Holomorphic BF theory and Geometric Langlands

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4d QFT and Geometric Langlands

- Global Geometric Langlands
    - Equivalence of categories
    - Flat connections on Bun[C,G] <—> Bundles on Flat[C,G']
- 4d A- and B- twisted N=4 SYM
    - 4d Topological Field Theories, related by S-duality
    - Attach categories Z_A[C,G] <—> Z_B[C,G'] to Riemann surface C

Holomorphic vs topological

- GL categories use complex structure of C
  - Technical? Bundle ~ unitary flat connections
  - Interesting objects use complex structure of C
- Defects in 4d TFT can be holomorphic on C
  - Holomorphic defects may be used to build a description of Z[C]
  - Holomorphic defects may be used to build nice objects in Z[C]

2d vs 4d primer

- B-twisted 2d theory of maps into M
  - Z[pt]: Derived category of coherent sheaves
- 4d $Z_B[C,G]$ ~ 2d Z[pt] for M = Flat[C,G]
- A-twisted 2d theory of maps into T*M
  - Z[pt]: Derived category of D-modules on M
- 4d $Z_B[C,G]$ ~ 2d Z[pt] for T*M ~ Higgs[C,G]

2d vs 4d

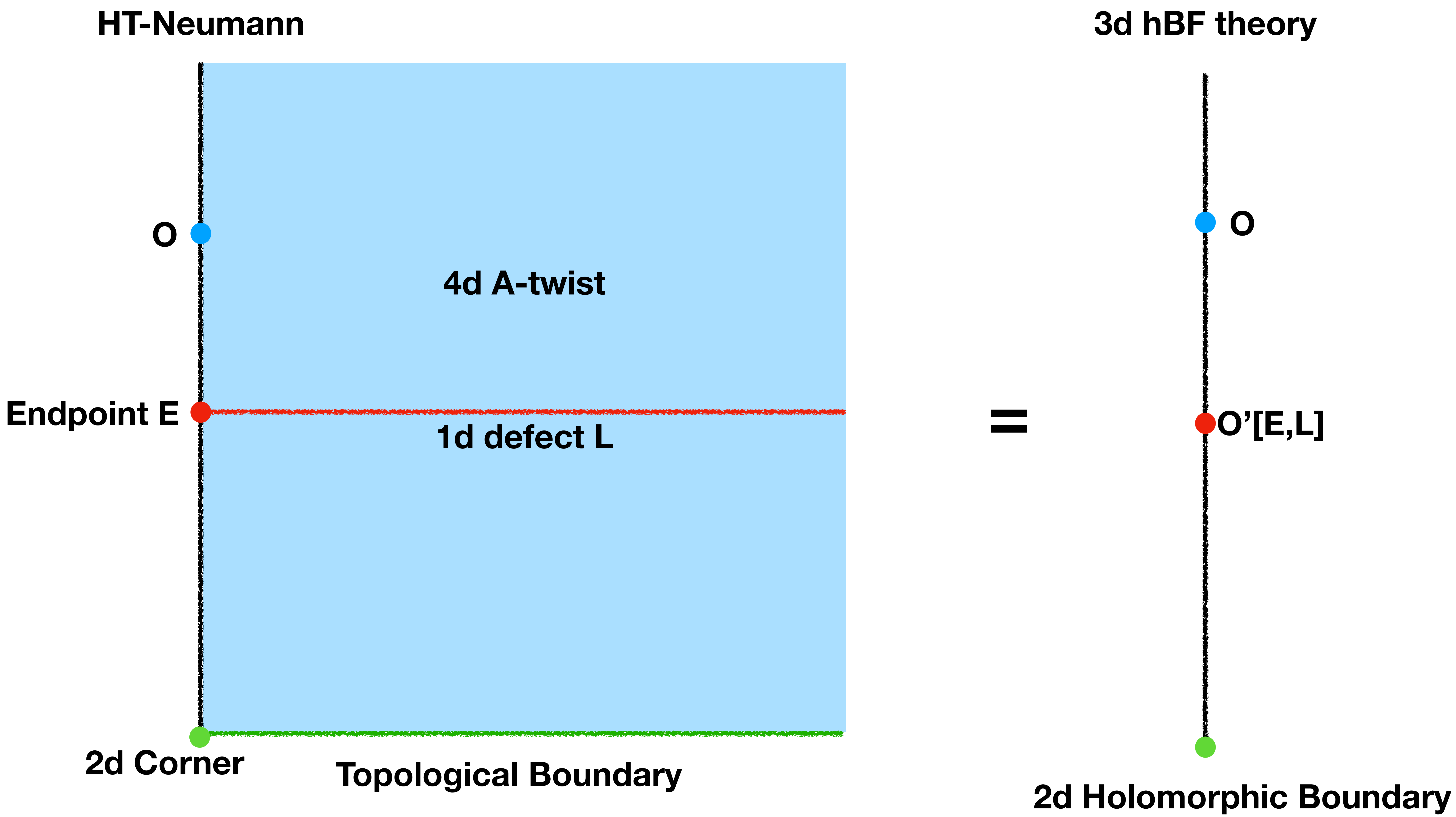
- 2d is poor approximation of 4d
- 4d is harder, but mathematically doable: semiclassical or perturbative
- 4d has extra structure: locality on C
- Special elements in Z[C], some known S-duals
  - 3d topological boundaries
  - 3d holomorphic-topological boundaries
  - $Z[U_3]$, C boundary of U_3

Half-BPS boundaries

- Preserve 3d N=4 in 4d N=4
- “Standard embedding”: 4d A (B) twist —> 3d A (B) twist
  - topological boundary condition
- “Alternative embedding”: holomorphic topological boundary
  - inherited from holomorphic twist
  - requires deformation, rarely available

A-twist, HT-Neumann

- 4d half space wt HT-Neumann <===> perturbative 3d HT BF theory
  - Many 4d calculations reduce to 3d BF theory
- Analogue of ``canonical coisotropic” brane in 2d
  - Phase space of BF theory <===> Higgs[C,G]
- 1d ’t Hooft lines ending on HT-Neu <===> local ops in 3d HT BF
  - Hecke correspondences on Higgs[C,G]


HT-Neumann
O
4d A-twist
Endpoint E
1d defect L
2d Corner
Topological Boundary
=
3d hBF theory
O
O'[E,L]
2d Holomorphic Boundary

B-twist, HT-Nahm

- Forces flat connection on C to be an “oper”
  - SL2: $D_t s = 0$ $\qquad D_{\bar{z}} s = 0$ $\qquad s \wedge D_z s = 1 \to D_z^2 s + t(z) s = 0$
  - Other groups: holomorphic-topological s with constraints on derivatives
- 1d Wilson lines ending on HT-Nahm
  - SL2, Fundamental: s(z) endpoint
  - Irrep R (derived Satake)—> D-module S_R on C depending on t(z), etc.

S-duality I

- HT-Neumann and HT-Nahm should be S-dual
  - Same spaces of local operators, endpoint of lines, etc.
- A: P[B] gauge-invariant local operators (and fermionic partners)
  - Quantize to $\hat{P}\left[\frac{\delta}{\delta A_{\bar{z}}}\right]$ quantum Hitchin Hamiltonians (“)
  - Not obvious, loop corrections could have obstructed
- B: t(z), …. functions on opers. BD: same as quantum Hitchin Hamiltonians

S-duality II

- Endpoint of lines?
- B: S_R
- A: Phase space is $T^*Gr_G/G[[z]]$
  - Quantize to equivariant D-modules on affine Grassmanian
  - Functor to endpoints as Hecke transformations
  - Match as D-modules on C, HT factorization module structure?

Critical Kac-Moody

- Dirichlet boundary conditions for 3d BF theory
    - Lifts to HT-Neumann - topological Dirichlet corner
- Supports Kac-Moody at critical level: $B(z) \to \frac{\delta}{\delta A_{\bar{z}}} \to J(z)$
- HT-Neumann local ops <—> Center of critical Kac-Moody
- Boundary ’t Hooft endpoints <—> D-modules on Gr_G
    - spectral flow modules
- ’t Hooft endpoints <—> Averaged spectral flow modules

Topological Nahm

- Hitchin section boundary condition for 3d BF theory
  - SL2: $D_{\bar{z}} v = 0$ $v \wedge \Phi v = 1$
- Chiral algebra: DS reduction of critical KM
  - Classical W-algebra. See t(z) directly
- S-dual to structure sheaf of Flat[C,G']
  - Corner: functions on Oper manifold
- Separation Of Variables: Dirichlet <—> Nahm + line defects

Boundary lines

- Boundary Wilson lines in HT-Neu <—> Wilson lines in 3d BF
  - Universal bundle on Higgs[G,C] —> Weyl modules for cKM
- Boundary ’t Hooft lines in HT-Nahm
  - Oper with singularity of trivial monodromy
- Gaudin model, etc.

Local GL

- 2-categories Z[circle] of surface defects
- Surface defects meet HT-Neu along topological line
  - BF Wilson line with quantization of flag manifold or other
  - Higgs bundles with regular or wild ramification
- cKM modules? Surface - top Dirichlet lines?
- Dual Surface defects meet HT-Nahm along topological line
  - Ramified opers

More HT boundaries

- Interfaces between U(N) and U(M)
- A: BF theory for U(N|M)
- B: N=M $A_{\bar{z}} = A'_{\bar{z}}$ $A_z = A'_z + XY$ $D_{\bar{z}} X = 0$ $D_{\bar{z}} Y = 0$
- B: N>M Partial oper N -> (N-M) + M, identify MxM block
- Mirabolic GL?
- OSp generalizations

Topological boundaries

- Enriched Neumann: A(B)-twisted 3d N=4 with G action
- Conformal blocks of boundary VOA should give object in Z[C]
  - A: D-module on Bun, B: sheaf on Flat
- Many S-dual pairs from String theory
- More from spherical varieties?

Intermission

Analytic GL

- 4d A-twist on C x R x [0,1] with Holomorphic and anti-holomorphic Neu
  - Path integral $\int DBD\bar{B}DAD\bar{A}e^{\int \mathrm{Tr} BF_A dz - \mathrm{Tr}\bar{B}\bar{F}_A d\bar{z}}$
- Quantize BF phase space as a real symplectic manifold
  - Hilbert space $L^2[Bun(G,C)]$
- Action of quantum Hitchin and conjugate. Analysis to make self-adjoint.
- Action of Hecke operators from ’t Hooft along [0,1]
  - Choice of 2 endpoints

Eigenvalue problem

- Common eigenvectors of commuting Hamiltonians, Hecke ops
  - Eigenvalues? Use S-duality!
- B: Flat connections which are opers and anti-opers (aka real opers)
  - Hitchin eigenvalues are oper t(z) etc.
  - Hecke eigenvalues are $\langle s(z), \bar{s}(\bar{z}) \rangle$

States from boundaries

- Top boundary at C x 0 x [0,1]
- Topological Dirichlet gives delta-function distribution
- Enriched Neumann:
  - A: Non-chiral partition function of boundary VOA coupled to G bundle
  - B: Non-chiral partition function of boundary VOA coupled to real oper
- S-duality computes expansion coefficients in eigenfunctions
  - Topological Nahm = Whittaker coefficient distribution

Analytic GL over reals

- Combine HT-Neumann and 3d manifold $(C\times[0,1])/Z_2$
  - Reflection of segment and anti-holomorphic involution on C
- Phase space: cotangent to “real” bundles
- S-duality: HT-Nahm and 3d manifold $(C\times[0,1])/Z_2$
  - Important subtleties if fixed lines
- Opers compatible with involution

Appetizer: a Gaudin intertwiner

A peculiar formula

- Gaudin Hamiltonians for half-densities on $(\mathbb{C}P^1)^n$

$$H_i^{(a)} = \sum_{j \neq i} \frac{e_i f_j + 2h_i h_j + f_i e_j}{z_i - z_j} \qquad f_i = \partial_{a_i} \qquad h_i = a_i \partial_{a_i} + \frac{1}{2} \qquad e_i = -a_i^2 \partial_{a_i} - a_i$$

- Intertwining condition:

$$\boxed{\begin{aligned} H_i^{(a)} K(a,b,c) &= H_i^{(b)} K(a,b,c) = H_i^{(c)} K(a,b,c) \\ \bar{H}_i^{(a)} K(a,b,c) &= \bar{H}_i^{(b)} K(a,b,c) = \bar{H}_i^{(c)} K(a,b,c) \end{aligned}}$$

- Algebraic solution:

$$K(a,b,c) = \frac{1}{|\det_{n \times n} A|}$$

$$\boxed{\begin{aligned} A_j^i &= \frac{(a_i - a_j)(b_i - b_j)(c_i - c_j)}{z_i - z_j} \qquad i \neq j \\ A_i^i &= 0 \end{aligned}}$$

The SL(2) addition kernel

Gaudin -> Hitchin

- Genus 0, parabolic points z_i, Bun_0
  - Lines $(1, a_i)$ at parabolic points z_i, modulo SL(2)
  - Gauge-fix three points: a_1, a_2, a_3, etc.

$$\sqrt{d\mu_a} = |a_1 - a_2||a_1 - a_3||a_2 - a_3||da_4 da_5 \cdots da_{n+3}|$$

$$K(a,b,c) = \frac{1}{|\det A|}\sqrt{d\mu_a}\sqrt{d\mu_b}\sqrt{d\mu_c}$$

- K(a,b,c) intertwines Hitchin's Hamiltonians on $\mathrm{Bun}_0 \times \mathrm{Bun}_0 \times \mathrm{Bun}_0$
- Claim: also intertwines aGL Hecke operators!

The Lame’ addition kernel

Genus 0, four points

- Fix $z_1 = 0, z_2 = 1, z_3 = \infty, z_4 = z$
- Gauge-fix $a_1 = 0, a_2 = 1, a_3 = \infty, a_4 = a$ etcetera.
- Hitchin Hamiltonians -> Lame’ operator $\partial_a a(a-1)(a-z)\partial_a + a$

$$K(a,b,c) = \frac{1}{|\det A|}|da||db||dc|$$

$$\det A \sim 1 + \frac{a^2b^2c^2}{z^2} + \frac{(1-a)^2(1-b)^2(1-c)^2}{(1-z)^2} - 2\frac{abc}{z} - 2\frac{(1-a)(1-b)(1-c)}{1-z} - 2\frac{abc}{z}\frac{(1-a)(1-b)(1-c)}{1-z}$$

- Why addition: det A=0 is support of addition along smooth fibers of Hitchin system, with zero at a=z

$$x^2 = a(a-1)(a-z)u \qquad \omega = \frac{dadu}{x} = \frac{dadx}{a(a-1)(a-z)}$$

The SL(2) addition kernel

Genus 2 and higher, no parabolic points

$$K(a,b,c) = \frac{1}{\left|\det \bar{\partial}_{E_a \otimes E_b \otimes E_c \otimes K^{\frac{1}{2}}}\right|}$$

- E: rank 2 associated bundle
- Singular at Theta divisor where $E_a \otimes E_b \otimes E_c \otimes K^{\frac{1}{2}}$ has sections
- Parabolic points: build A from Green's function

$$K(a,b,c) = \frac{1}{\left|\det \bar{\partial}_{E_a \otimes E_b \otimes E_c \otimes K^{\frac{1}{2}}}\right|} \frac{1}{|\det A|}$$

3d Holomorphic BF theory

- BV Action $$\int dz\,\mathrm{Tr}\,b\left(\mathrm{d}c - \frac{1}{2}[c,c]\right) = \int dz\,\mathrm{Tr}\,B\,F_A + \cdots$$
    - b, c are forms with $dt, d\bar{z}$ components, differential d acting on that
    - B scalar is 0 form part of b
    - A gauge field is 1-form part of c
- HT twist of 3d N=2 SYM or HT Neumann for 4d A-twisted N=4 SYM

Perturbative local operators

- Tree level differential $(Q+d)b = [c,b]$ $(Q+d)c = \frac{1}{2}[c,c]$
- Relative Lie algebra cohomology calculation
  - Zero form components of $\mathrm{Tr}\,\mathcal{P}(b)$ $\mathrm{STr}\frac{\partial\mathcal{P}(b)}{\partial b}\partial c$
- Loop corrections?