Homework 5
Due: Apr. 24, 2023

1. Show that

(a)

$$\frac{\partial \overline{\boldsymbol{C}}}{\partial \boldsymbol{C}} = J^{-\frac{2}{3}}\mathbb{P}^T, \text{ with } \mathbb{P} = \mathbb{I} - \frac{1}{3}\boldsymbol{C}^{-1} \otimes \boldsymbol{C}.$$

(b) Show that $\mathbb{P}$ is a projector, i.e., $\mathbb{P}^2 = \mathbb{P}$.

(c) Show that the push-forward of $\mathbb{P}\colon \mathbf{A}$, **i.e.,** $\boldsymbol{F}(\mathbb{P}\colon \boldsymbol{A})\boldsymbol{F}^T$**,** is deviatoric, for any material tensor **A.**

2. Consider the strain energy $\Psi(\boldsymbol{E})$, which can be represented in terms of $\boldsymbol{F}$, i.e. $\Psi(\boldsymbol{E}) = \widehat{\Psi}(\boldsymbol{F})$. Let

$$\mathbb{A}_{iIjJ} := \frac{\partial^2 \widehat{\Psi}}{\partial F_{iI} \partial F_{jJ}}.$$

Show that $\mathbb{A}_{iIjJ} = \delta_{ij} S_{IJ} + F_{iK} F_{jL} \mathbb{C}_{IKJL}$.

3. For the description of isotropic hyperelastic materials at finite strains consider the strain-energy function $\Psi = \Psi(I_1, I_2, I_3)$ in the *coupled* form, with the principal invariants $I_a, a = 1, 2, 3$.

(a) Show the following general form of the elasticity tensor in terms of the three principal invariants,

$$\begin{aligned}\mathbb{C} &= 2\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{C}} = 4\frac{\partial^2 \Psi(I_1, I_2, I_3)}{\partial \boldsymbol{C} \partial \boldsymbol{C}} \\ &= \delta_1 \boldsymbol{I} \otimes \boldsymbol{I} + \delta_2(\boldsymbol{I} \otimes \boldsymbol{C} + \boldsymbol{C} \otimes \boldsymbol{I}) + \delta_3(\boldsymbol{I} \otimes \boldsymbol{C}^{-1} + \boldsymbol{C}^{-1} \otimes \boldsymbol{I}) \\ &\quad + \delta_4 \boldsymbol{C} \otimes \boldsymbol{C} + \delta_5(\boldsymbol{C} \otimes \boldsymbol{C}^{-1} + \boldsymbol{C}^{-1} \otimes \boldsymbol{C}) \\ &\quad + \delta_6 \boldsymbol{C}^{-1} \otimes \boldsymbol{C}^{-1} + \delta_7 \boldsymbol{C}^{-1} \odot \boldsymbol{C}^{-1} + \delta_8 \mathbb{I},\end{aligned}$$

with the coefficients $\delta_1, \ldots, \delta_8$ defined by

$$\delta_1 = 4\left(\frac{\partial^2 \Psi}{\partial I_1 \partial I_1} + 2I_1 \frac{\partial^2 \Psi}{\partial I_1 \partial I_2} + \frac{\partial \Psi}{\partial I_2} + I_1^2 \frac{\partial^2 \Psi}{\partial I_2 \partial I_2}\right),$$

$$\delta_2 = -4\left(\frac{\partial^2 \Psi}{\partial I_1 \partial I_2} + I_1 \frac{\partial^2 \Psi}{\partial I_2 \partial I_2}\right),$$

$$\delta_3 = 4\left(I_3 \frac{\partial^2 \Psi}{\partial I_1 \partial I_3} + I_1 I_3 \frac{\partial^2 \Psi}{\partial I_2 \partial I_3}\right), \qquad \delta_4 = 4\frac{\partial^2 \Psi}{\partial I_2 \partial I_2},$$

$$\delta_5 = -4I_3 \frac{\partial^2 \Psi}{\partial I_2 \partial I_3}, \quad \delta_6 = 4\left(I_3 \frac{\partial \Psi}{\partial I_3} + I_3^2 \frac{\partial^2 \Psi}{\partial I_3 \partial I_3}\right),$$

$$\delta_7 = -4I_3 \frac{\partial \Psi}{\partial I_3}, \quad \delta_8 = -4\frac{\partial \Psi}{\partial I_2}.$$

In the above,

$$\frac{\partial \boldsymbol{C}^{-1}}{\partial \boldsymbol{C}} = -\boldsymbol{C}^{-1} \odot \boldsymbol{C}^{-1},$$

and

$$-(C^{-1} \odot C^{-1})_{ABCD} = -\frac{1}{2}(C_{AC}^{-1} C_{BD}^{-1} + C_{AD}^{-1} C_{BC}^{-1}).$$

(b) Particularize the coefficients $\delta_a, a = 1, \ldots, 8$, for Mooney-Rivilin, neo-Hookean, Blatz and Ko models.

4. Consider a compressible isotropic material characterized by the strain-energy function in the decoupled form of $\Psi(\lambda_1, \lambda_2, \lambda_3) = \Psi_{\text{vol}}(J) + \Psi_{\text{iso}}(\bar{\lambda}_1\, \bar{\lambda}_2, \bar{\lambda}_3)$, with volume ratio $J = \lambda_1 \lambda_2\, \lambda_3$ and the modified principal stretches $\bar{\lambda}_a = J^{-1/3} \lambda_a, a = 1,2,3$. The associated decoupled structure of the elasticity tensor $\mathbb{C}$ in the material description is given in as $\mathbb{C}(\lambda_1, \lambda_2, \lambda_3) = \mathbb{C}_{\text{vol}} + \mathbb{C}_{\text{iso}}$, with the *volumetric* contribution $\mathbb{C}_{\text{vol}}$.

(a) Read Example 2.8 of Holzapfel book. Make sure you understand the derivative of principal stretches with respect to the deformation tensor.

(b) Read the proof of (6.180) of Holzapfel book.

(c) Show that the spectral form of the isochoric contribution $\mathbb{C}_{\text{iso}}$ may be given by

$$\mathbb{C}_{\text{iso}} = \sum_{a,b=1}^{3} \frac{1}{\lambda_b} \frac{\partial S_{\text{iso a}}}{\partial \lambda_b} \boldsymbol{N}_a \otimes \boldsymbol{N}_a \otimes \boldsymbol{N}_b \otimes \boldsymbol{N}_b$$

$$+ \sum_{\substack{a,b=1 \\ a \neq b}}^{3} \frac{S_{\text{iso b}} - S_{\text{iso a}}}{\lambda_b^2 - \lambda_a^2} (\boldsymbol{N}_a \otimes \boldsymbol{N}_b \otimes \boldsymbol{N}_a \otimes \boldsymbol{N}_b + \boldsymbol{N}_a \otimes \boldsymbol{N}_b \otimes \boldsymbol{N}_b \otimes \boldsymbol{N}_a),$$

where $S_{\text{iso a}} = (\partial \Psi_{\text{iso}} / \partial \lambda_a)/\lambda_a$, $a = 1,2,3$, denote the principal values of the second Piola-Kirchhoff stress tensor $\boldsymbol{S}_{\text{iso}}$.

(d) In order to specify the elasticity tensor given above, consider Ogden's model, i.e.,

$$\Psi_{\text{iso}}(\bar{\lambda}_1\, \bar{\lambda}_2, \bar{\lambda}_3) = \sum_{a=1}^{3} \sum_{p=1}^{N} (\mu_p / \alpha_p)(\bar{\lambda}_a^{\alpha_p} - 1).$$

Show that

$$\frac{1}{\lambda_b} \frac{\partial S_{\text{iso a}}}{\partial \lambda_b} = \frac{J^{-1/3}}{\lambda_b} \sum_{c=1}^{3} \frac{\partial S_{\text{iso a}}}{\partial \bar{\lambda}_c} \left(\delta_{cb} - \frac{1}{3} \bar{\lambda}_c \bar{\lambda}_b^{-1} \right)$$

$$= \begin{cases} \lambda_a^{-2} \lambda_b^{-2} \displaystyle\sum_{p=1}^{N} \mu_p \alpha_p \left(\left(\frac{1}{3} - \frac{2}{\alpha_p} \right) \bar{\lambda}_a^{\alpha_p} + \left(\frac{1}{9} + \frac{2}{3\alpha_p} \right) \sum_{c=1}^{3} \bar{\lambda}_c^{\alpha_p} \right) & \text{for } a = b, \\ \lambda_a^{-2} \lambda_b^{-2} \displaystyle\sum_{p=1}^{N} \mu_p \alpha_p \left(-\frac{1}{3} \bar{\lambda}_a^{\alpha_p} - \frac{1}{3} \bar{\lambda}_b^{\alpha_p} + \frac{1}{9} \sum_{c=1}^{3} \bar{\lambda}_c^{\alpha_p} \right) & \text{for } a \neq b. \end{cases}$$