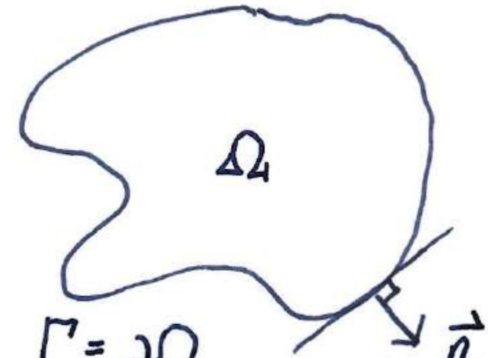
Review of Linear problems

1. Heat conduction — A scalar problem posed in 1/2/3D

$$1/2/3D = n_{sd}$$

n_{sd}: # of spatial dim.



$\Gamma = \partial\Omega$ is smooth.

$n_{sd} = 2$. $\quad 1 \le i, j, k, l \le n_{sd}$

$$x = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix} = \{x_i\}$$

$$n = \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \{n_i\}$$

temperature $u : \bar{\Omega} \to \mathbb{R} \qquad \bar{\Omega} = \Omega \cup \Gamma$

heat flux $\vec{q} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \qquad \kappa = [\kappa_{ij}] = \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix}$

Gen. Fourier's law: $q_i = - \underbrace{\kappa_{ij} u_{,j}}_{\text{Summation convention.}}$

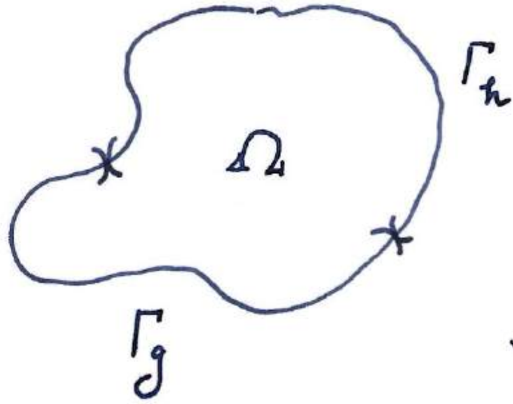
- κ is symmetric, i.e., $\kappa_{ij} = \kappa_{ji}$
- κ is positive definite, i.e., $\vec{c}^T \kappa \vec{c} \geq 0$ for all vectors $\vec{c} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$

  $\vec{c}^T \kappa \vec{c} = 0$ implies $\vec{c} = \vec{0}$.

- $\kappa = \kappa(\vec{x})$ is inhomogeneous / heterogeneous.

otherwise, κ is homogeneous.

- κ is isotropic if $\kappa = \underset{\text{scalar}}{\kappa}[\delta_{ij}] = \begin{bmatrix} \kappa & 0 \\ 0 & \kappa \end{bmatrix}$



$\Gamma_h \cap \Gamma_g = \phi \quad \overline{\Gamma_h \cup \Gamma_g} = \Gamma$

non-overlaping subdivision of Γ.

$f : \Omega \to \mathbb{R}$ heat supply per unit volume

$g : \Gamma_g \to \mathbb{R}$ prescribed boundary temp.

$h : \Gamma_h \to \mathbb{R}$ " " heat flux.

<u>Strong form of the boundary-value problem</u>

$$(S)\left\{\begin{array}{l} \text{Given } f : \Omega \to \mathbb{R},\ h : \Gamma_h \to \mathbb{R},\ g : \Gamma_g \to \mathbb{R} \\ \quad \kappa : \Omega \to \mathbb{R}^{n_{sd} \times n_{sd}},\ \text{find } u : \bar{\Omega} \to \mathbb{R} \text{ such that} \\ \nabla \cdot \vec{q} = q_{i,i} = -\left(\kappa_{ij} u_{,j}\right)_{,i} = f \quad \text{in } \Omega \\ \qquad\qquad u = g \quad \text{on } \Gamma_g \\ -\vec{q} \cdot \vec{n} = -q_i n_i = \kappa_{ij} u_{,j} n_i = h \quad \text{on } \Gamma_h . \end{array}\right.$$

Neumann boundary condition.

Dirichlet boundary condition

Remark 1: We present the math problem with heat conduction as the background. Yet, the problem is rather general, and we call it the "elliptic boundary-value problem."

For example,

		u	$\vec{q}$	κ	f	$q = -\kappa \nabla u$
heat conduction $\longleftrightarrow$	energy conservation :	temp.	heat flux	heat conductivity	heat source	Fourier's law
deformation of an elastic bar $\longleftrightarrow$	linear momentum conservation :	disp	stress	Young's modulus	body force	Hooke's law

See. Baker, Carey, Oden, P. 44.

Remark 2: There are other type of boundary conditions, e.g.,

$$\lambda u - q_i n_i = h.$$

or.

$$-q_i n_i = -\beta (u - u_{ref}).$$

is known as the Robin boundary condition.

Remark 3: For a precise statement, we need to provide the spaces to which the functions belong. See Hughes book Appendix 1.1 & Arbogast Bona Chap. 8.

Some function spaces:

$$L_2 \doteq L_2(\Omega) = \left\{ w : \int_0^1 w^2 \, dx < \infty \right\}$$

$$\left(\begin{array}{l} H^k = H^k(\Omega) = \{ w : w \in L_2 \;\; w_{,i} \in L_2, \; \cdots, \; w_{,\underbrace{ij\cdots l}_{k \text{ indices.}}} \in L_2 \} \\ \text{e.g. } n_{sd} = 2. \;\; H^2(\Omega) = \{ w : w \in L_2, \; w_{,x} \in L_2 \;\; w_{,y} \in L_2 \\ \qquad\qquad\qquad w_{,xx} \in L_2 \;\; w_{,xy} \in L_2 \;\; w_{,yy} \in L_2 \} \end{array} \right.$$

- $L_2 = H^0$ apparently
- We denote $H_0^1(\Omega) = \{ w : w \in H^1(\Omega) \;\; w = 0 \text{ on } \partial\Omega \}$

Divergence theorem:

$$\int_\Omega \nabla \cdot \vec{q} \, d\Omega = \int_{\partial\Omega} \vec{q} \cdot \vec{n} \, d\Gamma$$

$$\text{or} \quad \int_\Omega q_{i,i} \, d\Omega = \int_{\partial\Omega} q_i n_i \, d\Gamma$$

Integration-by-parts:

$$\int_\Omega f \, \nabla \cdot \vec{g} \, d\Omega = - \int_\Omega \nabla f \cdot \vec{g} + \int_{\partial\Omega} f \vec{g} \cdot \vec{n} \, d\Gamma$$

$$\text{or} \quad \int_\Omega f \, g_{i,i} \, d\Omega = - \int_\Omega f_{,i} g_i \, d\Omega + \int_{\partial\Omega} f g_i n_i \, d\Gamma$$

$\mathcal{S} := \{ u : u \in H^1(\Omega), \; u|_{\Gamma_g} = g \}$ is the trial solution space

$\mathcal{V} := \{ w : w \in H^1(\Omega), \; w|_{\Gamma_g} = 0 \}$ is the test/weighting function space.

Weak or variational form of the BV problem

(W) Given $f : \Omega \to \mathbb{R}$, $h : \Gamma_h \to \mathbb{R}$, $g : \Gamma_g \to \mathbb{R}$, and $\kappa : \Omega \to \mathbb{R}^{n_{sd} \times n_{sd}}$, find $u \in \mathcal{S}$ such that for all $w \in \mathcal{V}$

weak or generalized solution.

$$a(w, u) = (w, f) + (w, h)_{\Gamma_h}$$

Variational eqn. or equation of virtual work.

where

$$a(w, u) = \int_\Omega w_{,i} \kappa_{ij} u_{,j} \, d\Omega$$

$$(w, f) = \int_\Omega w f \, d\Omega$$

$$(w, h)_{\Gamma_h} = \int_{\Gamma_h} w \, h \, d\Gamma.$$

also called virtual disp. in mechanics

The equivalence of (S) and (W)

Proposition a : let u be a solution of (S), then u is a solution of (W).

Proof: u is a solution of (S), we have

$$(\kappa_{ij} u_{,j})_{,i} + f = 0$$

$$\Rightarrow \int_\Omega w (\kappa_{ij} u_{,j})_{,i} + w f \, d\Omega = 0 \quad \text{for } \forall w \in \mathcal{V}.$$

$= $ integration-by-parts $\Rightarrow$

$$-\int_{\Omega} w_{,i}\, \kappa_{ij}\, u_{,j}\, d\Omega + \int_{\partial\Omega} w\, \kappa_{ij}\, u_{,j}\, n_i\, d\Gamma + \int_{\Omega} w f\, d\Omega = 0$$

$$\int_{\partial\Omega} w\, \kappa_{ij}\, u_{,j}\, n_i\, d\Gamma = \int_{\Gamma_h} w\, h\, d\Gamma$$ because $w \in \mathcal{V}$ and u satisfies the Neumann B.C.

$$\Rightarrow \quad a(w, u) = (w, f) + (w, h)_{\Gamma_h}.$$

Thus. u solves (W).

$\blacksquare$

proposition b: let u be a solution of (W) (and u is smooth enough for second derivatives), then u is a solution of (S).

Proof: Let u be a solution of (W),

$$\int_{\Omega} w_{,i}\, \kappa_{ij}\, u_{,j}\, d\Omega = \int_{\Omega} w f\, d\Omega + \int_{\Gamma_h} w\, h\, d\Gamma.$$

$\parallel \leftarrow$ use of int.-by-parts & we assume $u_{,ji}$ exists or u is twice differentiable

$$-\int_{\Omega} w\, (\kappa_{ij}\, u_{,j})_{,i}\, d\Omega + \int_{\underset{\partial\Omega}{\overset{\parallel}{\Gamma}}} w\, \kappa_{ij}\, u_{,j}\, n_i\, d\Gamma.$$

$$\Rightarrow \int_{\Omega} w\left[(\kappa_{ij}\, u_{,j})_{,i} + f\right] d\Omega + \int_{\Gamma_h} w\left[h - \kappa_{ij}\, u_{,j}\, n_i\right] d\Gamma = 0.$$

$\uparrow$ Euler-Lagrange eqn of the weak / variational formulation.

Now, since $u \in \mathcal{S}$, $u = g$ on Γ_g is satisfied already due to the construction of the trial solution space. We only need to show $(\kappa_{ij} u_{,j})_{,i} + f = 0$ in Ω &

$$(\kappa_{ij} u_{,j}) n_i - h = 0 \quad \text{on } \Gamma_h .$$

We can choose $\hat{w} = \phi \left((\kappa_{ij} u_{,j})_{,i} + f \right)$ with $\phi > 0$ in Ω and $\phi = 0$ on $\Gamma = \partial\Omega$. Of course, $\hat{w} \in \mathcal{V}$, and we may insert $\hat{w}$ into E.-L. :

$$\int_\Omega \phi \left[(\kappa_{ij} u_{,j})_{,i} + f \right]^2 d\Omega = 0$$

$$\Rightarrow (\kappa_{ij} u_{,j})_{,i} + f = 0. \quad \text{in } \Omega.$$

Next, choosing $\hat{w} = \psi (h - \kappa_{ij} u_{,j} n_i)$ with $\psi > 0$ on $\bar{\Omega}$, we can establish $(\kappa_{ij} u_{,j}) n_i = h$ on Γ_h.

$\blacksquare$

Remark 1: We established (S) $\Leftrightarrow$ (W), under the assumption that the weak solution is twice differentiable.

Remark 2: The <u>Dirichlet</u> B.C. is built into the def. of $\mathcal{S}$, and B.C. of this type is called <u>essential</u> boundary conditions;

the <u>Neumann</u> B.C. is built implicitly in the variational eqn, and B.C. of this type is called <u>natural</u> boundary conditions.

Remark 3: The technique used after the E.-L. eqn is known as the fundamental lemma of the calculus of variations, which transform a weak form to its corresponding strong form. With this procedure, one may get the mathematical features of a weak form through the Euler-Lagrange equations.

Remark 4: $a(\cdot,\cdot)$ and $(\cdot,\cdot)$ are symmetric bilinear forms.

Symmetry: $a(w,u) = a(u,w)$

$$(w,u) = (u,w).$$

Bilinearity: linearity in both slots

$$a(c_1 w_1 + c_2 w_2, u) = c_1 a(w_1,u) + c_2 a(w_2,u)$$

$$a(w, c_1 u_1 + c_2 u_2) = c_1 a(w,u_1) + c_2 a(w,u_2).$$

Ref. Hughes. FEM book. Sec 1.1 – 1.4, Sec 2.1 – 2.3