Homework 1

Due: Feb. 27 2023

1. Consider the strong statement of the boundary-value problem in classical linear heat conduction in which the Neumann boundary condition is replaced by the following expression:

$$\lambda u - q_i n_i = h \qquad \text{on } \Gamma_h \tag{1}$$

Where $\lambda \geq 0$ is a given function of $\boldsymbol{x} \in \Gamma_h$. Generalized the weak formulation to include (1) as a natural boundary condition. Obtain an expression for the additional contribution to k_{ab}^e arsing from (1). Show that the matrix $\boldsymbol{K}$ is positive-definite.

The boundary condition (2.5.17) is equivalent to what is often called ***Newton's law of heat transfer;*** λ is called the ***coefficient of heat transfer***. This boundary condition applies to the case in which the heat flux is proportional to the difference of the surface temperatures of the body and surrounding medium, the latter formally represented by h/λ in (2.5.17).

2. Let

$$\Gamma_{int} = \left(\bigcup_{e=1}^{n_{el}} \Gamma^e\right) - \Gamma \qquad \text{(interior element boundaries)}$$

One side of Γ_{int} is (arbitrarily) designated to be the "+ side" and the other is the "- side". Let $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$ be unit normals to Γ_{int} which point in the plus and minus directions, respectively. Clearly $\boldsymbol{n}^+ = -\boldsymbol{n}^-$. Let q_i^+ and q_i^- denote the values of q_i obtained by approaching $\boldsymbol{x} \in \Gamma_{\text{int}}$ from + and - sides, respectively. The "jump" in $q_n = q_i n_i$ at $\boldsymbol{x}$ is defined to be

$$[q_n] = (q_i^+ - q_i^-) n_i^+ = q_i^+ n_i^+ + q_i^- n_i^-$$

As may be easily verified, the jump is invariant with respect to reserving the + and - designations.

Consider the weak formulation (i.e., (2.3.6)) and assume w and u are smooth on the element interiors but may experience discontinuities in gradient across element boundaries. (Functions of this type contain the standard C^0 finite element interpolations; see Chapter 3). Show that

$$0 = \sum_{e+1}^{n_{el}} \int_{\Omega_e} w(q_{i,i} - f)d\Omega - \int_{\Gamma_h} w(q_n + h)d\Gamma + \int_{\Gamma_{int}} w[q_n]d\Gamma$$

From which the Euler-Lagrange conditions may be readily deduced:

i. $q_{i,i} = f \;\; in \;\; \bigcup_{e=1}^{n_{el}} \Omega^e$

ii. $-q_n = h$ on Γ_h

iii. $[\mathrm{q_n}] = 0$ on Γ_{int}

As may be seen, (i) is the heat equation on the element interiors and (iii) is a continuity condition across element boundaries on the heat flux. Contrast the present results with those obtained assuming w and u are globally smooth. The Galerkin finite element formulation obtains an approximate solution to (i) through (iii).