

**Homework 7**  
**Due: May 21, 2023**

1. Consider the same problem given in Problem 3 of homework 6. Modify the problem into a system of first-order ordinary differential equations as follows,

$$\mathbf{M}\dot{\mathbf{v}} + \mathbf{K}\mathbf{d} = \mathbf{0},$$

$$\mathbf{v} - \dot{\mathbf{d}} = \mathbf{0}.$$

The generalized- $\alpha$  scheme for first order ODE problems is slightly different from the one presented in class. It can be stated as follows. At the time interval  $(t_n, t_{n+1})$ , given  $\dot{\mathbf{v}}_n$ ,  $\dot{\mathbf{d}}_n$ ,  $\mathbf{v}_n$ ,  $\mathbf{d}_n$ , determine  $\dot{\mathbf{v}}_{n+1}$ ,  $\dot{\mathbf{d}}_{n+1}$ ,  $\mathbf{v}_{n+1}$ ,  $\mathbf{d}_{n+1}$ , such that the following system of equations is satisfied.

$$\mathbf{M}\dot{\mathbf{v}}_{n+\alpha_m} + \mathbf{K}\mathbf{d}_{n+\alpha_f} = \mathbf{0},$$

$$\mathbf{v}_{n+\alpha_f} - \dot{\mathbf{d}}_{n+\alpha_m} = \mathbf{0},$$

$$\mathbf{v}_{n+1} = \mathbf{v}_n + (1 - \gamma)\Delta t \dot{\mathbf{v}}_{n+1} + \gamma\Delta t \dot{\mathbf{v}}_n,$$

$$\mathbf{d}_{n+1} = \mathbf{d}_n + (1 - \gamma)\Delta t \dot{\mathbf{d}}_{n+1} + \gamma\Delta t \dot{\mathbf{d}}_n,$$

and the intermediate terms are defined in the usual way, that is,

$$(\cdot)_{n+\alpha_m} = (1 - \alpha_m)(\cdot)_n + \alpha_m(\cdot)_{n+1},$$

$$(\cdot)_{n+\alpha_f} = (1 - \alpha_f)(\cdot)_n + \alpha_f(\cdot)_{n+1}.$$

The parameters are represented in terms of the spectral radius of the amplification matrix at the highest mode,  $\rho_\infty$ , as follows,

$$\alpha_m = \frac{3 - \rho_\infty}{2(1 + \rho_\infty)}, \quad \alpha_f = \frac{1}{1 + \rho_\infty}, \quad \gamma = \frac{1}{1 + \rho_\infty}.$$

- Give an implementation for the algorithm.
- Using the same material parameters and initial conditions, numerically study the problem with  $\rho_\infty = 0, 0.5, 1$ . Compare the results with the generalized- $\alpha$  scheme in HW6 in terms of the velocity overshoot.

2. Consider a multiplicative modification of the stress as follows,

$$\mathbf{S}_{\text{alg}} = \frac{\phi(\mathbf{C}_{n+1}) - \phi(\mathbf{C}_n)}{\mathbf{S}_m : \mathbf{Z}_n} \mathbf{S}_m,$$

where  $\mathbf{S}_m = \mathbf{S}(\mathbf{C}_{n+1/2})$ ,  $\mathbf{Z}_n = (\mathbf{Z}_{n+1} - \mathbf{Z}_n)/2$ . Show that this algorithmic stress satisfies the directionality property, and it thus leads to an energy-stable scheme for nonlinear elastodynamics. Different from the discrete gradient formula given in class, this one multiplies a scalar in front of  $\mathbf{S}_m$ . This algorithmic stress was originally provided in A. Chorin, et al., "Product formulas and numerical algorithms", Communications on Pure and Applied Mathematics, 32:205-256, 1978.

3. Consider the standard solid model. The strain can be represented in terms of the stress history in the convolution representation

$$\varepsilon(t) = \int_{-\infty}^t J(t-s) \dot{\sigma}(s) ds.$$

Show that the creep function

$$J(t) = \frac{1}{E_{\infty}} \left( 1 - \frac{E}{E_0} e^{-\frac{E_{\infty}}{\tau E_0} t} \right).$$