Redistributing quantum fields

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1 State redistribution

Consider a pure state $|\rho\rangle_{ACBD}$ of the tensor product of finite-dimensional systems, where Alice holds AC and Bob holds B. The goal is for Alice to give C to Bob, so that Alice holds only A and Bob holds CB, while maintaining all correlations in the global pure state. To achieve this, Alice may send a quantum system C_0 to Bob, and Alice and Bob may consume and produce shared entanglement.

A **state redistribution protocol** for $|\rho\rangle^{ACBD}$ consists of an input entangled state $|\psi\rangle^{C_B C_B}$ that is consumed by the protocol, an output $|\psi\rangle^{C_A C_A}$ entangled state that is produced by the protocol, an encoding contraction

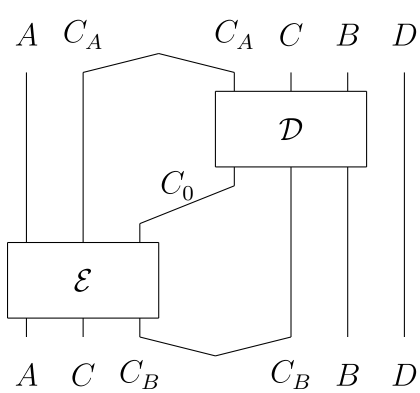
$$\mathcal{E} : ACC_B \to AC_AC_0$$

and a decoding

$$\mathcal{D} : C_0C_BB \to C_ACB$$

contraction. The protocol has error ϵ if

$$\left|\langle\psi|^{C_AC_A}\langle\rho|^{ACBD}\mathcal{D}\mathcal{E}|\psi\rangle^{C_BC_B}|\rho\rangle^{ACBD}\right|^2 \geq 1-\epsilon.$$



1.1 Redistributing i.i.d. states

For an i.i.d. source $|\rho\rangle_{ACBD}^{\otimes n}$ a rate pair (Q,E) is called **achievable** if there exists a sequence of protocols with

$$\frac{1}{n}\log|C_0| \to Q,$$
$$\frac{1}{n}(\log|C_A| - \log|C_B|) \to E$$

and $\epsilon \to 0$. The rate pairs

$$\begin{aligned} Q &\geq \frac{1}{2}I(C;D|A) = \frac{H(C|A)+H(C|B)}{2} \\ E &\geq \frac{1}{2}I(A;C) - \frac{1}{2}I(B;C) = \frac{H(C|B)-H(C|A)}{2} \end{aligned}$$

are achievable by [**DY08**, **YD09**], matching the outer bound of [**LD09**]. We outline an achievability proof, following [**Opp08**].

Let $\delta > 0$. For all sufficiently large n, there exists a δ-typical subspace $C_\delta^n \subset C^n$ with $\frac{1}{n}\log|C_\delta^n| \leq H(C)+\delta$ and $\operatorname{Tr} P_{C_\delta^n}\rho_C^{\otimes n} \geq 1-\epsilon$ for $\epsilon = \exp(-O(\delta^2 n))$. Randomly decomposing the typical subspace $C_\delta^n \simeq C_A C_0 C_B$ into subsystems satisfying

$$\lim \frac{1}{n}\log|C_A| \leq \frac{1}{2}I(C;A) - \delta$$

and

$$\lim \frac{1}{n}\log|C_B| \leq \frac{1}{2}I(C;B) - \delta$$

implies that $Q \geq \frac{1}{2}I(C;D|A)+\delta$,

$$\mathbb{E}P(\rho_{BD}^{\otimes n} \otimes \rho_{C_A}, \rho_{B^n C_A D^n}) \to 0,$$

and

$$\mathbb{E}P(\rho_{AD}^{\otimes n} \otimes \rho_{C_B}, \rho_{A^n C_B D^n}) \to 0.$$

Therefore

$$|\rho\rangle_{ACBD}^{\otimes n}|\Phi\rangle^{C_A C_A} \approxeq |\rho\rangle^{A^n C_0 B^n C_A C_B D^n} \approxeq |\rho\rangle_{ACBD}^{\otimes n}|\Phi\rangle^{C_B C_B}.$$

A protocol with error ϵ is then obtained from an asymptotic decomposition $C_\delta^n = C_A C_0 C_B$ with

$$\rho^{AC_A D} \approx \rho^{AD} \otimes \rho^{C_A}$$

and

$$\rho^{BC_B D} \approx \rho^{BD} \otimes \rho^{C_B}$$

because Uhlmann gives two natural isometries

$$\begin{aligned} U_A &: AC_A C_0 \to ACC_B \\ U_B &: C_0 C_B B \to C_A CB, \end{aligned}$$

where $U_A|\rho\rangle \simeq |\rho\rangle|\Phi\rangle^{C_B C_B}$ and $U_B|\rho\rangle \simeq |\Phi\rangle^{C_1}|\rho\rangle$. After Alice encodes, the state is

$$U_A^\dagger|\Phi\rangle^{C_B C_B}|\rho\rangle,$$

then Bob decodes to get

$$U_B U_A^\dagger|\Phi\rangle^{C_B C_B}|\rho\rangle \approx |\rho\rangle|\Phi\rangle^{C_A C_A}.$$

1.2 $Q = 0$

If $Q = 0$ there can be subtleties in finite dimensions, such as when $|\rho\rangle^{ACBD} = |\rho\rangle^{AC}|\rho\rangle^{BD}$. If the shared resource is maximal entanglement, we note that a sublinear amount of communication is still required if $|\rho\rangle^{AC}$ is non-maximal. On the other hand, entanglement concentration does not require any communication. Most generally, any state with $I(C;D|A) = 0$ is, up to local unitaries, of the form

$$\sum_x \sqrt{p_x}|x\rangle^{A'}|x\rangle^{B'}|\rho_x\rangle^{A_C B_C C}|\rho_x\rangle^{A_D B_D D} \sum_x \sqrt{p_x}|\rho_x\rangle^{A_x^C B_x^C C}|\rho_x\rangle^{A_x^D B_x^D D}$$

with $A = \bigoplus_x A_x^C A_x^D$, $B = \bigoplus_x B_x^C B_x^D$.

1.3 Distances

Note that $\|X\|_1 \le \sqrt{d}\|X\|_2$.

$$1 - F(\rho,\sigma) \le \frac{1}{2}\|\rho - \sigma\|_1 \le \sqrt{1 - F(\rho,\sigma)^2}$$

for $F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$. In particular,

$$\begin{aligned} |\langle\psi|\phi\rangle|^2 &\ge 1 - \epsilon \\ &\Rightarrow \\ \frac{1}{2}\Big\||\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|\Big\|_1 &\le \sqrt{\epsilon} \\ &\Rightarrow \\ |\langle\psi|\phi\rangle| &\ge 1 - \sqrt{\epsilon}. \end{aligned}$$

It may be better to work with the **purified distance** $P(\rho,\sigma) = \sqrt{1 - F(\rho,\sigma)^2}$, which still satisfies

$$P(\rho,\sigma)^2 \le \frac{1}{2}\|\rho - \sigma\|_1 \le P(\rho,\sigma)$$

even on subnormalized states. For pure states

$$\epsilon = P(|\rho\rangle, |\sigma\rangle) = \sin(\theta)$$

and

$$F(|\rho\rangle, |\sigma\rangle) = \sqrt{1 - \epsilon^2} = \cos(\theta_v) = \cos(\theta_s/2)$$

where e.g. θ_v measures photon polarization and θ_s is the spin polarization angle in the Bloch sphere.

The distance from a density matrix ρ_{CD} to the product of its marginals is

$$\|\rho_{CD} - \rho_C \otimes \rho_D\|_2^2 = \operatorname{Tr}\rho_{CD}^2 + \operatorname{Tr}\rho_C^2 \operatorname{Tr}\rho_D^2 - 2\operatorname{Tr}\rho_{CD}(\rho_C \otimes \rho_D).$$

The distance to the product of a maximally mixed state on C and the marginal on D takes the simpler form

$$\left\|\rho_{CD} - \frac{I_C}{d_C} \otimes \rho_D\right\|_2^2 = \operatorname{Tr}\rho_{CD}^2 - \frac{\operatorname{Tr}\rho_D^2}{d_C},$$

to which one can often reduce by inserting $\rho_C^{\pm 1/2}$s in appropriate places.

1.4 Decoupling

Lemma 1.1. *Let U be a Haar random unitary on $C = C_0C_1$ and consider the random state $\rho_{CE}(U) = U\rho_{CE}U^\dagger$. The variance of the reduced state $\rho_{C_1E}(U) = \operatorname{Tr}_{C_0}\rho_{CE}(U)$ satisfies*

$$\mathbb{E}\left\|\rho_{C_1E}(U) - \frac{I_{C_1}}{d_{C_1}} \otimes \rho_E\right\|_2^2 \leq \frac{\operatorname{Tr}\rho_{CE}^2}{d_{C_0}}.$$

Proof. We begin by calculating

$$\mathbb{E}\left\|\rho_{C_1E}(U) - \frac{I_{C_1}}{d_{C_1}} \otimes \rho_E\right\|_2^2 = \mathbb{E}\Big(\operatorname{Tr}\big(\rho_{C_1E}(U) - \mathbb{E}\rho_{C_1E}(U)\big)^2\Big) = \mathbb{E}\operatorname{Tr}\rho_{C_1E}(U)^2 - \frac{\operatorname{Tr}\rho_E^2}{d_{C_1}}.$$

Let $\Sigma_C = P_C^+ - P_C^- : C^{\otimes 2} \to C^{\otimes 2}$ be the swap operator, where $P_C^\pm$ are the projectors onto the symmetric and antisymmetric subspaces. Any $\mathcal{O} \in \operatorname{End}(C^{\otimes 2})$ satisfies

$$\begin{aligned}
\mathbb{E}(U\otimes U)\mathcal{O}(U\otimes U)^{-1} &= \frac{2\operatorname{Tr}\mathcal{O}P_C^+}{d_C^2 + d_C}P_C^+ + \frac{2\operatorname{Tr}\mathcal{O}P_C^-}{d_C^2 - d_C}P_C^- \\
&= \frac{\operatorname{Tr}\mathcal{O}\Big(I_C^{\otimes 2} - \frac{1}{d_C}\Sigma_C\Big)}{d_C^2 - 1}I_C^{\otimes 2} + \frac{\operatorname{Tr}\mathcal{O}\Big(\Sigma_C - \frac{1}{d_C}I_C^{\otimes 2}\Big)}{d_C^2 - 1}\Sigma_C.
\end{aligned}$$

For $\mathcal{O} = I_{C_0}^{\otimes 2} \otimes \Sigma_{C_1}$, this gives

$$\mathbb{E}(I_{C_0}^{\otimes 2} \otimes \Sigma_{C_1})(U) = \frac{d_{C_0}^2 d_{C_1} - d_{C_1}}{d_C^2 - 1}I_C^{\otimes 2} + \frac{d_{C_0}d_{C_1}^2 - d_{C_0}}{d_C^2 - 1}\Sigma_C.$$

Therefore

$$\mathbb{E}\operatorname{Tr}\rho_{C_1E}(U)^2 \quad = \quad \operatorname{Tr}\rho_{CE}^{\otimes 2}(\mathbb{E}\Sigma_{C_1}(U) \otimes \Sigma_E) = \frac{d_{C_0}^2 d_{C_1} - d_{C_1}}{d_C^2 - 1}\operatorname{Tr}\rho_C^2 + \frac{d_{C_0}d_{C_1}^2 - d_{C_0}}{d_C^2 - 1}\operatorname{Tr}\rho_{CE}^2.$$

Plugging in to the initial bound gives

$$\begin{aligned}
\mathbb{E}\left\|\rho_{C_1E} - \frac{I_{C_1}}{d_1} \otimes \rho_E\right\|_2^2 &= \left(\frac{(d_{C_0}^2 - 1)d_{C_1}}{d_C^2 - 1} - \frac{1}{d_{C_1}}\right)\operatorname{Tr}\rho_C^2 + \frac{d_{C_0}(d_{C_1}^2 - 1)}{d_C^2 - 1}\operatorname{Tr}\rho_{CE}^2 \\
&\leq \frac{\operatorname{Tr}\rho_{CE}^2}{d_{C_0}},
\end{aligned}$$

by the inequalities $\frac{(d_{C_0}^2-1)d_{C_1}}{d_C^2-1} \leq \frac{1}{d_{C_1}}$ and $\frac{d_{C_0}(d_{C_1}^2-1)}{d_C^2-1} \leq \frac{1}{d_{C_0}}$.

$\square$

Therefore there exists a deterministic U such that

$$\begin{aligned}
\left\|\rho_{C_1E}(U) - \frac{I_{C_1}}{d_{C_1}} \otimes \rho_E\right\|_1 &\leq \sqrt{d_{C_1}d_E}\left\|\rho_{C_1D}(U) - \frac{I_{C_1}}{d_1} \otimes \rho_E\right\|_2 \\
&\leq \sqrt{d_{C_1}d_E\frac{\operatorname{Tr}\rho_{CE}^2}{d_{C_0}}} = \frac{\sqrt{d_C d_E \operatorname{Tr}\rho_{CE}^2}}{d_{C_0}}
\end{aligned}$$

which is $\approx \frac{2^{\frac{n}{2}I(C;E)}}{d_{C_0}}$ on i.i.d. states and $\to 0$ if $\liminf \frac{1}{n}\log d_{C_0} > \frac{1}{2}I(C;E)$.

1.5 Decoupling subalgebras

Instead of decoupling a subsystem we can decouple a subspace [**HHWY08**]. Given a state ρ_{CE} and a decomposition $C = C_0 \oplus C_1$, the POVM

$$\sqrt{\frac{d_C}{d_{C_0}}} P_{C_0} U dU$$

produces an ensemble of states $\rho_{C_0 E}(U)$ satisfying the analogous bound

$$\mathbb{E}_U \left\| \rho_{C_0 E}(U) - \frac{1}{d_{C_0}} I_{C_0} \otimes \rho_E \right\|_2^2 \leq \operatorname{Tr} \rho_{CE}^2.$$

Then the trace norm distance is bounded by

$$\mathbb{E}_U \left\| \rho_{C_0 E}(U) - \frac{1}{d_{C_0}} I_{C_0} \otimes \rho_E \right\|_1^2 \leq \frac{1}{d_{C_0} d_E} \operatorname{Tr} \rho_{CE}^2.$$

On tensor power states, the rhs $\to 0$ exponentially with n provided that $\limsup \frac{1}{n} \log d_{C_0}$ is less than the coherent information $I_c = H(E|C)$.

Both these decoupling results should follow from a common one. In the subsystem case, there is an injective algebra map $\operatorname{End}(C_0) \to \operatorname{End}(C)$ acting as $\mathcal{O}_{C_0} \mapsto \mathcal{O}_{C_0} \otimes I_{C_1}$. Given a direct sum decomposition $C_0 \oplus C_1$, the corresponding conditional expectation $\operatorname{End}(C) \to \operatorname{End}(C_0) \oplus \operatorname{End}(C_1)$ is a homomorphism, as are the components $\operatorname{End}(C) \to \operatorname{End}(C_i)$.

$\operatorname{End}(C_0)$ is the intersection of a left ideal $\operatorname{Hom}(C_0, C)$ and a right ideal $\operatorname{Hom}(C, C_0)$.

Let $\mathcal{A}$ be a C*-algebra. Then we can define a norm on linear functionals, therefore on states, as

$$\|\nu\| = \sup\{|\nu(a)| : a \in \mathcal{A}, \|a\| \leq 1\}.$$

Suppose that $\mathcal{A}$ is furthermore a von Neumann algebra. Then it has a **predual** $\mathcal{A}_*$ with $\mathcal{A} \simeq \mathcal{A}_*^*$.

2 Reverse Shannon

The capacity $\max_{p(x)} I(X;Y)$ of a classical channel $p(y|x)$ characterizes the rate at which it can simulate perfect bit channels, as well as the rate at which it can be simulated using perfect bit channels, in the presense of common randomness. The entanglement-assisted quantum capacity $\frac{1}{2}\max_{\rho_A} I(A;B)$ of a quantum channel $A \to B$ characterizes the rate at which it can simulate perfect qubit channels. This is also the rate at which it can be simulated by perfect qubit channels when assisted with arbitrary entangled states.

To formulate this in more detail, let $\mathcal{N}_{D\to C}$ be a quantum channel with Stinespring extension $\mathcal{U}_{D\to CA}$. The goal is for Alice and Bob to jointly carry the isometry $\mathcal{U}_{D\to CA}^{\otimes n}$, such that Alice holds the environment systems A^n, with vanishing error as $n \to \infty$. The simplest setting with a complete solution allows Alice and Bob to send noiseless qubits $[q \to q]$ and to use perfect EPR states $[qq]$, while only requiring good simulation on i.i.d. inputs $\rho_A^{\otimes n}$. The set of achievable rate pairs (Q, E) for this task is

$$Q \geq \frac{1}{2} I(D;C), \;\; Q + E \geq H(C).$$

At the corner point, qubits are minimized, while up the diagonal boundary qubits are traded for ebits. At the corner point, the protocol is reversible and we get the resource *equality*

$$\langle \mathcal{U}_{D\to CA}, \rho_D \rangle = \frac{1}{2} I(C;D)[q \to q] + \frac{1}{2} I(C;A)[qq].$$

This task was originally called fully quantum reverse Shannon in [**Dev06**], then as state splitting, the reverse of state merging. The idea is that Alice is given half of a purification $|\psi\rangle_{DD}$ of ρ_D, then she locally applies the isometry to turn the state into $|\psi\rangle_{ACD}$, reducing the simulation task to giving C to Bob.

On the other hand, if EPR entanglement is a free resource, we can charge instead for the rate R of classical communication and see that the simulation cost becomes $I(C;D)$. Maximizing this over all ρ_D shows that the entanglement assisted classical capacity equals its reverse Shannon capacity, so the existence of a quantum reverse Shannon protocol will imply a strong converse for entanglement assisted classical communication.

2.1 Achievability

To achieve the corner point, Alice first turns her copies of $|\rho\rangle_{DD}$ into $|\rho\rangle_{ACD} = \mathcal{U}^{D\to CA}|\rho\rangle_{DD}$ after which she must redistribute the C systems to Bob. Equivalently we might view this as state redistribution on a family of states of the form where the isometry is fixed.

Let $U : C_\delta \to C_0 C_1$ be a random unitary decomposition of the typical subspace $C_\delta \subset C^n$, with $C_1 \lesssim \frac{n}{2} I(C;A)$ qubits, $C_0 \gtrsim \frac{n}{2} I(C;D)$ qubits and $C_\delta \gtrsim nH(C)$ qubits. Then with high probability, $\rho_{D^n C_1} \simeq \rho_D^{\otimes n} \otimes \pi_{C_1}$. Let $|\Phi\rangle_{C_1 C_1}$ be maximally entangled and note that $|\rho\rangle_{D^n C_1 \boldsymbol{C_0 A^n}}$ and $|\Phi\rangle_{\boldsymbol{C_1} C_1} |\rho\rangle_{D^n \boldsymbol{C_\delta A^n}}$ are purifications of $\rho_D^{\otimes n} \otimes \pi_{C_1}$, where purifying systems are boldfaced. Therefore there exists a partial isometry $V : \boldsymbol{C_0 A^n} \to \boldsymbol{C_1 C_\delta A^n}$, defined on the support of $\rho_{C_0 A^n}$, such that

$$V|\rho\rangle_{D^n \boldsymbol{C_0} C_1 \boldsymbol{A^n}} \simeq |\rho\rangle_{D\boldsymbol{CA}}^{\otimes n} |\Phi\rangle_{C_1 \boldsymbol{C_1}}.$$

Note that it is sufficient to use a partial isometry defined on the support of $\rho_{C_0 A_\delta}$, whose rank grows at the rate

$$\frac{H(\rho_{C_0 A^n})}{n} = \frac{H(\Phi_{C_1} \otimes \rho_{C_\delta A^n})}{n} = \frac{H(\rho_{D^n C_1})}{n} \gtrsim \frac{H(A) + H(C) + H(D)}{2}.$$

To redistribute the C systems to Bob, Alice first compresses C^n onto C' and then applies $V^{-1} : C_1 C_\delta A^n \to C_0 A^n$, where C_1 holds her half of the entanglement. Then she sends C_0 to Bob. After receiving C_0, Bob applies $U^{-1} : C_0 C_1 \to C_\delta$, where C_1 is a second copy holding Bob's half of the entanglement, after which he holds C.

2.2 Non-tensor-power sources

We consider states

$$|\rho\rangle^{A^n C^n D} = U_{F\to AC}^{\otimes n} |\rho\rangle^{D^n D^n}.$$

that can be created over n uses of a the isometric extension $U_{D\to AC}$ of a channel $D \to C$. Then the communication cost is $\frac{1}{2} I(D^n; C^n | A^n) = \frac{1}{2} I(D^n; C^n)$ and the entanglement cost is $\frac{1}{2} I(A^n; C^n)$.

If we aim to minimimize the number of qubits sent for tensor power sources, the entanglement cost is $\frac{1}{2} I(B; E)$ ebits. If we change the rules and only allow Alice to send bits, the entanglement cost changes to $H(B)$.

A superposition of tensor power sources, on the other hand, requires superpositions of different amounts of maximal (EPR) entanglement. It suffices for Alice and Bob to share a large enough embezzling state

$$|\varphi_N\rangle \propto \sum_{n=1}^{N} \frac{1}{\sqrt{n}} |n\rangle|n\rangle,$$

i.e. for large enough N.

The series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - p^{-s})^{-1}$$

converges absolutely for $\mathrm{Re}(s) > 1$. Its completion

$$\Lambda(s) = \pi^{s/2} \Gamma(s/2) \zeta(s)$$

satisfies the functional equation $\Lambda(1 - s) = \Lambda(s)$, so $\zeta(s)$ can be analytically continued everywhere except for a pole at $s = 1$. What does this have to do with the infinite embezzling state

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} |n\rangle|n\rangle?$$

This state is not normalizable, but we would like to somehow make sense of it as a limit

$$\lim_{s\to 1} \frac{\sum_{n=1}^{\infty} n^{-s/2} |n\rangle|n\rangle}{\sqrt{|\zeta(s)|}}.$$

Is it a state on an algebra, such as

$$\bigotimes_p \mathbb{C}^{p\times p}???$$

The answer to this seems to be yes via the Bost-Connes system. A more "basic physics" interpretation uses the Casimir effect.

First steps: Removing finitely many local factors $(1-p^{-s})^{-1}$ shouldn't change the state - this is directly analogous to the functional equation for partial zeta functions (where the local factors at finitely many primes are left out). Interpret as perfect embezzlement of arbitrary entangled states (can pull out maximal entanglement in any dimension). Formulate single-shot state merging with such resource states.

Key insight: The "magic" of state redistribution/splitting is that the entanglement is used to "teleport" part of the state that is suitably decoupled from other systems. Use TT Theory to allow such renormalizable states to prove a "clean" state redistribution theorem, will yield a fundamental primitive for quantum communication with clear connections to nonlocal games and which may also provide new tools for existence proofs in QFT.

3 Topological entanglement entropy

3.1 Spurious topological entanglement entropy

Interpretation via boundary conditions.

Kim et al [**KLL+23**]

4 C^*-algebras

4.1 States on C^*-algebras

A C^***-algebra** is a Banach $*$-algebra $\mathcal{A}$ with $\|a\|\|a^*\| = \|aa^*\|$ for every $a \in \mathcal{A}$. Equivalently, it is a $\mathbb{C}$-algebra with involution * of the second kind, closed under the norm

$$\|a\| = \sup_{\pi} \|\pi(a)\|,$$

where the sup is over all $*$-homomorphisms $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. We do not consider non-unital C^*-algebras.

A **von Neumann algebra** is a C^*-algebra $\mathcal{A}$ that is closed under any and thus all of the following: the w.o.t. (weak operator topology, weakest s.t. all $\langle \psi, a\phi \rangle$ are continuous), the s.o.t. (strong operator topology) or the ultraweak (a.k.a. weak-$*$) topology. Equivalently, a C^*-algebra is a von Neumann algebra iff it has a **predual** $\mathcal{A}_*$, consisting of the ultraweakly continuous linear functionals on $\mathcal{A}$, such that $\mathcal{A}$ is the Banach space dual of $\mathcal{A}_*$.

A **weight** on a C^*-algebra is a linear functional $\omega\colon \mathcal{A} \to \mathbb{C}$ taking the cone $\mathcal{A}_+ = \{aa^\dagger : a \in \mathcal{A}\}$ of positive-semidefinite operators to a subset of $[0, \infty]$, i.e. s $\omega(\mathcal{A}_+) \subset [0, \infty]$. A **state** is a weight ω normalized such that $\omega(1) = 1$, a **trace** is a weight with $\omega(aa^\dagger) = \omega(a^\dagger a)$ for all $a \in \mathcal{A}$ and a **tracial state** is a trace that is also a state.

The set of states is convex and equals the weak*-closure of the extreme points, or pure states. Given a state φ on $\mathcal{A}$, the GNS Theorem guarantees the existence of a Hilbert space $\mathcal{H}$, a $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and a vector $|\varphi\rangle \in \mathcal{H}$ such that $\pi(\mathcal{A})|\varphi\rangle \subset \mathcal{H}$ is dense and $\rho(a) = \langle\varphi|a|\varphi\rangle$ for all $a \in \mathcal{A}$. Furthermore, φ is pure iff π is an irreducible representation (see e.g. [**Mur14**] Theorem 5.1.6).

Type-II and type-III factors have a unique indecomposable representation up to isomorphism; in particular this means they have no irreducible representations, hence no pure states. Rather, projections in their commutant give isomorphism subrepresentations.

4.2 Tomita-Takesaki theory

Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra with a cyclic and separating vector $|\psi\rangle \in \mathcal{H}$. Then $\mathcal{A}|\psi\rangle \subset \mathcal{H}$ is dense and the map $a \mapsto a|\psi\rangle$ has trivial kernel. Such can be obtained from the weak-$*$ completion of a representation of a C^*-algebra with respect to a faithful state.

Let S_0 be the unbounded linear operator defined on $\mathcal{A}|\psi\rangle$ by $S_0 : a|\psi\rangle \mapsto a^\dagger|\psi\rangle$. Its closure

$$S_\psi = \overline{S_0} = S_0^{**}$$

is known as the **Tomita operator**. The **modular operator** $\Delta_\psi = S_\psi^* S_\psi$ is positive definite; the corresponding polar decomposition

$$S_\psi = J\Delta_\psi^{1/2}$$

defines the **modular conjugation** J, an antilinear isometry that satisfies $J = J^{-1} = J^*$ and exchanges the algebra with its commutant $\mathcal{A}' = J\mathcal{A}J$ under conjugation. In particular, the modular operator of $\mathcal{A}'$ is

$$S_\psi^* = \Delta_\psi^{1/2} J = J\Delta_\psi^{-1/2},$$

i.e. $S'_\psi = S^*_\psi$, $J' = J$ and $\Delta'_\psi = J\Delta_\psi J = \Delta_\psi^{-1}$.

The modular operator defines a 1-parameter family $\sigma_t^\psi(a) = \Delta_\psi^{it} a \Delta_\psi^{-it}$ of **modular automorphisms** of $\mathcal{A}$. It turns out that J only depends on ψ up to an inner automorphism.

Connes showed that given another faithful state φ, then there are unitaries u_t such that $\sigma_t^\varphi(a) = u_t \sigma_t^\psi(a) u_t^{-1}$. They should satisfy $u_{s+t} = u_s \sigma_s^\psi(u_t)$. This means that the action σ_t^φ of $\mathbb{R}$ on $\mathcal{A}$ is obtained by twisting the action of σ_t^ψ by a 1-cocycle u_t.

The **relative Tomita operator** is the closed operator $S_{\psi|\varphi}$ defined on a dense subspace by

$$S_{\psi|\varphi} a|\psi\rangle = a^\dagger|\varphi\rangle.$$

The polar decomposition $S_{\psi|\varphi} = J_{\psi|\varphi}\Delta_{\psi|\varphi}^{1/2}$ determines the **relative modular conjugation** $J_{\psi|\varphi}$ and **relative modular operator** $\Delta_{\psi|\varphi}$. The **relative entropy**

$$S_{\psi|\varphi}(\mathcal{A}) = -\langle\psi|\log\Delta_{\psi|\varphi}|\psi\rangle$$

between states in a region $\mathcal{A}$.

For example, let $A = \mathbb{C}^{2\times 2}$, let $\rho = \left(\begin{smallmatrix}\lambda_0 & 0\\ 0 & \lambda_1\end{smallmatrix}\right)$ with $\lambda_0 \geq \lambda_1 > 0$, let $\lambda = \lambda_0/\lambda_1$ and consider the tracial state $\varphi : \mathbb{C}^{2\times 2} \to \mathbb{C}$ with $\varphi(a) = \operatorname{Tr} a\rho$. The GNS construction embeds $\iota : \mathcal{A} \hookrightarrow \mathcal{A}\otimes\mathcal{A}$ via $\iota(a) = a\otimes 1$ such that the pure state

$$|\varphi\rangle = \sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle = \begin{pmatrix}\sqrt{\lambda_0}\\ 0\\ 0\\ \sqrt{\lambda_1}\end{pmatrix}$$

satisfies $\varphi(a) = \langle\varphi|a\otimes 1|\varphi\rangle$ for all $a \in \mathcal{A}$. If $C : \alpha|ij\rangle \mapsto \bar{\alpha}|ij\rangle$ is complex conjugation in the computational basis and $\Sigma : |ij\rangle \mapsto |ji\rangle$ is the swap gate, then

$$J = C\Sigma, \quad \Delta = \begin{pmatrix}1 &&& \\ & \lambda && \\ && \lambda^{-1} & \\ &&& 1\end{pmatrix}, \quad S = C\begin{pmatrix}1 &&& \\ && \lambda^{-1/2} & \\ & \lambda^{1/2} && \\ &&& 1\end{pmatrix} = J\Delta^{1/2}.$$

Then $J(a\otimes 1)J = 1\otimes\bar{a}$, whereas

$$\Sigma(a\otimes 1)|\varphi\rangle = \Sigma(a\otimes 1)\Sigma|\varphi\rangle = (1\otimes a)|\varphi\rangle.$$

4.3 Bost-Connes system

Let Γ be a group. A subgroup $\Gamma_0 \subset \Gamma$ is called **almost normal** if each double coset in $\Gamma_0\backslash\Gamma/\Gamma_0$ is a union of finitely many left (equivalently, right) cosets, i.e.

$$\Gamma_0\gamma\Gamma_0 = \bigcup_{i=1}^{L(\gamma)} \gamma_{L,i}\Gamma_0 = \bigcup_{i=1}^{R(\gamma)} \Gamma_0\gamma_{R,i}$$

for bi-invariant functions L, R on Γ. Another equivalent condition is that the **commensurator**

$$\left\{\gamma \in \Gamma : [\Gamma_0 : g\Gamma_0 g^{-1} \cap \Gamma_0] < \infty\right\}$$

of the pair (Γ, Γ_0) is all of Γ.

Many interesting examples take Γ_0 to be a finitely generated (or does it need to be arithmetic?) subgroup of the rational points Γ of an algebraic group. For instance, $\Gamma = \mathrm{GL}_2(\mathbb{Q})$ and $\Gamma_0 = \mathrm{SL}_2(\mathbb{Z})$ is relevant in number theory, and $\Gamma = P(\mathbb{Q})$ and $\Gamma_0 = P(\mathbb{Z})$ is relevant for the Bost-Connes system [**BC95**], where $P \subset \mathrm{GL}_2$ is the group of matrices fixing the line spanned by $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, for which

$$\Gamma = P(\mathbb{Q}) = \left\{\left(\begin{smallmatrix} 1 & b \\ 0 & d \end{smallmatrix}\right) : b \in \mathbb{Q}, d \in \mathbb{Q}^\times\right\}, \ \Gamma_0 = P(\mathbb{Z}) = \left\{\left(\begin{smallmatrix} 1 & b \\ 0 & \pm 1 \end{smallmatrix}\right) : b \in \mathbb{Z}\right\}.$$

A right-invariant function $f \in \mathbb{C}(\Gamma)^{\Gamma_0} \simeq \mathbb{C}(\Gamma/\Gamma_0)$ and a left-invariant function $g \in {}^{\Gamma_0}\mathbb{C}(\Gamma) \simeq \mathbb{C}(\Gamma_0\backslash\Gamma)$ have a well-defined convolution product $f * g \in \mathbb{C}(\Gamma)$, given by

$$(f * g)(\gamma) = \sum_{\Gamma_0\eta\in\Gamma_0\backslash\Gamma} f(\gamma\eta^{-1})g(\eta).$$

If f is also left invariant, then so is $f * g$. Similarly, if g is right invariant, so is $f * g$. The bi-invariant functions are therefore closed under convolution, as is the Hecke algebra $H = \mathbb{C}(\Gamma_0\backslash\Gamma/\Gamma_0)_0$, corresponding to the bi-invariant functions supported on finitely many double cosets.

Let

$$\mathcal{H} = \ell^2(\Gamma_0\backslash\Gamma) = \left\{\psi : \Gamma_0\backslash\Gamma \to \mathbb{C} \text{ such that } \sum_{\Gamma_0\gamma\in\Gamma_0\backslash\Gamma} |\psi(\gamma)|^2 < \infty\right\}$$

be the Hilbert space of functions on the right cosets. Convolution defines a left action $h(\psi) = h * \psi$ of $h \in H$ on $\psi \in \mathcal{H}$, giving an injective $*$-homomorphism $H \hookrightarrow \mathcal{B}(\mathcal{H})$. The C^*-algebra completion $C^*(H)$ of H for the induced norm

$$\|h\| = \sup_{\psi\in\mathcal{H}} \frac{\langle\psi, h\psi\rangle}{\langle\psi, \psi\rangle}$$

determines an injective $*$-homomorphism $C^*(H) \hookrightarrow \mathcal{B}(\mathcal{H})$.

Let $L(\gamma)$ be the number of left cosets in $\Gamma_0\gamma\Gamma_0$ and let $R(\gamma)$ be the number of right cosets. Then $L, R \in \mathbb{C}(\Gamma_0\backslash\Gamma/\Gamma_0)$ and $L(\gamma^{-1}) = R(\gamma)$. There is a 1-parameter subgroup $\{\sigma_t : t \in \mathbb{R}\} \subset \mathrm{Aut}(C^*(H))$ given by

$$\sigma_t(h)(\gamma) = \left(\frac{L(\gamma)}{R(\gamma)}\right)^{-it} h(\gamma).$$

A thermal state of the C^*-dynamical system $(C^*(H), \sigma_t)$ at inverse temperature $\beta > 0$ is a state φ on $C^*(H)$ satisfying the KMS condition: For any $a, b \in C^*(H)$, there is a bounded analytic function F on the strip $0 \leq \operatorname{Im} \tau \leq \beta$ such that for all $t \in \mathbb{R}$,

$$\varphi(a\sigma_t(b)) = F(t), \;\; \varphi(\sigma_t(b)a) = F(t + i\beta).$$

Bost-Connes proved that at high temperatures $0 < \beta \leq 1$, there is a unique KMS state, which is a "factor state" on the hyperfinite factor R_∞ of type III_1. At low temperatures $\beta > 1$, they found that the thermal states form a Choquet simplex whose extreme points φ_v correspond to complex embeddings $v : \mathbb{Q}^{ab} \hookrightarrow \mathbb{C}$ of the maximal abelian extension $\mathbb{Q}^{ab}$ of $\mathbb{Q}$. The partition function is the Riemann zeta function. What does this mean???

By a "factor state" I think they mean that the weak closure $C^*(H)''$ in the GNS representation associated to the thermal state has trivial center. Is there a sense in which this is like the thermofield double?

4.4 Examples of almost normal subgroups

Does Γ_0 need to be arithmetic, or even finitely presented? I don't think that $G(\mathbb{Q})$ is finitely generated, but it is finitely presented in a certain sense. I know that arithmetic implies finitely presented - are thin groups finitely presented? I would think so as their relations should come from the defining equations of the group... Or maybe this makes no sense... all you get is the product - not clear how knowing the relations would help - all are subgroups of a GL_n.

5 Redistributing fields

5.1 Haag-Kastler axioms

Let M be a globally hyperbolic spacetime, i.e. a time-oriented smooth Lorentzian manifold with a Cauchy hypersurface (through which each maximal differentiable timelike curve passes exactly once). The usual example is Minkowski space $\mathbb{R}^{1,D-1}$ with the Lorentzian metric

$$ds^2 = -dt^2 + dx_1^2 + \cdots + dx_{D-1}^2$$

and and $t = 0$ Cauchy hypersurface $S = \{(0, x) : x \in \mathbb{R}^{D-1}\}$.

Given an open subset $U \subset M$, let U' be the subset of points in M that are spacelike separated from every point in the closure $\overline{U}$ of U. Then U' is a causally complete open set and U'' is the causal closure of U. In Minkowski space, the main examples are causal diamonds.

We are interested in C^*-algebra-valued functions $\mathcal{O}$ on the open sets of M satisfying the following physically motivated axioms:

- $U \subset V \Rightarrow \mathcal{O}(U) \subset \mathcal{O}(V)$ (precosheaf)
- $\mathcal{O}(U'') = \mathcal{O}(U)$ (causal closure reconstruction)
- U and V spacelike separated $\Leftrightarrow [\mathcal{O}(U), \mathcal{O}(V)] = 0$ (locality)

Given a subset $\mathcal{A} \subset \mathcal{O}(M)$, let

$$\mathcal{A}' = \{b \in \mathcal{O}(M) : [a, b] = 0 \text{ for all } a \in \mathcal{A}\}$$

denote its commutant in the global algebra.

Some theories [**BW75**, **Wit20**] satisfy the stronger condition

$$\mathcal{O}(U') = \mathcal{O}(U)' \quad \text{(Haag duality)}$$

but others do not. [**LRT78**, **DL84**, **Sch19**, **HO19**, **Wit18**]. In the GNS representation, the algebras $\mathcal{O}(U)$ in such theories are von Neumann algebras.

Generalizations have been considered [**Cam07**].

Let $A_0 C_0$ be Alice's neighboring regions in the past and let $C_1 B_1$ be Bob's in the future. We seek to factor the automorphism $\mathrm{id}_{C_0 C_1} \in \mathcal{O}(C_0 C_1)$ moving C_0 to C_1 as a product

$$\alpha_{C_0 C_1} = \alpha_{C_1 B_1} \, \mathrm{id}_{C_0' C_1'} \, \alpha_{A_0 C_0},$$

where $C_0' \subset C_0$ and $C_1' \subset C_1$.

5.2 von Neumann algebras as resources

Let $\mathcal{O}_0 \subset \left(\mathbb{C}^{2\times 2}\right)^{\infty}$ be the subalgebra of local operators with finite support, for which all but finitely many terms equal I_2. For each sequence $\lambda_1, \lambda_2, \ldots \in [0,1]$, let

$$\Psi_\lambda = \Psi_{\lambda_1} \otimes \Psi_{\lambda_2} \otimes \cdots \in \left(\mathbb{C}^{2\times 2}\right)^{\otimes\infty},$$

where

$$\Psi_{\lambda_i} = \begin{pmatrix} \sqrt{\frac{1}{1+\lambda_i}} & 0 \\ 0 & \sqrt{\frac{\lambda_i}{1+\lambda_i}} \end{pmatrix}.$$

Then $\mathcal{H}_{\lambda,0} = \mathcal{O}_0 \Psi_\lambda \mathcal{O}_0$ is spanned by by infinite tensor products

$$\Psi = \Psi_1 \otimes \Psi_2 \otimes \cdots$$

of finite support, with $\Psi_i \neq \Psi_{\lambda_i}$ finitely often. Define a Hermitian form

$$\langle \Phi, \Psi \rangle = \prod_{i \in S \supset \mathrm{supp}} \operatorname{Tr} \Phi_i^\dagger \Psi_i$$

on $\mathcal{H}_{\lambda,0}$ by truncating anywhere outside the union of the supports and consider the von Neumann algebra $\mathcal{A}_\lambda \subset \mathcal{B}(\mathcal{H}_\lambda)$ generated by the left $\mathcal{O}_0$-action $a|\Phi\rangle = |a\Phi\rangle$. Its commutant $\mathcal{A}'_\lambda$ is the von Neumann algebra generated by the right action $a|\Phi\rangle = |\Phi a^T\rangle$ of $\mathcal{O}_0$.

The state $\langle \Psi_\lambda | a | \Psi_\lambda \rangle$ is tracial iff $\lambda_i = 1$ for all i iff it is the type-II_1 hyperfinite factor R. For $0 < \lambda < 1$, it gives the hyperfinite type III_λ factor iff the sequence converges to λ. If the λs converge to 0 it is either type I_∞ or III_0, and if the λs are dense in an appropriate sense, it gives the hyperfinite type III_1 factor.

Note that

$$\langle \Phi_{\lambda_i} | a_i b_i | \Phi_{\lambda_i} \rangle = \langle \Phi_{\lambda_i} | b_i a_i | \Phi_{\lambda_i} \rangle$$

for all a_i, b_i iff $\lambda_i = 1$, so that

$$\langle \Phi_\lambda | ab | \Phi_\lambda \rangle = \langle \Phi_\lambda | ba | \Phi_\lambda \rangle$$

for all a, b iff $\lambda_i = 1$ for all i. However, if $\lambda_i \neq 1$ for any i, it will no longer be a trace.

Let $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{B}(\mathcal{H})$ be von Neumann algebras on a separable Hilbert space $\mathcal{H}$. The commutants satisfy $\mathcal{A}'_1 \subset \mathcal{A}'_0$.

5.3 Perfect embezzlement

CLP [**CLP17**] shows that there exists a pure state $|\Psi\rangle^{\mathcal{H}}$ on a separable Hilbert space $\mathcal{H}$ and commuting unitaries $U \in \mathrm{U}(A \otimes \mathcal{H})$, $V \in \mathrm{U}(\mathcal{H} \otimes B)$ such that $UV|\Psi\rangle = |\psi_1\rangle^{AB}|\Psi\rangle^{\mathcal{H}}$. They give two proofs: The first shows this is equivalent to the existence of a certain state $\rho : \mathrm{U}_{\mathrm{nc}}(2) \otimes_{\max} \mathrm{U}_{\mathrm{nc}}(2) \to \mathbb{C}$ on the max-tensor product of the universal C^*-algebras of 2×2 unitaries. By the GNS construction, there then exists a Hilbert space $\mathcal{H}$ a $*$-homomorphism

$$\pi : \mathrm{U}_{\mathrm{nc}}(2) \otimes_{\max} \mathrm{U}_{\mathrm{nc}}(2) \to \mathcal{B}(\mathcal{H})$$

and a cyclic vector $|\psi\rangle \in \mathcal{H}$ realizing that state as $\langle \psi | a | \psi \rangle$. An explicit protocol is also given there.

5.4 Split property

Subsets $A \subset AC$ of Minkowski space satisfy the **split property** if there exists a type-I factor $\mathcal{N}$ such that

$$\mathcal{O}_A \subset \mathcal{N} \subset \mathcal{O}_{AC}.$$

This implies there exists a unitary $U : \mathcal{H} \to \mathcal{H}_1 \otimes \mathcal{H}_2$ such that

$$U\mathcal{O}_A U^\dagger \subset U\mathcal{N}U^\dagger = \mathcal{B}(\mathcal{H}_1) \otimes I_{\mathcal{H}_2}.$$

For B spacelike separated from AC, then

$$U\mathcal{O}_B U^{-1} \subset U\mathcal{N}' U^\dagger = I_{\mathcal{H}_1} \otimes \mathcal{B}(\mathcal{H}_2).$$

The split property can fail when there are higher-form symmetries. For a 2+1D TQFT, the group of abelian anyons is its 1-form symmetry group. Even Maxwell theory has 1-form symmetry group $\mathbb{Z}/2$ generated by the fermion. So?

5.5 Quantum correlations

Consider the following sets of correlations $\{p(a,b|x,y)\}$ achievable via increasingly powerful families of strategies:

- C_c = classical strategies
- C_q = finite-dimensional quantum strategies
- C_{qs} = infinite-dimensional tensor product quantum
- C_{qa} = limits of finite-dimensional quantum
- C_{qc} = commuting-operator infinite-dimensional quantum

They satisfy the following inequalities:

$$C_c \subsetneq C_q \subsetneq C_{qs} \subsetneq C_{qa} = \overline{C_q} \subseteq C_{qc}.$$

Slofstra [**Slo19**] showed that $C_{qs} \neq C_{qc}$ by embedding finitely presented groups into the solution groups of linear system games. Slofstra modified [**Slo19a**] his embedding theorem to establish the intermediate separation $C_{qs} \neq \overline{C_q}$, after which Coladangelo and Stark [**CS18**] proved showed the other intermediate separation $C_q \neq C_{qs}$; each implies that the set of quantum correlations is not closed, i.e. $C_q \neq \overline{C}_q$. As shown by [**Fri12**, **JNP+11**], $C_{qa} \neq C_{qc}$ iff there is a counterexample to Connes's embedding conjecture (that any type-II$_1$ factor embeds in an ultrapower of the hyperfinite type-II factor R_1). Such a counterexample is claimed [**JNV+20**] to exist.

We would like to interpret these sets in the context of QFT.

6 References