

Mixed Integer Programming Formulations

Lecture 1

Juan Pablo Vielma

Massachusetts Institute of Technology

IPCO Summer School
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<https://github.com/juan-pablo-vielma/IPCO-2016-Summer-School>



MIP Formulation Techniques

- Special Ordered Sets of Type 2 (SOS2)

$$-\lambda \in \mathbb{R}_+^{n+1}, \quad \sum_{i=1}^{n+1} \lambda_i = 1$$

$$\text{at most 2 } \lambda_i > 0 \text{ and } \lambda_i > 0 \wedge \lambda_j > 0 \Rightarrow |i - j| \leq 1$$

$$0 \leq \lambda_1 \leq \underbrace{y_1}_{\substack{\text{Bounds}}} \quad 2(n+1)$$

$$\sum_{i=1}^5 \lambda_i = 1$$

$$0 \leq \lambda_2 \leq y_1 + y_2$$

$$y \in \{0, 1\}^4, \quad \sum_{i=1}^4 y_i = 1$$

$$0 \leq \lambda_3 \leq y_2 + y_3$$

- Minimum # of (general) inequalities?

$$0 \leq \lambda_4 \leq y_3 + y_4$$

– Integral formulation.

$$0 \leq \lambda_5 \leq y_4$$

– Non-integral formulation.

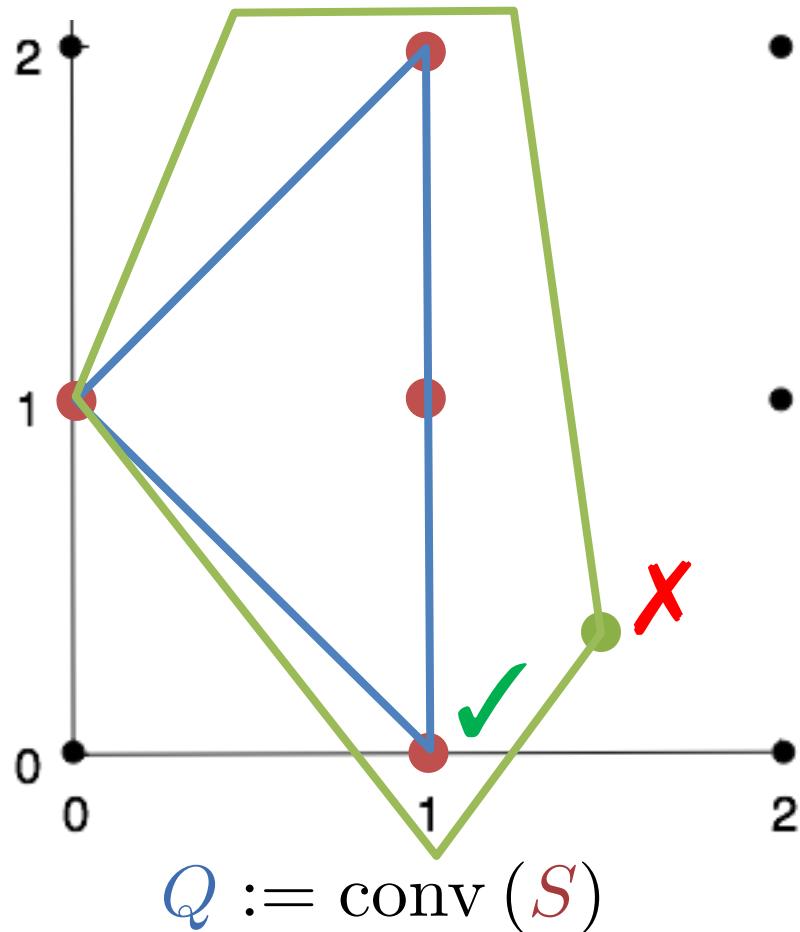
Bounds

General Inequalities

Pure Integer Formulations

- Pure Integer $S \subseteq \mathbb{Z}^d$ s.t. $\text{conv}(S) \cap \mathbb{Z}^d = S$

$$P \cap \mathbb{Z}^d = S \quad (P \subseteq \mathbb{R}^d)$$



- Formulation:
 - LP relaxation P
 - Integrality
- Formulation strength:
 - $P = \text{convex hull of } S$
 - or equivalently P has integral extreme points
- Size:
 - # of inequalities
 - no need for auxiliary variables (non-extended)

Outline

- MIP formulations of disjunctive constraints
 - Differences with pure integer
 - Sizes, strength, two classes of formulations
 - Towards optimal formulations through encodings and embeddings
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- Relaxation complexity = smallest formulation
 - Embedding complexity = smallest integral formulation
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- Practical formulation techniques
 - Nonlinear MIP?
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- Julia / JuMP tutorial

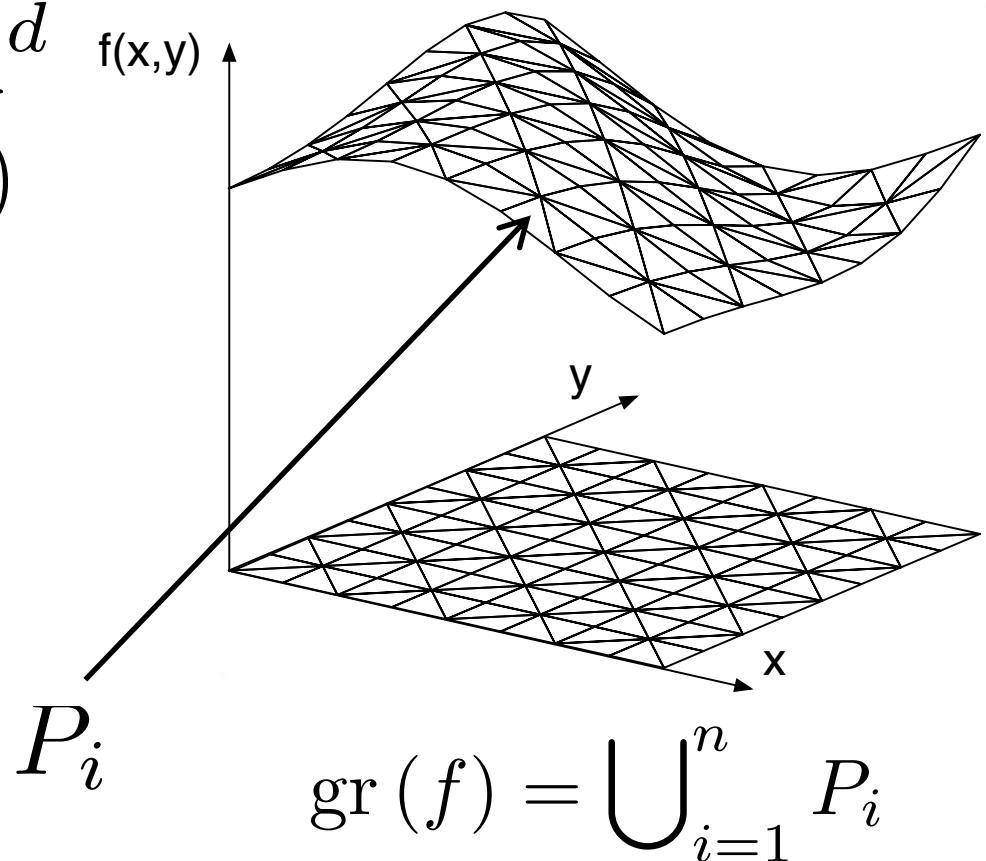
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(Linear) Mixed 0-1 Integer Formulations

- Modeling Finite Alternatives = Unions of Polyhedra

$$\begin{aligned} & x \in \bigcup_{i=1}^n P_i \subseteq \mathbb{R}^d \\ \min & \quad \sum_{j=1}^m f_j(x_j, y_j) \\ \text{s.t.} & \quad (x, y) \in X \\ & \quad P_1 \quad \text{blue triangle} \\ & \quad P_2 \quad \text{pink hexagon} \\ & \quad P_3 \quad \text{green rectangle} \end{aligned}$$



Size of Smallest 0-1 Formulation for $x \in \bigcup_{i=1}^n P_i$

- Standard **ideal (integral) extended** formulation for

$P_i = \{x \in \mathbb{R}^d : A^i x \leq b^i\}$ (Balas, Jeroslow and Lowe):

$$A^i \mathbf{x}^i \leq b^i y_i \quad \forall i \in \{1, \dots, n\}$$

$$\sum_{i=1}^n \mathbf{x}^i = x, \quad \mathbf{x}^i \in \mathbb{R}^d \quad \forall i \in \{1, \dots, n\}$$

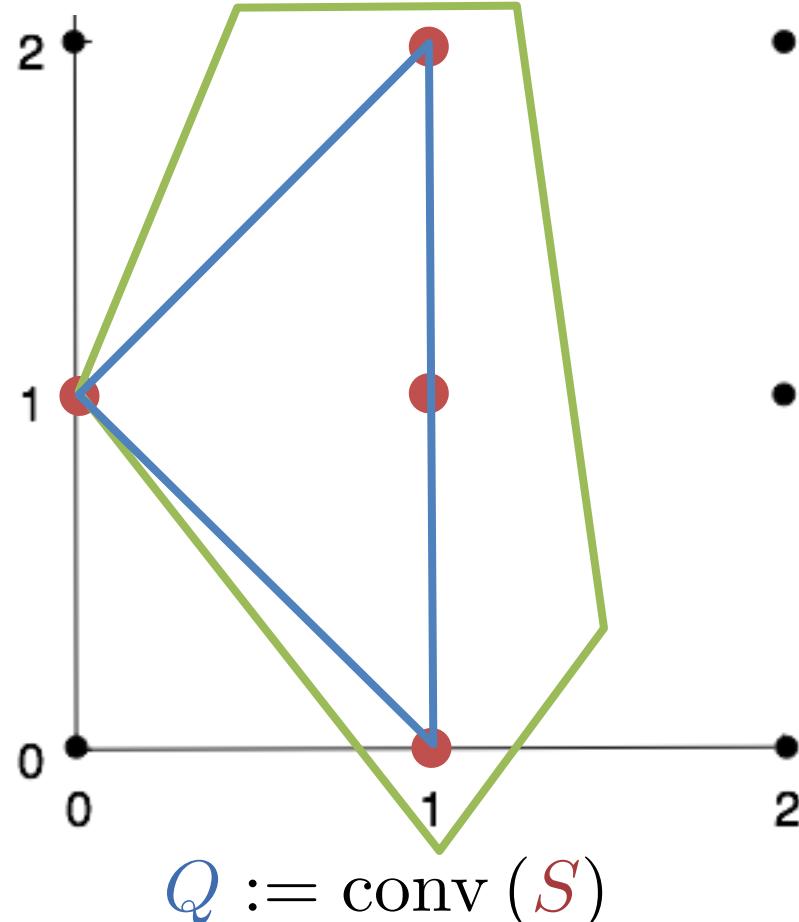
$$\sum_{i=1}^n y_i = 1, \quad y \in \{0, 1\}^n$$

- What about non-**extended** (i.e. no **variables copies**) ?
- What about non-**ideal**? (i.e. **some** fractional extreme pts.)?
- What about **precise** lower/upper bounds on size?

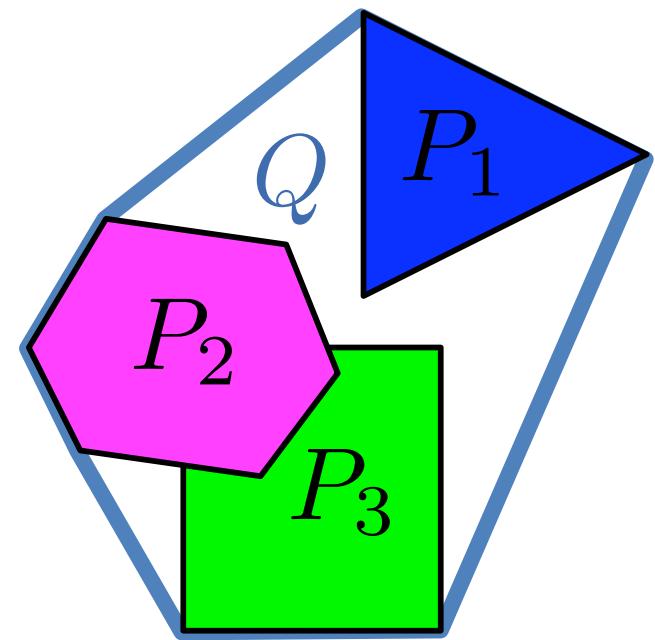
IP v/s MIP 1: Auxiliary Variables

- Pure Integer $S \subseteq \mathbb{Z}^d$
- Mixed-Integer $S = \bigcup_{i=1}^n P_i \subseteq \mathbb{R}^d$

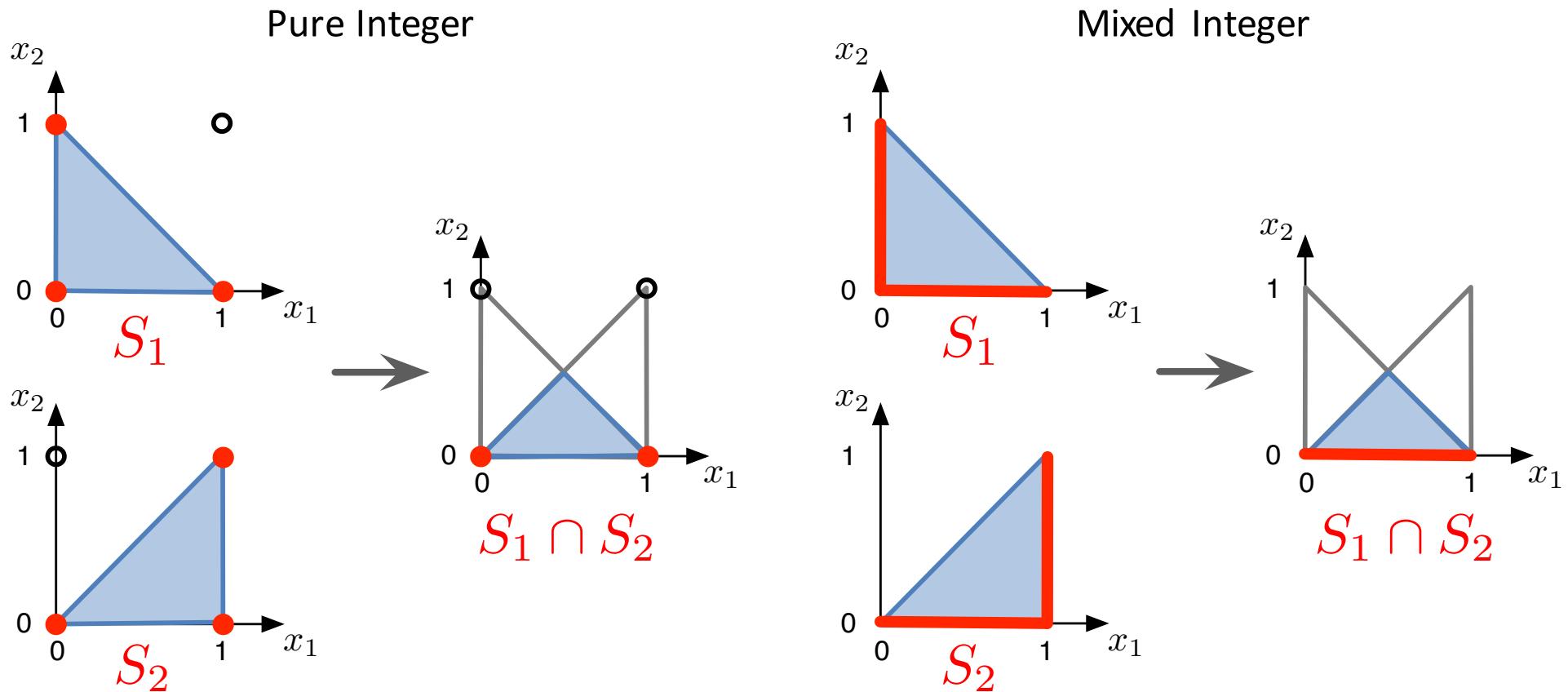
$$P \cap \mathbb{Z}^d = S \quad (P \subseteq \mathbb{R}^d)$$



$$Q := \text{conv}(S) ?$$



Convex Hull and Combining Formulations



- Mixed Integer needs some 0-1 auxiliary variables

MIP Needs Integer Auxiliary Variables

- $S = [0, 1] \cup [2, 3]$

$$\begin{aligned} 2y_1 \leq x_1 &\leq 1 + 2y_1 \\ 0 \leq y_1 &\leq 1 \end{aligned}$$

P

$$y_1 \in \mathbb{Z}$$

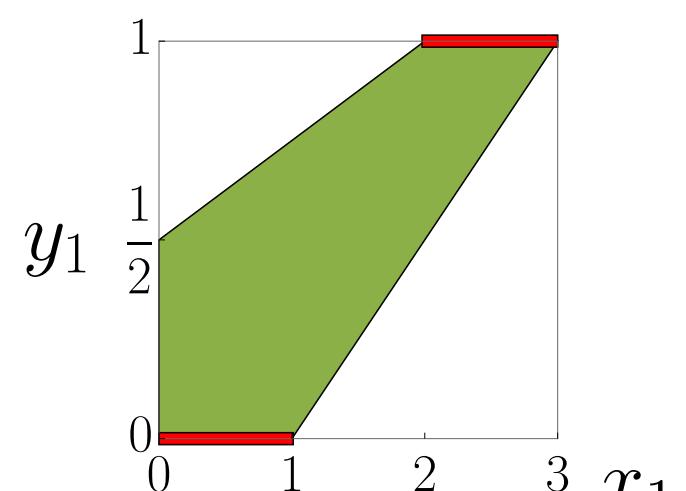
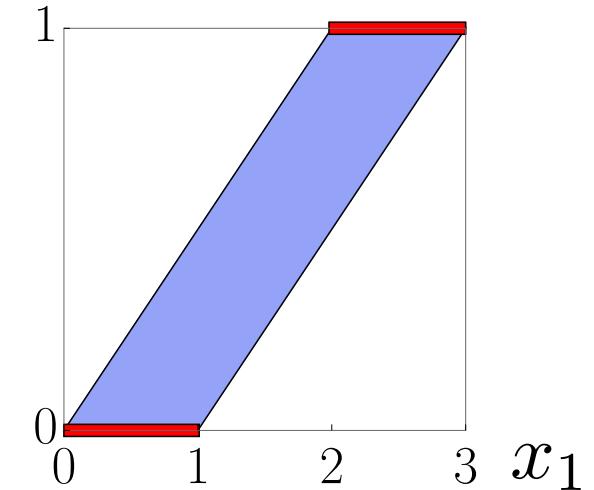
$$\begin{aligned} 4y_1 - 2 \leq x_1 &\leq 1 + 2y_1 \\ 0 \leq y_1 &\leq 1 \end{aligned}$$

P

$$y_1 \in \mathbb{Z}$$

- $S \subseteq \mathbb{R}^n, P \subseteq \mathbb{R}^{n+k}$

$$\text{Proj}_x(P \cap (\mathbb{R}^n \times \mathbb{Z}^k)) = S$$



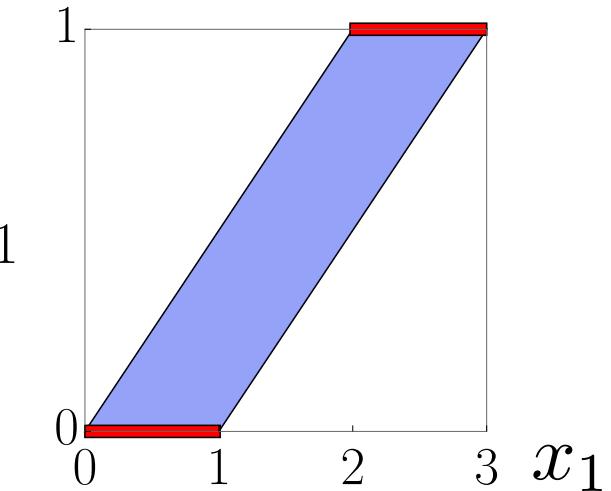
For MIP convex hull (sharp) \neq integral (ideal)

- $S = [0, 1] \cup [2, 3]$

- $$\begin{aligned} 2y_1 \leq x_1 &\leq 1 + 2y_1 \\ 0 \leq y_1 &\leq 1 \end{aligned}$$

$$y_1 \in \mathbb{Z}$$

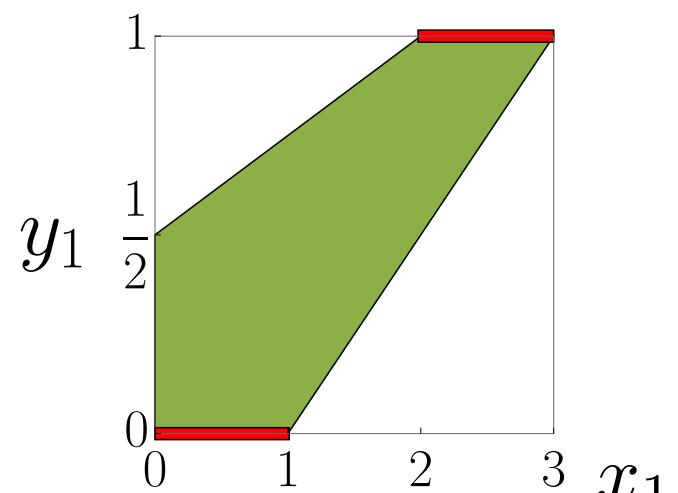
P



- $$\begin{aligned} 4y_1 - 2 \leq x_1 &\leq 1 + 2y_1 \\ 0 \leq y_1 &\leq 1 \end{aligned}$$

$$y_1 \in \mathbb{Z}$$

P



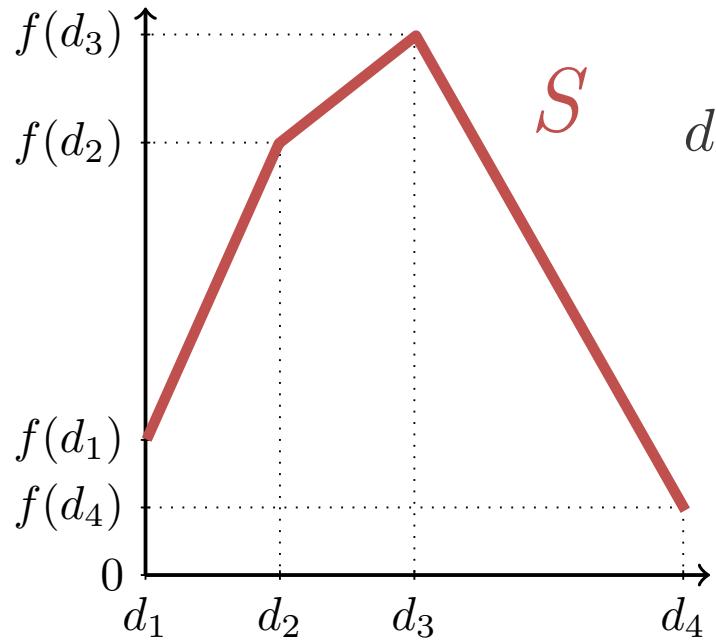
- $\text{Proj}_x(P) = \text{conv}(S)$



- $\text{ext}(P) \subseteq \mathbb{R}^n \times \mathbb{Z}^k$

$$\text{Proj}_x(P \cap (\mathbb{R}^n \times \mathbb{Z}^k)) = S$$

Naïve Formulation for Piecewise Linear Functions

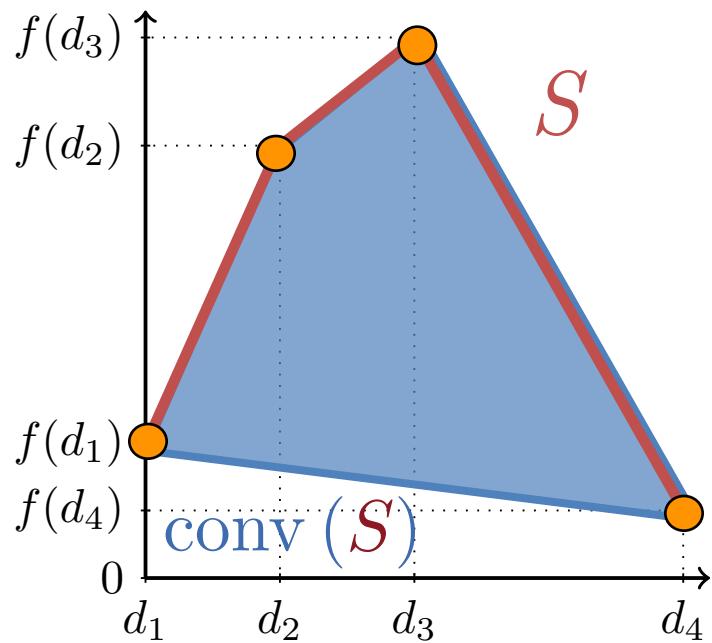


$$\begin{aligned}
 d_i - (d_i - d_1) (1 - y_i) &\leq x & \forall i \in [n] \\
 d_{i+1} + (d_{i+1} - d_k) (1 - y_i) &\geq x & \forall i \in [n] \\
 m_i x + c_i - \underline{M}_i (1 - y_i) &\leq z & \forall i \in [n] \\
 m_i x + c_i + \overline{M}_i (1 - y_i) &\geq z & \forall i \in [n] \\
 \sum_{i=1}^k y_i &= 1 \\
 y &\in \{0, 1\}^n
 \end{aligned}$$

$$f(x) = \begin{cases} m_1 x + c_1 & x \in [d_1, d_2] \\ & \vdots \\ m_k x + c_k & x \in [d_k, d_{k+1}] \end{cases} \quad \begin{aligned} \underline{M}_i &:= \max_{j=1}^{n+1} \{m_i d_j + c_i - f(d_j)\} \\ \overline{M}_i &:= \max_{j=1}^{n+1} \{f(d_j) - m_i d_j - c_i\} \end{aligned}$$

Exercise: Show it is not always ideal or sharp

Better Formulation (CC) \Leftrightarrow SOS2



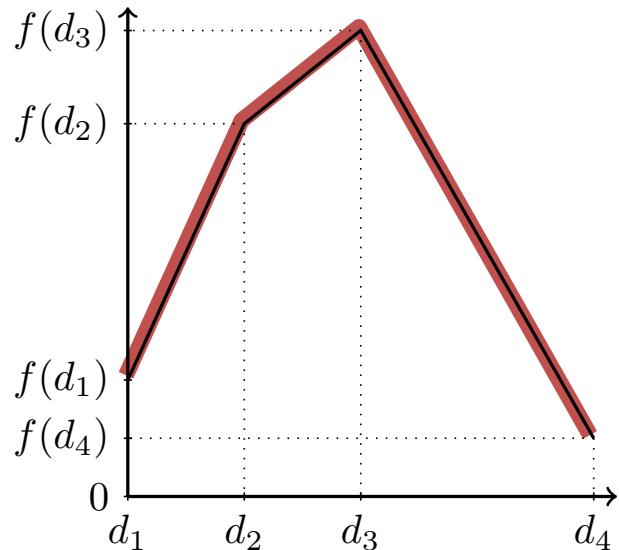
$$\begin{aligned}
 & \sum_{i=1}^{n+1} \lambda_i d_i = x \\
 & \sum_{i=1}^{n+1} \lambda_i f(d_i) = z \\
 & \sum_{i=1}^{n+1} \lambda_i = 1 \\
 & \lambda_1 \leq y_1 \\
 & \lambda_i \leq y_{i-1} + y_i \quad \forall 2 \leq i \leq n \\
 & \lambda_{n+1} \leq y_k
 \end{aligned}$$

$$f(x) = \begin{cases} m_1x + c_1 & x \in [d_1, d_2] \\ & \vdots \\ m_kx + c_k & x \in [d_k, d_{k+1}] \end{cases} \quad \begin{aligned} & \sum_{i=1}^n y_i = 1 \\ & y \in \{0, 1\}^n \end{aligned}$$

Is always sharp. Exercise: Show it is not ideal.

Non-Extended Formulation for PWL Functions

$$S = \text{gr}(f) = \bigcup_{i=1}^n \left\{ (x, z) \in \mathbb{R}^2 : \begin{array}{l} d_i \leq x \leq d_{i+1} \\ m_i x + c_i = z \end{array} \right\} \quad \text{BigM Formulation:}$$

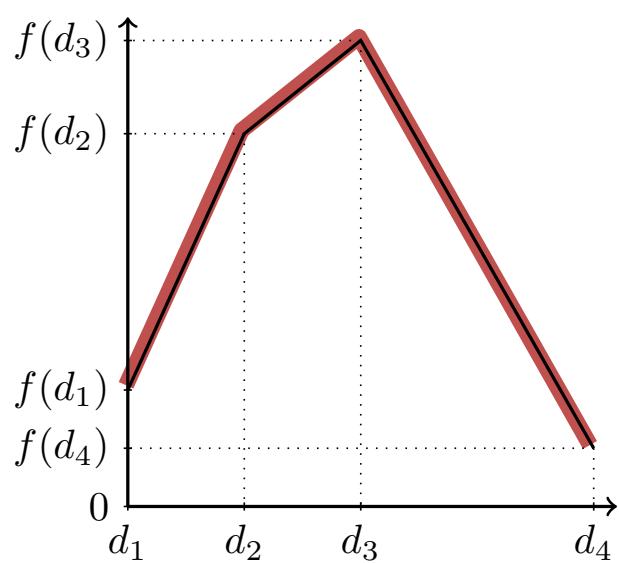


$$\begin{aligned}
 d_i - (d_i - d_1)(1 - y_i) &\leq x & \forall i \in [n] \\
 d_{i+1} + (d_{i+1} - d_k)(1 - y_i) &\geq x & \forall i \in [n] \\
 m_i x + c_i - \underline{M}_i(1 - y_i) &\leq z & \forall i \in [n] \\
 m_i x + c_i + \overline{M}_i(1 - y_i) &\geq z & \forall i \in [n] \\
 \sum_{i=1}^n y_i &= 1 \\
 y &\in \{0, 1\}^n
 \end{aligned}$$

- Some (0-1) auxiliary variables needed for a valid formulation
 - For now n 0-1auxiliary variables that add exactly to 1

Extended Formulation for PWL Functions

$$S = \text{gr}(f) = \bigcup_{i=1}^n \left\{ (x, z) \in \mathbb{R}^2 : \begin{array}{l} d_i \leq x \leq d_{i+1} \\ m_i x + c_i = z \end{array} \right\} \quad \text{MC Formulation:}$$



$$d_i y_i \leq x^i \leq d_{i+1} y_i \quad \forall i \in [n]$$

$$m_i x^i + c_i y_i = z^i \quad \forall i \in [n]$$

$$\sum_{i=1}^n x^i = x$$

$$\sum_{i=1}^n z^i = z$$

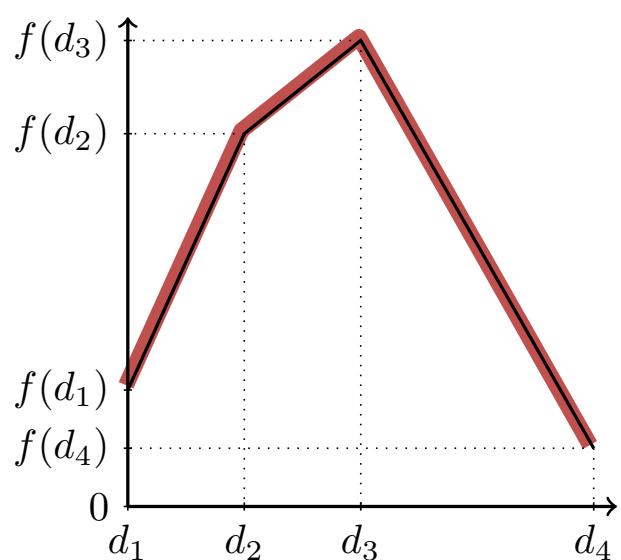
$$\sum_{i=1}^n y_i = 1$$

$$\mathbf{y} \in \{0, 1\}^n$$

- Some continuous auxiliary variables only used to construct strong formulations

What Are The Natural Original Variables?

$$S = \text{gr}(f) = \bigcup_{i=1}^n \left\{ (x, z) \in \mathbb{R}^2 : \begin{array}{l} d_i \leq x \leq d_{i+1} \\ m_i x + c_i = z \end{array} \right\} \quad \text{CC Formulation:}$$

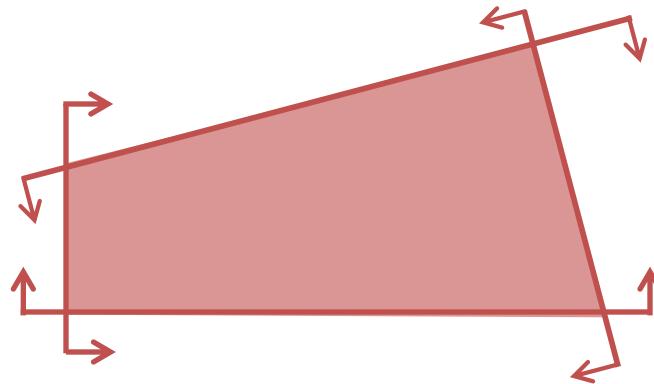


$$\begin{aligned} & \sum_{i=1}^{n+1} \lambda_i d_i = x \\ & \sum_{i=1}^{n+1} \lambda_i f(d_i) = z \\ & \sum_{i=1}^{n+1} \lambda_i = 1 \\ & \lambda_1 \leq y_1 \\ & \lambda_i \leq y_{i-1} + y_i \quad \forall 2 \leq i \leq n \\ & \lambda_{n+1} \leq y_k \\ & \sum_{i=1}^n y_i = 1 \\ & \mathbf{y} \in \{0, 1\}^n \end{aligned}$$

Two Types of Polytopes

- *\mathcal{H} -polyhedron* iff $\exists A \in \mathbb{Q}^{m \times d}$ and $b \in \mathbb{Q}^m$ s.t.

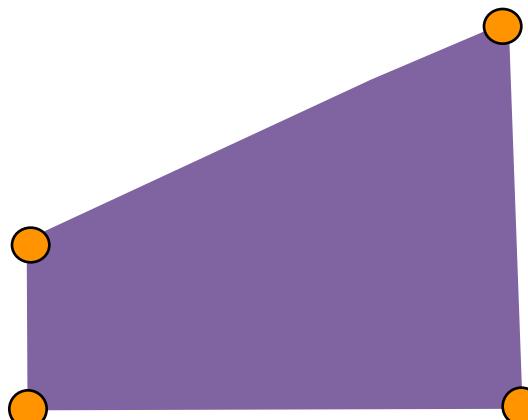
$$P = \{x \in \mathbb{Q}^d : Ax \leq b\}$$



- *\mathcal{V} -polyhedron* iff \exists finite sets $V \subseteq \mathbb{Q}^d$ s.t.

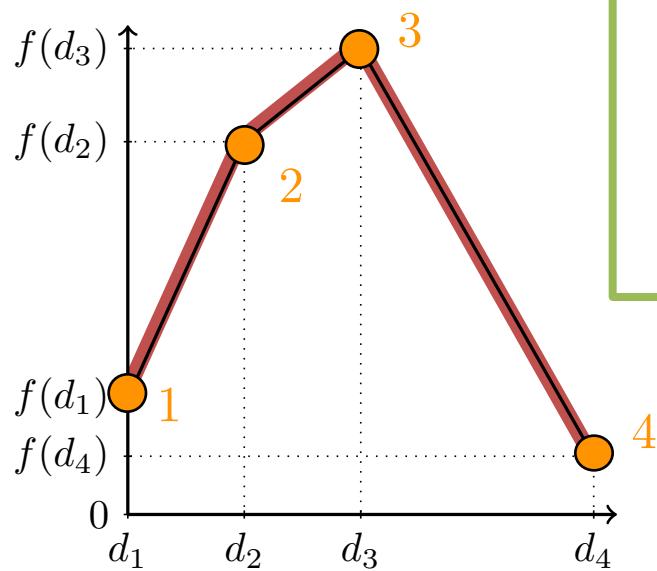
$$P = \text{conv}(V)$$

can take $V = \text{ext}(P)$



Vertex / Extreme Point Variables Also Natural

$$S = \text{gr}(f) = \bigcup_{i=1}^n \left\{ (x, z) \in \mathbb{R}^2 : \begin{array}{l} d_i \leq x \leq d_{i+1} \\ m_i x + c_i = z \end{array} \right\} \quad \text{CC Formulation:}$$



$$\boxed{\begin{aligned} \sum_{i=1}^{n+1} \lambda_i d_i &= x \\ \sum_{i=1}^{n+1} \lambda_i f(d_i) &= z \end{aligned}}$$

Linear Transformation

$$\sum_{i=1}^{n+1} \lambda_i = 1$$

$$\lambda_1 \leq y_1$$

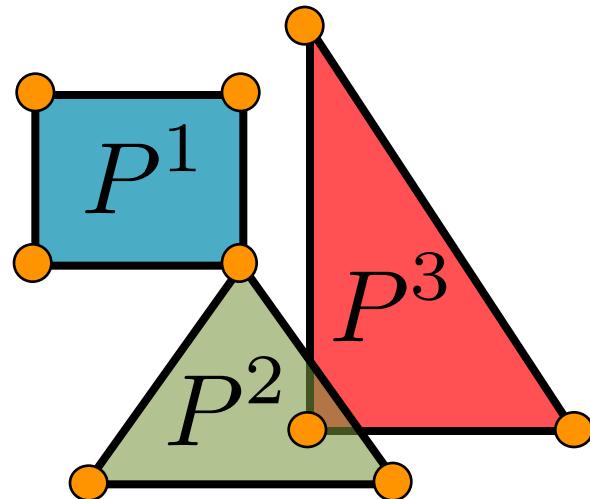
$$\lambda_i \leq y_{i-1} + y_i \quad \forall 2 \leq i \leq n$$

$$\lambda_{n+1} \leq y_k$$

$$\sum_{i=1}^n y_i = 1$$

$$\mathbf{y} \in \{0, 1\}^n$$

Linear Transformation for \mathcal{V} -Formulation



$$V := \bigcup_{i=1}^n \text{ext}(P^i)$$

$$\Delta^V := \left\{ \lambda \in \mathbb{R}_+^V : \sum_{v \in V} \lambda_v = 1 \right\}$$

$$Q^i := \left\{ \lambda \in \Delta^V : \lambda_v \leq 0 \quad \forall v \notin \text{ext}(P^i) \right\}$$

$$x \in \bigcup_{i=1}^n P^i$$



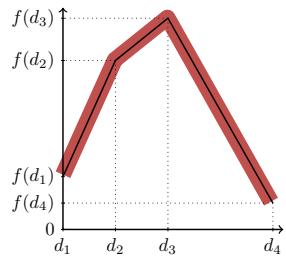
$$x = \sum_{v \in V} v \lambda_v$$

$$\lambda \in \bigcup_{i=1}^n Q^i$$

Dependent on specific
data from polytopes

Purely Combinatorial

Combinatorial Structure v/s Data



$$S = \text{gr}(f) = \bigcup_{i=1}^n \left\{ (x, z) \in \mathbb{R}^2 : \begin{array}{l} d_i \leq x \leq d_{i+1} \\ m_i x + c_i = z \end{array} \right\}$$

BigM Formulation:

$$\begin{aligned} d_i - (d_i - d_1)(1 - y_i) &\leq x \\ d_{i+1} + (d_{i+1} - d_k)(1 - y_i) &\geq x \\ m_i x + c_i - \underline{M}_i(1 - y_i) &\leq z \\ m_i x + c_i + \overline{M}_i(1 - y_i) &\geq z \end{aligned}$$

$$\sum_{i=1}^n y_i = 1$$

$$\mathbf{y} \in \{0, 1\}^n$$

$$\begin{aligned} \forall i \in [n] \\ \forall i \in [n] \\ \forall i \in [n] \\ \forall i \in [n] \end{aligned}$$

CC Formulation:

$$\begin{aligned} \sum_{i=1}^{n+1} \lambda_i d_i &= x \\ \sum_{i=1}^{n+1} \lambda_i f(d_i) &= z \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^{n+1} \lambda_i &= 1 \\ \lambda_1 &\leq y_1 \\ \lambda_i &\leq y_{i-1} + y_i \quad \forall 2 \leq i \leq r \end{aligned}$$

$$\begin{aligned} \lambda_{n+1} &\leq y_k \\ \sum_{i=1}^n y_i &= 1 \end{aligned}$$

$$\mathbf{y} \in \{0, 1\}^n$$

Constructing Integral Non-Extended Formulations

- Add inequalities to simple non-extended formulations to make them integral

$$Ax + Dy \leq d \quad \longrightarrow \quad (x, y) \in \text{conv} \left(\{(x, y) \in \mathbb{R}^d \times \mathbb{Z}^n : Ax + Dy \leq b\} \right)$$
$$y \in \mathbb{Z}^n \qquad \qquad \qquad y \in \mathbb{Z}^n$$

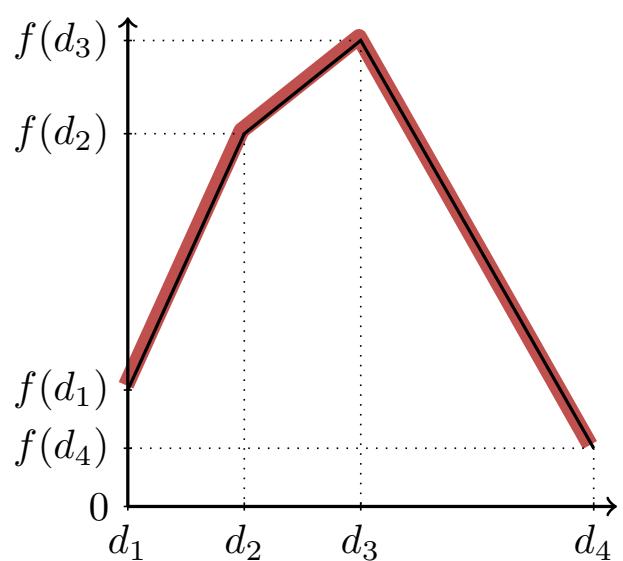
$$x = L\lambda \qquad \qquad x = L\lambda$$

$$A\lambda + Dy \leq d \quad \longrightarrow \quad (\lambda, y) \in \text{conv} \left(\{(\lambda, y) \in \mathbb{R}^V \times \mathbb{Z}^n : A\lambda + Dy \leq b\} \right)$$
$$y \in \mathbb{Z}^n \qquad \qquad \qquad y \in \mathbb{Z}^n$$

Purely Combinatorial / Data Independent = Often Simpler

Making CC Ideal

$$S = \text{gr}(f) = \bigcup_{i=1}^n \left\{ (x, z) \in \mathbb{R}^2 : \begin{array}{l} d_i \leq x \leq d_{i+1} \\ m_i x + c_i = z \end{array} \right\} \quad \text{ICC Formulation:}$$



$$\sum_{i=1}^{n+1} \lambda_i d_i = x$$

$$\sum_{i=1}^{n+1} \lambda_i f(d_i) = z$$

$$\sum_{i=1}^{n+1} \lambda_i = 1$$

$$\sum_{i=1}^l \lambda_i \leq \sum_{i=1}^l y_i \quad \forall l \in [n-1]$$

$$\sum_{i=l+2}^{n+1} \lambda_i \leq \sum_{i=l+1}^n y_i \quad \forall l \in [n-1]$$

$$\sum_{i=1}^n y_i = 1$$

$$y \in \{0, 1\}^n$$

Sometimes Does Not Work Well

$$\sum_{v \in V} v \lambda_v = x$$

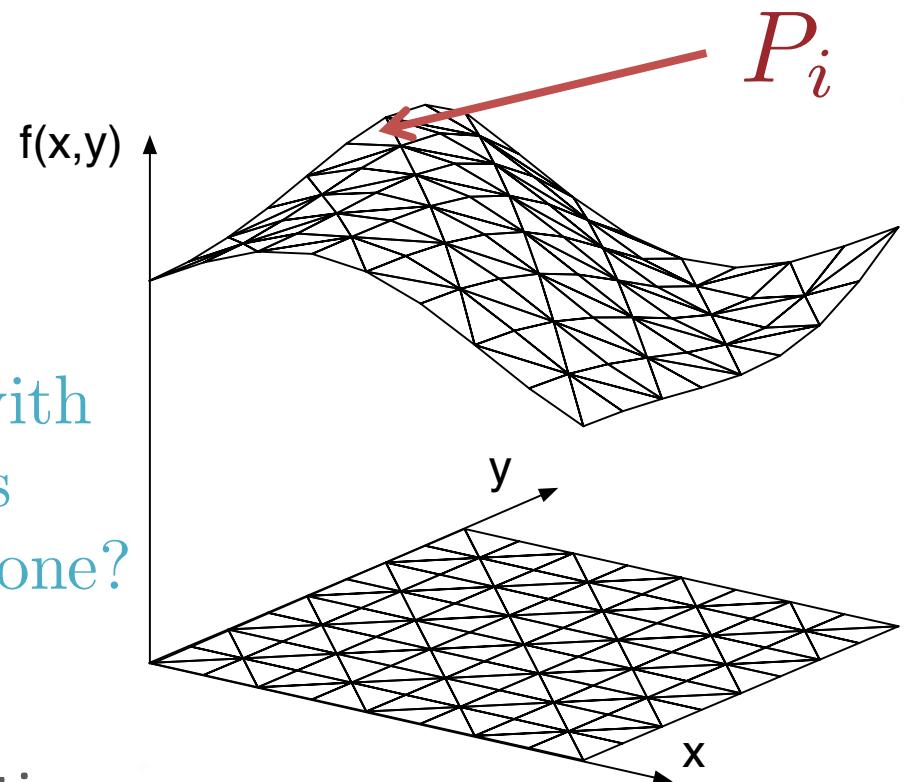
$$\sum_{v \in V} \lambda_v = 1$$

$$\lambda_v \leq \sum_{i: v \in \text{ext}(P_i)} y_i$$

$$\sum_{i=1}^n y_i = 1$$

$$y \in \{0, 1\}^n, \quad \lambda \in \mathbb{R}_+^V$$

Why stick with
0-1 variables
that add to one?



- Can also make ideal
 - Number of additional inequalities

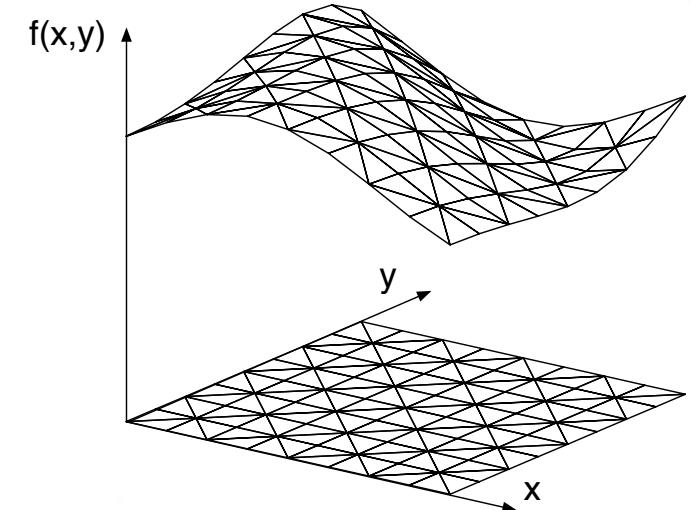
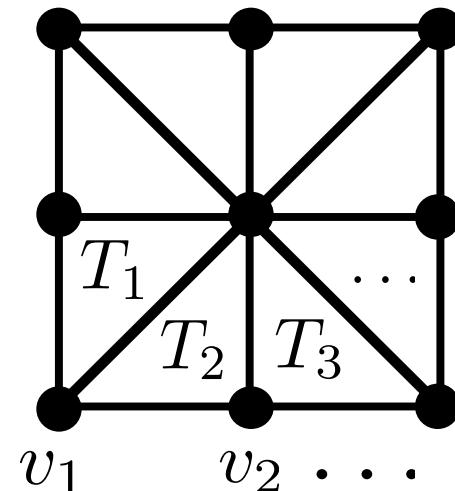
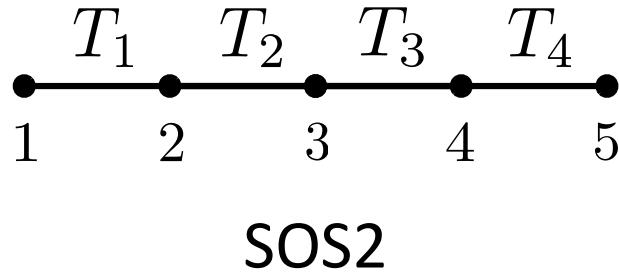
$$\begin{pmatrix} 2\sqrt{n/2} \\ \sqrt{n/2} \end{pmatrix}$$

$$\text{gr } (f) = \bigcup_{i=1}^n P_i$$

Encodings and Embeddings

Examples = \mathcal{V} -polytopes = Faces of Simplex

- $\Delta^V := \left\{ \lambda \in \mathbb{R}_+^V : \sum_{v \in V} \lambda_v = 1 \right\},$
- $P_i = \left\{ \lambda \in \Delta^V : \lambda_v = 0 \quad \forall v \notin T_i \right\}$
- $\lambda \in \bigcup_{i=1}^n P_i$
- $T_i = \text{cliques of a graph}$



Piecewise Linear Functions

Formulations and encodings

$$e^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, e^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, e^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, e^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{y} \in \{0, 1\}^4, \sum_{i=1}^5 \lambda_i = 1$$

$$\sum_{i=1}^4 y_i = 1$$

$$\begin{aligned} 0 \leq \lambda_1 &\leq y_1 \\ 0 \leq \lambda_2 &\leq y_1 + y_2 \\ 0 \leq \lambda_3 &\leq y_2 + y_3 \\ 0 \leq \lambda_4 &\leq y_3 + y_4 \\ 0 \leq \lambda_5 &\leq y_4 \end{aligned}$$

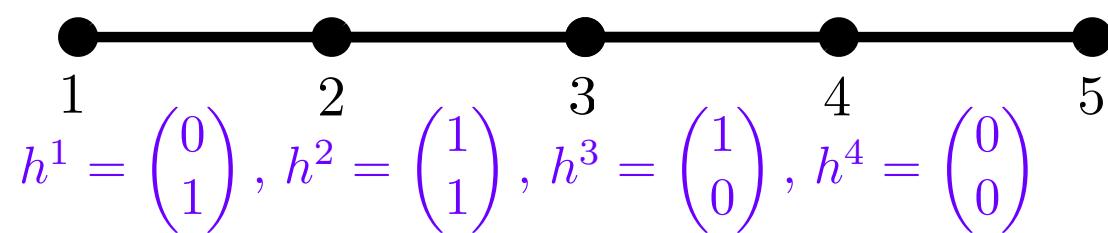
$\leftarrow Q = \text{LP relaxation} \rightarrow$

$$\begin{aligned} (\lambda, \mathbf{y}) \in Q \cap (\mathbb{R}^5 \times \mathbb{Z}^4) \\ \Updownarrow \\ \mathbf{y} = e^i \wedge \lambda \in P_i \end{aligned}$$

$$P_i := \{\lambda \in \Lambda^5 : \lambda_j = 0 \quad j \notin \{i, i+1\}\}$$

Unary Encoding

Binary Encoded Ideal Formulation for SOS2



$$\textcolor{violet}{y} \in \{0, 1\}^2, \sum_{i=1}^5 \lambda_i = 1$$



$Q = \text{LP relaxation}$

$$\begin{aligned} 0 \leq \lambda_1 + \lambda_5 &\leq \textcolor{violet}{1} - y_1 \\ 0 \leq \lambda_3 &\leq \textcolor{violet}{y}_1 \\ 0 \leq \lambda_4 + \lambda_5 &\leq \textcolor{violet}{1} - y_2 \\ 0 \leq \lambda_1 + \lambda_2 &\leq \textcolor{violet}{y}_2 \end{aligned}$$

$$(\lambda, \textcolor{violet}{y}) \in Q \cap (\mathbb{R}^5 \times \mathbb{Z}^2)$$



$$\textcolor{violet}{y} = h^i \wedge \lambda \in P_i$$

$$P_i := \{\lambda \in \Lambda^5 : \lambda_j = 0 \quad j \notin \{i, i+1\}\}$$

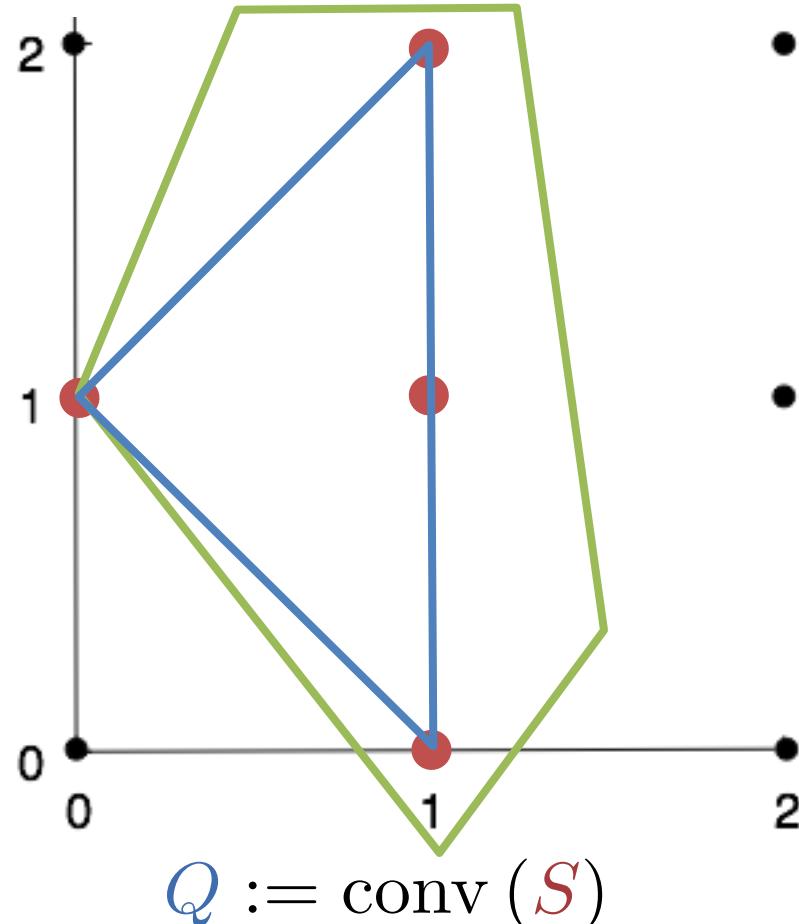
Binary Encoding



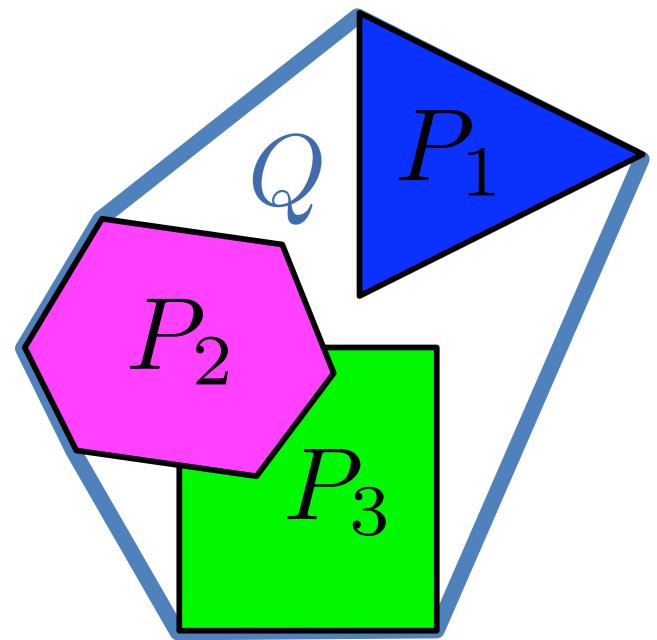
Non-Extended Formulations

- Pure Integer $S \subseteq \mathbb{Z}^d$
- Mixed-Integer $S = \bigcup_{i=1}^n P_i \subseteq \mathbb{R}^d$

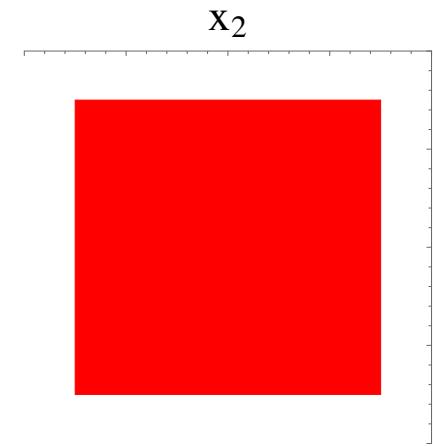
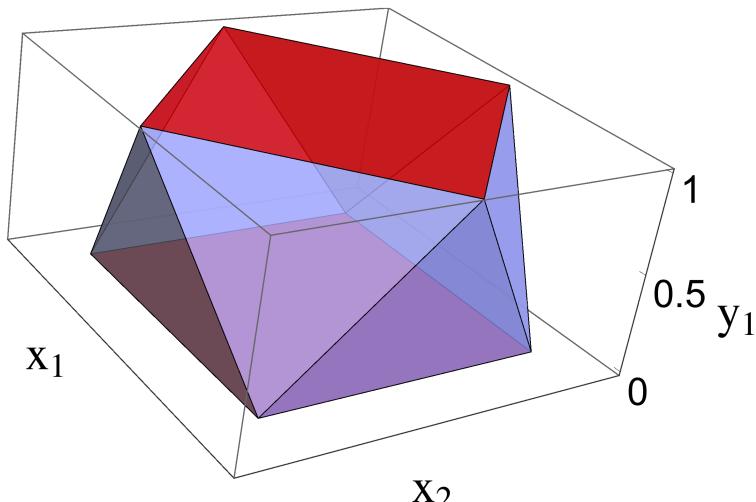
$$P \cap \mathbb{Z}^d = S \quad (P \subseteq \mathbb{R}^d)$$



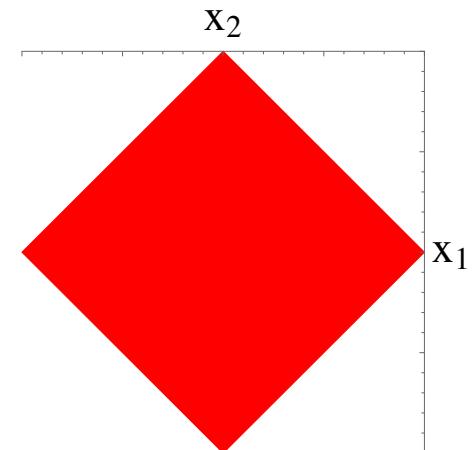
$$Q := \text{conv}(S) ?$$



Embedding Formulation = Ideal non-Extended



P_1



P_2

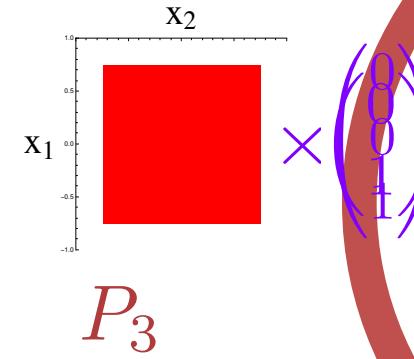
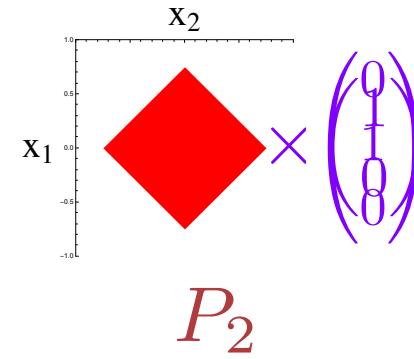
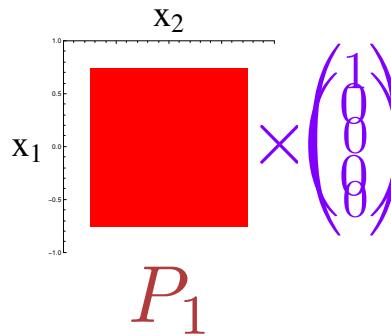
$$Q(H) := \text{conv} \left(\bigcup_{i=1}^n P_i \times \{h^i\} \right)$$

$$(x, y) \in Q \cap (\mathbb{R}^d \times \mathbb{Z}^k) \iff y = h^i \wedge x \in P_i$$

$$\text{ext}(Q) \subseteq \mathbb{R}^d \times \mathbb{Z}^k \quad H := \{h^i\}_{i=1}^n \subseteq \{0, 1\}^k, \quad h^i \neq h^j$$

Alternative Encodings

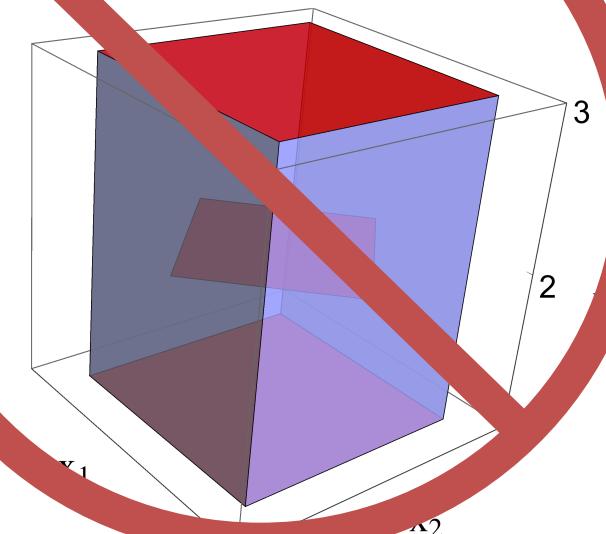
- “Only” use 0-1 encodings



- Options for 0-1 encodings:
 - Traditional or **Unary** encoding

$$H = \left\{ y \in \{0, 1\}^n : \sum_{i=1}^n y_i = 1 \right\}$$
$$= \{\mathbf{e}^i\}_{i=1}^n$$

$$\bigcup_{i=1}^n P_i \times \{i\}$$



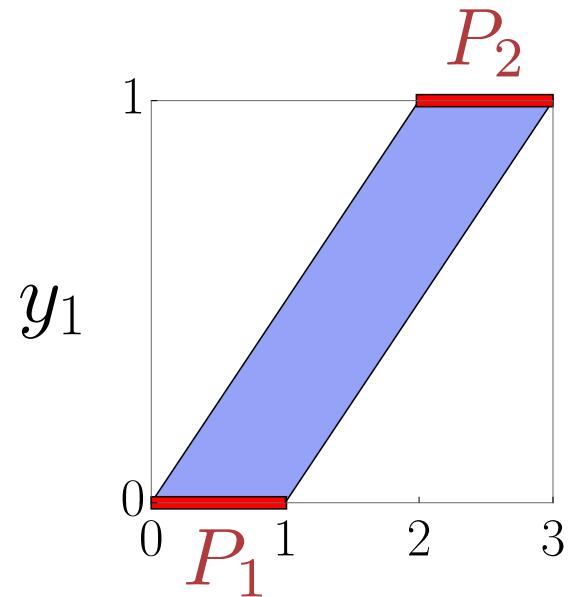
$$\mathbf{e}_j^i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- Binary** encodings: $H \equiv \{0, 1\}^{\log_2 n}$
- Others (e.g. **incremental** encoding \equiv unary)

Complexity of Family of Polyhedra $\mathcal{P} := \{P_i\}_{i=1}^n$

- Embedding complexity = smallest **ideal** formulation

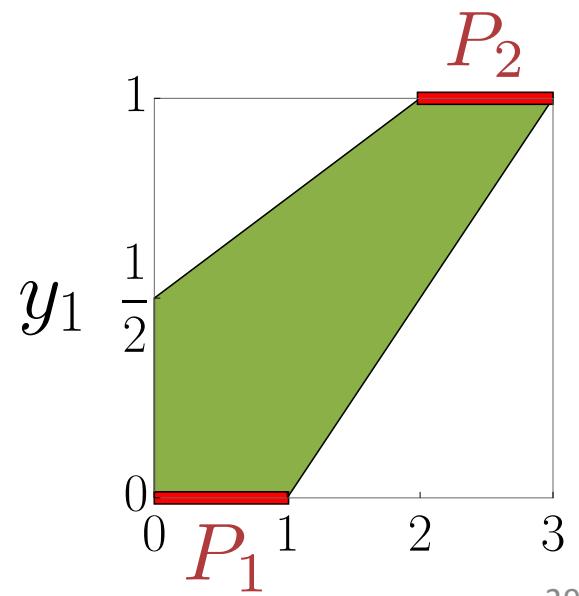
$$\text{mc}_{\mathbf{G}}(\mathcal{P}) := \min_{\mathbf{H}} \{\text{size}(\mathbf{Q}(\mathbf{H}))\}$$



- Relaxation complexity = smallest formulation

$$\text{rc}_{\mathbf{G}}(\mathcal{P}) := \min_{\mathbf{Q}, \mathbf{H}} \{\text{size}(\mathbf{Q})\}$$

$$(x, \mathbf{y}) \in \mathbf{Q} \cap (\mathbb{R}^d \times \mathbb{Z}^k) \quad \Leftrightarrow \quad \mathbf{y} = \mathbf{h}^i \quad \wedge \quad x \in P_i$$

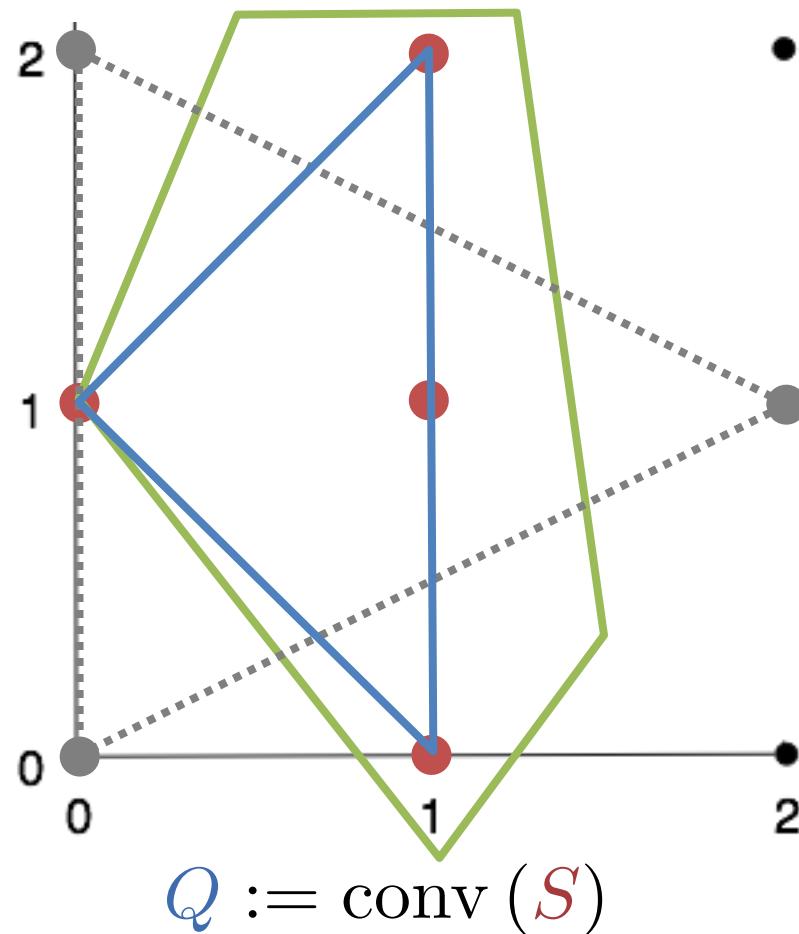


Relaxation Complexity

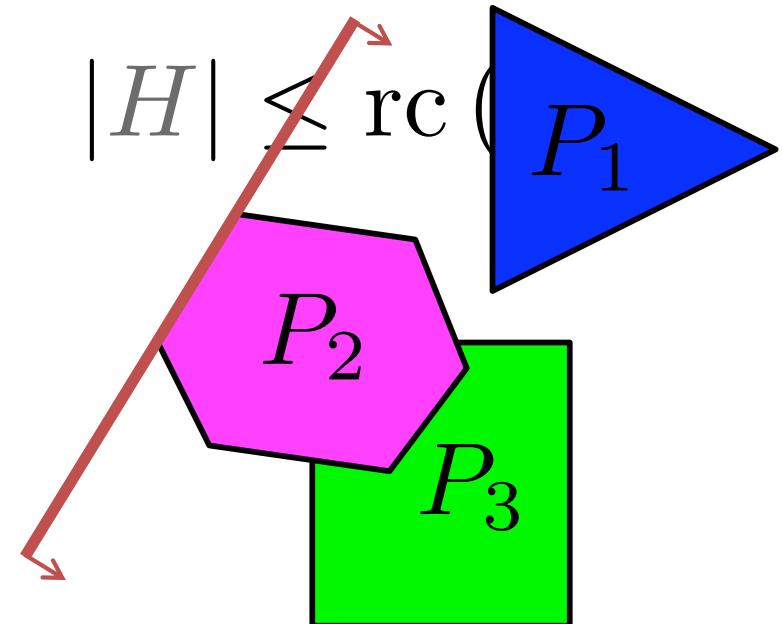
Relaxation Complexity: Pure v/s Mixed Integer

- Pure Integer $S \subseteq \mathbb{Z}^d$
- Mixed-Integer $S = \bigcup_{i=1}^n P_i \subseteq \mathbb{R}^d$

$$P \cap \mathbb{Z}^d = S \quad (P \subseteq \mathbb{R}^d)$$



Hiding set H

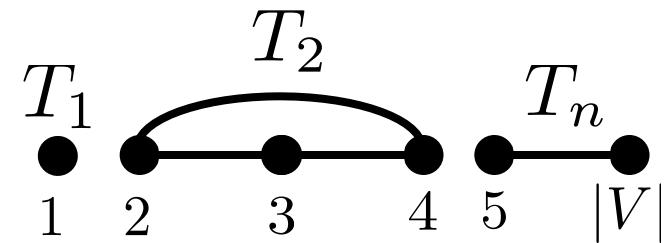


Bounds on Relaxation Complexity

- Disjoint Case : $T_i \cap T_j = \emptyset$

$$\text{rc}_G(\mathcal{P}) = 2$$

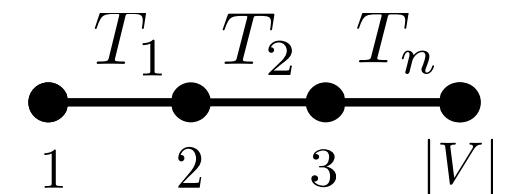
$$2 \leq \text{rc}(\mathcal{P}) \leq 2 + |V| + n$$



- SOS2 constraints : $T_i = \{i, i + 1\}$

$$2 \leq \text{rc}_G(\mathcal{P}) \leq 4$$

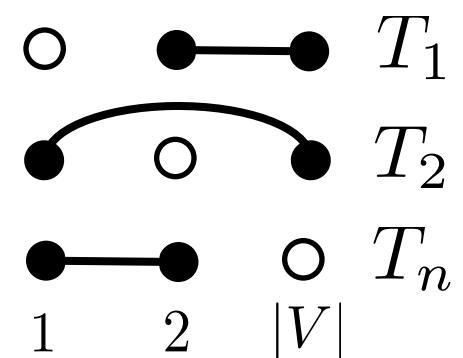
$$2 \leq \text{rc}(\mathcal{P}) \leq 5 + 2n$$



- SOS(-1) constraints : $T_i = V \setminus \{i\}$

$$\text{mc}_G(\mathcal{P}) = \text{rc}_G(\mathcal{P}) = n$$

$$n \leq \text{rc}(\mathcal{P}) \leq \text{mc}(\mathcal{P}) \leq 3n$$



Formulation for Disjoint Case



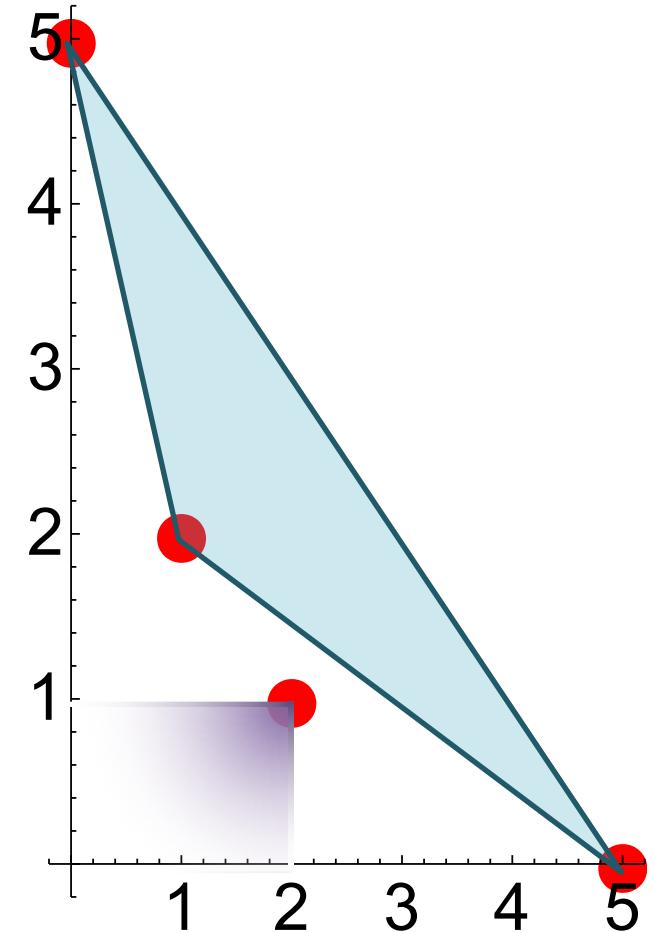
$$\sum_{i=1}^n p^i (\lambda_{2i-1} + \lambda_{2i}) \leq \sum_{i=1}^n p^i y_i$$

$$\sum_{i=1}^{2n} \lambda_i = 1, \quad \lambda \in \mathbb{R}_+^{2n}$$

$$\sum_{i=1}^n y_i = 1, \quad y \in \mathbb{R}_+^n$$

$$p^i \in \mathbb{R}_+^2, \quad \text{conv} \left(\{p^j\}_{j \neq i} \right) \not\leq p^i$$

- Polynomial sized coefficients:
 - $p^i \in \mathbb{Z}_+^2, \quad \|p^i\|_\infty \leq 5^{\lceil(n-2)/2\rceil}$
- **80** fractional extreme points for $n = 5$



Embedding Complexity

Embedding Complexity for SOS2

- Unary encoding:

$$\text{size}_G(Q(H)) = 2(n - 1), \quad \text{size}(Q(H)) = 2n$$

- **Smallest Binary** encoding:

$$\text{size}_G(Q(H)) = 2 \lceil \log_2 n \rceil,$$

$$2 + 2 \lceil \log_2 n \rceil \leq \text{size}(Q(H)) \leq n + 1 + 2 \lceil \log_2 n \rceil$$

- Adding lower bounds:

$$\text{mc}_G(\mathcal{P}) = 2 \lceil \log_2 n \rceil,$$

$$n + 1 \leq \text{mc}(\mathcal{P}) \leq n + 1 + 2 \lceil \log_2 n \rceil$$

Binary Encoded Formulation for SOS2

$$h^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, h^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, h^3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, h^4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Gray code = 1 bit difference from i to i+1

$$\sum_{i=1}^5 \lambda_i = 1$$

Q = LP relaxation

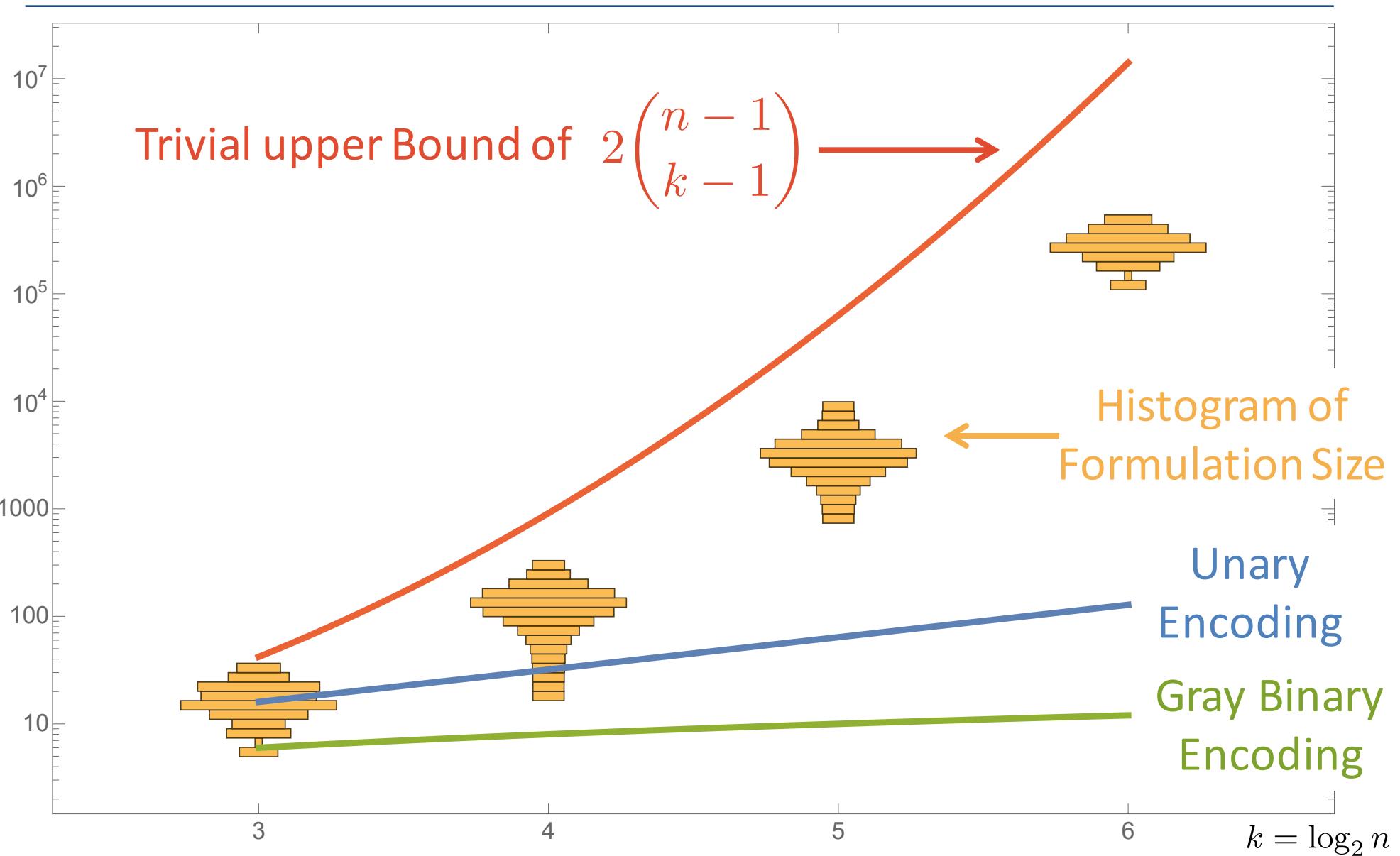
$$\begin{aligned} 0 \leq \lambda_1 + \lambda_5 &\leq 1 - y_1 \\ 0 \leq \lambda_3 &\leq y_1 \\ 0 \leq \lambda_4 + \lambda_5 &\leq 1 - y_2 \\ 0 \leq \lambda_1 + \lambda_2 &\leq y_2 \end{aligned}$$

$$\begin{aligned} (\lambda, y) \in Q \cap (\mathbb{R}^5 \times \mathbb{Z}^2) \\ \Updownarrow \\ y = h^i \wedge \lambda \in P_i \end{aligned}$$

$$P_i := \{\lambda \in \Lambda^5 : \lambda_j = 0 \quad j \notin \{i, i+1\}\}$$

Binary Encoding

General Inequalities for all Binary Encodings



Formulations and Complexity for Triangulations

- Lower bound:

$$\left(\sqrt{n/2} + 1\right)^2 \leq \text{mc}(\mathcal{P})$$

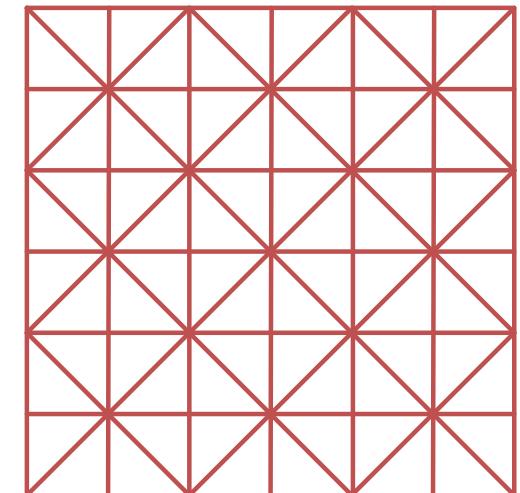
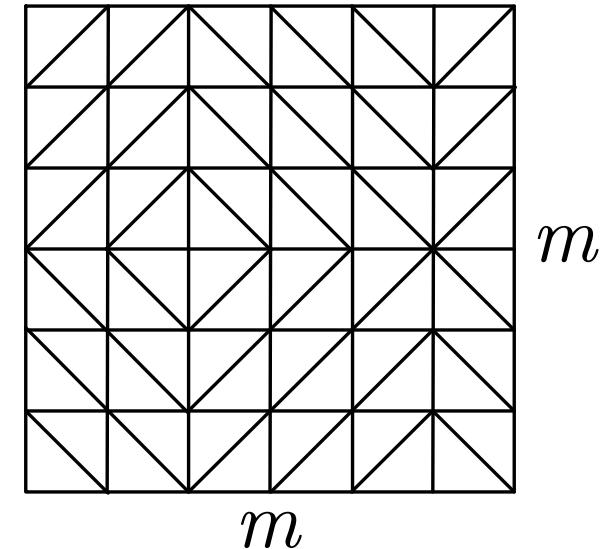
- Size of unary formulation is:

$$\text{mc}(\mathcal{P}) \leq \binom{2\sqrt{n/2}}{\sqrt{n/2}} + \left(\sqrt{n/2} + 1\right)^2$$

- Small binary formulation for union jack triangulation of size:

$$\text{mc}(\mathcal{P}) \leq 4 \log_2 \sqrt{n/2} + 2 + \left(\sqrt{n/2} + 1\right)^2$$

$$n = 2m^2$$



Next Lecture

- Practical construction technique 1:
 - Independent branching formulations for V-polyhedra
- Practical construction technique 2:
 - Case study for H-polyhedra
 - Encoding and alternative representations of disjunctions

References

- V., “Mixed integer linear programming formulation techniques”, SIAM Review 57, 2015. pp. 3–57.
- Kaibel and Weltge, “Lower Bounds on the Sizes of Integer Programs without Additional Variables”, Math. Program. 154, 2015. pp. 407–425.
- V. “Embedding Formulations and Complexity for Unions of Polyhedra”, 2015. arXiv:1506.01417
- V. “Relaxation Complexity for Special Ordered Sets”, 2016. Working Paper.

Extended Formulations for $x \in \bigcup_{i=1}^n P_i$

$$P_i = \{x \in \mathbb{R}^d : A^i x \leq b^i\}$$

$$\begin{aligned} A^i x^i &\leq b^i y_i \quad \forall i \in [n] \\ \sum_{i=1}^n x^i &= x \\ \sum_{i=1}^n y_i &= 1 \\ y &\in \{0, 1\}^n \end{aligned}$$

\mathcal{H} -formulation

$$\begin{aligned} \sum_{i=1}^n \sum_{v \in \text{ext}(P_i)} v \lambda_v^i &= x \\ \sum_{v \in \text{ext}(P^i)} \lambda_v^i &= y_i \quad \forall i \in [n] \\ \sum_{i=1}^n y_i &= 1 \\ \lambda^i &\in \mathbb{R}_+^{\text{ext}(P_i)} \\ y &\in \{0, 1\}^n \end{aligned}$$

\mathcal{V} -formulation

- Both formulations are ideal and use copies of variables

Simple Non-Extended \mathcal{H} -formulation

$$P^i = \{x \in \mathbb{R}^d : A^i x \leq b^i\} \rightarrow P^i = \{x \in \mathbb{R}^d : Dx \leq d^i\}$$

$$D = \begin{bmatrix} A^1 \\ \vdots \\ A^k \end{bmatrix}, \quad d^i \text{ appropriately constructed (Big-Ms)}$$

$$Dx \leq \sum_{i=1}^n d^i y_i$$

$$\sum_{i=1}^n y_i = 1$$

$$y \in \{0, 1\}^n$$

- Usually not ideal or sharp, but often stronger than standard Big-M (Sharp under some conditions)

Simple Non-Extended \mathcal{V} -formulation

$$V := \bigcup_{i=1}^n \text{ext}(P_i)$$

$$\sum_{v \in V} v \lambda_v = x$$

$$\sum_{v \in V} \lambda_v = 1$$

$$\lambda_v \leq \sum_{i: v \in \text{ext}(P_i)} y_i$$

$$\sum_{i=1}^n y_i = 1$$

$$y \in \{0, 1\}^n, \quad \lambda \in \mathbb{R}_+^V$$

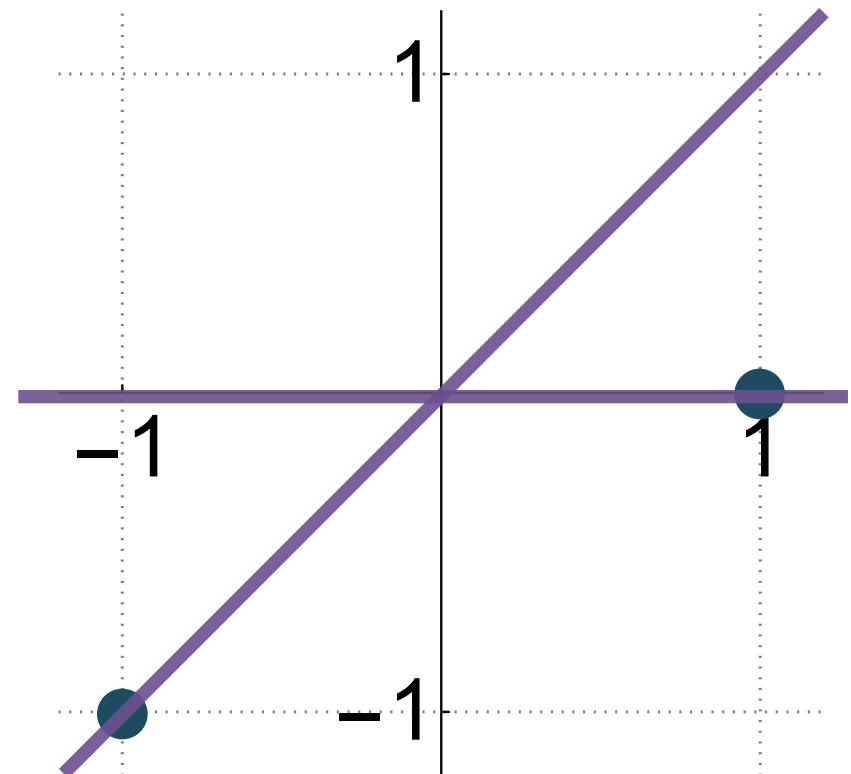
- Usually not ideal, but automatically sharp

Embedding Formulation for SOS2: Part 1

- From encodings to hyperplanes:

$$\begin{aligned} & \left\{ h^i \right\}_{i=1}^n \\ & \downarrow \\ & c^i = h^{i+1} - h^i \\ & \downarrow \\ & \left\{ c^i \right\}_{i=1}^{n-1} \\ & \text{Hyperplanes spanned by} \\ & \downarrow \\ & \left\{ b^i \cdot y = 0 \right\}_{j=1}^L \end{aligned}$$

$$h^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, h^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, h^3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



Embedding Formulation for SOS2: Part 2

$$\{b^i \cdot y = 0\}_{j=1}^L$$

$$Q(H) = L(H) := \text{aff}(H) - h^1$$

$$\begin{aligned} (b^j \cdot h^1) \lambda_1 + \sum_{i=2}^n \min \{b^j \cdot h^i, b^j \cdot h^{i-1}\} \lambda_i + (b^j \cdot h^n) \lambda_{n+1} &\leq b^j \cdot y \quad \forall j \\ - (b^j \cdot h^1) \lambda_1 - \sum_{i=2}^n \max \{b^j \cdot h^i, b^j \cdot h^{i-1}\} \lambda_i - (b^j \cdot h^n) \lambda_{n+1} &\leq -b^j \cdot y \quad \forall j \end{aligned}$$

$$\sum_{i=1}^{n+1} \lambda_i = 1, \quad \lambda \in \mathbb{R}_+^{n+1}$$

$$y \in L(H)$$

- # general inequalities = $2 \times \# \text{ of hyperplanes}$