1.) prove:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

- assumption 1: $A \cup (B \cap C)$

- if $\underline{x \in A}$, then $x \in A \cup B$ and $x \in A \cup C$
$\equiv$ $x \in (A \cup B) \cap (A \cup C)$

- if $x \in B \cap C$, then $x \in B$, thus $x \in A \cup B$
- if $x \in B \cap C$, then $x \in C$, thus $x \in A \cup C$
$\equiv$ $x \in (A \cup B) \cap (A \cup C)$

Based on such statements, we can draw out:
if $A \cup (B \cap C)$, then $x \in (A \cup B) \cap (A \cup C)$

- assumption 2: $(A \cup B) \cap (A \cup C)$

- if $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, thus we can conclude $x \in A \cup (B \cap C)$

- if $x \notin A$, then $x \in B$ and $x \in C$, thus $x \in B \cap C$, meaning we can introduce $x \in A \cup (B \cap C)$.

Based on such statements, we can draw out:

if $(A \cup B) \cap (A \cup C)$, then $x \in A \cup (B \cap C)$

$\Rightarrow$ Combine each conclusion from both assumption 1 and assumption 2:

if $A \cup (B \cap C)$, then $x \in (A \cup B) \cap (A \cup C)$

$+$

if $(A \cup B) \cap (A \cup C)$, then $x \in A \cup (B \cap C)$

so we get

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

2.) Suppose $A \subseteq B$ and $C \subseteq D$,

Show $A \times C \subseteq B \times D$

Cartesian product:

$$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$$

meaning for any $(x, y) \in A \times C =$

$x \in A \wedge y \in C$

Since $A \subseteq B$ and $C \subseteq D$, and $x \in A$, so $x \in B$
Since $A \subseteq B$ and $C \subseteq D$, and $y \in C$, so $y \in D$
Meaning, if $(x,y) \in A \times C$, then $(x,y) \in B \times D$

thus, $A \times C \subseteq B \times D$ given $A \subseteq B$ and $C \subseteq D$

3.) Prove that $A - (B \cap C) = (A - B) \cup (A - C)$
- assumption 1: $A - (B \cap C)$
- if $x \in A - (B \cap C)$, then $x \in A \wedge x \notin B \cap C$
  meaning $x \in A \wedge (x \notin B \vee x \notin C)$.
  Therefore, $\Rightarrow$ $x \in A$ and $x \notin B$, so $x \in A - B$
  $x \in A$ and $x \notin C$, so $x \in A - C$
  when put together, $x \in (A-B) \cup (A-C)$
  in conclusion, if $A - (B \cap C)$, then $(A-B) \cup (A-C)$

- assumption 2: $(A - B) \cup (A - C)$
- if $x \in (A-B) \cup (A-C)$, then it means:
  if $x \in (A-B)$ $\vee$ if $(A-C)$

$x \in A$, but $x \notin B$ — $x \in A$, but $x \notin C$

$\Rightarrow$ $x \in A$, but $x \notin$ (B and C)

$\equiv \quad x \in A$ and $x \notin (B \cap C)$

thus $\quad x \in A - (B \cap C)$

in conclusion, if $(A - B) \cup (A - C)$,

then $A - (B \cap C)$

Put conclusion from assumption 1 and 2 together, where

if $A - (B \cap C)$, then $(A - B) \cup (A - C)$

$+$

in conclusion, if $(A - B) \cup (A - C)$,

then $A - (B \cap C)$

means that $A - (B \cap C) = (A - B) \cup (A - C)$

$$A - (B \cap C) = (A - B) \cup (A - C)$$

4.) An ordered pair $(a, b)$ can be defined as the set $\{\{a\}, \{a, b\}\}$.

Show that $(a, b) = (c, d)$ iff

$a = c$ and $b = d$

Based on the definition of ordered pair,

Pair $(a, b)$ makes the set $\{\{a\}, \{a, b\}\}$.

Pair $(c, d)$ makes the set $\{\{c\}, \{c, d\}\}$.

So if $(a, b) = (c, d)$, then

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$$

given $(a, d)$,

$\Rightarrow$ if $a = b$, then it makes $\{\{b\}, \{b, d\}\}$

$\Rightarrow$ if $a = d$, then it makes $\{\{d\}, \{d, d\}\}$

$\Rightarrow$ if $a = c$, then it makes $\{\{c\}, \{c, d\}\}$

meaning, ONLY if $a=c$, $\{\{c\}, \{c, d\}\}$

given $(c, b)$,

$\Rightarrow$ if $b = a$, then it makes $\{\{c\}, \{c, a\}\}$

$\Rightarrow$ if $b = c$, then it makes $\{\{c\}, \{c, c\}\}$

$\Rightarrow$ if $b = d$, then it makes $\{\{c\}, \{c, d\}\}$

meaning, ONLY if $b=d$, $\{\{c\}, \{c, d\}\}$

since ONLY if $a=c$, $\{\{c\}, \{c, d\}\}$

$+$

ONLY if $b=d$, $\{\{c\}, \{c, d\}\}$

$\Rightarrow$ if and only if $a=c$ and $b=d$, $\{\{c\}, \{c, d\}\}$

5.) Given an arbituary relation R, suppose we compute two relations:

- $R_1$, the reflexive closure of the transitive closure of R
- $R_2$, the transitive closure of the reflexive closure of R

Prove or disprove: $R_1 = R_2$ for all R

Since both reflexive closure and transitive closure adds minimum amount of pairs required to satisfy its respective requirements to the set by definition, doing a closure of a closure on the same set R would produce the same result no matter of which closure occurs first.
For example,
assuming

$R = \{ (1,2), (2,3), (3,4) \},$

- the transitive closure of R is
  $R^t = \{ (1,2) (2,3) (3,4), (1,3), (2,4), (1,4) \}$
  thus $R_1$ (reflexive closure of $R^t$) is

$R_1 = \{ (1,1) (2,2), (3,3), (4,4), (1,2), (2,3), (3,4)$

$(1,3),(2,4),(1,4)\}$

- the reflexive closure of R is

$R^R \{ (1,1),(2,2),(3,3),(4,4),(1,2),(2,3),(3,4)\}$

thus $R_2$ (transitive closure of $R^R$) is

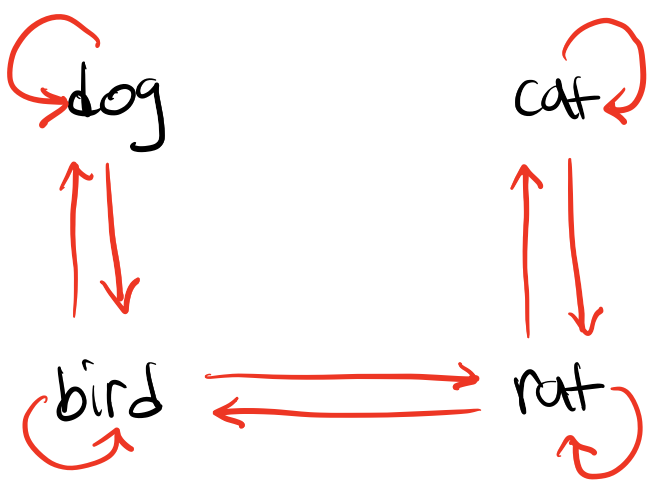
$R_2 = \{(1,1)(2,2),(3,3),(4,4),(1,2),(2,3),(3,4)$
$(1,3),(2,4),(1,4)\}$

thus $R_1 = R_2$ for all $R$

6.) Let $A = \{$ cat, dog, bird, rat $\}$
and R be a relation on A defined by

$\{(x,y) \mid x$ and $y$ have at least one letter in common$\}$

a.)



b.)

$R = \{ (cat, cat), (dog, dog), (bird, bird),$
$(rat, rat), (cat, rat), (rat, cat),$
$(dog, bird), (bird, dog), (bird, rat)$
$(rat, bird) \}$

- Reflexive: since $\forall x \ (x,x) \in R$, R is reflexive.

- Symmetric: since $(x,y) \in R \rightarrow (y,x) \in R$,
  R is symmetric.

- Transitive:
  by definition $(a,b) \in R \wedge (b,c) \in R \rightarrow$
  $(a,c) \in R$

  since $(cat, rat) \wedge (rat, bird) \in R$
  but $(cat, bird) \notin R$,
  R is not transitive.

7.) Given a relation $R$ on set $A$, Prove that if $R$ is transitive, so is $R^{-1}$

- Definition of inverse relations:
  If $R$ is from $x$ to $y$, then $R^{-1}$ is from $y$ to $x$, and

$$R^{-1} = \{(y, x) : (x, y) \in R\}$$

So, assume $x, y, z \in A$ and $(x, y) \in R^{-1} \wedge (y, z) \in R^{-1}$.

By definition of inverse relations,

$(x, y) \in R^{-1} \rightarrow (y, x) \in R$

$(y, z) \in R^{-1} \rightarrow (z, y) \in R$

given $(y, x) \in R$ and $(z, y) \in R$, and $R$ is transitive, $(z, x) \in R$

Since we have $(z, x) \in R$ and by definition of inverse relations, we have $(x, z) \in R^{-1}$. meaning,

$(z, x) \in R \rightarrow (x, z) \in R^{-1}$

Since we know
$(x, y) \in R^{-1}$ and $(y, z) \in R^{-1}$
from assumption and we have
$(x, z) \in R^{-1}$,

by definition of transivity,

$(x, y) \in R^{-1} \wedge (y, z) \in R^{-1} \rightarrow (x, z) \in R^{-1}$

$R^{-1}$ is transitive.
therefore, $R^{-1}$ is transitive if $R$ is transitive.

8.) Suppose $R$ and $S$ are symmetric relations on a set $A$. Prove that $R \circ S$ is symmetric iff $R \circ S = S \circ R$

assume: $a, b, c \in A$ and $R \circ S = S \circ R$

- if $(a, c) \in R \circ S$, then
  $\exists b \; (a, b) \in R \; (b, c) \in S$

Knowing R and S are symmetric, we know
$(c, b) \in S$ and $(b, a) \in R$
meaning,
$(c, a) \in S \circ R = (c, a)\ R \circ S$

and since we have $(c, a)\ R \circ S$
$(a, c) \in R \circ S \longrightarrow (c, a)\ R \circ S$
which satisfies $R \circ S$ symmetric requirement.

- Knowing $R \circ S$ is symmetric,
  if we assume
  $(c, a) \in R \circ S \leftrightarrow (a, c) \in R \circ S$
  then
  $(a, c) \in R \circ S \longleftrightarrow (c, a) \in S \circ R$
  which concludes
  $R \circ S \subseteq S \circ R$

on the other hand, knowing $R \circ S$ is symmetric
if we assume
$(a, c)\ S \circ R \leftrightarrow (c, a) \in R \circ S$
then
$(c, a) \in R \circ S \longleftrightarrow (x, c) \in R \circ S$
which concludes
$S \circ R \subseteq R \circ S$

since we have
$R \circ S \subseteq S \circ R$ and $S \circ R \subseteq R \circ S$,
we now know
$$R \circ S = S \circ R$$

9.)

Symmetry : if $(a,b) \in R \rightarrow (b,a) \in R$
Anti-Symmetric : if $(a,b) \in R \wedge (b,a) \in R \rightarrow a=b$

if we have some $R = \{(x,y) \mid x=y\}$,
since we know $x=y$,
if $(x,y) \in R$, then $(y,x) \in R$.
So by definition, $R$ is Symmetric since
$$\underline{(x,y) \in R} \longrightarrow \underline{(y,x) \in R}$$
at the same time,
given $x=y$ means $\underline{(x,y) \in R} \wedge \underline{(y,x) \in R}$
$R$ is antisymmetric as
$$(x,y) \in R \wedge (y,x) \in R \longrightarrow x=y$$

10.) Find all equivalence relations on $\{1, 2, 3\}$

given $A = \{1, 2, 3\}$
and its cartesian product $(A \times A)$ to make $R$,
$R = \{(1,1), (2,2), (3,3),$
$(1,2), (2,1), (2,3),$
$(3,2), (1,3), (3,1)\}$

equivalence relation requires the relation to be reflexive, symmetric, and transitive.

$R_1 = \{(1,1), (2,2), (3,3), (1,2), (2,1), (1,3),$
$(3,1), (2,3), (3,2)\}$
satisfies equivalence ✓

$R_2 = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$
satisfies equivalence ✓

$R_3 = \{(1,1), (2,2), (3,3), (1,3), (3,1)\}$
satisfies equivalence ✓

$R_4 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$
satisfies equivalence ✓

$R_5 = \{(1,1), (2,2), (3,3\}$

satisfies equivalence ✓

all other combinations of sets on R does not satisfy equivalence, thus there are 5 equivalence relations.