Homework 2

Due March 30 2021

1. The three-point Gaussian quadrature is given as follows. The quadrature points are $-\sqrt{3/5}$, 0, and $\sqrt{3/5}$; the corresponding quadrature weights are $5/9$, $8/9$, and $5/9$. Verify that its algebraic accuracy is 5.

2. Consider the mesh shown in Figure 1. Setup the IEN, ID, and LM arrays for this mesh. Notice that the essential boundary condition is specified on nodes 5, 6, 7, and 8.

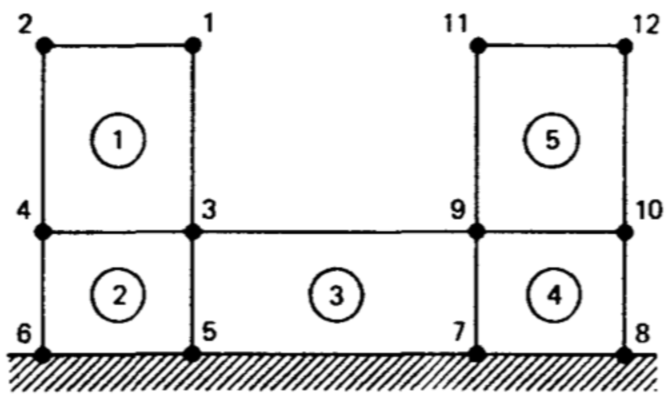

Figure 1: A finite element mesh consisting of five bilinear quadrilateral elements.

3. Consider the 10-node tetrahedral element shown in Figure 2. Derive the shape functions and verify that they satisfy the interpolation property, that is, $N_a(r_b, s_b, t_b, u_b) = \delta_{ab}$ with $u_b = 1 - r_b - s_b - t_b$.

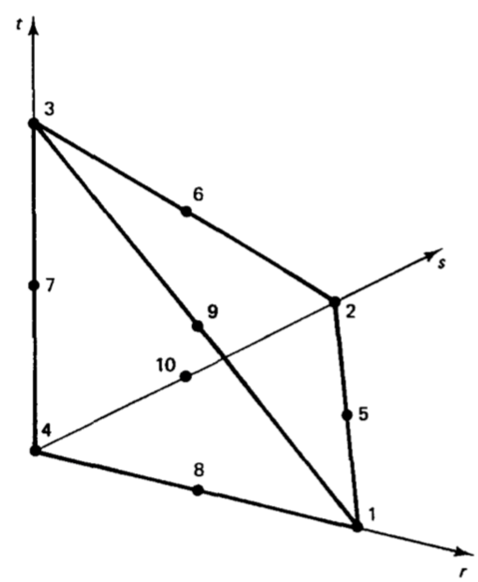

Figure 2: Quadratic tetrahedral element in the reference domain.

4. Consider a one-dimensional quadratic three-node element shown in Figure 3. The shape functions are given as follows.

$$N_1(\xi) = \frac{1}{2}\xi\left(\xi - 1\right), \quad N_2(\xi) = 1 - \xi^2, \quad N_3(\xi) = \frac{1}{2}\xi\left(\xi + 1\right).$$

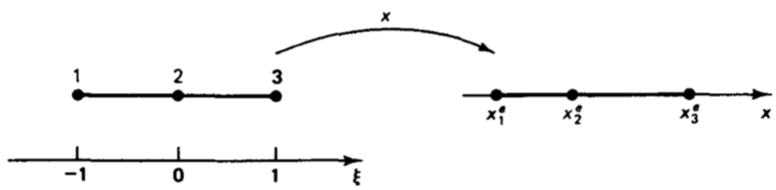

Figure 3: Quadratic element.

(a) Assume f is constant and let $x_2^e = (x_1^e + x_3^e)/2$. Determine the exact expression for

$$f_a^e = \int_{x_1^e}^{x_3^e} N_a f dx, \quad a = 1, 2, 3.$$

(b) Assume f is constant, but make no assumption on the location of x_2^e other than $x_1^e < x_2^e < x_3^e$. Determine the lowest-order Gaussian quadrature formula which exactly integrates f^e, and why?

(c) Assume $x_2^e = (x_1^e + x_3^e)/2$. Determine the lowest-order quadrature formula for the element stiffness matrix, and why?

(d) Modify the MATLAB code developed in class by using the quadratic three-node element, with the assumption that $x_2^e = (x_1^e + x_3^e)/2$.

(e) Let the manufactured solution be $u = \sin(x)$. Design the source term and the boundary conditions. Perform calculations with equally spaced nodes for $4, 6, 8, 12$ and 16 elements.

(f) Let $e := u^h - u$ be the error in the finite element approximation. Calculate

$$|e|_1 := \left(\int_0^1 (e_{,x})^2 \, dx \right)^{\frac{1}{2}},$$

and

$$|u|_1 := \left(\int_0^1 (u_{,x})^2 \, dx \right)^{\frac{1}{2}},$$

plot $\log\left(|e|_1/|u|_1\right)$ versus $\log(h) = -\log(n_{en})$. What is the slope of the curve?