Geometric Structures in 4d N=2 class S theories

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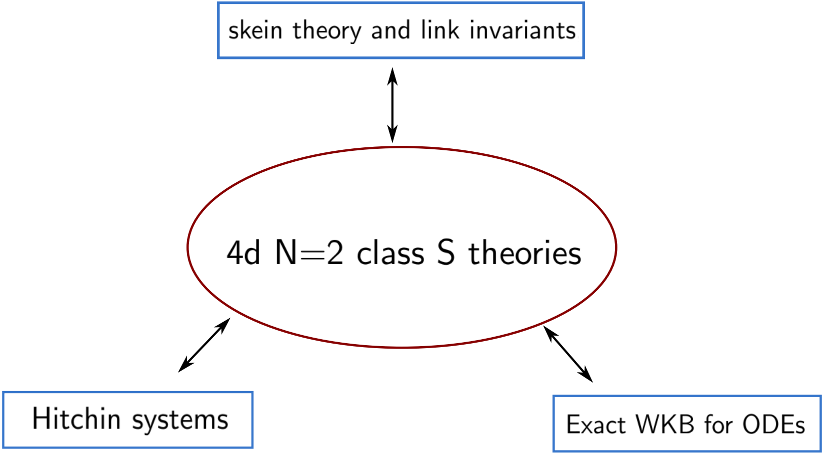
Outlook


skein theory and link invariants
4d N=2 class S theories
Hitchin systems
Exact WKB for ODEs

4d N=2 theories

- (3+1)-dim supersymmetric QFTs with 8 supercharges
- 4d N=2 supersymmetric multiplets:

vector multiplet

$$\begin{tikzcd} & \lambda_\alpha \arrow[r] & A_\mu \\ \phi \arrow[ur] \arrow[r] & \tilde{\lambda}_\alpha \arrow[ur] & \end{tikzcd}$$

hypermultiplet

$$\begin{tikzcd} & q \arrow[r, leftrightarrow] & \psi \\ \tilde{\psi}^\dagger \arrow[ur, leftrightarrow] \arrow[r, leftrightarrow] & \tilde{q}^\dagger \arrow[ur, leftrightarrow] & \end{tikzcd}$$

- Lagrangian for G gauge theory with matter.
- Many 4d N=2 theories do not admit a Lagrangian description, with a geometric origin. [Katz-Klemm-Vafa],[Gaiotto],[Gaiotto-Moore-Neitzke],...

Coulomb branch of 4d N=2 theories

- Coulomb branch: $\langle q \rangle = \langle \tilde{q} \rangle = 0$, Higgs branch: $\langle \phi \rangle = 0$
- At a generic point on the Coulomb branch $\mathcal{B}$, the low energy effective theory is 4d N=2 $U(1)^r$ gauge theory. [Seiberg-Witten] (here r=1)
- Central charge and BPS states

  $\mathcal{H}_u$: 1-particle Hilbert space on $\mathbb{R}^3$, vacua at $\infty \leftrightarrow u \in \mathcal{B}$

electro-magnetic and flavor charge

$$\mathcal{H}_u = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{u,\gamma} \qquad \gamma = (n_e, n_m, n_f)$$

central charge

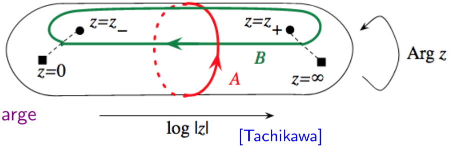
$$E_\gamma \geqslant |Z_\gamma| \qquad Z_\gamma = \underbrace{n_e a}_{\text{elec.period}} + \underbrace{n_m a_D}_{\text{mag.period}} + \underbrace{n_f \mu}_{\text{mass}}$$

The Seiberg-Witten curve

- The low energy dynamics is encoded in Seiberg-Witten curve.
- Example: SW curve for 4d $N = 2$ pure $SU(2)$ Yang-Mills is

$$\widetilde{C}_{\mathrm{SU}(2)}: \ \Lambda^2 z + \frac{\Lambda^2}{z} = x^2 - u,$$

with SW differential $\lambda = \frac{x}{z} dz$. ($u \in$ CB, Λ: strong scale)



electro-magnetic charge
$\gamma \in H_1(\widetilde{C}, \mathbb{Z})$

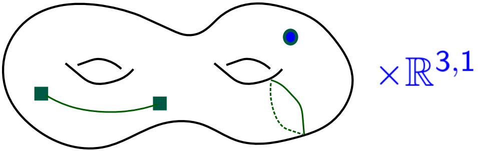
[Tachikawa]

Electro-magnetic periods: $a \sim \oint_A \lambda, \ a_D \sim \oint_B \lambda$

Gauge coupling $\tau(a) = \frac{\partial a_D}{\partial a}$

4d N=2 theories of class S

- Class S theories $\mathcal{T}[\mathfrak{g}, C]$ are 4d N=2 supersymmetric theories originating from twisted compactification of a 6d $(2,0)$ theory of type $\mathfrak{g}$ ($\{A, D, E\}$) on a Riemann surface C with appropriate decorations (punctures or twisted lines) [Gaiotto],[Gaiotto-Moore-Neitzke]



4d N=2 theories of class S: Coulomb branch

- 6d tensor branch $\rightarrow$ Coulomb branch $\mathcal{B}$ of class S theory.
  $\mathcal{B}$ is parameterized by meromorphic d_k-differentials on C

$$\mathcal{B} \subset \bigoplus_{k=1}^{r} H^0\left(C, K_C^{\otimes d_k}\left(\sum_i p_{d_k}^{(i)} z_i\right)\right)$$

[Gaiotto],[Gaiotto-Moore-Neitzke],[Chacaltana-Distler-Tachikawa],[Chacaltana-Distler-Trimm-Zhu] ...

- The Seiberg-Witten curve $\widetilde{C}$ is a branched covering of C, embedded in T^*C.

4d N=2 theories of class S: an example

4d N=2 pure SU(2) Yang-Mills revisited:

- The Riemann surface $C_{SU(2)}$: $\mathbb{CP}^1$ with 2 irregular punctures
- The Seiberg-Witten curve

$$\widetilde{C}_{\mathrm{SU(2)}} : \lambda^2 - \phi_2(z) = 0,\ \phi_2(z) = \left(\frac{\Lambda^2}{z} + \frac{u}{z^2} + \frac{\Lambda^2}{z^3} \right) dz^2$$

The Seiberg-Witten differential $\lambda = ydz$
y: fiber coordinate of T^*C

Relation to Hitchin systems

Further compactifying $\mathcal{T}[\mathfrak{g}, C]$ on S^1_R, the low energy effective theory is a 3d $N = 4$ sigma model with target space $M_H(G, C)$.

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- Starting from 6d and changing the order of compactification on $C \times S^1_R$ [Gaiotto-Moore-Neitzke], $M_H(G, C)$ is identified with the moduli space of solutions to Hitchin's equations:

$$F_A + R^2 \left[\Phi, \bar{\Phi}\right] = 0,$$
$$\bar{\partial}_A \Phi = 0, \quad \partial_A \bar{\Phi} = 0.$$

$\partial + A$ is a G-connection in a top. trivial G-bundle $V \to C$, $\Phi \in \Omega^{1,0}(\mathrm{End}\, V)$ is the Higgs field.

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- Seiberg-Witten curve $\longleftrightarrow$ spectral curve, characteristic of Φ, 4d Coulomb branch $\longleftrightarrow$ Hitchin base (Casimirs of Φ)

Relation to Hitchin systems

$M_H(G, C)$ is hyperkähler, has a $\mathbb{CP}^1$-worth of complex structures J_ζ. Different J_ζ expose different features of $M_H(G, C)$:

[Hitchin], [Simpson], [Biquard-Boalch], [Gaiotto-Moore-Neitzke]...

- $\zeta = 0$: (M_H, J_0) diff. to moduli space of Higgs bundles M_{Higgs}, which is a complex integrable system: $M_{\text{Higgs}} \to \mathcal{B}$ with generic fiber being compact tori.

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$$\widetilde{C} = \{(z \in C, \lambda \in T_z^* C) : \text{Det}\,(\Phi(z) - \lambda) = 0\} \subset T^* C$$

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  $$\widetilde{C} = \{(z \in C, \lambda \in T_z^* C) : \text{Det}\,(\Phi(z) - \lambda) = 0\} \subset T^* C$$

- $\zeta \in \mathbb{C}^\times$: Hitchin's equations indicate $\partial + \mathcal{A}$ is flat, with

  $$\mathcal{A} := \frac{R}{\zeta}\Phi + A + R\zeta\bar{\Phi}$$

  (M_H, J_ζ) diff. to a moduli space of flat $G_{\mathbb{C}}$-connections on C.

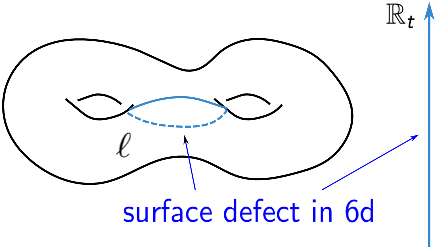
Line defects in class S theories

$\mathcal{T}[\mathfrak{g}, C]$ admits families of line defects $\mathbb{L}(\zeta)$ extending along $\mathbb{R}^t$-direction, where $\zeta \in \mathbb{C}^\times$ parametrizes preserved supercharges.

[Kapustin],[Kapustin-Saulina],[Drukker-Morrison-Okuda],[Drukker-Gaiotto-Gomis],[Drukker-Gomis-Okuda-Teschner]
[Gaiotto-Moore-Neitzke],[Córdova-Neitzke],[Aharony-Seiberg-Tachikawa],[Moore-Royston-van den Bleeken]...

$\mathbb{L}(\zeta, \mathfrak{p}, \mathcal{R})$ depends on path $\mathfrak{p}$ on C (up to homotopy), carrying representation $\mathcal{R}$ of $\mathfrak{g}$.

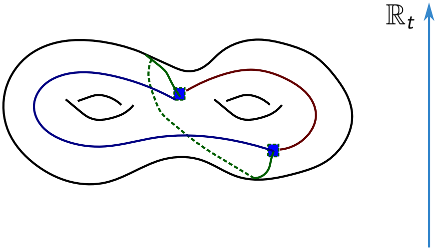
- $\mathfrak{g} = A_1$, C only has regular punctures:
  $\mathfrak{p}$ is a non-self-intersecting closed curve on C.



Line defects in class S theories

- C contains irregular punctures:
  $\mathfrak{p}$ corresponds to integral laminations [Fock-Goncharov],[Gaiotto-Moore-Neitzke], collection of paths either closed or open with ends on marked points corres. to Stokes directions at irregular punctures.
- In general $\mathfrak{p}$ could contain junctions, where paths carrying different $\mathcal{R}_i$ meet, associated with certain $\mathfrak{g}$-invariant tensor.

[Sikora],[Le],[Xie],[Saulina],[Coman-Gabella-Teschner],[Tachikawa-Watanabe],[Gabella]...



Line defects and Hitchin system

Upon circle compactification, the vacuum expectation values of $\mathbb{L}(\zeta)$ wrapping S^1_R are J_ζ-holomorphic functions on $M_{\text{flat}}(G_{\mathbb{C}}, C)$.

[Gaiotto-Moore-Neitzke]

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[Gaiotto-Moore-Neitzke]

- $\mathfrak{g} = A_1$, C has only regular punctures:

$$\begin{aligned}\langle \mathbb{L}(\zeta, \mathfrak{p}, \mathcal{R}) \rangle &= \mathrm{Tr}_{\mathcal{R}} \mathrm{Hol}_{\mathfrak{p}} \left(\frac{R\Phi}{\zeta} + A + R\zeta\bar{\Phi} \right) \\ &= \mathrm{Tr}_{\mathcal{R}} \mathrm{Hol}_{\mathfrak{p}} \mathcal{A}(\zeta).\end{aligned}$$

- In general, compute parallel transport of $\mathcal{A}(\zeta)$ along paths, contract together via $\mathfrak{g}$-invariant tensors.

The UV-IR map for line defects

A useful way to study $\mathbb{L}(\zeta)$ in class S theories, is deforming to a point u on the Coulomb branch $\mathcal{B}$ and follow the defect into IR.

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A useful way to study $\mathbb{L}(\zeta)$ in class S theories, is deforming to a point u on the Coulomb branch $\mathcal{B}$ and follow the defect into IR.

The IR limit of $\mathbb{L}(\zeta)$ is a superposition of supersymmetric line defects in the abelian theory, with integer coefficients in this superposition given by framed BPS index $\underline{\overline{\Omega}}(\mathbb{L}(\zeta), \gamma, u)$. [Gaiotto-Moore-Neitzke],[Córdova-Neitzke],[Cirafici-Del Zotto], [Coman-Gabella-Teschner],[Moore-Royston-van den Bleeken],[Ito-Okuda-Taki],[Galakhov-Longhi-Moore],[Brennan-Dey-Moore],...

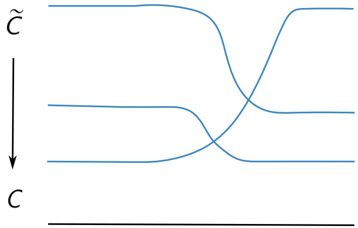
The UV-IR map for line defects:

$$\mathbb{L}(\zeta) \rightsquigarrow \sum_{\gamma} \underline{\overline{\Omega}}(\mathbb{L}(\zeta), \gamma, u) X_{\gamma}(\zeta)$$

$X_{\gamma}(\zeta)$ represent IR Wilson-'t Hooft lines with charge γ.

The IR line defects

Recall that, a point u on the Coulomb branch $\mathcal{B}$ corresponds to a branched covering $\widetilde{C} \to C$ (spectral curve/Seiberg-Witten curve):



Geometrically the IR line defect $X_\gamma(\zeta)$ correspond to loops $\tilde{\mathfrak{p}} \subset \widetilde{C}$ in class $\gamma \in H_1(\widetilde{C}, \mathbb{Z})$ (γ: IR charge).

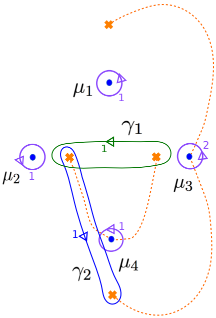
The UV-IR map $\leftrightarrow$ Uplift of $\mathfrak{p} \subset C$ to combinations of $\tilde{\mathfrak{p}} \subset \widetilde{C}$.

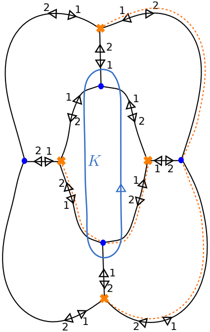
[Gaiotto-Moore-Neitzke]

The UV-IR map for line defects: an example

4d N=2 $SU(2)$ gauge theory with $N_f = 4$ fundamental hypermultiplets.
C is a four-punctured sphere. (punctures: blue, branch points: orange)

$$\widetilde{C} = \{\lambda : \lambda^2 + \phi_2 = 0\} \subset T^*C, \quad \phi_2 = -\frac{z^4 + 2z^2 - 1}{2(z^4 - 1)^2} dz^2.$$





The UV line defect labeled by $K \subset C$

7 IR line defects

$$X_{-\gamma_2-\mu_2+\mu_3} + X_{-\gamma_2-\mu_1-\mu_4} + X_{\gamma_1+\mu_1-\mu_4} + X_{-\gamma_1-\mu_1+\mu_4} + X_{\gamma_1-\gamma_2+\mu_1-\mu_4} + X_{\gamma_1-\gamma_2-\mu_2+\mu_3-2\mu_4} + X_{\gamma_1-2\gamma_2-\mu_2+\mu_3-2\mu_4},$$

The IR line defects and Hitchin system

Upon circle compactification, the VEV $\mathcal{X}_\gamma(\zeta)$ of $X_\gamma(\zeta)$ wrapping S^1_R are local Darboux coordinates on $M_{\text{flat}}(G_{\mathbb{C}}, C)$: Fock-Goncharov, complexified Fenchel-Nielsen, or more general spectral coordinates.

[Fock-Goncharov],[Fenchel-Nielsen],[Gaiotto-Moore-Neitzke],[Nekrasov-Rosly-Shatashvili],

[Hollands-Neitzke],[Hollands-Kidwai],[Allegretti],[Nikolaev],[Jeong-Nekrasov],[Coman-Longhi-Teschner] ...

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[Hollands-Neitzke],[Hollands-Kidwai],[Allegretti],[Nikolaev],[Jeong-Nekrasov],[Coman-Longhi-Teschner] ...

- $\mathcal{X}_\gamma$ has distinguished asymptotic behavior as $\zeta \to 0$ and $\zeta \to \infty$; has discontinuities across "BPS walls" controlled by Kontsevich-Soibelman symplectomorphisms.
  $\mathcal{X}_\gamma$ are solutions to a Riemann-Hilbert problem.

  [Gaiotto-Moore-Neitzke],[Gaiotto],[Bridgeland],[Barbieri],[Bridgeland-Barbieri-Stoppa]...

The UV-IR map as the trace map

- UV: $\langle \mathbb{L}(\zeta) \rangle$ are J_ζ-holomorphic trace functions on $M_{\text{flat}}(G_{\mathbb{C}}, C)$.
- IR: $\mathcal{X}_\gamma(\zeta) := \langle X_\gamma(\zeta) \rangle$ are Darboux-coordinates on $M_{\text{flat}}(G_{\mathbb{C}}, C)$.

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- The UV-IR map

$$\mathbb{L}(\zeta) \rightsquigarrow \sum_\gamma \underline{\overline{\Omega}}(\mathbb{L}(\zeta), \gamma) X_\gamma(\zeta)$$

then implies the trace map:

$$\mathrm{Tr}_{\mathcal{R}} \mathrm{Hol}_{\mathfrak{p}} \mathcal{A}(\zeta) = \sum_\gamma \underline{\overline{\Omega}}(\mathbb{L}(\zeta), \gamma) \mathcal{X}_\gamma(\zeta)$$

$\mathcal{A}(\zeta)$: flat $G_{\mathbb{C}}$-connection on C, $\mathfrak{p}$: path on C.

Line defects OPE

- The algebra structure on the space of J_ζ-holomorphic functions corresponds to line defects operator products (OPE):

$$\langle \mathbb{L}_1(\zeta)\mathbb{L}_2(\zeta)\rangle = \langle \mathbb{L}_1(\zeta)\rangle\langle \mathbb{L}_2(\zeta)\rangle$$

- This algebra structure admits a quantization via skein algebras.

  [Reshetikhin-Turaev],[Turaev],[Witten],[Alday-Gaiotto-Gukov-Tachikawa-Verlinde],[Gaiotto-Moore-Neitzke],

  [Drukker-Gomis-Okuda-Teschner],[Tachikawa-Watanabe],[Coman-Gabella-Teschner],[Gabella]...

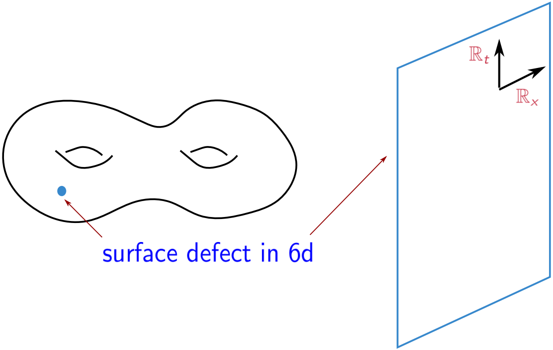
- This leads to a quantization of the UV-IR map (trace map).

  [c.f. my talk next Monday]

Intermediate Summary

- Introduction to 4d N=2 class S theories
- Relation to Hitchin systems
- Line defects in class S theories and the UV-IR map / trace map

Surface defects in 4d $N = 2$ class-S theories



[Gukov-Witten],[Alday-Gaiotto-Gukov-Tachikawa-Verlinde],[Gaiotto],[Gaiotto-Moore-Neitzke], [Gaiotto-Gukov-Seiberg],...

- canonical surface defects preserve 2d $(2, 2)$ susy $\subset$ 4d N=2 susy
- $z \in C \leftrightarrow$ marginal chiral deformation parameter for the defect theory

Surface defects in class-S theories: IR picture

Deforming into Coulomb branch (CB) of the 4d $N = 2$ theory:

- The 4d bulk is described by an effective abelian theory.
- The surface defect has a set of massive vacua, fibered over C to form a space of vacua $\widetilde{C} \to C$:
  1-form $\lambda = x dz$, $x \leftrightarrow$ VEV of twisted chiral ring operator

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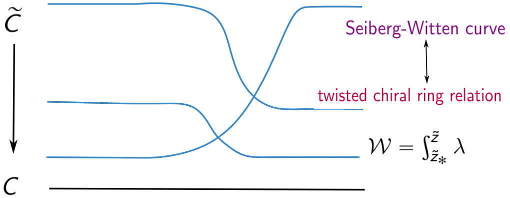
The vacua structure encodes the bulk Seiberg-Witten geometry.



Surface defects and Schrödinger equations

- Consider the Seiberg-Witten curve of certain 4d $N = 2$ theory:
$$\widetilde{C}: \; x^2 + P(z) = 0,$$
Promoting x (momentum) and z (position) to Heisenberg operators $\rightsquigarrow$ Schrödinger equation:
$$\left[\partial_z^2 + \hbar^{-2} P(z,\hbar)\right] \psi(z) = 0$$
- Turning on the Nekrasov-Shatashvili limit of Ω-background along the surface defect quantizes the Seiberg-Witten curve into Schrödinger equations or higher rank analogue, in a canonical way.
[Nekrasov-Shatashvili],[Nekrasov],[Jeong],[Jeong-Nekrasov],[Jeong-Lee-Nekrasov],...
- This can also be derived through the conformal limit [Gaiotto] of the Hitchin moduli space or from the AGT-correspondence.
[Alday-Gaiotto-Tachikawa],[Alday-Gaiotto-Gukov-Tachikawa-Verlinde],...

Exact WKB for Schrödinger equations

WKB ansatz: $\psi(z) = \exp\left(\hbar^{-1}\int_{z_0}^{z}\lambda(z')dz'\right) \rightarrow [\partial_z^2 + \hbar^{-2}P(z)]\psi(z) = 0$

$\lambda(z)$ obeys the Ricatti equation

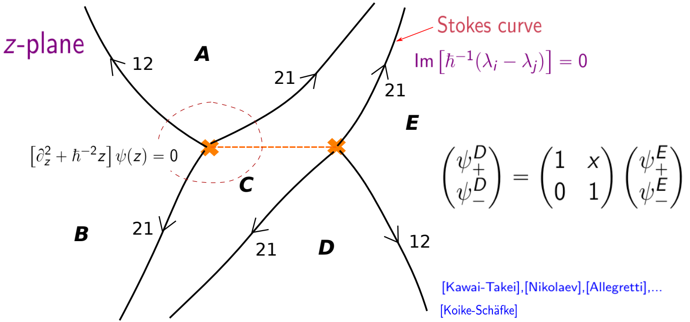
$$\lambda(z)^2 + P(z) + \hbar\partial_z\lambda(z) = 0.$$

Build a formal series solution λ^{formal} in powers of $\hbar$,

$$\text{order-}\hbar^0 : \ (\lambda^{(0)})^2 + p(z) = 0, \quad \text{classical SW curve}$$

Choose a branch labeled by $i \in \{\pm\} \rightsquigarrow$ 2 formal solutions $\lambda_\pm^{\text{formal}}$
$\longrightarrow$ Two formal solutions $\psi_\pm^{\text{formal}}(z,\hbar)$ as series in $\hbar$.

Exact WKB for Schrödinger equations



- Borel resummation gives two actual solutions $\psi_\pm(z)$ within each region, where the solutions jump across a Stokes curve.
- Stokes curves $\leftrightarrow$ soliton spectrum of surface defects [Gaiotto-Moore-Neitzke]

The Voros symbol

The Voros symbol: $\mathcal{X}_\gamma(\hbar) \in \mathbb{C}^\times$, $\gamma \leftrightarrow$ 1-cycles of Seiberg-Witten curve

- $\mathcal{X}_\gamma(\hbar)$ captures the Borel resummed WKB periods:

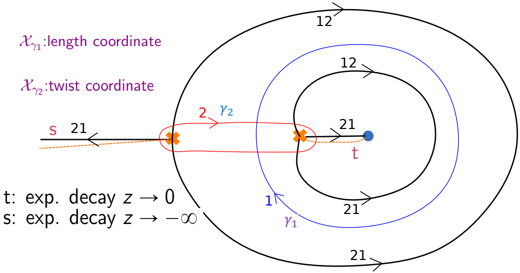
$$\Pi_\gamma(\hbar) := \oint_\gamma \lambda^{\text{formal}}(\hbar) dz = \sum_{n=0}^{\infty} \Pi_\gamma^{(n)} \hbar^n$$

- $\mathcal{X}_\gamma(\hbar)$ expressed as Wronskians of distinguished local solutions:
  - ⋆ asymptotically decaying solutions as z approaches a singularity
  - ⋆ eigenvectors of the monodromy around a loop
- $\mathcal{X}_\gamma(\hbar)$ encodes exact quantization conditions for spectral problems.

The Voros symbol: modified Mathieu operator

$$[-\hbar^2 \partial_x^2 + 2\cosh(x) - 2E]\psi(x) = 0 \quad (E > 1)$$

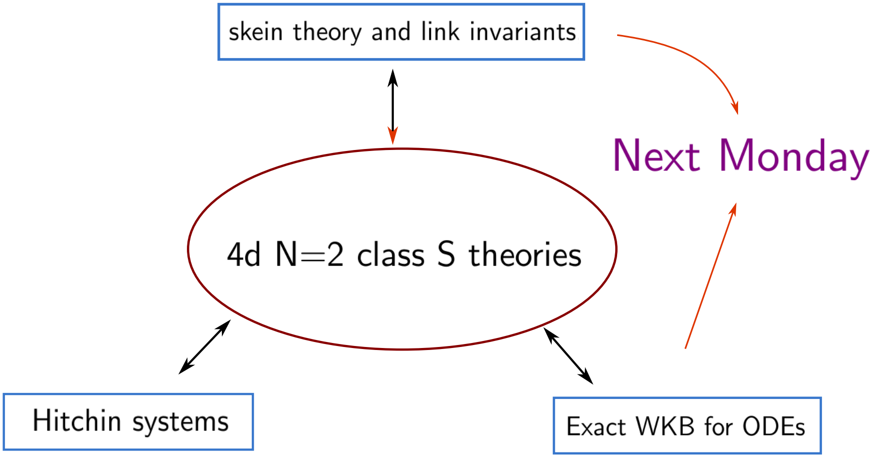
$$z = -e^{-x} \to \left[\hbar^2 \partial_z^2 + \left(\frac{1}{z^3} + \frac{1}{z} + \frac{2E + 0.25\hbar^2}{z^2}\right)\right]\tilde{\psi}(z) = 0.$$ SU(2) SYM



bound states: s prop. to $t \to \mathcal{X}_{\gamma_2} = 1$ (exact quantization condition)

[Mironov-Morozov],[He-Miao],[Basar-Dunne],[Dunne-Ünsal],[Codesido-Marino-Schiappa],[Hollands-Neitzke],...

Summary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Next Monday
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Thank You and Stay Safe!