Modeling and Solving Discrete Optimization Problems in Practice

Juan Pablo Vielma

Massachusetts Institute of Technology

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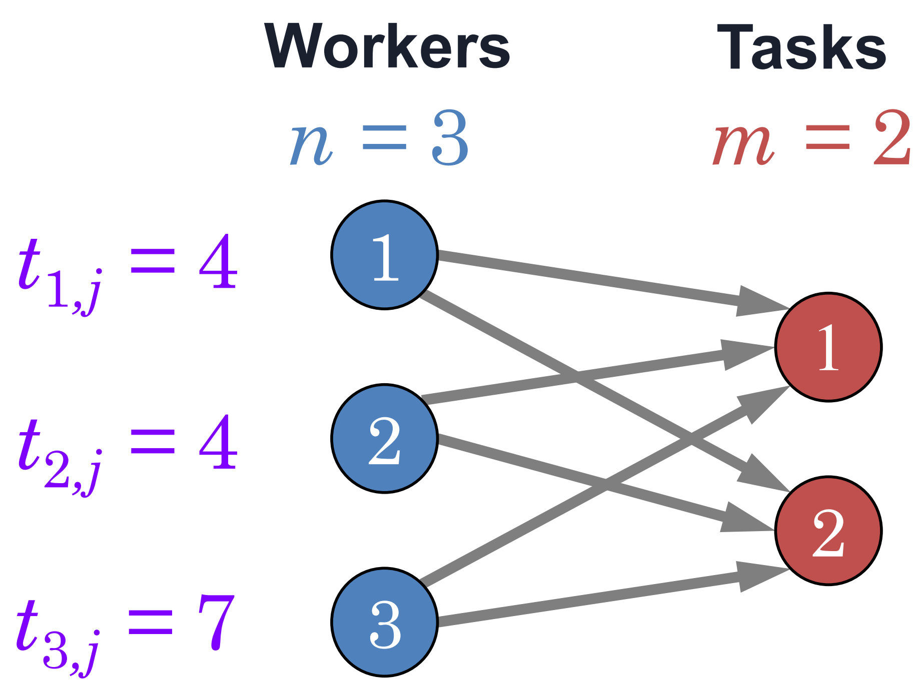
Combinatorial Example: Assignment Problem

- Assign n workers to m tasks to complete all tasks
- At most one task per worker
- Worker i takes $t_{i,j}$ hours to complete task j
- Minimize total time worked

- Graph:
  - Worker and task **nodes**
  - **Arcs** between worker and task **nodes**



Combinatorial Example: Assignment Problem

- Assign n workers to m tasks to complete all tasks
- At most one task per worker
- Worker i takes $t_{i,j}$ hours to complete task j
- Minimize total time worked
- Variables: $x_{i,j} = 1$ if worker i is assigned to task j and 0 o.w.

$$\min \quad \sum_{i=1}^{n} \sum_{j=1}^{m} t_{i,j} x_{i,j}$$

$$s.t.$$

$$\sum_{j=1}^{m} x_{i,j} \leq 1 \qquad \forall i \in \{1, \ldots, n\}$$

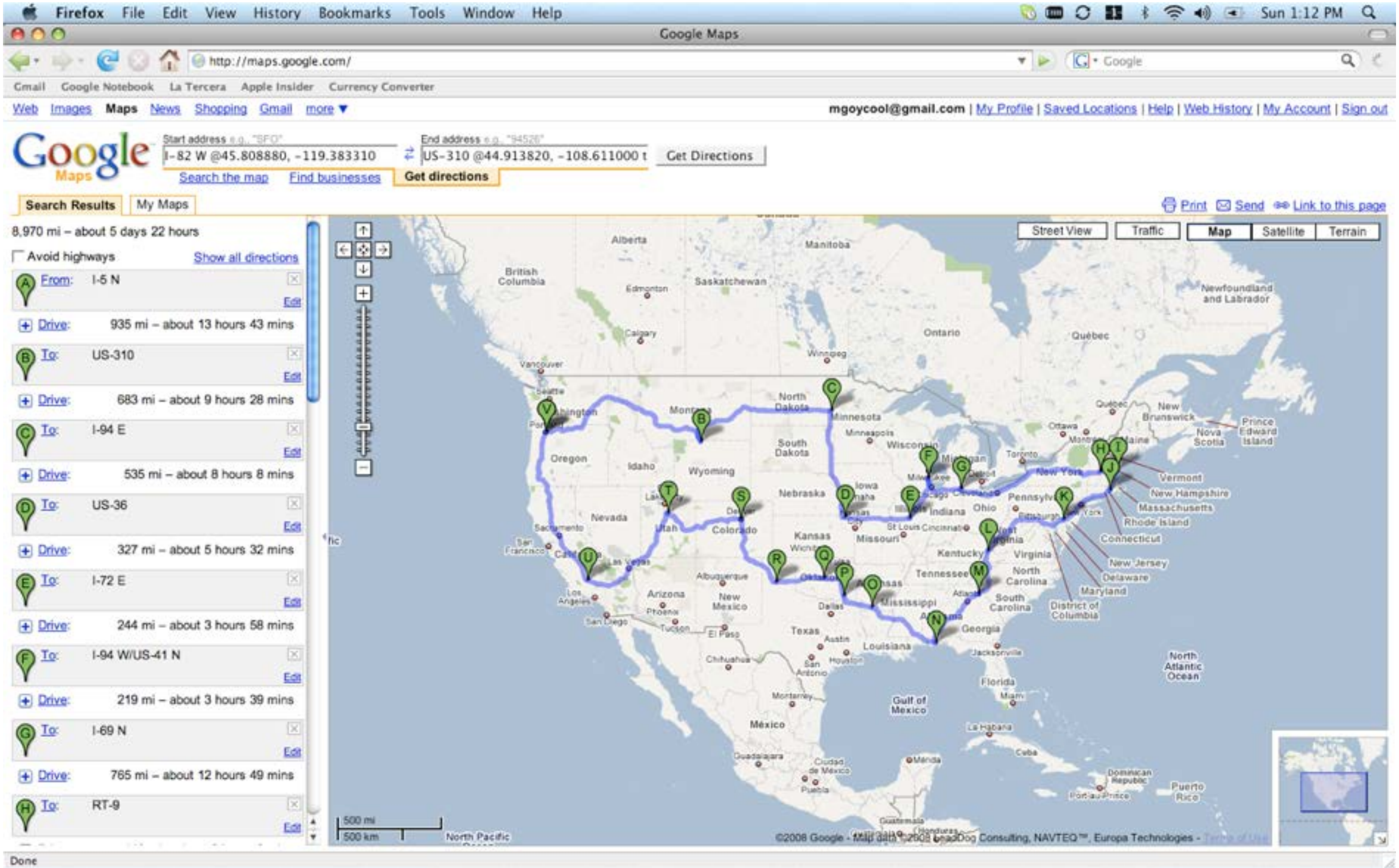
Worker constraints

$$\sum_{i=1}^{n} x_{i,j} \geq 1 \qquad \forall j \in \{1, \ldots, m\}$$

Task constraints

$$x_{i,j} \in \{0, 1\} \qquad \forall i \in \{1, \ldots, n\},\ j \in \{1, \ldots, m\}$$

Traveling Salesman Problem : Visit all Cities Once



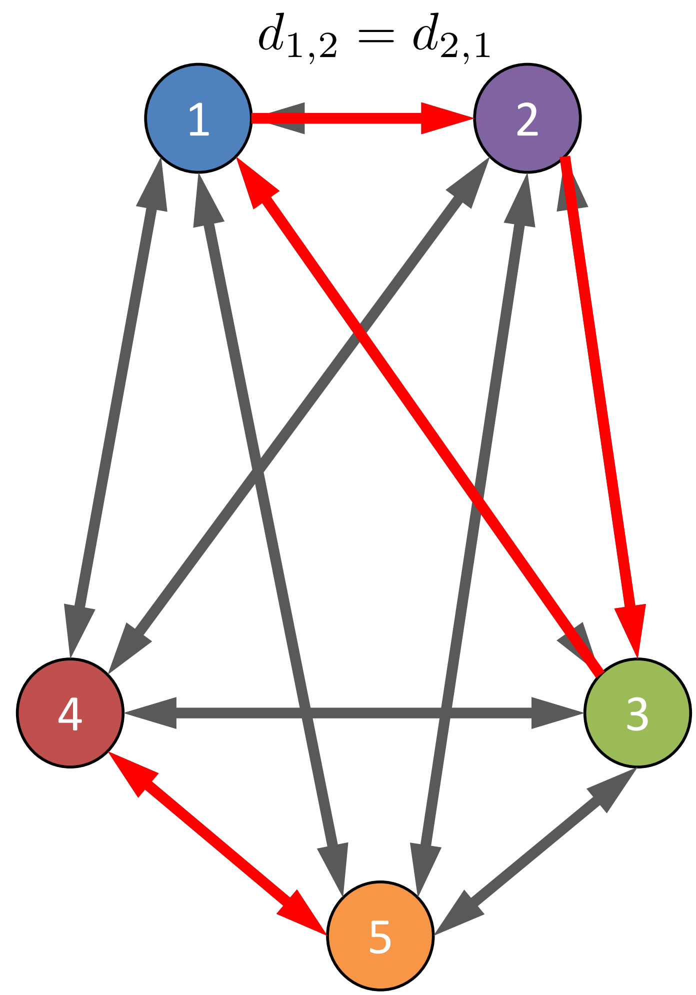
Formulation for Traveling Salesman Problem

$$[n] := \{1, \ldots, n\}$$

$$\min \sum_{i,j=1}^{n} d(i,j) x_{i,j}$$

$s.t.$

$$\begin{aligned} \sum_{j=1}^{n} x_{i,j} &= 1 & \forall i \in [n] \\ \sum_{i=1}^{n} x_{i,j} &= 1 & \forall j \in [n] \\ x_{i,i} &= 0 & \forall i \in [n] \\ x_{i,j} &\in \{0,1\} & \forall i,j \in [n] \end{aligned}$$

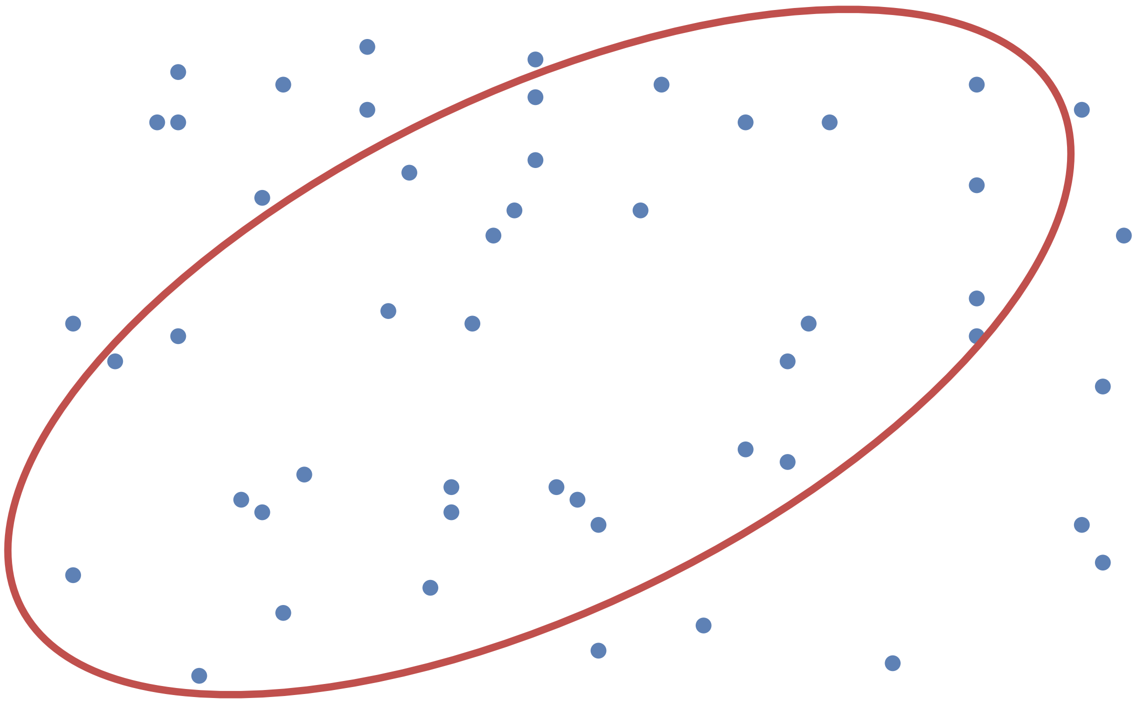


Homework Question 1: Add missing constraints

Hint: You will need around 2^n inequalities

Mixed Integer Programming (MIP)

- Discrete and continuous variables or combinatorial constraints on continuous variables.
- Example: Find minimum volume ellipsoid that contains 90% of data points



MIP & Daily Fantasy Sports

DRAFT KINGS

The Greater Boston FOOD BANK

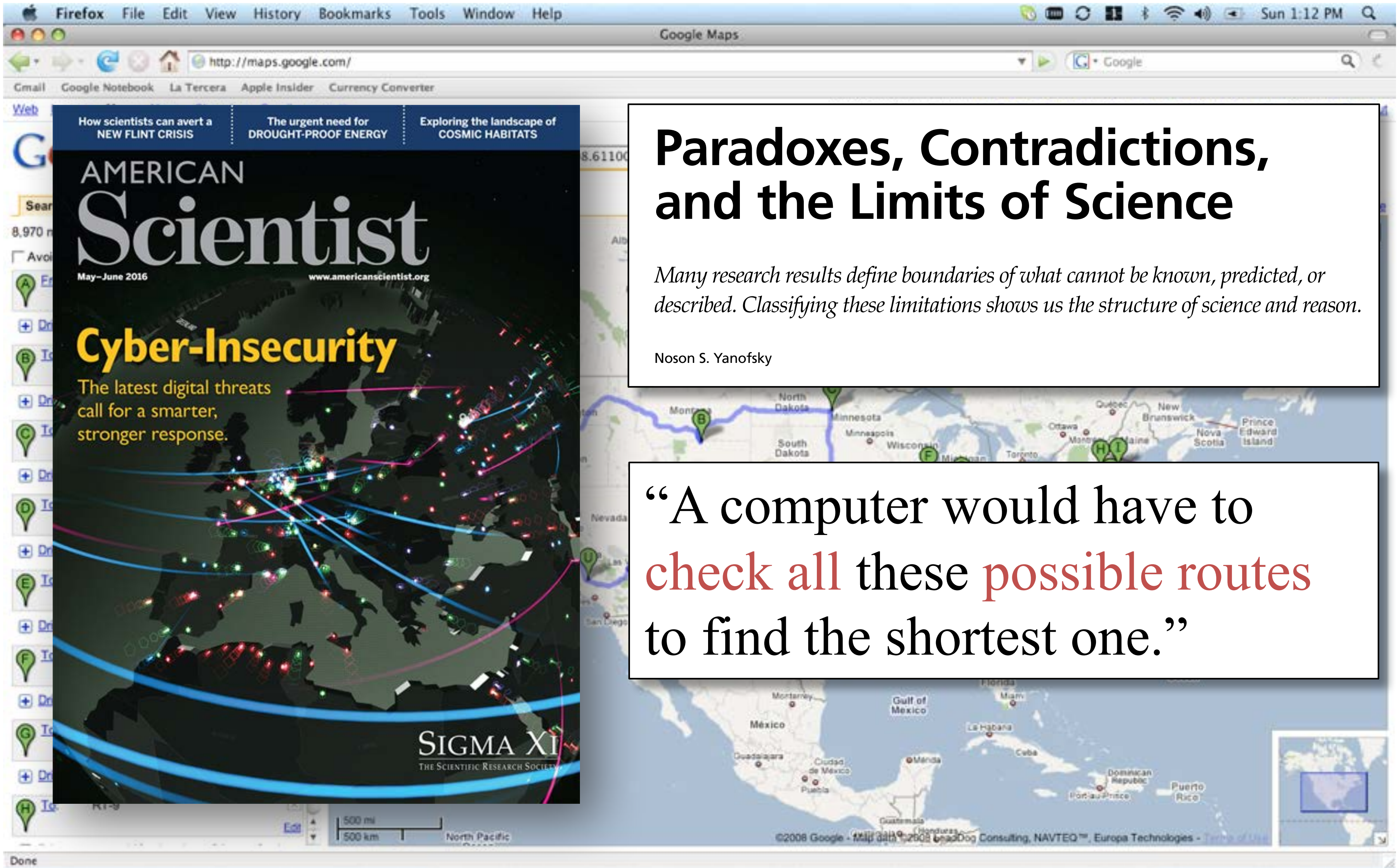
> $15K

Download Code from Github:

https://github.com/dscotthunter/Fantasy-Hockey-IP-Code

http://arxiv.org/pdf/1604.01455v1.pdf

How hard is MIP: Traveling Salesman Problem ?



Paradoxes, Contradictions, and the Limits of Science

Many research results define boundaries of what cannot be known, predicted, or described. Classifying these limitations shows us the structure of science and reason.

Noson S. Yanofsky

“A computer would have to check all these possible routes to find the shortest one.”

MIP = Avoid (Complete) Enumeration

- Number of tours for 49 cities $= 48!/2 \approx 10^{60}$
- Fastest supercomputer $\approx 10^{17}$flops
- Assuming one floating point operation per tour:
  $> 10^{35}$ years $\approx 10^{25}$ times the age of the universe!
- How long does it take on an iphone?
  - < 1 sec ! Dantzig, Fulkerson and Johnson ✍️ in 54’
  - Even <u>theoretically hard</u> MIPs “can” be solved:
    - Open-source solvers: GLPK, CBC, etc.
    - Commercial: Gurobi, CPLEX, etc.
    - Modeling Language: JuMP julia

Easy MIP through julia & JuMP

- julia : general purpose programming language
  - download https://julialang.org/downloads/ then click or run from command line
- JuMP : modeling language for optimization
- GNU GLPK : Open-source MIP solver
  - ```
    julia> Pkg.add("JuMP"); Pkg.add("GLPKMathProgInterface")
    ```
- Can also try JuliaBox on web
  - https://www.juliabox.com/
    ```

Easy MIP through julia & JuMP

- Assignment problem:

$$\min \quad \sum_{i=1}^{n} \sum_{j=1}^{m} t_{i,j} x_{i,j}$$

$$s.t.$$

$$\begin{aligned}
\sum_{j=1}^{m} x_{i,j} &\leq 1 && \forall i \in \{1,\ldots,n\} \\
\sum_{i=1}^{n} x_{i,j} &\geq 1 && \forall j \in \{1,\ldots,m\} \\
x_{i,j} &\in \{0,1\} && \forall i \in \{1,\ldots,n\},\ j \in \{1,\ldots,m\}
\end{aligned}$$

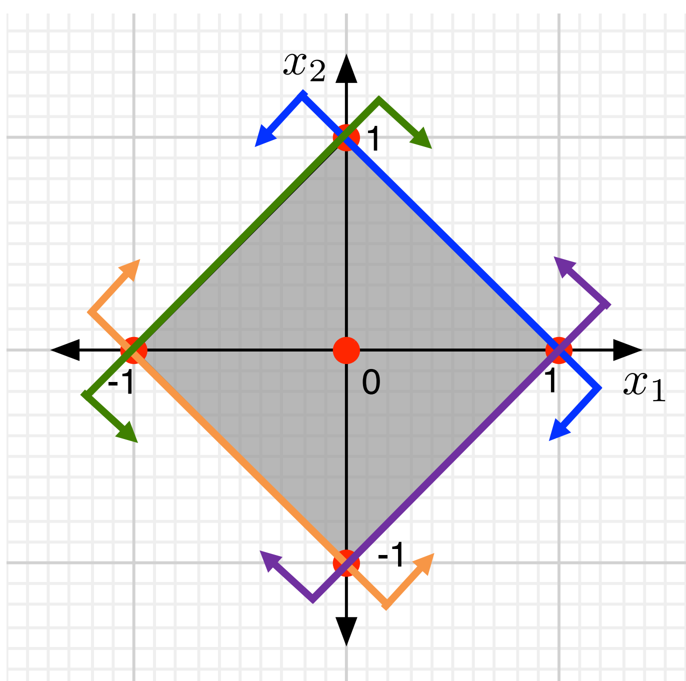
```
model = Model(solver=GLPKSolverMIP());
@variable(model, x[1:n,1:m], Bin);
@objective(model, Min, sum(t[i,j]*x[i,j] for i in 1:n, j in 1:m));
@constraint(model, [i=1:n], sum(x[i,j] for j in 1:m) <= 1);
@constraint(model, [j=1:m], sum(x[i,j] for i in 1:n) >= 1);
```

Homework Question 2: Solve problem with random cost
Complete file in website.

Solving MIPs: Step 1 = Linear Programming

$$
\begin{aligned}
\max \quad & x_2 \\
s.t. \quad & x_1 + x_2 \leq 1 \\
& -x_1 - x_2 \leq 1 \\
& +x_1 - x_2 \leq 1 \\
& -x_1 + x_2 \leq 1 \\
& x_1, x_2 \in \mathbb{Z}
\end{aligned}
$$

← Linear Programming (LP) Relaxation



- Solving LPs is easy in theory and practice.
- One reason = LP duality
  - Suppose I guess optimum $x_1 = 0$ and $x_2 = 1$.
  - How do I prove that for all solutions of LP $x_2 \leq 1$?

$$
\begin{array}{rl}
 & (1/2) \times (\;\; x_1 + x_2 \leq 1) \\
+ & (1/2) \times (-x_1 + x_2 \leq 1) \\
\hline
 & x_2 \leq 1
\end{array}
$$

Solving MIPs: Step 1 = Linear Programming

$$
\begin{aligned}
\max \quad & x_2 \\
s.t. \quad & x_1 + x_2 \leq 1 \\
& -x_1 - x_2 \leq 1 \\
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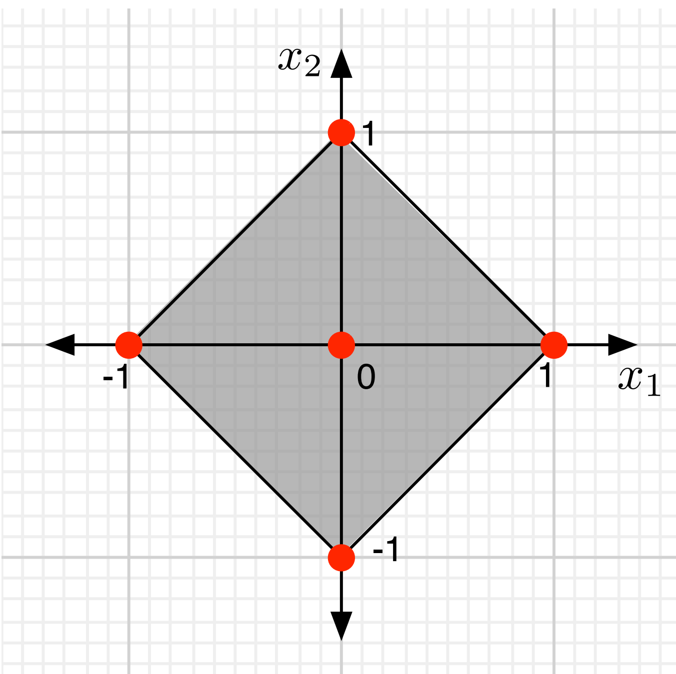
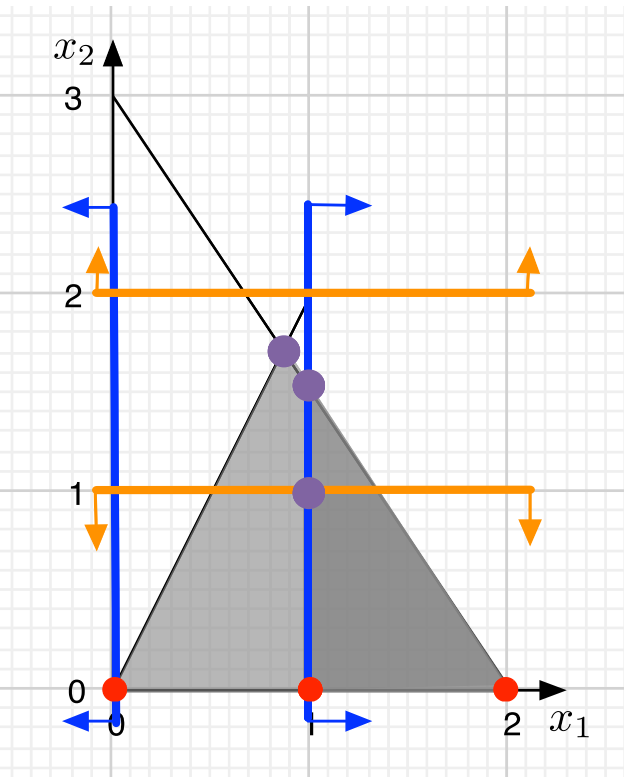
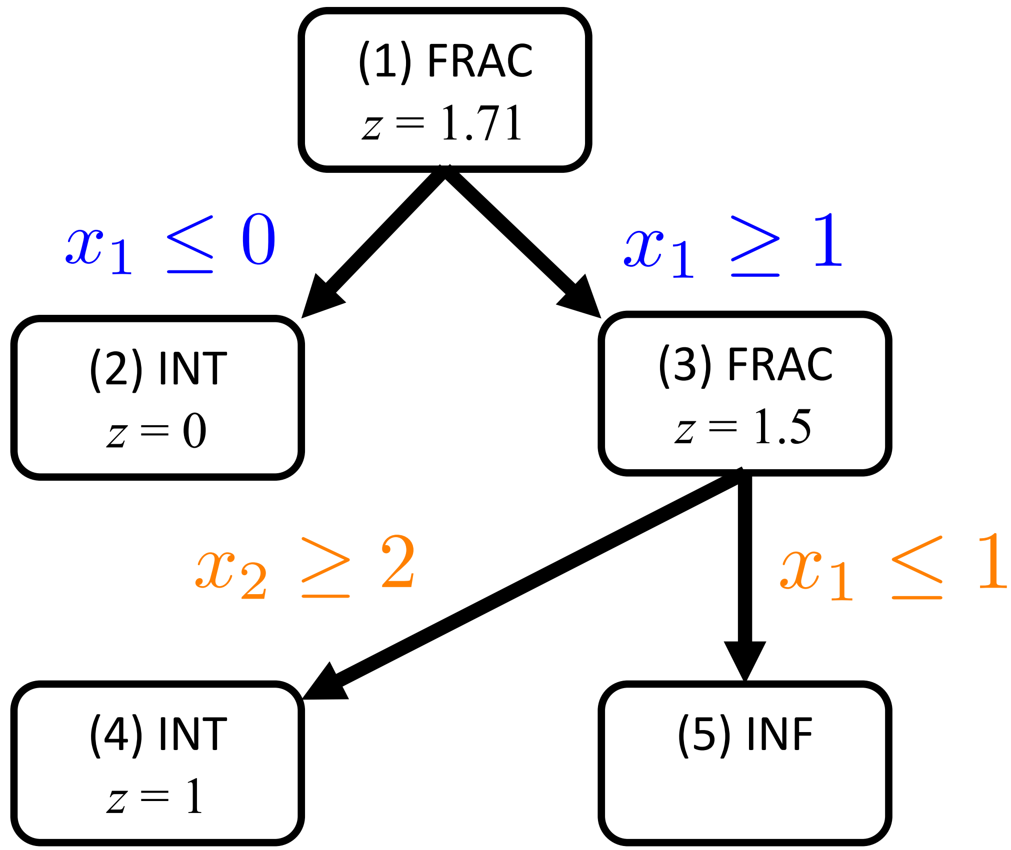
← Linear Programming (LP) Relaxation



- LP relaxation always gives a (upper) bound on the MIP:
  - If solution of LP is "integer" then you solved the MIP
  - LP solvers return "corner" solution, which fixes "multiple optima" (e.g. $\max x_1 + x_2$)
  - Homework Question 3: Solve LP relaxation of assignment problem with JuMP. Is solution integer?

Solving MIPs: Step 2 = Branch-and-Bound





$$
\begin{aligned}
\max z := x_2 & \\
3x_1 + 2x_2 &\leq 6 \\
-2x_1 + x_2 &\leq 0 \\
x_1,\, x_2 &\geq 0
\end{aligned}
$$

$$x_1,\, x_2 \in \mathbb{Z}$$

← Linear Programming (LP) Relaxation

Homework Question 4:
Prove $x_2 \leq 12 / 7$ for LP Relaxation.

Modern MIP Solvers = B&B++

- Really branch-and-cut:
  - Use cuts to improve LP relaxation.
- Elaborate heuristics: Rounding +++
- Preprocessing: fixing variables by logical implications.
- Advanced management of B&B tree.
- Extensive tuning of parameters and techniques.

Cutting Plane Example: Chátal-Gomory Cuts

$$P := \left\{ x \in \mathbb{R}^2 \; : \; \begin{array}{r} x_1 + x_2 \le 3, \\ 5x_1 - 3x_2 \le 3 \end{array} \right\}$$

$$\subseteq$$

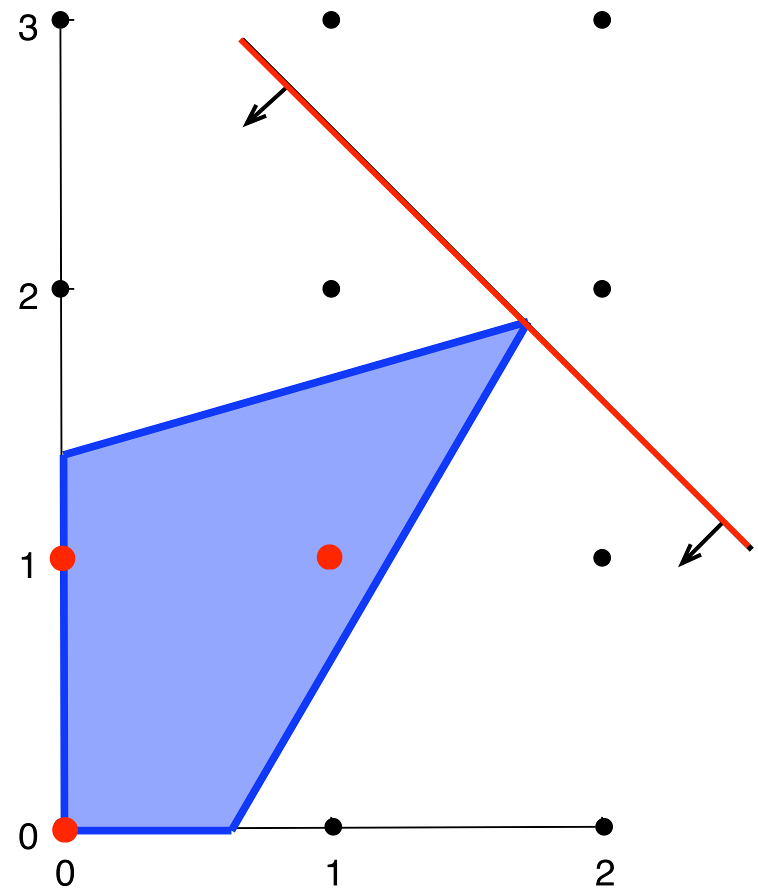
$$H := \{ x \in \mathbb{R}^2 \; : \; \underbrace{4x_1 + 3x_2}_{\in \mathbb{Z}} \le 10.5 \}$$

$$\text{if } x \in \mathbb{Z}^2 \; \Big\downarrow$$

$$4x_1 + 3x_2 \le \lfloor 10.5 \rfloor$$

Valid for $H \cap \mathbb{Z}^2$

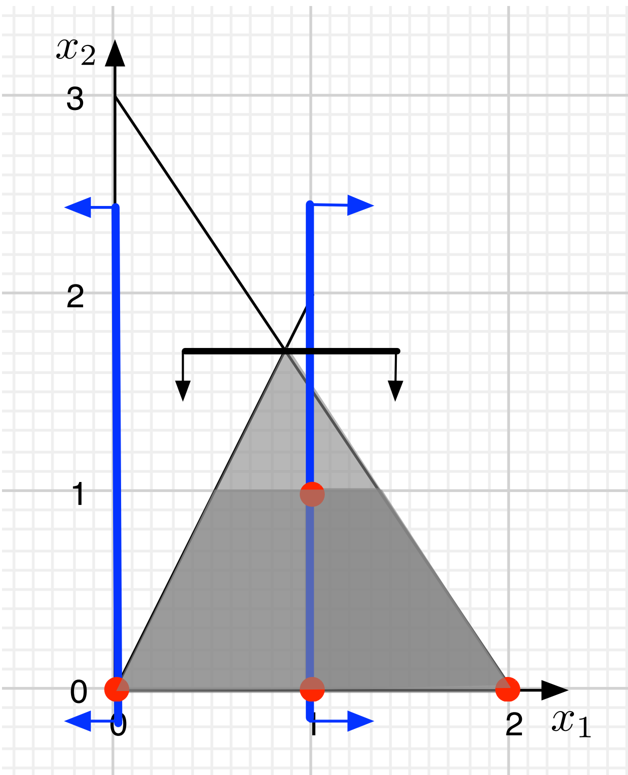
Valid for $P \cap \mathbb{Z}^2$



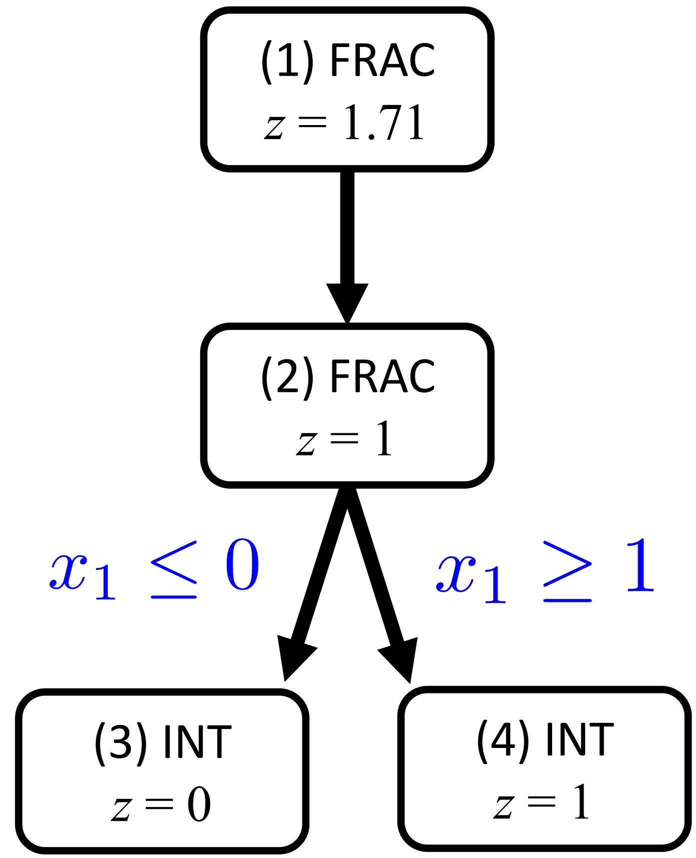
$$\begin{array}{rl} & (27/8)(\;\; x_1 + \;\; x_2 \le 3) \\ + & (\;\, 1/8)(5x_1 - 3x_2 \le 3) \\ \hline \end{array}$$

$$\Rightarrow 4x_1 + 3x_2 \le 10.5$$

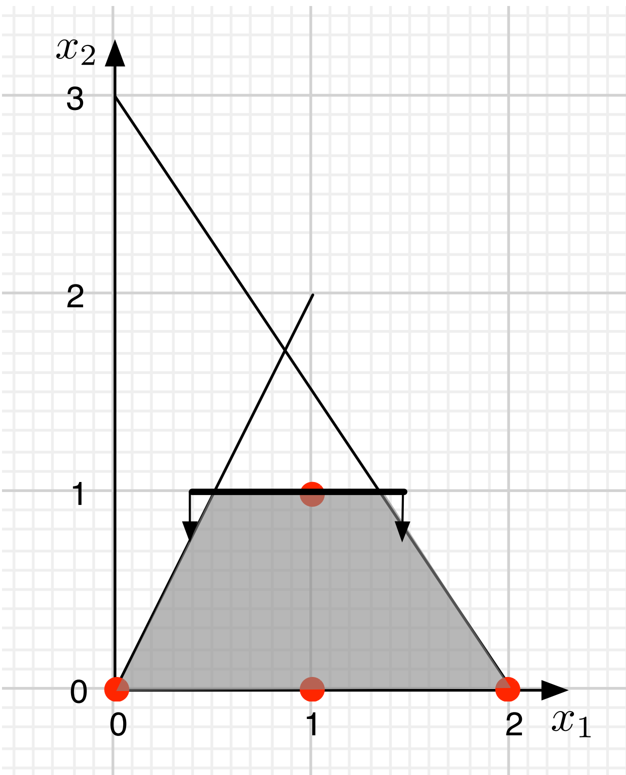
Branch-and-Bound and Cuts (Branch-and-Cut)



$$
\begin{array}{rl}
\max z := x_2 & \\
3x_1 + 2x_2 \le 6 & x_2 \le \lfloor 1.71 \rfloor = 1 \\
-2x_1 + x_2 \le 0 & \\
x_1,\ x_2 \ge 0 & \\
x_1,\ x_2 \in \mathbb{Z} &
\end{array}
$$



Branch-and-Bound and Cuts (Branch-and-Cut)



Homework Question 5:
Add two more Chátal-Gomory cuts so the LP relaxation with all cuts solves the MIP.

Hint:

$$
\begin{array}{rl}
 & (1/3)(3x_1 + 2x_2 \le 6) \\
+ & (1/3)(\qquad\quad x_2 \le 1) \\
\hline
\end{array}
$$

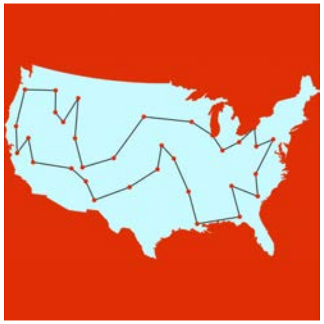
and

$$
\begin{array}{rl}
 & (1/2)(-2x_1 + x_2 \le 0) \\
+ & (1/2)(\qquad\quad x_2 \le 1) \\
\hline
\end{array}
$$

$$
\begin{aligned}
\max z := x_2 & \\
3x_1 + 2x_2 &\le 6 \quad x_2 \le \lfloor 1.71 \rfloor = 1 \\
-2x_1 + x_2 &\le 0 \\
x_1,\, x_2 &\ge 0 \\
x_1,\, x_2 &\in \mathbb{Z}
\end{aligned}
$$

No Enumeration = Keep Adding Cuts

- Number of tours for 49 cities $= 48!/2 \approx 10^{60}$
- Fastest supercomputer $\approx 10^{17}$flops
- Assuming one floating point operation per tour:
  $> 10^{35}$ years $\approx 10^{25}$ times the age of the universe!
- How long does it take on an iphone?
  - < 1 sec ! Dantzig, Fulkerson and Johnson ✍️ in 54'
  - This is how DFJ solved the problem by hand in 54'
  - In practice Branch-and-Cut is better.
  - More details in Concord TSP App
    - Cutting plane tutorial for TSP
    - http://www.math.uwaterloo.ca/tsp/iphone/



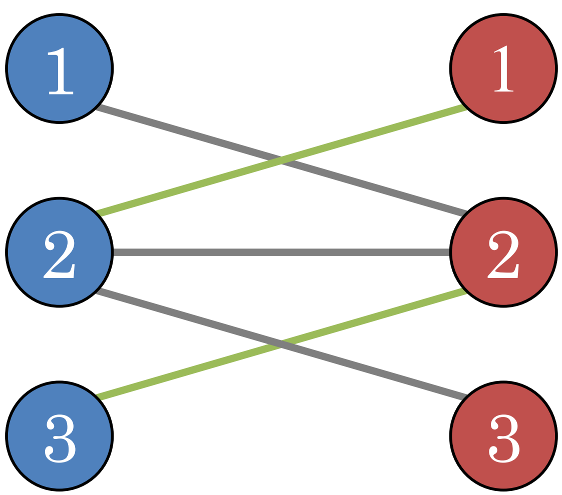
Easy Problems : LP Relaxation Always Integral

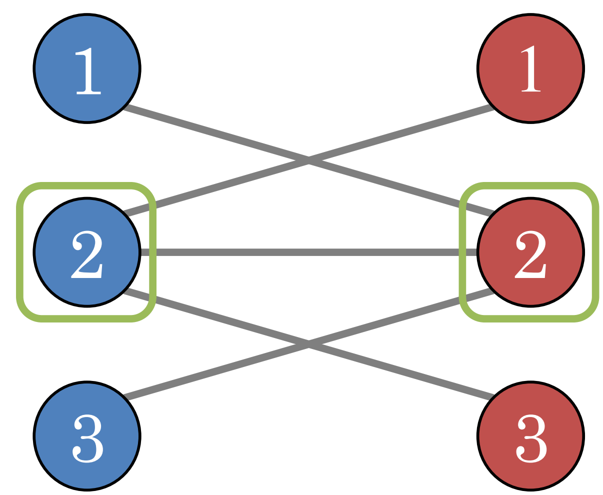
Consequence of LP duality: Kőnig's theorem

- Largest Matching
  – Pick edges, at most one edge per node

=

- Smallest Node Cover
  – Pick nodes that touch all edges



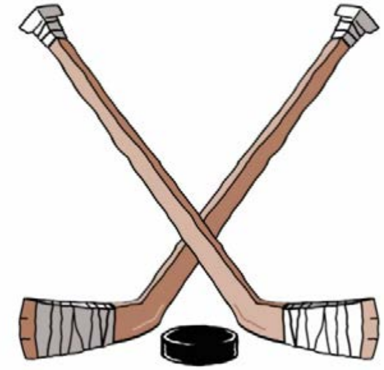


Classes and Links

- julia , JuMP and Optimization
  - https://github.com/JuliaOpt/JuMP.jl
  - http://www.juliaopt.org
- 15.053 Optimization Methods in Business Analytics
  - Modeling and computation
  - Instructor: James B. Orlin
  - Spring 2018: http://mit.edu/15.053/www/
- 18.453 Combinatorial Optimization
  - Theory and algorithms
  - Instructor: Michel Goemans
  - Spring **2017** : http://www-math.mit.edu/~goemans/18453S17/18453.html

MIP & Daily Fantasy Sports

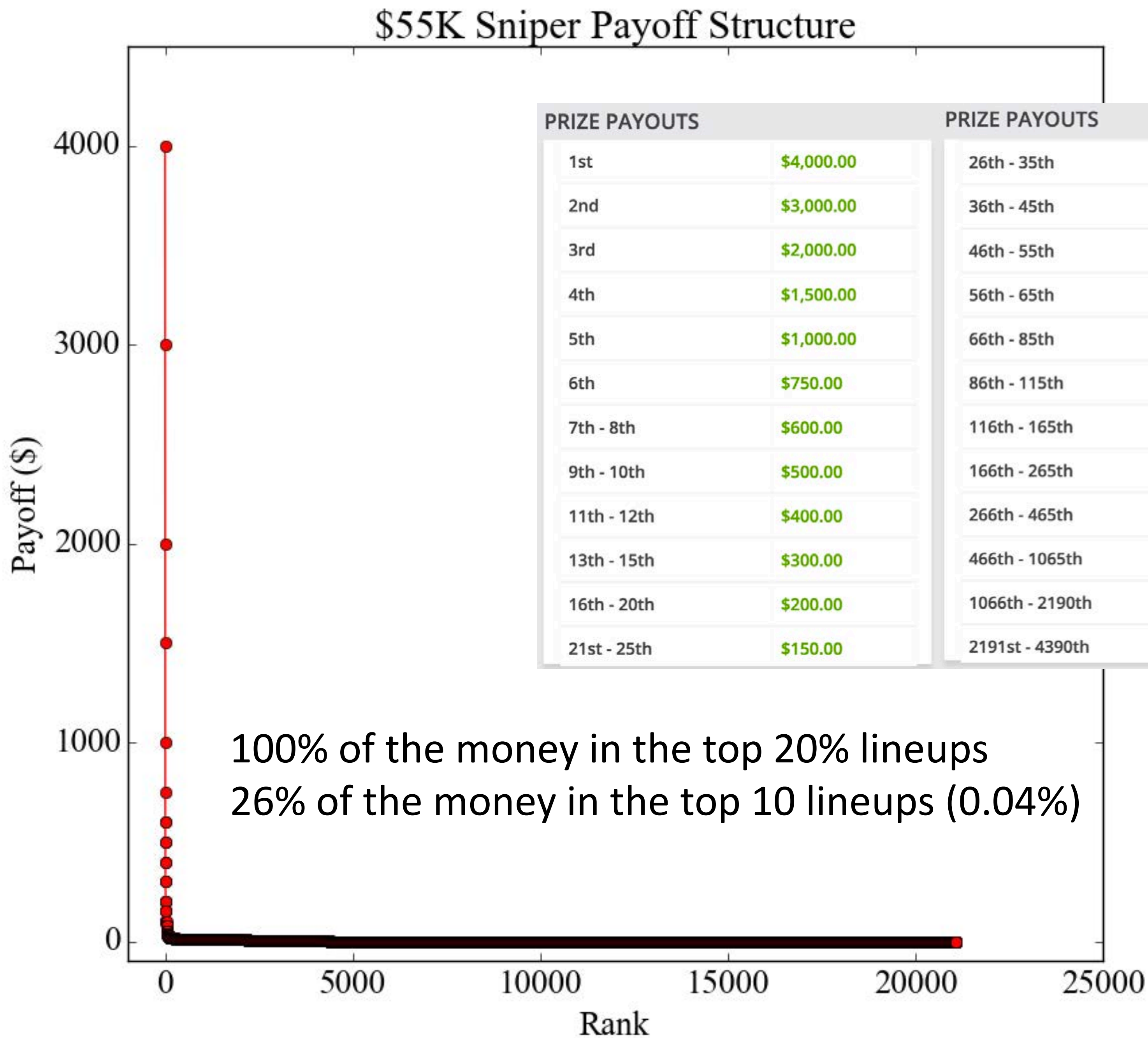
Example Entry



LINEUP

Avg. Rem. / Player: $0
Rem. Salary: $0

POS	PLAYER	OPP	FPPG	SALARY
C	Jussi Jokinen	Fla@Anh	3.1	$5,300
C	Brandon Sutter	Pit@Van	3.0	$4,400
W	Nikolaj Ehlers	Wpg@Tor	3.9	$4,800
W	Daniel Sedin	Pit@Van	3.8	$6,400
W	Radim Vrbata	Pit@Van	3.4	$5,800
D	Brian Campbell	Fla@Anh	2.6	$4,100
D	Morgan Rielly	Wpg@Tor	3.5	$4,200
G	Corey Crawford P	StL@Chi	6.3	$7,800
UTIL	Blake Wheeler	Wpg@Tor	4.8	$7,200



PRIZE PAYOUTS

1st	$4,000.00
2nd	$3,000.00
3rd	$2,000.00
4th	$1,500.00
5th	$1,000.00
6th	$750.00
7th - 8th	$600.00
9th - 10th	$500.00
11th - 12th	$400.00
13th - 15th	$300.00
16th - 20th	$200.00
21st - 25th	$150.00

PRIZE PAYOUTS

26th - 35th	$100.00
36th - 45th	$75.00
46th - 55th	$50.00
56th - 65th	$40.00
66th - 85th	$30.00
86th - 115th	$25.00
116th - 165th	$20.00
166th - 265th	$15.00
266th - 465th	$12.00
466th - 1065th	$10.00
1066th - 2190th	$8.00
2191st - 4390th	$6.00

100% of the money in the top 20% lineups
26% of the money in the top 10 lineups (0.04%)

Building a Lineup

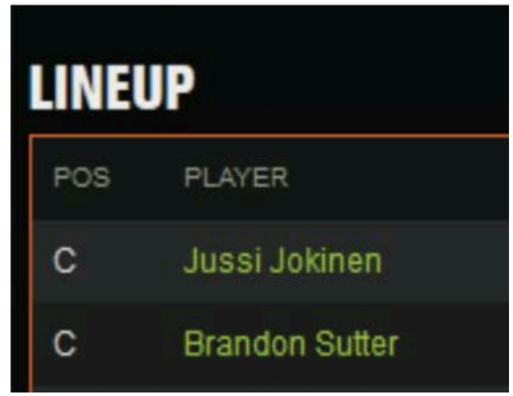
MIP Formulation

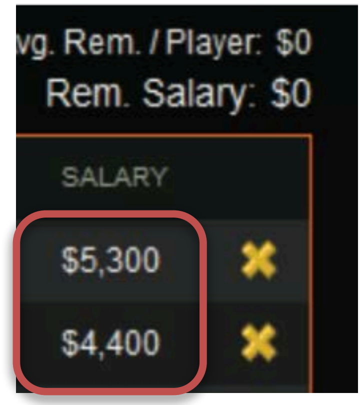
- L lineups : indexed by l
- 9 players per lineup: indexed by p
- Decision variables

$$x_{pl} = \begin{cases} 1, & \text{if player } p \text{ in lineup } l \\ 0, & \text{otherwise} \end{cases}$$

Basic Feasibility

- Basic constraints:
  - 9 different players
  - Salary less than $50,000





c_p

$$\sum_{p=1}^{N} c_p x_{pl} \leq \$50,000, \quad \text{(budget constraint)}$$

$$\sum_{p=1}^{N} x_{pl} = 9, \quad \text{(lineup size constraint)}$$

$$x_{pl} \in \{0,1\}, \quad 1 \leq p \leq N.$$

Position Feasibility

- Between 2 and 3 centers
- Between 3 and 4 wingers
- Between 2 and 3 defensemen
- 1 goalie

Position constraints

$$2 \leq \sum_{p \in C} x_{pl} \leq 3, , \quad \text{(center constraint)}$$

$$3 \leq \sum_{u \in W} x_{pl} \leq 4, \quad \text{(winger constraint)}$$

$$2 \leq \sum_{u \in D} x_{pl} \leq 3, \quad \text{(defensemen constraint)}$$

$$\sum_{u \in G} x_{pl} = 1 \quad \text{(goalie constraint)}$$

Team Feasibility

- At least 3 different NHL teams

Team constraints

$$t_i \leq \sum_{p \in T_i} x_{pl}, \quad \forall\, i \in \{1, \ldots, N_T\}$$

$$\sum_{i=1}^{N_T} t_i \geq 3,$$

$$t_i \in \{0,\ 1\}, \quad \forall\, i \in \{1, \ldots, N_T\}.$$

Maximize Points

- Forecasted points for player p: f_p

Score type	Points
Goal	3
Assist	2
Shot on Goal	0.5
Blocked Shot	0.5
Short Handed Point Bonus (Goal/Assist)	1
Shootout Goal	0.2
Hat Trick Bonus	1.5
Win (goalie only)	3
Save (goalie only)	0.2
Goal allowed (goalie only)	-1
Shutout Bonus (goalie only)	2

Table 1 **Points system for NHL contests in DraftKings.**

Points Objective Function

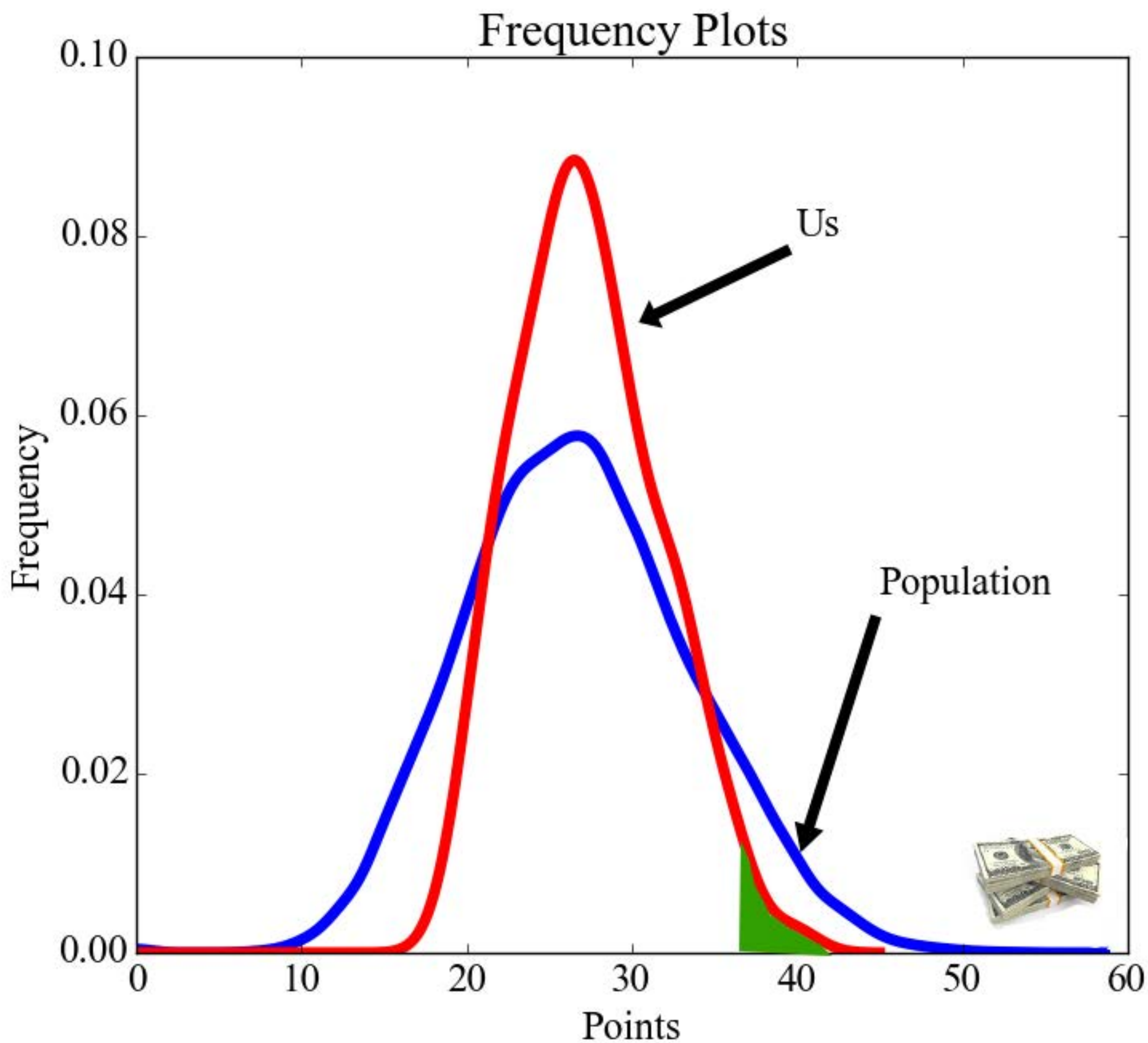
$$\sum_{p=1}^{N} f_p x_{pl}$$

Lineup

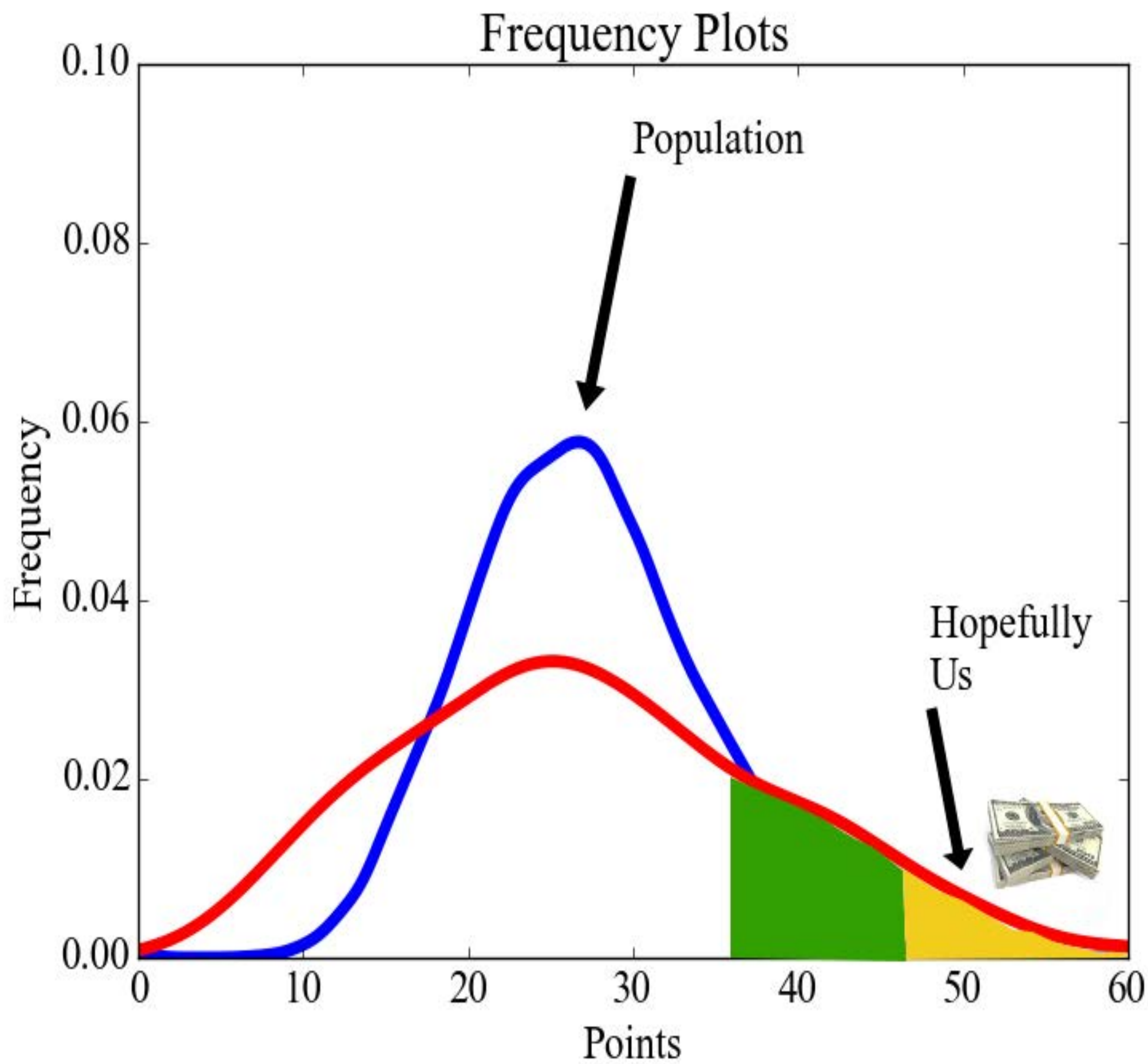
Projections:	5.4	2.5	3.4	3.0	3.2	4.2	3.5	3.4	5.7
	$9500	$2700	$4600	$3800	$4600	$6400	$5200	$5100	$8000
	W	UTIL	D	D	C	C	W	W	G

23 points on average

Need > 38 points for a chance to win



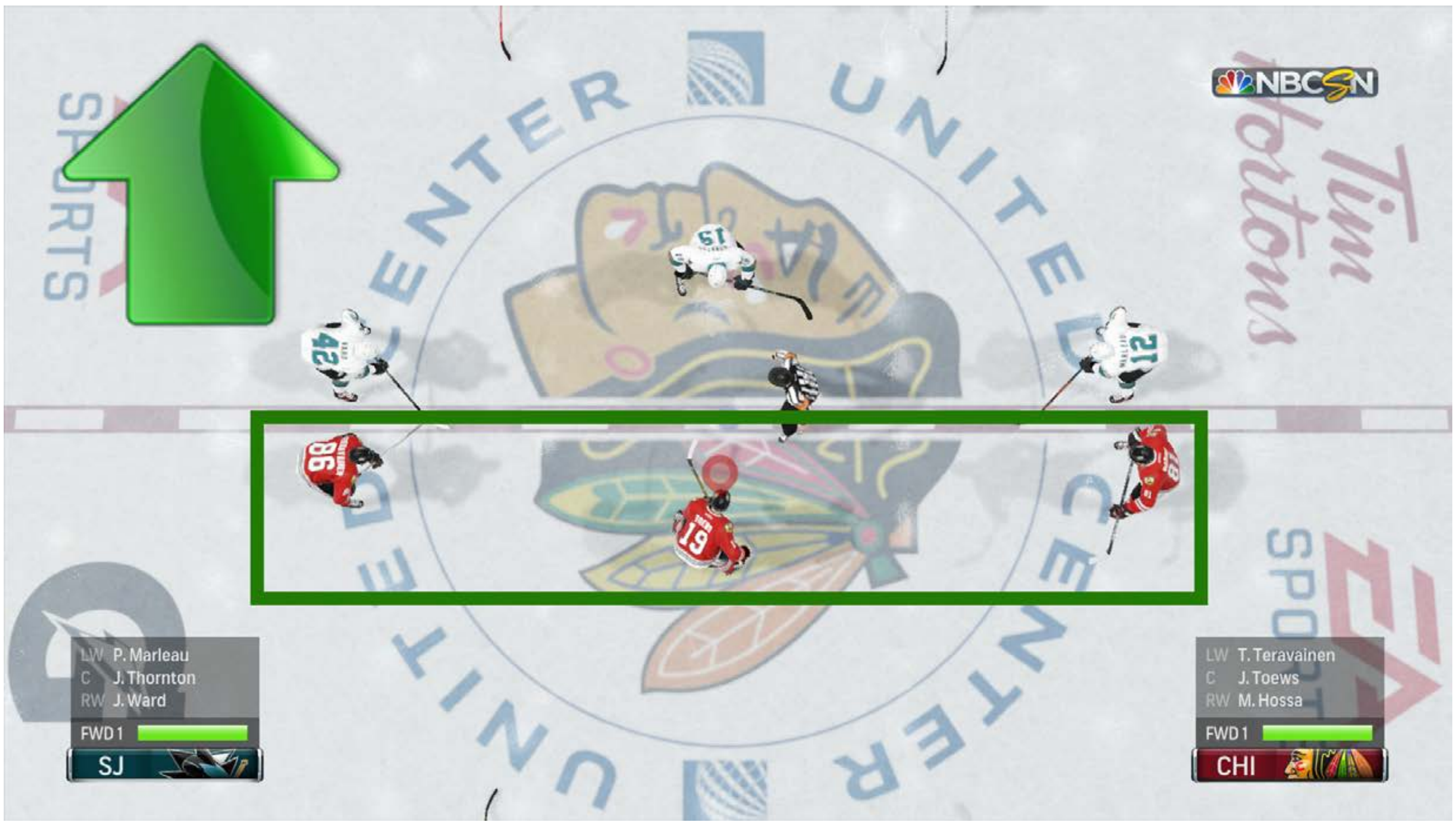
Increase variance to have a chance



Structural Correlations : Teams

Structural Correlations : Lines

- Goal = 3 pt, assist = 2 pt



Structural Correlations : Lines = Stacking

- At least 1 complete line (3 players per line)
- At least 2 partial lines (at least 2 players per line)

1 complete line constraint

$$3v_i \leq \sum_{p \in L_i} x_{pl}, \qquad \forall i \in \{1, \ldots, N_L\}$$

$$\sum_{i=1}^{N_L} v_i \geq 1$$

$$v_i \in \{0,1\}, \qquad \forall i \in \{1, \ldots, N_L\}.$$

2 partial lines constraint

$$2w_i \leq \sum_{p \in L_i} x_{pl}, \qquad \forall i \in \{1, \ldots, N_L\}$$

$$\sum_{i=1}^{N_L} w_i \geq 2$$

$$w_i \in \{0,1\}, \qquad \forall i \in \{1, \ldots, N_L\}.$$

Structural Correlations : Goalie Against Opposing Players

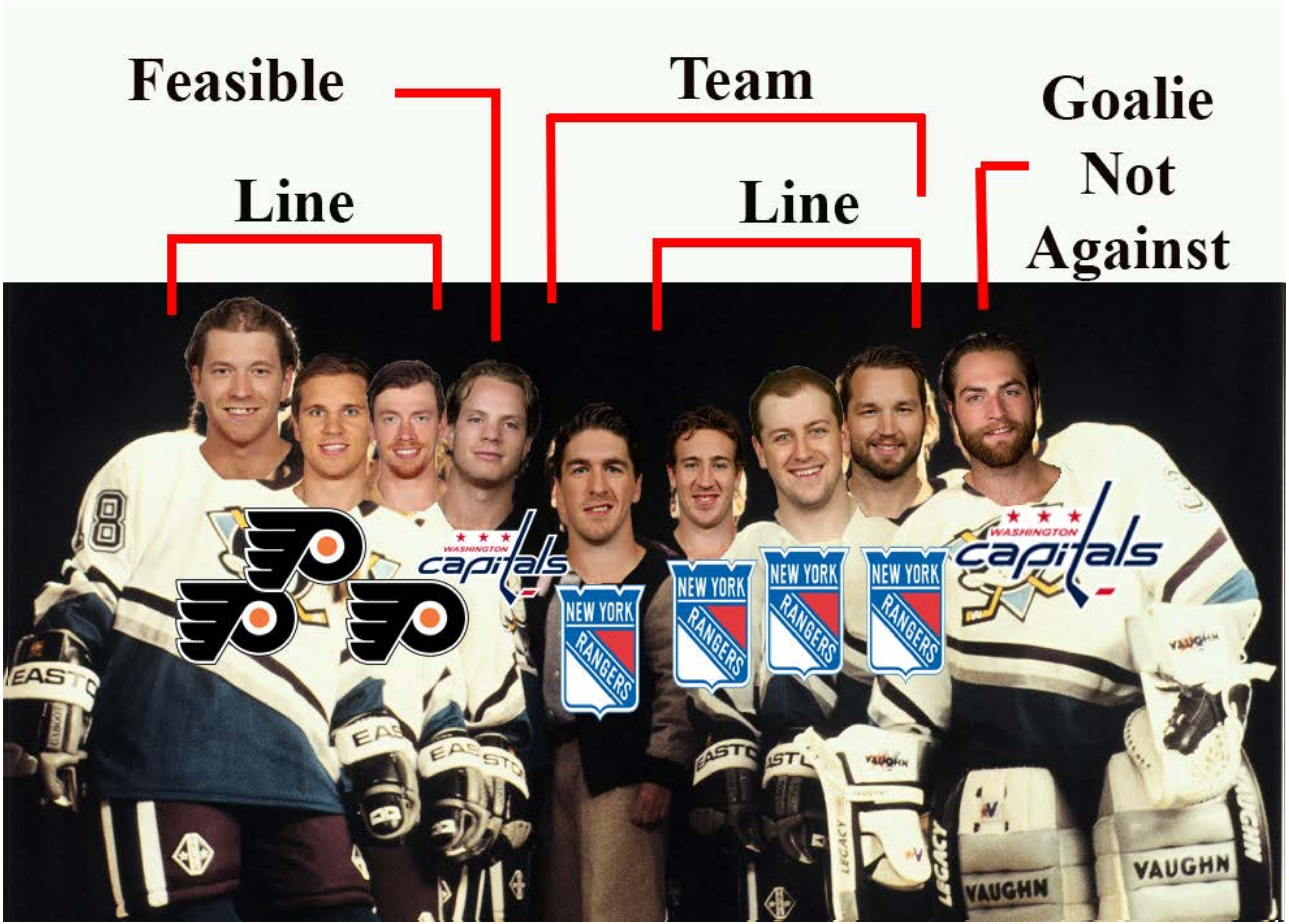
Structural Correlations : Goalie Against Opposing Players

- No skater against goalie

No skater against goalie constraint

$$6x_{pl} + \sum_{q \in Opponents_p} x_{ql} \leq 6, \quad \forall p \in G$$

Good, but not great chance



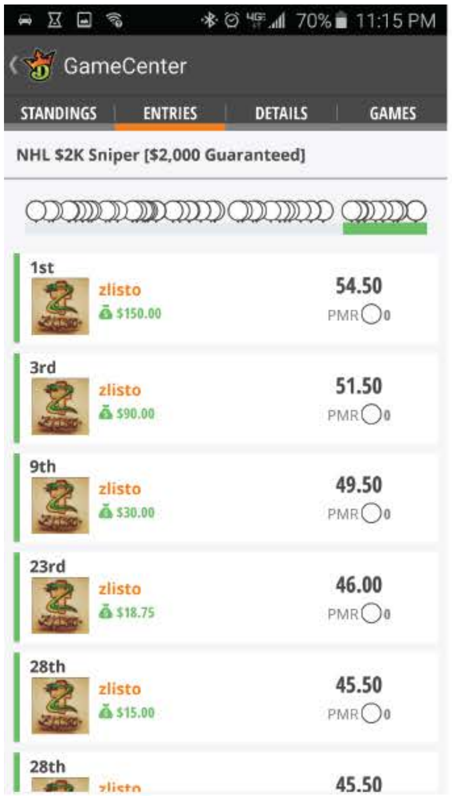
Play many diverse Lineups

- Make sure lineup l has no more than γ players in common with lineups 1 to l-1

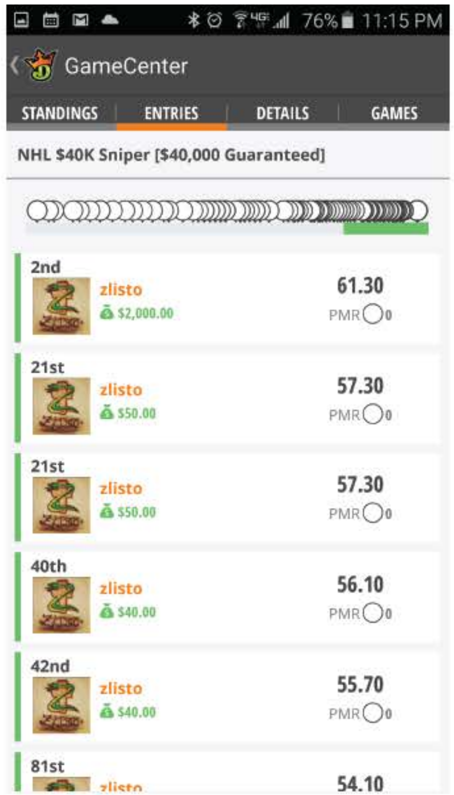
Diversity constraint

$$\sum_{p=1}^{N} x^*_{pk} x_{pl} \leq \gamma, k = 1, \ldots, l-1$$

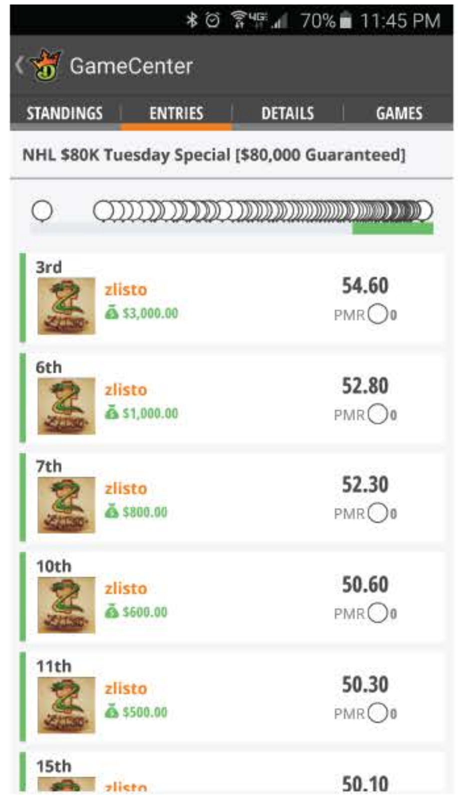
Were we able to do it?



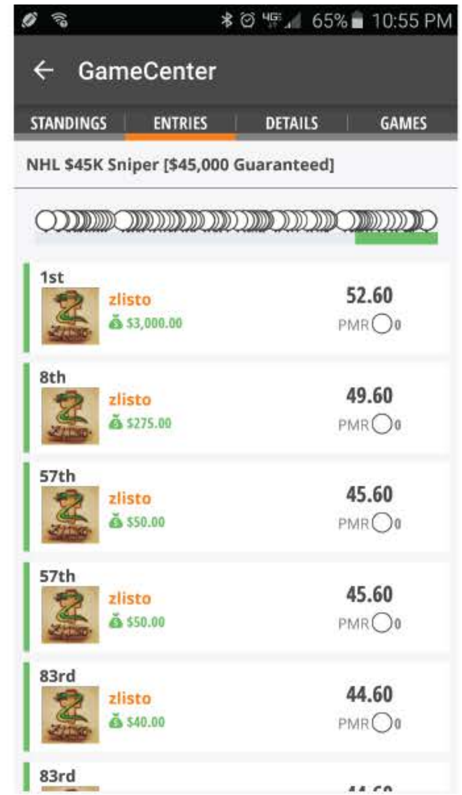
November 15, 2015



November 16, 2015



November 17, 2015



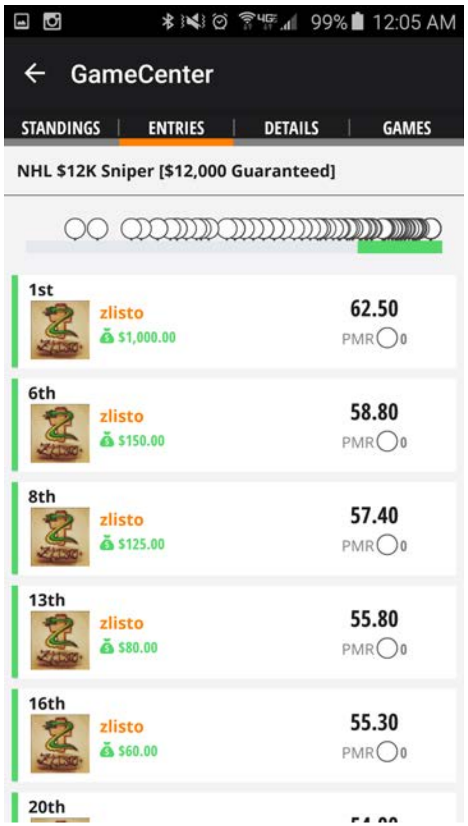
November 23, 2015

200 lineups

Policy Change

200 lineups -> 100 lineups

Were we able to continue it?



December 12, 2015

100 lineups

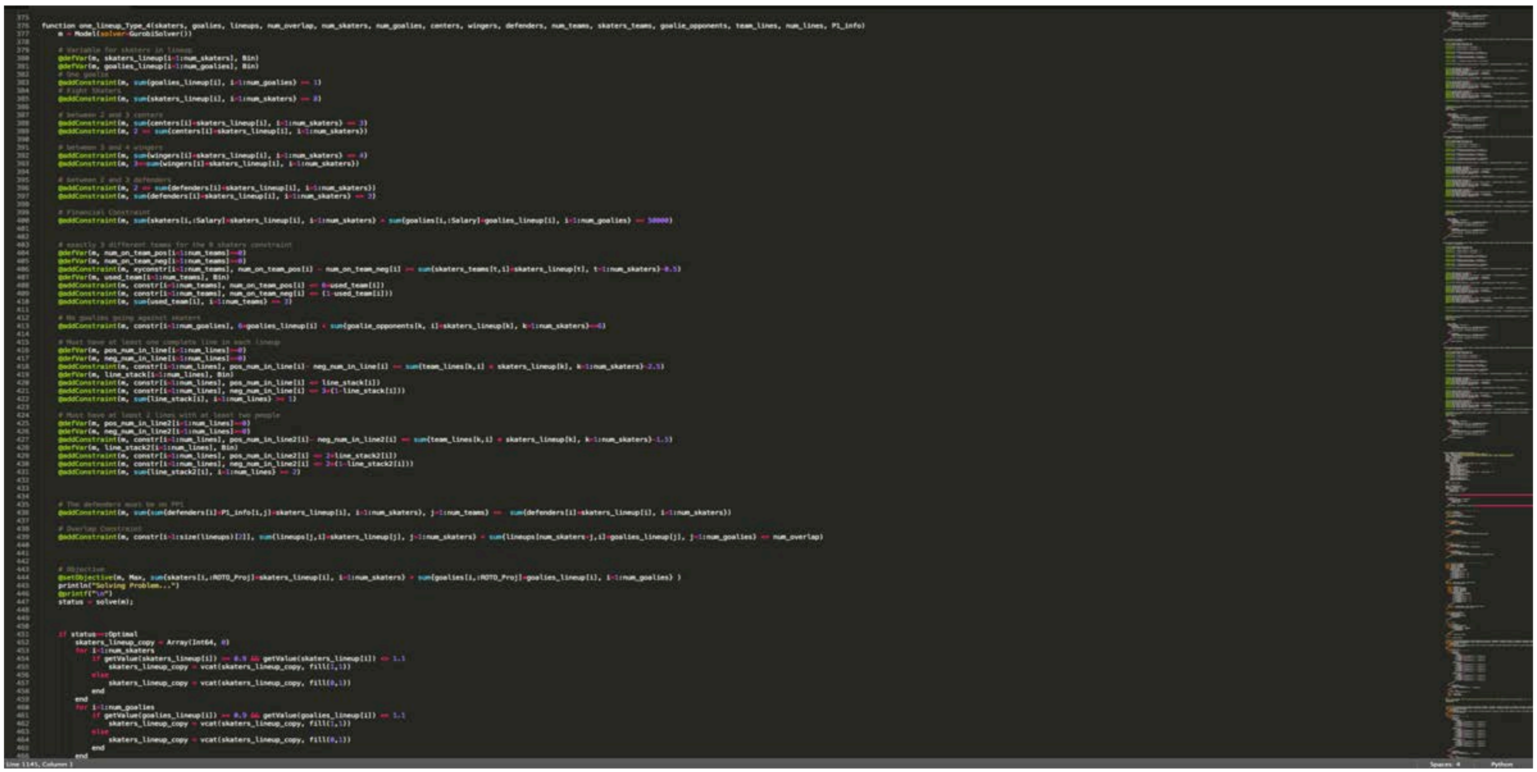
The Greater Boston FOOD BANK®

> $15K

How can you do it?

Download Code from Github:

https://github.com/dscotthunter/Fantasy-Hockey-IP-Code



http://arxiv.org/pdf/1604.01455v1.pdf

Performance Time < 30 Minutes