Homework 3

Due April 13 2021

1. Let $\boldsymbol{S}$ be a second-rank tensor and $\boldsymbol{T}$ be a symmetric second-rank tensor. Prove that

$$S_{ij}T_{ij} = S_{(ij)}T_{ij} = S_{(ij)}T_{(ij)}.$$

2. (Euler-Lagrange condition of the Galerkin formulation)

   (a) Consider the weak-form problem of the one-dimensional model problem,

$$\int_0^1 w_{,x}u_{,x}dx = \int_0^1 wf dx + w(0)h,$$

in which w and u are assumed to have sufficient smoothness *globally*. One can show its equivalence to the strong-form problem via considering its Euler-Lagrange condition. Now we consider the corresponding Galerkin formulation, where the test function and trial solution are represented by piecewise linear functions. Therefore, they are smooth only in the element interiors, but are only C^0 across element boundaries. Because of this, we can only perform integration-by-parts in an element-by-element fashion for the Galerkin formulation. Show that

$$\begin{aligned} 0 = &\sum_{A=1}^{n} \int_{x_A}^{x_{A+1}} w^h \left(u^h_{,xx} + f \right) dx + w^h(0) \left(u^h_{,x}(0^+) + h \right) \\ &+ \sum_{A=2}^{n} w^h(x_A) \left(u^h_{,x}(x_A^+) - u^h_{,x}(x_A^-) \right). \end{aligned}$$

Further, show that the above Euler-Lagrange condition suggests that

$$\begin{aligned} &u^h_{,xx} + f = 0, \quad \text{for } x \in (x_A, x_{A+1}), \text{ and } A = 1, 2, \cdots, n, \\ &- u^h_{,x}(0^+) = h, \\ &u^h_{,x}(x_A^-) = u^h_{,x}(x_A^+), \quad \text{for } A = 2, 3, \cdots, n. \end{aligned}$$

   (b) Consider the weak-form problem for elastostatics,

$$\int_\Omega w_{(i,j)}\sigma_{ij} d\Omega = \int_\Omega w_i f_i d\Omega + \int_{\Gamma_h} w_i h_i d\Gamma.$$

If the test and trail function spaces are constructed by C^0 basis functions, show that

$$0 = \sum_{e=1}^{n_{el}} \int_{\Omega^e} w_i^h \left(\sigma^h_{ij,j} + f_i \right) d\Omega - \int_{\Gamma_h} w_i^h \left(\sigma^h_{ij} n_j - h_i \right) d\Gamma - \int_{\Gamma_{\text{int}}} w_i^h [\![\sigma_{ij} n_j]\!] d\Gamma,$$

wherein

$$\Gamma_{\text{int}} := \cup_{e=1}^{n_{el}} \partial\Omega^e - \partial\Omega.$$

From the above Euler-Lagrange equation, one may readily conclude that the Galerkin formulation implies

$$\sigma_{ij,j} + f_i = 0, \qquad \text{in } \cup_{e=1}^{n_{el}} \Omega^e,$$

$$\sigma_{ij} n_j = h_i, \qquad \text{on } \Gamma_h,$$
$$[\![\sigma_{ij} n_j]\!] = 0, \qquad \text{on } \Gamma_{\text{int}}.$$

3. (Voigt notation)

   (a) Show that $\boldsymbol{\sigma}^{\text{vect}} = \boldsymbol{D}\boldsymbol{\epsilon}^{\text{vect}}$ is equivalent to $\sigma_{ij} = c_{ijkl}\sigma_{kl}$.

   (b) For isotropic materials, setup the matrix $\boldsymbol{D}$.

4. Use quadratic and linear shape functions to design shape functions for the six-node quadrilateral depicted in the following figure.