# The Chvátal-Gomory Closure of a Strictly Convex Body is a Rational Polyhedron


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# Outline

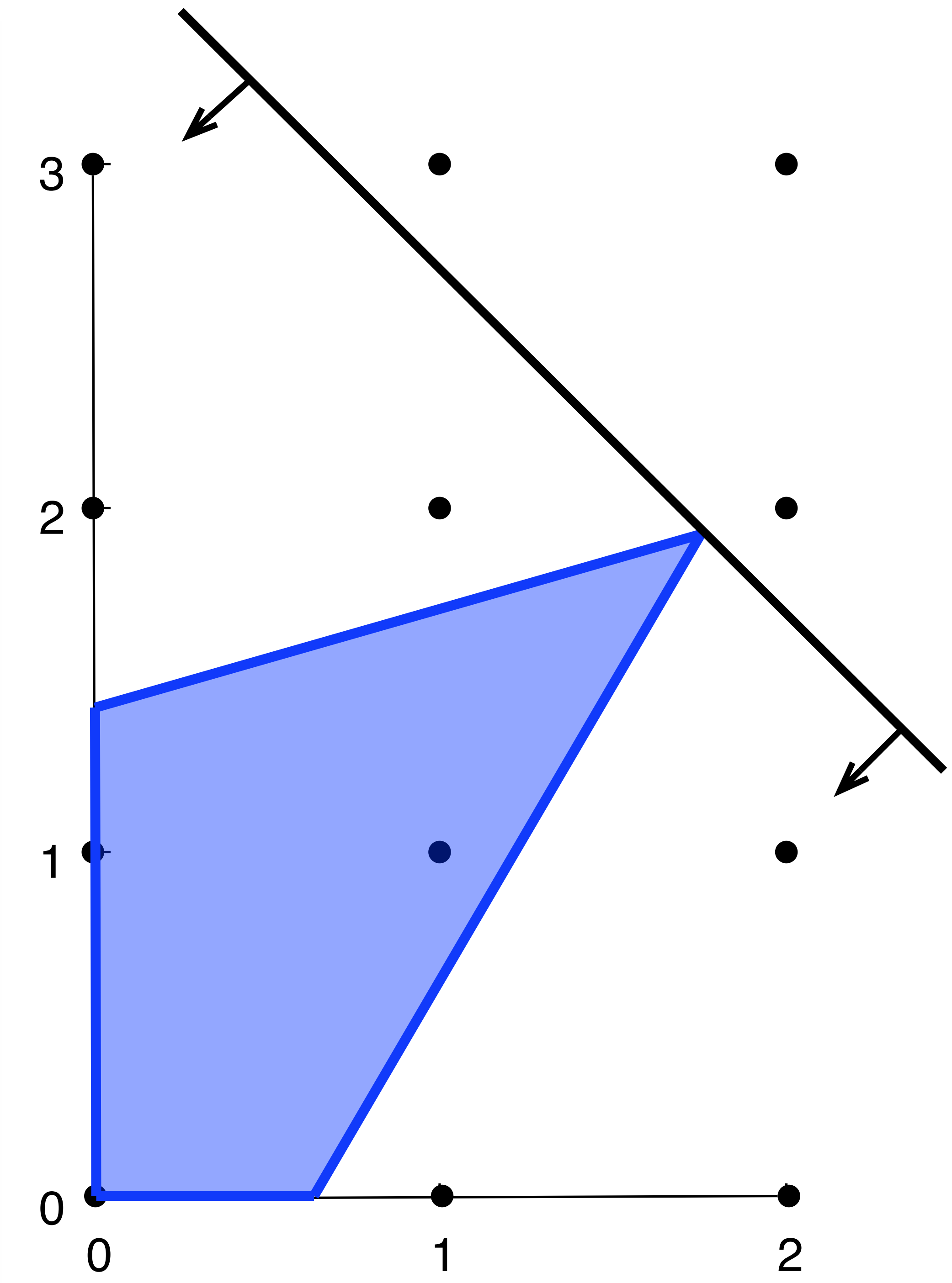
- Introduction
- Proof:
  - Step 1
  - Step 2
- Conclusions and Future Work

# CG Cuts for Rational Polyhedra

$$P := \left\{ x \in \mathbb{R}^2 \ : \ \begin{aligned} x_1 + x_2 &\leq 3, \\ 5x_1 - 3x_2 &\leq 3 \end{aligned} \right\}$$

$$\cap\!|$$

$$H := \{x \in \mathbb{R}^2 \ : \ 4x_1 + 3x_2 \leq 10.5\}$$



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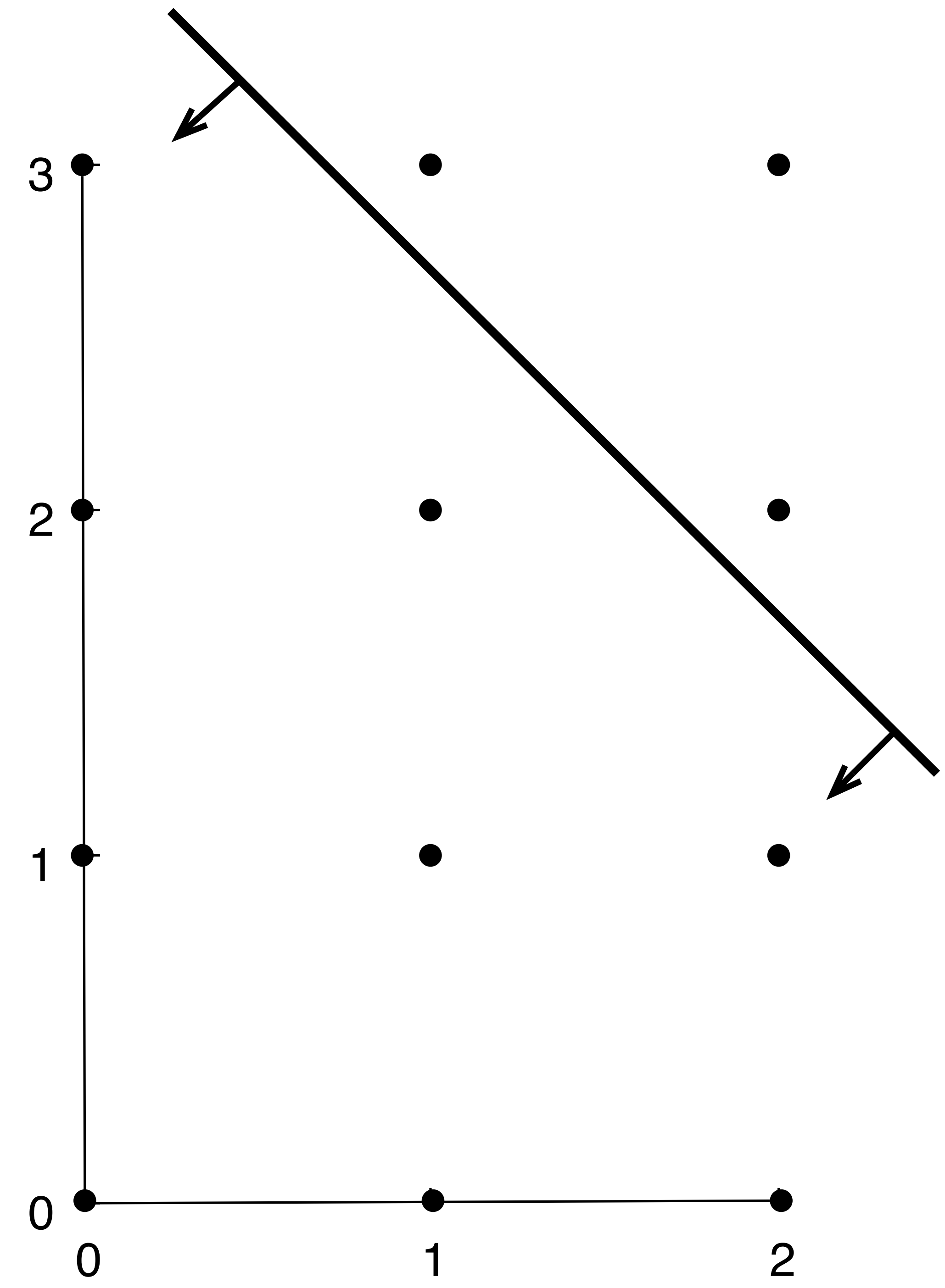

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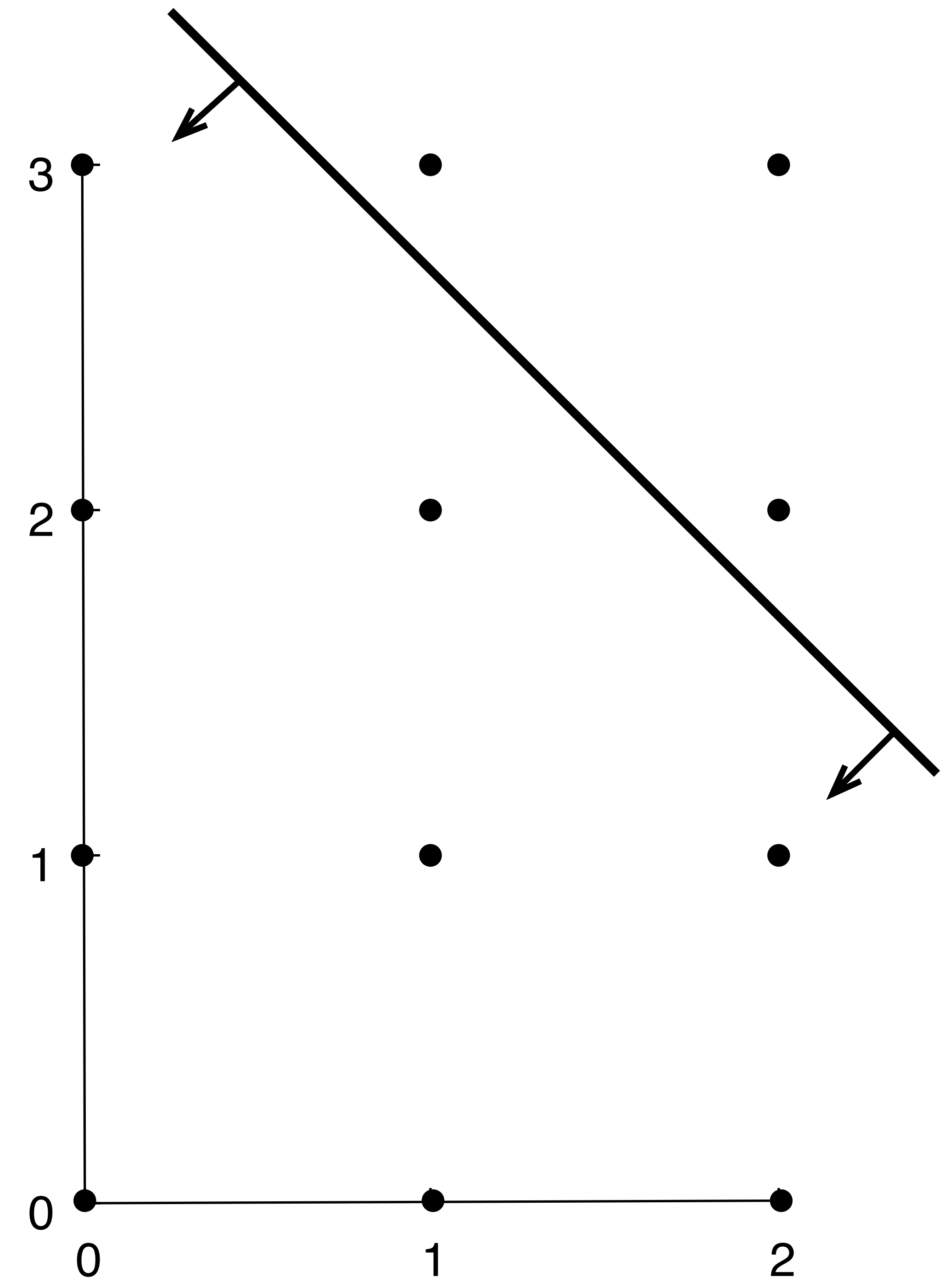
if $x \in \mathbb{Z}^2$



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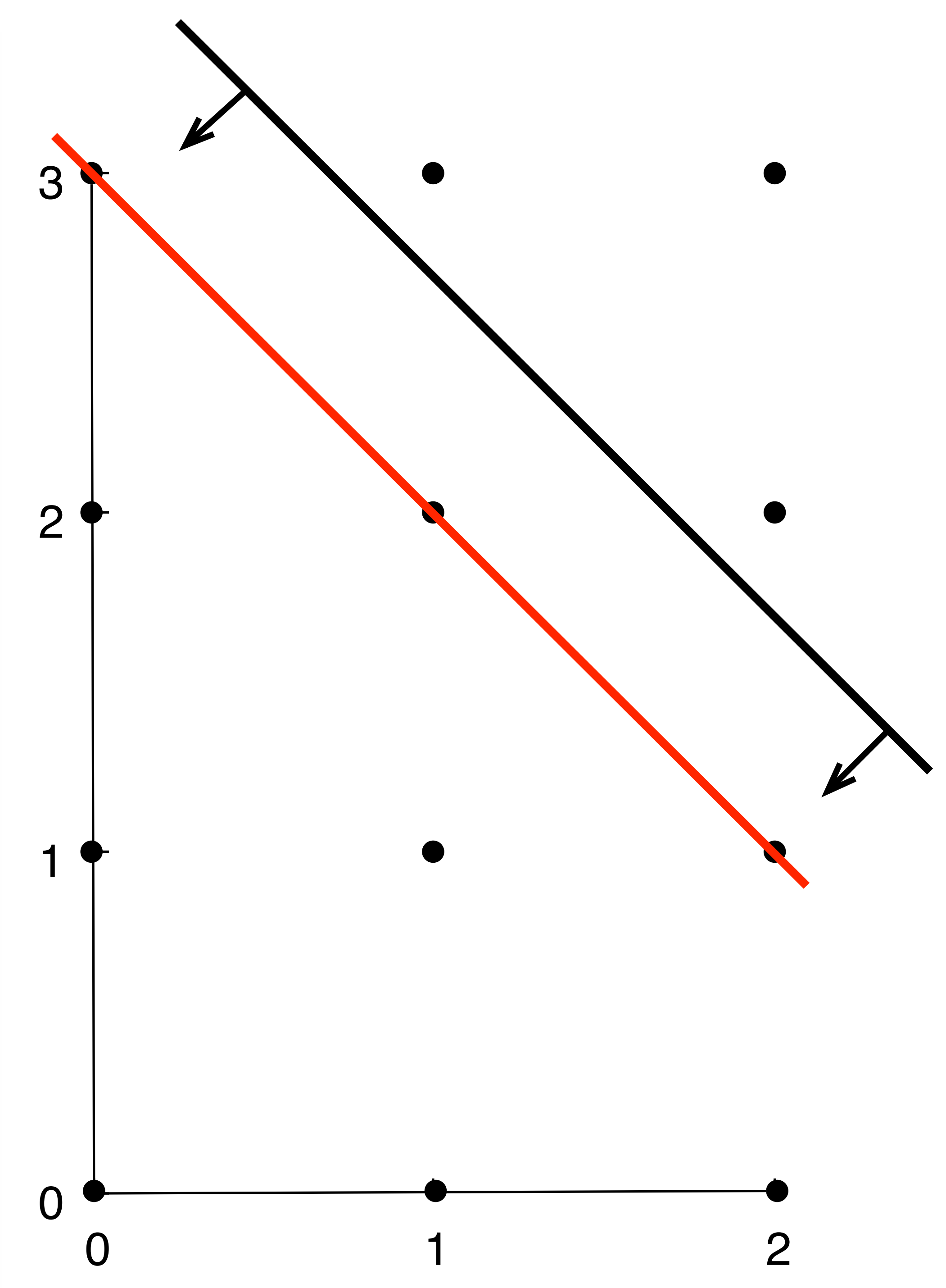
$$\text{Valid for } H \cap \mathbb{Z}^2$$



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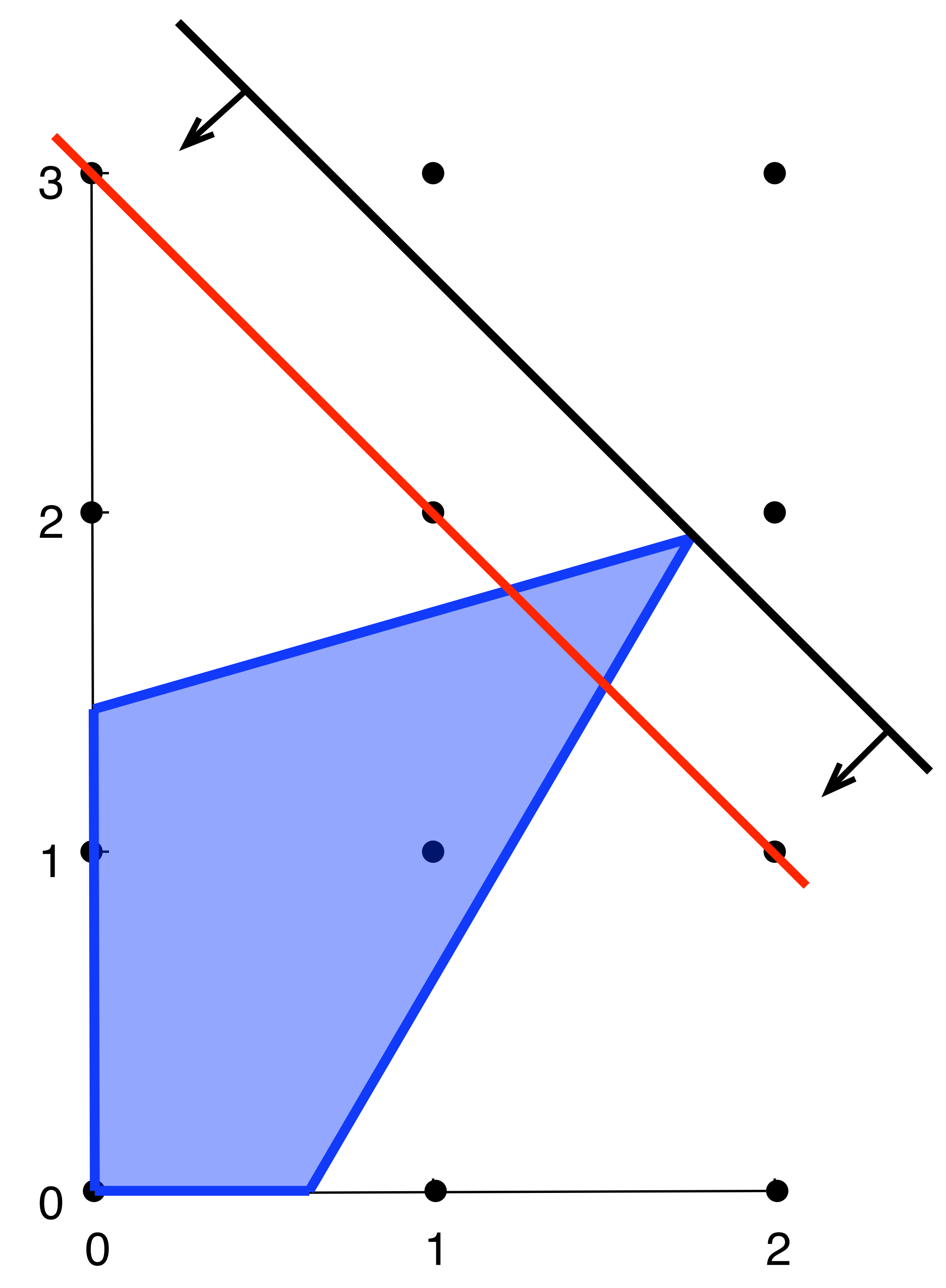
Valid for $H \cap \mathbb{Z}^2$

Valid for $C \cap \mathbb{Z}^2$



# Importance of CG Cuts

- 3 main papers (Chvátal, Gomory, Schrijver)
  - +1,200 citations.
- First pure cutting plane algorithm for Integer Programming.
- CG cuts yield Matching Polytope.
- Cutting plane proofs.
- Still practical computationally.

# CG Cuts for General Convex Sets

$$\sigma_C(a) := \sup\{\langle a, x\rangle \ : \ x \in C\}$$

$$C = \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n \ : \ \langle a, x\rangle \le \sigma_C(a)\}$$

$$\rotatebox{-90}{$\subseteq$}$$

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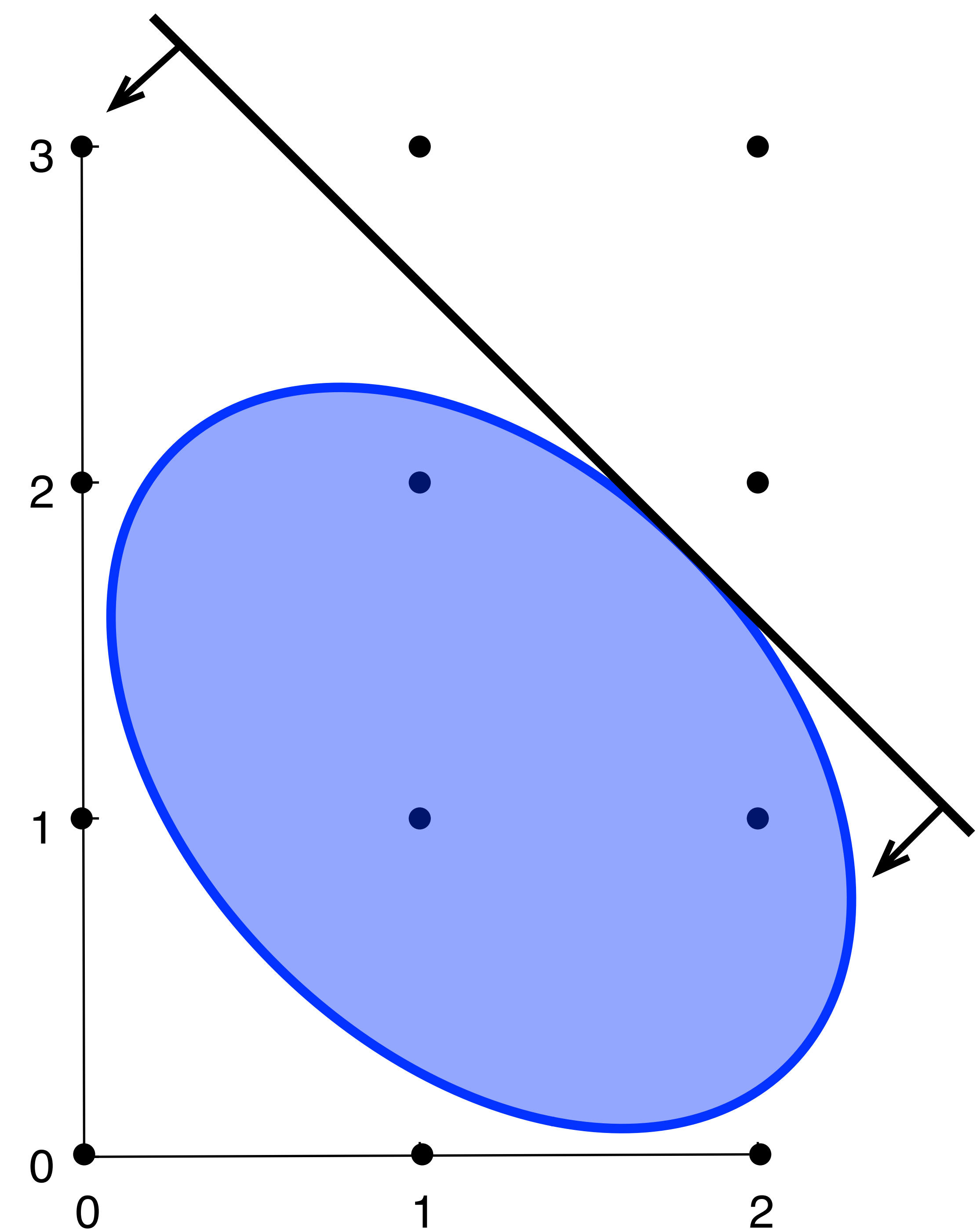

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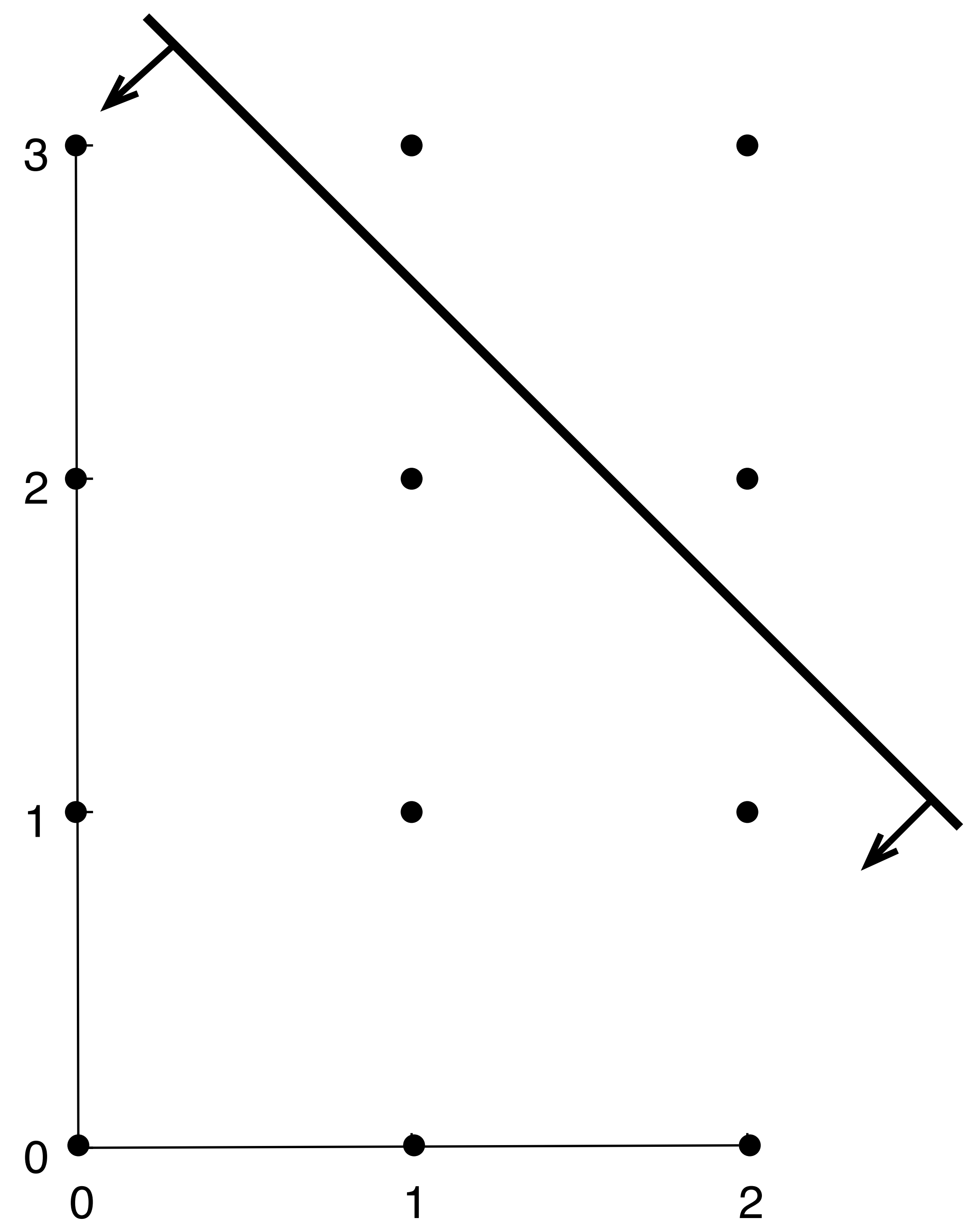

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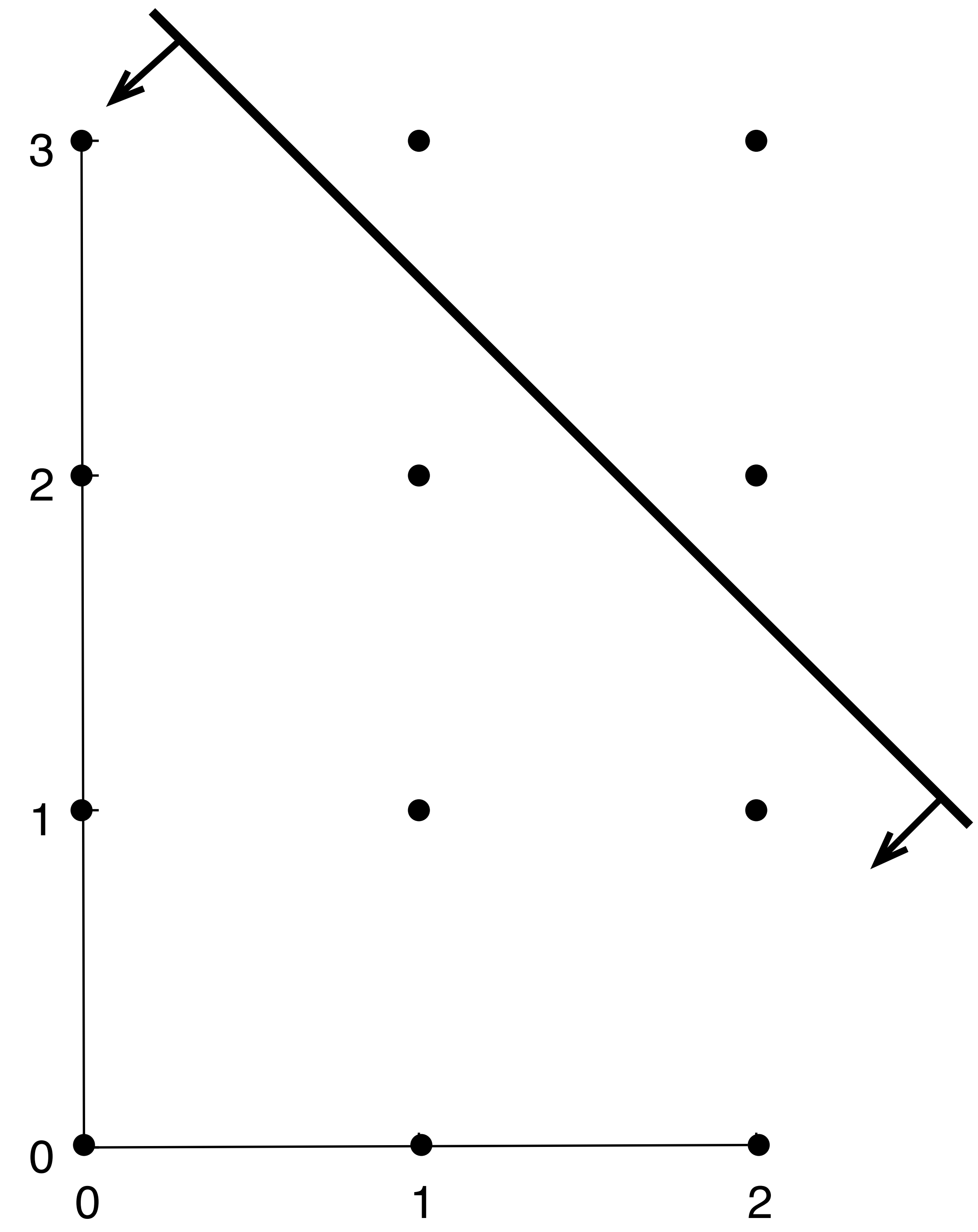

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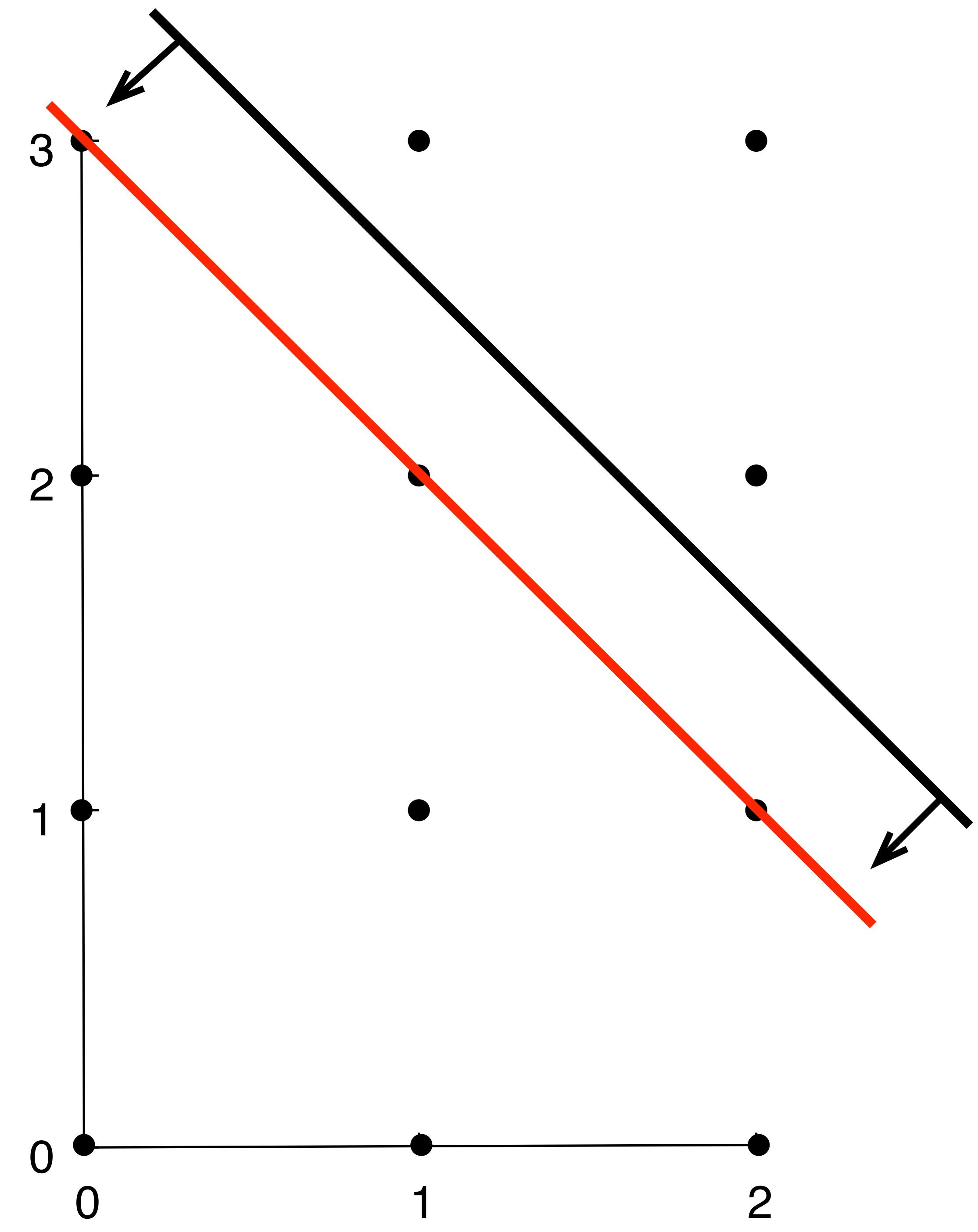
$$\langle a, x\rangle \leq \lfloor \sigma_C(a) \rfloor$$

Valid for $H \cap \mathbb{Z}^n$



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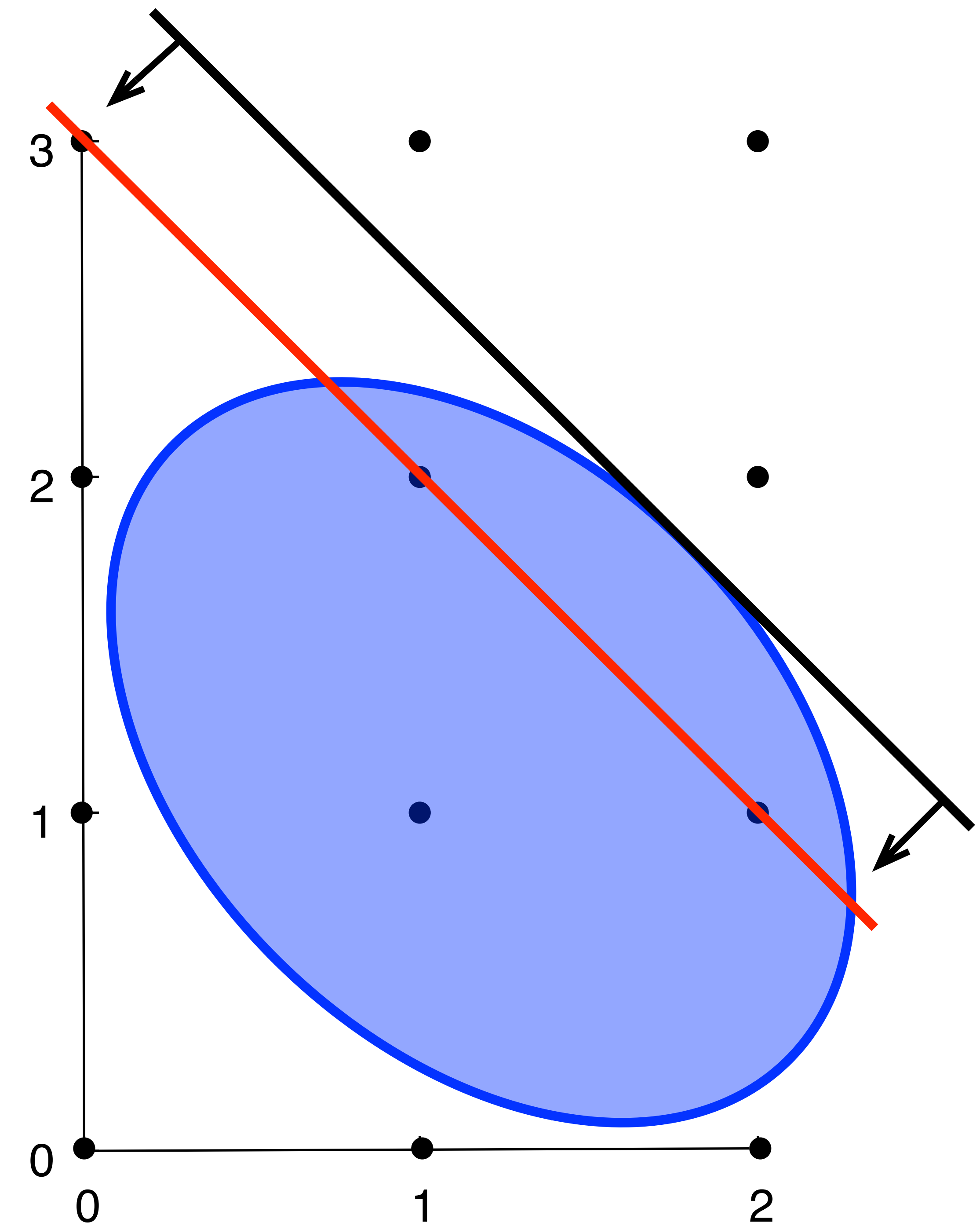

# CG Closure = Add all CG Cuts

$$\mathtt{CGC}(C) := \bigcap_{a \in \mathbb{Z}^n} \{x \in \mathbb{R}^n \ : \ \langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor\}$$

- Not necessarily a polyhedron, remember:

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- $\texttt{CGC}(C)$ is a polyhedron if $C$ is:
  - a rational polyhedra (Schrijver, 1980).
  - a “rational” ellipsoid (Dey and V. 2010).

# Finite Number of Important CG Cuts

Theorem: There exists finite $S \subseteq \mathbb{Z}^n$ such that

$$\mathtt{CGC}(C) = \bigcap_{a \in S} \{x \in \mathbb{R}^n \ : \ \langle a, x \rangle \leq \lfloor \sigma_C(a) \rfloor\}$$

# Proof Outline for $\mathrm{bd}(C) \cap \mathbb{Z}^n = \emptyset$

Theorem: There exists finite $S \subseteq \mathbb{Z}^n$ such that

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- Step 1: There exists finite $S^1 \subseteq \mathbb{Z}^n$ such that
  - $\mathtt{CGC}(S^1, C) \subseteq \mathrm{int}(C)$

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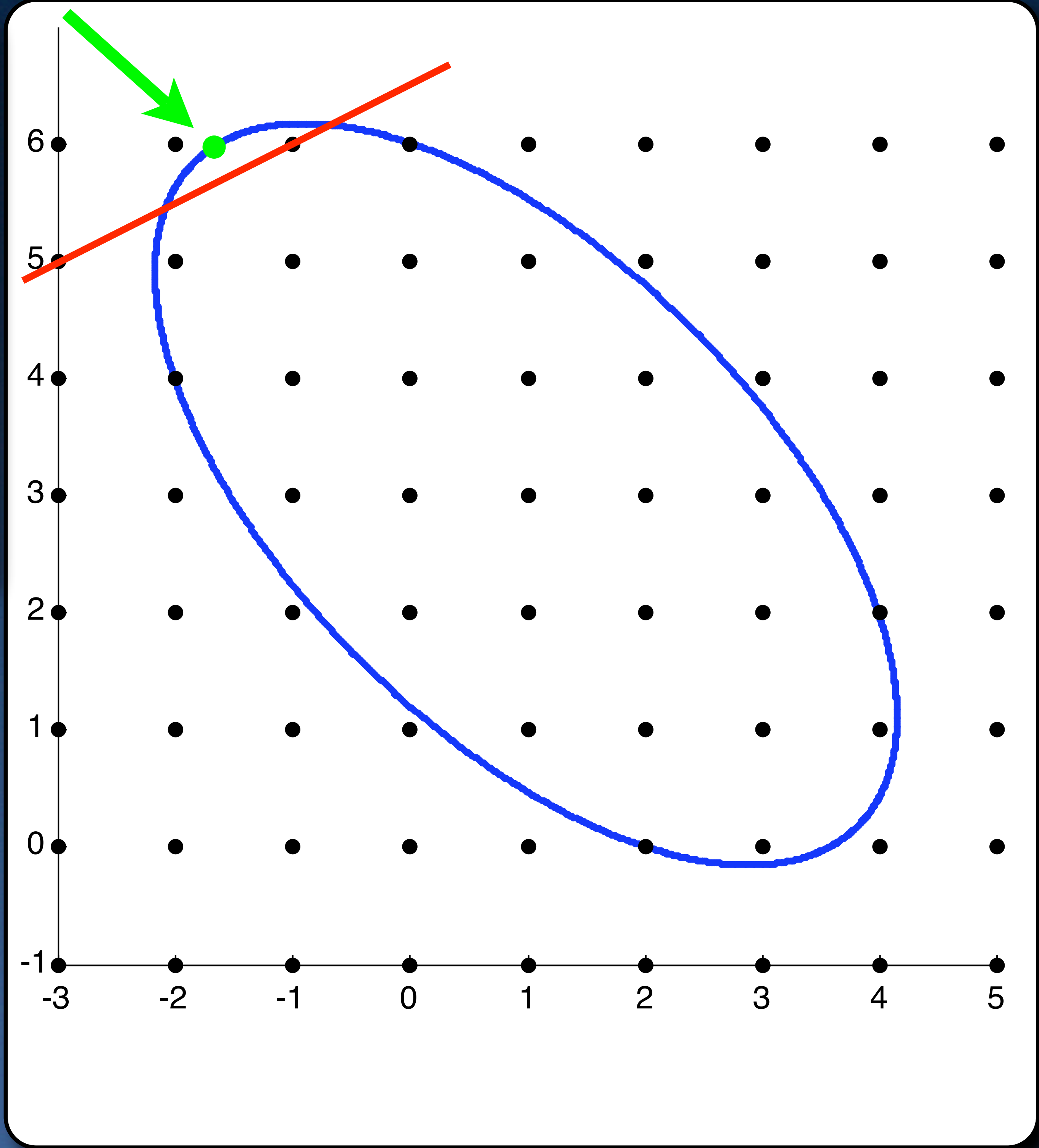
- Step 1: There exists finite $S^1 \subseteq \mathbb{Z}^n$ such that
  - $\mathtt{CGC}(S^1, C) \subseteq \mathrm{int}(C)$
- Step 2: There exists finite $S^2 \subseteq \mathbb{Z}^n$ such that
  - $\mathtt{CGC}(C) = \mathtt{CGC}(S^1, C) \cap \mathtt{CGC}(S^2, C)$

# Separate non-integral points in $\mathrm{bd}(C)$

$u \in \mathrm{bd}(C) \setminus \mathbb{Z}^n \qquad \exists\, a^u \in \mathbb{Z}^n$

$$\langle a^u, u \rangle > \lfloor \sigma_C(a^u) \rfloor$$



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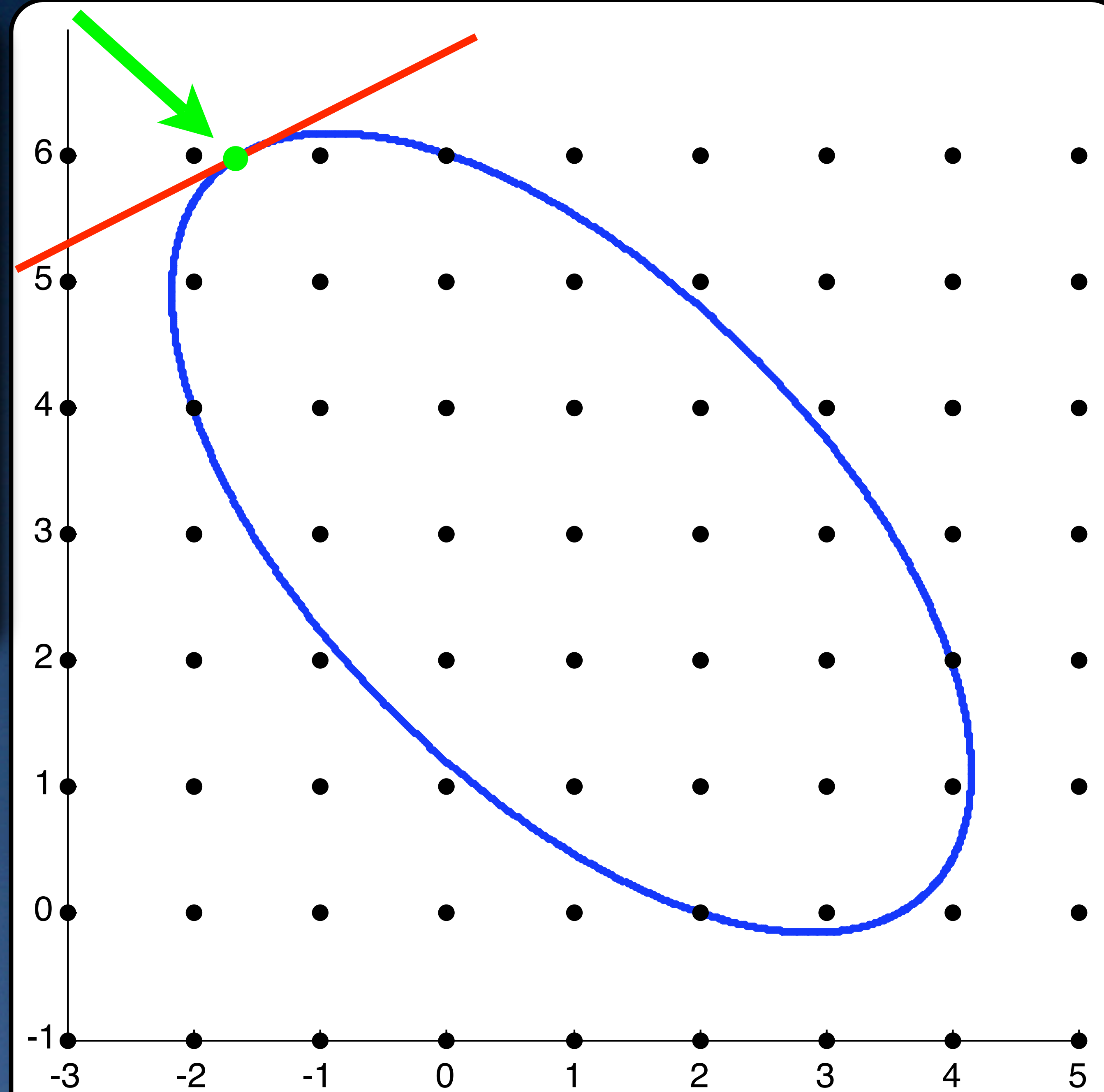
$$\langle s(u), u \rangle = \sigma_C(s(u))$$



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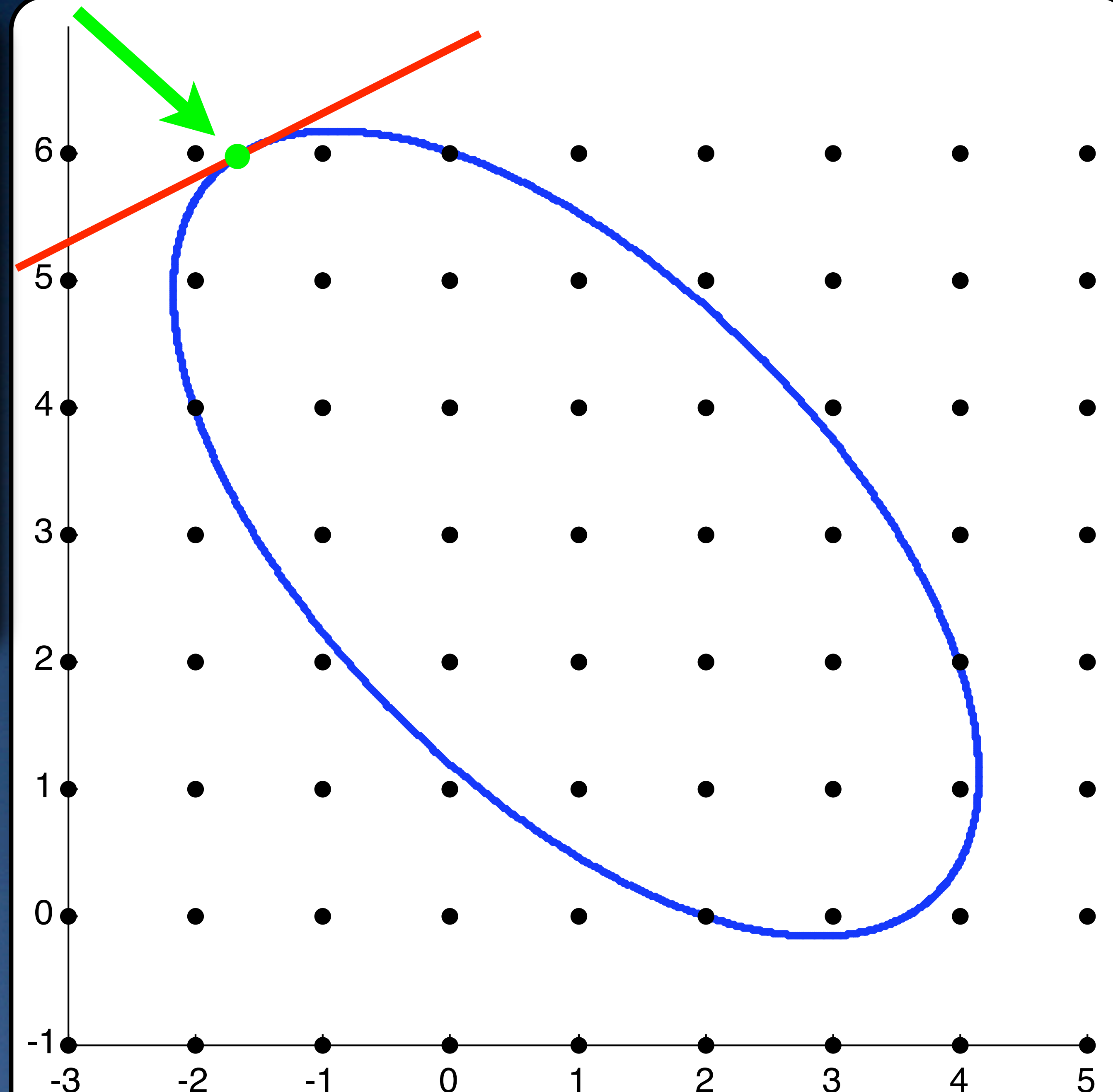

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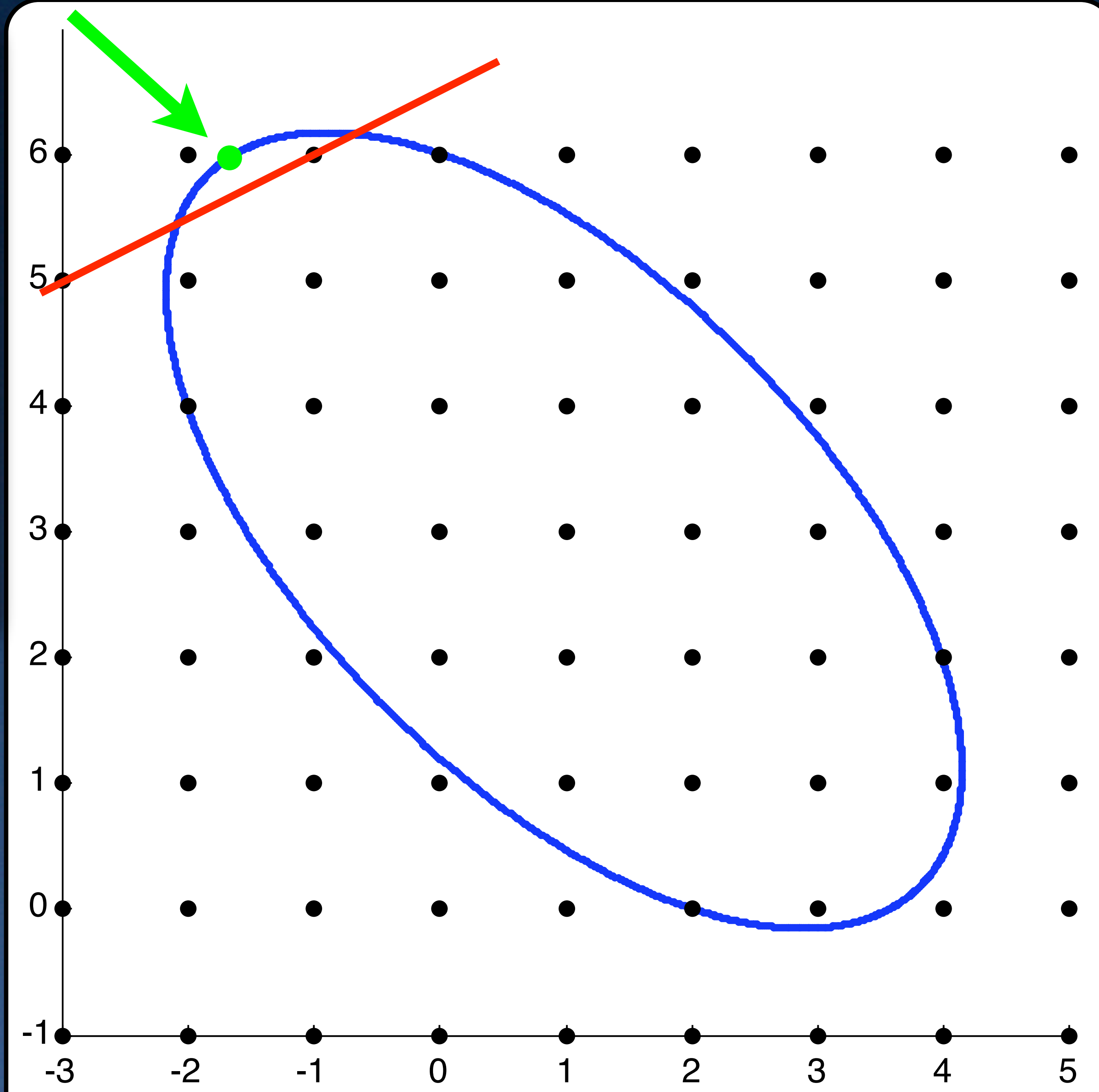

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$\lambda s(u) \notin \mathbb{Z}^n \quad \forall \lambda > 0 :$

$\lambda s(u) \in \mathbb{Z}^n \Rightarrow \sigma_C(\lambda s(u)) \in \mathbb{Z} :$

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$$C = \left\{ x \in \mathbb{R}^2 \;:\; \sqrt{x_1^2 + x_2^2} \leq 1 \right\}$$

$$u = (1/2, \sqrt{3}/2)^T \in \mathrm{bd}(C)$$

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$\lambda s(u) \in \mathbb{Z}^n \Rightarrow \sigma_C(\lambda s(u)) \in \mathbb{Z}:$

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$$C = \left\{ x \in \mathbb{R}^2 \; : \; \sqrt{x_1^2 + x_2^2} \leq 5 \right\}$$

$$u = (25/13, 60/13)^T \in \mathrm{bd}(C)$$

$$s(u) = (5, 12)^T,\ \sigma_C(s(u)) = 65$$

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$$\frac{s^i}{\|s^i\|} \xrightarrow{i \to \infty} \frac{s(u)}{\|s(u)\|}$$

$$\lim_{i \to \infty} \langle s^i, u\rangle - \lfloor \sigma_C(s^i) \rfloor > 0$$

Diophantine approx. of $s(u)$

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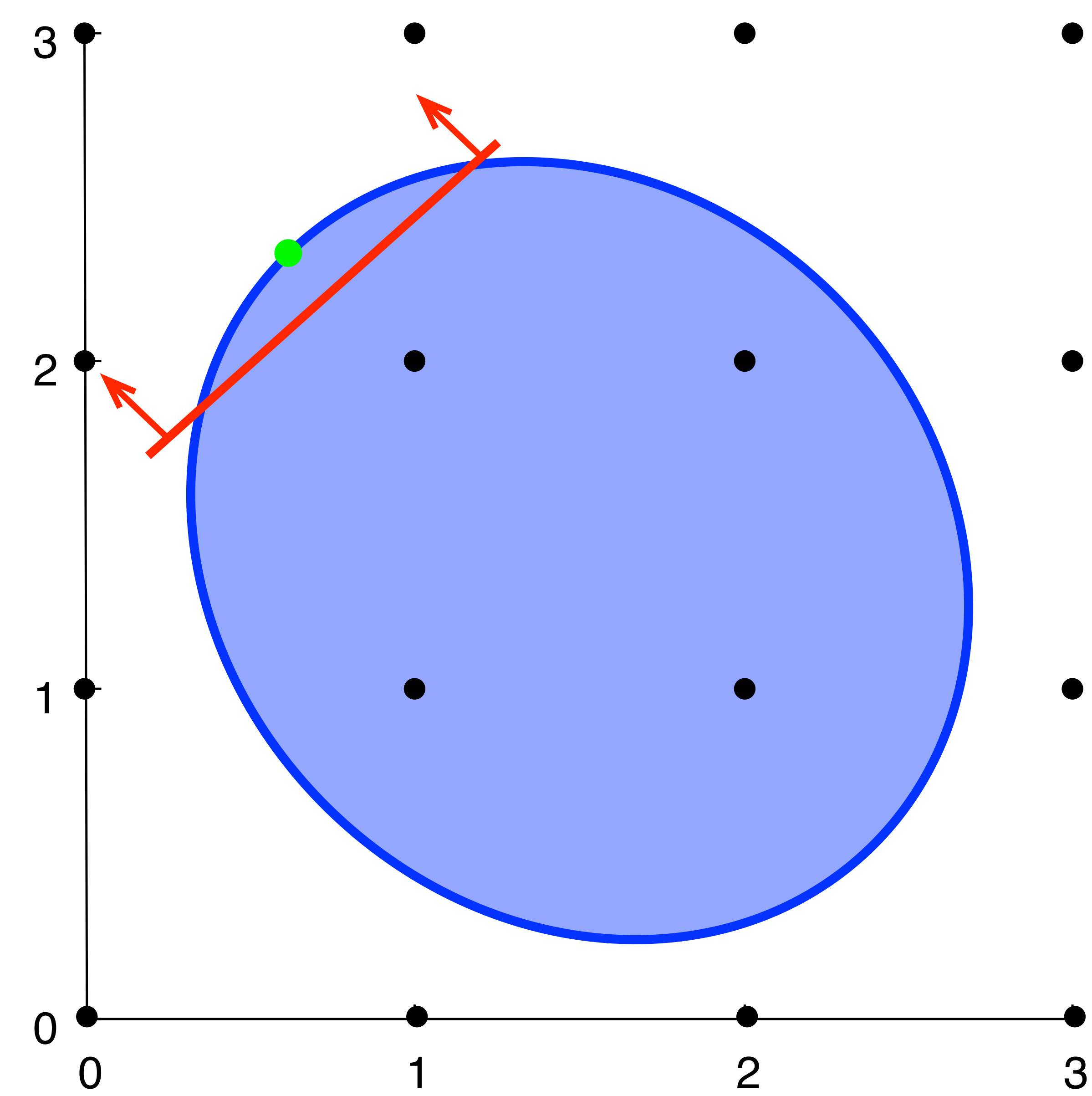
# Compactness Argument

$K := \mathrm{bd}(C)$

$S_u := \{x : \langle a^u, x\rangle > \lfloor \sigma_C(a^u) \rfloor\}$



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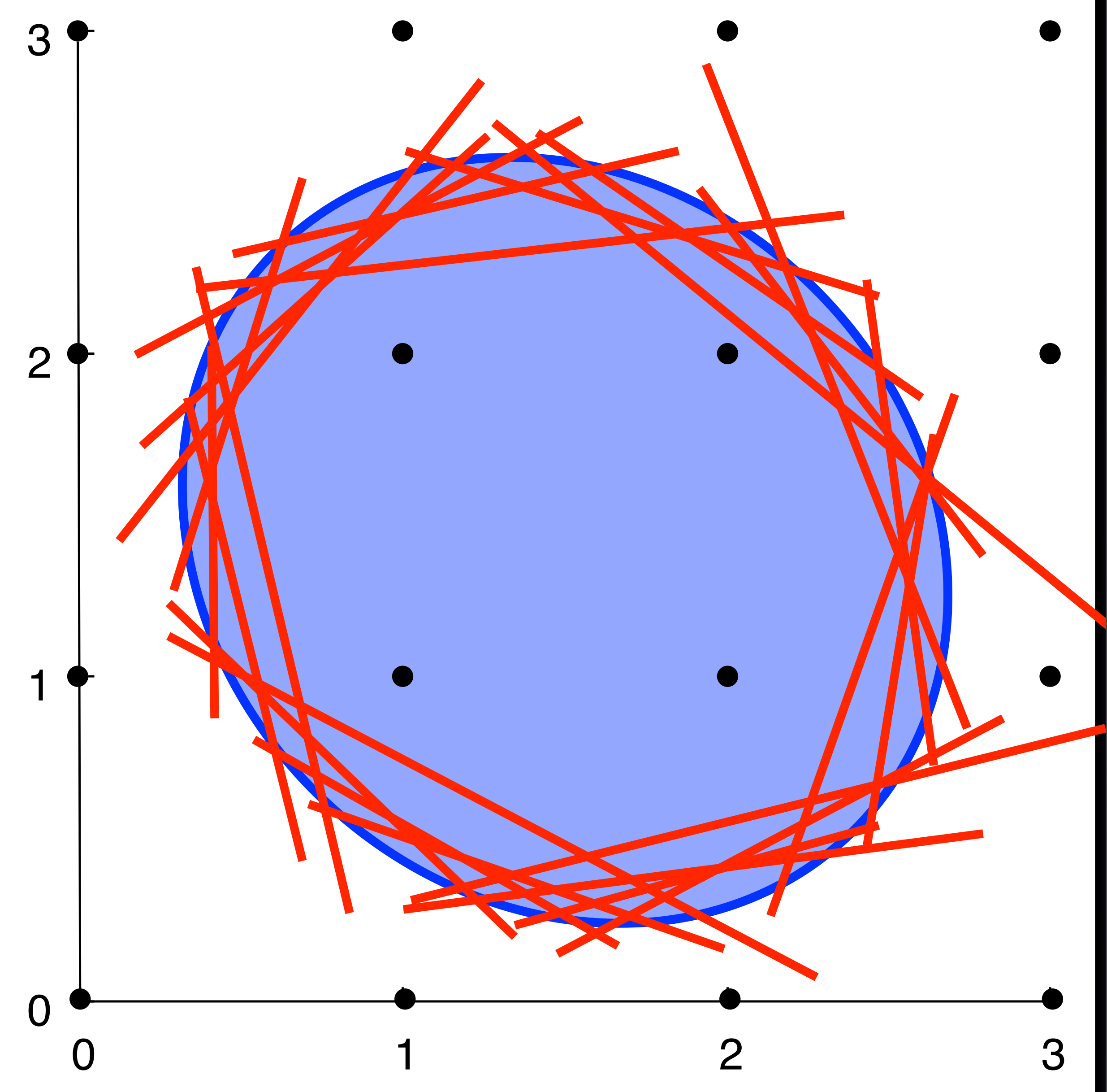
$$K \subset \bigcup_{u \in K} S_u$$



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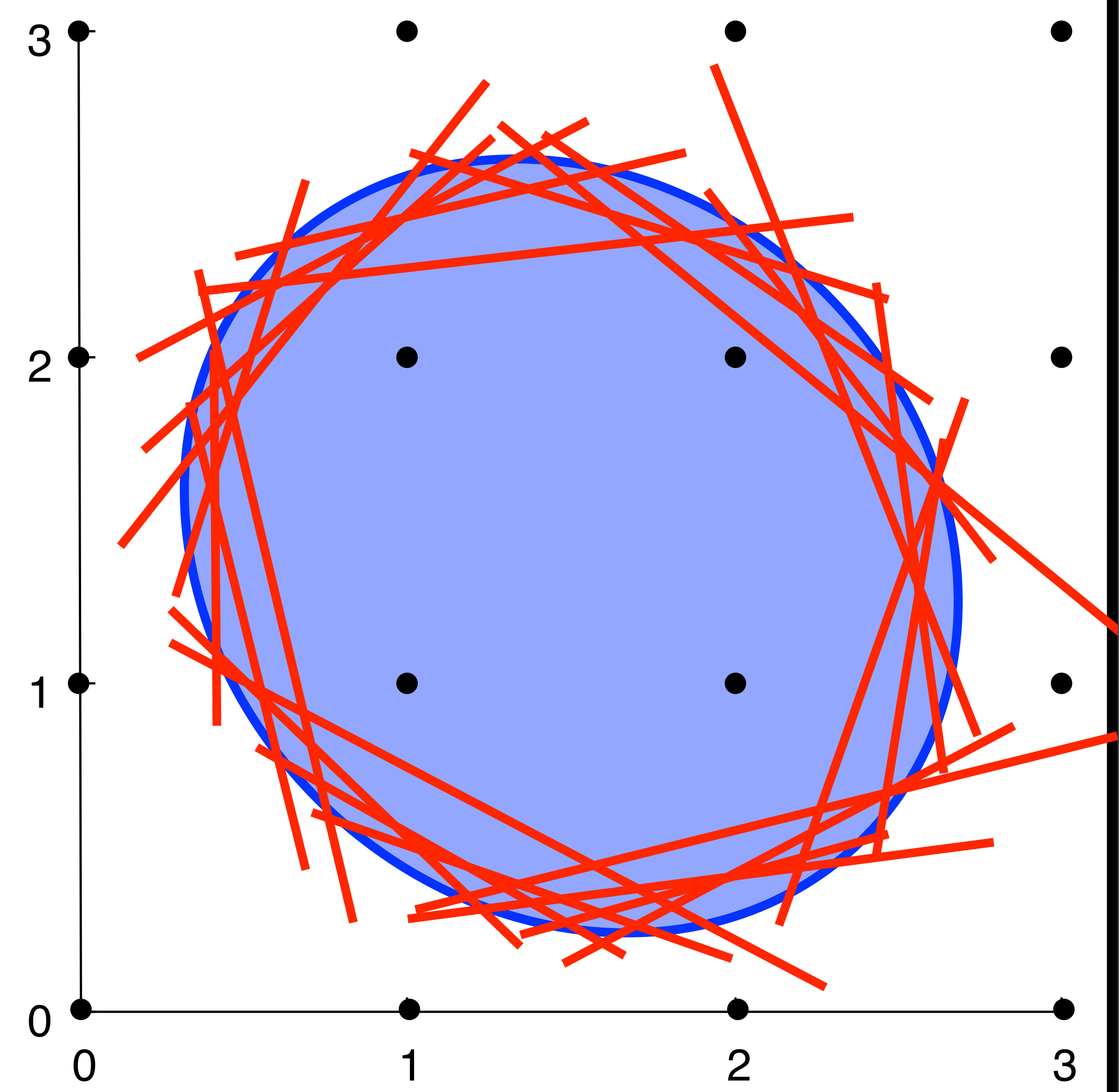
compact $\longrightarrow$ $K \subset \bigcup_{u \in K} S_u$



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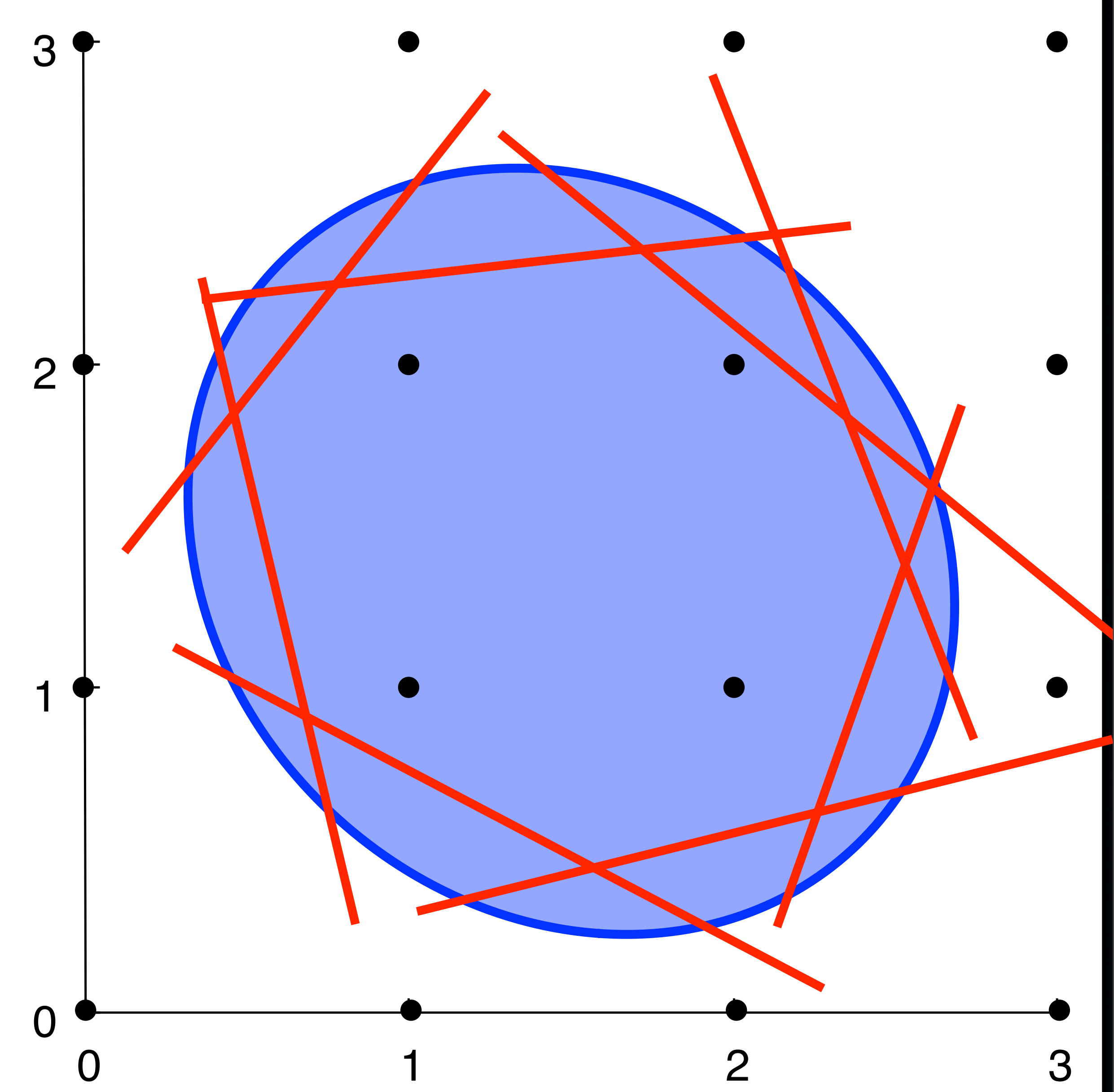
$\downarrow$

$K \subset \bigcup_{i=1}^{m} S_{u^i}$



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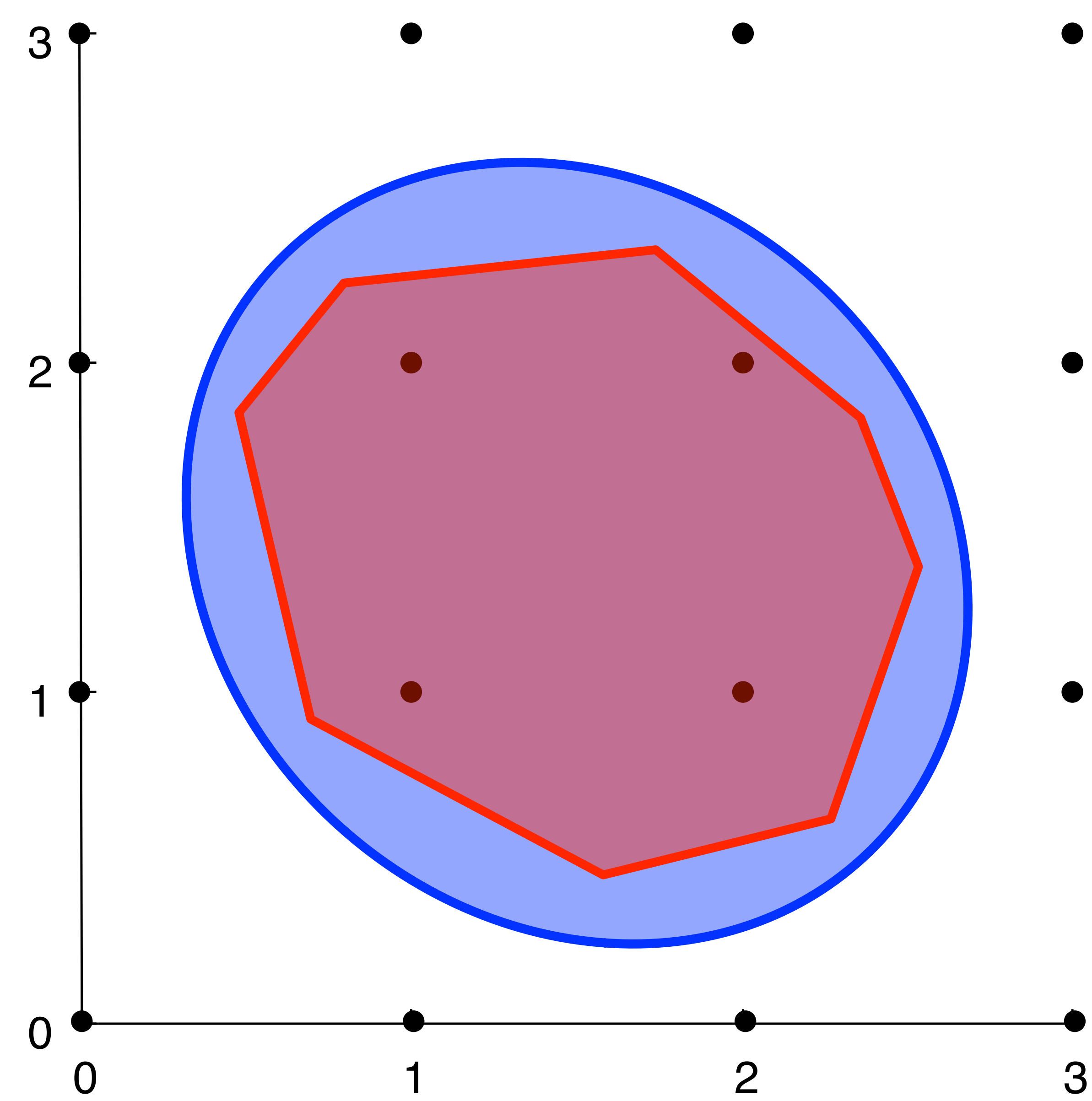
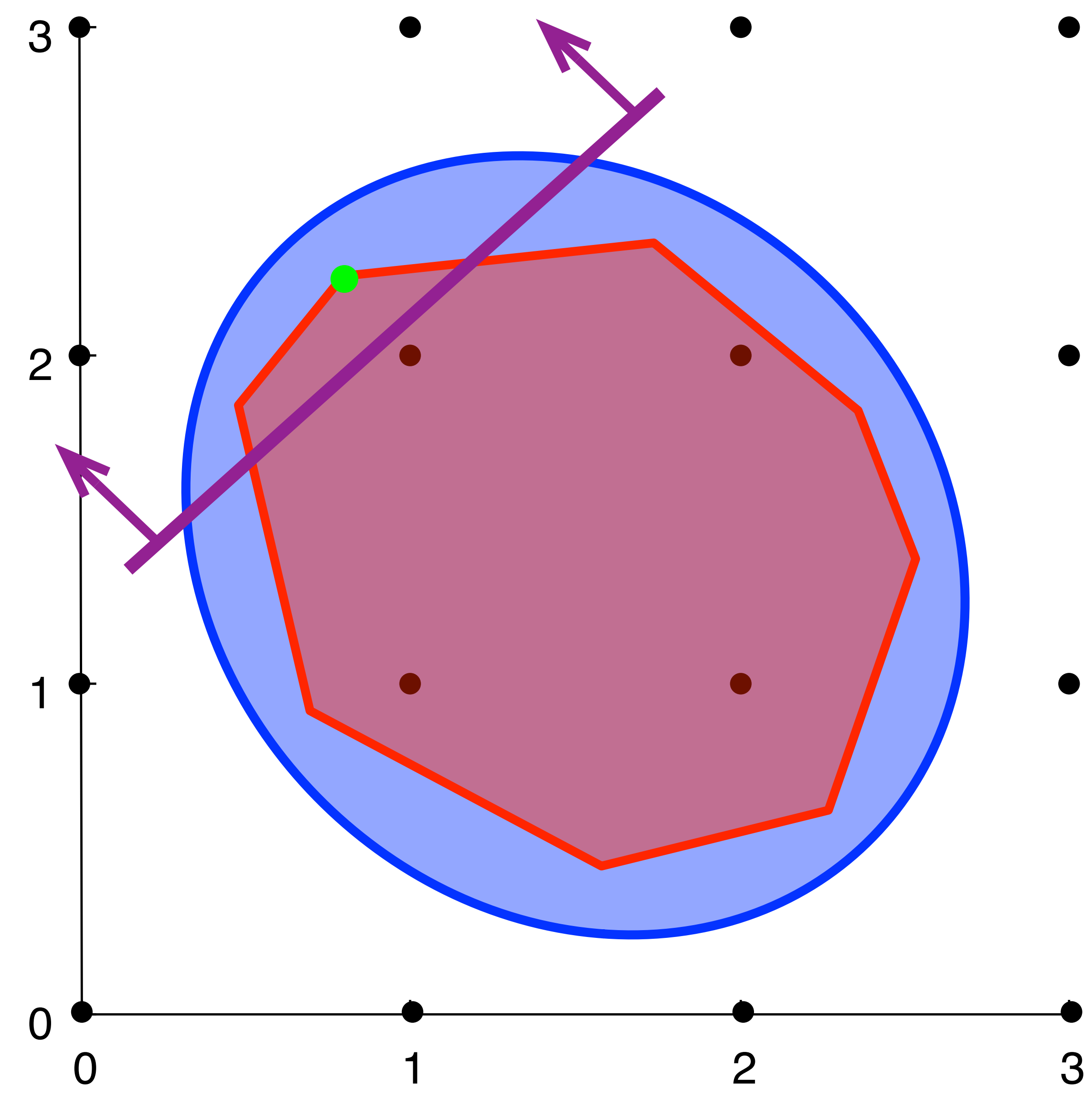
$$S^1 = \bigcup_{i=1}^{m} \left\{ a^{u^i} \right\}$$



# Step 2 : Separate $\mathtt{CGC}(S^1, C) \setminus \mathtt{CGC}(C)$

$$V := \mathrm{Ext}\left(\mathtt{CGC}(S^1, C)\right) \setminus \mathbb{Z}^n$$

$$\langle a, v \rangle > \lfloor \sigma_C(a) \rfloor$$



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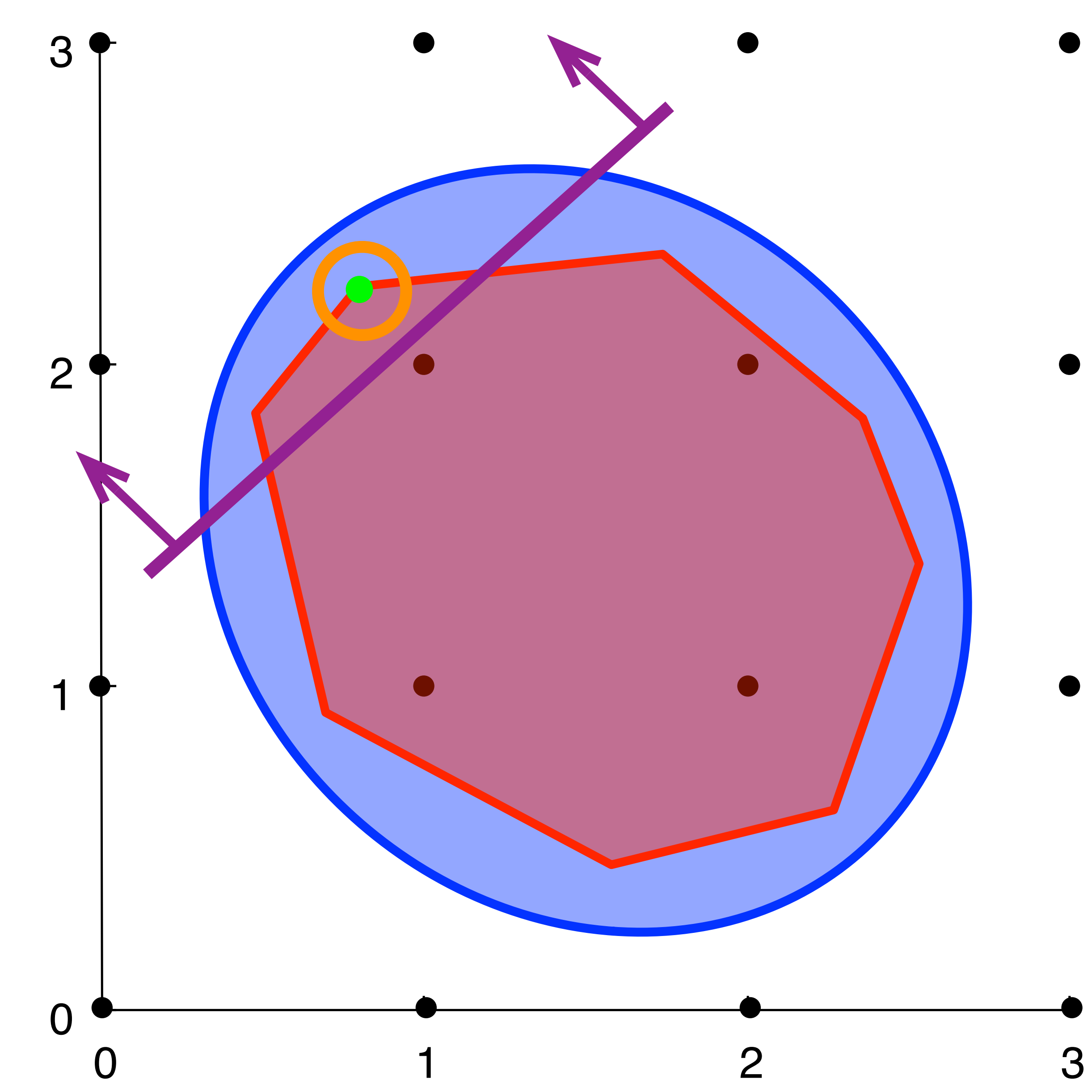
$$\exists \varepsilon > 0 \quad \varepsilon B^n + v \subset C \quad \forall v \in V$$



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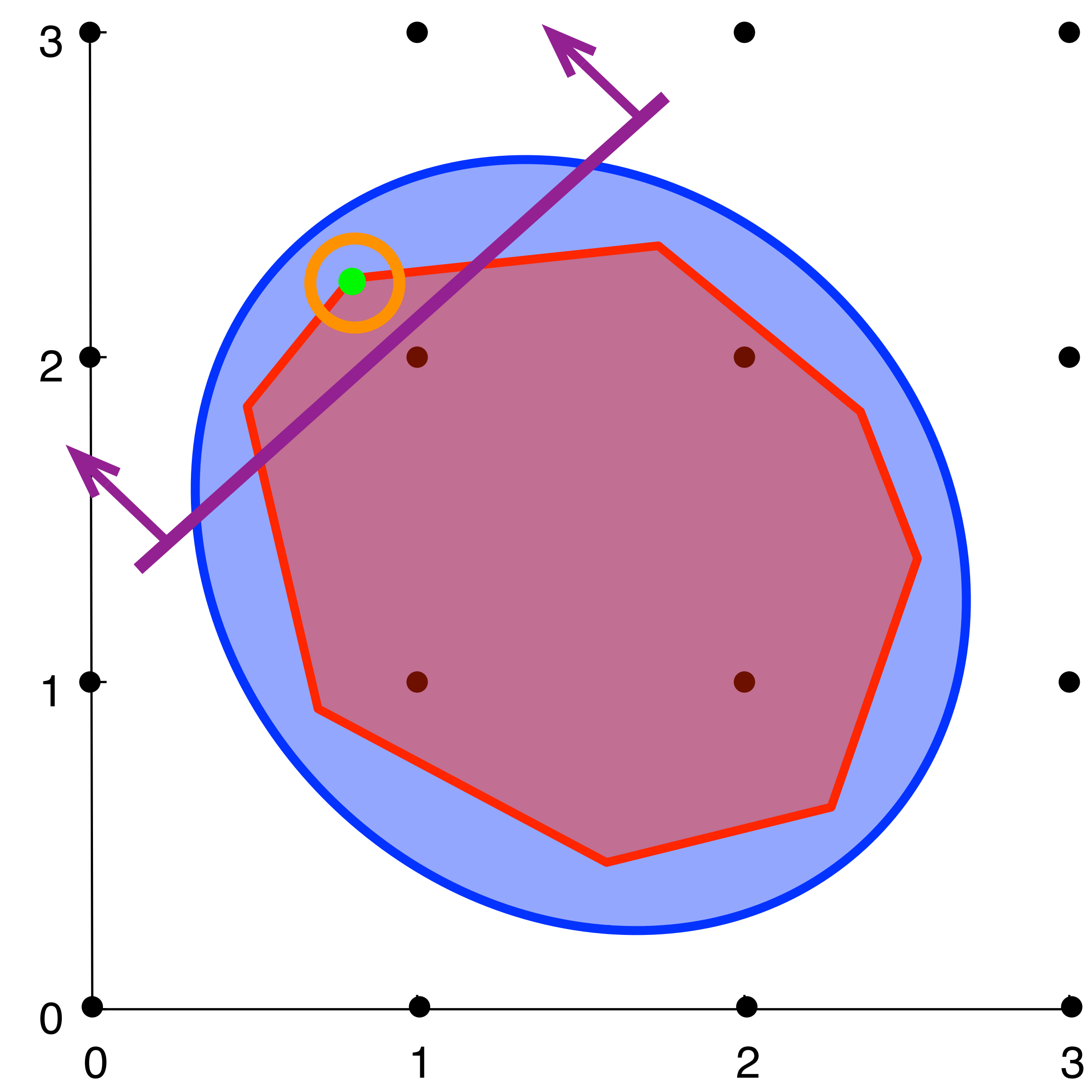
$$\|a\| \geq \frac{1}{\varepsilon} \Rightarrow$$

$$\begin{aligned} \lfloor \sigma_C(a) \rfloor &\geq \sigma_C(a) - 1 \\ &\geq \sigma_{v+\varepsilon B^n}(a) - 1 \\ &= \langle v, a \rangle + \varepsilon \|a\| - 1 \\ &\geq \langle v, a \rangle \end{aligned}$$



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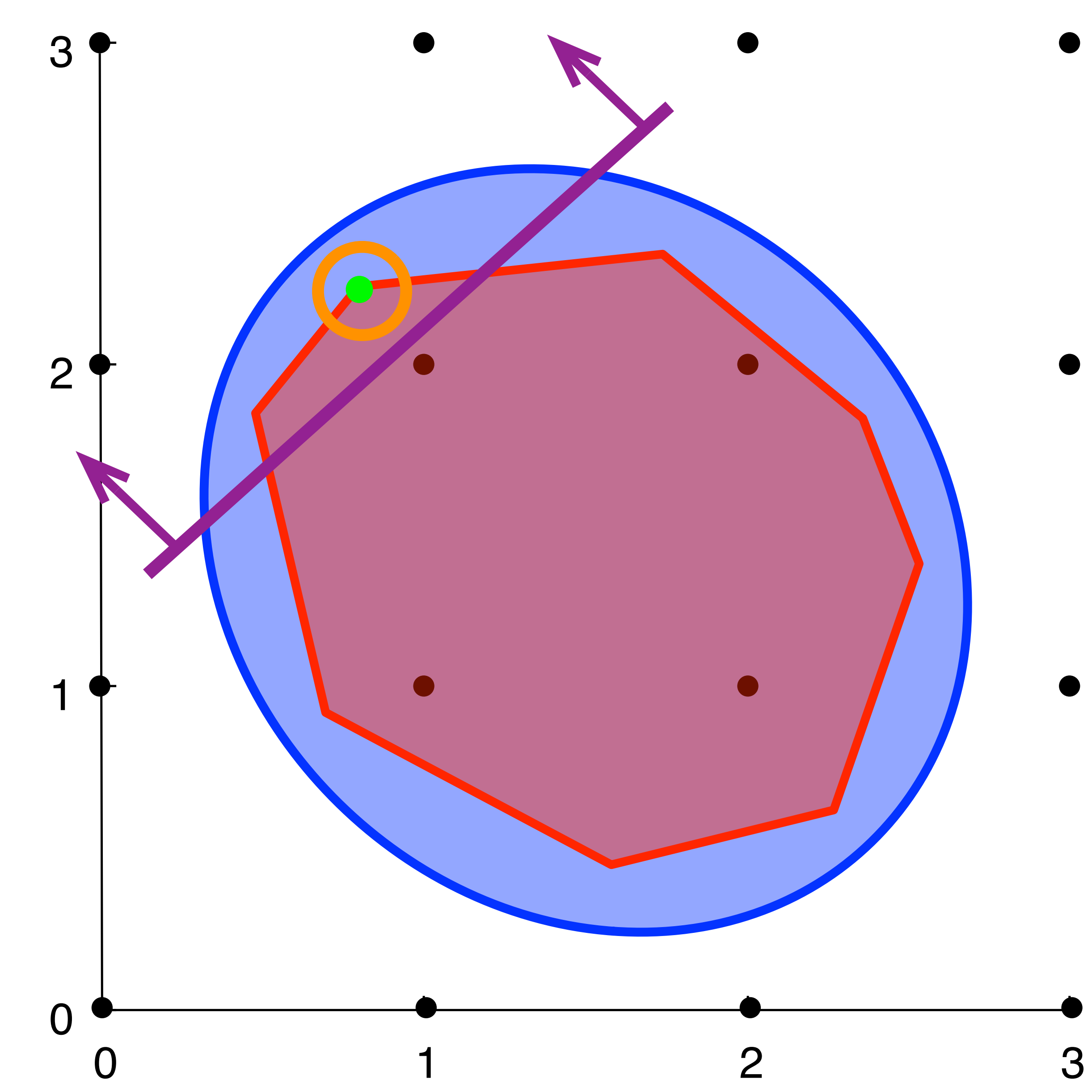
$$S^2 = (1/\varepsilon) B \cap \mathbb{Z}^n$$



# Conclusions and Future Work

- Non-Constructive because of compactness argument in step 1.
- General compact convex sets including non-rational polytopes done:
  - Dadush, Dey and V. MB39: Monday 11:00, C - Room 9B, Level 3
- Open Problems:
  - Simpler Proof (Circle in $\mathbb{R}^2$ ?).
  - Constructive/Algorithmic proof.