# Mixed Integer Programming Formulations
# Lecture 1

# Non-Extended

Juan Pablo Vielma

Massachusetts Institute of Technology

IPCO Summer School

University of Liège, Belgique

May, 2016.

https://github.com/juan-pablo-vielma/IPCO-2016-Summer-School

# MIP Formulation Techniques


- Special Ordered Sets of Type 2 (SOS2)
  - $\lambda \in \mathbb{R}^{n+1}_{+}, \quad \sum_{i=1}^{n+1} \lambda_i = 1$
  - at most 2 $\lambda_i > 0$ and $\lambda_i > 0 \wedge \lambda_j > 0 \Rightarrow |i - j| \leq 1$

$$\overbrace{\phantom{0 \leq \lambda_1 \leq y_1}}^{2(n+1)}$$

$$0 \leq \lambda_1 \leq y_1$$

$$0 \leq \lambda_2 \leq y_1 + y_2$$

$$0 \leq \lambda_3 \leq y_2 + y_3$$

$$0 \leq \lambda_4 \leq y_3 + y_4$$

$$0 \leq \lambda_5 \leq y_4$$

Bounds

General Inequalities

$$\sum_{i=1}^{5} \lambda_i = 1$$

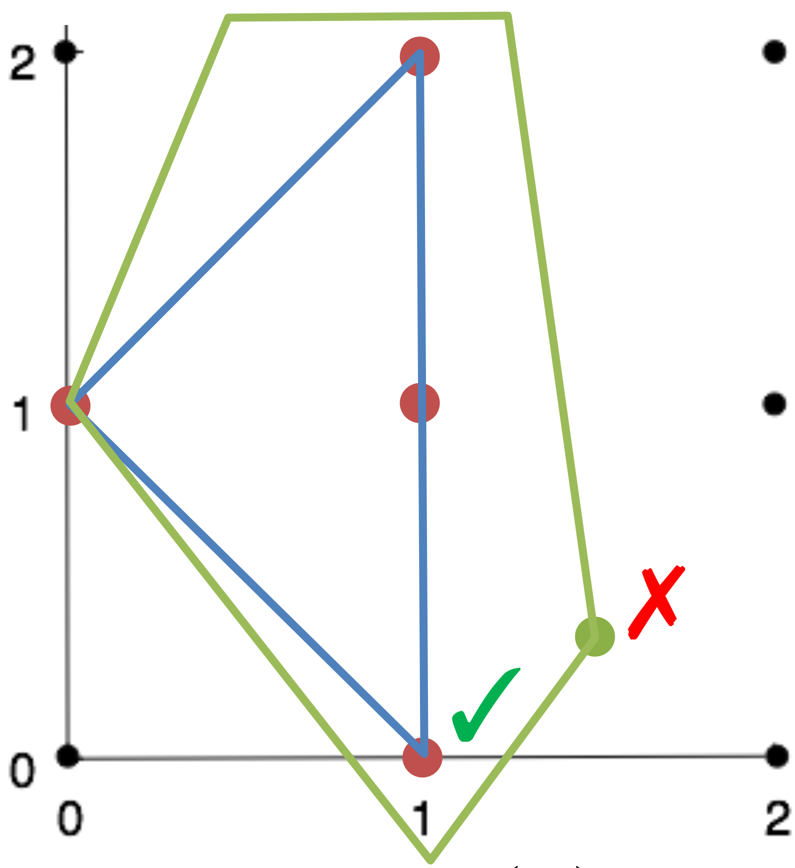
$$y \in \{0,1\}^4, \quad \sum_{i=1}^{4} y_i = 1$$

- Minimum # of (general) inequalities?
  - Integral formulation.
  - Non-integral formulation.

# Pure Integer Formulations

- Pure Integer $S \subseteq \mathbb{Z}^d$ s.t. $\operatorname{conv}(S) \cap \mathbb{Z}^d = S$

$$P \cap \mathbb{Z}^d = S \; (P \subseteq \mathbb{R}^d)$$




$$Q := \operatorname{conv}(S)$$

- Formulation:
  - LP relaxation $P$
  - Integrality
- Formulation strength:
  - $P$ = convex hull of $S$
  - or equivalently $P$ has **integral** extreme points
- Size:
  - # of inequalities
  - no need for auxiliary variables (non-extended)

# Outline

- MIP formulations of disjunctive constraints
  - Differences with pure integer
  - Sizes, strength, two classes of formulations
- Towards optimal formulations through encodings and embeddings
- Relaxation complexity = smallest formulation
- Embedding complexity = smallest integral formulation
- Practical formulation techniques
- Nonlinear MIP?
- Julia / JuMP tutorial

1

3

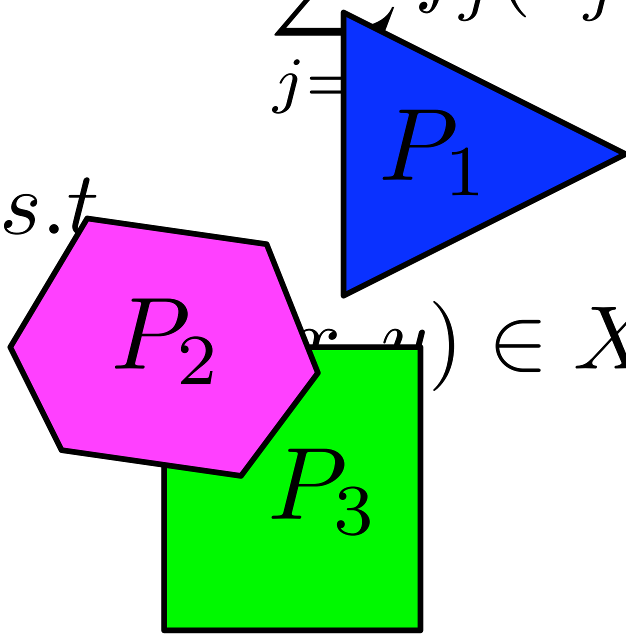
# (Linear) Mixed 0-1 Integer Formulations

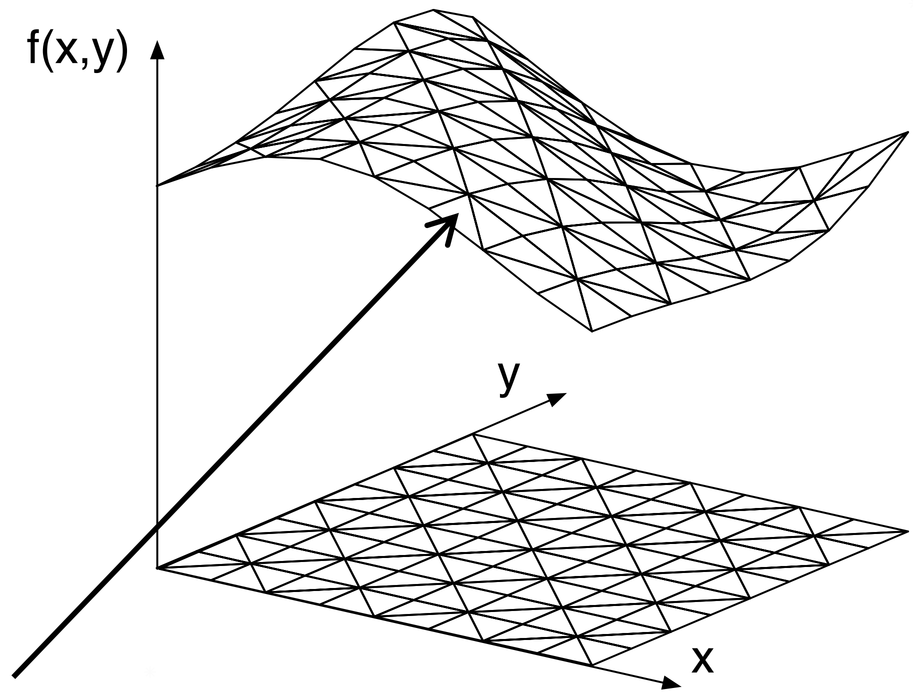
- Modeling Finite Alternatives = Unions of Polyhedra

$$x \in \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^d$$

$$\min \sum_{j=1}^{m} f_j(x_j, y_j)$$

$$s.t. \quad (x, y) \in X$$




$P_i$

$$\mathrm{gr}(f) = \bigcup_{i=1}^{n} P_i$$

# Size of Smallest 0-1 Formulation for $x \in \bigcup_{i=1}^{n} P_i$

- Standard **ideal (integral)** **extended** formulation for $P_i = \{x \in \mathbb{R}^d \ : \ A^i x \le b^i\}$ (Balas, Jeroslow and Lowe):

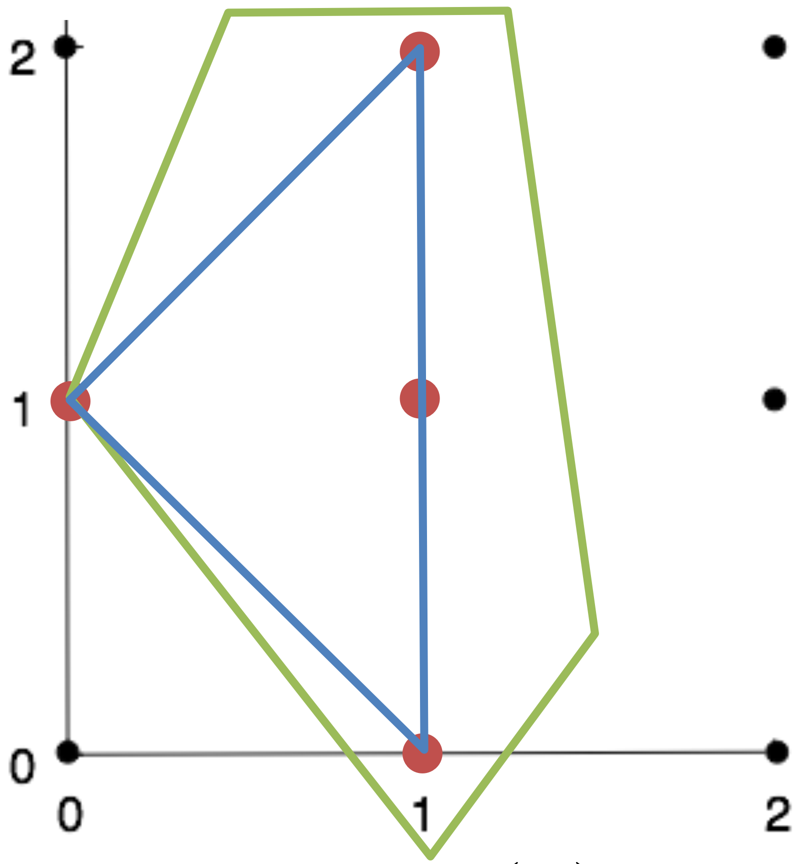
$$
\begin{aligned}
A^i x^i &\le b^i y_i & & \forall i \in \{1,\ldots,n\} \\
\sum_{i=1}^{n} x^i &= x, & x^i \in \mathbb{R}^d & \quad \forall i \in \{1,\ldots,n\} \\
\sum_{i=1}^{n} y_i &= 1, & y \in \{0,1\}^n &
\end{aligned}
$$

- What about non-extended (i.e. no variables copies) ?
- What about non-ideal? (i.e. **some** fractional extreme pts.)?
- What about **precise** lower/upper bounds on size?

# IP v/s MIP 1: Auxiliary Variables

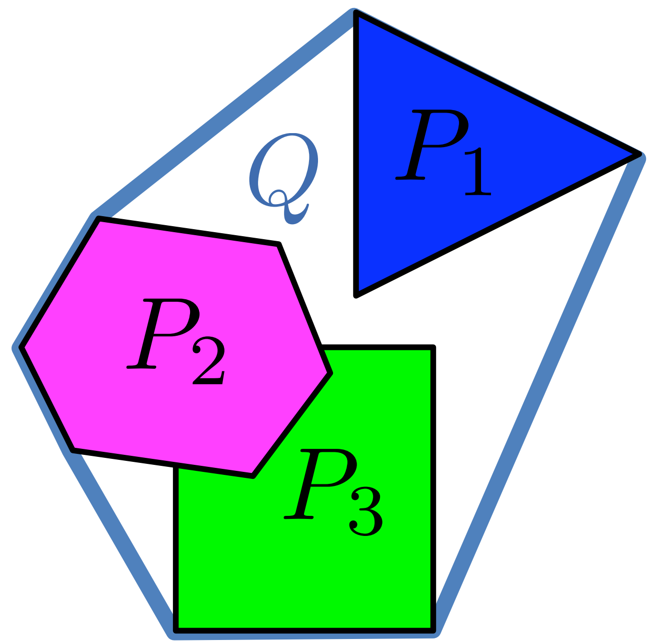
- Pure Integer $S \subseteq \mathbb{Z}^d$
- Mixed-Integer $S = \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^d$

$$P \cap \mathbb{Z}^d = S \; (P \subseteq \mathbb{R}^d)$$

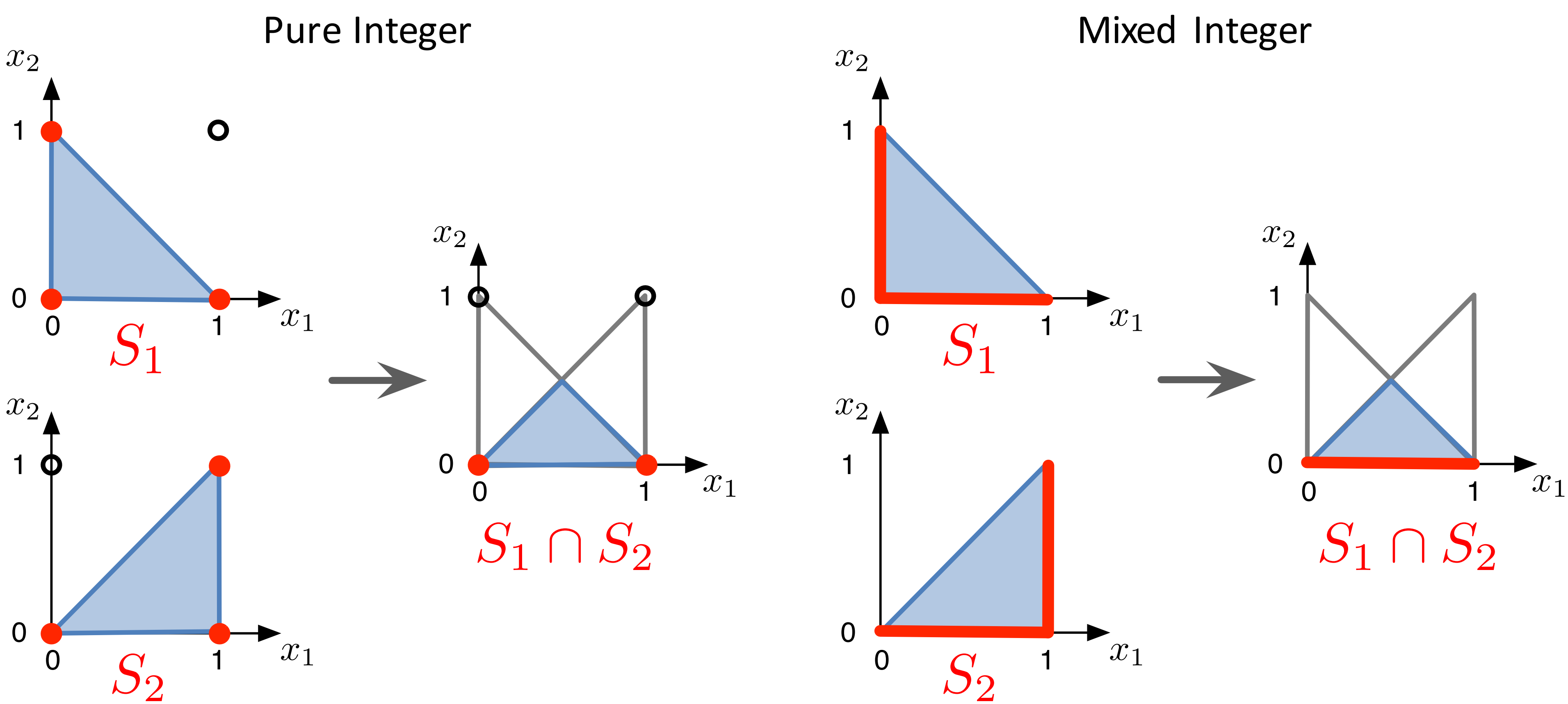



$$Q := \operatorname{conv}(S)$$

$$Q := \operatorname{conv}(S)?$$



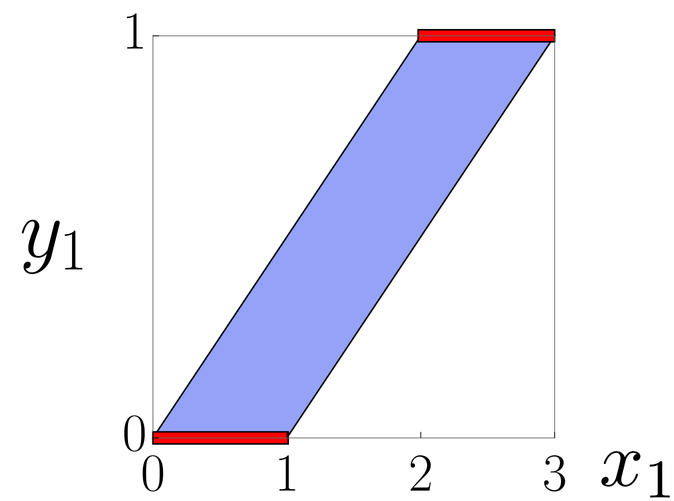
# Convex Hull and Combining Formulations



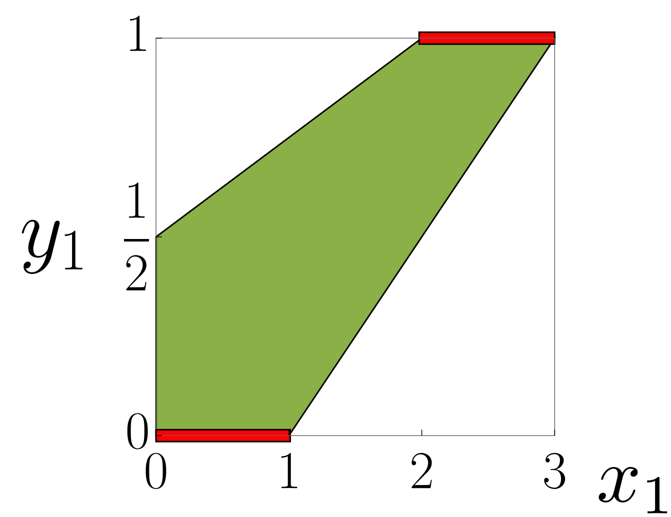

- Mixed Integer <u>needs</u> some 0-1 auxiliary variables

# MIP Needs Integer Auxiliary Variables

- $S = [0,1] \cup [2,3]$
- $$\boxed{\begin{array}{c} 2y_1 \le x_1 \le 1 + 2y_1 \\ 0 \le y_1 \le 1 \end{array}}\ P$$ $$y_1 \in \mathbb{Z}$$




- $$\boxed{\begin{array}{c} 4y_1 - 2 \le x_1 \le 1 + 2y_1 \\ 0 \le y_1 \le 1 \end{array}}\ P$$ $$y_1 \in \mathbb{Z}$$

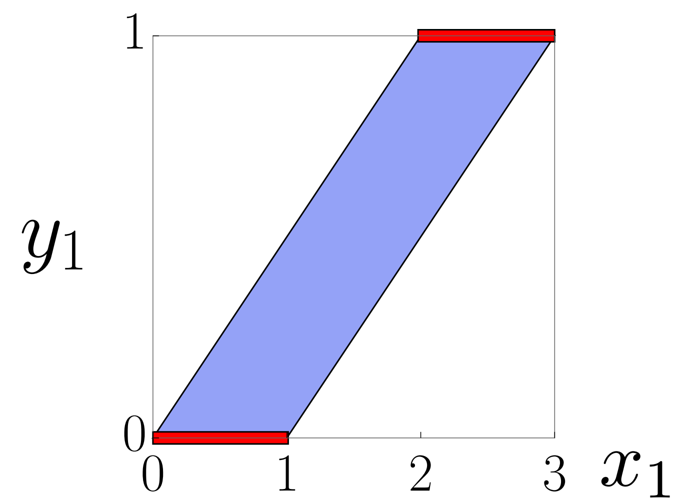


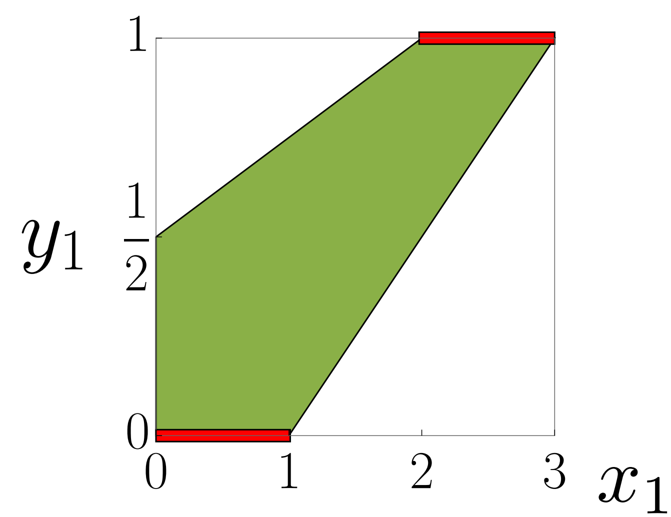

- $S \subseteq \mathbb{R}^n, \quad P \subseteq \mathbb{R}^{n+k}$

$$\mathrm{Proj}_x \left(P \cap \left(\mathbb{R}^n \times \mathbb{Z}^k\right)\right) = S$$

# For MIP convex hull (sharp) ≠ integral (ideal)

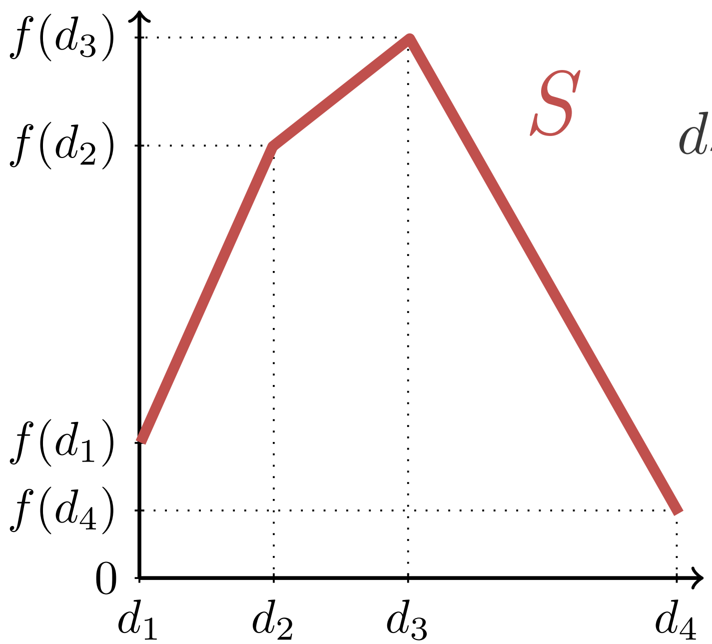
- $S = [0,1] \cup [2,3]$
- $$\boxed{\begin{aligned} 2y_1 \le x_1 &\le 1 + 2y_1 \\ 0 \le y_1 &\le 1 \end{aligned}} \quad P$$
  $$y_1 \in \mathbb{Z}$$
- $$\boxed{\begin{aligned} 4y_1 - 2 \le x_1 &\le 1 + 2y_1 \\ 0 \le y_1 &\le 1 \end{aligned}} \quad P$$
  $$y_1 \in \mathbb{Z}$$
- $\operatorname{Proj}_x(P) = \operatorname{conv}(S)$
- $\operatorname{ext}(P) \subseteq \mathbb{R}^n \times \mathbb{Z}^k$




$\operatorname{Proj}_x\left(P \cap \left(\mathbb{R}^n \times \mathbb{Z}^k\right)\right) = S$

# Naïve Formulation for Piecewise Linear Functions




$$
\begin{aligned}
d_i - (d_i - d_1)(1 - y_i) &\leq x & \forall i \in [n] \\
d_{i+1} + (d_{i+1} - d_k)(1 - y_i) &\geq x & \forall i \in [n] \\
m_i x + c_i - \underline{M}_i (1 - y_i) &\leq z & \forall i \in [n] \\
m_i x + c_i + \overline{M}_i (1 - y_i) &\geq z & \forall i \in [n]
\end{aligned}
$$

$$\sum_{i=1}^{k} y_i = 1$$

$$y \in \{0, 1\}^n$$

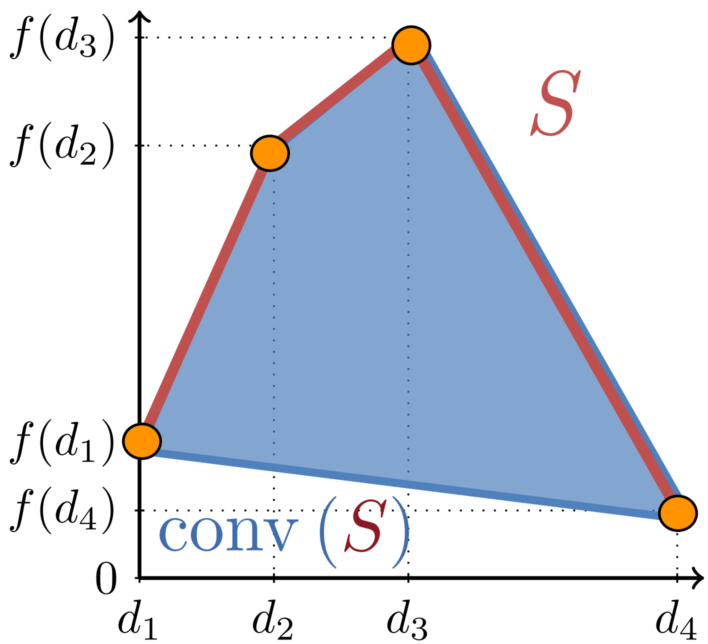
$$f(x) = \begin{cases} m_1 x + c_1 & x \in [d_1, d_2] \\ & \vdots \\ m_k x + c_k & x \in [d_k, d_{k+1}] \end{cases}$$

$$\underline{M}_i := \max_{j=1}^{n+1} \{ m_i d_j + c_i - f(d_j) \}$$

$$\overline{M}_i := \max_{j=1}^{n+1} \{ f(d_j) - m_i d_j - c_i \}$$

Exercise: Show it is not always ideal or sharp

# Better Formulation (CC) <=> SOS2




$$f(x) = \begin{cases} m_1 x + c_1 & x \in [d_1, d_2] \\ & \vdots \\ m_k x + c_k & x \in [d_k, d_{k+1}] \end{cases}$$

$$\sum_{i=1}^{n+1} \lambda_i d_i = x$$

$$\sum_{i=1}^{n+1} \lambda_i f(d_i) = z$$

$$\sum_{i=1}^{n+1} \lambda_i = 1$$

$$\lambda_1 \leq y_1$$

$$\lambda_i \leq y_{i-1} + y_i \quad \forall 2 \leq i \leq n$$

$$\lambda_{n+1} \leq y_k$$

$$\sum_{i=1}^{n} y_i = 1$$

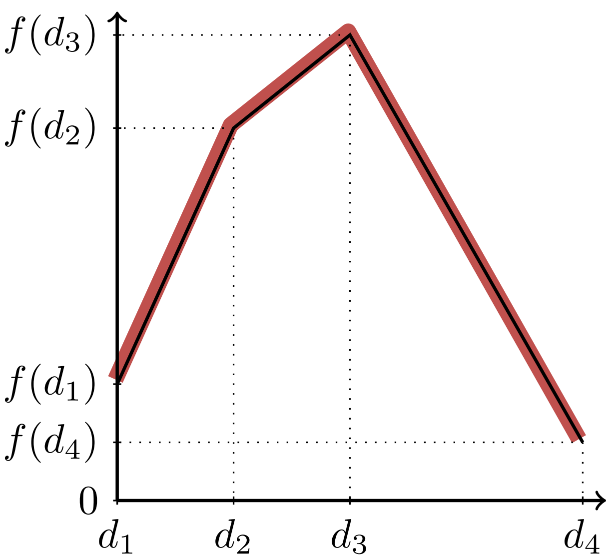
$$y \in \{0,1\}^n$$

Is always sharp. Exercise: Show it is not ideal.

# Non-Extended Formulation for PWL Functions

$$S = \operatorname{gr}(f) = \bigcup_{i=1}^{n} \left\{ (x,z) \in \mathbb{R}^2 \ : \ \begin{array}{r} d_i \le x \le d_{i+1} \\ m_i x + c_i = z \end{array} \right\}$$

BigM Formulation:



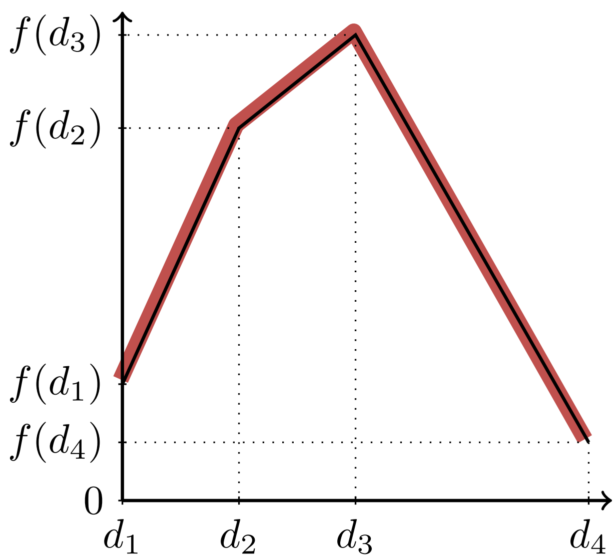

$$\begin{aligned} d_i - (d_i - d_1)(1 - y_i) &\le x && \forall i \in [n] \\ d_{i+1} + (d_{i+1} - d_k)(1 - y_i) &\ge x && \forall i \in [n] \\ m_i x + c_i - \underline{M}_i (1 - y_i) &\le z && \forall i \in [n] \\ m_i x + c_i + \overline{M}_i (1 - y_i) &\ge z && \forall i \in [n] \\ \sum_{i=1}^{n} y_i &= 1 \\ y &\in \{0,1\}^n \end{aligned}$$

- Some (0-1) auxiliary variables needed for a valid formulation
  - For now n 0-1auxiliary variables that add exactly to 1

# Extended Formulation for PWL Functions

$$S = \operatorname{gr}(f) = \bigcup_{i=1}^{n} \left\{ (x, z) \in \mathbb{R}^2 \; : \; \begin{array}{r} d_i \le x \le d_{i+1} \\ m_i x + c_i = z \end{array} \right\}$$

MC Formulation:



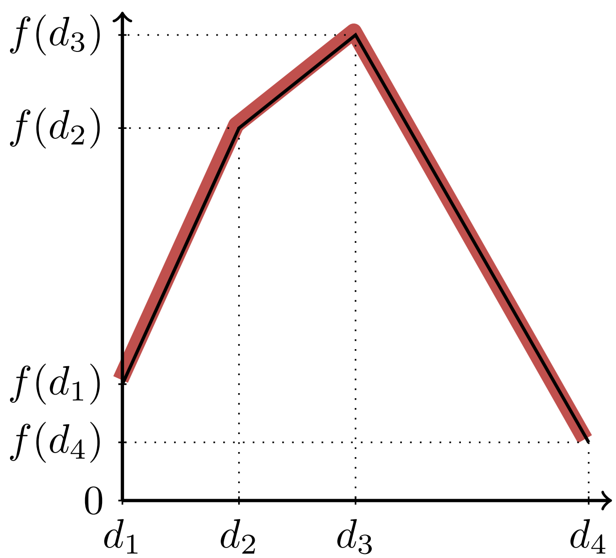

$$\begin{aligned} d_i y_i \le x^i &\le d_{i+1} y_i && \forall i \in [n] \\ m_i x^i + c_i y_i &= z^i && \forall i \in [n] \\ \sum\nolimits_{i=1}^{n} x^i &= x \\ \sum\nolimits_{i=1}^{n} z^i &= z \\ \sum\nolimits_{i=1}^{n} y_i &= 1 \\ y &\in \{0,1\}^n \end{aligned}$$

- Some continuous auxiliary variables only <u>used</u> to construct strong formulations

# What Are The Natural Original Variables?

$$S = \mathrm{gr}\,(f) = \bigcup_{i=1}^{n} \left\{ (x,z) \in \mathbb{R}^2 \; : \; \begin{array}{r} d_i \leq x \leq d_{i+1} \\ m_i x + c_i = z \end{array} \right\}$$

CC Formulation:




$$\sum_{i=1}^{n+1} \lambda_i d_i = x$$

$$\sum_{i=1}^{n+1} \lambda_i f(d_i) = z$$

$$\sum_{i=1}^{n+1} \lambda_i = 1$$

$$\lambda_1 \leq y_1$$

$$\lambda_i \leq y_{i-1} + y_i \quad \forall 2 \leq i \leq n$$

$$\lambda_{n+1} \leq y_k$$

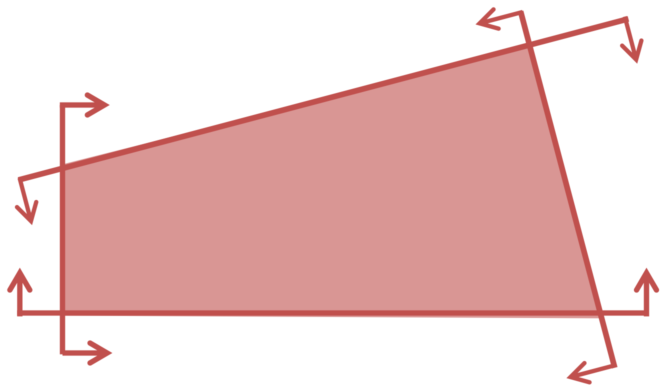
$$\sum_{i=1}^{n} y_i = 1$$

$$y \in \{0,1\}^n$$

# Two Types of Polytopes

- $\mathcal{H}$*-polyhedron* iff $\exists\ A \in \mathbb{Q}^{m\times d}$ and $b \in \mathbb{Q}^{m}$ s.t.

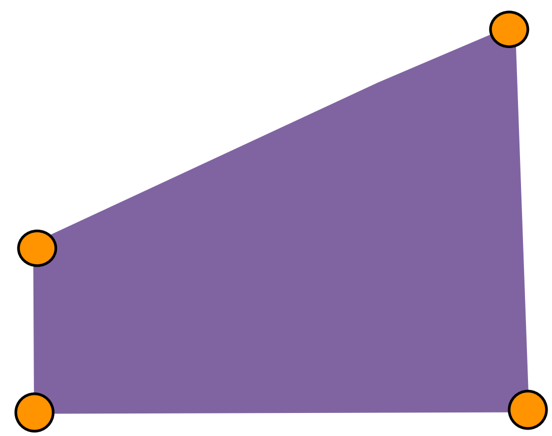
$$P = \{x \in \mathbb{Q}^d \ : \ Ax \leq b\}$$



- $\mathcal{V}$*-polyhedron* iff $\exists$ finite sets $V \subseteq \mathbb{Q}^d$ s.t.

$$P = \operatorname{conv}(V)$$

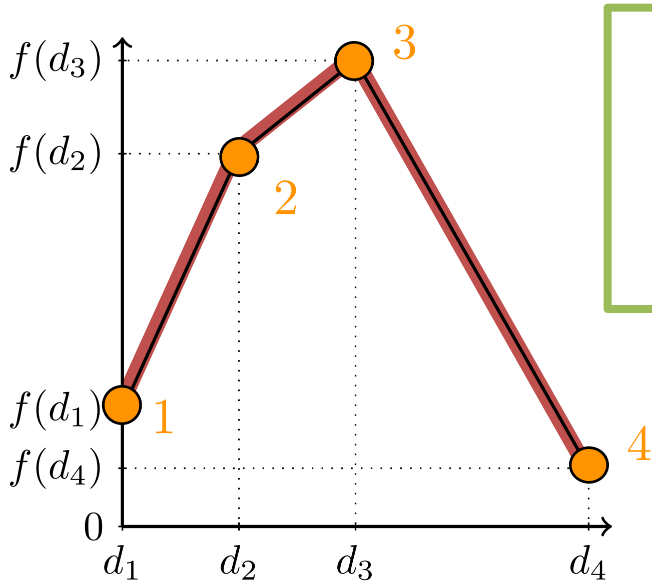
can take $V = \operatorname{ext}(P)$



# Vertex / Extreme Point Variables Also Natural

$$S = \mathrm{gr}\left(f\right) = \bigcup_{i=1}^{n} \left\{ (x,z) \in \mathbb{R}^2 \; : \; \begin{array}{r} d_i \leq x \leq d_{i+1} \\ m_i x + c_i = z \end{array} \right\}$$

CC Formulation:




$$\boxed{\begin{aligned} \sum\nolimits_{i=1}^{n+1} \lambda_i d_i &= x \\ \sum\nolimits_{i=1}^{n+1} \lambda_i f\left(d_i\right) &= z \end{aligned}}$$

Linear Transformation

$$\sum\nolimits_{i=1}^{n+1} \lambda_i = 1$$

$$\lambda_1 \leq y_1$$

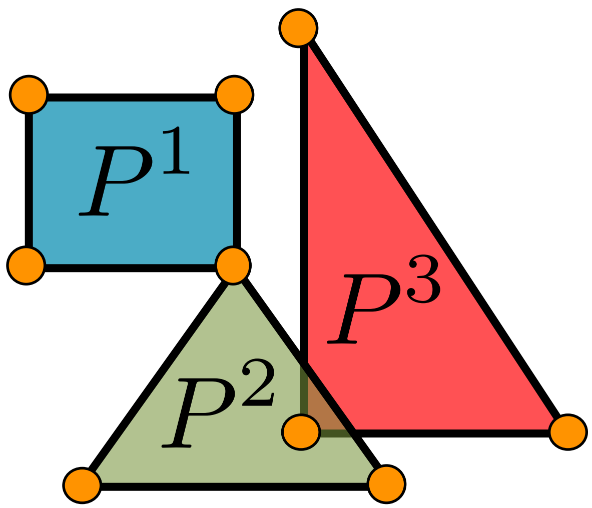
$$\lambda_i \leq y_{i-1} + y_i \qquad \forall 2 \leq i \leq n$$

$$\lambda_{n+1} \leq y_k$$

$$\sum\nolimits_{i=1}^{n} y_i = 1$$

$$y \in \{0,1\}^n$$

# Linear Transformation for $\mathcal{V}$-Formulation




$$V := \bigcup_{i=1}^{n} \operatorname{ext}\left(P^i\right)$$

$$\Delta^V := \left\{\lambda \in \mathbb{R}_+^V \; : \; \sum_{v \in V} \lambda_v = 1\right\}$$

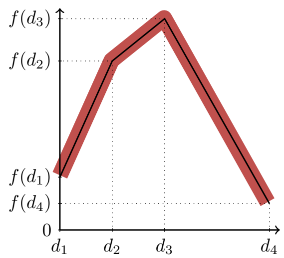
$$Q^i := \left\{\lambda \in \Delta^V \; : \; \lambda_v \leq 0 \quad \forall v \notin \operatorname{ext}\left(P^i\right)\right\}$$

$$x \in \bigcup_{i=1}^{n} P^i \quad \longleftrightarrow \quad \begin{array}{l} x = \sum_{v \in V} v \lambda_v \\ \lambda \in \bigcup_{i=1}^{n} Q^i \end{array}$$

Dependent on specific data from polytopes

Purely Combinatorial

# Combinatorial Structure v/s Data




$$S = \mathrm{gr}\left(f\right) = \bigcup_{i=1}^{n} \left\{ (x,z) \in \mathbb{R}^2 \ : \begin{array}{r} d_i \leq x \leq d_{i+1} \\ m_i x + c_i = z \end{array} \right\}$$

## BigM Formulation:

$$\boxed{\begin{array}{r} d_i - \left(d_i - d_1\right)\left(1 - y_i\right) \leq x \\ d_{i+1} + \left(d_{i+1} - d_k\right)\left(1 - y_i\right) \geq x \\ m_i x + c_i - \underline{M}_i\left(1 - y_i\right) \leq z \\ m_i x + c_i + \overline{M}_i\left(1 - y_i\right) \geq z \end{array}} \quad \begin{array}{l} \forall i \in [n] \\ \forall i \in [n] \\ \forall i \in [n] \\ \forall i \in [n] \end{array}$$

$$\sum_{i=1}^{n} y_i = 1$$

$$y \in \{0,1\}^n$$

## CC Formulation:

$$\boxed{\begin{array}{r} \sum_{i=1}^{n+1} \lambda_i d_i = x \\ \sum_{i=1}^{n+1} \lambda_i f\left(d_i\right) = z \end{array}}$$

$$\boxed{\begin{array}{rl} \sum_{i=1}^{n+1} \lambda_i = 1 & \\ \lambda_1 \leq y_1 & \\ \lambda_i \leq y_{i-1} + y_i & \forall 2 \leq i \leq n \\ \lambda_{n+1} \leq y_k & \\ \sum_{i=1}^{n} y_i = 1 & \\ y \in \{0,1\}^n & \end{array}}$$

# Constructing Integral Non-Extended Formulations

- Add inequalities to simple non-extended formulations to make them integral

$$\begin{aligned} Ax + Dy &\leq d \\ y &\in \mathbb{Z}^n \end{aligned} \longrightarrow \begin{aligned} &(x,y) \in \operatorname{conv}\left(\left\{(x,y) \in \mathbb{R}^d \times \mathbb{Z}^n \;:\; Ax + Dy \leq b\right\}\right) \\ &y \in \mathbb{Z}^n \end{aligned}$$

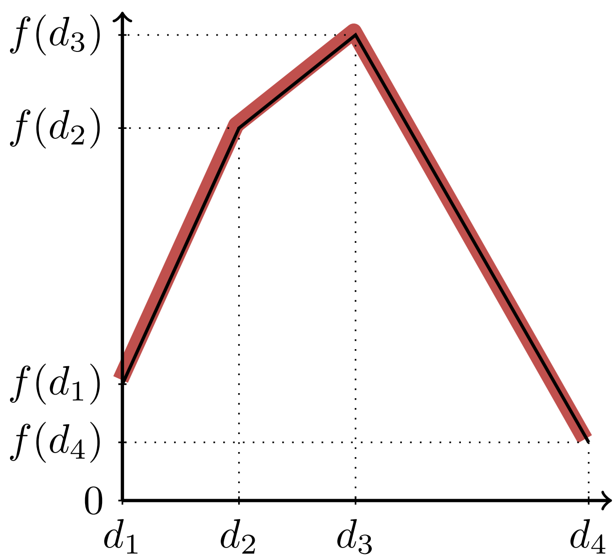
$$\begin{aligned} x &= L\lambda \\ A\lambda + Dy &\leq d \\ y &\in \mathbb{Z}^n \end{aligned} \longrightarrow \begin{aligned} &x = L\lambda \\ &(\lambda,y) \in \operatorname{conv}\left(\left\{(\lambda,y) \in \mathbb{R}^V \times \mathbb{Z}^n \;:\; A\lambda + Dy \leq b\right\}\right) \\ &y \in \mathbb{Z}^n \end{aligned}$$

Purely Combinatorial / Data Independent = Often Simpler

# Making CC Ideal

$$S = \mathrm{gr}\,(f) = \bigcup_{i=1}^{n} \left\{ (x,z) \in \mathbb{R}^2 \; : \; \begin{array}{r} d_i \leq x \leq d_{i+1} \\ m_i x + c_i = z \end{array} \right\}$$

ICC Formulation:




$$\sum_{i=1}^{n+1} \lambda_i d_i = x$$

$$\sum_{i=1}^{n+1} \lambda_i f\,(d_i) = z$$

$$\sum_{i=1}^{n+1} \lambda_i = 1$$

$$\sum_{i=1}^{l} \lambda_i \leq \sum_{i=1}^{l} y_i \qquad \forall l \in [n-1]$$

$$\sum_{i=l+2}^{n+1} \lambda_i \leq \sum_{i=l+1}^{n} y_i \qquad \forall l \in [n-1]$$

$$\sum_{i=1}^{n} y_i = 1$$

$$y \in \{0,1\}^n$$

# Sometimes Does Not Work Well

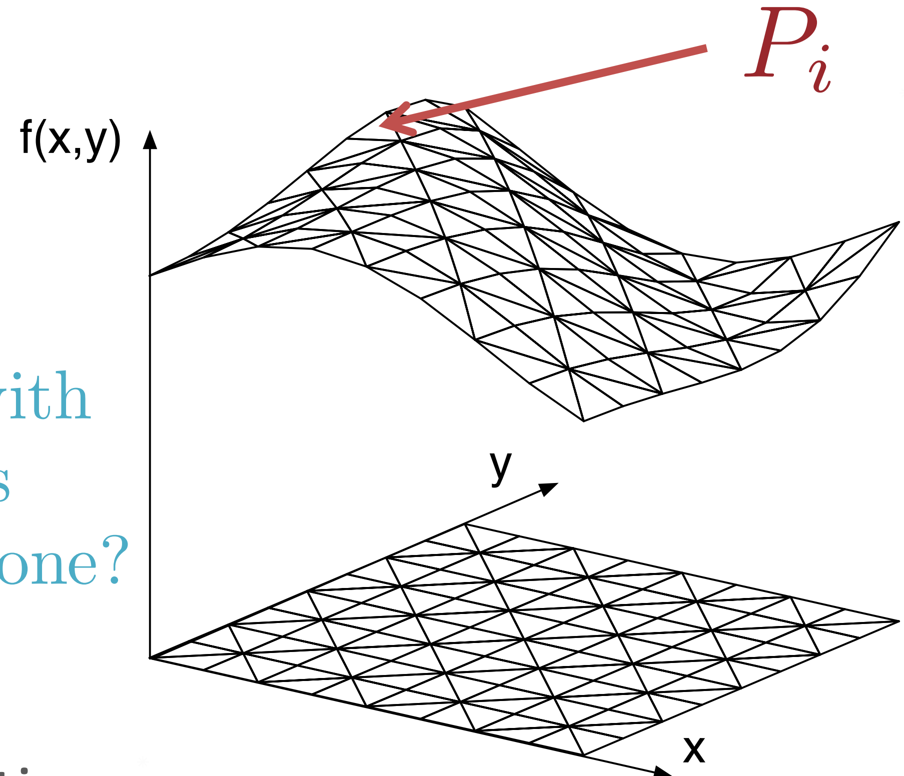
$$\sum_{v\in V} v\lambda_v = x$$

$$\sum_{v\in V} \lambda_v = 1$$

$$\lambda_v \leq \sum_{i:v\in \operatorname{ext}(P_i)} y_i$$

$$\boxed{\begin{array}{c} \sum_{i=1}^{n} y_i = 1 \\ y \in \{0,1\}^n, \quad \lambda \in \mathbb{R}_+^V \end{array}}$$

Why stick with 0-1 variables that add to one?




- Can also make ideal
  - Number of additional inequalities

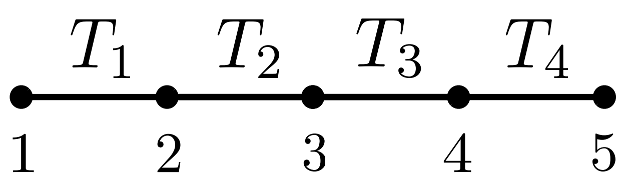
$$\binom{2\sqrt{n/2}}{\sqrt{n/2}}$$

$$\operatorname{gr}(f) = \bigcup_{i=1}^{n} P_i$$

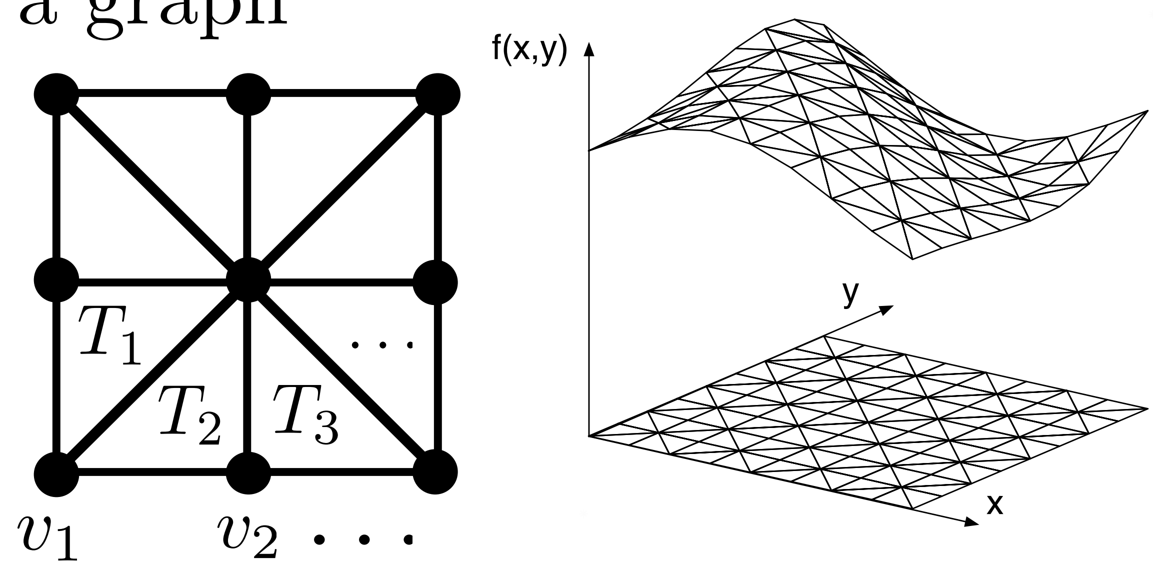
# Encodings and Embeddings

# Examples = $\mathcal{V}$-polytopes = Faces of Simplex

- $\Delta^V := \left\{ \lambda \in \mathbb{R}^V_+ \, : \, \sum_{v \in V} \lambda_v = 1 \right\},$
- $P_i = \{ \lambda \in \Delta^V \, : \, \lambda_v = 0 \quad \forall v \notin T_i \}$
- $\lambda \in \bigcup_{i=1}^n P_i$
- $T_i$ = cliques of a graph

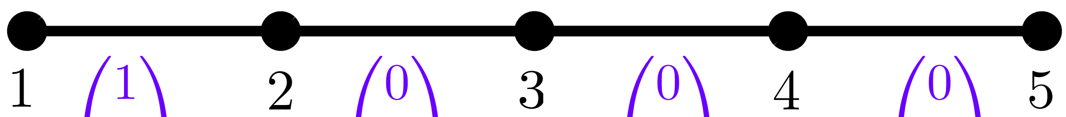



SOS2




Piecewise Linear Functions

# Formulations and encodings




$$e^1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ e^2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ e^3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ e^4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$y \in \{0,1\}^4, \quad \boxed{\begin{aligned} \sum_{i=1}^{5} \lambda_i &= 1 \\ \sum_{i=1}^{4} y_i &= 1 \end{aligned}}$$

$Q$ = LP relaxation

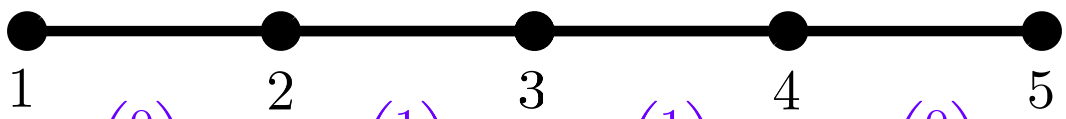
$$\boxed{\begin{aligned} 0 \le \lambda_1 &\le y_1 \\ 0 \le \lambda_2 &\le y_1 + y_2 \\ 0 \le \lambda_3 &\le y_2 + y_3 \\ 0 \le \lambda_4 &\le y_3 + y_4 \\ 0 \le \lambda_5 &\le y_4 \end{aligned}}$$

$$\boxed{\begin{gathered} (\lambda, y) \in Q \cap \left(\mathbb{R}^5 \times \mathbb{Z}^4\right) \\ \Updownarrow \\ y = e^i \ \wedge \ \lambda \in P_i \end{gathered}}$$

Unary Encoding

$$P_i := \{\lambda \in \Lambda^5 \ : \ \lambda_j = 0 \quad j \notin \{i, i+1\}\}$$

# Binary Encoded **Ideal** Formulation for SOS2




$h^1 = \binom{0}{1},\ h^2 = \binom{1}{1},\ h^3 = \binom{1}{0},\ h^4 = \binom{0}{0}$

$y \in \{0,1\}^2,\ \boxed{\sum_{i=1}^{5} \lambda_i = 1}$

$Q$ = LP relaxation

$$
\boxed{
\begin{aligned}
0 \leq \lambda_1 + \lambda_5 &\leq 1 - y_1 \\
0 \leq \lambda_3 \qquad\quad &\leq y_1 \\
0 \leq \lambda_4 + \lambda_5 &\leq 1 - y_2 \\
0 \leq \lambda_1 + \lambda_2 &\leq y_2
\end{aligned}
}
$$

$$
\boxed{
\begin{gathered}
(\lambda, y) \in Q \cap \left(\mathbb{R}^5 \times \mathbb{Z}^2\right) \\
\Updownarrow \\
y = h^i \ \wedge\ \lambda \in P_i
\end{gathered}
}
$$

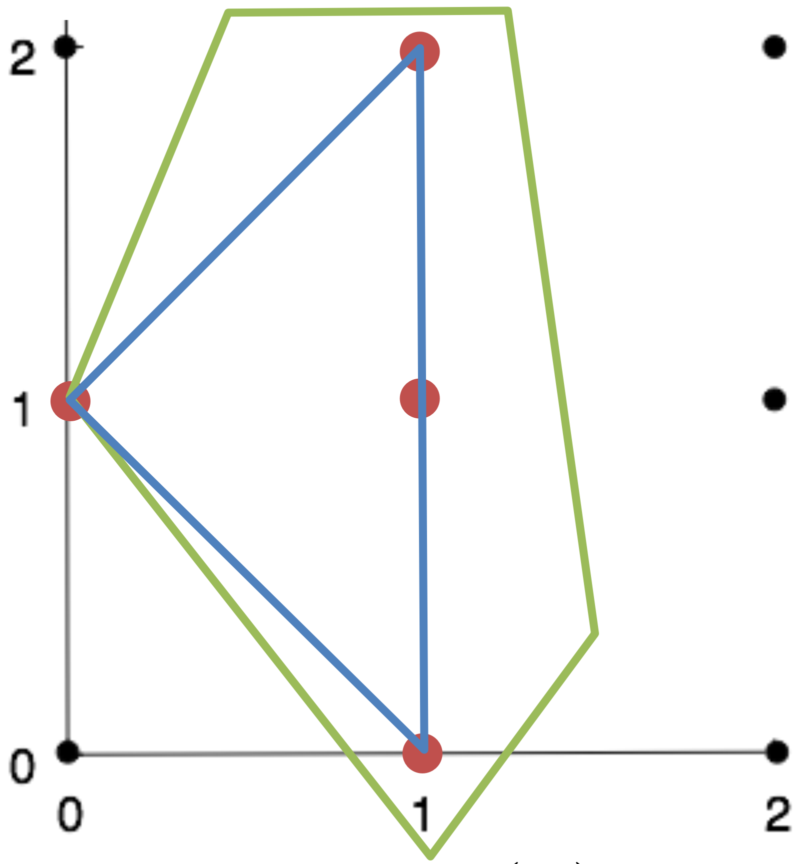
Binary Encoding

$$P_i := \left\{\lambda \in \Lambda^5 \ : \ \lambda_j = 0 \quad j \notin \{i, i+1\}\right\}$$

# Non-Extended Formulations

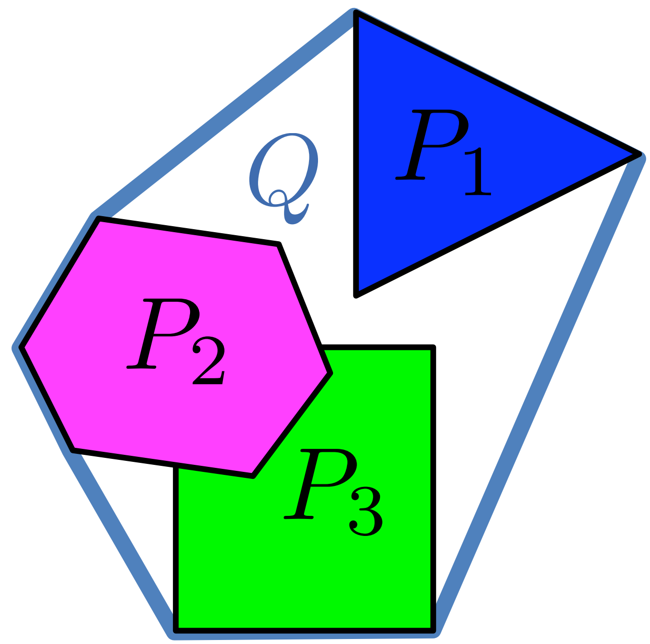
- Pure Integer $S \subseteq \mathbb{Z}^d$
- Mixed-Integer $S = \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^d$

$$P \cap \mathbb{Z}^d = S \ (P \subseteq \mathbb{R}^d)$$

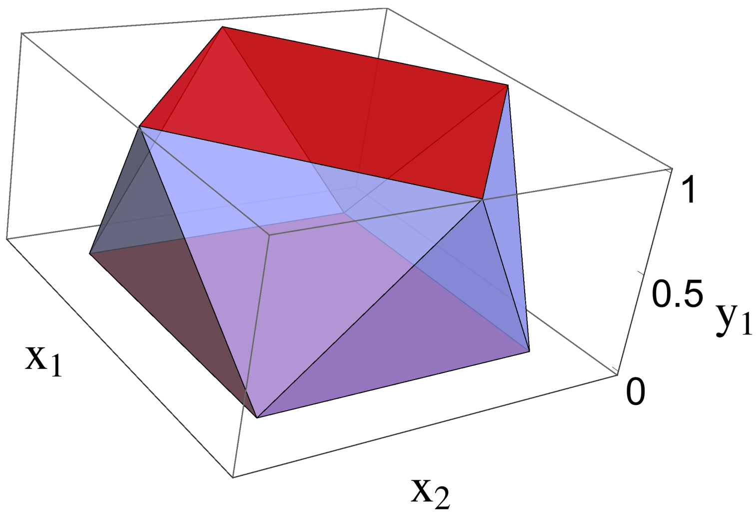


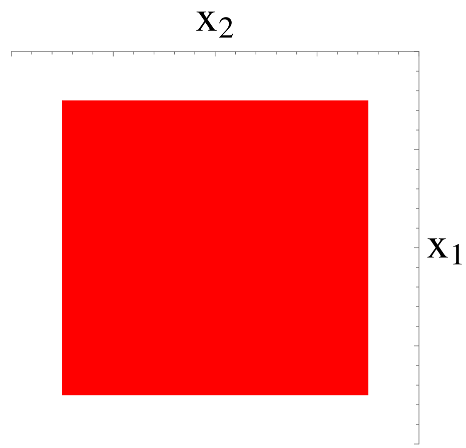

$$Q := \mathrm{conv}\,(S)$$

$$Q := \mathrm{conv}\,(S)?$$

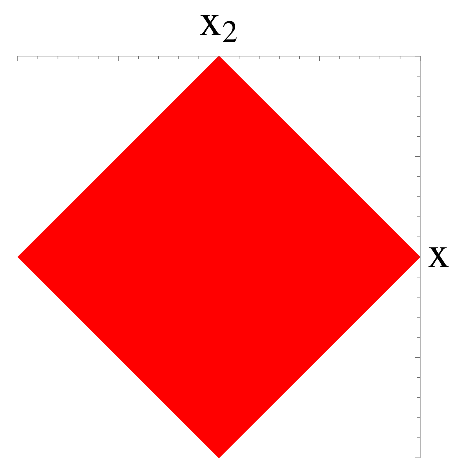


# Embedding Formulation = Ideal non-Extended




$P_1$




$P_2$

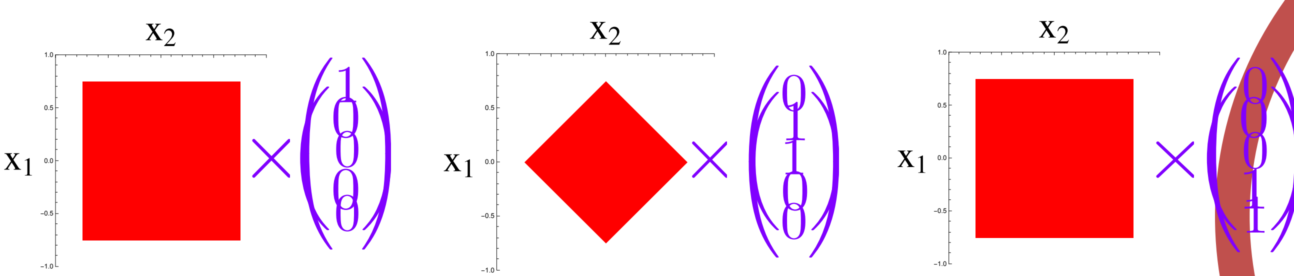
$$Q(H) := \operatorname{conv}\left(\bigcup_{i=1}^{n} P_i \times \{h^i\}\right)$$

$$(x, y) \in Q \cap \left(\mathbb{R}^d \times \mathbb{Z}^k\right) \quad \Leftrightarrow \quad y = h^i \ \wedge \ x \in P_i$$

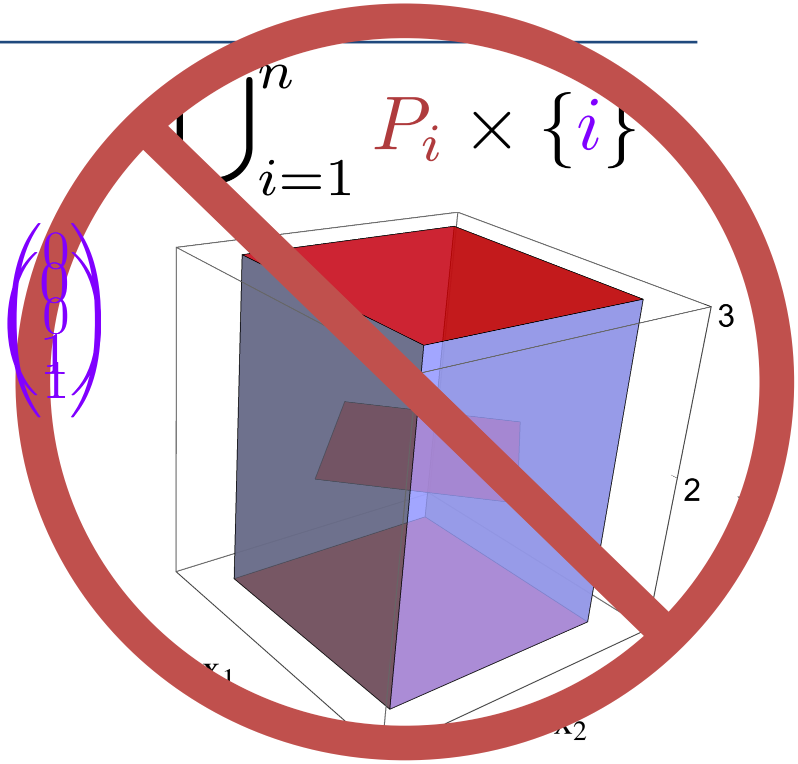
$$\operatorname{ext}(Q) \subseteq \mathbb{R}^d \times \mathbb{Z}^k \qquad H := \{h^i\}_{i=1}^{n} \subseteq \{0,1\}^k, \quad h^i \neq h^j$$

# Alternative Encodings

- "Only" use 0-1 encodings




$P_1$ $\quad$ $P_2$ $\quad$ $P_3$




- Options for 0-1 encodings:
  - Traditional or **Unary** encoding

$$H = \left\{ y \in \{0,1\}^n \;:\; \sum_{i=1}^{n} y_i = 1 \right\}$$

$$= \{\mathbf{e}^i\}_{i=1}^{n}$$

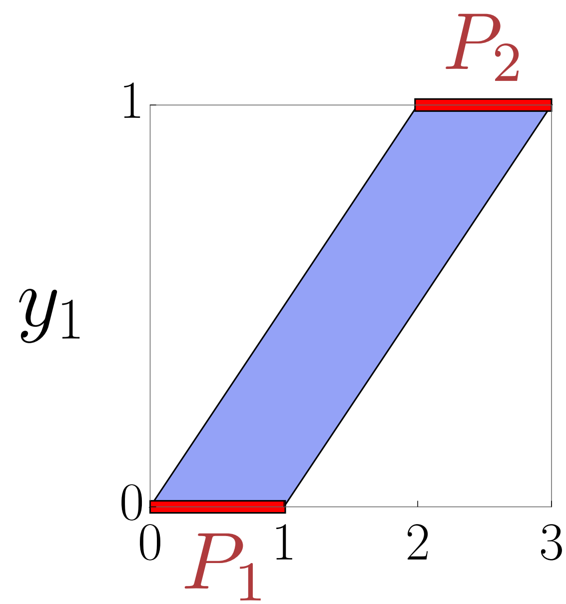
$$\mathbf{e}^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

  - **Binary** encodings: $H \equiv \{0,1\}^{\log_2 n}$
  - Others (e.g. **incremental** encoding $\equiv$ unary)

# Complexity of Family of Polyhedra $\mathcal{P} := \{P_i\}_{i=1}^n$

- Embedding complexity = smallest **ideal** formulation

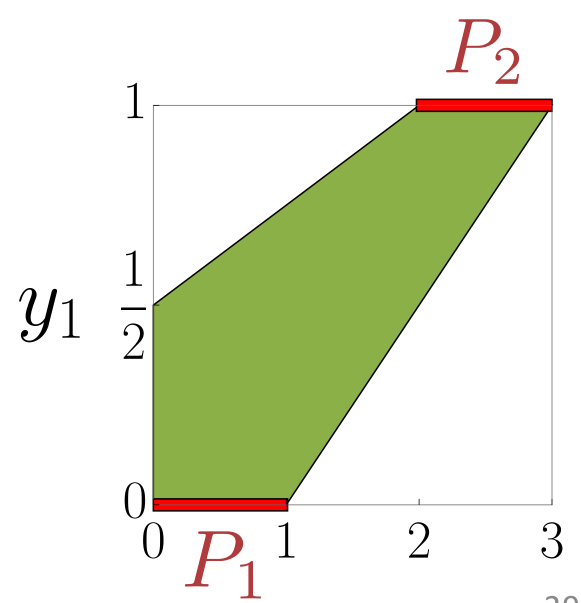
$$\mathrm{mc}(\mathcal{P}) := \min_H \{\mathrm{size}(Q(H))\}$$




- Relaxation complexity = smallest formulation

$$\mathrm{rc}(\mathcal{P}) := \min_{Q,H} \{\mathrm{size}(Q)\}$$

$$(x, y) \in Q \cap (\mathbb{R}^d \times \mathbb{Z}^k) \quad \Leftrightarrow \quad y = h^i \ \wedge \ x \in P_i$$

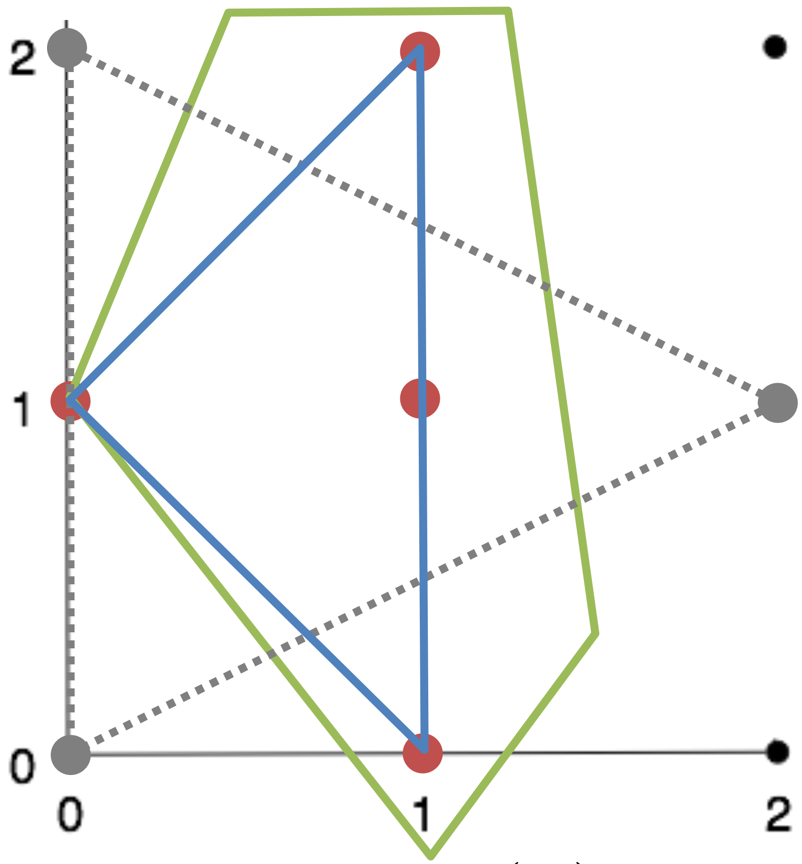


# Relaxation Complexity

# Relaxation Complexity: Pure v/s Mixed Integer

- Pure Integer $S \subseteq \mathbb{Z}^d$
- Mixed-Integer $S = \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^d$

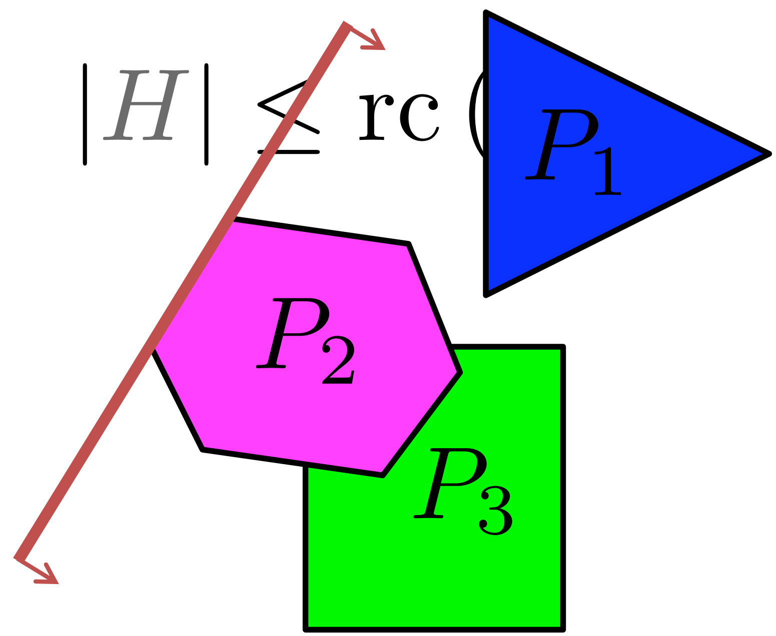
$$P \cap \mathbb{Z}^d = S \;\; (P \subseteq \mathbb{R}^d)$$




$$Q := \operatorname{conv}(S)$$

Hiding set $H$

$|H| \leq \operatorname{rc}($

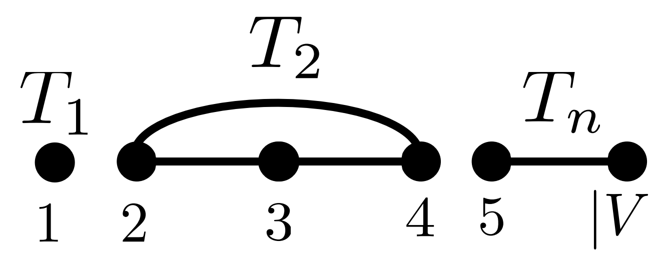


# Bounds on Relaxation Complexity

- Disjoint Case : $T_i \cap T_j = \emptyset$

$$\mathrm{rc}_G\,(\mathcal{P}) = 2$$

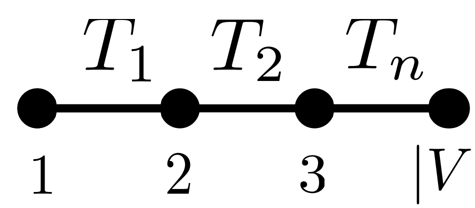
$$2 \leq \mathrm{rc}\,(\mathcal{P}) \;\; \leq 2 + |V| + n$$




- SOS2 constraints : $T_i = \{i, i+1\}$

$$2 \leq \mathrm{rc}_G\,(\mathcal{P}) \leq 4$$

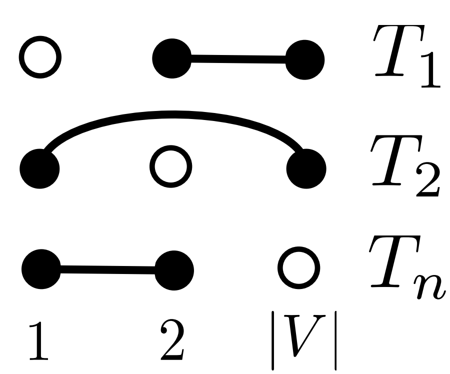
$$2 \leq \mathrm{rc}\,(\mathcal{P}) \;\; \leq 5 + 2n$$



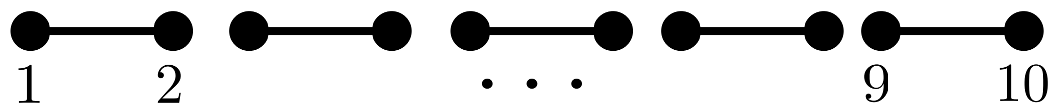

- SOS(-1) constraints : $T_i = V \setminus \{i\}$

$$\mathrm{mc}_G\,(\mathcal{P}) = \mathrm{rc}_G\,(\mathcal{P}) = n$$

$$n \leq \mathrm{rc}\,(\mathcal{P}) \quad \leq \mathrm{mc}\,(\mathcal{P}) \leq 3n$$

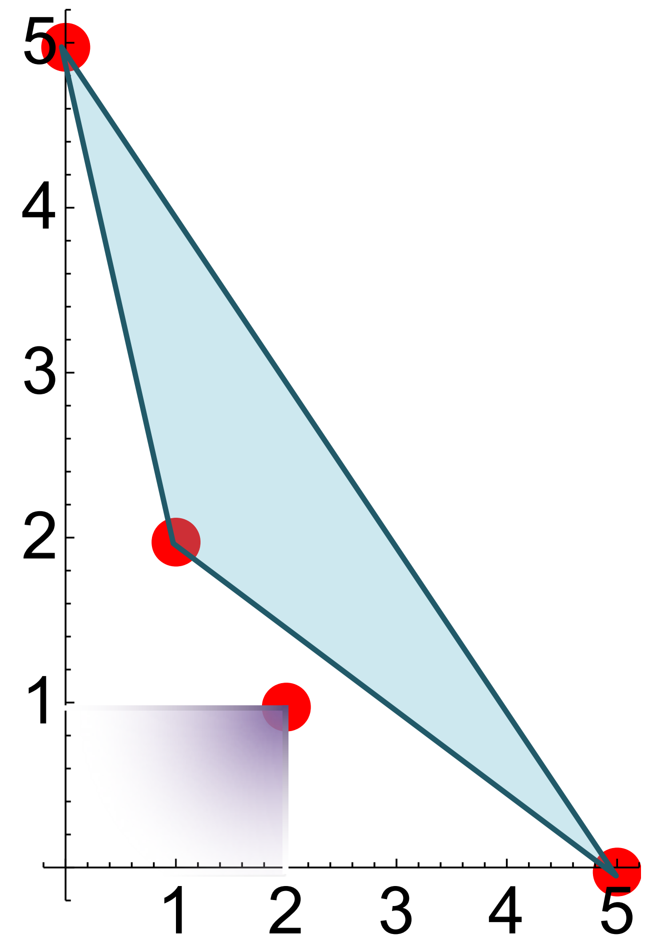


# Formulation for Disjoint Case




$$\sum_{i=1}^{n} p^i (\lambda_{2i-1} + \lambda_{2i}) \leq \sum_{i=1}^{n} p^i y_i$$
$$\sum_{i=1}^{2n} \lambda_i = 1, \quad \lambda \in \mathbb{R}_+^{2n}$$
$$\sum_{i=1}^{n} y_i = 1, \quad y \in \mathbb{R}_+^{n}$$

$$p^i \in \mathbb{R}_+^2, \quad \operatorname{conv}\left(\left\{p^j\right\}_{j \neq i}\right) \not\leq p^i$$




- Polynomial sized coefficients:
  - $p^i \in \mathbb{Z}_+^2, \quad \left\|p^i\right\|_\infty \leq 5^{\lceil (n-2)/2 \rceil}$
- 80 fractional extreme points for n = 5

# Embedding Complexity

# Embedding Complexity for SOS2

- Unary encoding:

$$\mathrm{size}_G\left(Q\left(H\right)\right) = 2(n-1), \quad \mathrm{size}\left(Q\left(H\right)\right) = 2n$$

- **Smallest Binary** encoding:

$$\mathrm{size}_G\left(Q\left(H\right)\right) = 2\left\lceil \log_2 n \right\rceil,$$

$$2 + 2\left\lceil \log_2 n \right\rceil \leq \mathrm{size}\left(Q\left(H\right)\right) \leq n + 1 + 2\left\lceil \log_2 n \right\rceil$$

- Adding lower bounds:

$$\mathrm{mc}_G\left(\mathcal{P}\right) = 2\left\lceil \log_2 n \right\rceil,$$

$$n + 1 \leq \mathrm{mc}\left(\mathcal{P}\right) \leq n + 1 + 2\left\lceil \log_2 n \right\rceil$$

# Binary Encoded Formulation for SOS2

$$1 \qquad 2 \qquad 3 \qquad 4 \qquad 5$$

$$h^1 = \binom{0}{1},\ h^2 = \binom{1}{1},\ h^3 = \binom{1}{0},\ h^4 = \binom{0}{0}$$

**Gray code = 1 bit difference from i to i+1**

$$\sum_{i=1}^{5} \lambda_i = 1$$

$Q$ = LP relaxation

$$\begin{aligned} 0 \leq \lambda_1 + \lambda_5 &\leq 1 - y_1 \\ 0 \leq \lambda_3 \qquad &\leq y_1 \\ 0 \leq \lambda_4 + \lambda_5 &\leq 1 - y_2 \\ 0 \leq \lambda_1 + \lambda_2 &\leq y_2 \end{aligned}$$

$$(\lambda, y) \in Q \cap \left(\mathbb{R}^5 \times \mathbb{Z}^2\right)$$

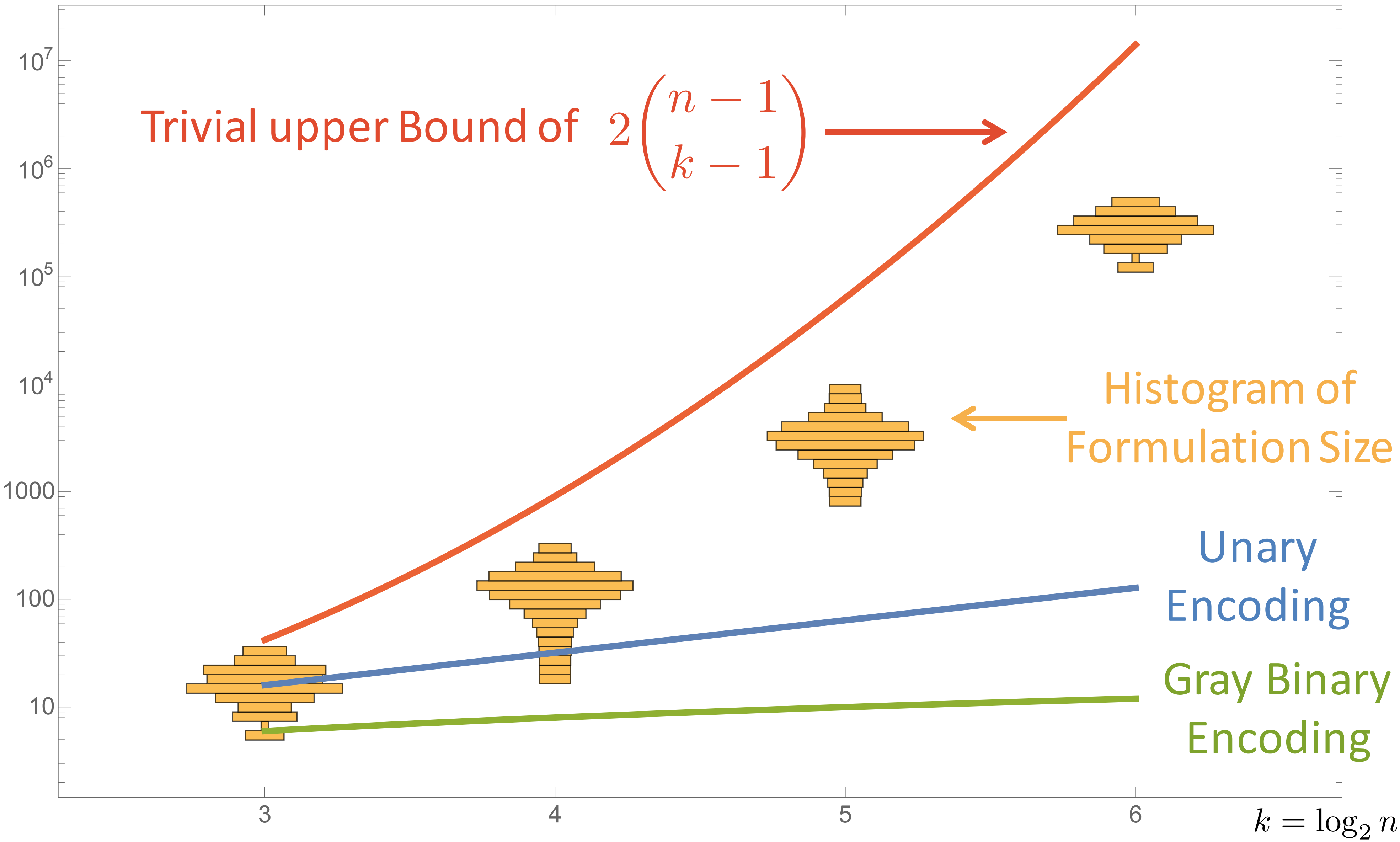
$$\Updownarrow$$

$$y = h^i \ \wedge \ \lambda \in P_i$$

Binary Encoding

$$P_i := \{\lambda \in \Lambda^5 \,:\, \lambda_j = 0 \quad j \notin \{i, i+1\}\}$$

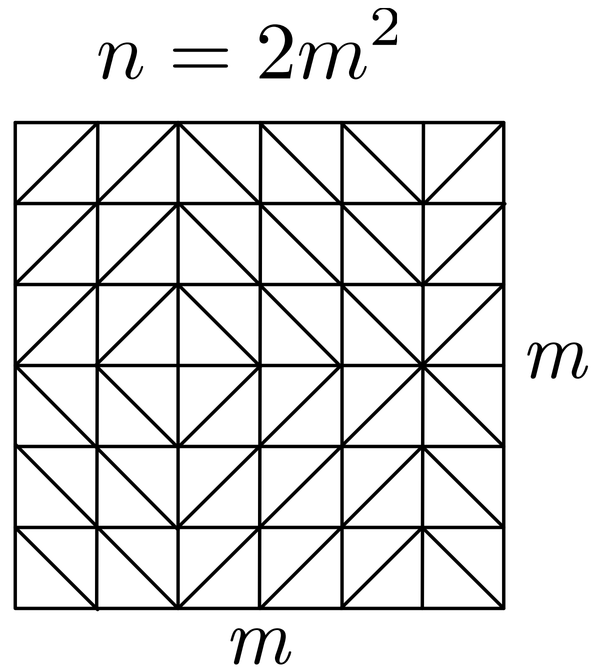
# # General Inequalities for all Binary Encodings



# Formulations and Complexity for Triangulations

- Lower bound:

$$\left(\sqrt{n/2}+1\right)^2 \leq \mathrm{mc}\left(\mathcal{P}\right)$$

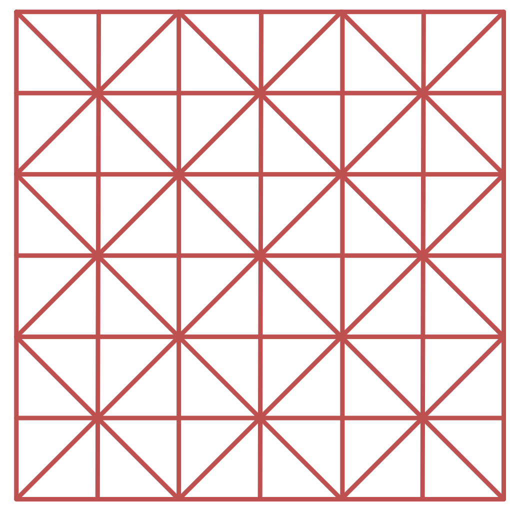



- Size of unary formulation is:

$$\mathrm{mc}\left(\mathcal{P}\right) \leq \binom{2\sqrt{n/2}}{\sqrt{n/2}} + \left(\sqrt{n/2}+1\right)^2$$

- Small binary formulation for union jack triangulation of size:

$$\mathrm{mc}\left(\mathcal{P}\right) \leq 4\log_2\sqrt{n/2}+2+\left(\sqrt{n/2}+1\right)^2$$



# Next Lecture

- Practical construction technique 1:
  - Independent branching formulations for V-polyhedra
- Practical construction technique 2:
  - Case study for H-polyhedra
  - Encoding and alternative representations of disjunctions

# References


- V., “Mixed integer linear programming formulation techniques”, SIAM Review 57, 2015. pp. 3—57.
- Kaibel and Weltge, “Lower Bounds on the Sizes of Integer Programs without Additional Variables”, Math. Program. 154, 2015. pp. 407—425.
- V. “Embedding Formulations and Complexity for Unions of Polyhedra”, 2015. arXiv:1506.01417
- V. “Relaxation Complexity for Special Ordered Sets”, 2016. Working Paper.

# Extended Formulations for $x \in \bigcup_{i=1}^{n} P_i$

$$P_i = \{x \in \mathbb{R}^d \,:\, A^i x \leq b^i\}$$

$$\begin{aligned} A^i x^i &\leq b^i y_i \quad \forall i \in [n] \\ \sum_{i=1}^{n} x^i &= x \\ \sum_{i=1}^{n} y_i &= 1 \\ y &\in \{0,1\}^n \end{aligned}$$

$\mathcal{H}$-formulation

$$\begin{aligned} \sum_{i=1}^{n} \sum_{v \in \operatorname{ext}(P_i)} v \lambda_v^i &= x \\ \sum_{v \in \operatorname{ext}(P^i)} \lambda_v^i &= y_i \quad \forall i \in [n] \\ \sum_{i=1}^{n} y_i &= 1 \\ \lambda^i &\in \mathbb{R}_+^{\operatorname{ext}(P_i)} \\ y &\in \{0,1\}^n \end{aligned}$$

$\mathcal{V}$-formulation

- Both formulations are ideal and use copies of variables

# Simple Non-Extended $\mathcal{H}$-formulation

$$P^i = \{x \in \mathbb{R}^d \,:\, A^i x \leq b^i\} \to P^i = \{x \in \mathbb{R}^d \,:\, Dx \leq d^i\}$$

$$D = \begin{bmatrix} A^1 \\ \vdots \\ A^k \end{bmatrix}, \quad d^i \text{ appropriately constructed (Big-Ms)}$$

$$Dx \leq \sum_{i=1}^{n} d^i y_i$$

$$\sum_{i=1}^{n} y_i = 1$$

$$y \in \{0,1\}^n$$

- Usually not ideal or sharp, but often stronger than standard Big-M (Sharp under some conditions)

# Simple Non-Extended $\mathcal{V}$-formulation

$$V := \bigcup_{i=1}^{n} \operatorname{ext}(P_i)$$

$$\begin{aligned} \sum_{v \in V} v \lambda_v &= x \\ \sum_{v \in V} \lambda_v &= 1 \\ \lambda_v &\le \sum_{i : v \in \operatorname{ext}(P_i)} y_i \\ \sum_{i=1}^{n} y_i &= 1 \end{aligned}$$

$$y \in \{0,1\}^n, \quad \lambda \in \mathbb{R}_+^V$$

- Usually not ideal, but automatically sharp

# Embedding Formulation for SOS2: Part 1

- From encodings to hyperplanes:

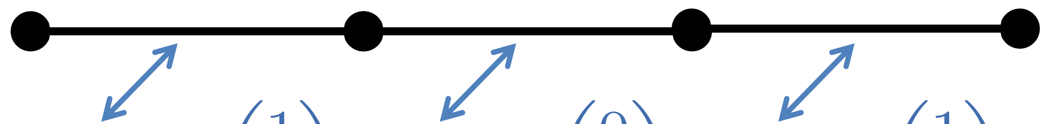
$$\left\{h^i\right\}_{i=1}^{n}$$

$$c^i = h^{i+1} - h^i$$

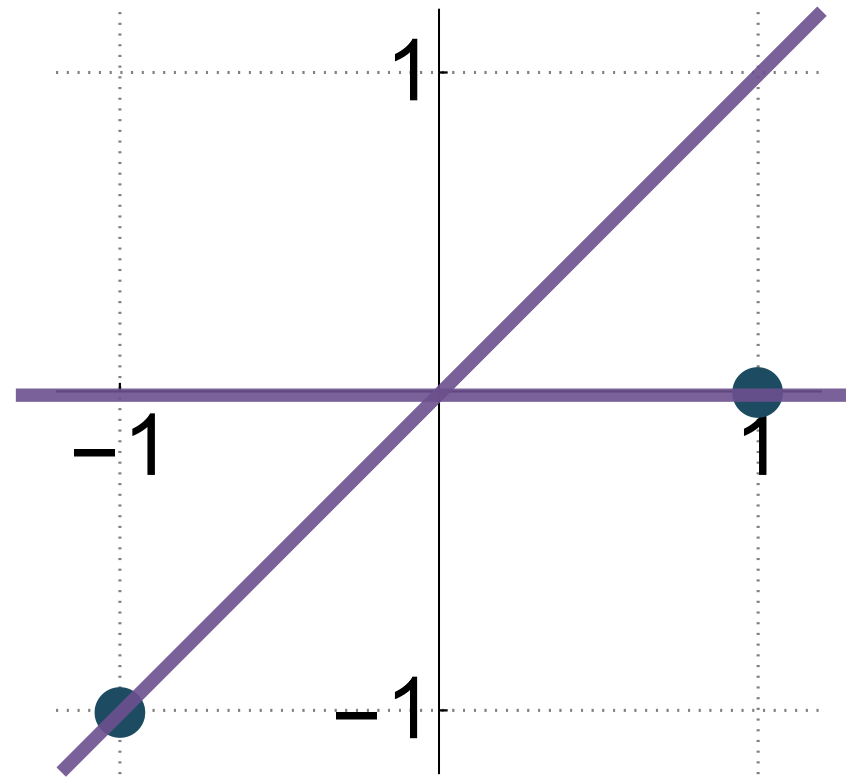
$$\left\{c^i\right\}_{i=1}^{n-1}$$

Hyperplanes spanned by

$$\left\{b^i \cdot y = 0\right\}_{j=1}^{L}$$



$$h^1 = \begin{pmatrix}1\\1\end{pmatrix},\ h^2 = \begin{pmatrix}0\\0\end{pmatrix},\ h^3 = \begin{pmatrix}1\\0\end{pmatrix}$$



# Embedding Formulation for SOS2: Part 2

$$\left\{b^i \cdot y = 0\right\}_{j=1}^{L}$$

$$L\left(H\right) := \mathrm{aff}\left(H\right) - h^1$$

$$Q(H) =$$

$$\begin{aligned}
\left(b^j \cdot h^1\right)\lambda_1 + \sum_{i=2}^{n} \min\left\{b^j \cdot h^i, b^j \cdot h^{i-1}\right\}\lambda_i + \left(b^j \cdot h^n\right)\lambda_{n+1} &\leq \quad b^j \cdot y \quad \forall j \\
-\left(b^j \cdot h^1\right)\lambda_1 - \sum_{i=2}^{n} \max\left\{b^j \cdot h^i, b^j \cdot h^{i-1}\right\}\lambda_i - \left(b^j \cdot h^n\right)\lambda_{n+1} &\leq -b^j \cdot y \quad \forall j \\
\sum_{i=1}^{n+1} \lambda_i = 1, \quad \lambda &\in \mathbb{R}_+^{n+1} \\
y &\in L(H)
\end{aligned}$$

- # general inequalities = 2 × # of hyperplanes