Differential formulas in orthogonal coordinate systems

General coordinates (u_1, u_2, u_3)

$$\mathbf{e}_{u_i} = \frac{\nabla u_i}{|\nabla u_i|} = \frac{\partial \mathbf{r}}{\partial u_i} \Big/ \left|\frac{\partial \mathbf{r}}{\partial u_i}\right|$$

Let $\mathbf{F}(P) = F_x\mathbf{e}_x + F_y\mathbf{e}_y + F_z\mathbf{e}_z = F_{u_1}\mathbf{e}_{u_1} + F_{u_2}\mathbf{e}_{u_2} + F_{u_3}\mathbf{e}_{u_3}$ be vector valued. Let $u(P)$ be scalar valued.

Set $h_i = \left|\dfrac{\partial \mathbf{r}}{\partial u_i}\right| = \dfrac{1}{|\nabla u_i|} = \sqrt{\left(\dfrac{\partial x}{\partial u_i}\right)^2 + \left(\dfrac{\partial y}{\partial u_i}\right)^2 + \left(\dfrac{\partial z}{\partial u_i}\right)^2}$, $i = 1, 2, 3.$

Then

(*i*) $$F_{u_i} = h_i^{-1}\left(F_x \frac{\partial x}{\partial u_i} + F_y \frac{\partial y}{\partial u_i} + F_z \frac{\partial z}{\partial u_i}\right),\ i = 1, 2, 3.$$

(*vector component relationship*)

(*iia*) $d\mathbf{r} = h_1 du_1 \mathbf{e}_{u_1} + h_2 du_2 \mathbf{e}_{u_2} + h_3 du_3 \mathbf{e}_{u_3}$ (*displacement vector*)

(*b*) $ds^2 = |d\mathbf{r}|^2 = h_1{}^2 du_1{}^2 + h_2{}^2 du_2{}^2 + h_3{}^2 du_3{}^2$ (*arc length element*)

(*c*) $h_1 h_2\, du_1 du_2,\quad h_2 h_3\, du_2 du_3,\quad h_3 h_1\, du_3 du_1$ (*surface elements*)

(*d*) $dV = h_1 h_2 h_3\, du_1 du_2 du_3$ (*volume element*)

(*iii*) $$\operatorname{grad} u = \nabla u = \sum_{i=1}^{3} \frac{1}{h_i} \frac{\partial u}{\partial u_i} \mathbf{e}_{u_i}$$

(*iv*) $$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \frac{\partial}{\partial u_i}\left(\frac{h_1 h_2 h_3}{h_i} F_{u_i}\right)$$

(*v*) $$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_{u_1} & h_2 \mathbf{e}_{u_2} & h_3 \mathbf{e}_{u_3} \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 F_{u_1} & h_2 F_{u_2} & h_3 F_{u_3} \end{vmatrix}$$

(*vi*) $$\Delta u = \nabla^2 u = \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \frac{\partial}{\partial u_i}\left(\frac{h_1 h_2 h_3}{h_i^2} \frac{\partial u}{\partial u_i}\right)$$

Cartesian (rectangular) coordinates (x, y, z)

$h_1 = h_2 = h_3 = 1$

(*ii*) $ds^2 = dx^2 + dy^2 + dz^2$

(*iii*) $\operatorname{grad} u = \dfrac{\partial u}{\partial x}\mathbf{e}_x + \dfrac{\partial u}{\partial y}\mathbf{e}_y + \dfrac{\partial u}{\partial z}\mathbf{e}_z$ $\quad(\mathbf{e}_x = \mathbf{i},\ \mathbf{e}_y = \mathbf{j},\ \mathbf{e}_z = \mathbf{k})$

(*iv*) $$\operatorname{div} \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

(*v*) $$\operatorname{curl} \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{e}_z$$

(*vi*) $$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Translated and rotated coordinates (ξ, η, ζ)

$$\begin{cases} x=\xi_0+a_{11}\xi+a_{12}\eta+a_{13}\zeta \\ y=\eta_0+a_{21}\xi+a_{22}\eta+a_{23}\zeta \\ z=\zeta_0+a_{31}\xi+a_{32}\eta+a_{33}\zeta \end{cases} \quad (a_{ij}) \text{ orthogonal matrix}$$

$h_1=h_2=h_3=1$

$$(i) \quad \begin{cases} F_\xi=a_{11}F_x+a_{21}F_y+a_{31}F_z \\ F_\eta=a_{12}F_x+a_{22}F_y+a_{32}F_z \\ F_\zeta=a_{13}F_x+a_{23}F_y+a_{33}F_z \end{cases}$$

(iia) $d\mathbf{r}=d\xi\mathbf{e}_\xi+d\eta\,\mathbf{e}_\eta+d\zeta\mathbf{e}_\zeta$

(b) $ds^2=d\xi^2+d\eta^2+d\zeta^2$

(c) $d\xi d\eta,\ d\eta d\zeta,\ d\zeta d\xi$

(d) $dV=d\xi d\eta d\zeta$

$$(iii) \quad \text{grad } u=\frac{\partial u}{\partial \xi}\mathbf{e}_\xi+\frac{\partial u}{\partial \eta}\mathbf{e}_\eta+\frac{\partial u}{\partial \zeta}\mathbf{e}_\zeta$$

$$(iv) \quad \text{div } \mathbf{F}=\frac{\partial F_\xi}{\partial \xi}+\frac{\partial F_\eta}{\partial \eta}+\frac{\partial F_\zeta}{\partial \zeta}$$

$$(v) \quad \text{curl } \mathbf{F}=\left(\frac{\partial F_\zeta}{\partial \eta}-\frac{\partial F_\eta}{\partial \zeta}\right)\mathbf{e}_\xi+\left(\frac{\partial F_\xi}{\partial \zeta}-\frac{\partial F_\zeta}{\partial \xi}\right)\mathbf{e}_\eta+\left(\frac{\partial F_\eta}{\partial \xi}-\frac{\partial F_\xi}{\partial \eta}\right)\mathbf{e}_\zeta$$

$$(vi) \quad \Delta u=\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}+\frac{\partial^2 u}{\partial \zeta^2}$$

Cylindrical coordinates (ρ, φ, z)
(**Polar coordinates in the plane:** Neglect terms with z.)

Coordinate transformations

$x=\rho\cos\varphi,\ \ y=\rho\sin\varphi,\ \ z=z$

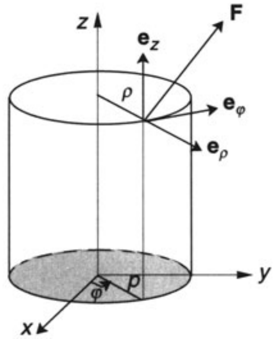
$\rho=\sqrt{x^2+y^2}$, $\varphi=\tan^{-1}\dfrac{y}{x}$ (suitable branch), $z=z$

$h_1=1,\ h_2=\rho,\ h_3=1$

Basis vector relationship

$$\begin{cases} \mathbf{e}_x=\mathbf{e}_\rho\cos\varphi-\mathbf{e}_\varphi\sin\varphi \\ \mathbf{e}_y=\mathbf{e}_\rho\sin\varphi+\mathbf{e}_\varphi\cos\varphi \\ \mathbf{e}_z=\mathbf{e}_z \end{cases}$$

$$\begin{cases} \mathbf{e}_\rho=\mathbf{e}_x\cos\varphi+\mathbf{e}_y\sin\varphi \\ \mathbf{e}_\varphi=-\mathbf{e}_x\sin\varphi+\mathbf{e}_y\cos\varphi \\ \mathbf{e}_z=\mathbf{e}_z \end{cases}$$



(i) Vector component relationship:

$$\begin{cases} F_x = F_\rho \cos\varphi - F_\varphi \sin\varphi \\ F_y = F_\rho \sin\varphi + F_\varphi \cos\varphi \\ F_z = F_z \end{cases} \Leftrightarrow \begin{cases} F_\rho = F_x \cos\varphi + F_y \sin\varphi \\ F_\varphi = -F_x \sin\varphi + F_y \cos\varphi \\ F_z = F_z \end{cases}$$

(iia) $d\mathbf{r} = \mathbf{e}_\rho d\rho + \mathbf{e}_\varphi \rho\, d\varphi + \mathbf{e}_z dz$ (*displacement vector*)

(iib) $ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2$ (*arc length element*)

(iic) $\rho d\rho d\varphi,\ \rho d\varphi dz,\ dz d\rho$ (*surface elements*)

(iid) $dV = \rho d\rho d\varphi dz$ (*volume element*)

(iii) $$\text{grad } u = \nabla u = \frac{\partial u}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial u}{\partial \varphi}\mathbf{e}_\varphi + \frac{du}{dz}\mathbf{e}_z$$

(iv) $$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\varphi}{\partial \varphi} + \frac{\partial F_z}{\partial z}$$

(v) $$\text{curl } \mathbf{F} = \text{rot } \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{1}{\rho}\frac{\partial F_z}{\partial \varphi} - \frac{\partial F_\varphi}{\partial z}\right)\mathbf{e}_\rho + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right)\mathbf{e}_\varphi + $$
$$+ \frac{1}{\rho}\left(\frac{\partial(\rho F_\varphi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \varphi}\right)\mathbf{e}_z$$

(vi) $$\Delta u = \nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\varphi\varphi} + u_{zz}$$

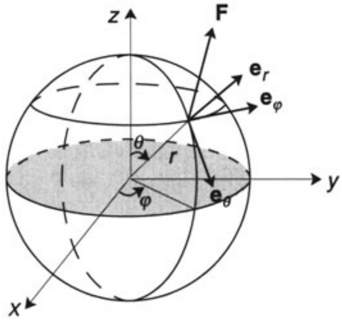
Spherical coordinates (r, θ, φ)

Coordinate transformations

$x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$

$r = \sqrt{x^2+y^2+z^2}$, $\theta = \arccos(z/\sqrt{x^2+y^2+z^2})$, $\varphi = \tan^{-1}(y/x)$ (suitable branch)

$h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$



Basis vector relationship

$$\begin{cases} \mathbf{e}_x = \mathbf{e}_r \sin\theta\cos\varphi + \mathbf{e}_\theta \cos\theta\cos\varphi - \mathbf{e}_\varphi \sin\varphi \\ \mathbf{e}_y = \mathbf{e}_r \sin\theta\sin\varphi + \mathbf{e}_\theta \cos\theta\sin\varphi + \mathbf{e}_\varphi \cos\varphi \\ \mathbf{e}_z = \mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta \end{cases}$$

$$\begin{cases} \mathbf{e}_r = \mathbf{e}_x \sin\theta\cos\varphi + \mathbf{e}_y \sin\theta\sin\varphi + \mathbf{e}_z \cos\theta \\ \mathbf{e}_\theta = \mathbf{e}_x \cos\theta\cos\varphi + \mathbf{e}_y \cos\theta\sin\varphi - \mathbf{e}_z \sin\theta \\ \mathbf{e}_\varphi = -\mathbf{e}_x \sin\varphi + \mathbf{e}_y \cos\varphi \end{cases}$$

(i) Vector component relationship:

$$\begin{cases} F_x = F_r \sin\theta\cos\varphi + F_\theta \cos\theta\cos\varphi - F_\varphi \sin\varphi \\ F_y = F_r \sin\theta\sin\varphi + F_\theta \cos\theta\sin\varphi + F_\varphi \cos\varphi \\ F_z = F_r \cos\theta - F_\theta \sin\theta \end{cases} \Leftrightarrow$$

$$\begin{cases} F_r = F_x \sin\theta\cos\varphi + F_y \sin\theta\sin\varphi + F_z \cos\theta \\ F_\theta = F_x \cos\theta\cos\varphi + F_y \cos\theta\sin\varphi - F_z \sin\theta \\ F_\varphi = -F_x \sin\varphi + F_y \cos\varphi \end{cases}$$

(*iia*) $d\mathbf{r}=\mathbf{e}_r dr+\mathbf{e}_\theta r d\theta+\mathbf{e}_\varphi r\sin\theta d\varphi$ (*displacement vector*)
(*iib*) $ds^2=dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2$ (*arc length element*)
(*iic*) $rdrd\theta$, $r^2\sin\theta d\theta d\varphi$, $r\sin\theta d\varphi dr$ (*surface elements*)
(*iid*) $dV=r^2\sin\theta dr d\theta d\varphi$ (*volume element*)

(*iii*) $\text{grad } u=\nabla u=\frac{\partial u}{\partial r}\mathbf{e}_r+\frac{1}{r}\frac{\partial u}{\partial \theta}\mathbf{e}_\theta+\frac{1}{r\sin\theta}\frac{du}{d\varphi}\mathbf{e}_\varphi$

(*iv*) $\text{div }\mathbf{F}=\nabla\cdot\mathbf{F}=\frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(F_\theta\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial F_\varphi}{d\varphi}$

(*v*) $\text{curl }\mathbf{F}=\text{rot }\mathbf{F}=\nabla\times\mathbf{F}=\frac{1}{r\sin\theta}\left(\frac{\partial(F_\varphi\sin\theta)}{\partial\theta}-\frac{\partial F_\theta}{\partial\varphi}\right)\mathbf{e}_r+$

$$+\frac{1}{r\sin\theta}\left(\frac{\partial F_r}{\partial\varphi}-\sin\theta\frac{\partial(rF_\varphi)}{\partial r}\right)\mathbf{e}_\theta+\frac{1}{r}\left(\frac{\partial(rF_\theta)}{\partial r}-\frac{\partial F_r}{\partial\theta}\right)\mathbf{e}_\varphi$$

(*vi*) $\Delta u=\nabla^2u=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2u}{\partial\varphi^2}=$

$$=\frac{\partial^2u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\left[\frac{\partial}{\partial\xi}\left((1-\xi^2)\frac{\partial u}{\partial\xi}\right)+\frac{1}{1-\xi^2}\frac{\partial^2u}{\partial\varphi^2}\right]\quad\text{if } \xi=\cos\theta$$

11.3 Line Integrals

Differential forms

f, g: $\mathbf{R}^n\to\mathbf{R}$, h: $\mathbf{R}\to\mathbf{R}$. Differential form:

$$\omega=fdg=f\left(\frac{\partial g}{\partial x_1}dx_1+\ldots+\frac{\partial g}{\partial x_n}dx_n\right)$$

1. $d(af+bg)=a\,df+b\,dg$	2. $d(fg)=f\,dg+g\,df$
3. $d\left(\frac{f}{g}\right)=\frac{g\,df-f\,dg}{g^2}$	4. $d(h(f))=h'(f)df$

Exact differential forms

In $\mathbf{R}^2$: $\omega=P\,dx+Q\,dy$ is *exact* if there exists $\Phi(x, y)$ called *primitive function* of ω such that $\Phi'_x=P$, $\Phi'_y=Q$, (i.e. $\omega=d\Phi$).

Test: $P\,dx+Q\,dy$ exact $\Leftrightarrow$ $P'_y=Q'_x$, (in a simply connected domain).

In $\mathbf{R}^3$: $\omega=P\,dx+Q\,dy+R\,dz$ is *exact* if there exists $\Phi(x, y, z)$ [*primitive function*] such that $\Phi'_x=P$, $\Phi'_y=Q$, $\Phi'_z=R$, (i.e. $\omega=d\Phi$).

Test: $P\,dx+Q\,dy+R\,dz$ exact $\Leftrightarrow$ $\text{curl}(P, Q, R)=\mathbf{0}$ $\Leftrightarrow$ $P'_y=Q'_x$, $P'_z=R'_x$, $Q'_z=R'_y$, (in a simply connected domain).