Best Constants for the Hardy-Littlewood Maximal Theorem

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Notation

- We say $f \in L^p(\mathbb{R}^n)$ if $\|f\|_p < \infty$ where
$$\|f\|_p := \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}.$$
- We say $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ if
$$\int_K |f(x)| \, dx < \infty$$
for all compact $K \subseteq \mathbb{R}^n$.
- For a measurable $E \subseteq \mathbb{R}^n$, we denote the Lebesgue measure of E by $|E|$.

Notation

- For a measurable $E \subseteq \mathbb{R}^n$ and a measurable function f, define
$$\fint_E f(x)\,dx := \frac{1}{|E|}\int_E f(x)\,dx.$$
- We will write $\{f > \lambda\}$ to mean
$$\{x \in \mathbb{R}^n \mid f(x) > \lambda\}.$$
- For any $p > 1$, let
$$p' := \frac{p}{p-1}.$$

The Hardy-Littlewood Maximal Function

Definition

For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we define the (uncentered) Hardy-Littlewood Maximal function as

$$Mf(x) = \sup_{Q \ni x} \fint_Q |f(y)|\, dy$$

where we take the supremum over all cubes containing x.

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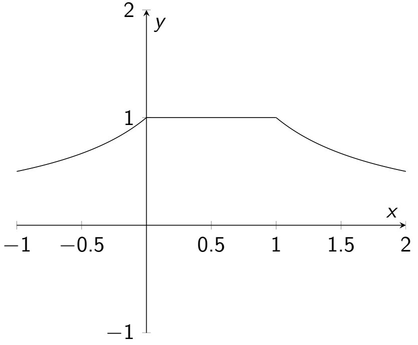
- Introduced by Hardy and Littlewood in 1930
- Ubiquitous in Harmonic Analysis

Example: $f = \chi_{[0,1]}$

On $\mathbb{R}$, let $f(x) = \chi_{[0,1]}$. Then

$$Mf(x) = \begin{cases} \frac{1}{1-x}, & \text{if } x < 0, \\ 1, & \text{if } 0 \leq x < 1, \\ \frac{1}{x}, & \text{if } x \geq 1. \end{cases}$$

Graph:



Variants

- Centered Cubes:

$$M^c f(x) = \sup_{Q(x,r) \ni x} \fint_{Q(x,r)} |f(y)|\, dy = \sup_{r>0} \frac{1}{(2r)^n} \int_{Q(x,r)} |f(y)|\, dy$$

where $Q(x,r) = [x_1 - r, x_1 + r] \times \cdots \times [x_n - r, x_n + r]$

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- Uncentered Balls:

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$$\mathcal{M}^c f(x) = \sup_{B(x,r)\ni x} \fint_{B(x,r)} |f(y)|\, dy$$

$$= \sup_{r>0} \frac{\Gamma(n/2+1) r^n}{\pi^{n/2}} \int_{B(x,r)} |f(y)|\, dy$$

where $B(x, r)$ is the ball of radius r centered at x.

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- Dyadic:

$$M^d f(x) = \sup_{\substack{Q\in\mathcal{D}\\ Q\ni x}} \fint_Q |f(y)|\, dy$$

  where we take the supremum over all dyadic cubes containing x and

$$\mathcal{D} = \left\{ 2^k \left([0,1)^n + \vec{m}\right) \,\middle|\, k \in \mathbb{Z},\ \vec{m} \in \mathbb{Z}^n \right\}.$$

Variants

- These variants are rarely equal; however, they are equivalent up to a constant. That is, there exist constants $c_1, c_2 > 0$ such that

$$c_1 M_1 f(x) \leq M_2 f(x) \leq c_2 M_1 f(x)$$

where M_1 and M_2 denote any of the variants of the Hardy-Littlewood maximal operator.

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- For example, $M^c f(x) \leq Mf(x)$ trivially. On the other hand,

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(y)|\,dy &\leq \frac{1}{|Q|}\int_{Q(x,2r)} |f(y)|\,dy \\ &= \frac{2^n}{|Q(x,2r)|}\int_{Q(x,2r)} |f(y)|\,dy \leq 2^n M^c f(x)\end{aligned}$$

where Q is a cube containing x and $r = \ell(Q)$.

Boundedness

Theorem (Hardy-Littlewood Maximal Theorem)

The Hardy-Littlewood Maximal Operator is weak-type (1,1) and strong-type (p,p) for $1 < p \leq \infty$; that is, there exist nonnegative constants C_1 and C_p such that

$$\textit{Weak-Type } (1,1)\text{:} \quad |\{x \in \mathbb{R}^n \mid Mf(x) > \lambda\}| \leq \frac{C_1}{\lambda}\|f\|_1$$

$$\textit{Strong-Type } (p,p)\text{:} \quad \|Mf\|_p \leq C_p\|f\|_p \quad \forall p \in (1,\infty].$$

Proof Outline.

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Weak-Type $(1,1)$: $$|\{x \in \mathbb{R}^n \mid Mf(x) > \lambda\}| \leq \frac{C_1}{\lambda}\|f\|_1$$

Strong-Type (p,p): $$\|Mf\|_p \leq C_p\|f\|_p \quad \forall p \in (1, \infty].$$

Proof Outline.

- Prove weak-type $(1,1)$ using Vitali Covering.
- $p = \infty$ case is trivial.
- weak-type $(1,1)$ + $L^\infty \implies$ strong-type (p,p) for $1 < p \leq \infty$ or directly prove it

Proof - Weak-Type (1,1)

Lemma (Vitali Covering Lemma)

Given a collection of cubes in $\mathbb{R}$, say $\{Q_1, Q_2, \ldots, Q_n\}$, there exists $S \subseteq \{1, 2, \ldots, n\}$ such that $\{Q_j\}_{j \in S}$ is disjoint and

$$\bigcup_{j=1}^{n} Q_j \subseteq \bigcup_{j \in S} 3Q_j.$$

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- *Proof of weak-type* $(1,1)$. Let $K \subseteq \{x \mid Mf(x) > \lambda\}$ be a (non-empty) compact subset. Then $K \subseteq \bigcup_{x \in K} Q_x$ where

$$\fint_{Q_x} |f(y)|\, dy > \lambda \quad \forall x \in K. \tag{1}$$

  By compactness, there exists a finite subcollection such that $K \subseteq \bigcup_{j=1}^{n} Q_j$ where each Q_j satisfy (1). By the Vitali Covering Lemma, there exists a further subcollection $\{Q_j\}_{j \in S}$ such that $K \subseteq \bigcup_{j \in S} 3Q_j$.

Proof - Weak-Type (1,1)

- Then

$$
\begin{aligned}
|K| &\leq \sum_{j\in S} |3Q_j| \\
&= \sum_{j\in S} 3^n \, |Q_j| \\
&< 3^n \sum_{j\in S} \frac{1}{\lambda} \int_{Q_j} |f(y)| \, dy \\
&= \frac{3^n}{\lambda} \int_{\bigcup_{j\in S} Q_j} |f(y)| \, dy \quad \left[\leq \frac{3^n}{\lambda} \int_{\{Mf>\lambda\}} |f(y)| \, dy\right] \\
&\leq \frac{3^n}{\lambda} \|f\|_1
\end{aligned}
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Taking the supremum over all compact sets K finishes the proof.

Proof - Weak-Type (1,1)

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Taking the supremum over all compact sets K finishes the proof.

- *Note*. We can take $C_1 = 3^n$.

Proof - Strong-Type (∞, ∞)

- When $p = \infty$,

$$\fint_Q |f(y)|\, dy \leq \fint_Q \|f\|_\infty \, dy = \|f\|_\infty.$$

Taking the supremum over all cubes shows

$$Mf(x) \leq \|f\|_\infty \quad \forall x \in \mathbb{R}^n.$$

Thus, $\|Mf\|_\infty \leq \|f\|_\infty$.

Proof - Strong Type (p, p)

- By the weak-type $(1, 1)$ inequality,

$$\begin{aligned}
\|Mf\|_p^p = \int_{\mathbb{R}^n} |Mf(x)|^p \, dx &= p \int_0^\infty \lambda^{p-1} \, |\{x \mid Mf(x) > \lambda\}| \, dx \, d\lambda \\
&\leq 3^n p \int_0^\infty \lambda^{p-2} \int_{\{Mf > \lambda\}} |f(x)| \, dx \, d\lambda \\
&= 3^n p \int_{\mathbb{R}^n} |f(x)| \int_0^{Mf(x)} \lambda^{p-2} \, d\lambda \, dx \\
&= 3^n \frac{p}{p-1} \int_{\mathbb{R}^n} |f(x)| Mf(x)^{p-1} \, dx \\
&= 3^n p' \|f\|_p \|Mf\|_p^{p-1}.
\end{aligned}$$

Then

$$\|Mf\|_p \leq 3^n p' \|f\|_p.$$

Proof - Strong Type (p, p)

- By the weak-type $(1,1)$ inequality,

$$\begin{aligned}\|Mf\|_p^p = \int_{\mathbb{R}^n} |Mf(x)|^p \, dx &= p \int_0^\infty \lambda^{p-1} |\{x \mid Mf(x) > \lambda\}| \, dx \, d\lambda \\ &\leq 3^n p \int_0^\infty \lambda^{p-2} \int_{\{Mf > \lambda\}} |f(x)| \, dx \, d\lambda \\ &= 3^n p \int_{\mathbb{R}^n} |f(x)| \int_0^{Mf(x)} \lambda^{p-2} \, d\lambda \, dx \\ &= 3^n \frac{p}{p-1} \int_{\mathbb{R}^n} |f(x)| Mf(x)^{p-1} \, dx \\ &= 3^n p' \|f\|_p \|Mf\|_p^{p-1}.\end{aligned}$$

  Then

$$\|Mf\|_p \leq 3^n p' \|f\|_p.$$

- *Note*. We can take $C_p = 3^n p'$.

Best Constants

- *Question* : What are the best constants C_1 and C_p?
  We will write $\|M\|_{L^{1,\infty}(\mathbb{R}^n)}$ and $\|M\|_{L^p(\mathbb{R}^n)}$.

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- *Question* : What are the best constants C_1 and C_p?
  We will write $\|M\|_{L^{1,\infty}(\mathbb{R}^n)}$ and $\|M\|_{L^p(\mathbb{R}^n)}$.
- Finding the best constant requires two steps
  - The upperbound:

$$\|Mf\|_p \leq \|M\|_{L^p(\mathbb{R}^n)} \|f\|_p, \quad \forall f \in L^p.$$

  - The lowerbound:

$$\|Mf_0\|_p = \|M\|_{L^p(\mathbb{R}^n)}, \quad \text{where } \|f_0\|_p = 1$$

or

$$\lim_{k\to\infty} \|Mf_k\|_p = \|M\|_{L^p(\mathbb{R}^n)} \quad \text{where } \|f_k\|_p = 1.$$

What is Known: Dimension $n = 1$, Uncentered

For $p = 1$,

$$\|M\|_{L^{1,\infty}(\mathbb{R})} = 2.$$

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Theorem (Grafakos and Montgomery Smith, 1997)

For $p > 1$, $\|M\|_{L^p(\mathbb{R})}$ is the largest root of

$$(p-1)x^p - px^{p-1} - 1$$

For example,

$$\|M\|_{L^2(\mathbb{R})} = 1 + \sqrt{2}.$$

What is Know: Dimension $n = 1$, Centered

Theorem (Melas, 2003)

$\|M^c\|_{L^{1,\infty}(\mathbb{R}^n)}$ is the largest root of $12x^2 - 22x + 5$, which is

$$\|M^c\|_{L^{1,\infty}(\mathbb{R}^n)} = \frac{11 + \sqrt{61}}{12} \approx 1.5675.$$

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For $p > 1$, the best constant is unknown.

What is Known: Dyadic

For the dyadic maximal function, the constants are known:

$$\left\| M^d \right\|_{L^{1,\infty}(\mathbb{R}^n)} = 1$$

and

$$\left\| M^d \right\|_{L^p(\mathbb{R}^n)} = \frac{p}{p-1} =: p'.$$

What is Known: Dyadic

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$$\left\| M^d \right\|_{L^{1,\infty}(\mathbb{R}^n)} = 1$$

and

$$\left\| M^d \right\|_{L^p(\mathbb{R}^n)} = \frac{p}{p-1} =: p'.$$

- The bound (supposedly) goes back to Doob (1953) and Burkholder (1984).
- The sharpness is due to Melas (2005) and Slavin, Stokolos, and Vasyunin (2008).

What is Known: Higher Dimensions, $p = 1$

- Almost nothing is known in dimension $n \geq 2$.

What is Known: Higher Dimensions, $p = 1$

- Almost nothing is known in dimension $n \geq 2$.
- Stein and Strömberg (1983) showed that

$$\|M\|_{L^{1,\infty}(\mathbb{R}^n)} \leq O(n \log n)$$

  and

$$\|\mathcal{M}\|_{L^{1,\infty}(\mathbb{R}^n)} \leq O(n).$$

- Aldaz (2011) showed

$$\|M^c\|_{L^{1,\infty}(\mathbb{R}^n)} \nearrow \infty \quad \text{as } n \to \infty.$$

What is Known: Higher Dimensions, $p > 1$

- Grafakos and Montogomery-Smith (1997) showed

$$\|\mathcal{M}\|_{L^p(\mathbb{R}^n)} \approx c^n \quad \text{for some } c > 1.$$

- Bourgain (1987) showed that $\|M^c\|_{L^p(\mathbb{R}^n)}$ is independent of dimension when $p > \dfrac{3}{2}$.
- Stein and Strömberg (1983) showed that $\|\mathcal{M}^c\|_{L^p(\mathbb{R}^n)}$ is independent of dimension.

Summary

- Dimension $n = 1$

	$p = 1$	$p > 1$
Uncentered	2	root of $(p-1)x^p - px^{p-1} - 1$
Centered	$\frac{11+\sqrt{61}}{2}$	unknown

Summary

- Dimension $n = 1$

	$p = 1$	$p > 1$
Uncentered	2	root of $(p-1)x^p - px^{p-1} - 1$
Centered	$\frac{11+\sqrt{61}}{2}$	unknown

- Dimension $n \geq 2$

	$p = 1$	$p > 1$
Centered Cubes	$\nearrow \infty$ as $n \to \infty$	independent of n for $p > 3/2$
Centered Balls	unknown	independent of n
Uncentered Balls	unknown	exponential in n
Dyadic	1	p'

Uncentered: $p = 1$, $n = 1$ (Lemma)

Lemma (Garnett (1981))

Given a finite collection of intervals $\{I_1, I_2, \ldots, I_N\}$*, there exists two subcollections*

$$\{J_1, J_2, \ldots, J_m\} \quad \textit{and} \quad \{L_1, L_2, \ldots, L_n\}$$

such that each subcollection is pairwise disjoint and

$$\bigcup_{i=1}^{N} I_i = \left(\bigcup_{j=1}^{m} J_i \right) \cup \left(\bigcup_{\ell=1}^{n} L_\ell \right).$$

Uncentered: $p = 1$, $n = 1$ (Boundedness)

Let $K \subseteq \{Mf(x) > \lambda\}$ be compact. Then

$$K \subseteq \bigcup_{i=1}^{N} I_i$$

where

$$\frac{1}{|I_i|} \int_{I_i} |f(y)|\, dy > \lambda.$$

Write

$$\bigcup_{i=1}^{N} I_i = \left(\bigcup_{j=1}^{m} J_i \right) \cup \left(\bigcup_{\ell=1}^{n} L_\ell \right).$$

Then

$$|K| \leq \sum_{j=1}^{m} |J_j| + \sum_{\ell=1}^{n} |L_\ell| \leq \frac{1}{\lambda} \left(\sum_{j=1}^{m} \int_{J_j} |f(y)|\, dy + \sum_{\ell=1}^{n} \int_{L_\ell} |f(y)|\, dy \right)$$
$$\leq \frac{2}{\lambda} \int_{\mathbb{R}} |f(y)|\, dy.$$

Uncentered: $p = 1$, $n = 1$ (Sharpness)

Consider $f(x) = \chi_{[0,1]}$. For $\lambda < 1$,

$$\{x \mid Mf(x) > \lambda\} = \left(1 - \frac{1}{\lambda}, \frac{1}{\lambda}\right).$$

Thus,

$$\lambda \left|\{x \mid Mf(x) > \lambda\}\right| = \lambda \left|\left(1 - \frac{1}{\lambda}, \frac{1}{\lambda}\right)\right| = 2 - \lambda.$$

Then

$$\begin{aligned} 2 - \lambda = \lambda \left|\{x \mid Mf(x) > \lambda\}\right| \\ \le \|Mf\|_{L^{1,\infty}(\mathbb{R})} \|f\|_1 = \|Mf\|_{L^{1,\infty}(\mathbb{R})} \le 2. \end{aligned}$$

Dyadic: $p = 1$

If $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$ is such that

$$\left\{ x \in \mathbb{R}^n \,\middle|\, M^d f(x) > \lambda \right\} \neq \varnothing$$

then

$$\left\{ x \in \mathbb{R}^n \,\middle|\, M^d f(x) > \lambda \right\} = \bigcup_{k \in \mathbb{N}} Q_k$$

where $\{Q_k\}_{k \in \mathbb{N}}$ is a countable collection of dyadic cubes such that

$$\frac{1}{|Q_k|} \int_{Q_k} |f(y)| \, dy > \lambda.$$

Dyadic: $p = 1$ (Boundedness)

$$\begin{aligned}
\left|\left\{x \in \mathbb{R}^n \,\middle|\, M^d f(x) > \lambda\right\}\right| &= \sum_{k \in \mathbb{N}} |Q_k| \\
&\leq \sum_{k \in \mathbb{N}} \frac{1}{\lambda} \int_{Q_k} |f(y)|\, dy \\
&= \frac{1}{\lambda} \int_{\bigcup_k Q_k} |f(y)|\, dy \\
&= \frac{1}{\lambda} \int_{\{M^d f(x) > \lambda\}} |f(y)|\, dy \\
&\leq \frac{1}{\lambda} \|f\|_1.
\end{aligned}$$

Thus,

$$\left\|M^d\right\|_{L^{1,\infty}(\mathbb{R}^n)} \leq 1.$$

Dyadic: $p = 1$ (Sharpness)

For $\frac{1}{2} < \lambda < 1$, we have

$$\left\{ x \in \mathbb{R}^n \,\middle|\, M^d \left(\chi_{[0,1]} \right) (x) > \lambda \right\} = [0, 1].$$

Then

$$\begin{aligned} \lambda = \lambda \left| \left\{ x \in \mathbb{R}^n \,\middle|\, M^d f(x) > \lambda \right\} \right| &\leq \left\| M^d \right\|_{L^{1,\infty}(\mathbb{R}^n)} \left\| \chi_{[0,1]} \right\|_1 \\ &= \left\| M^d \right\|_{L^{1,\infty}(\mathbb{R}^n)} . \end{aligned}$$

Dyadic: $p > 1$ (Boundedness)

For $f \in L^p$ and $f \not\equiv 0$, write

$$\begin{aligned}
\left\|M^d f\right\|_p^p = \int_{\mathbb{R}^n} \left|M^d f(y)\right|^p dy &= p \int_0^\infty \lambda^{p-1} \left|\left\{M^d f > \lambda\right\}\right| d\lambda \\
&\leq p \int_0^\infty \lambda^{p-2} \int_{\{M^d f > \lambda\}} |f(y)|\, dy\, d\lambda \\
&= p \int_{\mathbb{R}^n} |f(x)| \int_0^{M^d f(x)} \lambda^{p-2}\, d\lambda\, dx \\
&= \frac{p}{p-1} \int_{\mathbb{R}^n} |f(x)| \left(M^d f(x)\right)^{p-1} dx \\
&\leq p' \|f\|_p \left\|M^d f\right\|_p^{p-1}.
\end{aligned}$$

Rearranging,

$$\left\|M^d f\right\|_p \leq p' \|f\|_p.$$

Uncentered: $p > 1$, $n = 1$ (Measure Theoretic Inequality)

Lemma

For any $E \subseteq \mathbb{R}$ with $|E| < \infty$, we have

$$\frac{1}{\lambda}\int_E f(x)\,dx + |\{x \mid f(x) > \lambda\}| \leq \frac{1}{\lambda}\int_{\{f>\lambda\}} f(x)|\,dx + |E|.$$

Uncentered: $p > 1$, $n = 1$ (Measure Theoretic Inequality)

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$$\frac{1}{\lambda}\int_E f(x)\,dx + |\{x \mid f(x) > \lambda\}| \leq \frac{1}{\lambda}\int_{\{f>\lambda\}} f(x)|\,dx + |E|.$$

Proof.

$$\begin{aligned}
\int_E f(x)\,dx - \lambda|E| &= \int_E (f(x) - \lambda)\,dx \\
&= \int_{E\cap\{f\leq\lambda\}} (f(x) - \lambda)\,dx + \int_{E\cap\{f>\lambda\}} (f(x) - \lambda)\,dx \\
&\leq \int_{\{f>\lambda\}} (f(x) - \lambda)\,dx \\
&= \int_{\{f>\lambda\}} f(x)\,dx - \lambda\left|\{x \mid f(x) > \lambda\}\right|.
\end{aligned}$$

Uncentered: $p > 1$, $n = 1$ (Lemma)

Lemma (Grafakos and Kinnunen (1998))

For $f \geq 0$ *and* $f \in L^1(\mathbb{R})$,

$$|\{x \mid Mf(x) > \lambda\}| + |\{x \mid f(x) > \lambda\}| \leq \frac{1}{\lambda}\int_{\{Mf > \lambda\}} f(x)\,dx + \frac{1}{\lambda}\int_{\{f > \lambda\}} f(x)\,dx.$$

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For $f \geq 0$ and $f \in L^1(\mathbb{R})$,

$$|\{x \mid Mf(x) > \lambda\}| + |\{x \mid f(x) > \lambda\}| \leq \frac{1}{\lambda} \int_{\{Mf > \lambda\}} f(x)\, dx + \frac{1}{\lambda} \int_{\{f > \lambda\}} f(x)\, dx.$$

Proof. Let $K \subseteq \{x \mid Mf(x) > \lambda\}$. Then $K \subseteq J \cup L$ where

$$|J| \leq \frac{1}{\lambda} \int_J f(x)\, dx \quad \text{and} \quad |L| \leq \frac{1}{\lambda} \int_L f(x)\, dx.$$

Then

$$\begin{aligned} |J \cup L| + |J \cap L| = |J| + |L| &\leq \frac{1}{\lambda} \int_J f(x)\, dx + \frac{1}{\lambda} \int_L f(x)\, dx \\ &= \frac{1}{\lambda} \int_{L \cup J} f(x)\, dx + \frac{1}{\lambda} \int_{L \cap J} f(x)\, dx. \end{aligned}$$

Uncentered: $p > 1$, $n = 1$ (Lemma)

By the previous inequality,

$$\begin{aligned}
|K| &+ |\{x \mid f(x) > \lambda\}| + |J \cap L| \\
&\leq |J \cup L| + |J \cap L| + |\{x \mid f(x) > \lambda\}| \\
&\leq \frac{1}{\lambda}\int_{J\cup L} f(x)\,dx + \frac{1}{\lambda}\int_{J\cap L} f(x)\,dx + |\{x \mid f(x) > \lambda\}| \\
&\leq \frac{1}{\lambda}\int_{J\cup L} f(x)\,dx + \frac{1}{\lambda}\int_{\{f>\lambda\}} f(x)\,dx + |J \cap L|.
\end{aligned}$$

Taking the supremum over all compact sets K gives

$$|\{x \mid Mf(x) > \lambda\}| + |\{x \mid f(x) > \lambda\}| \leq \frac{1}{\lambda}\int_{\{Mf>\lambda\}} f(x)\,dx + \frac{1}{\lambda}\int_{\{f>\lambda\}} f(x)\,dx.$$

Uncentered: $p > 1$, $n = 1$ (Boundedness)

Using the lemma,

$$
\begin{aligned}
&\int_{\mathbb{R}} Mf(x)^p \, dx + \int_{\mathbb{R}} f(x)^p \, dx \\
&= p \int_0^\infty \lambda^{p-1} \left(|\{Mf > \lambda\}| + |\{f > \lambda\}| \right) d\lambda \\
&\leq p \int_0^\infty \lambda^{p-2} \int_{\{Mf > \lambda\}} f(x) \, dx \, d\lambda + p \int_0^\infty \lambda^{p-2} \int_{\{f > \lambda\}} f(x) \, dx \, d\lambda \\
&= p' \int_{\mathbb{R}} f(x) Mf(x)^{p-1} \, dx + p' \int_{\mathbb{R}} f(x)^p \, dx \\
&\leq p' \|f\|_p \|Mf\|_p^{p-1} + p' \|f\|_p^p.
\end{aligned}
$$

Uncentered: $p > 1$, $n = 1$ (Boundedness)

Using the lemma,

$$\begin{aligned}
&\int_{\mathbb{R}} Mf(x)^p\,dx + \int_{\mathbb{R}} f(x)^p\,dx \\
&= p\int_0^\infty \lambda^{p-1}\left(|\{Mf > \lambda\}| + |\{f > \lambda\}|\right)d\lambda \\
&\leq p\int_0^\infty \lambda^{p-2}\int_{\{Mf>\lambda\}} f(x)\,dx\,d\lambda + p\int_0^\infty \lambda^{p-2}\int_{\{f>\lambda\}} f(x)\,dx\,d\lambda \\
&= p'\int_{\mathbb{R}} f(x)Mf(x)^{p-1}\,dx + p'\int_{\mathbb{R}} f(x)^p\,dx \\
&\leq p'\|f\|_p\|Mf\|_p^{p-1} + p'\|f\|_p^p.
\end{aligned}$$

Rearranging gives

$$(p-1)\left(\frac{\|Mf\|_p}{\|f\|_p}\right)^p - p\left(\frac{\|Mf\|_p}{\|f\|_p}\right)^{p-1} - 1 \leq 0.$$

Uncentered: $p > 1$, $n = 1$ (Sharpness)

Let A_p denote the largest root of

$$(p-1)x^p - px^{p-1} - 1.$$

For $f(x) = |x|^{-1/p}$, it turns out that

$$M\left(|x|^{-1/p}\right) = A_p|x|^{-1/p}.$$

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Since $f(x) \notin L^p$, we consider the family

$$f_\varepsilon(x) = |x|^{-1/p} \min\left(|x|^{-\varepsilon}, |x|^{\varepsilon}\right)$$

for $\varepsilon > 0$.