Generalised Langlands, VOAs, and (generalised) tau-functions

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Part I

Context:

Topologically twisted $N = 4$ QFT in $d = 3$ and $d = 4$

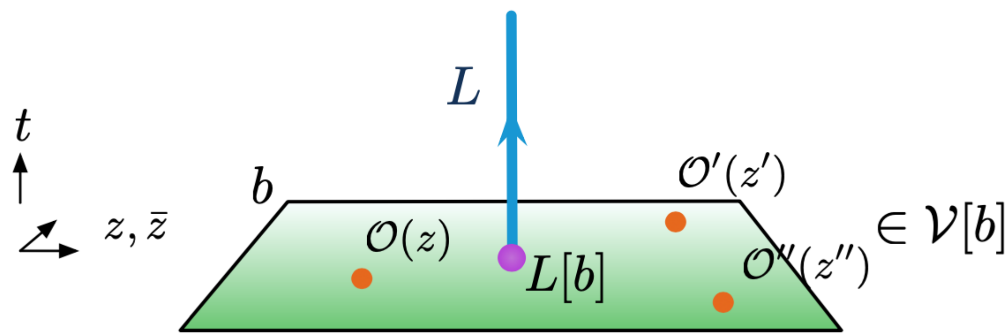
The following slides reflect my (limited) understanding of recent work of many colleagues, at the risk of errors and inaccuracies.

$N=4$, $d=3$ twisted SUSY QFT

$N=4$, $d=3$ SUSY QFT on manifolds with certain boundaries admit (A/B) twists which are **topological** in the 3d bulk, and **holomorphic** on the 2d boundary b.

Holomorphic boundary observables $\mathcal{O}(z)$ generate VOAs[1].

$\rightsquigarrow \cdots \rightsquigarrow$ expect rich generalisation of WZW-CS relationship, schematically[2],



assigning, in particular

- $T(C)$ – (dg) vector spaces (of VOA conformal blocks) to Riemann surfaces C,
- $T(D_x)$ – (dg) category (of line defects $\sim$ VOA representations),

where D_x is disc with puncture at x.

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[1] Costello-Gaiotto, Costello-Creutzig-Gaiotto; closely related: important developments by S. Gukov and collaborators.

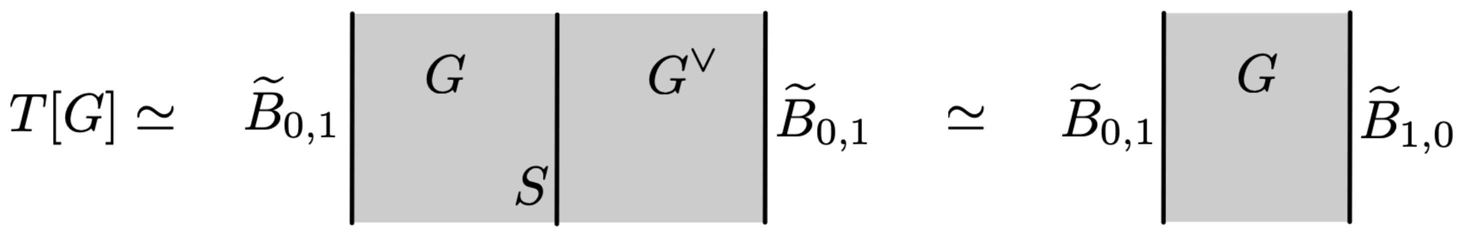
[2] Picture taken from Creutzig-Dimofte-Garner-Geer, arXiv:2112.01559

$N = 4$, $d = 4$ SUSY Yang-Mills theory (SYM)

Topological twists of $N = 4$ SYM form the basis for the gauge-theoretic approach to the geometric Langlands correspondence initiated in the work of Kapustin and Witten.

Interesting relations to $N = 4$, $d = 3$ twisted SUSY QFT emerge by considering $N = 4$, $d = 4$ SYM on $M^4 = M^2 \times C$, with $M^2 = I \times \mathbb{R}$. Depending on the boundary conditions on the ends of I one gets various $N = 4$, $d = 3$ QFT in the IR.

Example:[3] Theory $T[G]$, G: compact Lie Group, mostly $SU(n)$.



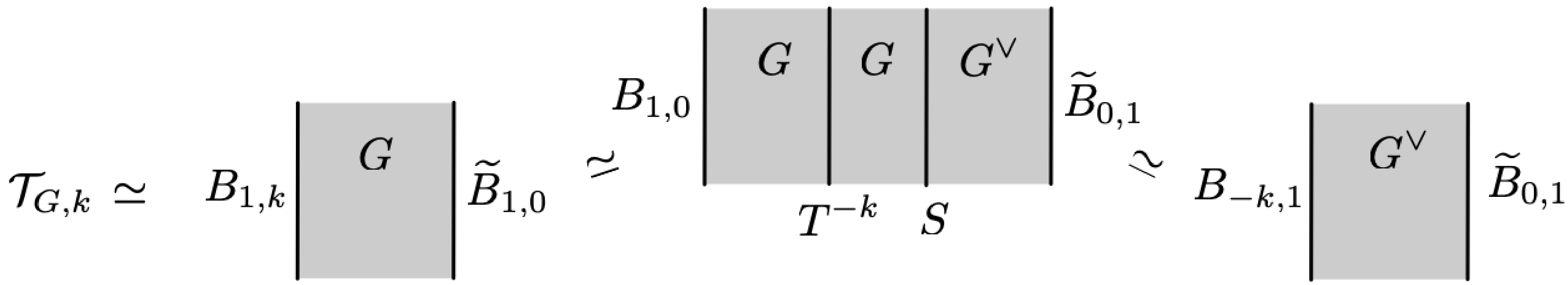
using S-duality interface, and boundary conditions for $N = 4$ SYM of following types:

- $\tilde{B}_{0,1}$ Dirichlet,
- $\tilde{B}_{1,0}$ S-dual of Dirichlet.

[3] Picture taken from Creutzig-Dimofte-Garner-Geer, arXiv:2112.01559

$3d$ Theories $\mathcal{T}_{G,k}$ from $4d$

Another interesting example was recently studied in Creutzig-Dimofte-Garner-Geer, arXiv:2112.01559, defining a family of $N = 4$, $d = 3$ QFT denoted $\mathcal{T}_{G,k}$, with[4]



The resulting $N = 4$, $d = 3$ QFT admit topological twists defining $\mathcal{T}^A_{G,k}$. Coming from $4d$: Induced by Kapustin-Witten's GL twist at $\Psi = 0$.

Creutzig-Dimofte-Garner-Geer argue that the boundary VOAs $\mathcal{D}_k(\mathfrak{g})$ for $\mathcal{T}^A_{G,k}$ are the Feigin-Tipunin logarithmic VOAs $\mathcal{FT}_k(\mathfrak{sl}_n)$, $\mathfrak{g}$: Lie algebra of G. Let $\mathcal{D}_{n,k} = \mathcal{D}_k(\mathfrak{sl}_n)$.

The argument is based on the idea of corner VOAs:

[4]Picture taken from Creutzig-Dimofte-Garner-Geer, arXiv:2112.01559

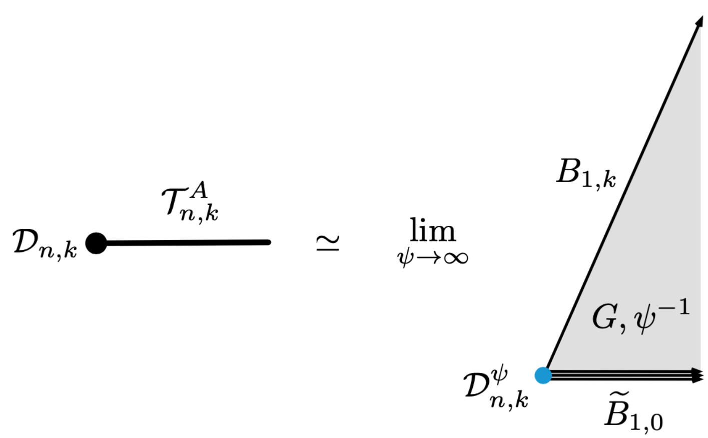
Corner VOAs

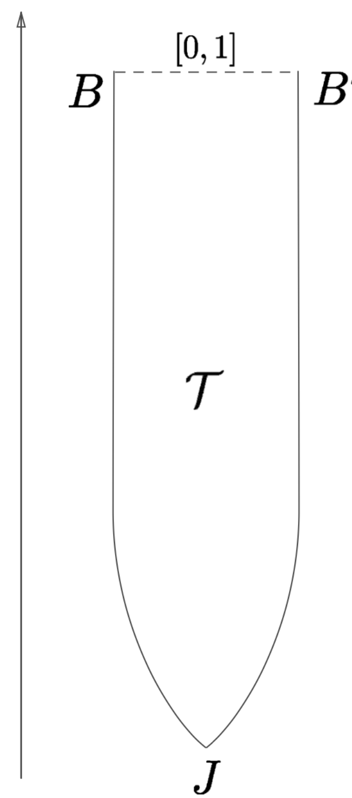
$N = 4$ SYM admits $3d$ boundaries with boundary conditions B. They can meet at $2d$ corners. Twist can make $4d$ bulk topological, leaving corners holomorphic. Holomorphic fields at corners $\rightsquigarrow$ corner VOA. (Gaiotto, Creutzig-Gaiotto, Rapcak-Gaiotto)

Expect effective $3d$ descriptions for corner configurations, with $3d$ theory determined by boundary conditions meeting at corner. Then corner VOA = boundary VOA.[5]

Right: Relation between conformal blocks in corner VOAs and states in $T(C) = \mathrm{Hom}(B, B')$.

Bottom: Example for relation of 3d boundary and 4d corner VOAs.



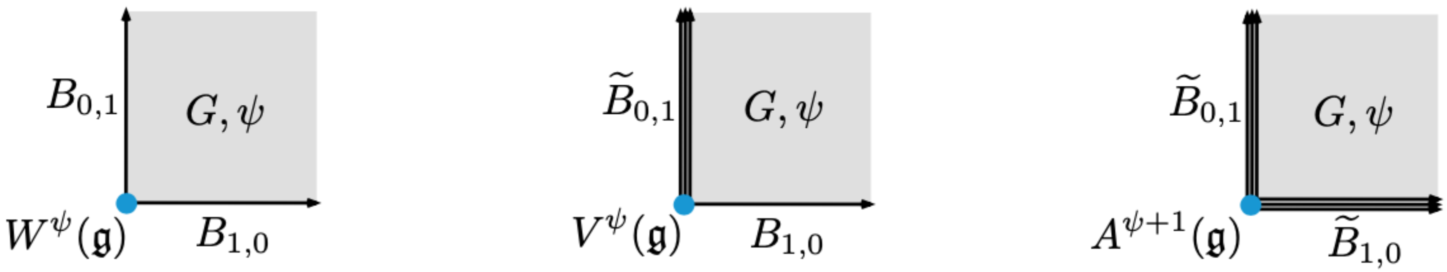


[5] Pictures taken from Frenkel-Gaiotto, arXiv:1805.00203 and Creutzig-Dimofte-Garner-Geer, arXiv:2112.01559

Corner VOAs

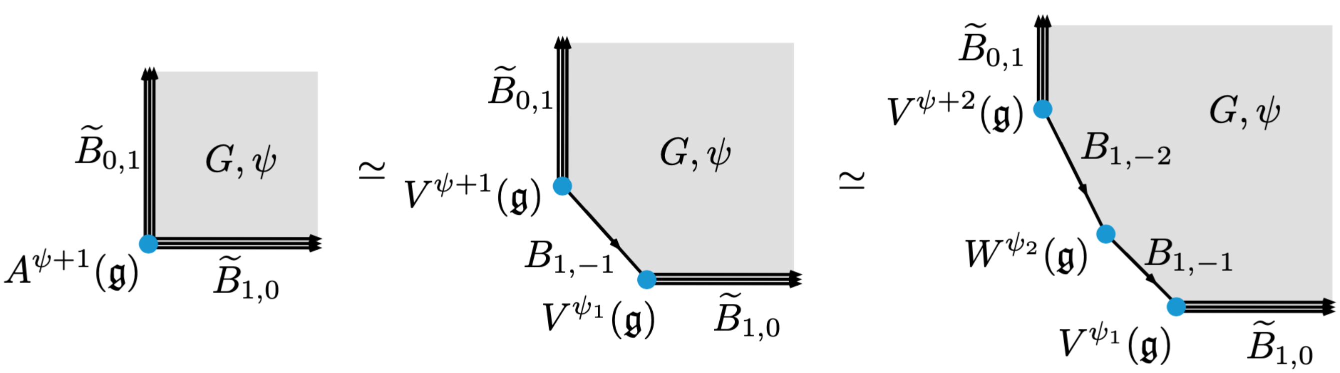
Useful toolkit for building corner VOAs: (Gaiotto, Creutzig-Gaiotto, Rapcak-Gaiotto)

Basic examples of corner VOAs:[6]



- $V^{\psi}(\mathfrak{g})$: affine VOA at level $\psi - h^{\vee}$,
- $W^{\psi}(\mathfrak{g}) \simeq W^{1/\psi}(\mathfrak{g})$: principal W-algebra of $\mathfrak{g}$.

Alternative representations from slicing:



[6] Pictures taken from Creutzig-Dimofte-Garner-Geer, arXiv:2112.01559

Feigin-Tipunin algebras as extensions of W-algebras

Boundary VOAs $\mathcal{D}_{n,k}$ of $\mathcal{T}^{A}_{SU(n),k}$ predicted by Creutzig-Dimofte-Garner-Geer to be Feigin-Tipunin algebras $\mathcal{FT}_k(\mathfrak{sl}_n)$. These algebras are representable as (Sugimoto)

$$\mathcal{FT}_k(\mathfrak{sl}_n) = \bigoplus_{\lambda \in Q^+} R_\lambda \otimes W^{1/k}_{\lambda,0}, \qquad \text{(FT)}$$

- Q^+: positive roots in root lattice Q of $\mathfrak{g} = \mathfrak{sl}_n$,
- R_λ: finite-dimensonal irreducible $\mathfrak{sl}_n$-modules with weight λ,
- $W^{\psi}_{\lambda,\mu}$ simple quotient of the W-algebra $W^{\psi}(\mathfrak{sl}_n)$-module with weight $\lambda - \psi\mu$.

Key feature:

- VOAs $\mathcal{FT}_k(\mathfrak{sl}_n)$ admit group of automorphisms $G^{\vee}_{\mathbb{C}} = PGL(n, \mathbb{C})$.
- Conformal blocks of $\mathcal{FT}_k(\mathfrak{sl}_n)$ admit twisting with $G^{\vee}_{\mathbb{C}}$-local systems.

Spaces of conformal blocks for generic twist expected to be $2^g k^{3g-3}$-dimensional, related to "semi-simplification" of non-semisimple TQFT from $U_q(\mathfrak{sl}_2)$ at $q = e^{\frac{\pi i}{k}}$. [7]

[7] Costantino/Geer/Patureau-Mirand.

A (trivial?) special case

Case $k = 1$: Right hand side makes perfect sense, but $\mathcal{D}_{n,1}$: lattice VOA V_Q.

However,

- V_Q has well-know super-VOA extension $\mathrm{FF}(n)$, containing V_Q via bosonisation.
- Relation (FT) for $k = 1$: Consequence of "bosonisation" formulae

$$\begin{aligned} \psi_s(z) &= e^{+\mathrm{i}\varphi_0(z)} V^{+s}_{1/2}(z) \\ \bar{\psi}_s(z) &= e^{-\mathrm{i}\varphi_0(z)} V^{-s}_{1/2}(z) \end{aligned} \qquad \Rightarrow \qquad \bar{\psi}_s(x)\psi_t(y) \sim \frac{\delta_{st}}{x-y},$$

where $V^{\pm}_{1/2}(z)$: Degenerate fields of $W^1(\mathfrak{sl}_2) = \mathrm{Vir}_{c=1}$, φ_0 auxilliary free boson.

Part II

Twisted free fermion conformal blocks, Virasoro algebra and tau-functions

Twisted free fermion conformal blocks

Free fermion CFT defined by VOA $\mathrm{FF}(n)$,

$$\bar{\psi}_s(x)\psi_t(y) \sim \frac{\delta_{st}}{x-y}, \qquad s,t = 1,\ldots,n,$$

associates 2^g-dimensional spaces of conformal blocks to Riemann surfaces $C = C_{g,n}$, characterised by functionals

$$G_{st}(x,y) := \left\langle\, \bar{\psi}_s(x)\psi_t(y) \,\right\rangle_C^{\mathrm{FF}}.$$

$\mathrm{FF}(n)$ has $GL(n,\mathbb{C})$-automorphism $\rightsquigarrow$ can define twisted conformal blocks, characterised by functionals satisfying

$$G_\rho(x,\gamma.y) = G_\rho(x,y)\cdot\rho(\gamma)$$

for given representation $\rho : \pi_1(C) \to G = GL(n,\mathbb{C})$.

Upshot: **Sheaf** with stalks $\mathrm{CB}_{\mathrm{FF}}(C)$ over $\mathrm{LocSys}_{G^\vee}(C)$.

Twisted free fermion conformal blocks II – Main claim[8]

Away from singularities, $\exists$ structure of **holomorphic line bundle** over $\mathrm{LocSys}_{G^\vee}(C)$.

- Cover $\mathrm{LocSys}_{G^\vee}(C)$ with charts U_α, coordinates of Fock-Goncharov (FG) or Fenchel-Nielsen (FN) type.
- Pick Darboux coordinates $(\sigma_\alpha^r, \eta_r^\alpha)$, $r = 1, \ldots, h$, $h := 3g - 3 + n$, in U_α.
- Define transition functions on $U_\alpha \cap U_\beta$ as the difference **generating functions**

$$G_{\alpha\beta}(\sigma_\alpha + \delta_r, \sigma_\beta) = e^{2\pi \mathrm{i} \eta_r^\alpha} G_{\alpha\beta}(\sigma_\alpha, \sigma_\beta),$$

$$G_{\alpha\beta}(\sigma_\alpha, \sigma_\beta + \delta_r) = e^{-2\pi \mathrm{i} \eta_r^\beta} G_{\alpha\beta}(\sigma_\alpha, \sigma_\beta).$$

$\rightsquigarrow \cdots \rightsquigarrow$ holomorphic line bundle $\mathcal{L}$ over $\mathrm{LocSys}_{G^\vee}(C) \times \mathcal{M}_{g,n}$.

Claim: Suitably normalised free fermion partition functions $\mathcal{T}_\alpha(\sigma_\alpha, \eta_\alpha; \tau) := \langle\, \mathrm{id} \,\rangle_C^{\mathrm{FF}}$ represent **holomorphic sections**, satisfying

$$\frac{\mathcal{T}_\alpha(\sigma_\alpha, \eta_\alpha; \tau)}{\mathcal{T}_\beta(\sigma_\beta, \eta_\beta; \tau)} = G_{\alpha\beta}(\sigma_\alpha, \sigma_\beta),$$

(i) $\mathcal{T}_\alpha(\sigma_\alpha + \delta_r, \eta_\alpha; \tau) = e^{2\pi \mathrm{i} \eta_r^\alpha} \mathcal{T}_\alpha(\sigma_\alpha, \eta_\alpha; \tau)$

(ii) $\mathcal{T}_\alpha(\sigma_\alpha, \eta_\alpha + \delta_r; \tau) = \mathcal{T}_\alpha(\sigma_\alpha, \eta_\alpha; \tau)$.

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[8] Coman-Longhi-J.T.

Twisted free fermion conformal blocks III

Note that (ii), (i) imply that $\mathcal{T}_\alpha(\sigma_\alpha, \eta_\alpha; \tau)$ admits expansion of the form

$$\mathcal{T}_\alpha(\sigma_\alpha, \eta_\alpha; \tau) = \sum_{\mathbf{n} \in \mathbb{Z}^h} e^{2\pi \mathrm{i}(\mathbf{n},\eta)} \mathcal{F}_\alpha(\sigma_\alpha - \mathbf{n}; \tau). \qquad \text{(GIL)}$$

When $(\sigma_\alpha, \eta_\alpha)$ are coordinates of FN-type, $\mathcal{F}_\alpha(\sigma_\alpha; \tau) \sim$ **conformal block** of $\mathrm{Vir}_{c=1}$.

(Discovered by Gamayun-Iorgov-Lisovyy (GIL); proofs for $g = 0$ Iorgov-Lisovyy-J.T., Bershtein-Shchechkin, Gavrylenko-Lisovyy, Nekrasov; sketch of generalisation to $g > 0$: Coman-Longhi-J.T.)

Note that $\mathcal{T}_\alpha$: **isomonodromic tau-functions** admitting Fredholm determinant representations[9] defining them rigorously as analytic objects.

Formula (GIL) can be understood as consequence of bosonisation relations

$$\psi_s(z) = e^{+\mathrm{i}\varphi_0(z)} V_{1/2}^{+s}(z), \qquad \bar{\psi}_s(z) = e^{-\mathrm{i}\varphi_0(z)} V_{1/2}^{-s}(z),$$

as will now be explained:

[9] Gavrylenko-Lisovyy, Cafasso-Gavrylenko-Lisovyy, Coman-Pomoni-J.T., Coman-Longhi-J.T.

Non-abelian bosonisation

Idea of proof of (GIL) by Iorgov-Lisovyy-J.T.:

- Gluing construction of conformal blocks $\rightsquigarrow$ Conformal blocks $\langle . \rangle_{\Lambda(\sigma)}^{\mathrm{Vir}_c}$, where $\Lambda(\sigma)$ is a collection of curves γ_r defining a pants decomposition, with representation V_{σ_r} assigned to cutting curve γ_r, $r = 1, \ldots, h$.

- On vector space generated by conformal blocks $\langle . \rangle_{\Lambda(\sigma)}^{\mathrm{Vir}_c}$ one can define **quantum monodromies** by considering analytic continuation of

$$\mathbf{G}_{\sigma}^{st}(x,y) = \left\langle\, V_{1/2}^{s}(x) V_{1/2}^{t}(y) \,\right\rangle_{\Lambda(\sigma)}^{\mathrm{Vir}} .$$

  with respect to y along closed curves on C defined by nullvector decoupling equations (BPZ).

- Quantum monodromies take the form

$$\mathbf{G}_{\sigma}(x, \gamma.y) = \mathbf{G}_{\sigma}(x,y) \cdot \overleftarrow{\mathsf{M}}_{\gamma}, \quad \mathsf{M}_{\gamma} = \begin{Bmatrix} \text{Laurent-Polynomial in } \mathsf{V}_r = e^{\partial_{\sigma_r}}, \\ \text{rational function in } \mathsf{U}_r = e^{2\pi \mathrm{i} \sigma_r}. \end{Bmatrix}$$

Non-abelian bosonisation II

Idea of proof of (GIL) by Iorgov-Lisovyy-J.T. (ctd.):

- Note that

$$\mathsf{V}_r\mathsf{U}_r = e^{2\pi\mathrm{i}}\mathsf{U}_r\mathsf{V}_r = \mathsf{U}_r\mathsf{V}_r, \qquad \mathsf{V}_r = e^{\partial_{\sigma_r}}, \quad \mathsf{U}_r = e^{2\pi\mathrm{i}\sigma_r}.$$

  Fourier-transformation in (GIL), applied to

$$G_{\rho_{\sigma,\eta}}(x,y) = \sum_{\mathbf{n}\in\mathbb{Z}^h} e^{2\pi\mathrm{i}(\mathbf{n},\eta)}\mathbf{G}_{\sigma-\mathbf{n}}(x,y), \qquad \text{(GIL')}$$

  diagonalises quantum monodromy:

$$\mathsf{V}_r \;\mapsto\; e^{2\pi\mathrm{i}\eta_r}, \qquad \mathsf{M}_\gamma \;\mapsto\; \rho_{\sigma,\eta}(\gamma),$$

  where $\rho_{\sigma,\eta}(\gamma)$: classical monodromy parameterised by FN-type coordinates (σ,η). It follows that $G_{\rho_{\sigma,\eta}}(x,y)$ has monodromies of twisted conformal block of $\mathrm{FF}(2)$.

- **Upshot:** Transformation in (GIL) relates conformal blocks of VOAs $\mathrm{Vir}_{c=1}$ and $\mathrm{FF}(2)$, with $G_{\rho_{\sigma,\eta}}(x,y)$ being the twisted free fermion two-point function.

Covering $\mathrm{LocSys}_{G^\vee}(C) \times \mathcal{M}(C)$

The FN type coordinates $(\sigma_\alpha, \eta_\alpha)$ associated to M_α won't cover $\mathrm{LocSys}_{G^\vee}(C)$, in general. Need to patch different charts together.

Claim: Changes of normalisation

$$\frac{\mathcal{T}_\alpha(\sigma_\alpha, \eta_\alpha; \tau)}{\mathcal{T}_\beta(\sigma_\beta, \eta_\beta; \tau)} = G_{\alpha\beta}(\sigma_\alpha, \sigma_\beta),$$

will preserve existence of expansions (GIL) iff

$$G_{\alpha\beta}(\sigma_\alpha + \delta_r, \sigma_\beta) = e^{2\pi\mathrm{i}\eta_r^\alpha} G_{\alpha\beta}(\sigma_\alpha, \sigma_\beta),$$
$$G_{\alpha\beta}(\sigma_\alpha, \sigma_\beta + \delta_r) = e^{-2\pi\mathrm{i}\eta_r^\beta} G_{\alpha\beta}(\sigma_\alpha, \sigma_\beta).$$

And indeed, there exist solutions to these equations canonically defined by the changes of coordinates $(\sigma_\alpha, \eta_\alpha) \leftrightarrow (\sigma_\beta, \eta_\beta)$.[10]

$\rightsquigarrow \cdots \rightsquigarrow$ holomorphic line bundle $\mathcal{L}$ over $\mathrm{LocSys}_{G^\vee}(C) \times \mathcal{M}_{g,n}$.

[10] Coman-Longhi-J.T., using crucial building block from Iorgov-Lisovyy-Tykhyy, arXiv:1308.4092.

Part III

Generalised tau-functions

and generalised geometric Langlands?

A collection of observations and conjectures.

Generalisation to $k > 1$?

Recall idea of proof of Iorgov-Lisovyy-J.T., modify constructions for $k > 1$:

- On vector space generated by conformal blocks $\langle . \rangle^{\mathrm{Vir}_c}_{\Lambda(\sigma)}$ one can define **quantum monodromies** by considering analytic continuation of

$$\mathbf{G}^{st}(x,y) = \big\langle \, V^s_{2,1}(x) V^t_{2,1}(y) \, \big\rangle^{\mathrm{Vir}}_{\Lambda(\sigma)} \, . \qquad \check{\mathbf{G}}^{st}(x,y) = \big\langle \, V^s_{1,2}(x) V^t_{1,2}(y) \, \big\rangle^{\mathrm{Vir}}_{\Lambda(\sigma)} \, .$$

  with respect to positions of degenerate fields $V^t_{2,1}(y)$ and $V^t_{1,2}(y)$.

- Quantum monodromies take the form

$$\mathbf{G}_\sigma(x,\gamma.y) = \mathbf{G}_\sigma(x,y) \cdot \overleftarrow{\mathsf{M}}_\gamma, \quad \mathsf{M}_\gamma = \left\{ \begin{matrix} \text{Laurent-Polynomial in } \mathsf{V}_r = e^{\frac{1}{k}\partial_{\sigma_r}}, \\ \text{rational in } \mathsf{U}_r = e^{2\pi \mathrm{i} \sigma_r},\ q = e^{\frac{\pi \mathrm{i}}{k}}. \end{matrix} \right\}$$

$$\check{\mathbf{G}}_\sigma(x,\gamma.y) = \check{\mathbf{G}}_\sigma(x,y) \cdot \overleftarrow{\mathsf{M}}^\vee_\gamma, \quad \mathsf{M}^\vee_\gamma = \left\{ \begin{matrix} \text{Laurent-Polynomial in } \check{\mathsf{V}}_r = e^{\partial_{\sigma_r}}, \\ \text{rational in } \check{\mathsf{U}}_r = e^{2\pi \mathrm{i} k\, \sigma_r}. \end{matrix} \right\}$$

Generalisation to $k > 1$, II

- Note that

$$\mathsf{V}_r\mathsf{U}_r = q^2\,\mathsf{U}_r\mathsf{V}_r, \qquad \mathsf{V}_r = e^{\frac{1}{k}\partial_{\sigma_r}}, \quad \mathsf{U}_r = e^{2\pi\mathrm{i}\sigma_r}, \quad q = e^{\frac{\pi\mathrm{i}}{k}}.$$

Modify Fourier-transformation in (GIL') to

$$\mathfrak{G}^{\mathbf{p}}_{\sigma,\eta}(x,y) = \sum_{\mathbf{n}\in\mathbb{Z}^h} e^{2\pi\mathrm{i}(\mathbf{n},\eta)}\mathbf{G}_{\sigma+\frac{\mathbf{p}}{k}-\mathbf{n}}(x,y), \qquad \mathbf{p}\in\mathbb{Z}_k^h. \qquad \text{(GIL")}$$

maps $\mathsf{V}_r \mapsto T_r$, with $T_r\,\mathfrak{G}^{\mathbf{p}}_{\sigma,\eta} = \mathfrak{G}^{\mathbf{p}+\delta_r}_{\sigma,\eta}$, and

$$\check{\mathsf{V}}_r \mapsto e^{2\pi\mathrm{i}\eta_r} \quad \Rightarrow \quad \boxed{\mathsf{M}^\vee_\gamma \mapsto \rho_{k\sigma,\eta}(\gamma),}$$

while M_γ mapped to operator-valued matrices on space of dimension k^{3g-3+n}.

Interpretation: $\left[\begin{array}{l}\text{Quantum monodromy} \mapsto \textbf{quantum} \text{ monodromy} \\ \text{on background of } \textbf{classical} \text{ local system } \rho_{\sigma,\eta}.\end{array}\right]$

Extension to coordinates of FG-type

Observation:[11] There exist canonical solutions to

$$G_{\alpha\beta}(\sigma_\alpha+\delta_r,\sigma_\beta)=e^{2\pi\mathrm{i}\eta_r^\alpha}G_{\alpha\beta}(\sigma_\alpha,\sigma_\beta),\quad G_{\alpha\beta}(\sigma_\alpha,\sigma_\beta+\delta_r)=e^{-2\pi\mathrm{i}\eta_r^\beta}G_{\alpha\beta}(\sigma_\alpha,\sigma_\beta).$$

for pairs $(\sigma_\alpha,\eta_\alpha),(\sigma_\beta,\eta_\beta)$ or mixed (FG/FN or FN/FG) type. $\rightsquigarrow$

- Extension of definition of $\mathcal{T}_\alpha$ to charts U_α with $(\sigma_\alpha,\eta_\alpha)\sim$ FG-type coordinates.
- Inversion of GIL-formula

$$\mathcal{T}_\alpha(\sigma_\alpha,\eta_\alpha;\tau)=\sum_{\mathbf{n}\in\mathbb{Z}^h}e^{2\pi\mathrm{i}(\mathbf{n},\eta)}\mathcal{F}_\alpha(\sigma_\alpha-\mathbf{n};\tau) \qquad \text{(GIL)}$$

  defines "wave-functions" $\mathcal{F}_\alpha$, and M_γ represented by quantum FG coordinates.
- FG-FG transition functions $G_{\alpha\beta}(\sigma_\alpha,\sigma_\beta)$,

$$x_r^\beta=x_r^\alpha\big(1+(x_s^\alpha)^{-\operatorname{sgn}(\epsilon_{rs})}\big)^{-\epsilon_{rs}},$$

  build from **classical** dilogarithm!

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[11] Coman-Longhi-J.T., based on crucial building block from Its-Lisovyy-Tykhyy

Root of unity phenomena in quantised cluster varieties

Quantum cluster varieties:

- Cluster variables X_r, relations $\mathsf{X}_r\mathsf{X}_s = q^{2\epsilon_{rs}}\mathsf{X}_s\mathsf{X}_r$.
- Change of variables maps

$$\mu_s(\mathsf{X}'_r) = \begin{cases} \mathsf{X}_r \prod_{\ell=1}^{|\epsilon_{rs}|}(1+q^{2\ell-1}\mathsf{X}_s^{-\operatorname{sgn}(\epsilon_{rs})})^{-\operatorname{sgn}(\epsilon_{rs})}, & r \neq s, \\ \mathsf{X}_r^{-1} & r = s. \end{cases}$$

- Quantum analogs of trace functions from quantum trace map.

Key observations

- If $q^k = -1$, $\exists$ large center generated by $\check{X}_r := X_r^k$, representable by $\check{X}_r \mapsto -x_r \,\mathrm{id}$,
- change of variables for center,

$$x'_r = x_r(1 + x_s^{-\operatorname{sgn}(\epsilon_{rs})})^{-\epsilon_{rs}},$$

  follows from $(u+v)^k = u^k + v^k$ for $uv = q^2 vu$.

Root of unity phenomena in quantised cluster varieties II

The following **conjecture** would represent a natural generalisation of the results known for $k = 1$ to $k > 1$:

> The spaces of conformal blocks for the VOAs $\mathcal{D}_{n,k}$ twisted by generic local systems carry a natural structure as holomorphic vector bundle with connection over $\mathrm{LocSys}_{G^\vee}(C) \times \mathcal{M}_{g,n}$.
>
> This structure is equivalent to the structure furnished by the **geometric quantisation** of $\mathrm{LocSys}_{G^\vee}(C)$ (Fock-Goncharov, to appear) in terms of cluster coordinates.

It should be very interesting to analyse the behaviour near singular loci where the twisting local system becomes degenerate, cf. Creutzig-Dimofte-Garner-Geer.

Generalisations of the geometric Langlands correspondence

Replacing $B_{0,1}$ by Dirichlet type boundary condition $\tilde{B}_{0,1}$ produces VOAs with affine Lie algebra symmetry and allows for twisting with holomorphic bundles.

Resulting picture shares many ingredients with the CFT-approach to the geometric Langlands correspondence (Beilinson-Drinfeld, Feigin-Frenkel):

- Conformal blocks of $V^0(\mathfrak{g}) \rightsquigarrow$ D-modules on Bun_G,
- twisting with $G^\vee$-local systems $\rightsquigarrow$ Hecke eigenvalue property.

Work of Kapustin-Witten, Gaiotto and many others has reproduced many aspects of the geometric Langlands correspondence from $N = 4$ SYM.

Note that away from critical one may define **quantum** monodromy of degenerate fields representing **Hecke modifications** as earlier in this talk, turning into a **classical** $G^\vee$-local system in the critical level limit.[12] All this indicates that the phenomena discussed in this talk form a part of several **generalisations** of the geometric Langlands correspondence arising in the context of $N = 4$ SYM.

[12] J.T., 2010