Homework 4
Due: Apr. 10 2023

1. Let $\boldsymbol{U}_\epsilon = \boldsymbol{u} + \epsilon \boldsymbol{w}$, where $\epsilon \in \mathbb{R}$. Note every member of the trial solution space S can be represented in the form of $\boldsymbol{U}_\epsilon$. for some test function $\boldsymbol{w} \in \boldsymbol{\mathcal{V}}$ and $\epsilon \in \mathbb{R}$. Define the potential energy by

$$I(\boldsymbol{U}_\epsilon) = \frac{1}{2} a(\boldsymbol{U}_\epsilon, \boldsymbol{U}_\epsilon) - (\boldsymbol{U}_\epsilon, \boldsymbol{l}) - (\boldsymbol{U}_\epsilon, \boldsymbol{h})_\Gamma,$$

Where $a(\cdot,\cdot)$ is the bilinear form for **linear elasticity**. Establish the following results.

a) The potential energy is stationary (i.e., $(dI(\boldsymbol{U}_\epsilon)/d\epsilon)|_{\epsilon=0} = 0$) if and only if the variational equation for linear elasticity is satisfied.

b) The potential energy is minimized at $\boldsymbol{u}$, that is, $I(\boldsymbol{u}) \leq I(\boldsymbol{u} + \epsilon \boldsymbol{w})$ for all $\boldsymbol{w} \in \boldsymbol{\mathcal{V}}$ and $\epsilon \in \mathbb{R}$. (*Hint*: Use part (a) and show that $(d^2 I(\boldsymbol{U}_\epsilon)/d\epsilon^2)|_{\epsilon=0} = a(\boldsymbol{w}, \boldsymbol{w}) \geq \boldsymbol{0}$.)

c) The approximate solution overestimates the potential energy, i.e., $I(\boldsymbol{u}^h) \geq I(\boldsymbol{u})$. (*Hint*: This follows immediately from $\boldsymbol{S}^h \subset \boldsymbol{S}$.)

2. Consider the uniform deformation given by the placement mapping

$$\varphi(X) = \begin{pmatrix} \frac{1}{4}(18 + 4X_1 + 6X_2) \\ \frac{1}{4}(14 + 6X_2) \end{pmatrix}.$$

a) Draw the deformed state of the body $\Omega_X = (-1,1)^2$.

b) Calculate $\boldsymbol{F}$, $\boldsymbol{F}^{-1}$, $\boldsymbol{F}^{\boldsymbol{T}}$, and $\boldsymbol{F}^{-\boldsymbol{T}}$.

c) Consider the vectors $(1,0)^T$ and $(0,1)^T$ in the initial configuration. Obtain their deformed counterparts in the current configuration.

3. Consider an $n \times n$ invertible matrix $\boldsymbol{A}$. Determine

$$\frac{\partial A^{-1}}{\partial A}, \quad \frac{\partial \det(A)}{\partial A}, \quad \frac{\partial \mathrm{cof}(A)}{\partial A}, \quad \frac{\partial \mathrm{tr}(A^2)}{\partial A}, \quad \frac{\partial \mathrm{dev}(A)}{\partial A}.$$

4. Let

$$\Phi(\boldsymbol{C}) = \frac{c}{2}\left(\mathrm{tr}(\widetilde{\boldsymbol{C}}) - 3\right) - \kappa(J\ln(J) - J + 1),$$

where c and κ constant shear and bulk moduli, respectively. Determine

$$S_{IJ}, \quad C_{IJKL}, \quad P_{iI}, \quad \sigma_{ij}, \quad A_{iIjJ}.$$