Field theory in $\frac{1}{2}\Omega$-background

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A few different set-ups:

6d $\mathcal{N}=1$ SYM

$\downarrow$ dimensionally reduce on $\mathbb{R}^4 \times T^2$, with $T^2 \to 0$

$T^\# \to \mathcal{M}_{Higgs}$

$\downarrow$

4d $\mathcal{N}=2$ SYM $\xrightarrow[IR]{}$ vacuum moduli space

* Coulomb $\mathcal{B}$
* Higgs

$\downarrow$ dimensionally reduce on $\mathbb{R}^3 \times S^1$, with $S^1 \to 0$

3d $\mathcal{N}=4$ SYM $\xrightarrow[IR]{}$ 3d σ-model into vacuum moduli space

* Coulomb
* Higgs $\mathbb{R}^{4\#}/G$

Also: 4d $\mathcal{N}=2$ SYM

$\downarrow$ reduce on $\mathbb{R}^3 \times S^1_R$

3d $\mathcal{N}=4$ SYM

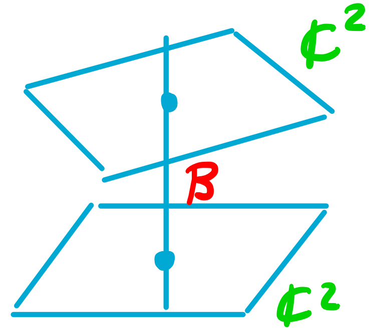
with $\mathcal{M}_C = \mathcal{M}^{\zeta}_{Hitchin}(R)$:

$$\begin{cases} F_A + R^2[\varphi, \bar{\varphi}] = 0 \\ \bar{\partial}_A \varphi = 0 \\ \partial_A \bar{\varphi} = 0 \end{cases}$$

A is a connection on $SU(k)$-bundle $E \to C$

and $\varphi \in \Omega^{1,0}(C, \operatorname{End} E)$ the Higgs field

Then: 5d $\mathcal{N}=1$ SYM



on $\mathbb{C}^2$-bundle over S^1_β:

$\beta \to 0$ $\downarrow$

$$(z_1, z_2, x_5) \to (z_1 e^{i\beta\epsilon_1}, z_2 e^{i\beta\epsilon_2}, x_5 + \beta)$$

also need $SU(2)_I$ rotation

$$\begin{pmatrix} e^{i\beta(\epsilon_1+\epsilon_2)/2} & 0 \\ 0 & e^{-i\beta(\epsilon_1+\epsilon_2)/2} \end{pmatrix}$$

to preserve susy

4d $\mathcal{N}=2$ SYM

in the Ω-background

$\mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2}$

and gauge rotation $e^{i\beta \underline{a}} \in G$

$\rightsquigarrow$

$$\log Z^{Nek}(\epsilon_{1,2}, \tau, \underline{a}) = \frac{1}{\epsilon_1\epsilon_2} \mathcal{F}_0(\tau, \underline{a}) + \text{terms less singular in } \epsilon_{1,2}$$

Twisting: 4d $\mathcal{N}=2$ theory has symmetry group

$$\underbrace{SU(2)_L \times SU(2)_R}_{SO(4)} \times SU(2)_I$$

R-symmetry

Poincare

Ω-deformation is compatible with Donaldson twist:

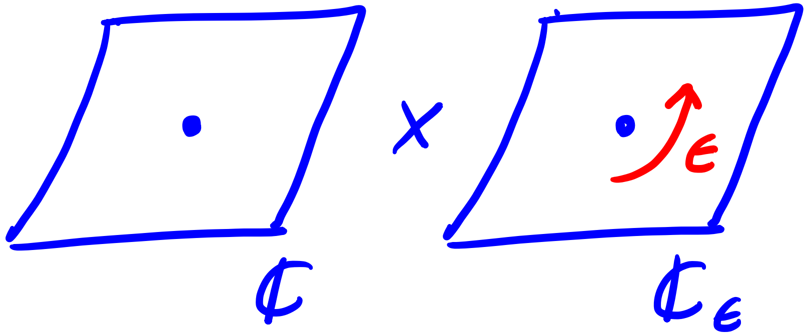
new Poincaré group $\quad SU(2)_L \times SU(2)_{diag}$

$$\cap$$

$$SU(2)_R \times SU(2)_I$$

(in general 1 scalar supercharge $\bar{Q}$)

Today we are interested in $\frac{1}{2}\Omega$- background $\mathbb{C} \times \mathbb{C}_\epsilon$:



[Nekrasov- Shatashvili]

on $\mathbb{C}^2$- bundle over S^1_β: $(z_1, z_2, x_5) \to (z_1, z_2 e^{i\beta\epsilon}, x_5 + \beta)$

with $\beta \to 0$

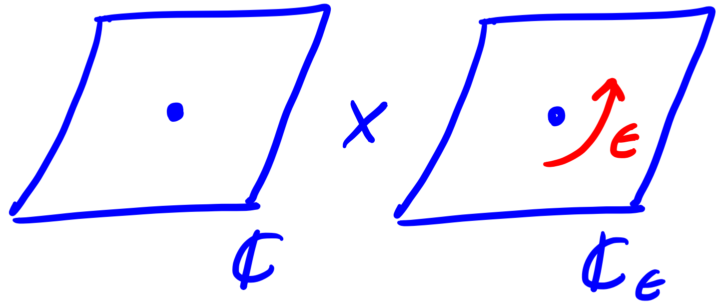
(note that the $\mathbb{C}_\epsilon$- bundle over S^1 is also known as $\mathbb{C} \times_q S^1$)

$\rightsquigarrow$ relation to 3d $\mathcal{N}=2$ theories

The $\frac{1}{2}\Omega$-background effectively defines

a 2d $\mathcal{N}=(2,2)$ theory on $\mathbb{C}$

with 4 supercharges

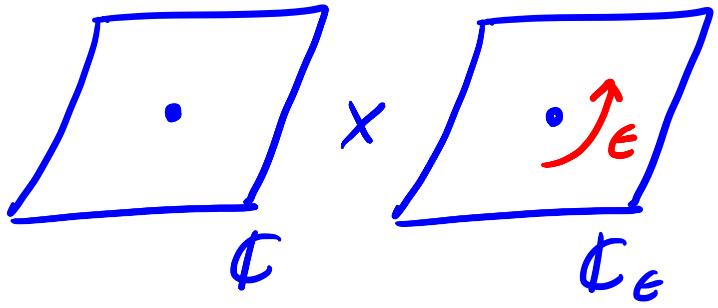


At low energies $E \ll |\epsilon|$, this theory is described by

r abelian vector multiplets coupled to an effective twisted superpotential

$$\frac{1}{\epsilon}\widetilde{W}^{\text{eff}}(\epsilon,\tau,\underline{a}) = \lim_{\epsilon_2\to 0} \epsilon_2 \log Z(\epsilon_{1,2},\tau,\underline{a}) = \frac{1}{\epsilon}F_0 + \dots$$

The $\frac{1}{2}\Omega$-background effectively defines a 2d $\mathcal{N}=(2,2)$ theory on $\mathbb{C}$.



Note: the Donaldson twist is <u>not</u> the appropriate twist here

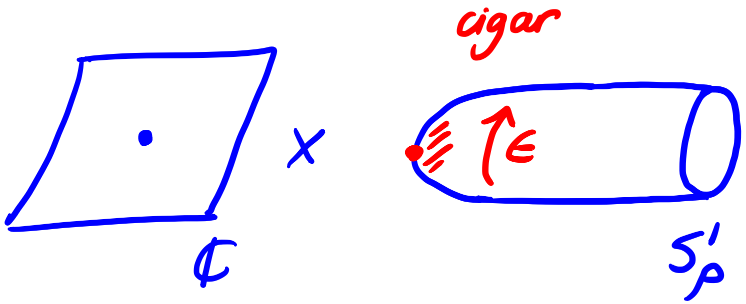
consider $U(1)_1 \times U(1)_2 \times U(1)_I$

$\subset SU(2)_L \times SU(2)_R \times SU(2)_I$

define twist wrt $U(1)_1 \times \mathrm{diag}\,(U(1)_2 \times U(1)_I)$

- local operators?
- boundary conditions?
- defects?

More insights from duality chain: [Nekrasov-Witten]

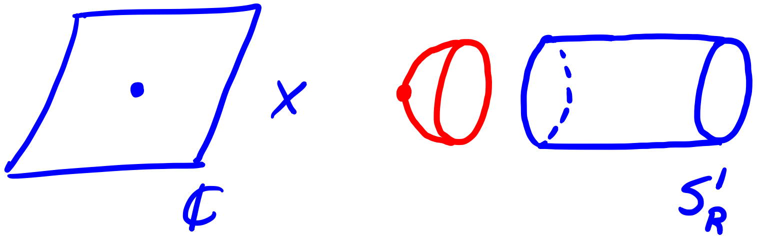


$$ds^2 = dr^2 + f(r)\,d\varphi^2$$

$f(r) \sim r^2 \qquad r \to 0$

$\quad\;\; \to \rho^2 \qquad r \to \infty$

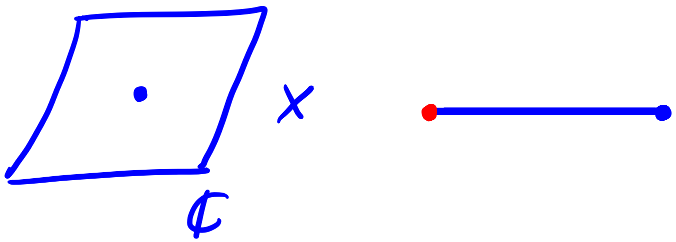
$\downarrow$ undo $\frac{1}{2}\Omega$-deformation away from origin cigar



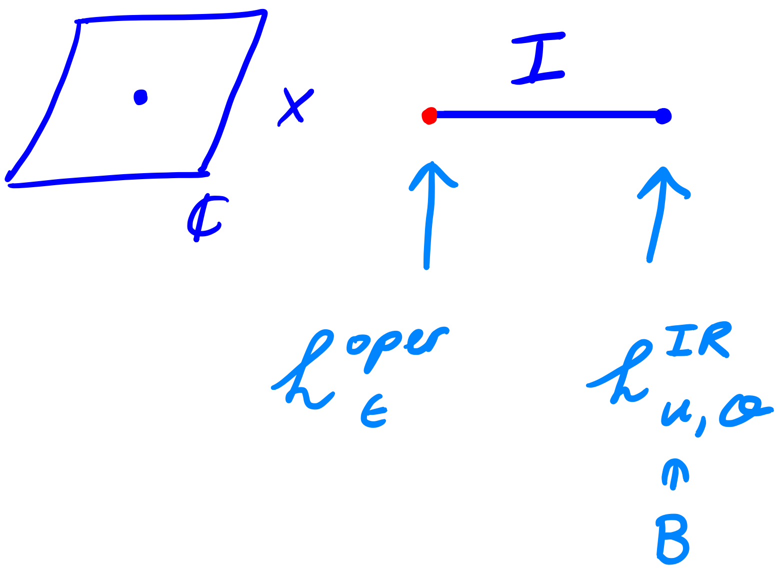
in return for a field definition

[Nekrasov-Witten]

$\downarrow$ compactify on S^1_R



in IR 3d sigma model with target $\mathcal{M}^{\zeta}_{Hitchin}(R)$ $\leftarrow$ $\zeta/R = \epsilon$



$\arg(\epsilon) = \vartheta \rightsquigarrow$ 4 supercharges preserved

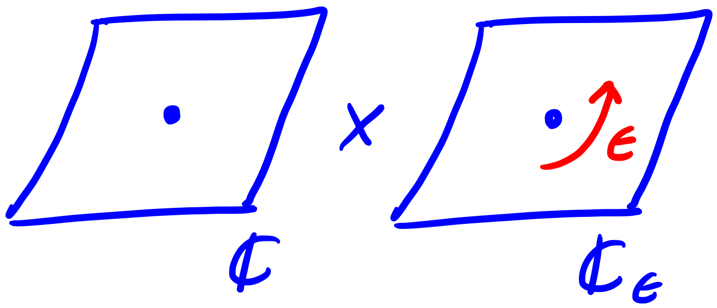
Note that boundary condition at ∞ in $\mathbb{C} \times \mathbb{C}_\epsilon$ becomes very explicit

$I \to 0$ $\downarrow$ [Kapustin-Rozansky-Saulina]

2d $\mathcal{N}=(2,2)$ sigma model into $\mathcal{B}$ with superpotential $W_{u,\vartheta}(\epsilon, \tau, \underline{a})$

= generating function for $\mathcal{L}^{\text{oper}}_{\epsilon} - \mathcal{L}^{\text{IR}}_{u,\vartheta}$

Thus, we find that the 2d $\mathcal{N}=(2,2)$ theory on $\mathbb{C} \times \mathbb{C}_\epsilon$ is governed in the IR by an object $W_{u,\mathfrak{o}}(\epsilon, \tau, \underline{a})$



- How is $W_{u,\mathfrak{o}}(\epsilon, \tau, \underline{a})$ defined?
- How is it related to $\frac{1}{\epsilon}\widetilde{W}^{eff}(\epsilon, \tau, \underline{a}) = \lim_{\epsilon_2 \to 0} \epsilon_2 \log Z(\epsilon_{1,2}, \tau, \underline{a})$?
- What are UV boundary conditions?

in a weakly coupled region

- $W_{u,\vartheta}(\epsilon, \tau, \underline{a})$ may be obtained* as the Borel resummation of the ϵ-expansion of $\widetilde{W}^{eff}(\epsilon, \tau, \underline{a})$ in the direction ϑ:

divergent $f(\epsilon) = \sum_{n=0}^{\infty} c_n \epsilon^n$ where $c_n \sim n!$

$\rightsquigarrow$ Borel transform $\mathcal{T}f(s) = \frac{c_n}{n!} s^n$

$\rightsquigarrow$ Borel sum $B_\vartheta f(\epsilon) = \int_0^{\infty e^{i\vartheta}} e^{-s}\, \mathcal{T}f(\epsilon s)\, ds$

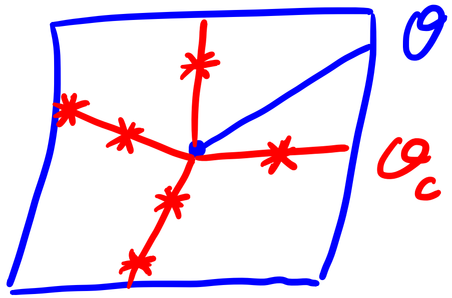
Might think of Borel sum as "transseries":

zero modes from inst + anti-inst

$$B_\vartheta f(t) = \sum \sum c_{k,\ell,n}\, t^n e^{-kc/t} \ln\left(\pm \frac{1}{t}\right)^{\ell}$$

non-pert instantons

- The Borel sum is not defined along "critical" directions ϑ_c where the Borel transform has singularities



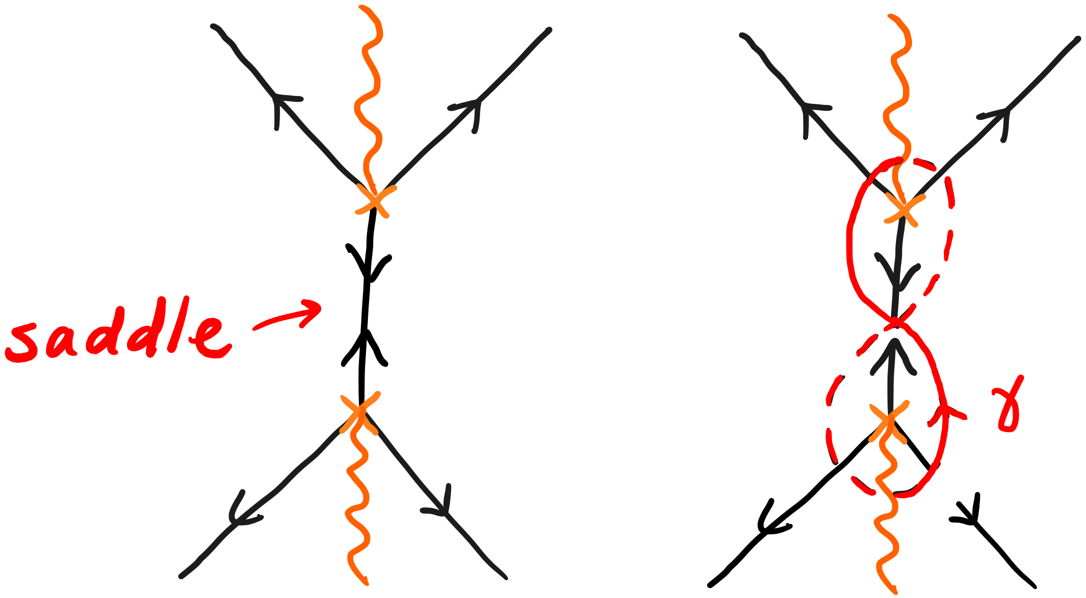
Borel plane

$\rightsquigarrow$ $W_{u,\vartheta}(\epsilon, \tau, \underline{a})$ is piece-wise constant in ϑ and "jumps" across the critical rays.

- The critical directions ϑ_c correspond to the phases

  $\vartheta_c = \arg Z_\gamma$ of BPS particles in the 4d $\mathcal{N}=2$ thy

These particles may be visualized as saddle trajectories in the spectral network $\mathcal{W}^{\vartheta}(u)$:

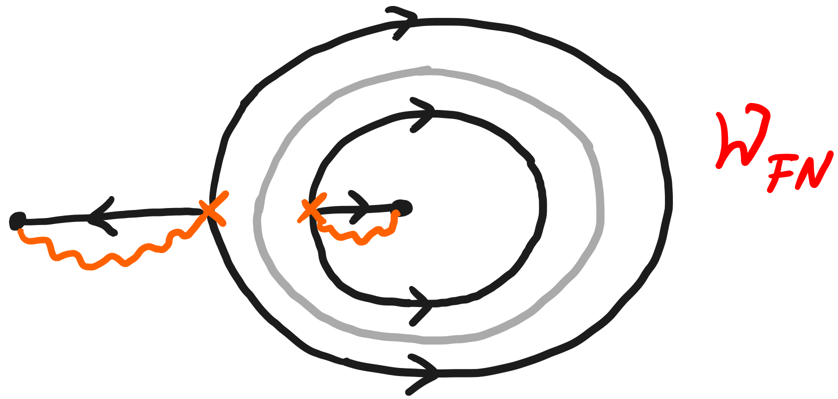


$Z_\gamma = \oint_\gamma \lambda$

Hence $W_{u,\vartheta}$ encodes the 4d particle spectrum!

- If the 4d $N=2$ theory has a weakly coupled description, there is a special phase ϑ_{FN} with $\vartheta_{FN} = \arg(W\text{-boson})$

- At this phase the spectral network $\mathcal{W}_{FN}$ includes a family of compact trajectories:

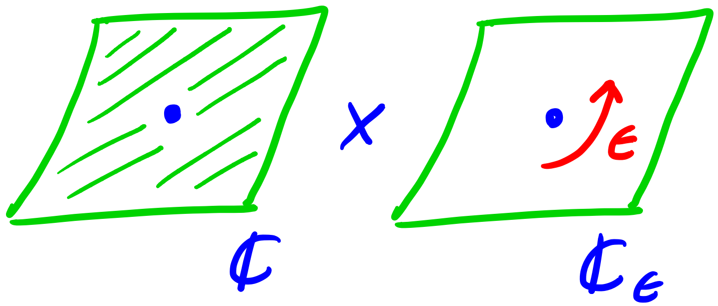


- We have $W_{u,\vartheta_{FN}}(\epsilon,\tau,\underline{a}) = \widetilde{W}^{\text{eff}}(\epsilon,\tau,\underline{a})$

- So far, we have implicitly assumed that the 4d $\mathcal{N}=2$ theory has a weakly coupled description as a gauge thy with gauge gp G coupled to some matter.

- We have also implicitly chosen a "standard Neumann" UV boundary condition at ∞ of $\mathbb{C} \times \mathbb{C}_\epsilon$.

  Neumann $A_{\parallel}$
  Dirichlet $A_{\perp}$



[Nekrasov-Witten]

- In the IR the boundary condition is then specified by a point $u \in \mathcal{B}$, a polarization of Σ_u and a phase ϑ.

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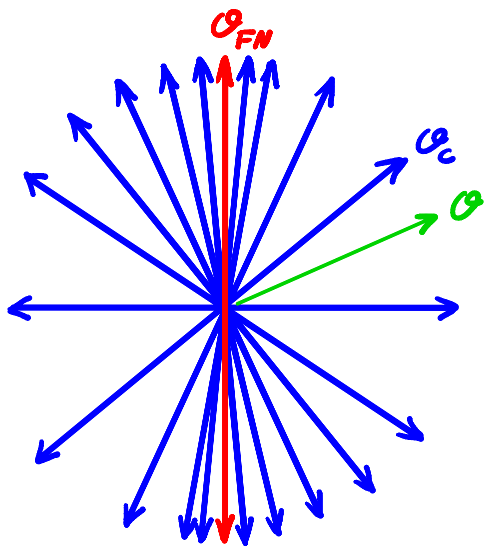
- The corresponding complex Lagrangian $\mathcal{L}^{IR}_{u,\vartheta} \subset \mathcal{M}_{Hitchin}$ is given by $\mathcal{X}^{\vartheta}_{B_j} = 1$, for the corresponding set of B-cycles B_j, where $\mathcal{X}^{\vartheta}_{\gamma_j}$ is a set of complex Darboux coordinates on $\mathcal{M}_{Hitchin}$

these coordinates may be found from $\mathcal{W}^{\vartheta}_u$ using $\mathcal{W}$-abelianization [Gaiotto-Moore-Neitzke], [Hollands-Neitzke], [Nikolaev]



- generic $\vartheta \longrightarrow$ FG-type coordinates
- $\vartheta = \vartheta_{FN} \longrightarrow$ FN-type coordinates

- The superpotential $W_{u,\vartheta}(\epsilon,\tau;\underline{a})$ may be formulated as

$$\mathcal{X}^{\vartheta}_{A_j}(\nabla^{oper}) \equiv -e^{\pi i a^{\vartheta}_j/\epsilon}$$

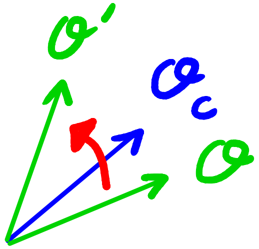
$$\mathcal{X}^{\vartheta}_{B_j}(\nabla^{oper}) \equiv e^{2\epsilon\, \partial W_{u,\vartheta}(\epsilon,\tau,\underline{a}^{\vartheta})/\partial a^{\vartheta}_j} \equiv e^{2\epsilon\, a^{\vartheta}_{D,j}}$$

- That is, $W_{u,\vartheta}(\epsilon,\tau,\underline{a})$ is the generating function of opers in the complex Darboux coord $\mathcal{X}^{\vartheta}_{\gamma}$ for $\vartheta=\vartheta_{FN}$ [Nekrasov-Rosly-Shatashvili] [H-Kidwai], [Nekrasov-Jeong]

- a^{ϑ}_j and $a^{\vartheta}_{D,j}$ are the Borel sums in the direction ϑ of the quantum periods $\frac{1}{\epsilon}\oint_{A_j/B_j} S(z,\epsilon)\,dz$ $\rightsquigarrow$ exact WKB method [Hollands-Neitzke]

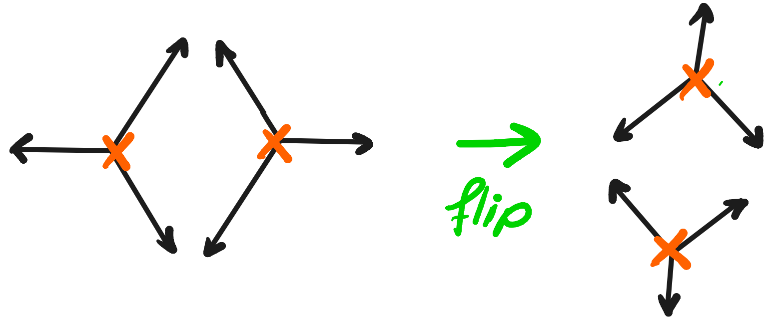
$S(z,\epsilon)$: sol'n Ricatti eqn

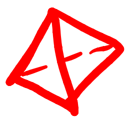
- Change of boundary condition, e.g. across isolated critical ray:



$$a^{\vartheta'} = a^{\vartheta}$$
$$a_D^{\vartheta'} = a_D^{\vartheta} + \log\left(1 - e^{\pi i a^{\vartheta}/\epsilon}\right)$$

$$\Delta W = -\frac{\epsilon}{2\pi i} \mathrm{Li}_2\left(e^{\pi i a^{\vartheta}/\epsilon}\right)$$



$\rightsquigarrow$ coupling the 3d $\mathcal{N}=2$  theory to the boundary

- Similar for any ϑ [Dimofte-Gaiotto-Veen], § 5.7 of 2109.14699

note: change polarization $\Sigma \rightsquigarrow$ e.m. duality transformation

$\rightsquigarrow$ generalized Legendre transformation on W

e.g. $W \to W' = \epsilon\, a' \cdot a + W$

- So far, we focussed on a weakly coupled region $\subset \mathcal{B}$
  However, the definition of $W_{u,\wp}(\epsilon, \tau, \underline{a})$ as a generating function of opers may be extended to all $\mathcal{B}$ and even to 4d $\mathcal{N}=2$ thys without a weakly coupled description

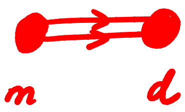
- $W_{u,\wp}(\epsilon, \tau, \underline{a})$ is also the solution to a Riemann-Hilbert problem specified by the corresponding BPS structure [Bridgeland], [Alim-Saha-Tulli-Teschner]

- $\exp W_{u,\wp}(\epsilon, \tau, \underline{a})$ is then a section of a "classical Chern-Simons" line bundle over the moduli space of flat connections

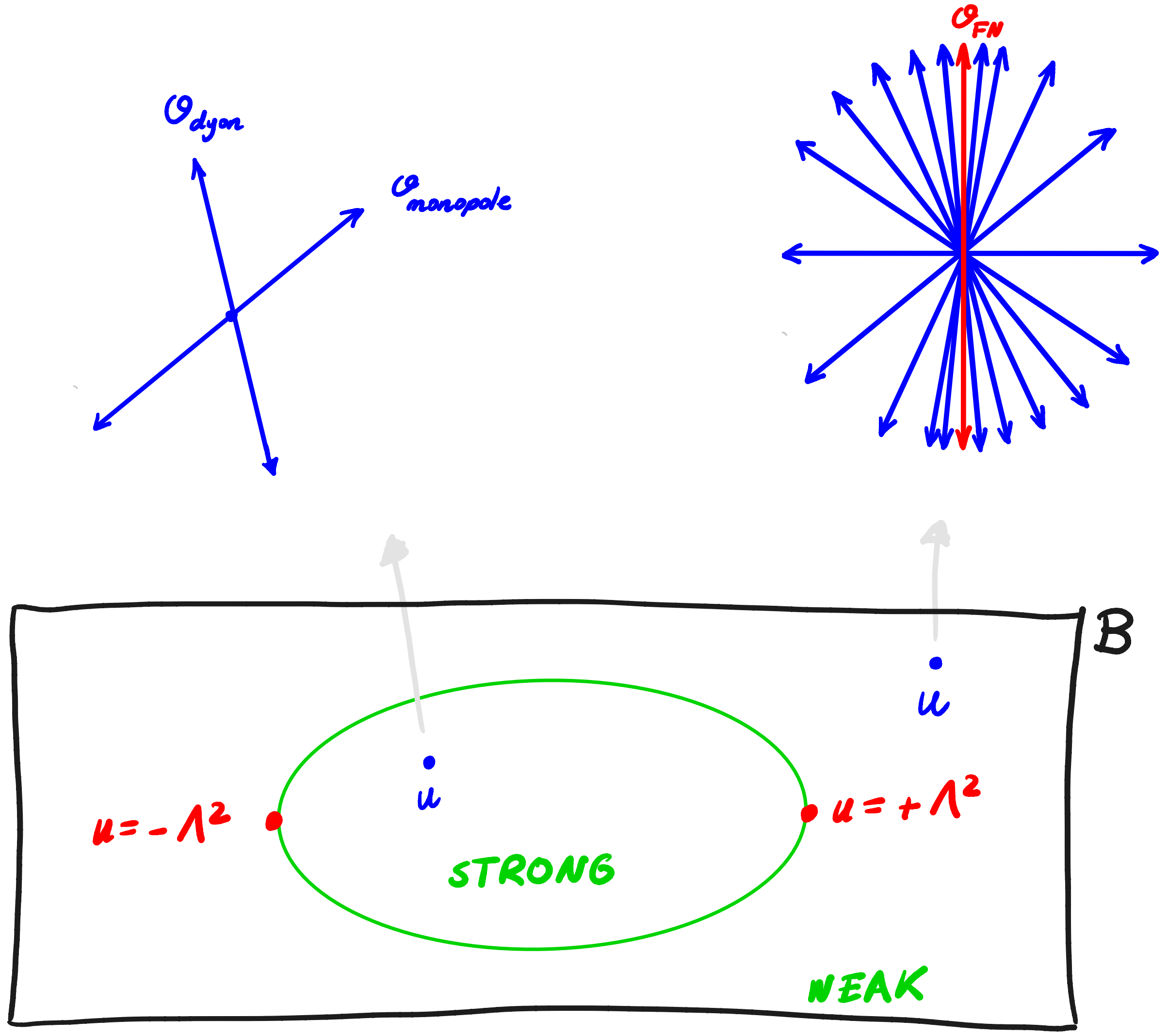
  [Neitzke], [Alexandrov-Persson-Pioline], [Coman-Longhi-Teschner], [Alim-Saha-Tulli-Teschner]

  the line bundle is defined by gluing between cluster charts using dilogs

• *example:* pure $SU(2)$

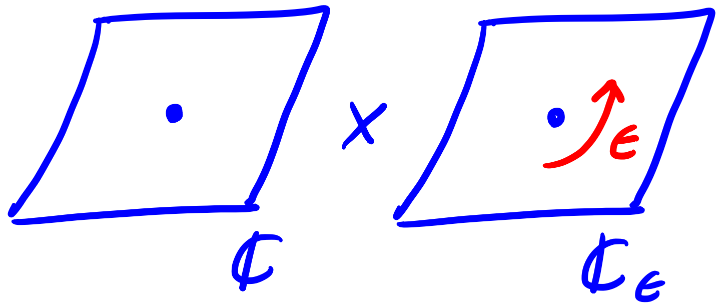


$$\begin{array}{c} W_{u,\vartheta} \in \mathfrak{h} \\ \downarrow \\ \mathcal{M}_{Hitchin} \times \mathbb{C}^*_\epsilon \longrightarrow \\ \cup \qquad \cup \\ U_\vartheta \qquad U_{\vartheta'} \end{array}$$



Going back to 4d $\mathcal{N}=2$ theory in $\frac{1}{2}\Omega$-background,

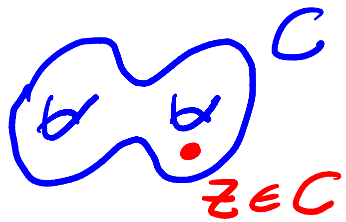
may enrich theory with defects:

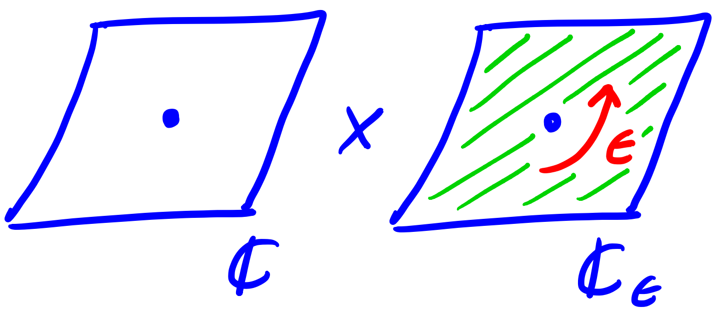


categorical description??

- line defects: ones introduced by Fei together with UV → IR map

- surface defects: simplest ones have moduli space C





corr 2d BPS states encoded as trajectories ending on z in spectral networks $\mathcal{W}_u^\vartheta$

their vevs $\psi(z)$ are solns to oper eqn $D_\epsilon \psi(z) = 0$

(open-closed)

Lift to 5d = 5d $\mathcal{N}=1$ theory on $\mathbb{C}\times(\mathbb{C}\times_q S^1)$ = topological string theory in NS-background

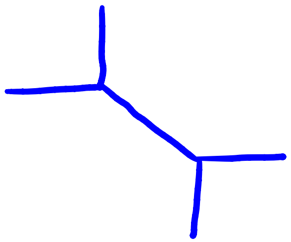
$q = e^{i\epsilon}$

surface defects on $\mathbb{C}\times_q S^1$: couple 3d $\mathcal{N}=2$ thy to 5d $\mathcal{N}=1$ thy

moduli space of surface defect : $\Sigma \subset \mathbb{C}^*_X \times \mathbb{C}^*_Y$

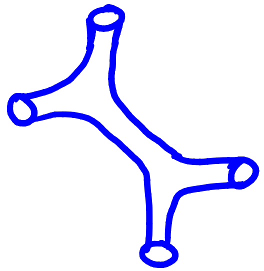
Σ = 5d SW curve = mirror curve

example :



"resolved conifold"

$\longleftrightarrow$



$$\Sigma:\ Y = \frac{1-QX}{1-X} \xrightarrow[1:1]{} C = \mathbb{C}^*_X$$

$$\lambda = \log Y \, d\log X$$

$$Q = e^{2\pi i t}$$

NS partition function: $\widetilde{W}^{\text{eff}}(\epsilon, t) = -\frac{1}{2}\sum_{k=1}^{\infty} \frac{Q^k}{k^2 \sin(k\epsilon/2)}$

surface defect ver: sol'n to $D_\Sigma \psi(X) = 0$

with $D_\Sigma = (1-X)e^{\epsilon\partial_x/2\pi} - (1-QX)$ and $X = e^{2\pi i x}$

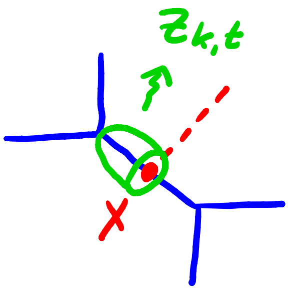
one sol'n: $\Psi_{GV}(\epsilon, x, t) = \frac{L(x,\epsilon)}{L(x+t,\epsilon)}$ with $L(x,\epsilon) = \prod_{j=0}^{\infty}(1-Xq^j)$

but more natural to consider the Borel sum $\Psi^{\Theta}(\epsilon, x, t) = B_\Theta \Psi^{\text{for}}(\epsilon, x, t)$

which encodes all solutions! *

[Garoufalides - Kashaev]

Indeed, Ψ^{ϑ} jumps in encode the 3d BPS particles coupled to the surface defect:



critical rays at $\vartheta^{3d} = \arg(Z^{3d})$

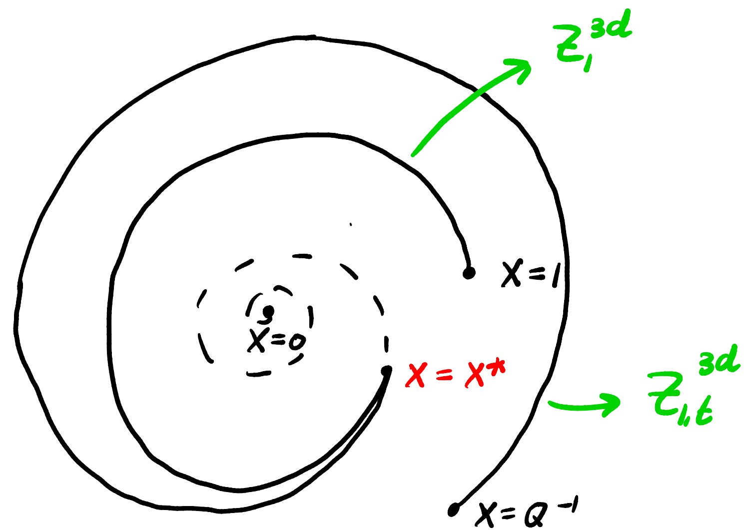
with either $Z_k^{3d} = \mp \frac{2\pi i}{\beta}(x+k)$

$Z_{k,t}^{3d} = \mp \frac{2\pi i}{\beta}(t+x+k)$

These may be visualized

in exponential spectral network:

[Banerjee-Longhi-Romo]
[Grassi-Hao-Neitzke]
[Alim-Hollands-Tulli]

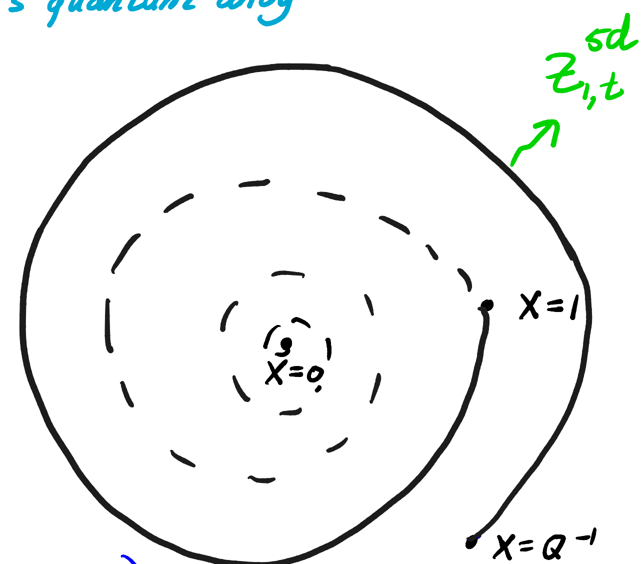


Two special cases: $\vartheta_{GV} = \pi/2$ and $\vartheta_{np} = 0$

$$\Psi^{\vartheta_{GV}} = \Psi_{GV} \quad \text{and} \quad \Psi^{\vartheta_{np}} = \frac{S_2(x/\hat{\varepsilon},1)}{S_2(x+t/\hat{\varepsilon},1)}$$

Fadeev's quantum dilog

$\tilde{W}^{\mathrm{eff},\vartheta}$ can be computed in two ways:



- $B^{\vartheta}\tilde{W}^{\mathrm{eff,pert}}$

critical rays at $\vartheta^{sd} = \arg Z^{sd}$

with $Z^{sd}_k = \pm \frac{2\pi i}{\beta}(t+k)$

corr to sd particles ($D2 + kD0$ or $-D2 + kD0$)

in sd $U(1)$ gauge thy

Again, two special cases:

[Alim-Hollands-Tulli] [Grassi-Hao-Neitzke]

similar to [Alim-Saha-Tulli-Teschner]

$$B_{\theta_{GV}} \widetilde{W}^{eff}_{sd} = \widetilde{W}^{eff}_{GV}$$

$$B_{\theta_{np}} \widetilde{W}^{eff}_{sd} = \widetilde{W}^{eff}_{np} = \widetilde{W}^{eff}_{GV}(\epsilon, t) + \widetilde{W}^{eff}_{GV}\left(\frac{4\pi^2}{\epsilon}, \frac{2\pi t}{\epsilon}\right)$$

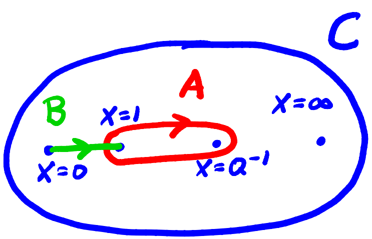
- Other method: (important to go to more intricate geometries)

compute Borel sums of quantum periods:

$$B_\theta \Pi_A = Q$$

$$B_\theta \Pi_B = \text{"}\frac{\Psi^\theta(x=0)}{\Psi^\theta(x=-i\infty)}\text{"} = S_2(x|\tilde{\epsilon},1)^{-1} \frac{\Psi^\theta(x=0)}{\Psi^\theta(x=-i\infty)}$$

$$= e^{-\frac{1}{2\pi}\partial_t \widetilde{W}^{eff,\theta}}$$



[Alim-Hollands-Tulli], non-pert generalization [Aganagic-Cheng-Dijkgraaf-Krefl-Vafa]

Conclusion:

- Both in 4d and 5d the NS partition function

$$Z_{NS} = \exp F_{NS} = \exp W_{NS}$$

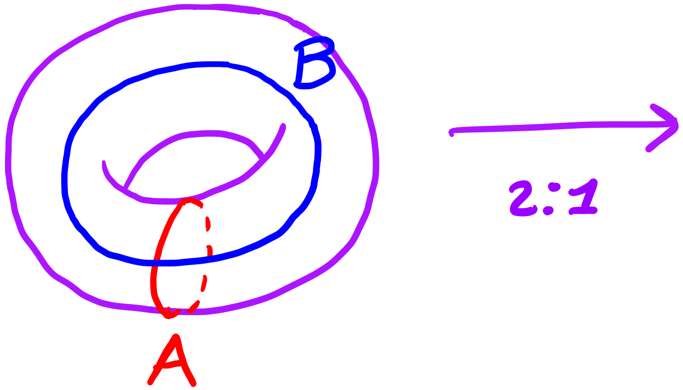
can be reformulated analytically using abelianization/exact WKB as a generating function of ϵ-opers/q-opers in terms of a special kind of (Fenchel-Nielsen) Darboux coordinates $\mathcal{X}_\gamma^{FN} / \mathcal{X}_\gamma^{GV}$

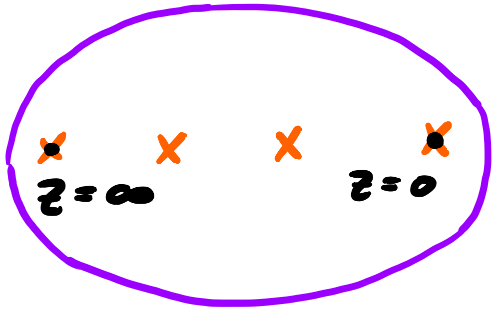
e.g. $$W_{NS}^{4d} = \frac{1}{2}\int dx \log \sqrt{\frac{[\psi_1', \psi_1''][\psi_1, \psi_2'']}{[\psi_1, \psi_1''][\psi_1', \psi_2']}}$$

- Equivalently, W_{NS} is the Borel sum of its ϵ-expansion in a critical direction $\vartheta_{FN} / \vartheta_{GV}$

- Other phases are interesting as well, and can be interpreted in terms of different boundary conditions $\frac{1}{2}\Omega$-background

Example: pure $SU(2)$ theory [H-R-S]





$C = \mathbb{P}^1 \setminus \{0, \infty\}$

$$\Sigma : w^2 = \frac{\Lambda^2}{z^3} - \frac{2u}{z^2} + \frac{\Lambda^2}{z}$$

u Coulomb par

Λ UV scale

family of opers parametrized by

Mathieu differential eqn:

$$D_\epsilon \psi(z) = \epsilon^2 \psi''(z) - \left(\frac{\Lambda^2}{z^3} - \frac{2u + \epsilon^2/4}{z^2} + \frac{\Lambda^2}{z} \right) \psi(z) = 0$$

(note: u just free par here)

Example : pure $SU(2)$ theory

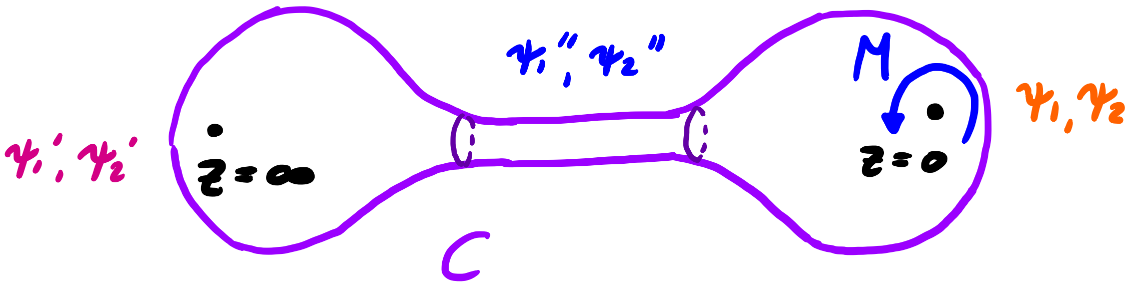
weak coupling $|\Lambda^2/u| \ll 1$:

ψ_1 asympt small (when $z \to 0$ along neg real axis)

ψ_1' " " " (when $z \to \infty$ " ")

ψ_2 small eigenvector of M

for $\vartheta = \arg(\epsilon) = \pi/2$ and $-u \gg \Lambda^2$



$$W_{NS}^{4d} = \frac{1}{2}\int dx \log \sqrt{\frac{[\psi_1', \psi_1''][\psi_1, \psi_2'']}{[\psi_1, \psi_1''][\psi_1', \psi_2']}} = \frac{a^2}{\epsilon}\log\left(\frac{\Lambda}{\epsilon}\right) - \frac{\epsilon}{2}\gamma\left(-\frac{a}{\epsilon}\right) - \frac{\epsilon}{2}\gamma\left(\frac{a}{\epsilon}\right)$$

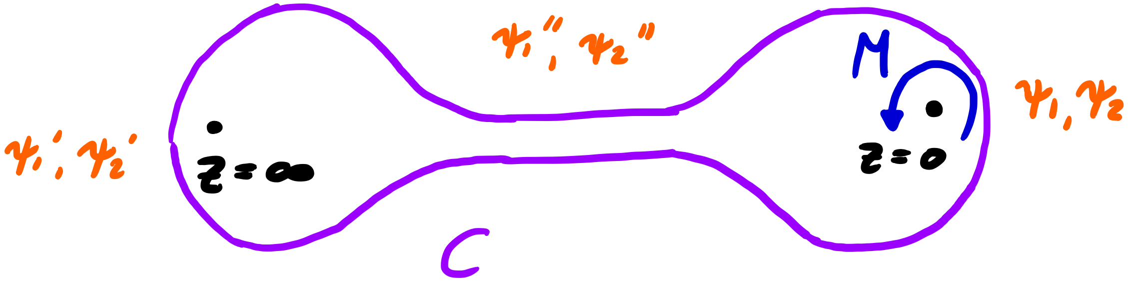
$$+ \frac{2\Lambda^4}{\epsilon(a^2-\epsilon^2)} + \frac{(5a^2+7\epsilon^2)\Lambda^8}{\epsilon(a^2-4\epsilon^2)(a^2-\epsilon^2)}$$

$$+ O(\Lambda^{12})$$

not very easy, but you can verify this in a perturbation in Λ [M-R-S]

Idea of computation: [H-R-S]

weak coupling $|\Lambda^2/u| \ll 1$:



- introduce new coordinate $v = z/\Lambda^2$ near $z=0$ and compute ψ_1, ψ_2 in v in series in Λ^2

- analytically continue ψ_1, ψ_2 to $v \to \infty$ while $z \sim 1$, and reorganize as series in Λ^2

- compute ψ_1'', ψ_2'' as eigenfunctions of M in series in Λ^2

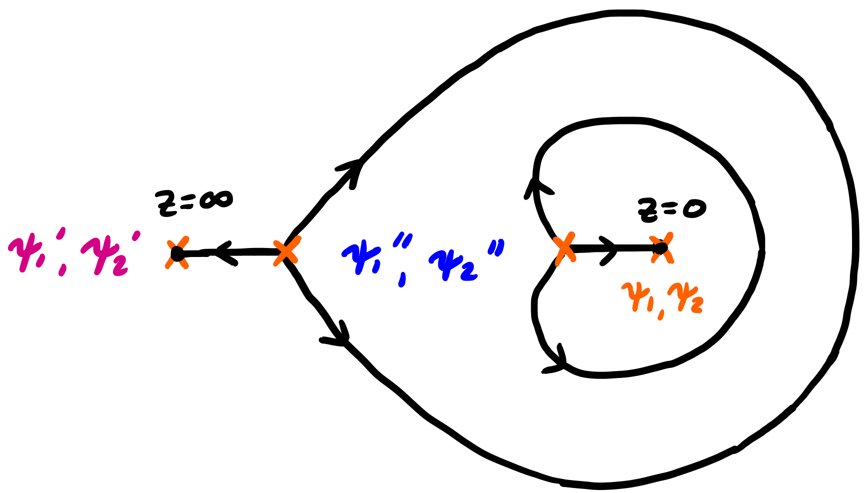
- then compare to ψ_1'', ψ_2''

similar to [H-Kidwai]

why $\vartheta = \arg(\epsilon) = \frac{\pi}{2}$?

special since W-bosons have $\arg(Z) = \frac{\pi}{2}$

spectral network or
Stokes graph at $\vartheta = \frac{\pi}{2}$:



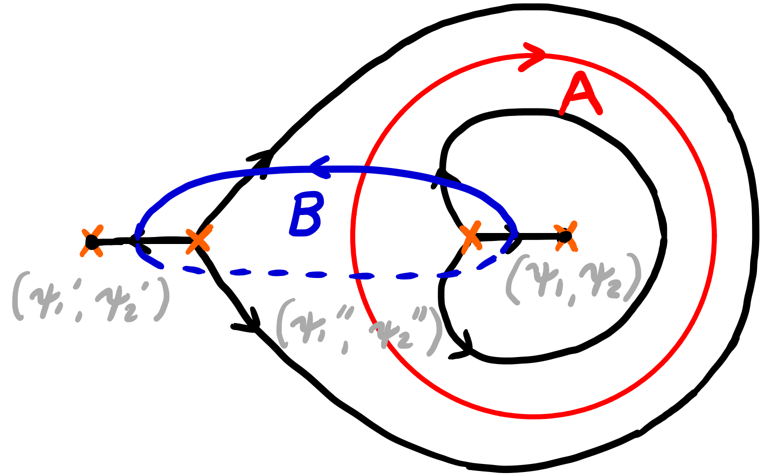
ψ's are Borel resummed solutions to $D_\epsilon \psi = 0$ in direction $\vartheta = \pi/2$

$\parallel$

ψ's form local bases for abelianization of D_ϵ

trajectories $e^{-i\vartheta}\sqrt{\frac{\Lambda^2}{z^3} - \frac{2u}{z^2} + \frac{\Lambda^2}{z}} \in \mathbb{R}^\times$

spectral network or
Stokes graph at $\theta = \frac{\pi}{2}$:



spectral coordinates /
exact quantum periods:

$$\mathcal{X}_A(\nabla^{oper}) = \frac{\text{eigenvalue } M}{\text{corr to } \psi_2''} \equiv -e^{\frac{\pi i a^{ex}_{\epsilon}}{\epsilon}}$$

$$\boxed{a^{ex}_{D,\epsilon} = \frac{\partial W^{4d}_{NS}}{\partial a^{ex}_{\epsilon}}}$$

$$\mathcal{X}_B(\nabla^{oper}) = \sqrt{\frac{[\psi_1', \psi_1''][\psi_1, \psi_2'']}{[\psi_1, \psi_1''][\psi_1', \psi_2']}} \equiv e^{2\epsilon a^{ex}_{D,\epsilon}}$$