Mixed Integer Programming Formulations

Lecture 2

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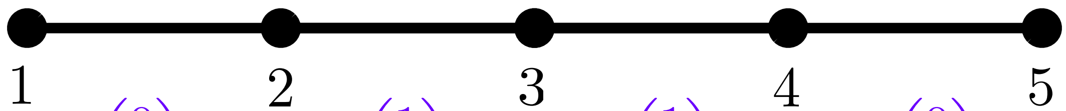
Outline

- Practical construction technique 1:
  - Independent branching formulations for V-polyhedra
- MINLP Formulations
- Practical construction technique 2:
  - Case study for H-polyhedra
  - Encoding and alternative representations of disjunctions

Binary Encoded **Ideal** Formulation for SOS2



$$h^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ h^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ h^3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ h^4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} &\sum_{i=1}^{5} \lambda_i = 1 \\ &0 \leq \lambda_1 + \lambda_5 \leq 1 - y_1 \\ &0 \leq \lambda_3 \leq y_1 \\ &0 \leq \lambda_4 + \lambda_5 \leq 1 - y_2 \\ &0 \leq \lambda_1 + \lambda_2 \leq y_2 \end{aligned}$$

$$y \in \{0,1\}^2,$$

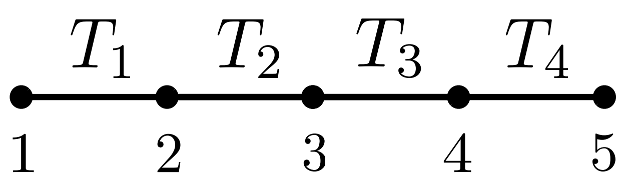
$$Q(H) := \operatorname{conv}\left(\bigcup_{i=1}^{n} P_i \times \{h^i\}\right)$$

$$P_i := \{\lambda \in \Lambda^5 \,:\, \lambda_j = 0 \quad j \notin \{i, i+1\}\}$$

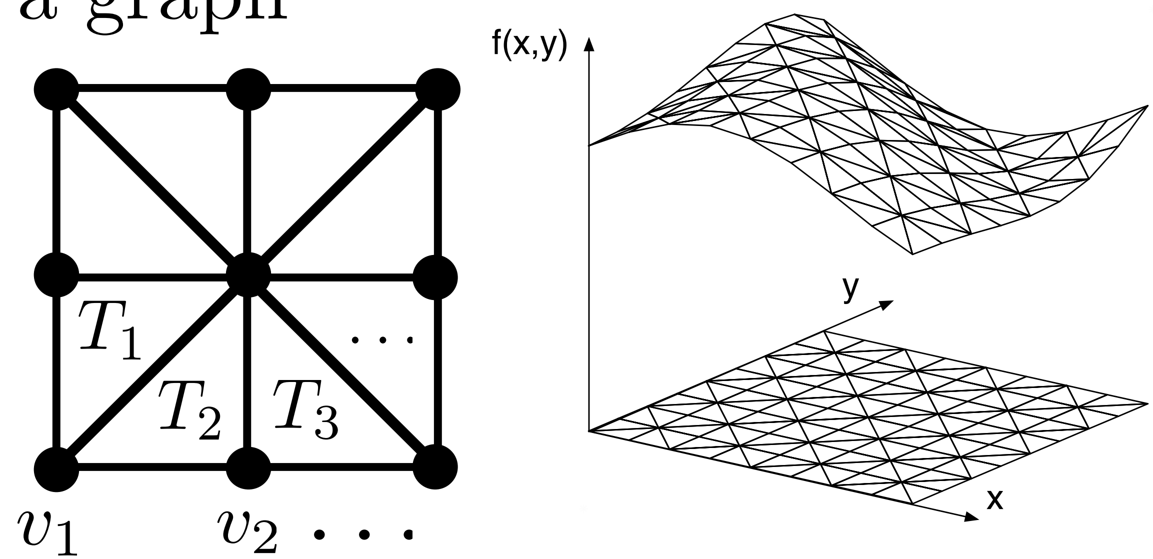
- Practical construction without convex hull

~~Examples~~ = $\mathcal{V}$-polytopes = Faces of Simplex

- $\Delta^V := \left\{\lambda \in \mathbb{R}^V_+ \; : \; \sum_{v \in V} \lambda_v = 1\right\},$
- $P_i = \left\{\lambda \in \Delta^V \; : \; \lambda_v = 0 \quad \forall v \notin T_i\right\}$
- $\lambda \in \bigcup_{i=1}^{n} P_i$
- $T_i =$ cliques of a graph

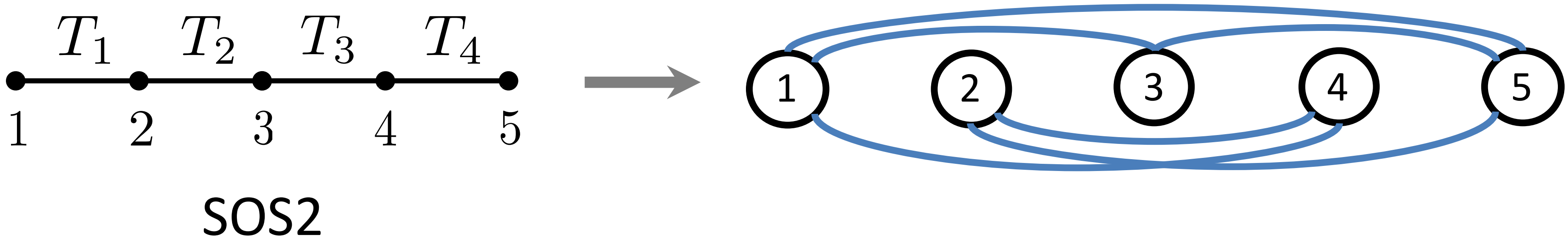


SOS2

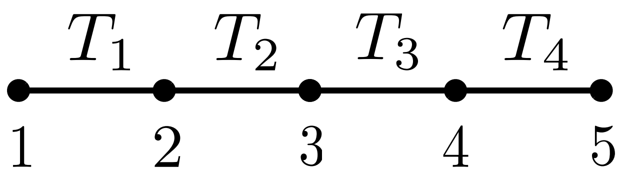


Piecewise Linear Functions

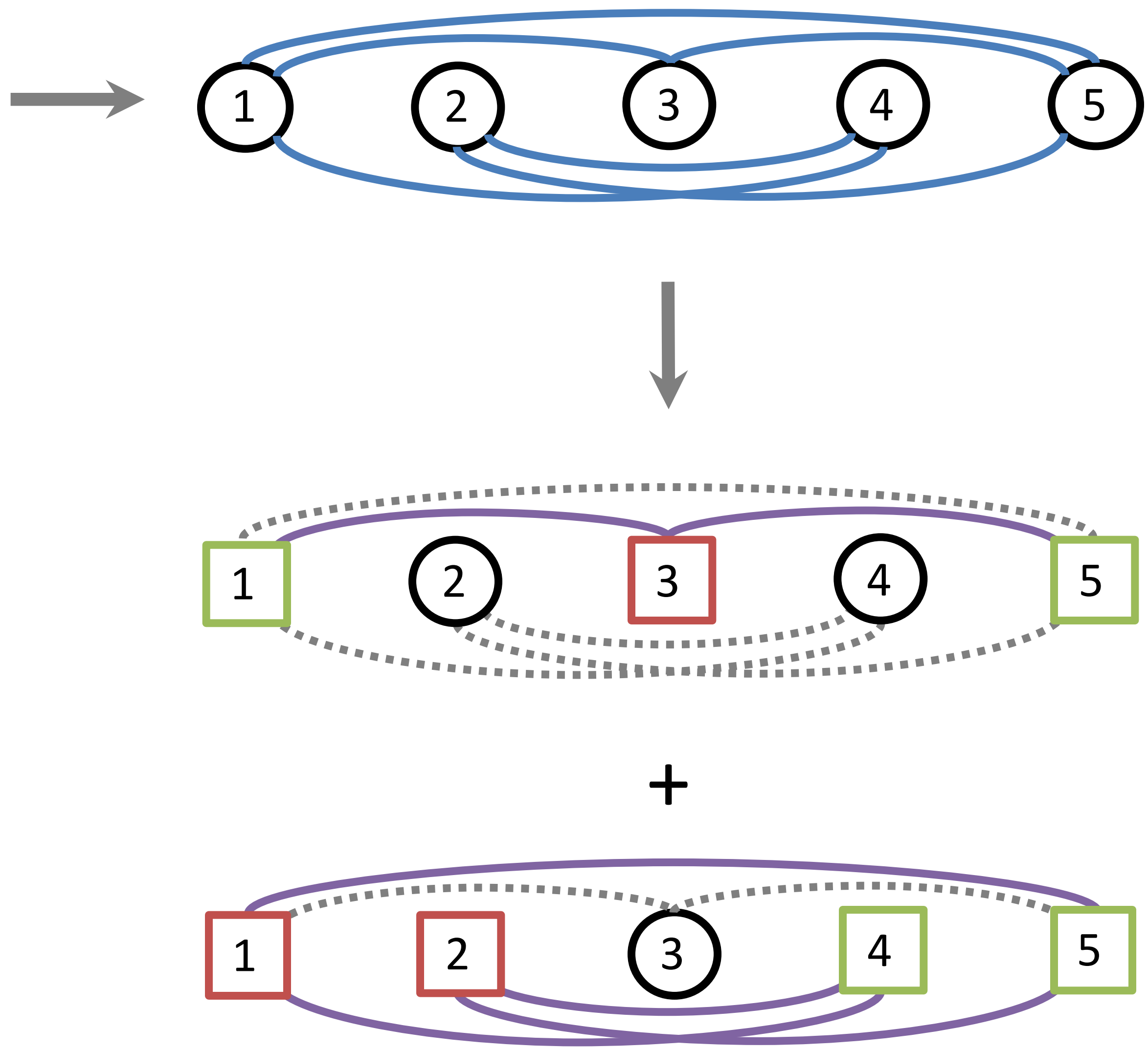
From Cliques to (Complement) Conflict Graph



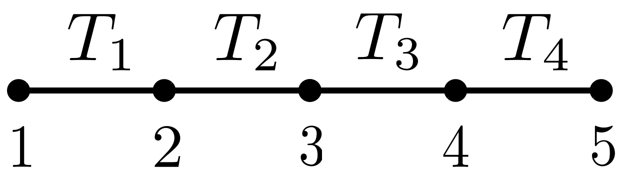
From Conflict Graph to Bi-clique Cover



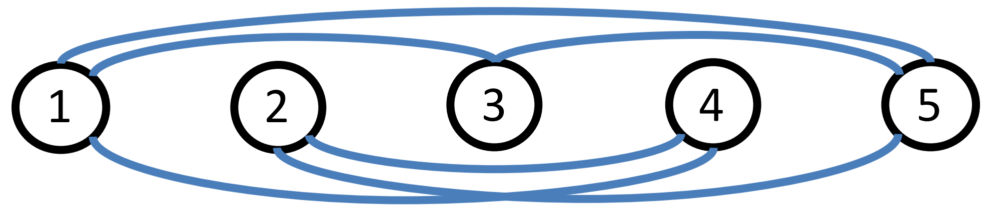
SOS2

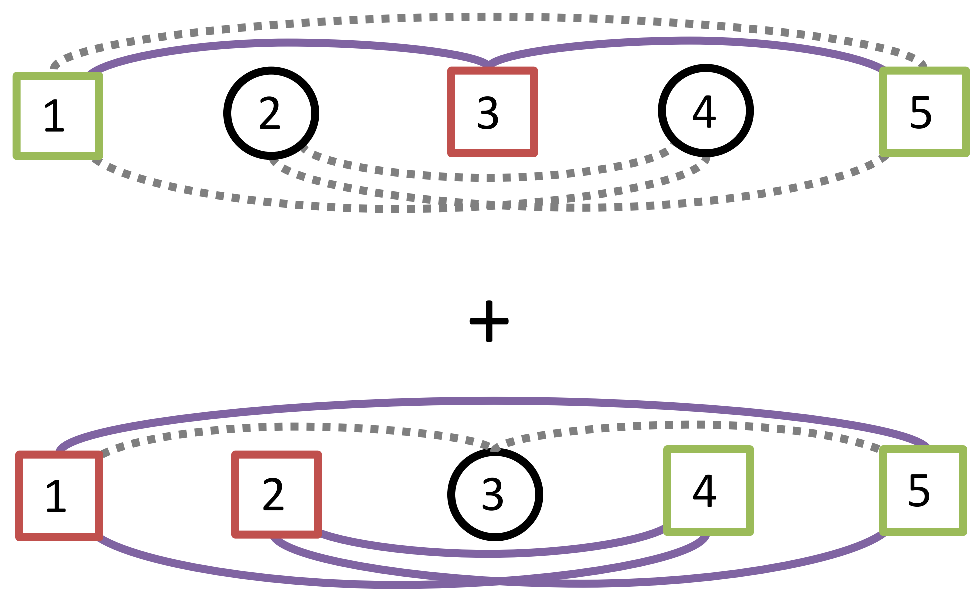


From Bi-clique Cover to Formulation



SOS2





$$
\begin{aligned}
0 &\le \lambda_1 + \lambda_5 \le 1 - y_1 \\
0 &\le \lambda_3 \le y_1 \\
0 &\le \lambda_4 + \lambda_5 \le 1 - y_2 \\
0 &\le \lambda_1 + \lambda_2 \le y_2
\end{aligned}
$$

Ideal Formulation from Bi-clique Cover

- Conflict Graph $G = (V, E)$

$$E = \{(u,v) \,:\, u,v \in V,\ u \neq v, \quad \nexists i \text{ s.t. } u,v \in T_i\}$$

- Bi-clique cover $\{(A^j, B^j)\}_{j=1}^{t}, \quad A^j, B^j \subseteq V$

$$\forall \{u,v\} \in E \quad \exists j \text{ s.t. } u \in A^j \ \wedge v \in B^j$$

$$\forall j,\ u \in A^j,\ v \in B^j \quad \{u,v\} \in E$$

- Formulation

$$\sum_{v \in A^j} \lambda_v \leq 1 - y_j \quad \forall j \in [t]$$

$$\sum_{v \in B^j} \lambda_v \leq y_j \quad \forall j \in [t]$$

$$y \in \{0,1\}^t$$

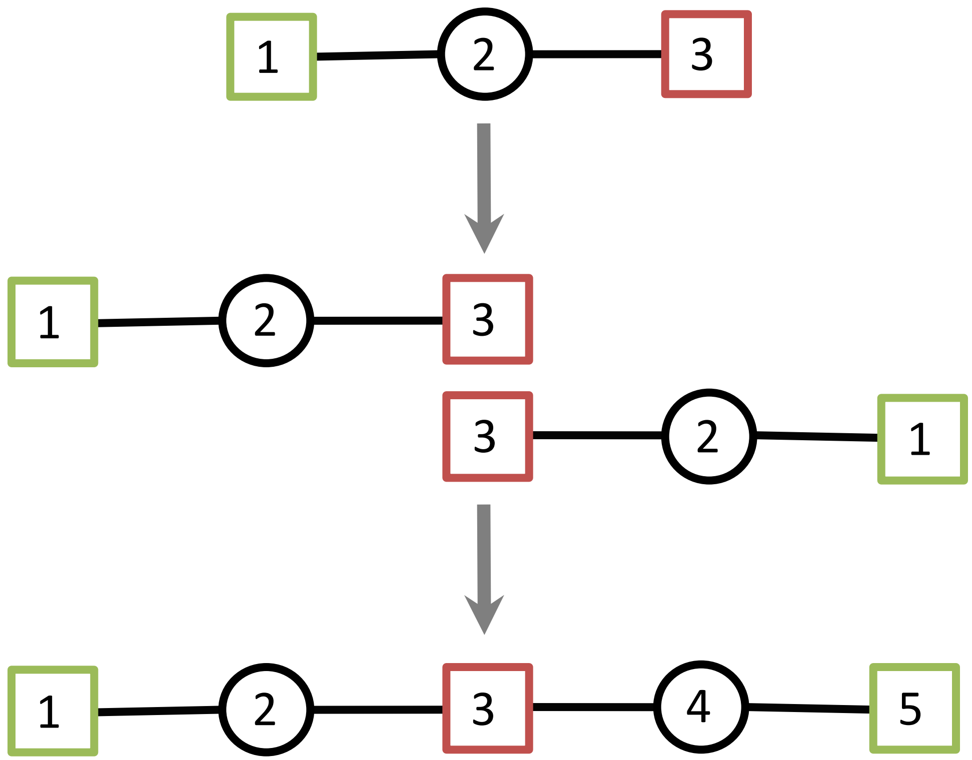
Recursive Construction of Cover for SOS2, Step 1

Base case $n=2^1$:

Step 1 recursion :

Reflect Graph / Cover

Stick Graph / Cover

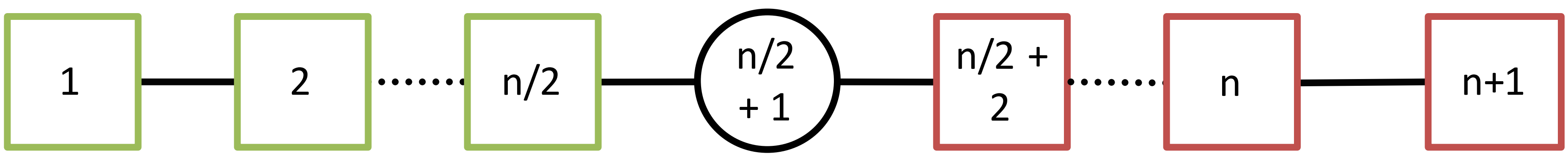


Repeat for all bi-cliques from 2^{k-1} to cover all edges within first and last half of conflict graph

Recursive Construction of Cover for SOS2, Step 2

Only edges missing are those between
first and last half of conflict graph

Step 2 : Add one more bi-clique

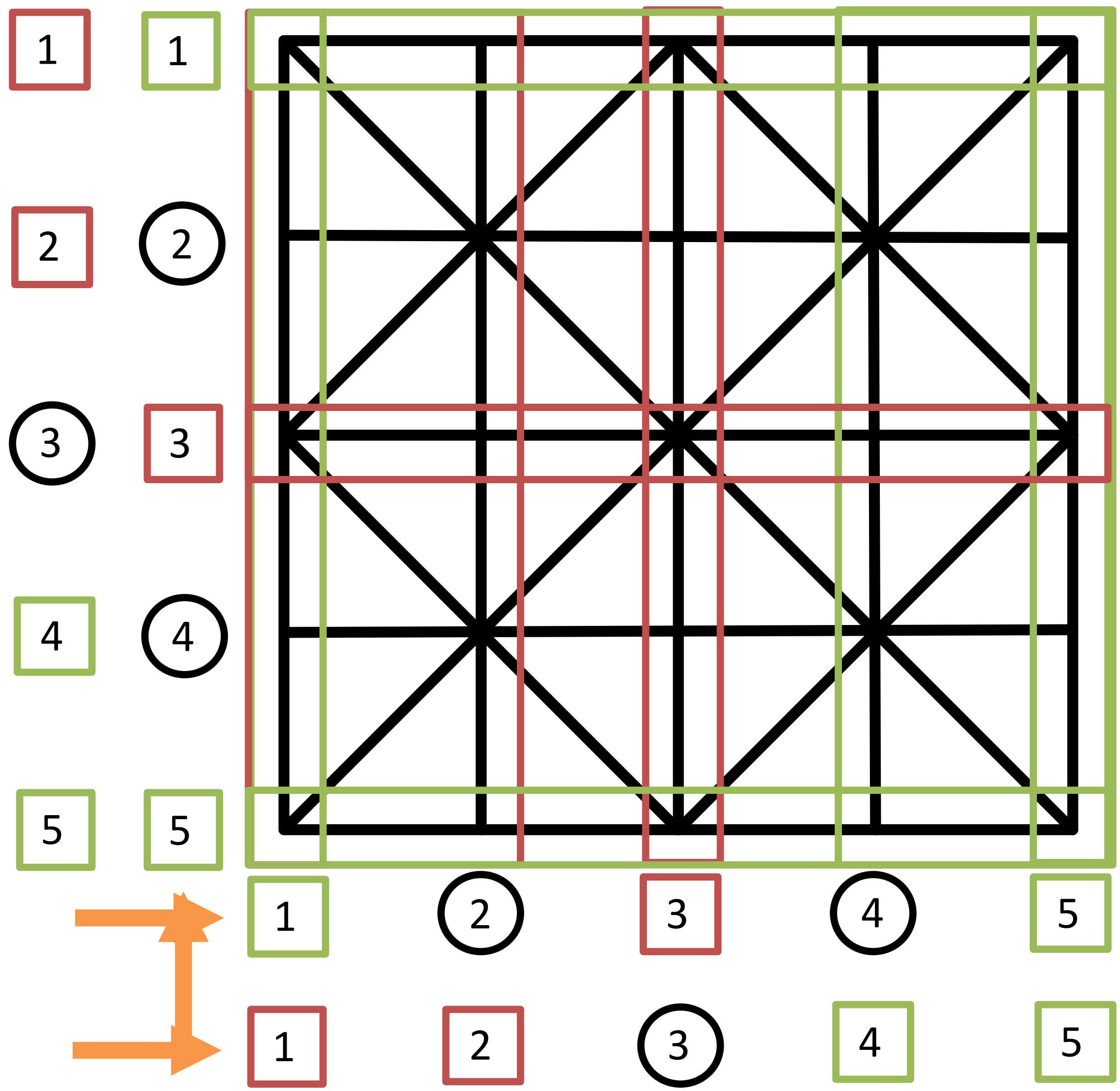


Cover has $\log_2 n$ bi-cliques.

For non-power of two just delete extra nodes.

Grid Triangulations: Step 1 = SOS2 for Inter-Box

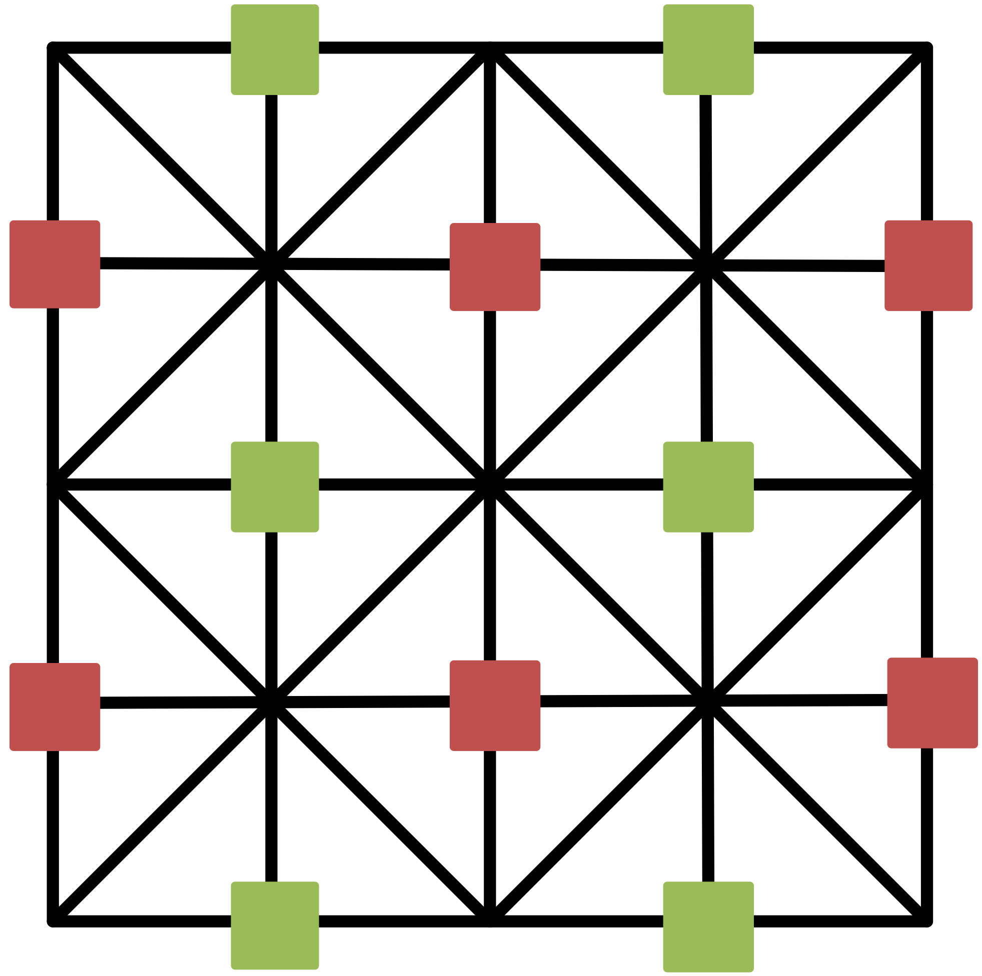
Covers all arcs between boxes



Grid Triangulations: Step 2 = Ad-hoc Intra-Box

Covers all arcs within boxes

Sometimes 1 additional cover

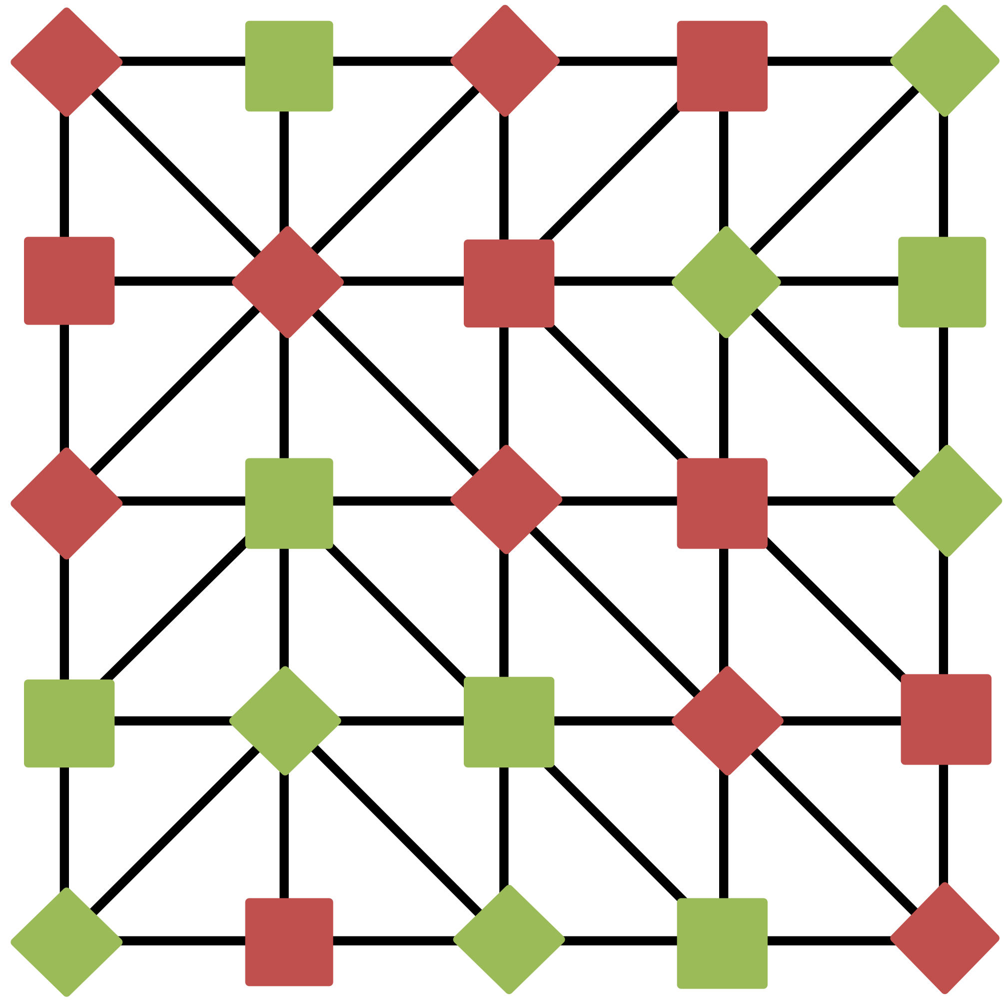


Grid Triangulations: Step 2 = Ad-hoc Intra-Box

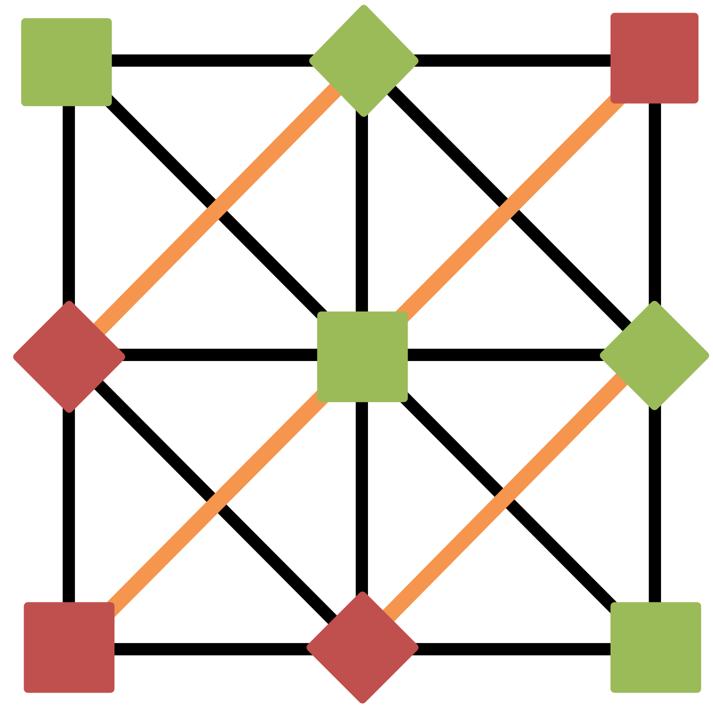
Sometimes **2** additional covers

Sometimes more, but always less than **9**

Simple rules to get (near) optimal in Fall '16

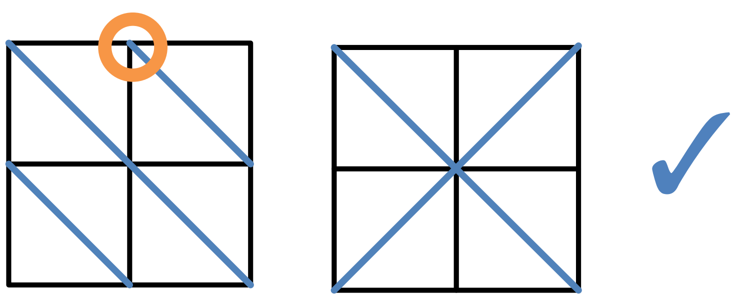


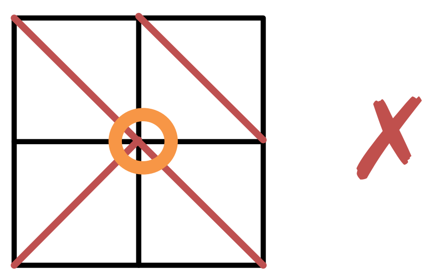
When 2 Additional Covers Work



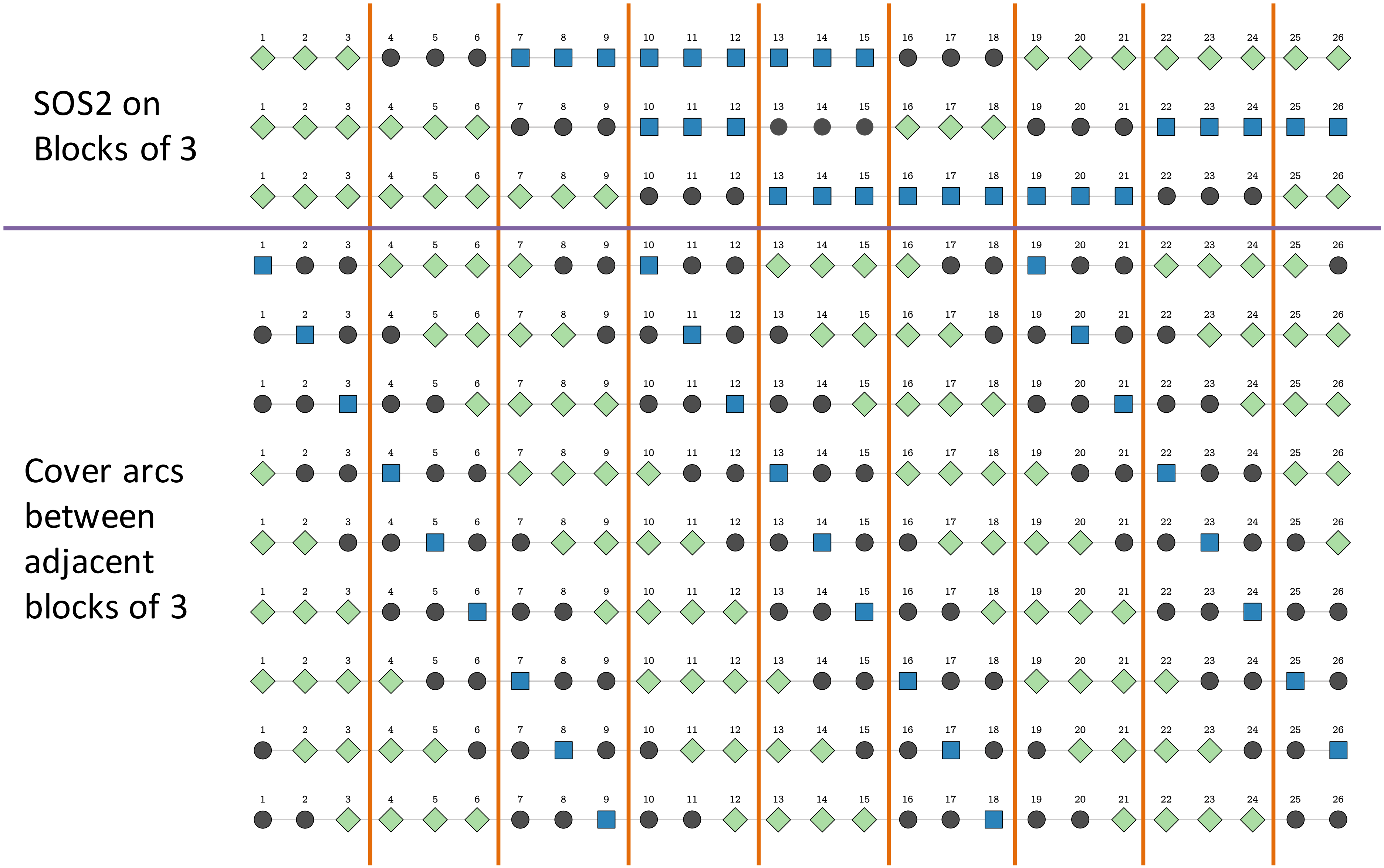
1. Add "Dual" Triangulation
2. Color vertices following diagonal arcs:
   - Keep color for original arcs
   - Change color for dual arcs
3. May need to repeat coloring once more

- Works for even degree outside boundary





More elaborate: SOS3(26)



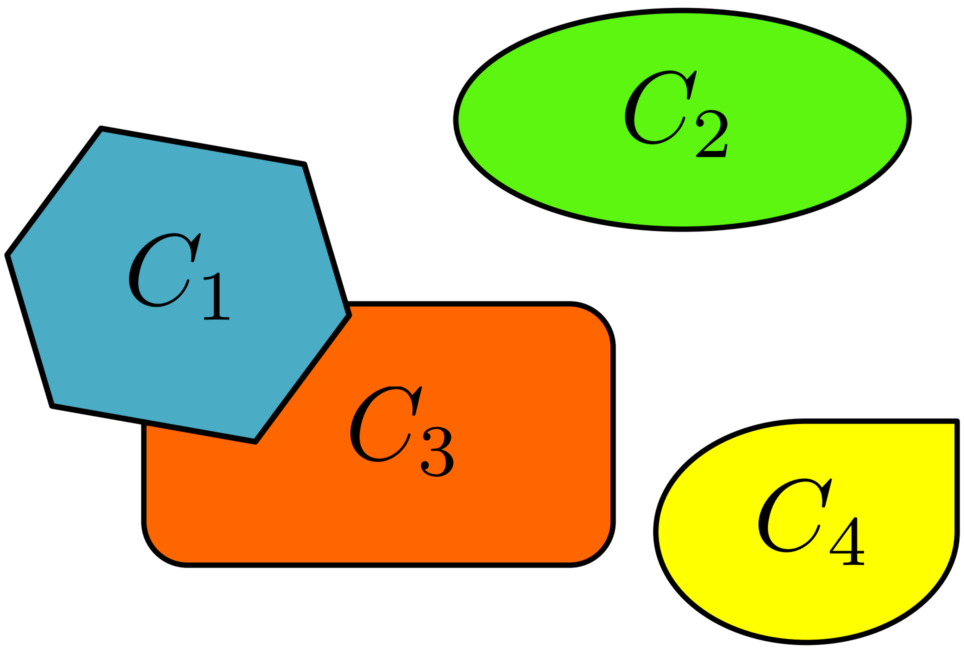
Summary for Independent Branching

- Other examples
  - ☑ Discretization of multilinear terms
  - ☐ General piecewise linear functions
  - ☒ Cardinality constraints
- References:
  - Huchette and V. “Small independent branching formulations for unions of V-polyhedra”, 2016. http://www.optimization-online.org/DB_HTML/2016/05/5454.html
  - V. “Embedding Formulations and Complexity for Unions of Polyhedra”, 2015. arXiv:1506.01417
  - More on Fall 2016.

Nonlinear Mixed <u>0-1</u> Integer Formulations

- Modeling Finite Alternatives = Unions of Convex Sets

$$x \in \bigcup_{i=1}^{n} C_i \subseteq \mathbb{R}^d$$



Extended Formulations: Perspective "v/s" Cones

$$C_i = \{x \in \mathbb{R}^d \ : \ f_i(x) \leq 0\}$$

$$\tilde{f}(x, y) = \begin{cases} y f(x/\boxed{y}) & \text{if } y > 0 \\ \lim_{\alpha \downarrow 0} \alpha f(x' - x + x/\alpha) & \text{if } y = 0 \\ +\infty & \text{if } y < 0 \end{cases}$$

$$\begin{array}{|lr|} \hline \tilde{f}_i\left(x^i, y_i\right) \leq 0 & \forall i \in [n] \\ \sum_{i=1}^n x^i = x & \\ \sum_{i=1}^n y_i = 1 & \\ y \in \{0,1\}^n & \\ x, \ x^i \in \mathbb{R}^d & \forall i \in [n] \\ \hline \end{array}$$

$$C_i = \left\{ x \in \mathbb{R}^d \ : \ \begin{array}{c} \exists u \in \mathbb{R}^{p_i} \text{ s.t.} \\ A^i x + D^i u - b \in K^i \end{array} \right\}$$

K^i closed convex cone

$$\begin{array}{|lr|} \hline A^i x^i + D^i u^i - b y_i \in K^i & \forall i \in [n] \\ \sum_{i=1}^n x^i = x & \\ \sum_{i=1}^n y_i = 1 & \\ y \in \{0,1\}^n & \\ x, \ x^i \in \mathbb{R}^d & \forall i \in [n] \\ u^i \in \mathbb{R}^{p_i} & \forall i \in [n] \\ \hline \end{array}$$

- Both formulations are ideal (extreme points of continuous relaxation satisfy integrality constraints)

Cones Can Mitigate Unintended Numerical Issues

- Let $C_i = \{x \in \mathbb{R}^2 \,:\, f_i(x) \leq 0\}$

  where $f_i(x) = (x_1 - a_i)^2 - x_2 - 1$

$$\tilde{f}_i(x, y) = \begin{cases} y(x_1 \boxed{/y} - a_i)^2 - x_2 - y & \text{if } y > 0 \\ -x_2 & \text{if } y = x_1 = 0 \\ +\infty & \text{if } o.w. \end{cases}$$

- Conic (SOCP) representation

$$C_i = \left\{ x \in \mathbb{R}^2 \,:\, \sqrt{(x_2 - a_i)^2 + 4x_1^2} \leq 2 + x_2 \right\}$$

$$\sqrt{\left(x_2^i - a_i y_i\right)^2 + 4\left(x_1^i\right)^2} \leq 2y_i + x_2^i$$

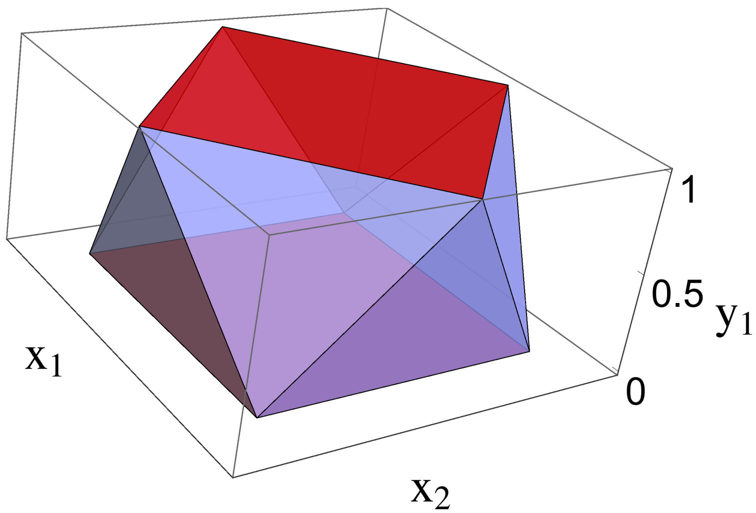
Conic = Really Extended

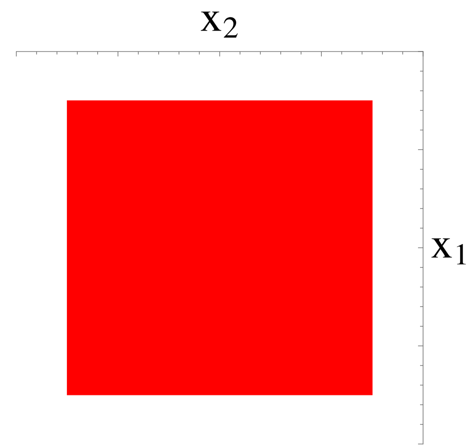
- Conic representation = additional auxiliary variables

$$C_i = \left\{ x \in \mathbb{R}^d \; : \; \begin{array}{r} \exists \boxed{u} \in \mathbb{R}^{p_i} \text{ s.t.} \\ A^i x + D^i \boxed{u} - b \in K^i \end{array} \right\}$$

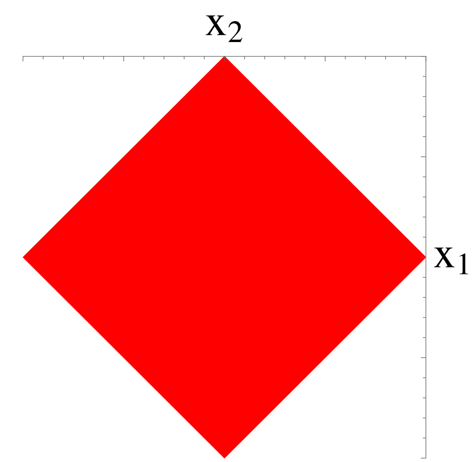
- Bad for NLP solvers, but good for polyhedral approximations in MINLP solvers:
  – Miles' talk on Wednesday, Session 3

Embedding Formulation = Ideal non-Extended





P_1



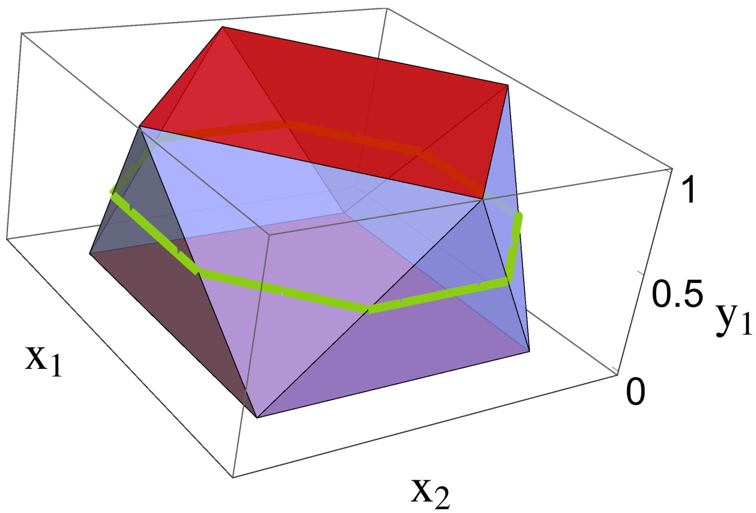
P_2

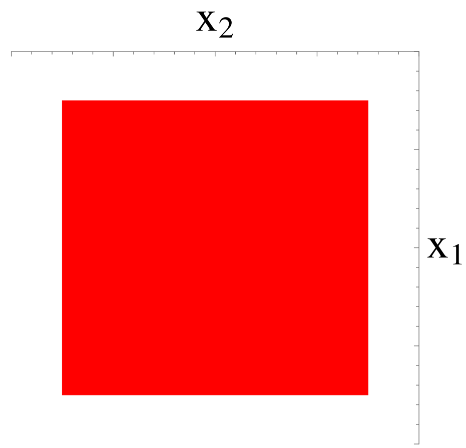
$$Q\,(H) := \operatorname{conv}\left(\bigcup_{i=1}^{n} P_i \times \{h^i\}\right)$$

$$(x, y) \in Q \cap \left(\mathbb{R}^d \times \mathbb{Z}^k\right) \quad \Leftrightarrow \quad y = h^i \;\wedge\; x \in P_i$$

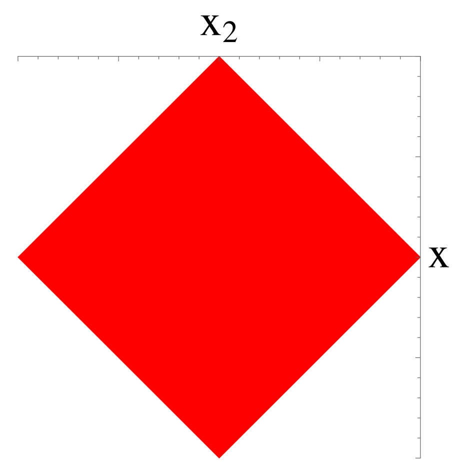
$$\operatorname{ext}(Q) \subseteq \mathbb{R}^d \times \mathbb{Z}^k \qquad H := \{h^i\}_{i=1}^{n} \subseteq \{0,1\}^k\,, \quad h^i \neq h^j$$

Unary Encoding, Minkowski Sum and Cayley Trick



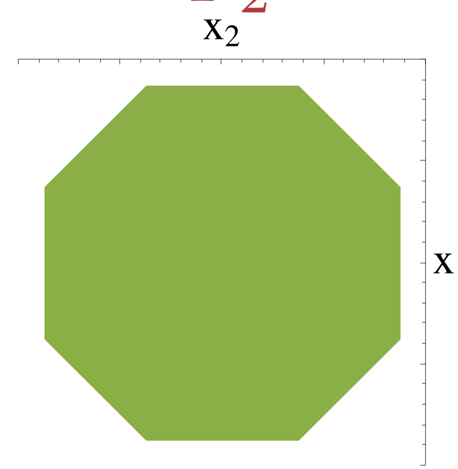


P_1



P_2

$$Q \cap \left(\mathbb{R}^2 \times \{0.5\}\right) \equiv P_1 + P_2 =$$



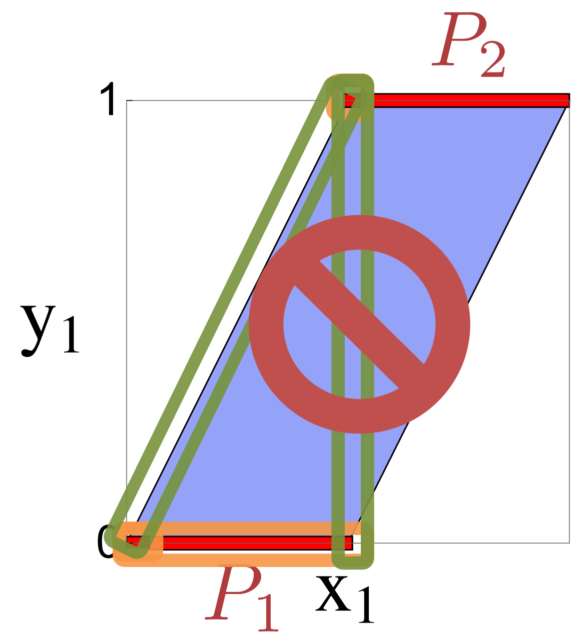
For traditional or unary encoding:

$$Q\,(H) \cap \left(\mathbb{R}^d \times \left\{\tfrac{1}{n} \sum_{i=1}^{n} \mathbf{e}^i \right\}\right) \equiv \sum_{i=1}^{n} P_i$$

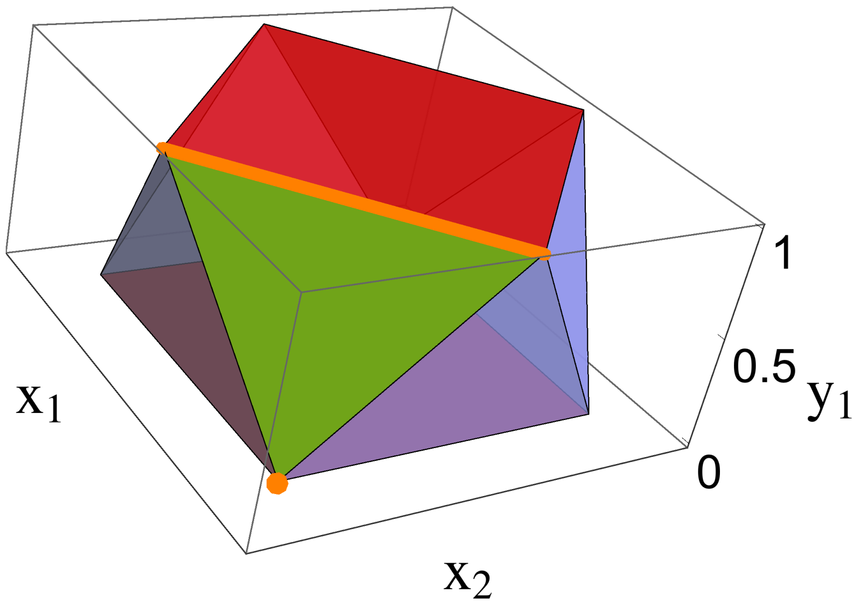
Faces for Unary Encoding: Good Minkowski Sum

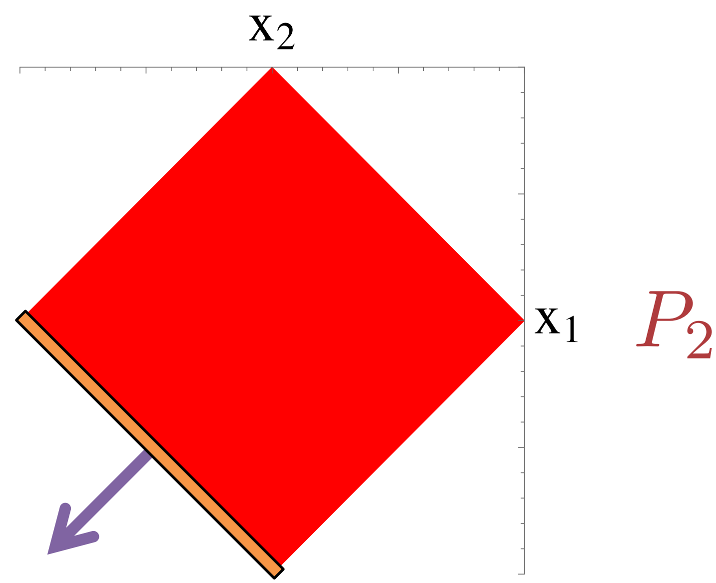
- Two types of facets (or faces):
  - $P_1 \times \{0\} \equiv y_i \geq 0$
  - $\operatorname{conv}\left(\left(F_1 \times 0\right) \cup \left(F_2 \times 1\right)\right)$

    F_i proper face of P_i
  - Not all combinations of faces
  - Which ones are valid?
    - Minkowski to the rescue!



Valid Combinations = Common Normals

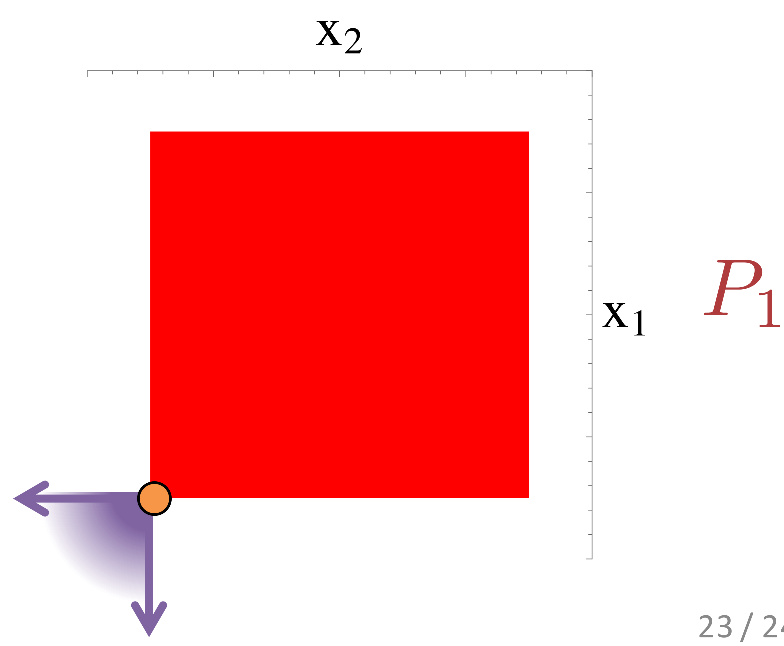




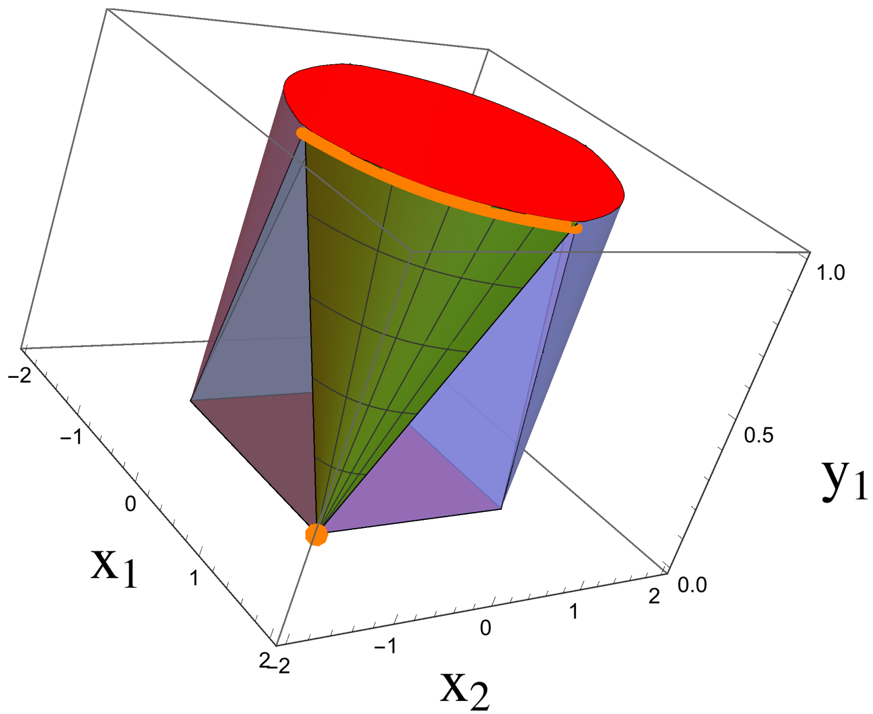
$$N\left(F_1\right) \cap N\left(F_2\right) \neq \emptyset$$

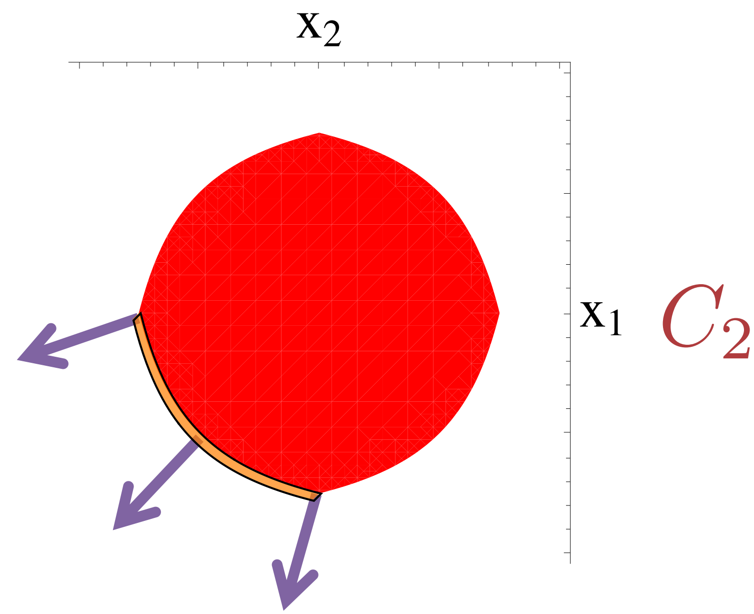
$$\Updownarrow$$

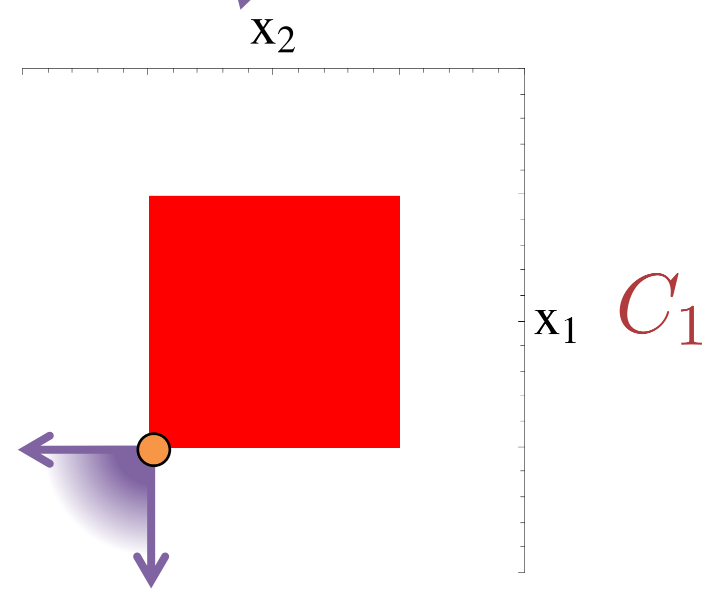
$$\operatorname{conv}\left(\left(F_1 \times 0\right) \cup \left(F_2 \times 1\right)\right)$$
$$\text{is face of } Q\left(H\right)$$



Unary Embedding for Unions of Convex Sets

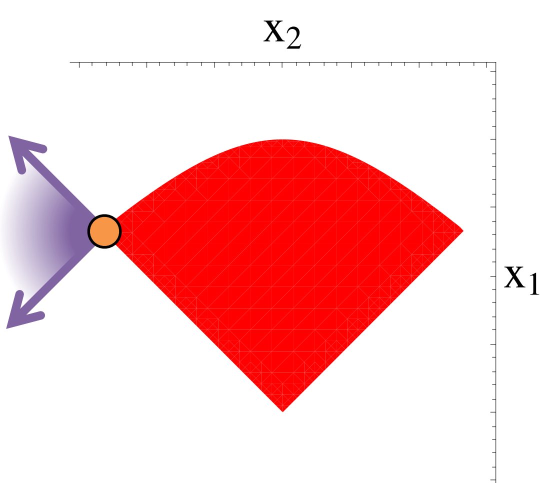






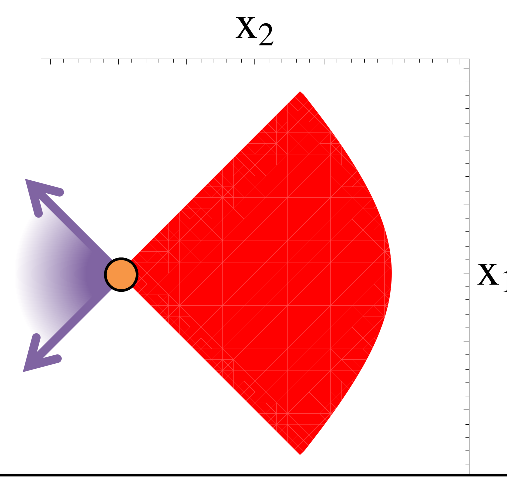
- Description of boundary of $Q(H)$ is easy if "normals condition" yields convex hull of 1 nonlinear constraint and point(s)

Example: Pizza Slices



$C_1:$

$$\sqrt{1+x_1^2} \leq 2 - x_2$$
$$x_1 - x_2 \leq 1$$
$$-x_2 - x_1 \leq 1$$



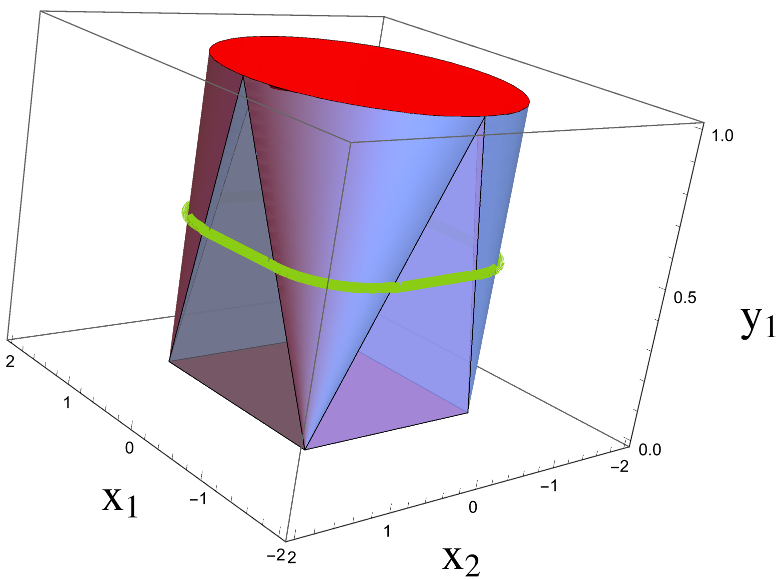
$C_2:$

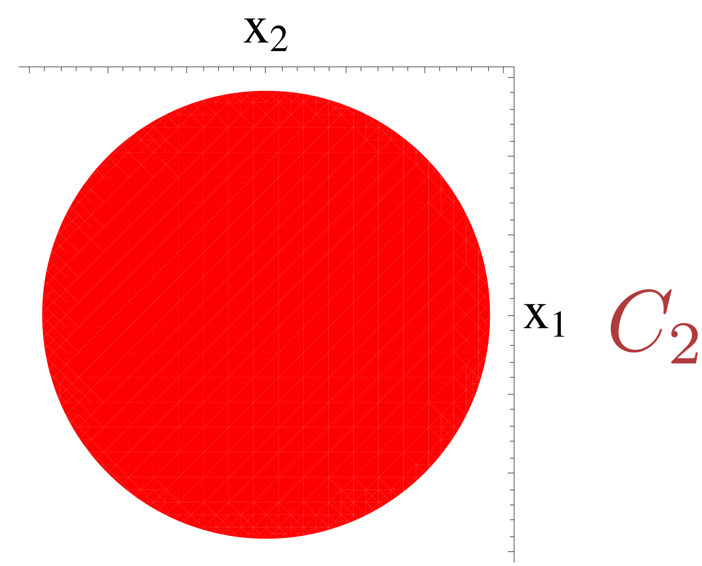
$$\sqrt{1+x_2^2} \leq 2 - x_1$$
$$x_2 - x_1 \leq 1$$
$$-x_2 - x_1 \leq 1$$

$$\sqrt{y_4^2 + \left(x_2 - \frac{1}{3}y_1 + \frac{1}{3}y_3\right)^2} \leq 2y_4 + x_1 + \frac{4}{3}y_1 + y_2 + \frac{4}{3}y_3$$

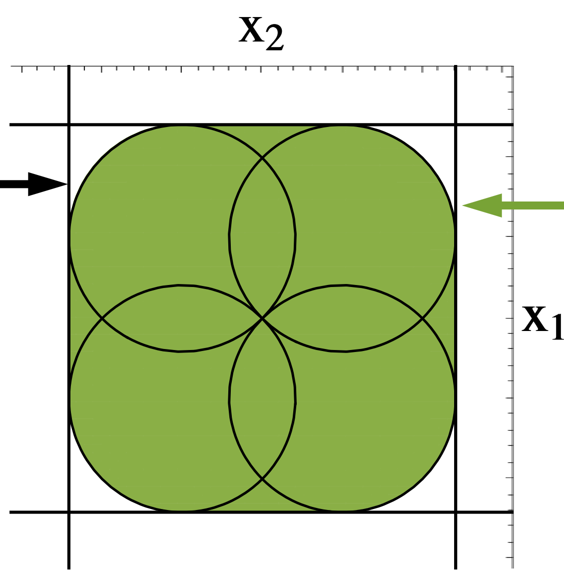
$\operatorname{conv}\left(\bigcup_{i=1}^{4}\left(C_i \times \{\mathbf{e}^i\}\right)\right)$ = 4 conic + 4 linear inequalities

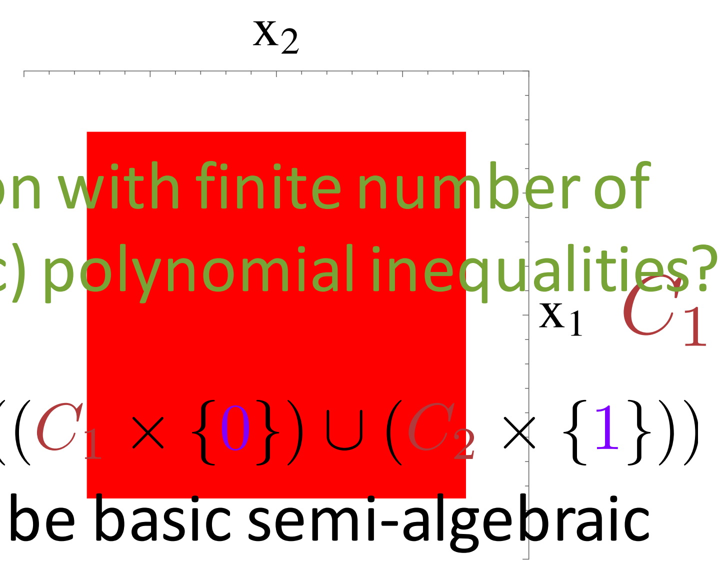
Bad Example: Representability Issues





Zariski closure of boundary?





Description with finite number of (quadratic) polynomial inequalities?

$$Q := \operatorname{conv}\left((C_1 \times \{0\}) \cup (C_2 \times \{1\})\right)$$

can fail to be basic semi-algebraic

Final Positive Results

- Unions of Homothetic Convex Bodies $C_i = \lambda_i C + b^i$ (all extreme points exposed)

$$\operatorname{conv}\left(\bigcup_{i=1}^n \left(C_i \times \{\mathbf{e}^i\}\right)\right) =$$

$$\gamma_C\left(x - \sum_{i=1}^n y_i b^i\right) \le \sum_{i=1}^n \lambda_i y_i$$

$$\sum_{i=1}^n y_i = 1$$

$$y \ge 0 \qquad\qquad \forall i \in [n]$$

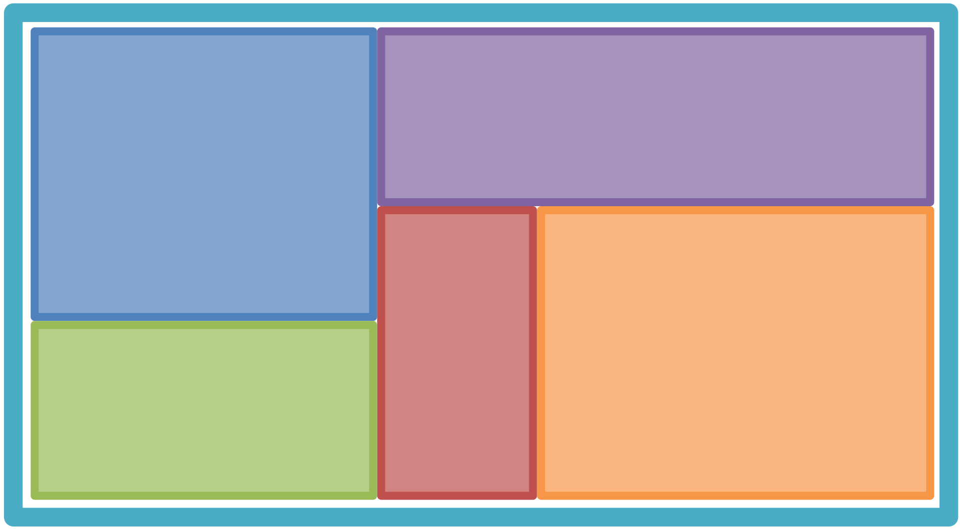
$$\gamma_C(x) := \inf\{\lambda > 0 \;:\; x \in \lambda C\}$$

- Generalizes polyhedral results from Balas '85, Jeroslow '88 and Blair '90

Summary for MINLP Formulations

- Use “really” extended formulations
- Non-extended formulations can be good or problematic
- References:

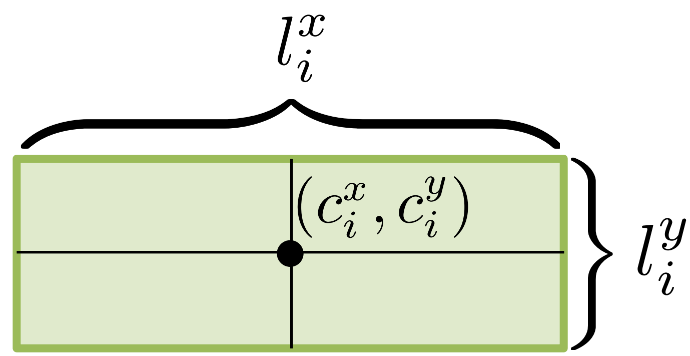
Floor Layout Problem



- Fit boxes in rectangular floor
- Shapes of boxes flexible, but must have min area and aspect ratio
- Minimize weighted Manhattan distance between centers
- Application in VLSI design

Notation and Simple Constraints

- Rectangular floor: $[0, L^x] \times [0, L^y]$ for $L^x,\ L^y > 0$
- Collection of N boxes: $\{\mathcal{B}_i\}_{i=1}^{N}$
- Characterize box $\mathcal{B}_i$ by
  - Center: (c_i^x, c_i^y)
  - Lengths: (ℓ_i^x, ℓ_i^y)



- Constraint 1: $\mathcal{B}_i$ must lie on the floor:
  - $\frac{1}{2}\ell_i^s \leqslant c_i^s \leqslant L^s - \frac{1}{2}\ell_i^s \quad \forall s \in \{x, y\}$
- Lower bound on width/height

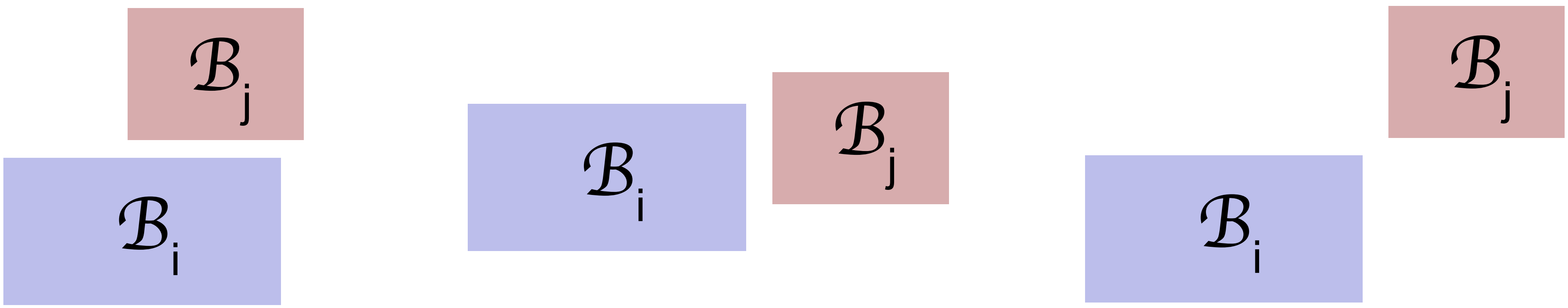
$$\ell_i^s \geqslant lb_i^s \qquad \forall s \in \{x, y\}$$

P_i

Non-Overlap = Disjunctive Constraints

DEFINITION 1. We say that $\mathcal{B}_i$ *precedes* $\mathcal{B}_j$ in direction s (denoted by $\mathcal{B}_i \leftarrow_s \mathcal{B}_j$) if

$$c_i^s + \frac{1}{2}\ell_i^s \leqslant c_j^s - \frac{1}{2}\ell_j^s.$$



- Disjunctive constraint: $\displaystyle\bigcup_{r=1}^{4} \{(c_i, l_i, c_j, l_j) \in P_i \times P_j \ : \ d_{i,j}^r\}$

$$d_{i,j}^1 = \mathcal{B}_i \leftarrow_y \mathcal{B}_j, \qquad d_{i,j}^2 = \mathcal{B}_i \leftarrow_x \mathcal{B}_j \qquad d_{i,j}^3 = \mathcal{B}_j \leftarrow_y \mathcal{B}_i \qquad d_{i,j}^4 = \mathcal{B}_j \leftarrow_x \mathcal{B}_i.$$

Formulation for 1 Pair of Boxes

$$\frac{1}{2}\ell_k^s \leqslant c_k^s \leqslant L^s - \frac{1}{2}\ell_k^s \qquad \forall s \in \{x, y\}, k \in \{i, j\}$$

$$c_i^y + \frac{1}{2}\ell_i^y \leqslant c_j^y - \frac{1}{2}\ell_j^y + L^y(1 - v_1), \qquad c_i^x + \frac{1}{2}\ell_i^x \leqslant c_j^x - \frac{1}{2}\ell_j^x + L^x(1 - v_2)$$

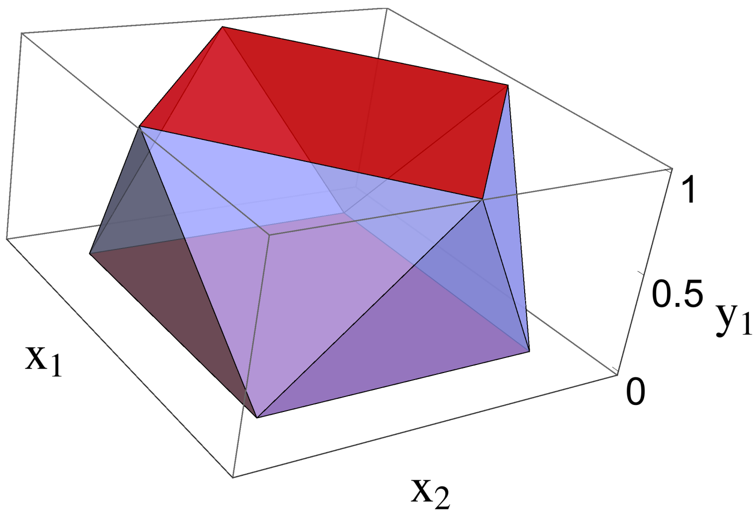
$$c_j^y + \frac{1}{2}\ell_j^y \leqslant c_i^y - \frac{1}{2}\ell_i^y + L^y(1 - v_3), \qquad c_j^x + \frac{1}{2}\ell_j^x \leqslant c_i^x - \frac{1}{2}\ell_i^x + L^x(1 - v_4)$$

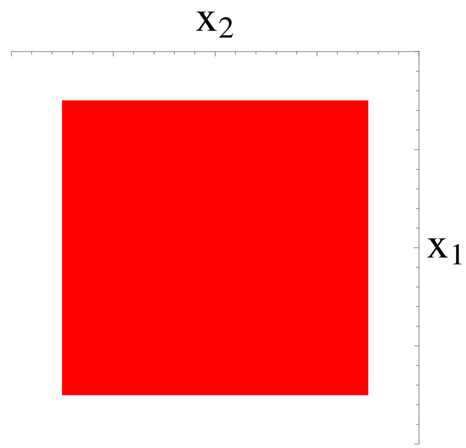
$$lb_k^s \leqslant \ell_k^s \qquad \forall s \in \{x, y\}, k \in \{i, j\}$$

$$\sum_{i=1}^{4} v_i = 1, \qquad v \in \{0, 1\}^4.$$

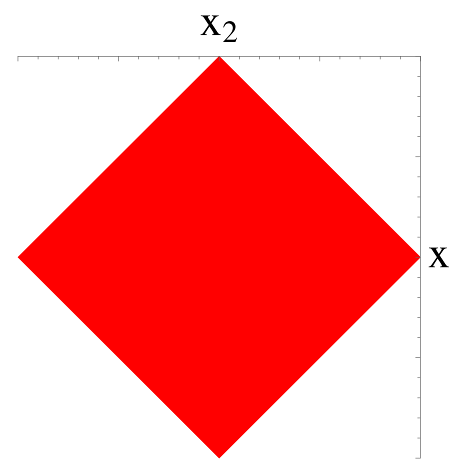
- Not ideal, but small

What About Embedding Formulations?





P_1



P_2

$$Q(H) := \operatorname{conv}\left(\bigcup_{i=1}^{n} P_i \times \{h^i\}\right)$$

$$(x, y) \in Q \cap \left(\mathbb{R}^d \times \mathbb{Z}^k\right) \quad \Leftrightarrow \quad y = h^i \;\wedge\; x \in P_i$$

$$\operatorname{ext}(Q) \subseteq \mathbb{R}^d \times \mathbb{Z}^k \qquad H := \{h^i\}_{i=1}^{n} \subseteq \{0,1\}^k, \quad h^i \neq h^j$$

Formulation for Unary Encoding

$$\frac{1}{2}\ell_p^s \boxed{+\, lb_q^s u_{q,p}^s} \leqslant c_p^s \leqslant L^s - \frac{1}{2}\ell_p^s \boxed{-\, lb_q^s u_{p,q}^s} \quad \forall s \in \{x,y\}, \{p,q\} = \{i,j\}$$

$$c_p^s + \frac{1}{2}\ell_p^s \leqslant c_q^s - \frac{1}{2}\ell_q^s + L^s(1 - u_{p,q}^s) \quad \forall s \in \{x,y\}, \{p,q\} = \{i,j\}$$

$$\ell_p^s \geqslant lb_p^s \quad \forall s \in \{x,y\}, p \in \{i,j\}$$

$$\boxed{u_{i,j}^x + u_{j,i}^x + u_{i,j}^y + u_{j,i}^y = 1}$$

$$u_{p,q}^s \in \{0,1\} \quad \forall s \in \{x,y\}, \{p,q\} = \{i,j\}.$$

- Ideal if $\quad lb_i^s + lb_j^s < L^s \ \mathit{for\ both}\ s \in \{x,y\}$

Formulation for One Binary Encoding

$$\frac{1}{2}\ell_p^s \leqslant c_p^s \leqslant L^s - \frac{1}{2}\ell_p^s \qquad \forall s \in \{x,y\}, p \in \{i,j\}$$

$$\ell_p^s \geqslant lb_p^s \qquad \forall s \in \{x,y\}, p \in \{i,j\}$$

$$c_i^y + \frac{1}{2}\ell_i^y \leqslant c_j^y - \frac{1}{2}\ell_j^y + L^y(w_1 + w_2)$$

$$c_i^x + \frac{1}{2}\ell_i^x \leqslant c_j^x - \frac{1}{2}\ell_j^x + L^x(1 - w_1 + w_2)$$

$$c_j^y + \frac{1}{2}\ell_j^y \leqslant c_i^y - \frac{1}{2}\ell_i^y + L^y(2 - w_1 - w_2)$$

$$c_j^x + \frac{1}{2}\ell_j^x \leqslant c_i^x - \frac{1}{2}\ell_i^x + L^x(1 + w_1 - w_2)$$

$$w \in \{0,1\}^2$$

- Not ideal

$$GB^4 \stackrel{\text{def}}{=} \{(0,0),(1,0),(0,1),(1,1)\}.$$

$$d_{i,j}^1 = \mathcal{B}_i \leftarrow_y \mathcal{B}_j, \qquad d_{i,j}^2 = \mathcal{B}_i \leftarrow_x \mathcal{B}_j, \qquad d_{i,j}^3 = \mathcal{B}_j \leftarrow_y \mathcal{B}_i, \qquad d_{i,j}^4 = \mathcal{B}_j \leftarrow_x \mathcal{B}_i.$$

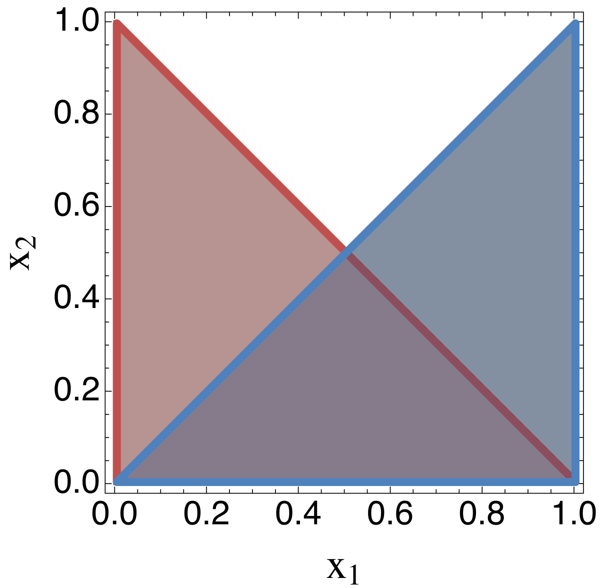
Topic: Alternative Disjunctions

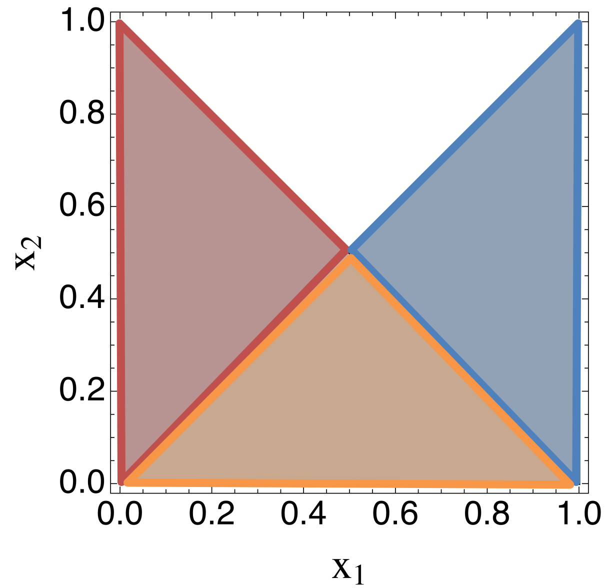
- Disjunction:

$$\{x \in [0,1]^2 \,:\, x_1 + x_2 \le 1\} \cup \{x \in [0,1]^2 \,:\, x_2 \le x_1\}$$

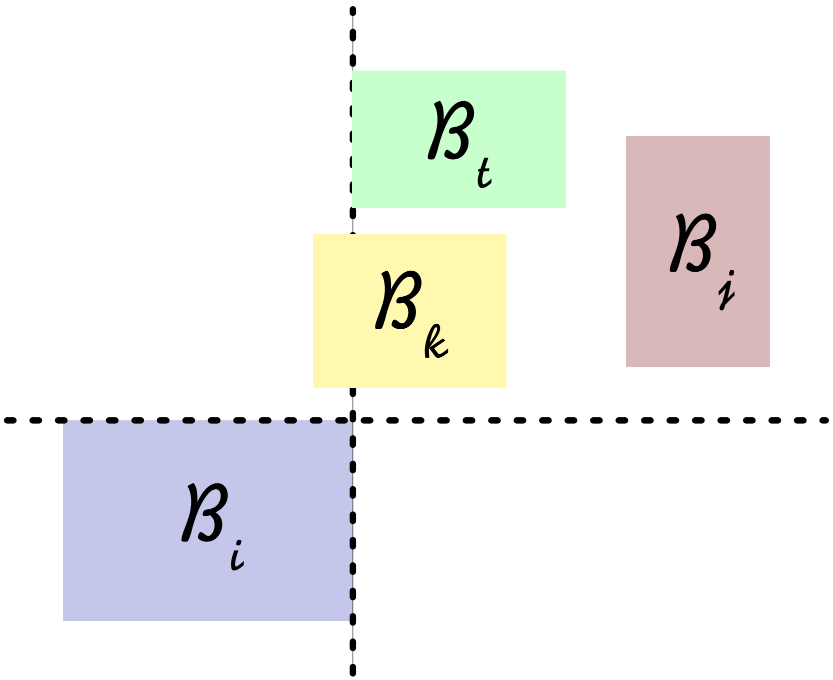
- Alternative Disjunction:

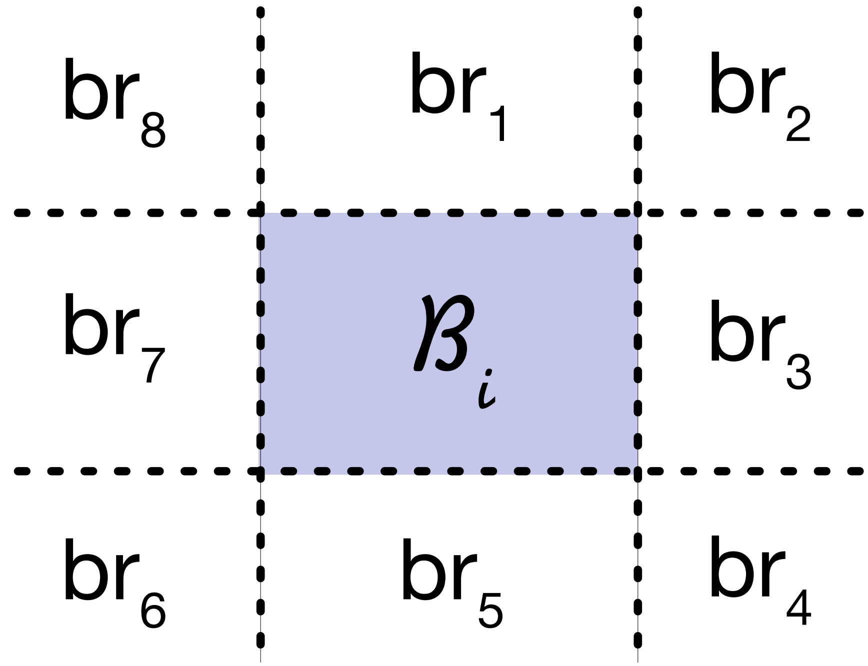
$$\left\{x \in [0,1]^2 \,:\, \begin{array}{r} x_1 + x_2 \le 1 \\ x_1 \le x_2 \end{array}\right\} \cup \left\{x \in [0,1]^2 \,:\, \begin{array}{r} x_1 + x_2 \le 1 \\ x_2 \le x_1 \end{array}\right\} \cup \left\{x \in [0,1]^2 \,:\, \begin{array}{r} x_1 + x_2 \ge 1 \\ x_2 \le x_1 \end{array}\right\}$$





Alternative Disjunctions for FLP





Definition 3. We say that $\mathcal{B}_i$ *does not precede* $\mathcal{B}_j$ (denoted by $\mathcal{B}_i \nleftarrow_s \mathcal{B}_j$) if $c_i^s + \frac{1}{2}\ell_i^s \geqslant c_j^s - \frac{1}{2}\ell_j^s$.

$$br_{i,j}^1 = (\mathcal{B}_i \leftarrow_y \mathcal{B}_j) \wedge (\mathcal{B}_i \nleftarrow_x \mathcal{B}_j) \wedge (\mathcal{B}_j \nleftarrow_x \mathcal{B}_i), \qquad br_{i,j}^2 = (\mathcal{B}_i \leftarrow_y \mathcal{B}_j) \wedge (\mathcal{B}_i \leftarrow_x \mathcal{B}_j)$$

$$br_{i,j}^3 = (\mathcal{B}_i \leftarrow_x \mathcal{B}_j) \wedge (\mathcal{B}_i \nleftarrow_y \mathcal{B}_j) \wedge (\mathcal{B}_j \nleftarrow_y \mathcal{B}_i), \qquad br_{i,j}^4 = (\mathcal{B}_i \leftarrow_x \mathcal{B}_j) \wedge (\mathcal{B}_j \leftarrow_y \mathcal{B}_i)$$

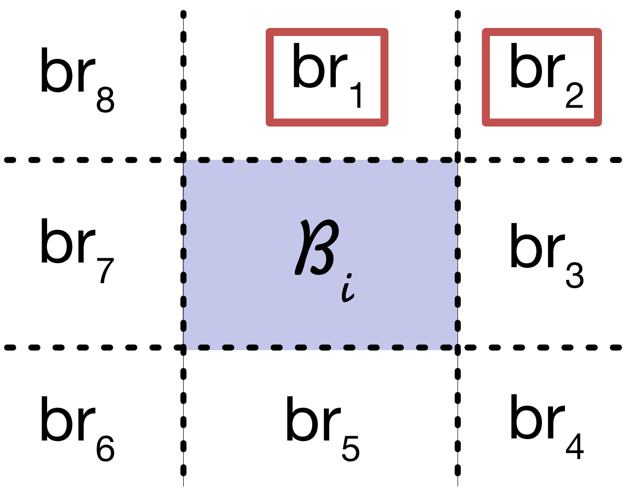
$$br_{i,j}^5 = (\mathcal{B}_j \leftarrow_y \mathcal{B}_i) \wedge (\mathcal{B}_i \nleftarrow_x \mathcal{B}_j) \wedge (\mathcal{B}_j \nleftarrow_x \mathcal{B}_i), \qquad br_{i,j}^6 = (\mathcal{B}_j \leftarrow_x \mathcal{B}_i) \wedge (\mathcal{B}_j \leftarrow_y \mathcal{B}_i)$$

$$br_{i,j}^7 = (\mathcal{B}_j \leftarrow_x \mathcal{B}_i) \wedge (\mathcal{B}_i \nleftarrow_y \mathcal{B}_j) \wedge (\mathcal{B}_j \nleftarrow_y \mathcal{B}_i), \qquad br_{i,j}^8 = (\mathcal{B}_j \leftarrow_x \mathcal{B}_i) \wedge (\mathcal{B}_i \leftarrow_y \mathcal{B}_j).$$

Alternative Disjunction + Ad-hoc Encoding

- Encoding: $C^8 \overset{\text{def}}{=} \{\boxed{\mathbf{e}^1}, \boxed{\mathbf{e}^1 + \mathbf{e}^2}, \mathbf{e}^2, \mathbf{e}^2 + \mathbf{e}^3, \mathbf{e}^3, \mathbf{e}^3 + \mathbf{e}^4, \mathbf{e}^4, \mathbf{e}^4 + \mathbf{e}^1\} \subseteq \{0,1\}^4$
- Formulation (“looks” ideal):

$$
\begin{aligned}
\frac{1}{2}\ell_p^s + lb_q^s z_{q,p}^s \leqslant c_p^s \leqslant L^s - \frac{1}{2}\ell_p^s - lb_q^s z_{p,q}^s \quad & \forall s \in \{x,y\}, \{p,q\} = \{i,j\} \\
c_p^s + \frac{1}{2}\ell_p^s \leqslant c_q^s - \frac{1}{2}\ell_q^s + L^s(1 - z_{p,q}^s) \quad & \forall s \in \{x,y\}, \{p,q\} = \{i,j\} \\
\ell_p^s \geqslant lb_p^s \quad & \forall s \in \{x,y\}, p \in \{i,j\} \\
\boxed{z_{i,j}^y} \boxed{+ z_{i,j}^x} + z_{j,i}^y + z_{j,i}^x \geqslant 1 \quad & \\
z_{i,j}^s + z_{j,i}^s \leqslant 1 \quad & \forall s \in \{x,y\} \\
z_{p,q}^s \in \{0,1\} \quad & \forall s \in \{x,y\}, \{p,q\} = \{i,j\} \\
c_p^s + \frac{1}{2}\ell_p^s + L^s z_{p,q}^s \geqslant c_q^s - \frac{1}{2}\ell_q^s + (lb_p^s + lb_q^s)(z_{i,j}^s + z_{j,i}^s) \quad & \forall s \in \{x,y\}, \{p,q\} = \{i,j\}.
\end{aligned}
$$



Summary for Formulations for FLP

- Alternative disjunction + ad-hoc encoding
  - Better than unary or binary encoding
  - Can provide an advantage over all known formulations (some are very elaborate)
  - Advantage is not consistent
- More details in:
  - Huchette, Dey and V., "Strong mixed-integer formulations for the floor layout problem", 2016, arXiv:1602.07760
  - Incorporating common constraints into disjunctions
  - Incorporating objective function into disjunctions