n)} D_{\max}^{1-\nu_n}(\phi_{A^nE^n}\otimes\tau_F\|\sigma_{A^nE^nF_n}) + 2\log|G_n| - \log(\varepsilon)\\
&\leq \min_{\sigma\in\mathsf{SEP}(A^n:E^n)} D_{\max}^{1-\nu_n}(\rho_{AE}^{\otimes n}\|\sigma_{A^nE^n}) + 2\log|G_n| - \log(\varepsilon)\\
&= \min_{\sigma\in\mathsf{SEP}(A^n:E^n)} D_{\max}^{\sqrt{1-2\nu_n-\nu_n^2}}(\rho_{AE}^{\otimes n}\|\sigma_{A^nE^n}) + 2\log|G_n| - \log(\varepsilon)\\
&\leq \min_{\sigma\in\mathsf{SEP}(A^n:E^n)} D_H^{2\nu_n+\nu_n^2}(\rho_{AE}^{\otimes n}\|\sigma_{A^nE^n}) + 2\log|G_n| - \log(\varepsilon(1-2\nu_n+\nu_n^2))
\end{aligned}$$

where we use (12) in the first line, (9) and continuity (7) in the second line with $\nu_n = 2\varepsilon + \delta_n$, non-lockability (6) in the third line, (10) and continuity (7) in the fourth line and (11) in the last line. Dividing both sides by n, taking the limit of large n and subsequently the liminf as ε goes to zero, we find that

$$I(A;B|E) \geq \tilde{D}_{\mathsf{ALL}}(\rho_{A:BE}) - D_{\mathsf{ALL}}(\rho_{A:E})$$

as required. □

A Hypothesis testing

For a compact convex centrally symmetric set C of POVM elements containing I and 0, define the $(\varepsilon, \mathsf{C})$-hypothesis-testing relative entropy as

$$D^{\varepsilon}_{\mathsf{C}}(\rho\|\sigma) = -\log\min\{\operatorname{Tr} M\sigma : M \in \mathsf{C}, \operatorname{Tr} M\rho \geq 1-\varepsilon\}.$$

When $\mathsf{C} = [0, I]$, it reduces to the ε-hypothesis-testing entropy $D^{\varepsilon}_{\mathsf{ALL}}(\rho\|\sigma) = D^{\varepsilon}_{H}(\rho\|\sigma)$ of [8], [9]. Let $\mathsf{M} = (\mathsf{M}_n)$ be a sequence of such sets of POVM elements on A^nB^n that are also closed under tensor products, partial traces and permutations. Then

$$\liminf_{n\to\infty} \frac{1}{n} \min_{\sigma_n \in \mathsf{SEP}(A:B)} D^{\varepsilon}_{M_n}(\rho^{\otimes n}_{AB}\|\sigma_n)$$

is the optimal type-II error exponent for distinguishing the null hypothesis $\rho^{\otimes n}_{AB}$ from the alternative hypothesis (the separable states) among all measurements from M with type-I error at most ε. The optimal rate with vanishing type-I error is given by

$$D_{\mathsf{M}}(\rho_{A:B}) = \liminf_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \min_{\sigma_n \in \mathsf{SEP}(A^n:B^n)} D^{\varepsilon}_{\mathsf{M}_n}(\rho^{\otimes n}\|\sigma_n),$$

or equivalently, by the limit as $\varepsilon \to 0$ over all $\varepsilon > 0$, as the limit is obtained by simply plugging in $\varepsilon = 0$. While each $D^{\varepsilon}_{H}(\rho^{\otimes n}_{AB})$ is continuous as a function of ε on the whole interval $[0, 1]$, the sequence does not converge uniformly on any interval containing 0, i.e. the limiting error rate is discontinuous at 0. Hence we want the liminf and not the limit.

Watrous gave a related quantity

$$D^{\varepsilon}_{W}(\rho\|\sigma) \quad = \quad -\inf\{\log\operatorname{Tr} X\sigma : 0 \leq X, \varepsilon X \leq I, \operatorname{Tr} X\rho \geq 1\}$$

that is defined for all $0 \leq \varepsilon \leq 1$ and interpolates between $D^0_W = D^0_{\max} = D_{\max}$ and $D^1_W = D^0_H = \operatorname{Tr} P_{\operatorname{im}(\rho)}\sigma$. Watrous proves (5.11 and 5.12) the following bounds:

$$D^{\sqrt{\varepsilon}}_{\max} \leq D^{\varepsilon}_{W} - \log\varepsilon(1-\varepsilon) \quad \text{and} \quad D^{\eta+\delta}_{W} \leq D^{\eta}_{\max} + \log\left(1 + \frac{\eta}{\delta}\right).$$

When $0 < \varepsilon \leq 1$,

$$D^{\varepsilon}_{W} = -\inf\left\{\log\frac{\operatorname{Tr}\sigma P}{\varepsilon} : 0 \leq P \leq I, \operatorname{Tr}\rho P \geq \varepsilon\right\} = D^{1-\varepsilon}_{H} + \log\varepsilon.$$

Therefore Watrous 5.11 gives the bound

$$D^{\sqrt{1-\varepsilon}}_{\max} \leq D^{\varepsilon}_{H} - \log\varepsilon. \tag{11}$$

Similarly, when applied to Watrous 5.12 with $\eta = 1 - 2\varepsilon$ and $\delta = \varepsilon$ this gives

$$D^{\varepsilon}_{H} \leq D^{1-2\varepsilon}_{\max} - \log\varepsilon. \tag{12}$$

B References

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