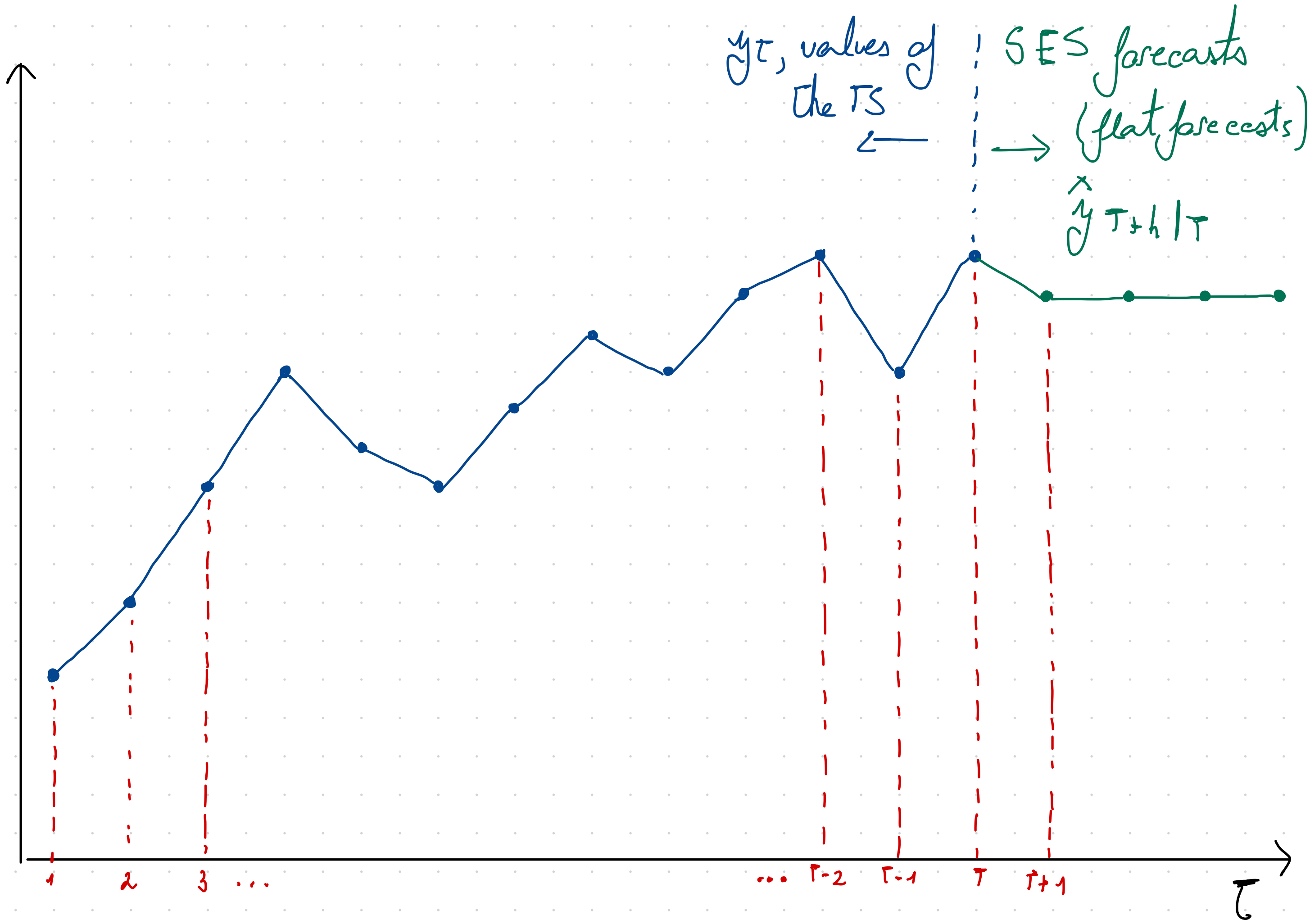
SIMPLE EXPONENTIAL SMOOTHING.



Geometric Succesion

→ 1, 2, 4, 8, 16, 32, 64... $r = 2$

→ 64, 32, 16, 8, 4, 2, 1... $r = 1/2$

GENERAL FORM

$$\left.\begin{aligned}
a_1 &= \\
a_2 &= r \cdot a_1 \\
a_3 &= r \cdot a_2 = r^2 \cdot a_1 \\
&\vdots \\
a_{n-1} &= r^{n-2} \cdot a_1 \\
a_n &= r^{n-1} \cdot a_1
\end{aligned}\right\} \quad r = \frac{a_n}{a_{n-1}} = r$$

Now a specific case $a_1 = \alpha \in (0, 1]$

$$r = \frac{a_n}{a_{n-1}} = (1-\alpha) \in (0, 1]$$

$$\begin{aligned}
a_1 &= \alpha \\
&\quad \cdot(1-\alpha) \\
a_2 &= \alpha(1-\alpha) \\
&\quad \cdot(1-\alpha) \\
a_3 &= \alpha(1-\alpha)^2 \\
&\vdots \\
a_n &= \alpha\underbrace{(1-\alpha)^{n-1}}
\end{aligned}$$

→ Because $1-\alpha < 1 \Rightarrow (1-\alpha)^{n-1} << 1$

Exponential decrease of a_n with $n \uparrow$

Sum of n terms of a geometric sequence

$$S_n = \alpha + \alpha(1-\alpha) + \alpha(1-\alpha)^2 + \dots + \overbrace{\alpha(1-\alpha)^{n-1}}^{a_n} \rightarrow (*)$$

$$(1-\alpha)\, S_n = \alpha(1-\alpha) + \alpha(1-\alpha)^2 + \alpha(1-\alpha)^3 + \dots + \alpha(1-\alpha)^n \rightarrow (**)$$

$$(*) - (**) = S_n - (1-\alpha)\, S_n = \alpha - \alpha(1-\alpha)^n$$

Rearranging: $\alpha S_n = \alpha - \alpha(1-\alpha)^n$

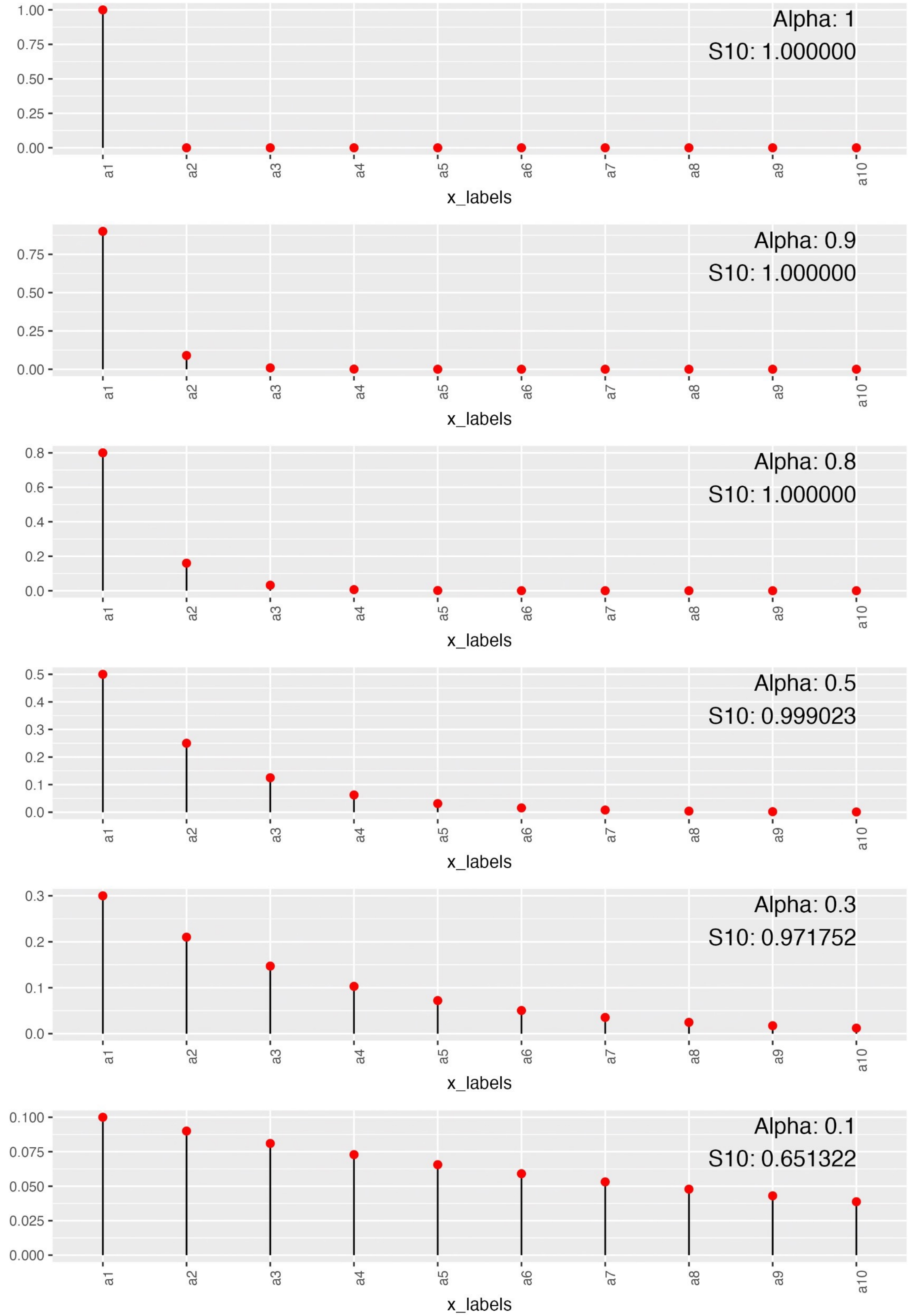
Since $\alpha \neq 0$ we may divide by α

$$\underbrace{S_n = 1 - (1-\alpha)^n}$$

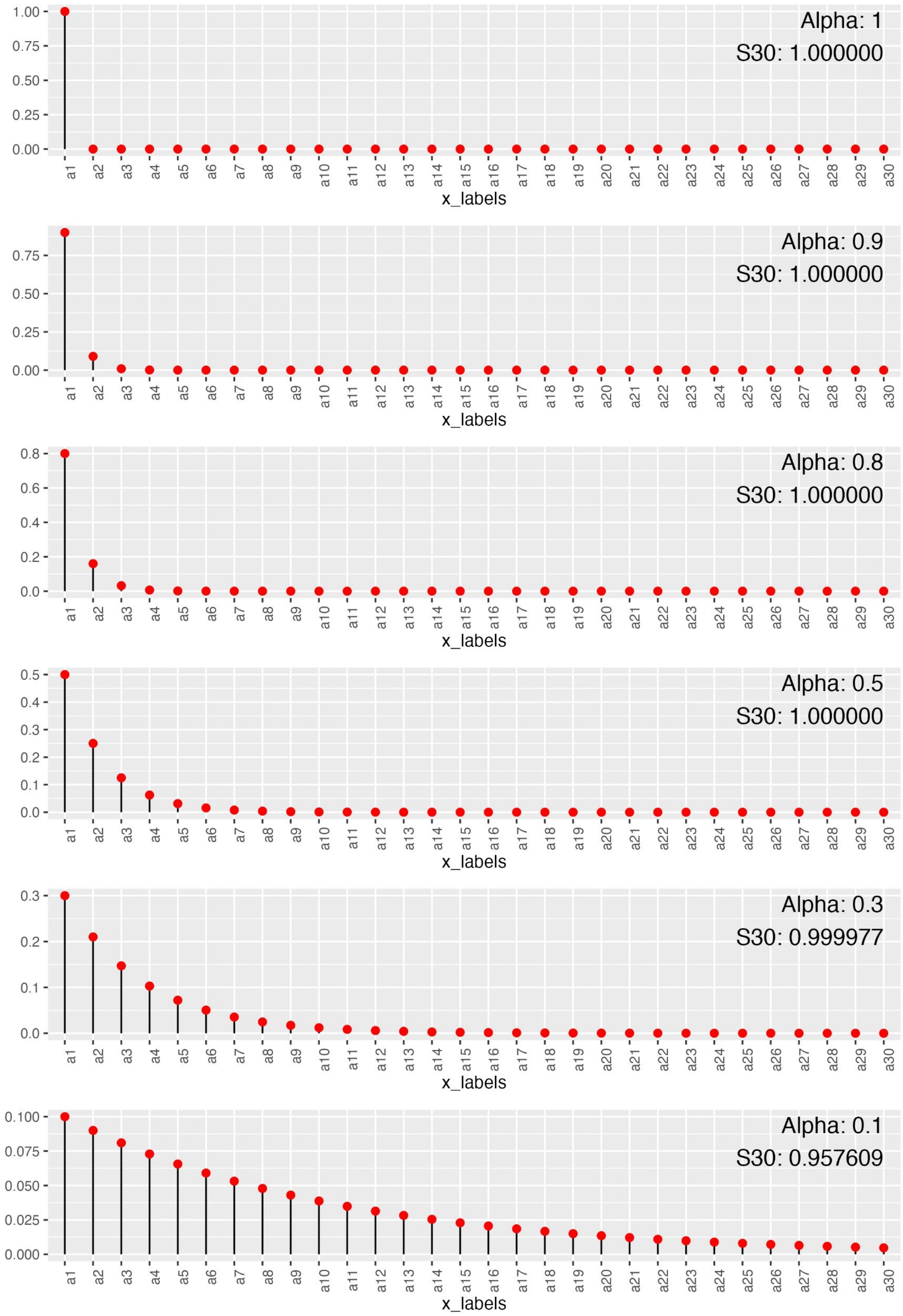
$\hookrightarrow$ If $n \to \infty \implies S_n \to 1$

In practice for a relatively small n, $S_n \sim 1$, since the α that results in the best forecasts will generally satisfy

$$\begin{cases} \alpha < 1 \ (\text{since } \alpha \in (0,1]) \\ \alpha \sim 1 \end{cases}$$



Alpha: 1
S10: 1.000000
Alpha: 0.9
S10: 1.000000
Alpha: 0.8
S10: 1.000000
Alpha: 0.5
S10: 0.999023
Alpha: 0.3
S10: 0.971752
Alpha: 0.1
S10: 0.651322
x_labels
a1
a2
a3
a4
a5
a6
a7
a8
a9
a10


Alpha: 1
S30: 1.000000
Alpha: 0.9
S30: 1.000000
Alpha: 0.8
S30: 1.000000
Alpha: 0.5
S30: 1.000000
Alpha: 0.3
S30: 0.999977
Alpha: 0.1
S30: 0.957609
x_labels
a1 a2 a3 a4 a5 a6 a7 a8 a9 a10 a11 a12 a13 a14 a15 a16 a17 a18 a19 a20 a21 a22 a23 a24 a25 a26 a27 a28 a29 a30

We have seen that

$$a_n = \alpha(1-\alpha)^{n-1} \rightarrow \text{Exp decrease}$$

for $r = 1-\alpha < 1 \Longrightarrow$

$$S_n = 1 - (1-\alpha)^n \rightarrow \underline{1}$$

→ Given this we may use the terms of the succession to produce a forecast that assigns more weight to recent observations and exponentially decaying weights to distant observations.

$$\hat{y}_{T+1|T} = \underbrace{\alpha}_{a_1} y_T + \underbrace{\alpha(1-\alpha)^{1}}_{a_2} y_{T-1} + \underbrace{\alpha(1-\alpha)^{2}}_{a_3} y_{T-2} + \dots + \underbrace{\alpha(1-\alpha)^{T-1}}_{a_T} y_{1}$$

$$1 = T-(T-1)$$

$$\hat{y}_{T+1|T} = \sum_{i=1}^{T} a_i \, y_{T-i+1} = \sum_{j=0}^{T-1} \alpha(1-\alpha)^j \cdot y_{T-j}$$

Since $S_n \rightarrow 1 \Rightarrow \sum_{j=0}^{T-1} \alpha(1-\alpha)^j y_{T-j}$ is a weighted average of the past observations that assigns greater weight to most recent observations and exponentially decaying weights to more distant observations.

Flat forecasts of SES

For SES, no Trend or seasonality will be modelled. The forecasts will be "flat", constant. Mathematically

$$\hat{y}_{T+h|T} = \hat{y}_{T+1|T} = \sum_{j=0}^{T-1} \underbrace{\alpha(\alpha-1)^j}_{\alpha \text{ UNKNOWN}} \underbrace{y_{T-j}}_{\text{VALUES OF THE TS}}$$

↳ VALUES OF THE TS ↳ KNOWN

α UNKNOWN

∠ PARAMETER OF MODEL → BEST FIT?

Fitting the model

Fitting the model means finding the value of α that results in a best fit to the data. → How?

No explicit formulas like in L.R.

To do this numerically we will

1. Guess α
2. Compute the fitted values $\hat{y}_t$
3. Compute the sum of squared errors.

$$\sum_{i=0}^{T} (y_i - \hat{y}_i)^2$$

4. Check if minimum of SSE has been reached

To do this we will use alternative forms for the equations

ALTERNATIVE FORM OF THE MODEL EQUATIONS

$$\rightarrow \hat{y}_{T+1|T} = \overbrace{\sum_{j=0}^{T-1} \alpha(1-\alpha)^{j} \cdot y_{T-j}}$$

$$\hat{y}_{T+1|T} = \alpha y_T + \overbrace{\alpha(1-\alpha) y_{T-1} + \alpha(1-\alpha)^2 y_{T-2} + \dots + \alpha(1-\alpha)^{T-1} y_{\underbrace{1}_{T-(T-1)}}}^{(*)}$$

$$\hat{y}_{\boxed{T}|\boxed{T-1}} = \alpha y_{T-1} + \alpha(1-\alpha) y_{T-2} + \alpha(1-\alpha)^2 y_{T-3} + \dots + \alpha(1-\alpha)^{T-2} y_{\underbrace{1}_{T-(T-1)}}$$

$$(1-\alpha)\hat{y}_{T|T-1} = \underbrace{\alpha(1-\alpha) y_1 + \alpha(1-\alpha)^2 y_{T-2} + \alpha(1-\alpha)^3 y_{T-3} + \dots + \alpha(1-\alpha)^{T-1} y_{\underbrace{1}_{T-(T-1)}}}_{(*)}$$

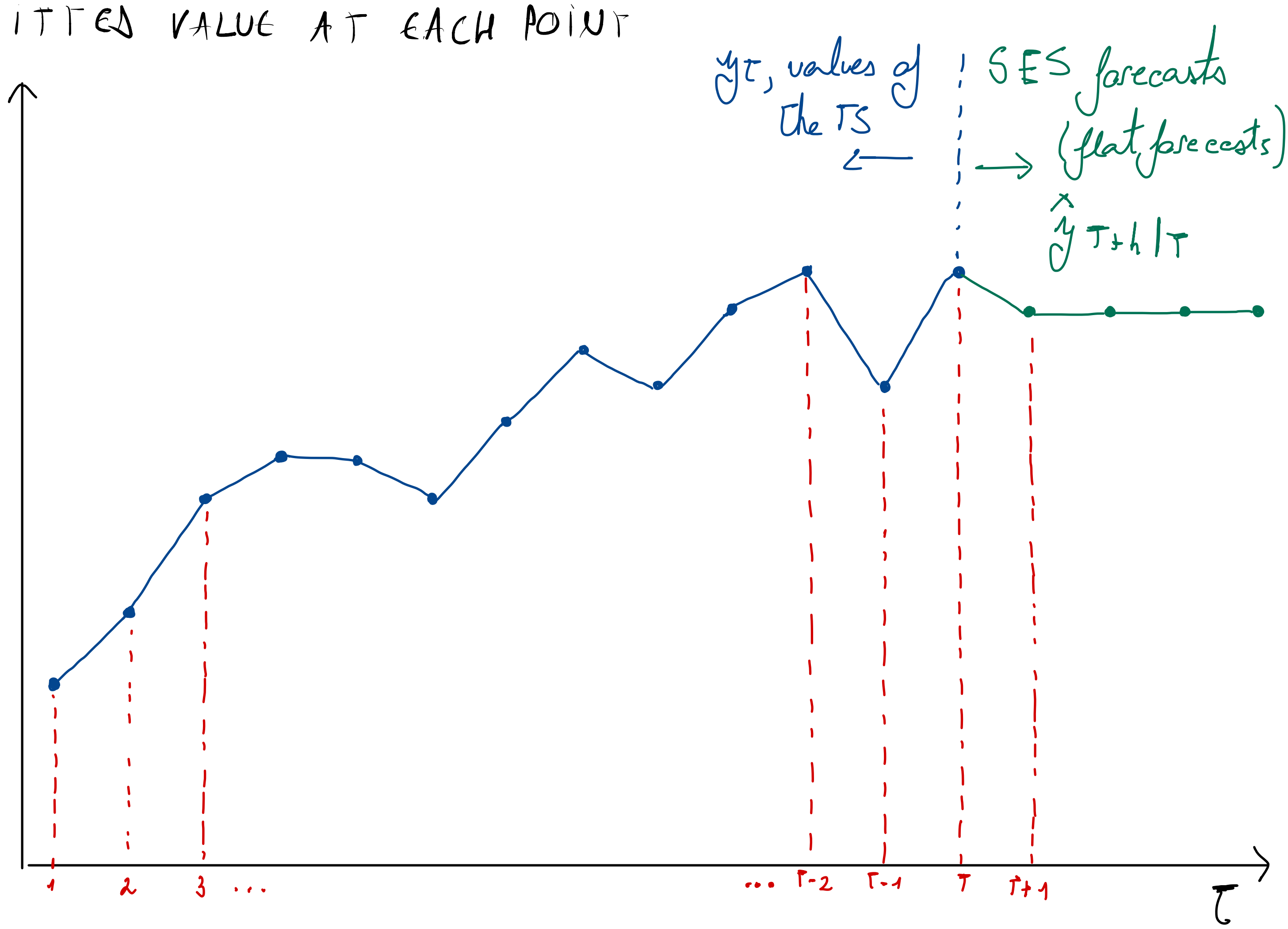
$$\hat{y}_{T+1|T} = \alpha y_T + (*) = \alpha y_T + (1-\alpha)\hat{y}_{T|T-1}$$

This yields: $\boxed{\hat{y}_{T+1|T} = \alpha y_T + (1-\alpha)\hat{y}_{T|T-1}}$

Equivalent to $\hat{y}_{T+1|T} = \sum_{j=0}^{T-1} \alpha(1-\alpha)^{j} \cdot y_{T-j}$ → No new information

Now let us take this equation and particularize at each of the points in our time series.

FITTED VALUE AT EACH POINT



$$\boxed{\hat{y}_{T+1|T} = \alpha y_T + (1-\alpha)\,\hat{y}_{T|T-1}}$$

$T+1$ $\quad \hat{y}_{T+1|T} = \alpha y_T + (1-\alpha)\,\hat{y}_{T|T-1}$

T $\quad \hat{y}_{T|T-1} = \alpha y_{T-1} + (1-\alpha)\,\hat{y}_{T-1|T-2}$

3 $\quad \hat{y}_{3|2} = \alpha y_2 + (1-\alpha)\,\hat{y}_{2|1}$

2 $\quad \hat{y}_{2|1} = \alpha y_1 + (1-\alpha)\,\hat{y}_{1|0}$

FITTED VALUE AT EACH POINT IN TERMS OF LEVELS

$$\boxed{\hat{y}_{T+1|T} = \alpha y_T + (1-\alpha)\hat{y}_{T|T-1}}$$

$$\hat{y}_{T+1|T} = \ell_T$$

$$
\begin{array}{l|l}
T+1 & \overbrace{\hat{y}_{T+1|T}}^{\ell_T} = \alpha y_T + (1-\alpha)\overbrace{\hat{y}_{T|T-1}}^{\ell_{T-1}} \\
T & \underbrace{\hat{y}_{T|T-1}}_{\ell_{T-1}} = \alpha y_{T-1} + (1-\alpha)\overbrace{\hat{y}_{T-1|T-2}}^{\ell_{T-2}} \\
3 & \overbrace{\hat{y}_{3|2}}^{\ell_2} = \alpha y_2 + (1-\alpha)\overbrace{\hat{y}_{2|1}}^{\ell_1} \\
2 & \overbrace{\hat{y}_{2|1}}^{\ell_1} = \alpha y_1 + (1-\alpha)\overbrace{\hat{y}_{1|0}}^{\ell_0}
\end{array}
$$

The level ℓ_T is an artifact we introduce to help us with the fitting process.

The level equation is obtained by replacing fitted values by levels in the equation above.

$$\underbrace{\hat{y}_{T+1|T}}_{\ell_T} = \alpha y_T + (1-\alpha)\underbrace{\hat{y}_{T|T-1}}_{\ell_{T-1}}$$

COMPONENT FORM OF EQUATIONS FOR SES

$$\boxed{\hat{y}_{T+1|T} = \sum_{j=0}^{T-1} \alpha(1-\alpha)^j \cdot y_{T-j}}$$

→ Weighted average form

$$\boxed{\begin{aligned} &\text{Level equation: } \ell_t = \alpha y_t + (1-\alpha)\ell_{t-1} \\ &\text{Forecast equation: } \hat{y}_{T+1|T} = \ell_T \end{aligned}}$$

→ Component form

↳ Given α and ℓ_0, this form provides explicit equations for the fitted values as follows:

① $\ell_1 = \alpha y_1 + (1-\alpha) \cdot \ell_0$ → everything given

$\hat{y}_{2|1} = \ell_1$

② $\ell_2 = \alpha \cdot y_2 + (1-\alpha) \cdot \ell_1$ ~ from previous step

$\hat{y}_{3|2} = \ell_2$

⋮

$T-1$ $\ell_{T-1} = \alpha y_{T-1} + (1-\alpha)\ell_{T-2}$ ~ from previous step

$\hat{y}_{T|T-1} = \ell_{T-1}$

Algorithm to fit SES model: