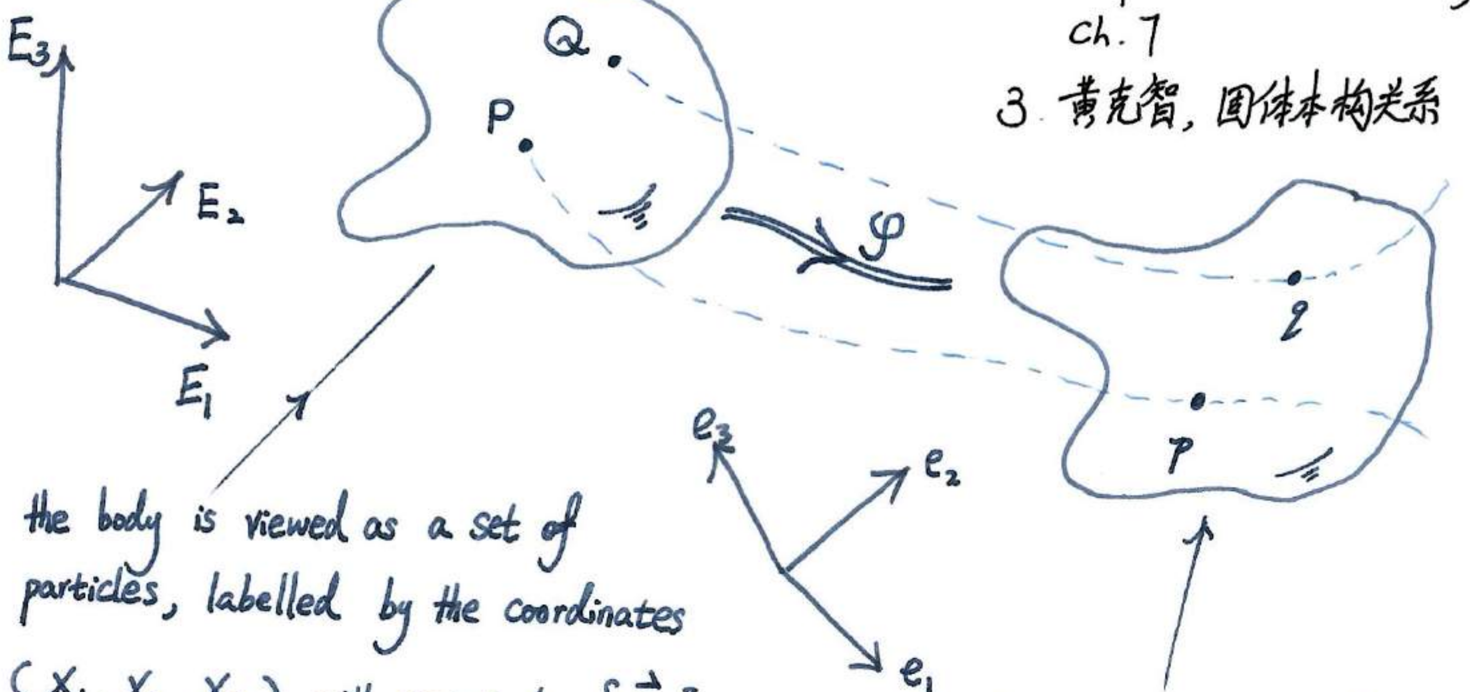
Finite-strain elasticity

References:

1. G. Holzapfel: Nonlinear Solid Mechanics
2. J. Simo & T.J.R. Hughes Computational Inelasticity ch. 7
3. 黄克智, 固体本构关系

- Kinematics: the study of motion and deformation without reference to the cause.



the body is viewed as a set of particles, labelled by the coordinates (X_1, X_2, X_3) with respect to $\{\vec{E}_I\}$ at time $t=0$.

material points

I, J, K, L

$(\ )_{,I} = \frac{\partial}{\partial X_I}$.

the current position of the particles at time t is (x_1, x_2, x_3) w.r.t. $\{\vec{e}_i\}$

spatial points i, j, k, l.

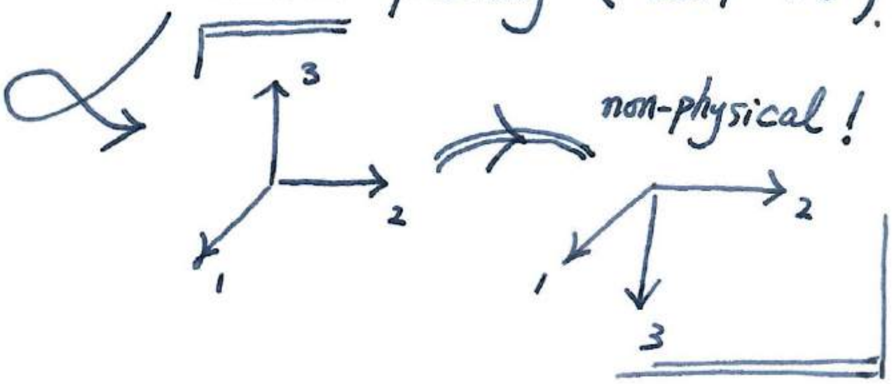
$(\ )_{,i} = \frac{\partial}{\partial x_i}$

$$\vec{x} = \varphi(\vec{X}, t)$$

→ placement: gives the configuration Ω_0 at time t

→ Requirement: • smooth (differentiable);

- one-to-one (except possibly at the boundary : contact)
- orientation - preserving ($\det F > 0$).



Displacement: $U_i(X, t) = \varphi_i(X, t) - \varphi_i(X, 0)$

$$= x_i - \delta_{iI} X_I$$

a vector on Ω_x

Note: the ambient space is Cartesian

rigid motion: $x = Q(t) X + c(t)$ ← rigid translation

rigid rotation: Q is proper orthogonal.

$\det(Q) = +1 \qquad Q^T Q = I.$

Deformation gradient:

$$dx_1 = x_{q_1} - x_p = \varphi_t(X_{Q_1}) - \varphi_t(X_p)$$

$$= \varphi_t(X_p + dX_1) - \varphi_t(X_p) = \frac{\partial \varphi_t}{\partial X}(X_p)\, dX_1.$$

We call $\frac{\partial \varphi_t}{\partial X} = \frac{\partial x}{\partial X} = F$ the <u>deformation gradient</u>.

$$F = F_{iI}\, \underset{\text{current/spatial Cartesian basis}}{\vec{e}_i} \otimes \underset{\text{initial material Cartesian basis}}{\vec{E}_I}$$

$$F^T = F_{iI}\, \vec{E}_I \otimes \vec{e}_i = (F^T)_{Ii}\, \vec{E}_I \otimes \vec{e}_i$$

$$F^{-1} = \frac{\partial X}{\partial x} = \frac{\partial X_I}{\partial x_i}\, \vec{E}_I \otimes \vec{e}_i$$

$$F^{-T} = \frac{\partial X_I}{\partial x_i}\, \vec{e}_i \otimes \vec{E}_I$$

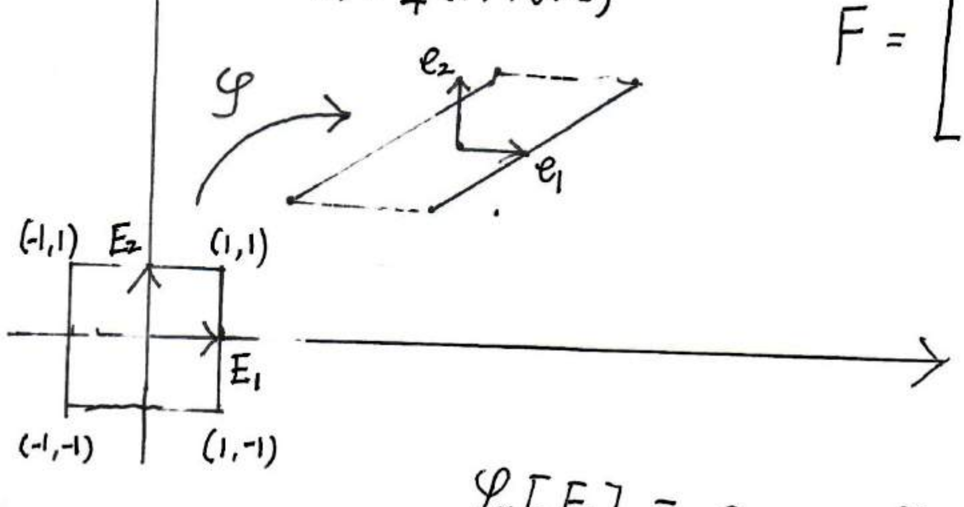
Two-point tensors: transform vectors of one configuration to vectors on another configuration

→ more general:

- push-forward $dx = F dX = \varphi_{t*}[dX]$
- pull back $dX = F^{-1} dx = \varphi_{t*}^{-1}[dx]$.

$x_1 = \frac{1}{4}(18 + 4X_1 + 6X_2)$

$x_2 = \frac{1}{4}(14 + 6X_2)$



$$F = \begin{bmatrix} 1 & 1.5 \\ 0 & 1.5 \end{bmatrix}$$

$$F^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & \frac{2}{3} \end{bmatrix}$$

$\varphi_*[E_1] = e_1$ $\quad$ $\varphi_*[E_2] = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$

$$\varphi_*^{-1}[e_1] = E_1 \qquad \varphi_*^{-1}[e_2] = \begin{bmatrix} -1 \\ \frac{2}{3} \end{bmatrix}$$

- Line, Area, and Volume change

$$\underline{dx = F\, dX}$$ line element

Lemma: $v, w \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 3}$, then

$$(Av) \times (Aw) = (\operatorname{cof} A)\,(v \times w)$$

if A^{-1} exists, $\operatorname{cof} A = \det A\, A^{-T}$.

Proof:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} M_{ij}$$

minor 余子式

Laplace expansion

$$(-1)^{i+j} M_{ij} = (\operatorname{cof} A)_{ij}$$

cofactor 代数余子式.

$$\Rightarrow \quad (\operatorname{cof} A)_{ij} A^T_{jk} = (\det A)\, \delta_{ik}$$

or if A is invertible, $\operatorname{cof} A = (\det A)\, A^{-T}$.

Levi-Civita symbol:

$$\det A = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} A_{1i} A_{2j} A_{3k} = \frac{1}{6} \varepsilon_{ijk} \varepsilon_{pqr} A_{ip} A_{jq} A_{kr}$$

$$v \times w = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} e_i v_j w_k.$$

$$(\mathrm{cof}A)_{ij}\,\underline{\underline{A_{ij}}} = 3\det A = \frac{1}{2}\varepsilon_{ipk}\varepsilon_{jqr}\underline{\underline{A_{ij}}}A_{pq}A_{kr}$$

$$\Rightarrow \quad (\mathrm{cof}A)_{ij} = \frac{1}{2}\varepsilon_{ipk}\varepsilon_{jqr}A_{pq}A_{kr}.$$

$$\Rightarrow (\mathrm{cof}A)_{ij}(v\times w)_j = (\mathrm{cof}A)_{ij}\,\varepsilon_{jmn}v_m w_n$$

$$= \frac{1}{2}\varepsilon_{ipk}\varepsilon_{jqr}\varepsilon_{jmn}v_m w_n\, A_{pq}A_{kr}$$

$$= \frac{1}{2}\varepsilon_{ipk}(\delta_{qm}\delta_{rn} - \delta_{qn}\delta_{rm})v_m w_n A_{pq}A_{kr}.$$

$$= \frac{1}{2}\varepsilon_{ipk}\cancel{(V_q w_r - V_r w_q)}\,(V_q W_r A_{pq}A_{kr} - V_r W_q A_{pq}A_{kr})$$

$$= \frac{1}{2}\varepsilon_{ipk}V_q W_r A_{pq}A_{kr} - \frac{1}{2}\varepsilon_{ipk}V_r W_q A_{pq}A_{kr}$$

$$= \varepsilon_{ipk}(A_{pq}V_q)(A_{kr}W_r).$$

∎

Now we may consider an area element:

$$dX_1 \times dX_2 = N\,dA.$$

orientation: normal vector.

after deformation: $dx_1 \times dx_2 = n\,da$

$$= (F dX_1)\times(F dX_2)$$

$$= (\mathrm{cof}F)(dX_1\times dX_2)$$

$$= (\mathrm{cof}F)\,N\,dA.$$

Jacobian $J := \det F.$

$$\boxed{n\,da = J\,F^{-T} N\,dA}$$

Nanson's formula.

F is invertible.

$$\boxed{\int_{\underset{\sim}{\mathcal{S}}} \cdots\; n\,da = \int_S \cdots\; J F^{-T} N\,dA.}$$

this is how one change the integration variable for integration on orientable surfaces. (e.g. traction integration).

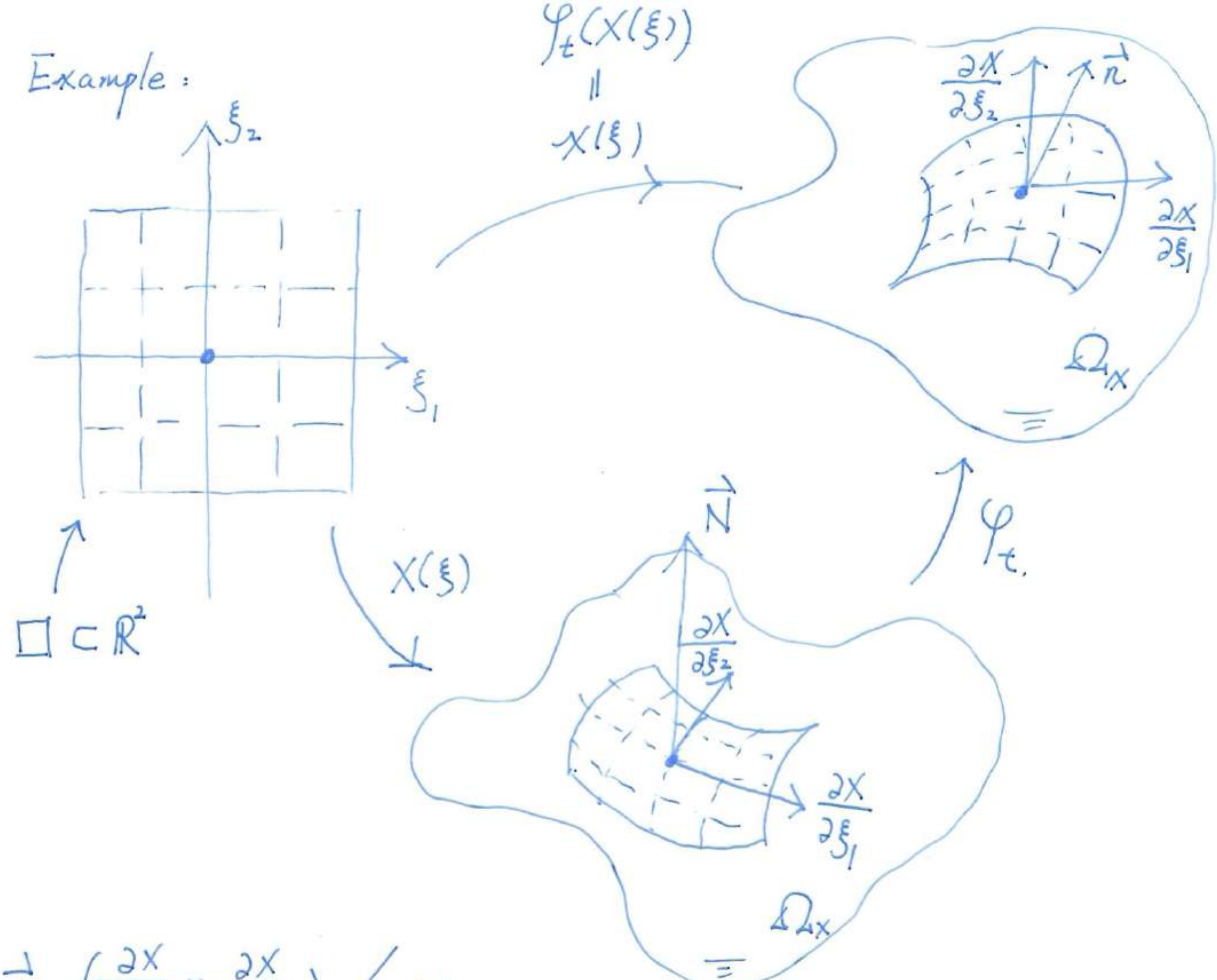
Consider a volume element $dX_1 \cdot (dX_2 \times dX_3)$

after deformation

$$
\begin{aligned}
& dx_1 \cdot (dx_2 \times dx_3) \\
&= F\,dX_1 \cdot (F\,dX_2 \times F\,dX_3) \\
&= dX_1 \cdot (F^T \operatorname{cof} F)(dX_2 \times dX_3) \\
&= J\; dX_1 \cdot (dX_2 \times dX_3)
\end{aligned}
$$

$$\boxed{\int_{v = \varphi_t(V)} \cdots\; dv = \int_V \cdots\; J\,dV}$$

Example :



$$\vec{N} = \left(\frac{\partial X}{\partial \xi_1} \times \frac{\partial X}{\partial \xi_2}\right) \Big/ \left\| \frac{\partial X}{\partial \xi_1} \times \frac{\partial X}{\partial \xi_2} \right\|$$

$$\vec{n} = \left(\frac{\partial x}{\partial \xi_1} \times \frac{\partial x}{\partial \xi_2}\right) \Big/ \left\| \frac{\partial x}{\partial \xi_1} \times \frac{\partial x}{\partial \xi_2} \right\|$$

$$\int_S \cdots dA = \int_\square \cdots \left\| \frac{\partial X}{\partial \xi_1} \times \frac{\partial X}{\partial \xi_2} \right\| d\xi_1 \, d\xi_2$$

$$\int_{\substack{s \\ \shortparallel \\ \varphi_t(S)}} \cdots da = \int_\square \cdots \left\| \frac{\partial x}{\partial \xi_1} \times \frac{\partial x}{\partial \xi_2} \right\| d\xi_1 \, d\xi_2$$

More, $\boxed{\frac{\partial J}{\partial F_{iI}} = \operatorname{cof} F_{iI} = J F^{-1}_{Ii}}$ ← Very useful !

Strain.

measures length & angle.

$$dx_1 \cdot dx_2 = F\, dX_1 \cdot F\, dX_2 = dX_1 \cdot \underbrace{(F^T F)}_{C} dX_2$$

$$C_{IJ} = (F^T)_{Ii} F_{iJ} = F_{iI} F_{iJ}$$

material tensor known as the *right Cauchy-Green deformation tensor*.

Alternatively, $dX_1 \cdot dX_2 = F^{-1} dx_1 \cdot F^{-1} dx_2$

$$= dx_1 \cdot \underbrace{(F^{-T} F^{-1})}_{b^{-1}} dx_2$$

$$b_{ij} = \cancel{(F^T)}\; F_{iI} F_{jI} \quad (b = FF^T)$$

spatial tensor known as the *left Cauchy-Green deformation tensor*,

or *Finger tensor*.

$$\frac{1}{2} (dx_1 \cdot dx_2 - dX_1 \cdot dX_2) = dX_1 \cdot E\, dX_2$$

$$E = \frac{1}{2} (C - I) = \frac{1}{2} (F^T F - I)$$

material tensor, *Green-Lagrange Strain tensor*

$$\frac{1}{2}\left(dx_1 \cdot dx_2 - dX_1 \cdot dX_2\right) = dx_1 \cdot e\, dx_2$$

$$e = \frac{1}{2}\left(I - b^{-1}\right)$$ spatial tensor, Euler-Almansi strain tensor

Remark 1:

$$2E = F^T F - I = \left[\left(\frac{\partial U}{\partial X}\right) + I\right]^T \left[\left(\frac{\partial U}{\partial X}\right) + I\right] - I$$

$$= \underbrace{\left(\frac{\partial U}{\partial X}\right)^T \left(\frac{\partial U}{\partial X}\right)}_{\text{quadratic / nonlinear}} + \underbrace{\left(\frac{\partial U}{\partial X} + \left(\frac{\partial U}{\partial X}\right)^T\right)}_{\text{Linear.}}$$

$$2\varepsilon = \frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial X}\right)^T$$ infinitesimal strain / Small strain.

linear appr. of E.

when deformation is small, we do not differentiate x and X.

For engineering materials, $E \sim 2 \times 10^{11}$ Pa.

$\sigma_Y \sim 2 \times 10^{8}$ Pa $\Rightarrow \sigma = E \underbrace{\frac{\partial U}{\partial X}}_{< 10^{-3} \ll 1}$

thus, it is acceptable to use ε.
as the quadratic term will vanish.

When there is rigid rotations.

$$\frac{\partial U}{\partial X} = Q - I \Rightarrow 2E = Q^T Q - I = 0$$

$$2\varepsilon = Q + Q^T - 2I.$$

Consider 2D rotations $Q = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$\mathcal{E} = (\cos\theta - 1)\, I. \simeq -\frac{\theta^2}{2} I.$ good for small θ.

extreme case $\theta = \pi/2.$ $\mathcal{E} = -I$ (bad!)

More on the deformation gradient:

Spectrum decomposition:

there exist $\{\vec{N}_a\}$ and $\{\vec{n}_a\}$ $a = 1, 2, 3$, and $\overset{\mathbb{R}}{\overset{\in}{\{\lambda_a\}}}$

$\{\vec{N}_a\}$ ↑ material, mutually orthogonal normalized

$\{\vec{n}_a\}$ ↑ spatial, mutually orthogonal, normalized

such that

$$\boxed{F = \sum_{a=1}^{3} \lambda_a \, \vec{n}_a \otimes \vec{N}_a}$$

principal stretches

principal spatial directions/axes

principal referential directions/axes

$$\boxed{F^{-1} = \sum_{a=1}^{3} \lambda_a^{-1} \, \vec{N}_a \otimes \vec{n}_a}$$

$$F \vec{N}_a = \sum_{b=1}^{3} \lambda_b \, \vec{n}_b (\vec{N}_b \cdot \vec{N}_a) = \lambda_a \vec{n}_a$$

$$F^{-1} \vec{n}_a = \sum_{b=1}^{3} \lambda_b^{-1} \vec{N}_b \; \vec{n}_b \cdot \vec{n}_a = \lambda_a^{-1} \vec{N}_a$$

$$C = \sum_{a=1}^{3} \lambda_a^2 \, \vec{N}_a \otimes \vec{N}_a \qquad b = \sum_{a=1}^{3} \lambda_a^2 \, \vec{n}_a \otimes \vec{n}_a$$

$$\Rightarrow \quad E = \frac{1}{2}\sum_{a=1}^{3} (\lambda_a^2 - 1) \vec{N}_a \otimes \vec{N}_a \qquad e = \frac{1}{2}\sum_{a=1}^{3} (1 - \lambda_a^{-2}) \vec{n}_a \otimes \vec{n}_a$$

the notion of strain can be generalized:

$$E^{(n)} = \frac{1}{n}\sum_{a=1}^{3} (\lambda_a^n - 1) \vec{N}_a \otimes \vec{N}_a \qquad e^{(n)} = \frac{1}{n}\sum_{a=1}^{3} (1 - \lambda_a^{-n}) \vec{n}_a \otimes \vec{n}_a .$$

and the logarithmic strain

$$E^{(0)} = \sum_{a=1}^{3} \ln \lambda_a \vec{N}_a \otimes \vec{N}_a \qquad e^{(0)} = \sum_{a=1}^{3} \ln \lambda_a \vec{n}_a \otimes \vec{n}_a .$$

Interesting features.

- Incompressibility : $\operatorname{tr} E^{(0)} = 0$
- Volumetric and ~~additive~~ isochoric parts are additively splited.

Thus, finit strain theory becomes similar to small strain theory.

In practice, we often form $C = F^T F$ and obtain $\{\vec{N}_a\}$ & $\{\lambda_a\}$ by performing eigen-decomposition. Then $\vec{n}_a = F\vec{N}_a / \lambda_a$.

Algorithm: W.M. Scherzinger & C.R. Dohrmann CMAME 197 (2008) 4007 – 4015.

To be open-sourced. → M^{3}C lab owns a C++ implementation.

Volumetric - Distortional decomposition

$$F = \underbrace{(J^{-1/3} I)}_{\text{volumetric part}} \bar{F}$$

volume-preserving or distortional part.

obviously $\bar{F} = J^{-1/3} F$.

also, modified deformation gradient.

$$\bar{F} = \sum_{a=1}^{3} \bar{\lambda}_a \cancel{\times} \vec{n}_a \otimes \vec{N}_a \qquad \bar{\lambda}_a = J^{-1/3} \lambda_a$$

modified principal stretches

Ref: J.C. Simo & R.L. Taylor. CMAME 85 (1991) 273-310

$$\bar{C} = \bar{F}^T \bar{F} = J^{-2/3} C$$

$$\bar{b} = \bar{F} \bar{F}^T = J^{-2/3} b.$$