# Analysis of the Random Walk with drift

Juan Garbayo

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# Random Walk Model (with drift)

$$x_t = \delta + \underbrace{x_{t-1}}_{} + w_t$$

## Recursive substitution

$$x_t = \delta + (\delta + x_{t-2} + w_{t-1}) + w_t =$$

$$= 2\delta + (\delta + x_{t-3} + w_{t-2}) + \underbrace{w_{t-1} + w_t}_{}$$

$3\delta$

$$= \dots$$

$$= t\delta + x_{t-t} + w_{t-(t-1)} + \underbrace{w_{t-(t-2)} + \dots + w_{t-1} + w_t}_{}$$

$$= \delta \cdot t + x_0 + \sum_{j=1}^{t} w_t$$

We proved that: $x_t = x_{t-1} + w_t = x_0 + \delta t + \sum_{j=1}^{t} w_t$

proven

Starting point of the random walk. We may set our coordinate system so that $x_0 = 0$

Without loss of generality

$$\boxed{x_t = x_{t-1} + w_t = \delta t + \sum_{j=1}^{t} w_t}$$

# RANDOM WALK MODEL → PROOF OF NON STATIONARITY

$$x_t = \cancel{x_0} + \cancel{\delta} t + \sum_{j=1}^{t} w_t \; ; \; Cov(x_s, x_t) = Cov\left[\sum_{j=1}^{s} w_s, \sum_{j=1}^{t} w_t\right] = \min\{s,t\} \cdot \sigma_w^2$$

$0$

ASSUME NO DRIFT

**NOTE:** by definition of w. noise: $cov(w_s, w_t) = \begin{cases} \sigma_w^2 & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases}$

PROPERTY: COVARIANCE OF LINEAR COMBINATIONS.

$$cov\left(\sum_{j=1}^{s} w_j, \sum_{k=1}^{t} w_k\right) = \sum_{j=1}^{s} \sum_{k=1}^{t} cov(w_j, w_k) =$$

$$= cov(w_1, w_1) + cov(\cancel{w_1}, w_2) + \dots + cov(\cancel{w_1}, w_t) +$$
$$+ cov(\cancel{w_2}, w_1) + cov(w_2, w_2) + \dots + cov(\cancel{w_2}, w_t) +$$
$$+ \dots +$$
$$+ cov(\cancel{w_s}, w_1) + cov(\cancel{w_s}, w_2) + \dots + \underbrace{cov(w_s, w_t)}_{0 \text{ if } s \neq t} =$$

$s$ rows

$t$ summands per row.

$$= \underbrace{cov(w_1, w_1)}_{\sigma_w^2} + \underbrace{cov(w_2, w_2)}_{\sigma_w^2} + \dots + \overbrace{cov(w_k, w_k)}^{\sigma_w^2} =$$

$k$ → $\min\{s,t\}$

$$= \underbrace{\min\{s,t\}}\cdot \sigma_w^2$$

→ DEPENDS on $s$ and $t$ and not solely on $|s-t|$ → $\boxed{\text{NOT STATIONARY}}$

# SOME MORE PROPERTIES OF THE RANDOM WALK:

## ① VARIANCE OF THE PROCESS INCREASES OVER TIME:

We have just proven: $Cov(x_s, x_t) = \min\{s, t\} \cdot \sigma_w^2$

For $s = t \rightarrow VAR(x_t) = \min\{t, t\} \cdot \sigma_w^2 = t \cdot \sigma_w^2$

$$VAR(x_t) = t \cdot \sigma_w^2$$

Another way to think of this:

$w_j$ are independent

$$VAR(x_t) = VAR\left[\delta t + \sum_{j=1}^{t} w_j\right] = VAR\left[\sum_{j=1}^{t} w_j\right] = \sum_{j=1}^{t} VAR(w_j) = t\sigma_w^2$$

$\delta t$ is deterministic.
/It is not a R.V.

## ② EXPECTATION OF THE RW

Linearity of E[]

$$E[x_t] = E\left[x_0 + \delta t + \sum_{j=1}^{t} w_j\right] = \underbrace{E[x_0]}_{\substack{x_0 \\ \text{(deterministic)}}} + \underbrace{E[\delta t]}_{\substack{\delta t \\ \text{(deterministic)}}} + \underbrace{E\left[\sum_{j=1}^{t} w_j\right]}_{=0}$$

LINEARITY OF EXPECTATION AND $E[w_j] = 0 \ \forall j$

$$\boxed{E[x_t] = x_0 + \delta t}$$

# Some properties of the variance and the covariance used before

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## COVARIANCE

$$\text{Cov}[X,Y] = E\Big[(\underbrace{X - E[X]}_{\tilde{X}})(\underbrace{Y - E[Y]}_{\tilde{Y}})\Big] = E[\tilde{X}\tilde{Y}]$$

$\tilde{X} = X - E[X]$, $\tilde{Y} = Y - E[Y]$ } → Deviations from their respective means.

ALTERNATIVE FORMULA (VERY USED)

$$\text{Cov}[X,Y] = E[XY] - E[X]E[Y]$$

Proof: simply develop formula

$$E\big[(X - E[X])(Y - E[Y])\big] = E\big[XY - XE[Y] - YE[X] + E[X]E[Y]\big]$$

$$= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] =$$

$$= E[XY] - E[X]E[Y]$$

COVARIANCE AND VARIANCE

$$\text{Cov}[X,X] = E\big[(X - E[X])^2\big] = \text{Var}(X)$$

SYMMETRY OF COVARIANCE:

$$\text{Cov}[X,Y] = \text{Cov}[Y,X] \rightarrow \text{follows from definition}$$

$$\text{Cov}[Y,X] = E\big[(Y - E[Y])(X - E[X])\big] = E\big[(X - E[X])(Y - E[Y])\big] = \text{Cov}(X,Y)$$

# BILINEARITY OF THE COVARIANCE

FROM REFERENCE [1]

The covariance operator is linear in both of its arguments.

**Proposition 122** *Let $a_1$ and $a_2$ be two constants. Let $X_1$, $X_2$ and $Y$ be three random variables such that* $\mathrm{Cov}[X_1, Y]$ *and* $\mathrm{Cov}[X_2, Y]$ *exist and are well-defined. Then,*

$$\mathrm{Cov}[a_1X_1 + a_2X_2, Y] = a_1\mathrm{Cov}[X_1, Y] + a_2\mathrm{Cov}[X_2, Y]$$

*and*

$$\mathrm{Cov}[Y, a_1X_1 + a_2X_2] = a_1\mathrm{Cov}[Y, X_1] + a_2\mathrm{Cov}[Y, X_2]$$

**Proof of 1** → Linearity of expectation

$$\mathrm{Cov}[a_1X_1 + a_2X_2, Y] = E\Big[\big(a_1X_1 + a_2X_2 - \overbrace{E(a_1X_1 + a_2X_2)}^{E[a_1X_1] + E[a_2X_2]}\big)\big(Y - E[Y]\big)\Big] =$$

$$= E\Big[\big(a_1X_1 - E[a_1X_1]\big)\big(Y - E[Y]\big) + \big(a_2X_2 - E[a_2X_2]\big)\big(Y - E[Y]\big)\Big] =$$

$$= E\Big[a_1\big(X_1 - E[X_1]\big)\big(Y - E[Y]\big) + a_2\big(X_2 - E[X_2]\big)\big(Y - E[Y]\big)\Big] =$$

$$= a_1 \underbrace{E\Big[\big(X_1 - E[X_1]\big)\big(Y - E[Y]\big)\Big]}_{\mathrm{Cov}(X_1, Y)} + a_2 \underbrace{E\Big[\big(X_2 - E[X_2]\big)\big(Y - E[Y]\big)\Big]}_{\mathrm{Cov}(X_2, Y)} =$$

**Proof of 2**

Symmetry $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$

$$\mathrm{Cov}(Y, a_1X_1 + a_2X_2) = \mathrm{Cov}(a_1X_1 + a_2X_2, Y) =$$

$$= a_1\mathrm{Cov}(X_1, Y) + a_2\mathrm{Cov}(X_2, Y).$$

# COVARIANCE OF LINEAR COMBINATIONS

$$\text{(1)} \quad \text{Cov}\left[\sum_{i=1}^{n} a_i X_i, Y\right] = \sum_{i=1}^{n} a_i \text{Cov}[X_i, Y]$$

$$\text{(2)} \quad \text{Cov}\left[Y, \sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i \text{Cov}[Y, X_i]$$

Reference [1]

**Proof of (1)** → apply bilinearity recursively.

- Bilinearity (proven before):

$$Cov(a_1 X_1 + a_2 X_2, Y) = a_1 Cov(X_1, Y) + a_2 Cov(X_2, Y)$$

Then

$$Cov\left[\sum_{i=1}^{n} a_i X_i, Y\right] = Cov\left[a_1 \cdot X_1 + \sum_{i=2}^{n} a_i X_i, Y\right] = a_1 Cov(X_1, Y) + Cov\left[\sum_{i=2}^{n} a_i X_i, Y\right]$$

$$= a_1 Cov(X_1, Y) + Cov\left[a_2 \cdot X_2 + \sum_{i=3}^{n} a_i X_i, Y\right] = a_1 Cov(X_1, Y) + a_2 Cov(X_2, Y) + Cov\left[\sum_{i=3}^{n} a_i X_i, Y\right]$$

$$= \dots = a_1 Cov(X_1, Y) + a_2 Cov(X_2, Y) + \dots + a_n \cdot Cov(X_n, Y) =$$

$$= \sum_{i=1}^{n} a_i Cov(X_i, Y)$$

**Proof of (2)**: by symmetry.

$$Cov\left[Y, \sum_{i=1}^{n} a_i X_i\right] \underset{\text{Symmetry}}{=} Cov\left[\sum_{i=1}^{n} a_i X_i, Y\right] \underset{\text{by (1)}}{=} \sum_{i=1}^{n} a_i \cdot Cov(X_i, Y)$$

## Covariance of Linear Combinations:

$$Cov(aX+bY,\ cW+dZ) = a\cdot c\, Cov(X,W) + a\cdot d\, Cov(X,Z) + b\cdot c\, Cov(Y,W) + bd\, Cov(Y,Z)$$

More generally: (from reference [2])

*If the random variables*

$$U = \sum_{j=1}^{m} a_j X_j \quad \text{and} \quad V = \sum_{k=1}^{r} b_k Y_k$$

*are linear combinations of (finite variance) random variables* $\{X_j\}$ *and* $\{Y_k\}$*, respectively, then*

$$③ \quad \operatorname{cov}(U,V) = \sum_{j=1}^{m}\sum_{k=1}^{r} a_j b_k \operatorname{cov}(X_j, Y_k). =$$

$$= a_1\left[\sum_{k=1}^{r} b_k\cdot cov(X_1, Y_k)\right] + a_2\left[\sum_{k=1}^{r} b_k\, cov(X_2, Y_k)\right] + \dots + a_m\left[\sum_{k=1}^{r} b_k\, cov(X_m, Y_k)\right]$$

Proof of ③

$$Cov\left[\sum_{j=1}^{m} a_j X_j,\ \sum_{k=1}^{r} b_k\cdot Y_k\right] \underset{\text{by } ①}{=} \sum_{j=1}^{m} a_j\, Cov\left[X_j,\ \sum_{k=1}^{r} b_k\cdot Y_k\right] \underset{\text{by } ②}{=} \sum_{j=1}^{m} a_j\left[\sum_{k=1}^{r} b_k\cdot Cov(X_j, Y_k)\right]$$

q.e.d

[1] - Taboga, M. (n.d.). *Lectures on probability theory and mathematical statistics* (3rd ed.)

[2] - Shumway, R. H., & Stoffer, D. S. (2017). *Time series analysis and its applications with R examples* (4th ed.). Springer.

[3] - Evans, M. J., & Rosenthal, J. S. (2015). Probability and statistics: The science of uncertainty. University of Toronto.

NOTE: both references [1] and [3] are excellent to review / learn probability and statistics theory.

Reference 3 is freely available in this Link, with solutions manual and the entirety of the book available as pdf.

Reference 1 is available as an online book in this Link ie

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