Finite Viscoelasticity

Ref. S. Reese & S. Govindjee, IJSS, vol 35: 3455–3482, 1998

$$\psi = \psi_{eq}(C) + \psi_{neq}(C_e)$$

C_e : the deformation in the spring of the Maxwell element.

$$F = F_e F_i \qquad \left(F_{iI} = F_{e_{i\alpha}} F_{i_{\alpha I}} \right)$$

then $\quad C_e = F_e^T F_e = F_i^{-T} C F_i^{-1}$.

$$\left(C_{e_{\alpha\beta}} = F^{-1}_{i_{I\alpha}} C_{IJ} F^{-1}_{i_{J\beta}} \right)$$

Then Clausius-Plank inequality gives

$$S : \tfrac{1}{2}\dot{C} - \dot{\psi} \geq 0$$

$$\parallel$$

$$S : \tfrac{1}{2}\dot{C} - \frac{\partial \psi_{eq}}{\partial C}\dot{C} - \frac{\partial \psi_{neq}}{\partial C_e}\frac{\partial C_e}{\partial C}\dot{C} - \frac{\partial \psi_{neq}}{\partial F_i}\dot{F}_i$$

$$\frac{\partial C_{e_{\alpha\beta}}}{\partial C_{MN}} = F^{-1}_{i_{I\alpha}} \mathbb{I}_{IJMN} F^{-1}_{i_{J\beta}} \Rightarrow \frac{\partial \psi_{neq}}{\partial C_{e_{\alpha\beta}}} \frac{\partial C_{e_{\alpha\beta}}}{\partial C_{MN}} \dot{C}_{MN}$$

$$= \frac{\partial \psi_{neq}}{\partial C_{e_{\alpha\beta}}} F^{-1}_{i_{I\alpha}} F^{-1}_{i_{J\beta}} \mathbb{I}_{IJMN} \dot{C}_{MN}$$

$$= \frac{\partial \psi_{neq}}{\partial C_{e\,\alpha\beta}} F^{-1}_{i\,I\alpha} F^{-1}_{i\,J\beta} \dot{C}_{IJ}$$

$$\frac{\partial \psi_{neq}}{\partial C_{e\,\alpha\beta}} \frac{\partial C_{e\,\alpha\beta}}{\partial F_{i\,\gamma L}} \dot{F}_{i\,\gamma L} = \frac{\partial \psi_{neq}}{\partial C_{e\,\alpha\beta}} \Big(- F^{-1}_{i\,I\gamma} F^{-1}_{i\,L\alpha} C_{IJ} F^{-1}_{i\,J\beta} - F^{-1}_{i\,I\alpha} C_{IJ} F^{-1}_{i\,J\gamma} F^{-1}_{i\,L\beta} \Big) \dot{F}_{i\,\gamma L}$$

$$\frac{\partial F^{-1}_{i\,I\alpha}}{\partial F_{i\,\gamma L}} = - F^{-1}_{I\gamma} F^{-1}_{L\alpha}$$

$$= \frac{\partial \psi_{neq}}{\partial C_{e\,\alpha\beta}} \Big(- C_{e\,\gamma\beta} F^{-1}_{i\,L\alpha} \dot{F}_{i\,\gamma L} - C_{e\,\alpha\gamma} F^{-1}_{i\,L\beta} \dot{F}_{i\,\gamma L} \Big)$$

$$F^{-1}_{Ii} \dot{F}_{jI} = l_{ij} \qquad = \frac{\partial \psi_{neq}}{\partial C_{e\,\alpha\beta}} \Big(- C_{e\,\gamma\beta}\, l_{i\,\gamma\alpha} - C_{e\,\alpha\gamma}\, l_{i\,\gamma\beta} \Big)$$

$$\Downarrow$$

$$F^{-1}_{i\,L\alpha} \dot{F}_{i\,\gamma L} = l_{i\,\gamma\alpha}.$$

Summary: $S = S_{eq} + S_{neq}$

$$S_{eq} = 2 \frac{\partial \psi_{eq}}{\partial C}$$

$$S_{neq} = 2 \frac{\partial \psi_{neq}}{\partial C}$$

$$= 2\, F_i^{-1} \frac{\partial \psi_{neq}}{\partial C_e} F_i^{-T}$$

Then we need $\frac{\partial \psi_{neq}}{\partial C_e} \left(l_i^T C_e + C_e l_i \right) \geq 0$

$$\overset{\text{Symmetry}}{\Leftrightarrow} \quad 2\frac{\partial \psi_{neq}}{\partial C_e} : (C_e\, l_i) \geq 0$$

index notation

$$2\frac{\partial \psi_{neq}}{\partial C_{e_{\alpha\beta}}} C_{e_{\alpha\gamma}} l_{i_{\gamma\beta}} \geq 0$$

$$\|$$

$$2 F_{e_{i\alpha}} \frac{\partial \psi_{neq}}{\partial C_{e\,\alpha\beta}} F_{e_{j\beta}} \quad F^{-1}_{e_{\delta i}} C_{e_{\alpha\gamma}} l_{i_{\gamma\beta}} F^{-1}_{e_{\eta j}}$$

$$\alpha \to \delta \qquad \gamma \to \eta$$

$$\Updownarrow$$

$$\underbrace{\left(2 F_e \frac{\partial \psi_{neq}}{\partial C_{e_{\alpha\beta}}} F_e^T\right)}_{\text{Spatial tensor denoted as } \tau_{neq}} : \left(F_e^{-T} C_e\, l_i\, F_e^{-1}\right) \geq 0$$

$$\|$$

$$\tau_{neq} : \left(F_e^{-T} C_e\, l_i\, F_e^{-1}\right)$$

Let $b_e = F_e F_e^T \Rightarrow b_e^{-1} = F_e^{-T} F_e^{-1} \Rightarrow F_e^{-1} = F_e^T b_e^{-1}$

$$F^{-1}_{e_{\beta j}} = F_{e_{l\beta}} b^{-1}_{e_{lj}}$$

$$\tau_{neq_{ij}} F_{e_{i\alpha}} l_{i_{\alpha\beta}} F^{-1}_{e_{\beta j}}$$

$$= \tau_{neq_{ij}} b^{-1}_{e_{lj}} \quad F_{e_{i\alpha}} l_{i_{\alpha\beta}} F_{e_{l\beta}}$$

$$\Rightarrow \left(\tau_{neq}\, b_e^{-1} \right) : \left(F_e\, l_i\, F_e^{T} \right) \geq 0$$

$$\begin{aligned}
\operatorname{Sym}\left(F_e\, l_i\, F_e^{T} \right) &= F_e\, \frac{l_i + l_i^{T}}{2}\, F_e^{T} \\
&= \frac{1}{2} F_e \left(\dot{F}_i F_i^{-1} + F_i^{-T} \dot{F}_i^{T} \right) F_e^{T} \\
&= \frac{1}{2} F F_i^{-1} \left(\dot{F}_i F_i^{-1} + F_i^{-T} \dot{F}_i^{T} \right) F_i^{T} F^{T} \\
&= \frac{1}{2} F \left(F_i^{-1} \dot{F}_i F_i^{-1} + F_i^{-T} \dot{F}_i^{T} F_i^{T} \right) F^{T} \\
&= -\frac{1}{2} F\, \dot{\overline{C_i^{-1}}}\, F^{T}.
\end{aligned}$$

Recall that b_e is a ~~stress-like~~ contravariant tensor,

$$\begin{aligned}
\varphi_*^{-1}(b_e) &= F^{-1} b_e F^{-T} = F^{-1} F_e F_e^{T} F^{-T} \\
&= F_i^{-1} F_i^{-T} = C_i^{-1}
\end{aligned}$$

Thus, $\operatorname{Sym}\left(F_e\, l_i\, F_e^{T} \right) = -\frac{1}{2} \mathcal{L}(b_e)$.

Assumption: ψ_{neq} is an isotropic tensor function.

$\Rightarrow$ τ_{neq} and b_e commute.

$\Rightarrow$ $\tau_{neq}\, b_e^{-1}$ is symmetric.

Finally, we demand.

$$-\tau_{neq} : \left(\tfrac{1}{2}\mathcal{L}(b_e)\, b_e^{-1}\right) \geq 0$$

A choice :

$$\boxed{-\tfrac{1}{2}\mathcal{L}(b_e)\, b_e^{-1} = \mathbb{V}^{-1} : \tau_{neq}}$$

→ viscosity tensor, isotropic, rank-4.

Numerical Integration :

Recall that $\chi_*^{-1}(b_e) = F^{-1} b_e F^{-T} = C_i^{-1}$

and $\dot{b}_e = \overline{F C_i^{-1} F^T}^{\bullet} = l b_e + b_e l^T + F \dot{\overline{C_i^{-1}}} F^T$

$$= l b_e + b_e l^T + \mathcal{L}(b_e).$$

Idea : from t_n to t_{n+1}, we deform elastically first, then correct its internal state viscously.

Step 1. elastic predictor

$$(b_e)_{trial} = F_{n+1}\, C_{i\,n}^{-1}\, F_{n+1}^T$$

Step 2. Viscous corrector

l is assumed to be zero.

$$\Rightarrow \quad \dot{b}_e = \mathcal{L}(b_e) = -\left(2\,\mathbb{V}^{-1} : \tau_{neq}\right) b_e$$

$$\Rightarrow \quad b_e = \exp\left(\int_{t_n}^{t_{n+1}} -2\,\mathbb{V}^{-1} : \tau_{neq}\, dt\right) b_{e\,trial}.$$

$$b_{e\,n+1} = \exp\left(-2\underbrace{(t_{n+1}-t_n)}_{\Delta t}\,\mathbb{V}^{-1} : \tau_{neq\,n+1}\right) b_{e\,trial}.$$

$\mathbb{V}^{-1}$ is isotropic:

$$\mathbb{V}^{-1} = \frac{1}{2\eta_D}\left(\mathbb{I} - \frac{1}{3} I \otimes I\right) + \frac{1}{9\eta_V} I \otimes I.$$

$$\lambda^2_{e_a\,n+1} = \exp\left(-\Delta t\left(\frac{1}{\eta_D}\operatorname{dev}(\tau_a) + \frac{2}{9\eta_V}\operatorname{tr}(\tau_{neq})\right)\right)\lambda^2_{a\,trial}$$

take $\ln$

$$\Rightarrow \quad \underline{\underline{\ln \lambda_{e\,a\,n+1}}} = -\Delta t\left(\frac{1}{2\eta_D}\underline{\underline{\operatorname{dev}(\tau_a)}} + \frac{1}{9\eta_V}\operatorname{tr}(\underline{\underline{\tau_{neq}}})\right) + \ln \lambda_{a\,trial}$$

unknowns at t_{n+1}

Remark: The integration rule for b_e is only first order accurate.

Remark: At each quadrature point, one need to solve the nonlinear eqn. for the principal values of b_e and τ_{neq} to determine the deformation state at t_{n+1}.