1.) If $f: R \to R$, and $f(x) = 4x^3 - 2$
show f is inj. and surj.

1a (injectivity).)
assume two xs where $x_1 = x_2$ within R
so that $f(x_1) = f(x_2)$

which means

$$f(x_1) = 4x_1^3 - 2 = f(x_2) = 4x_2^3 - 2$$

so that

$$4x_1^3 - 2 = 4x_2^3 - 2$$

$$= \quad x_1^3 = x_2^3 \quad \text{(add +2, /4 both sides)}$$

$$= \quad x_1 = x_2 \quad (\sqrt[3]{\ } \text{ both sides})$$

since $x_1 = x_2$, f is injective

1b (surjectivity).)

By definition of surjectivity,
assuming

$$f(x) = y$$

$$= \quad 4x^3 - 2 = y$$

$$= \quad x^3 = \frac{y+2}{4}$$

$$= \quad x = \sqrt[3]{\frac{y+2}{4}}$$

So there exists an x for each y in R

meaning, f is surjective.

2.)

given:

$$S = P(\mathbb{R})$$

$$f : \mathbb{R} \to S$$

$$f(x) = \{y \in \mathbb{R} : y^2 < x\}$$

meaning, $y = (\sqrt{x}, -\sqrt{x})$ only if $x \geq 0$

if $x < 0$, teere exists no y that can make $x < 0$ happen (empty set)

2a.) If $x_1 = -1$ and $x_2 = -2$,

$f(x_1) =$ empty set

$f(x_2) =$ empty set

since $f(x_1) = f(x_2)$,

f is not injective.

2b.) For example,

the set $\{3\}$ in S, or $P(R)$,

cannot be derived at all

given the function

$$f(x) = \{y \in R : y^2 < x\}$$

that only yields

$y = (\sqrt{x}, -\sqrt{x})$ only if $x \geq 0$ or

an empty set if $x < 0$

meaning,

every element of S or $P(R)$

will not be mapped.

Thus, f is not surjective.

3.) $f: A \to B$ and $g: B \to C$

so $g \circ f = A \to C$

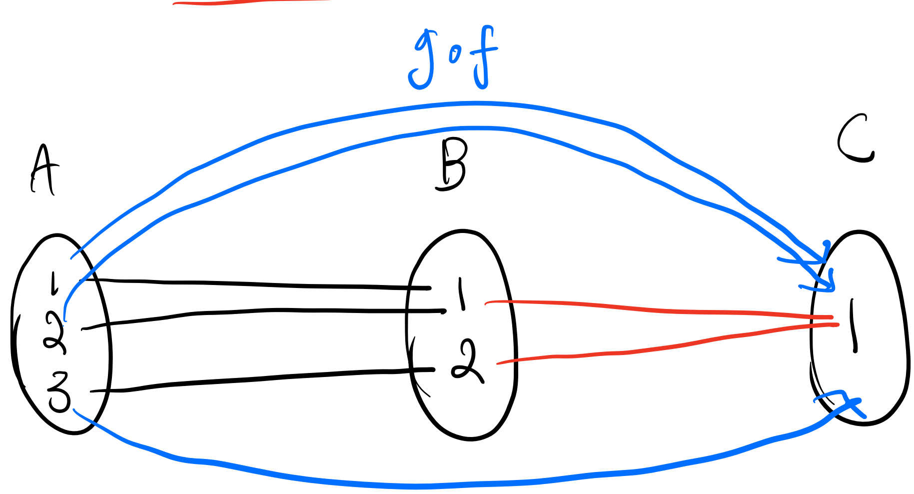
3a.) if f surj, g is not inj, $g \circ f$ is not injective

for example:

since f: surj, g is not inj

1. every element of B is mapped at least once

2. elements in C are mapped more than once



Since 3 elements in A are all mapped to 1 value in C,
$g \circ f$ which $= A \to C$
is not injective.

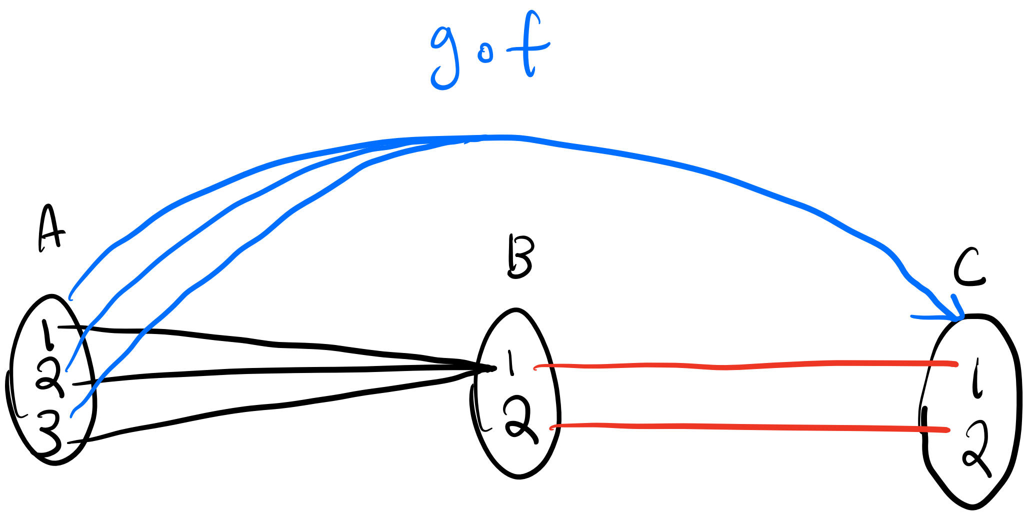
3b.) if f is not surj,
g is inj,
$g \circ f$ is not surjective

for example:

since f : not surj, g is inj

1.) every element in B is not mapped at least once

2.) elements in C are mapped at most once



Since every element in C is not mapped at least once in g∘f (A→C), g∘f is not surjective.

4.) Since $g, h : A \longrightarrow B$,

so $f \circ g = f \circ h$

Since f is injective,

there exists $f(x) = f(y)$

and $x = y$

meaning,

$$f \circ h\,(x) = f \circ g\,(x)$$

for some x

$$\equiv f(h(x)) = f(g(x))$$

$$\equiv h(x) = g(x)$$

therefore, $h = g$

which is $g = h$.

5.) for any $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

if $f(n)$, and $n = 0$

$$0^2 = \frac{0(0+1)(2(0)+1)}{6}$$

$\Rightarrow$ $0 = 0$ ✓ so $f(0)$ is true

based on this,

assume: $f(x)$ is true,

$$\sum_{k=0}^{x} k^2 = \frac{x(x+1)(2x+1)}{6}$$

then $f(x+1)$

$$\sum_{k=0}^{x+1} k^2 = \sum_{k=0}^{x} k^2 + (x+1)^2$$

$$= \frac{x(x+1)(2x+1)}{6} + (x+1)^2$$

$$= \frac{x(x+1)(2x+1) + 6(x+1)^2}{6}$$

$$= \frac{(x+1)\left(x(2x+1)+6(x+1)\right)}{6}$$

$$= \frac{(x+1)(x+2)(2(x+1)+1)}{6}$$

which is the same as

-1

$$\frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$

if $(n+1)$ were to be inserted into

$$\frac{n(n+1)(2n+1)}{6}$$

instead of n,
it would yield

② $$\frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$

since ① = ②

meaning, $f(n+1)$ is true
if $f(n)$ is true.

$$n! = \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) = n+1$$

6.) for any $n \in \mathbb{N}$ where $n \geq 1$

$$\prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) = n + 1$$

$f(n)$, $n = 1$

$$1 + \frac{1}{1} = 1 + 1$$

$$2 = 2$$

so $f(n)$ is true when $n = 1$

if $f(n+1)$

$$\prod_{k=1}^{n+1}\left(1+\frac{1}{k}\right)=\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)\cdot\left(1+\frac{1}{k+1}\right)$$

$$=(k+1)\left(1+\frac{1}{k+1}\right)$$

$$=k+\frac{k}{k+1}+1+\frac{1}{k+1}$$

$$=k+1+\frac{k+1}{k+1}$$

$$=k+1+1$$

$$=k+2$$

which is equivalent to

$= n+2$ ①

if $n+1$ were to be inserted into

$(n+1)$,

it would yield

② $n+2$

since ① = ②

meaning, $f(n+1)$ is true

if $f(n)$ is true.

7.)

The pattern of each row of pascal's triangle from row 1 - 6 are tee following:

row	sum
1	2
2	4
3	8
4	16
5	32
6	64

and tee change of sum in each row can be denoted by

$$\underline{\underline{2^{k}}}.$$

Using binomial theorem,

$$(x+y)^{n} = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k} = \sum_{k=0}^{n} \binom{n}{k} x^{k} y^{n-k}$$

we can replace:

$$(x+y)^{n} = 2^{k}$$

by making $x = 1$ and $y = 1$

and $n = k$

which would yield

$$2^k = \binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \dots$$
$$+ \binom{k}{k-2} + \binom{k}{k-1} + \binom{k}{k}$$

Meaning,

$$2^k = {}^kC_0 + {}^kC_1 + {}^kC_2 + \dots$$
$$+ {}^kC_{k-2} + {}^kC_{k-1} + {}^kC_k$$

So using this, if we try 2^6 where $k = 6$, it would yield 64, which is the sum of k^{th} row of Pascal's triangle.

Thus, 2^k is the formula for the sum of row k of Pascal's Triangle

8.) Let P_n be the nth pentagonial number.

attempting to prove
$$P_n = \frac{3n^2 - n}{2}$$ via induction

The first 5 pentagonial numbers are:

1, 5, 12, 22, 35

P(1) P(2) P(3) P(4) P(5)

as seen in the first 5 numbers,

$3n - 2$ points gets added

every iteration

1 thus,

$$P_n = P_{n-1} + 3n - 2$$

The first 5 pentagonical numbers are:

1, 5, 12, 22, 35

P(1) P(2) P(3) P(4) P(5)

if P(1)

$$P(1) = \frac{3(1)^2 - 1}{2} = 1$$

1

and

① $P_1 = P_{1-1} + 3(1) - 2$

$= 0 + 3 - 2 = 1$ ②

so since ① = ②

P(1) is proven.

and assuming for some p(k),

$$P(k) = \frac{3k^2 - k}{2}$$ is true,

Proving p(k+1)

① $P_{k+1} = P_k + 3(k+1) - 2$

$$= \frac{3k^2 - k}{2} + 3(k+1) - 2$$

$$= \frac{3k^2 - k + 6(k+1) - 4}{2}$$

$$= \frac{3k^2 + 5k + 2}{2}$$

$$= 3k^2 + 6k + 3 - (k+1)$$

$$= \frac{3(k+1)^2 - (k+1)}{2}$$

which is the equivalent of

$$= \frac{3(n+1)^2 - (n+1)}{2}$$

which is,

$$P(n) = \frac{3n^2 - n}{2}$$

if $n+1$ was inserted

$$P(n+1) = \frac{3(n+1)^2 - (n+1)}{2}$$

hence $P(n+1)$ is true when $P(n)$ is true.

and thus

$$P_n = \frac{3n^2 - n}{2}$$

is proven.

9.) the inductive proof is wrong as it involves the use of specific "horses in the middle," and if $n=2$ and there are

only two horses,
there are no
"horses in the middle"
which debunks the proof as
first and last horse
will not necessarily
be the same color
horses.

10.) prove for all $n \geq 1$,
it's possible to lay
tee tiles down to cover

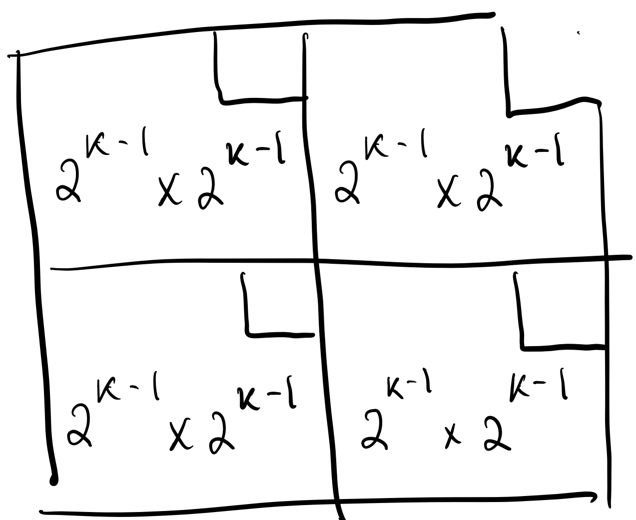
for base case:
$2^n \times 2^n$ where

$n=1$, we have
covered tee entire
board except one
corner square left
uncovered with one
L-shaped tile.

so base case is proven.

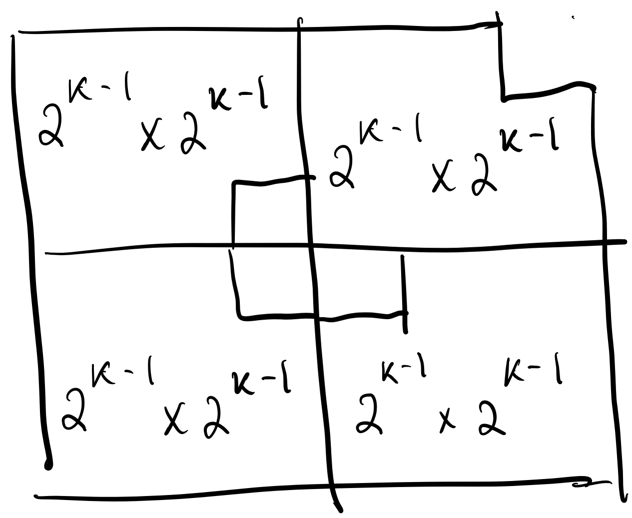
Assuming some number k,

Now for some $2^k \times 2^k$ board, there are four smaller boards, each with a size of $2^{k-1} \times 2^{k-1}$.

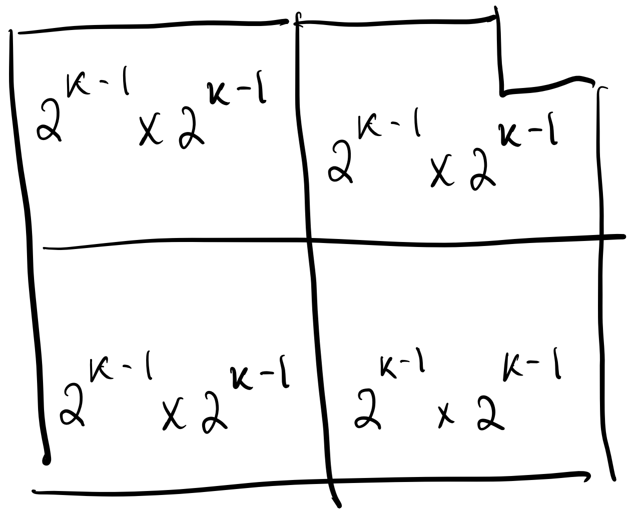


and has a one corner square missing.

now, if three of the four quadrants rotate such that...



and we can cover the L-shaped space in the middle with an L-shaped tile, leaving us with:



which proves that for all $n \geq 1$ it is possible to lay down L-shaped tiles to cover a $2^n \times 2^n$ square, with one corner square left uncovered.