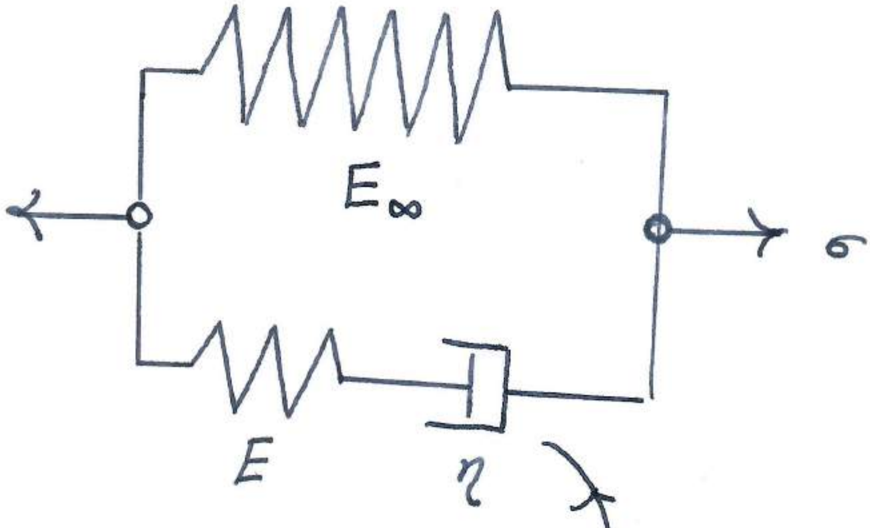
Viscoelasticity

Standard Solid Model:



dashpot

α : inelastic strain in the dashpot.

σ^v: viscous stress in the dashpot.

$$\sigma^v = \underbrace{\eta}_{\text{viscosity}} \frac{\partial}{\partial t}\alpha(t)$$

- $\sigma = \overbrace{E_\infty \underbrace{\varepsilon}_{\text{total strain}}}^{\text{linear elastic}} + \sigma^v$

- $\sigma^v = E(\varepsilon - \alpha)$

  strain can be <u>additively</u> decomposed.

$$\Rightarrow \quad \sigma = \underbrace{(E_\infty + E)}_{E_0 \,:\, \text{initial modulus}} \varepsilon - E\alpha \qquad \text{constitution}$$

$$\text{and} \quad (\sigma^v =) \quad \eta \dot{\alpha} = E(\varepsilon - \alpha)$$

$$\Rightarrow \quad \dot{\alpha} + \frac{E}{\eta}\alpha = \frac{E}{\eta}\varepsilon$$

$$\Rightarrow \quad \dot{\alpha} + \frac{1}{\tau}\alpha = \frac{1}{\tau}\varepsilon \qquad \tau := \frac{\eta}{E} \ \text{relaxation time}$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\left(\exp\left(\frac{t}{\tau}\right)\alpha\right) = \frac{1}{\tau}\varepsilon \, \exp\left(\frac{t}{\tau}\right)$$

$$\Rightarrow \quad \alpha(t) = \frac{1}{\tau}\int_{-\infty}^{t} \exp\left(-\frac{t-s}{\tau}\right)\varepsilon(s)\, ds$$

$$= \varepsilon(t) - \int_{-\infty}^{t} \exp\left(-\frac{t-s}{\tau}\right)\dot{\varepsilon}(s)\, ds$$

$$\Rightarrow \quad \sigma = E_\infty \varepsilon + E\int_{-\infty}^{t} \exp\left(-\frac{t-s}{\tau}\right)\dot{\varepsilon}(s)\, ds$$

$$= \int_{-\infty}^{t} \underbrace{\left[E_\infty + E\exp\left(-\frac{t-s}{\tau}\right)\right]}_{G(t-s) \ \text{relaxation function.}} \dot{\varepsilon}(s)\, ds$$

Relaxation test.

$$\varepsilon(t) = H(t)\,\varepsilon_0 = \begin{cases} 0 & \text{if } t<0 \\ \varepsilon_0 & \text{if } t\geq 0 \end{cases}$$

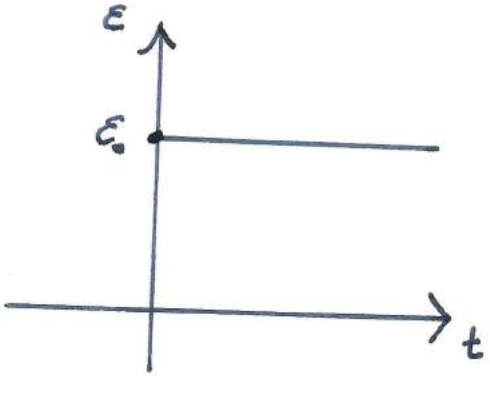
$$\Rightarrow \quad \sigma(t) = \int_{-\infty}^{t} G(t-s)\,\varepsilon_0\,\delta(s)\,ds = G(t)\,\varepsilon_0 \quad \text{for } t>0$$

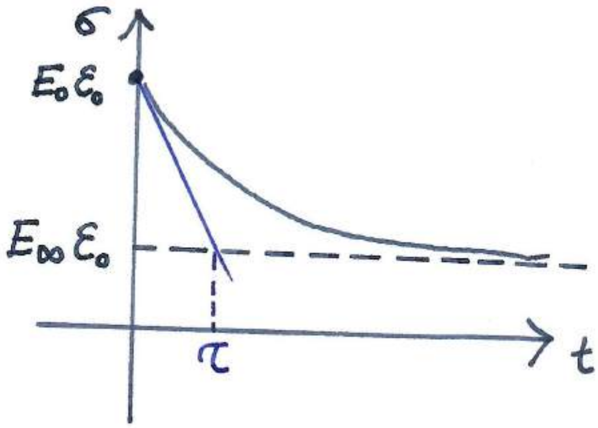
$$= \left[E_\infty + E\exp\left(-\frac{t}{\tau}\right)\right]\varepsilon_0$$

$$\dot{\sigma}(t) = -\frac{E}{\tau}\exp\left(-\frac{t}{\tau}\right)\varepsilon_0$$

$\sigma(0) = E_0\,\varepsilon_0$ $\qquad$ $\sigma(t) \to E_\infty\,\varepsilon_0$

$\dot{\sigma}(0) = -\frac{E}{\tau}\,\varepsilon_0$ $\qquad$ $\dot{\sigma}(0) \to 0$





Generalization — multiple Maxwell elements in paraller

$$\sigma = E_\infty\,\varepsilon + \sigma^v$$

$$\sigma^v = \sum_{i=1}^{N} E_i\left(\varepsilon - \alpha_i\right)$$

$$\Rightarrow \quad \sigma = E_0 \varepsilon - \sum_{i=1}^{N} E_i \alpha_i$$

$$\longrightarrow E_0 = E_\infty + \sum_{i=1}^{N} E_i$$

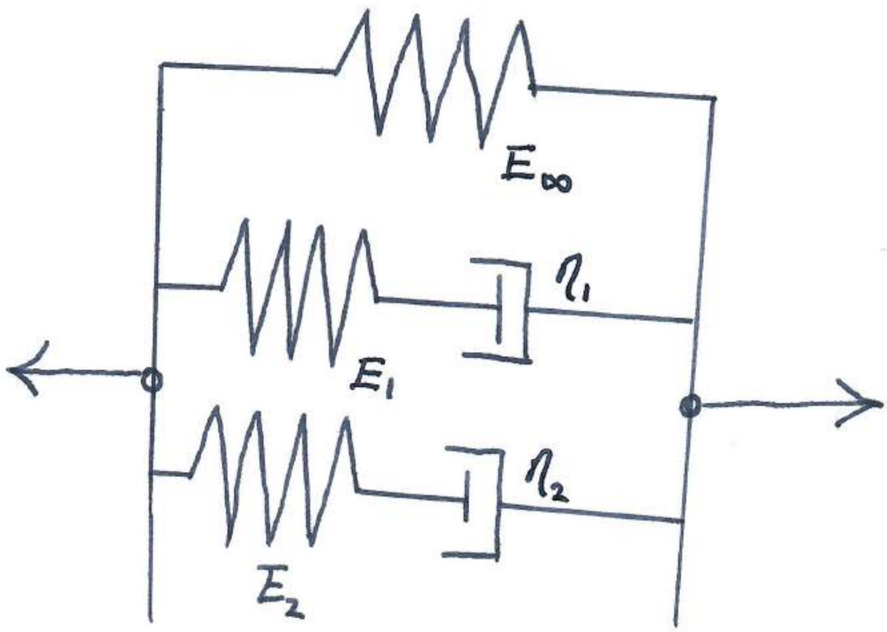
$$\eta_i \dot{\alpha}_i = E_i (\varepsilon - \alpha_i)$$

$$\Leftrightarrow \quad \dot{\alpha}_i + \frac{\alpha_i}{\tau_i} = \frac{\varepsilon}{\tau_i} \qquad \tau_i = \eta_i / E_i \quad i = 1, \dots, N.$$

$$\lim_{t \to -\infty} \alpha_i = 0$$

$$\alpha_i(t) = \varepsilon(t) - \int_{-\infty}^{t} \exp\left(-\frac{t-s}{\tau_i}\right) \dot{\varepsilon}(s)\, ds$$

$$\Rightarrow \sigma = \int_{-\infty}^{t} \underbrace{\left[E_\infty + \sum_{i=1}^{N} E_i \exp\left(-\frac{t-s}{\tau_i}\right) \right]}_{G(t-s)} \dot{\varepsilon}(s)\, ds$$



Stored energy: $\psi(\varepsilon, \alpha) = \frac{1}{2} E_\infty \varepsilon^2 + \frac{1}{2} \sum_{i=1}^{N} E_i (\varepsilon - \alpha_i)^2$

vector of $(\alpha_1, \cdots, \alpha_N)^T$.

Recall that Clausius-Plank inequality gives

$$\mathcal{D} := P : \dot{F} - \dot{\psi} - \eta \dot{\Theta} \geq 0$$

isothermal and small strain:

$$\mathcal{D} = \sigma : \dot{\varepsilon} - \dot{\psi}$$

$$= \sigma : \dot{\varepsilon} - \psi_{,\varepsilon} : \dot{\varepsilon} - \sum_i \psi_{,\alpha_i} \dot{\alpha}_i$$

$$\psi_{,\varepsilon} = E_\infty \varepsilon + \sum_{i=1}^{N} E_i (\varepsilon - \alpha_i) = \sigma$$

$$\psi_{,\alpha_i} = -E_i (\varepsilon - \alpha_i)$$

$$\Rightarrow \mathcal{D} = \sum_{i=1}^{N} \sigma_i^v \dot{\alpha}_i = \sum_{i=1}^{N} \eta_i (\dot{\alpha}_i)^2 \geq 0$$

due to the constitutive model for the dashpot. (inelasticity)

Characterization of equilibrium response.

Under a prescribed strain / stress, the device reaches <u>equilibrium</u> if no further changes in the dashpots take place. :

$$\dot{\alpha}_i \Big|_{eq} = 0 \quad \text{for} \quad i = 1, \cdots, N.$$

or.

$$\sigma_i^v \Big|_{eq} = 0 \quad \text{for} \quad i = 1, \cdots, N \quad (\text{i.e., the force vanishes})$$

$$\Rightarrow \quad \dot{\alpha}_i + \frac{\alpha_i}{\tau_i} = \frac{\varepsilon}{\tau_i} \quad \Rightarrow \quad \alpha_i \Big|_{eq} = \varepsilon \quad \text{in equilibrium}$$

$$\Rightarrow \quad \sigma \Big|_{eq} = E_\infty \varepsilon .$$

Remark: the variable $q_i = -\dfrac{\partial \psi}{\partial \alpha_i} = E_i (\varepsilon - \alpha_i) = \sigma_i^v$ is often used in place of α_i, in a lot works.