Robust Optimization for Risk Control in Enterprise-wide Optimization

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Uncertainty in Optimization Problems

$x \in \mathbb{R}^n$ Decision Variables

$L^x(\omega) : \Omega \to \mathbb{R}$ Random Variable

Average or Expected Value

$$\mathbb{E}\left(L^x\right)$$

No Risk Control

Probabilistic Constraint

$$\mathbb{P}\left(L^x \leq 1\right) \geq 0.95$$

Hard to Solve

Add Risk

Make Easy

Risk Measures

$$\mathbb{W}\left(L^x\right)$$

Equivalent

Robust Constraint

$$L^x(\omega) \leq 1 \quad \forall \omega \in R$$

Overview

- Risk Measures: Portfolio Optimization Example
- Robust and Probabilistic Constraints
- Meaning of Risk Measures and Robust Constraints
- Conclusions

Single Period Portfolio Optimization

$$\min \quad \mathbb{E}\left(1 - \sum_{i=1}^{20} \omega_i x_i\right)$$

$s.t.$

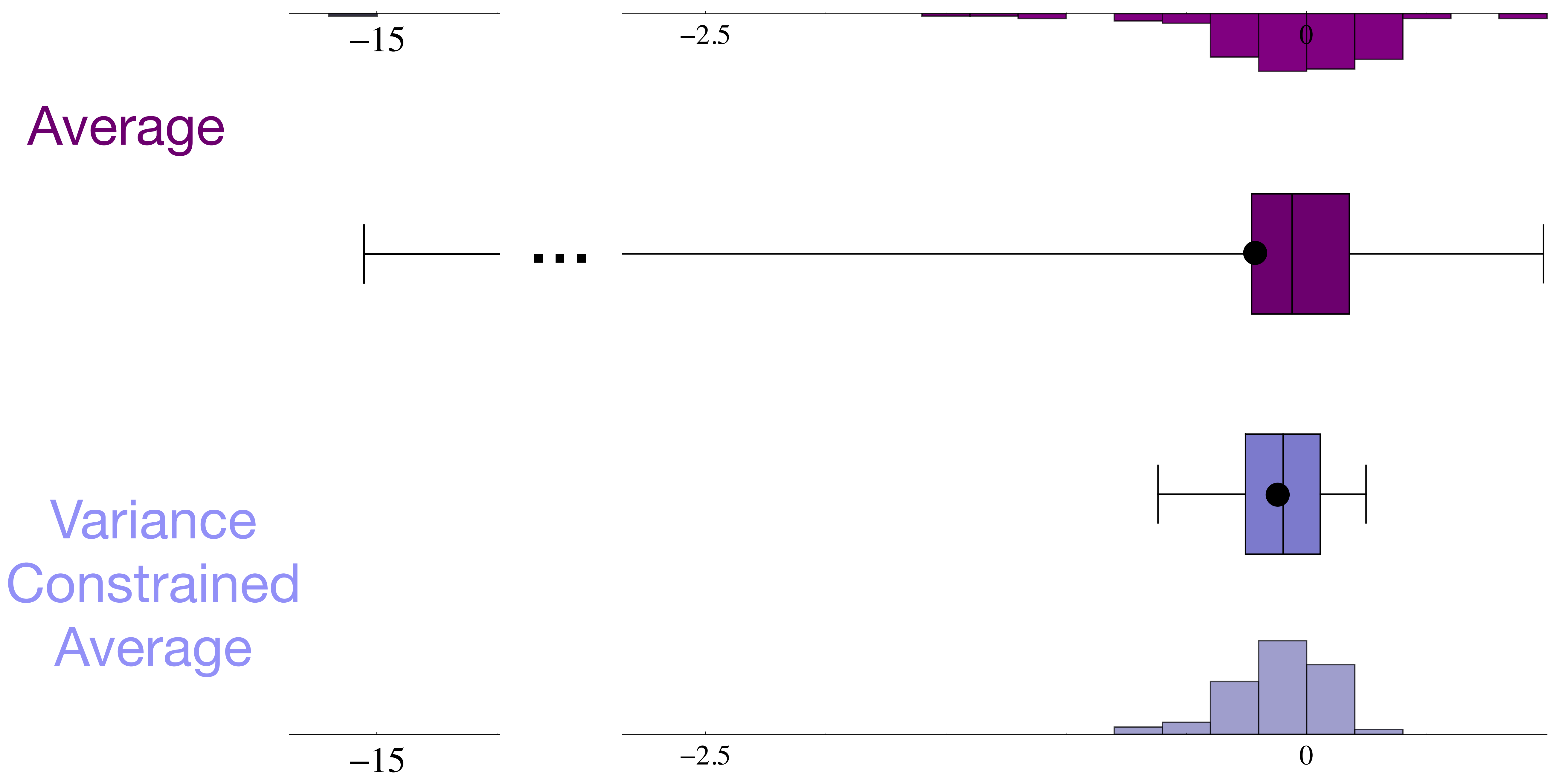
$$\sum_{i=1}^{20} x_i = 1$$

$$x_i \geq 0$$

$$\sigma^2\left(\sum_{i=1}^{20} \omega_i x_i\right) \leq k$$

- 20 possible assets
- x_i : millions in asset
- ω_i: (random) rate of return of asset i
- Pessimistic: Minimize average loss
- Risk control: Variance Constraint

Variance OK, but Penalizes Gains



Uncertain Rate of Return = Scenarios

- Data Driven Uncertainty:
  - $\omega \in \{\omega^1, \omega^2, \dots, \omega^{100}\} \subset \mathbb{R}^{20}$, $\mathbb{P}(\omega = \omega^s) = \frac{1}{100}$
- Loss is Random Variable:
  - $L^x(\omega) = 1 - \sum_{i=1}^{20} \omega_i x_i \in \{L_1^x, \dots, L_{100}^x\}$, $L_s^x = 1 - \sum_{i=1}^{20} \omega_i^s x_i$
  - Scalar Representation of $L^x(\omega)$:
    - $\mathbb{E}(L^x) = \frac{1}{100} \sum_{s=1}^{100} L_s^x$

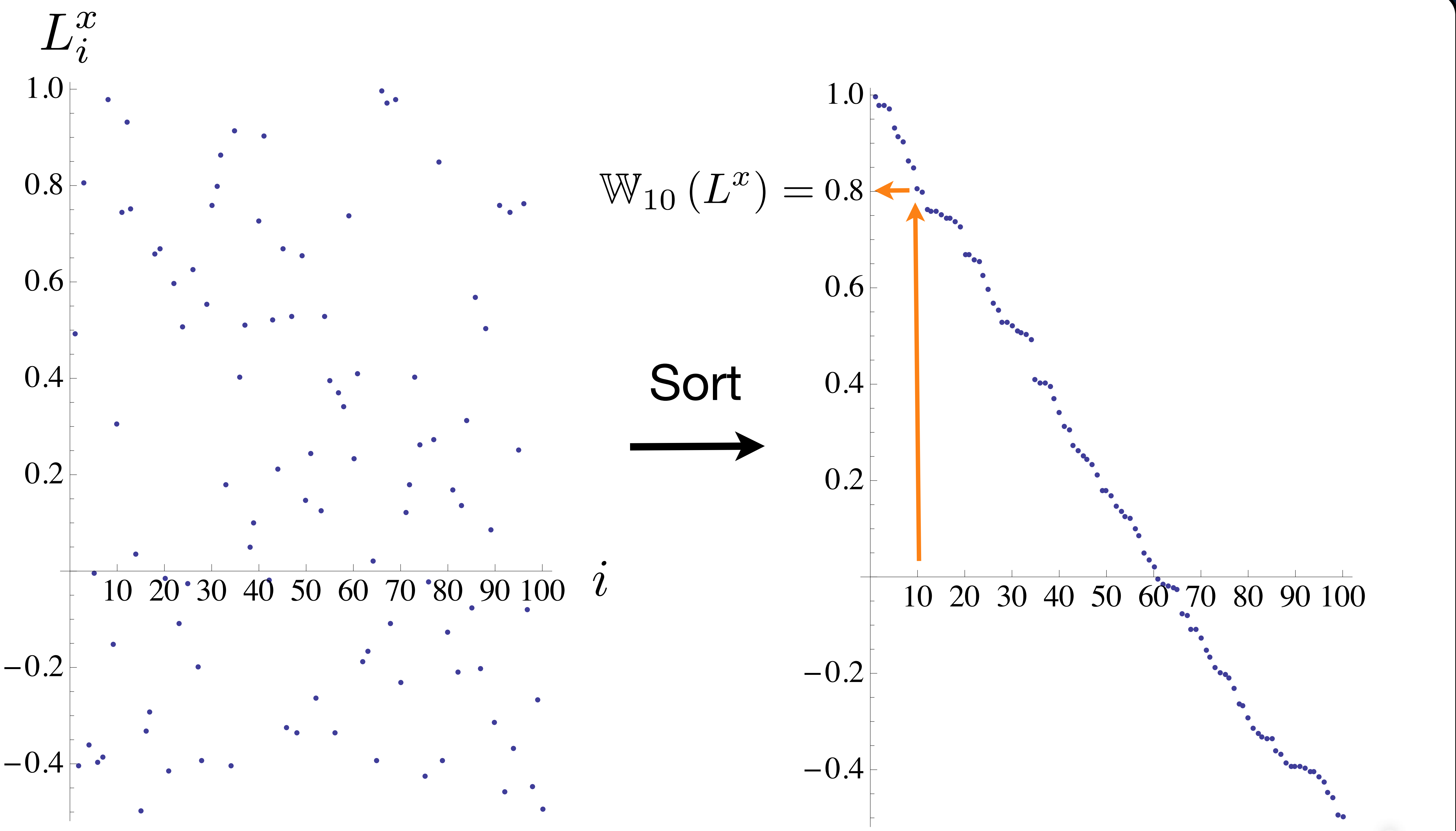
Including Risk in Scalar Representation

- Loss is Random Variable: $L^x \in \{L_1^x, \ldots, L_{100}^x\}$
- Pessimistic approach = Worst Case:

$$\mathbb{W}_1(L^x) := \max_{i=1}^{100} L_i^x$$

- Less pessimistic = k-th Worst Case:
  - Only k scenarios are equal or worse.

k-th Worst Case for k=10



A More Conservative k-th Worst Case

- Average: $\mathbb{E}\left(L^x\right) := \frac{1}{100}\sum_{s=1}^{100} L_s^x$

- k-th worst case:

$$\max_{i=1}^{100} L_i^x = \mathbb{W}_1\left(L^x\right) > \mathbb{W}_2\left(L^x\right) > \ldots > \mathbb{W}_{100}\left(L^x\right) = \min_{i=1}^{100} L_i^x$$

- What about average of k-th worst cases:

$$\mathbb{A}_k\left(L^x\right) := \frac{1}{k}\sum_{s=1}^{k} \mathbb{W}_k\left(L^x\right)$$

- We have: $\mathbb{A}_k\left(L^x\right) \geq \mathbb{W}_k\left(L^x\right)$

Two New Families of Problems

$$\min \quad f(x) := \mathbb{W}_k\left(L^x\right)$$

$$s.t.$$

$$\sum_{i=1}^{20} x_i = 1$$

$$x_i \geq 0$$

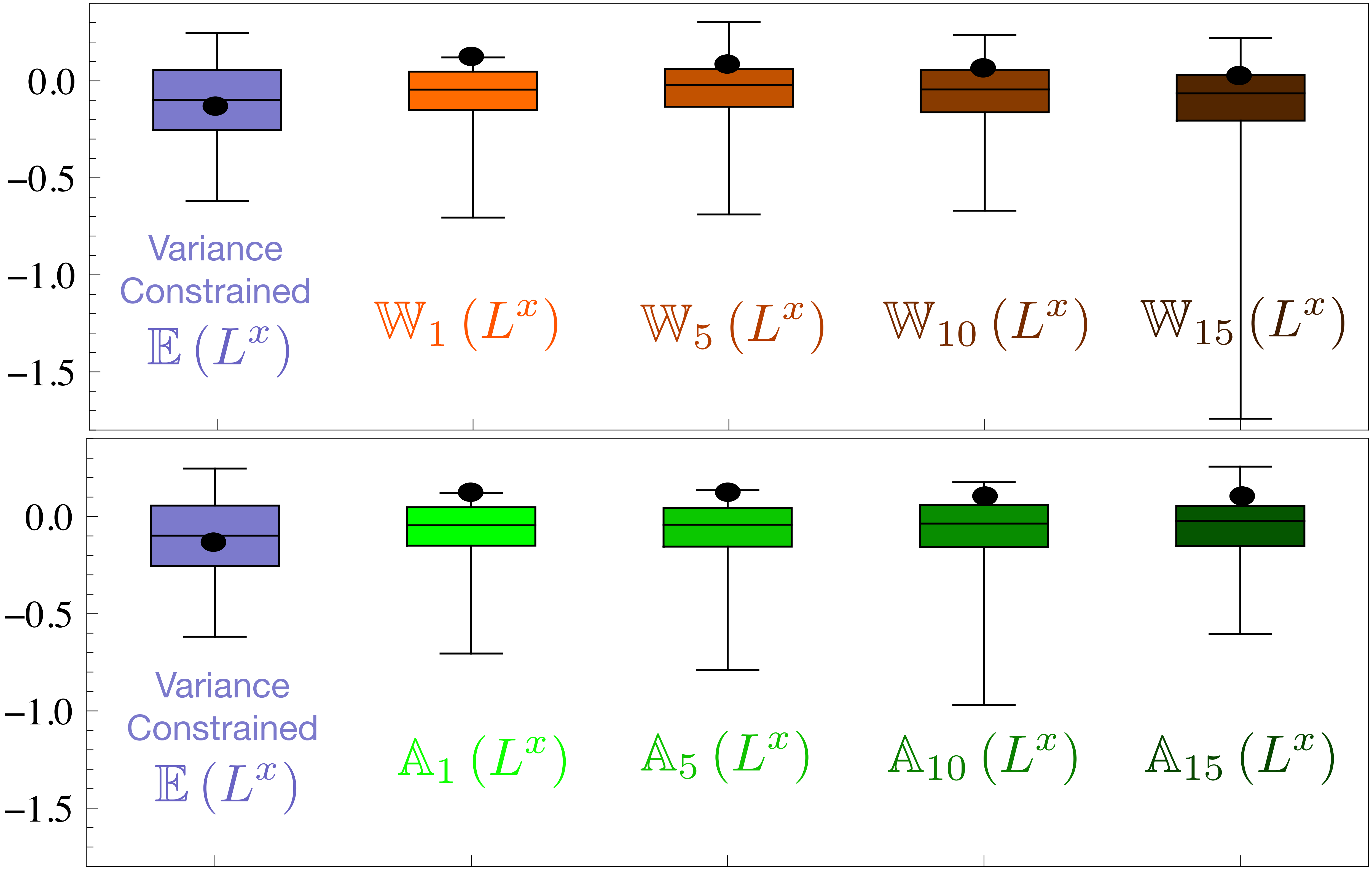
$$\min \quad g(x) := \mathbb{A}_k\left(L^x\right)$$

$$s.t.$$

$$\sum_{i=1}^{20} x_i = 1$$

$$x_i \geq 0$$

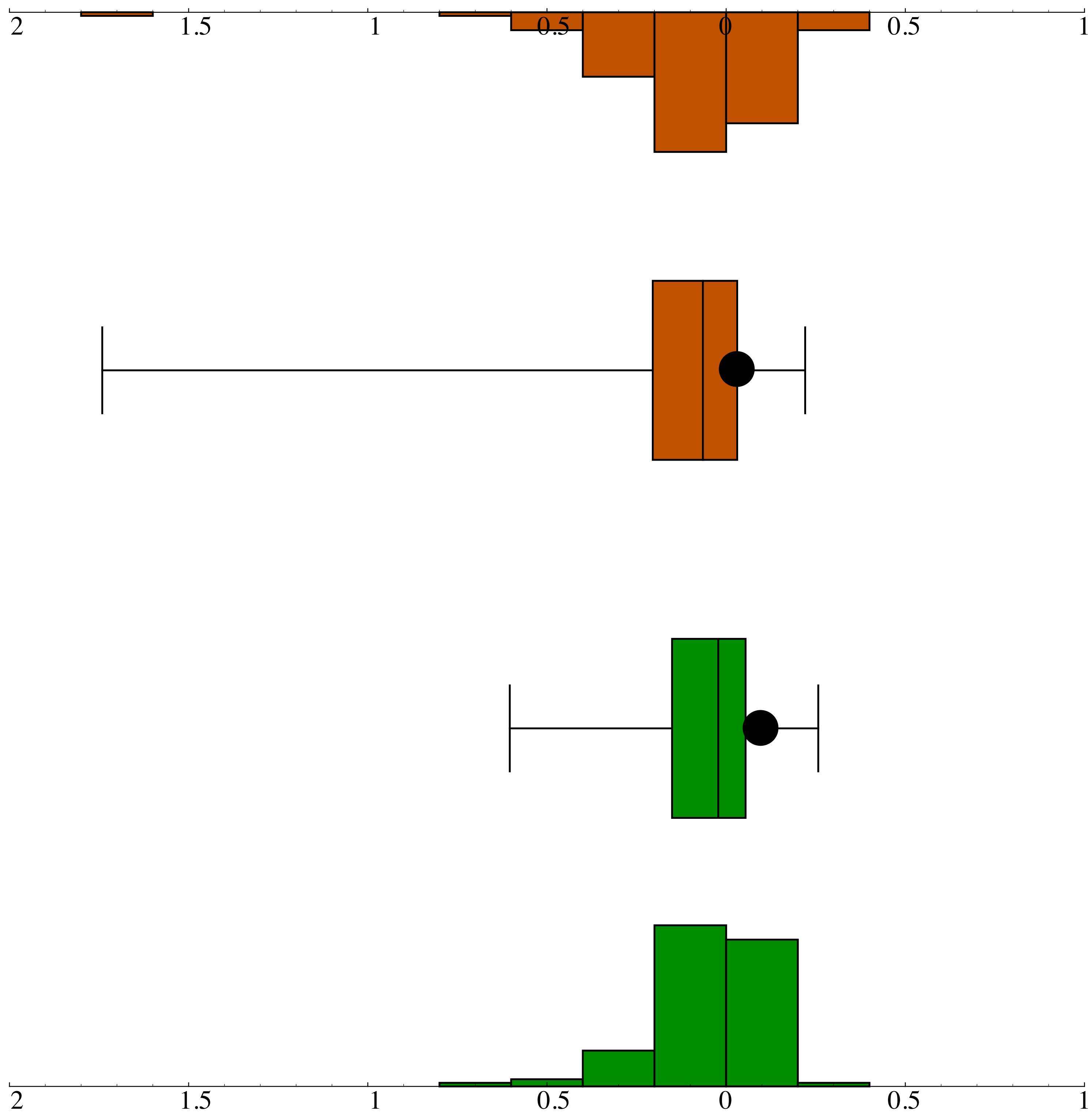
Behavior of Optimal Solutions



k-worst v/s average k-worst

$\mathbb{W}_{15}\left(L^{x}\right)$

$\mathbb{A}_{15}\left(L^{x}\right)$



Probabilistic Meaning

- k-worst = Value at Risk:

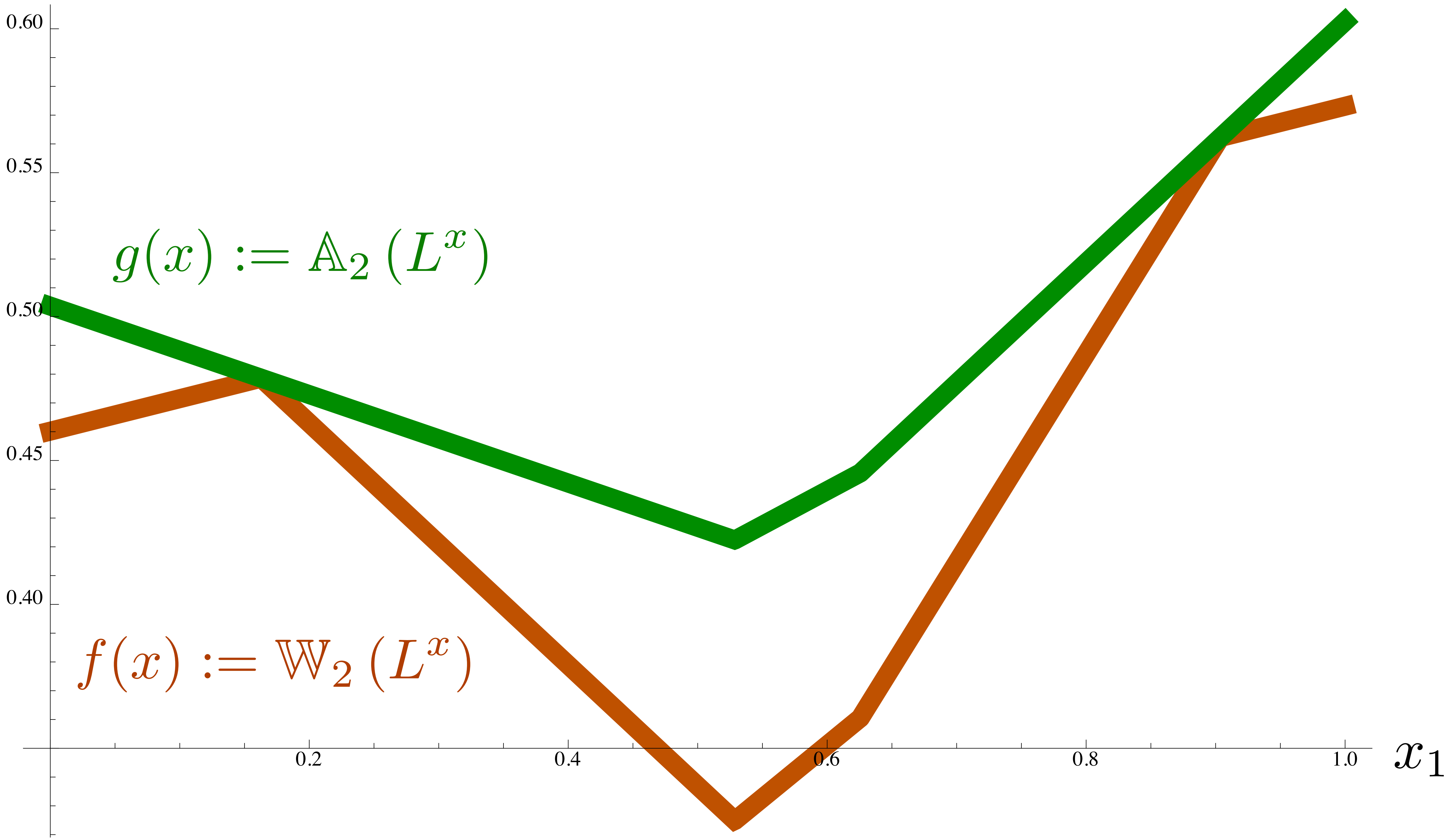
$$\mathbb{W}_k\left(L^x\right) = \inf\left\{t \,:\, \mathbb{P}(L^x \leq t) \geq 1 - \frac{k-1}{100}\right\}$$

$$=: V@R_{\frac{t-1}{100}}(L^x)$$

- Average k-worst = Average Value at Risk
  - Also known as Conditional Value at Risk

$$\mathbb{E}\left(L^x | L^x \geq V@R_{\frac{t-1}{100}}(L^x)\right)$$

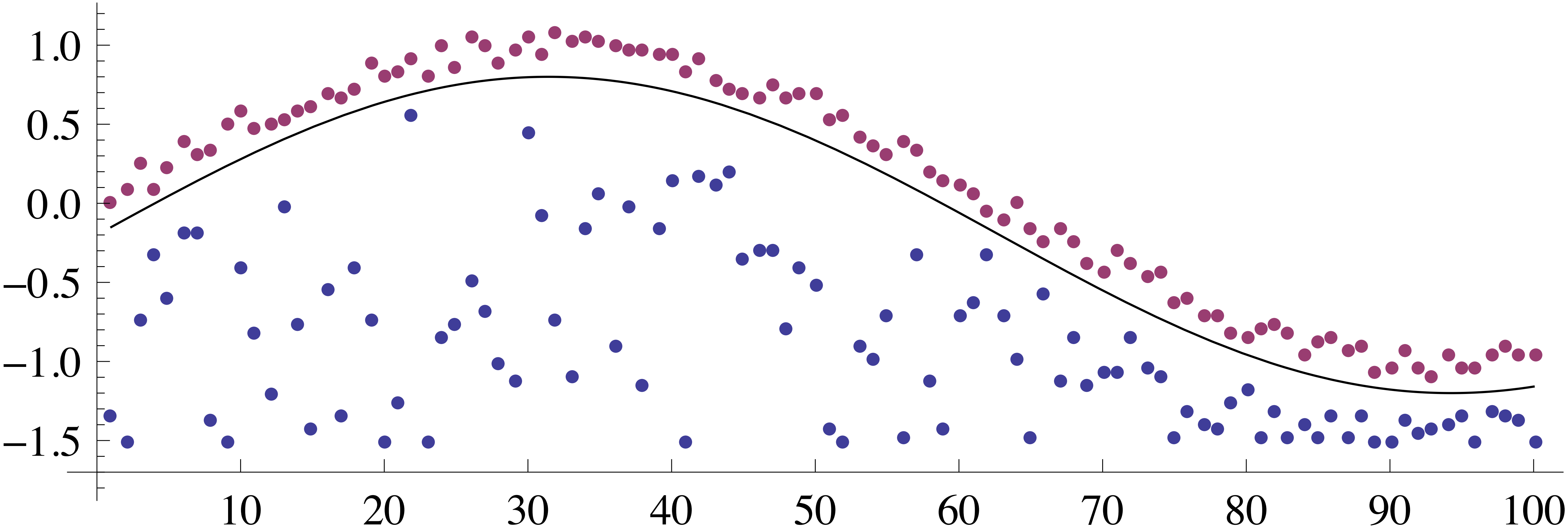
Shape of Objective Function for 2 Assets



Convexity and Coherence

- Average k-worst is a:
  - Coherent Risk Measure
    - Monotonic, Translation Invariant, Sub-additive, Positive Homogeneous.
  - Distortion/Spectral Risk Measure
    - Coherent + Co-monotonicity and Law Invariance

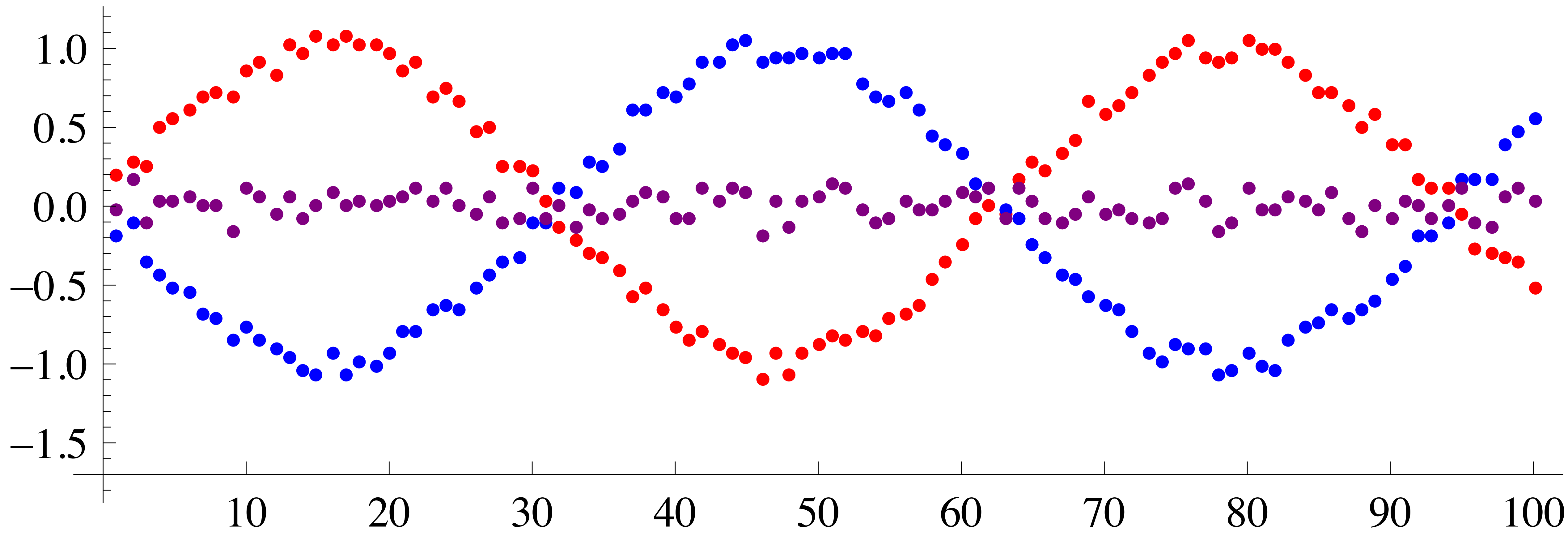
Monotonic



$$V_i^{x^1} \leq V_i^{x^2} \qquad \Rightarrow \qquad \mathbb{A}_k\left(L^{x^1}\right) \leq \mathbb{A}_k\left(L^{x^2}\right)$$

$$\forall i \in \{1, \ldots, 100\}$$

Sub-additive



$$\mathbb{A}_k\left(L^{x^1}+L^{x^2}\right) \leq \mathbb{A}_k\left(L^{x^1}\right)+\mathbb{A}_k\left(L^{x^2}\right)$$

$=$ → Co-monotonicity (Distortion)

Formulation for Average Worst Case

$$
\begin{aligned}
\min \quad & \mathbb{A}_k\left(L^x\right) \\
s.t. \quad & \\
& \sum_{i=1}^{20} x_i = 1 \\
& x_i \geq 0
\end{aligned}
$$

$$
t^* = \mathbb{W}_k\left(L^{x^*}\right)
$$

$$
\begin{aligned}
\min \quad & t + \frac{1}{k}\sum_{s=1}^{100} \alpha_s \\
s.t. \quad & \\
& \sum_{i=1}^{20} x_i = 1 \\
& x_i \geq 0 \\
& h\left(x, \omega^s\right) - t \leq \alpha_s \\
& \alpha_s \geq 0 \\
& t \in \mathbb{R}
\end{aligned}
$$

$$
h\left(x, \omega^s\right) := \sum_{i=1}^{20} \omega_i^s x_i
$$

Constraining with Risk Measures

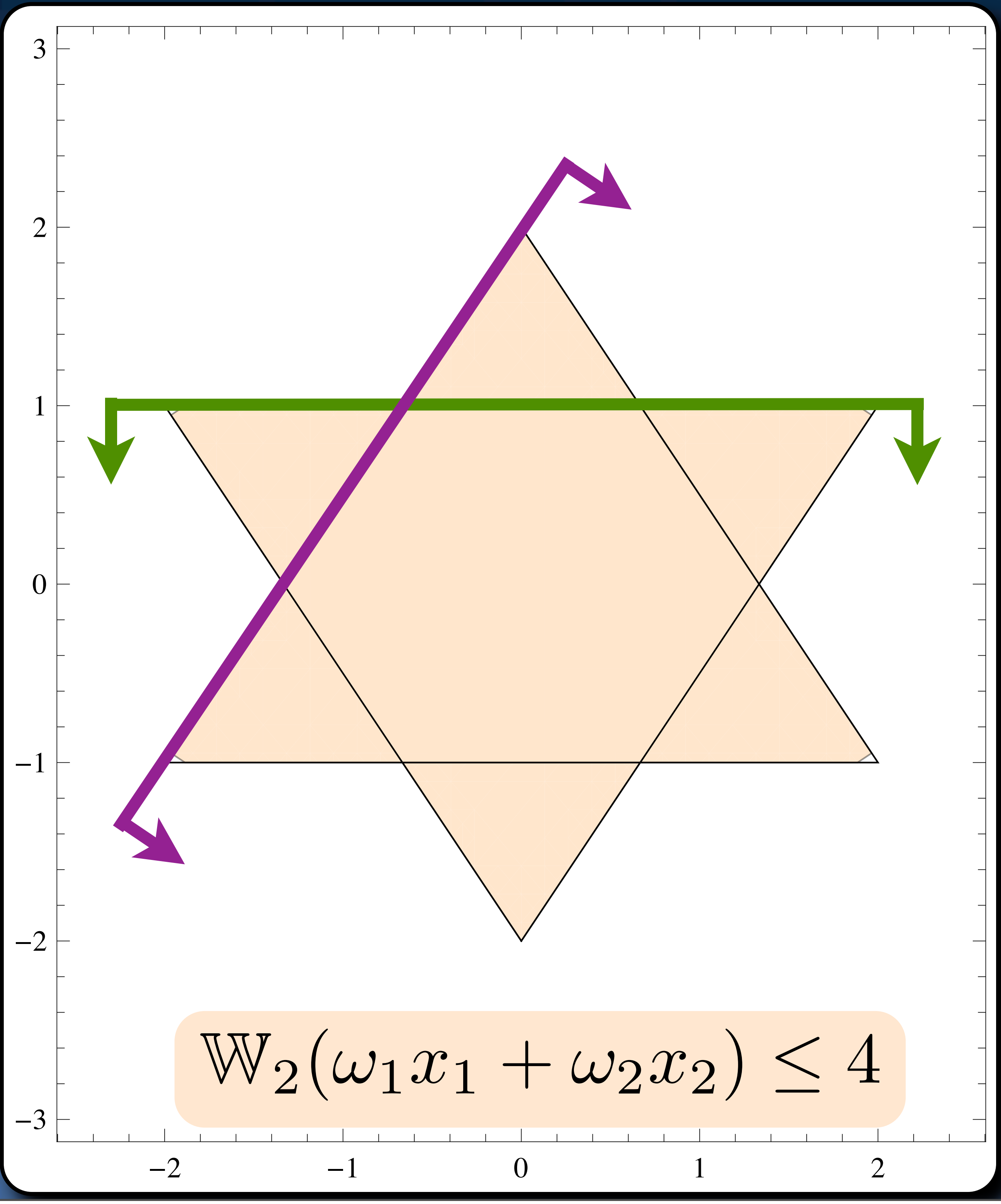
$$
\begin{aligned}
\min \quad & \mathbb{A}_k\left(L^x\right) \\
s.t. \quad & \\
& \sum_{i=1}^{20} x_i = 1 \\
& x_i \geq 0
\end{aligned}
\quad \Longleftrightarrow \quad
\begin{aligned}
\min \quad & z \\
s.t. \quad & \\
& \sum_{i=1}^{20} x_i = 1 \\
& x_i \geq 0 \\
& \mathbb{A}_k\left(L^x\right) \leq z
\end{aligned}
$$

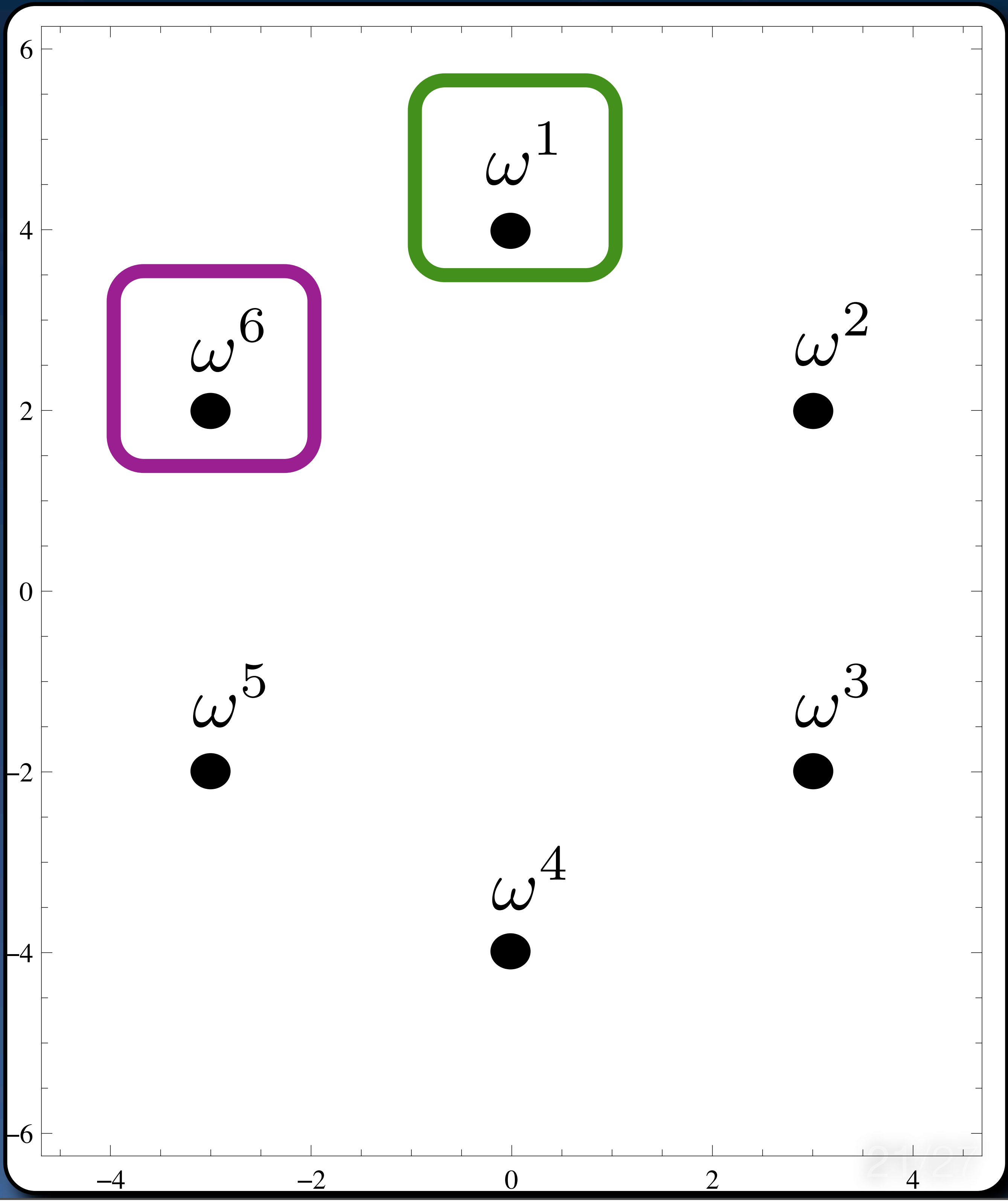
$$
\mathbb{A}_k\left(L^x\right) \leq 1 \quad \Longleftrightarrow \quad
\begin{aligned}
\mathbb{A}_k\left(L^x\right) &\leq z \\
z &= 1
\end{aligned}
$$

Probabilistic Constraints

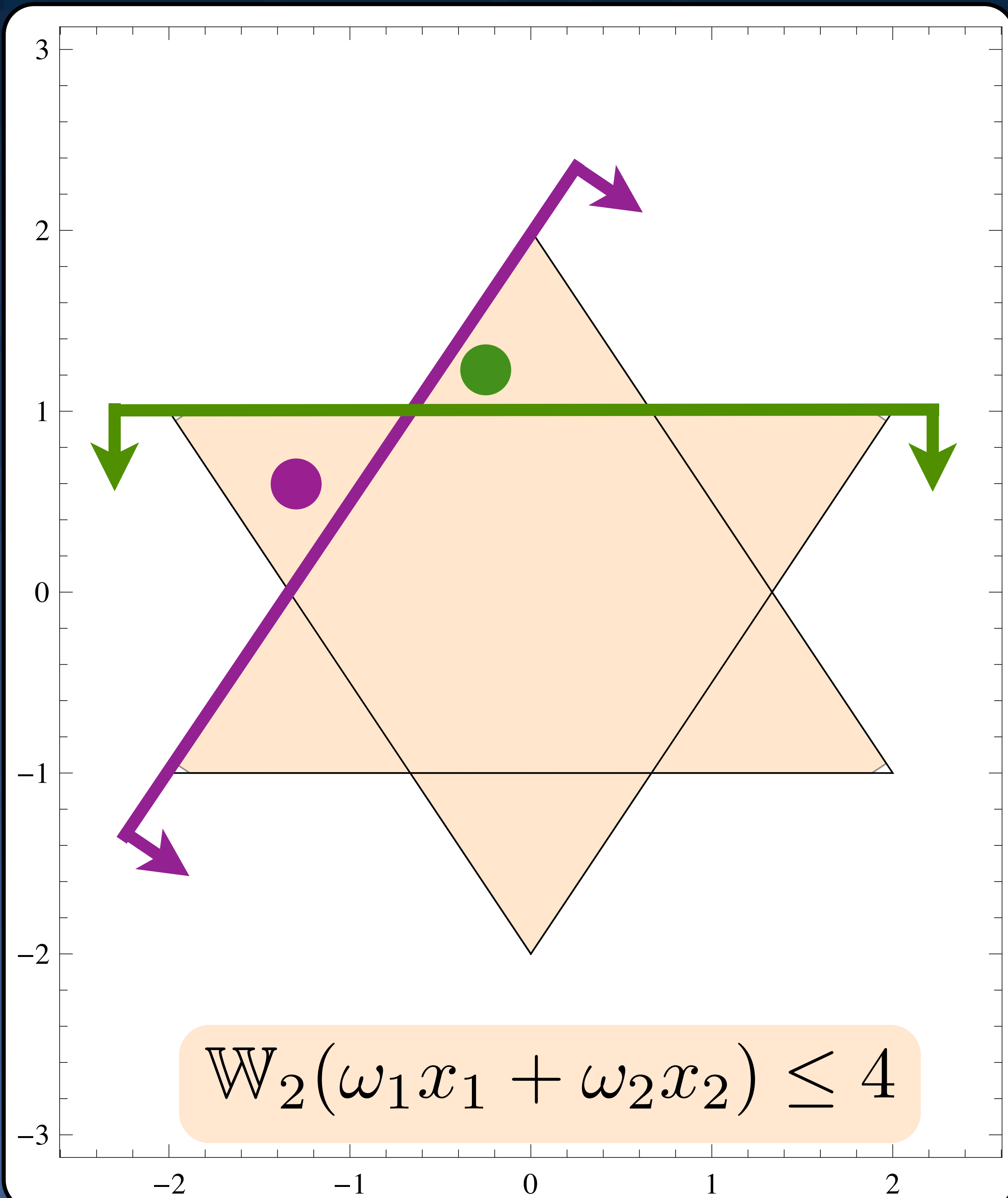
- $\omega \in \{\omega^1, \omega^2, \ldots, \omega^6\} \subset \mathbb{R}^2, \quad \mathbb{P}(\omega = \omega^s) = \dfrac{1}{6}$
- $\mathbb{P}(\omega_1 x_1 + \omega_2 x_2 \leq 4) \geq \dfrac{5}{6}$:
  - Satisfy 5 of 6 inequalities: $\{\omega_1^s x_1 + \omega_2^s x_2 \leq 4\}_{s=1}^6$
  - $\mathbb{W}_2(\omega_1 x_1 + \omega_2 x_2) \leq 4$
- $\mathbb{A}_2(\omega_1 x_1 + \omega_2 x_2) \leq \mathbb{W}_2(\omega_1 x_1 + \omega_2 x_2)$ so:
  - Conservative approximation:
    - $\mathbb{A}_2(\omega_1 x_1 + \omega_2 x_2) \leq 4$

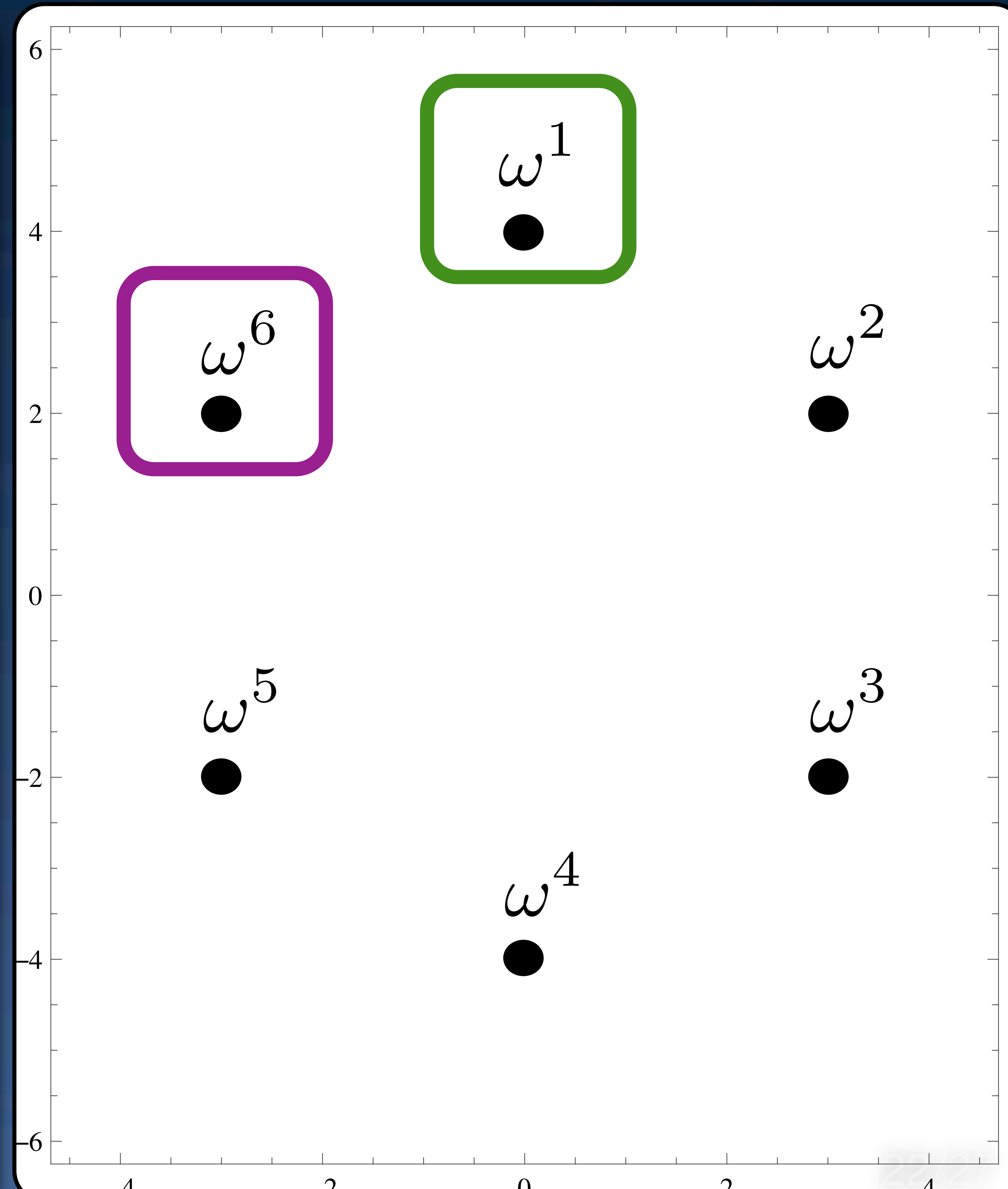
Satisfy 5 out of 6 Constraints



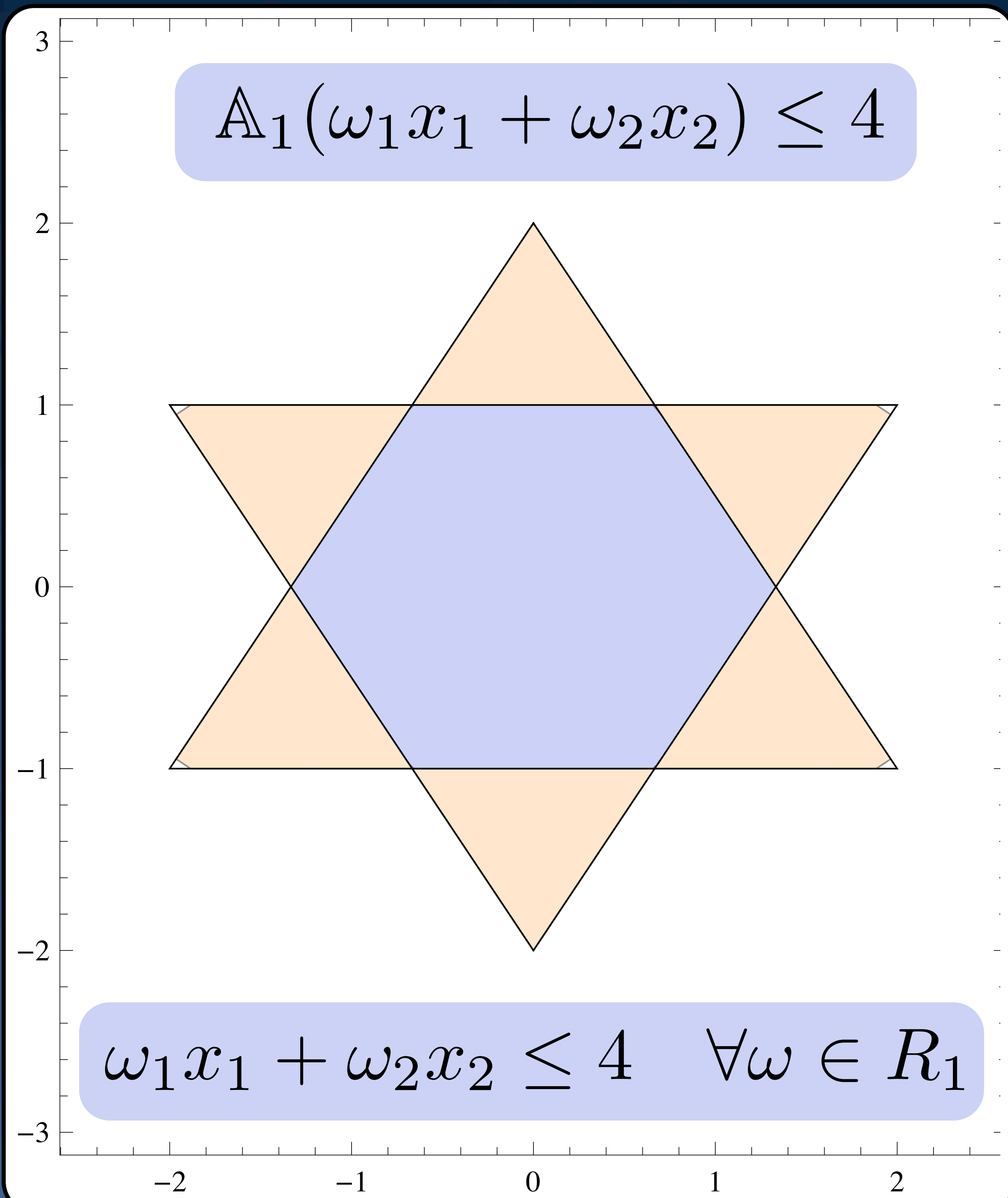


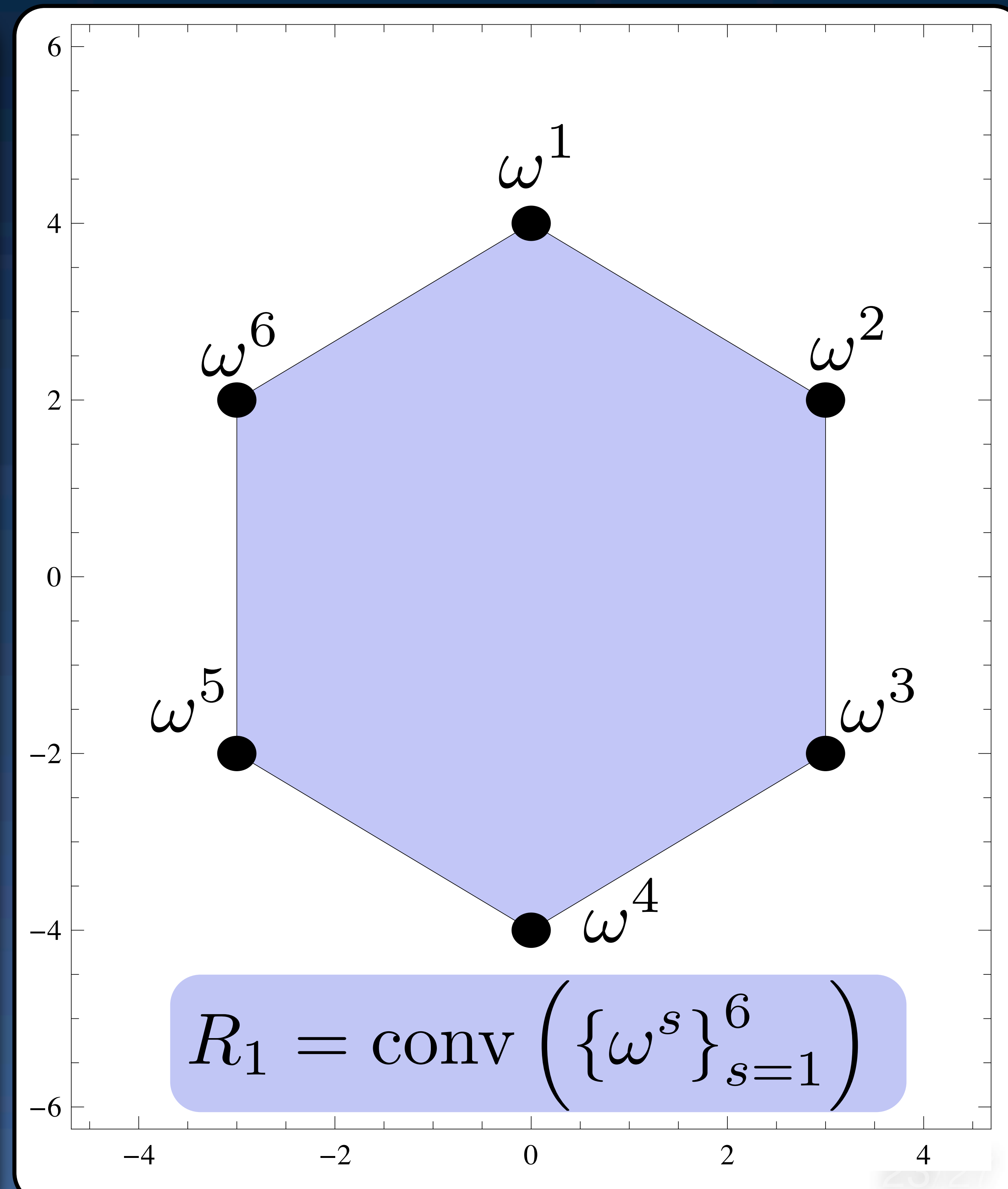
No Guarantee: Violate Each Constraint



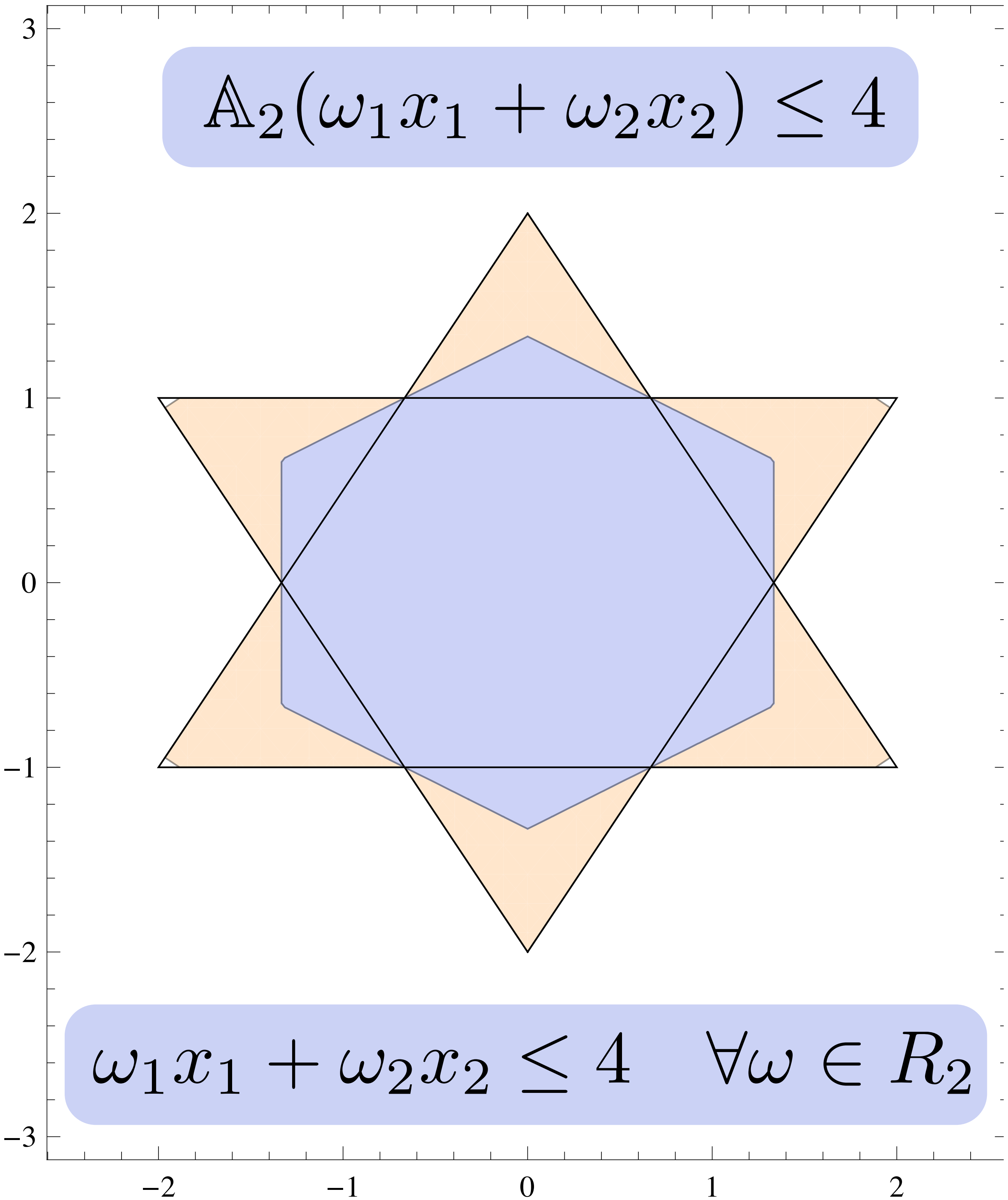


Guarantee to Satisfy All Constraints



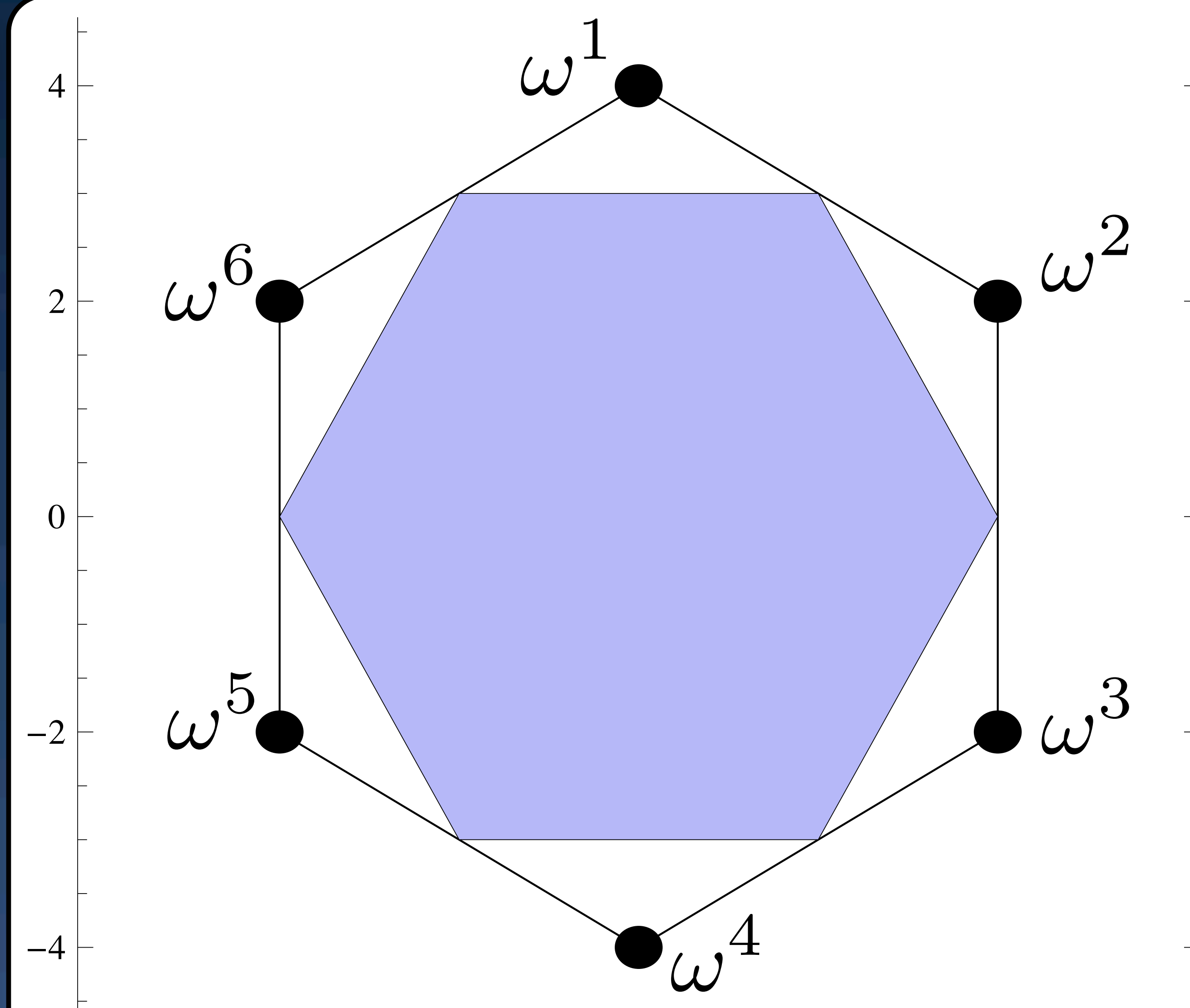


Weaker Guarantee: All 2-Averages



$$\mathbb{A}_2(\omega_1 x_1 + \omega_2 x_2) \le 4$$

$$\omega_1 x_1 + \omega_2 x_2 \le 4 \quad \forall \omega \in R_2$$



$$R_2 = \text{conv}\left(\left\{ \frac{1}{2}\omega^s + \frac{1}{2}\omega^t \right\}_{s,t=1}^{6} \right)$$

Coherent: Worst Case Expectation

- "Direct" from previous slide:

$$\mathbb{A}_2\left(\omega_1 x_1 + \omega_2 x_2\right) = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left(\omega_1 x_1 + \omega_2 x_2\right)$$

$$\mathcal{P} := \bigcup_{i\neq j}\left\{\mathbb{P} : \mathbb{P}(\omega = \omega^i) = \mathbb{P}(\omega = \omega^j) = \frac{1}{2}\right\}$$

- All Coherent Risk Measures are of this form

Distortion: Average of Worst Case

- Average Worst Case:

$$\mathbb{A}_k\left(L^x\right) := \frac{1}{k}\sum_{s=1}^{k} \mathbb{W}_k\left(L^x\right)$$

- General Distortion Risk Measure:

$$\mathbb{D}_p\left(L^x\right) := \sum_{s=1}^{S} p_i \mathbb{W}_k\left(L^x\right)$$

$$p \in \left\{\mathbb{R}_+^S : \sum_{s=1}^{S} p_i = 1, p_1 \geq p_2 \geq \ldots \geq p_S\right\}$$

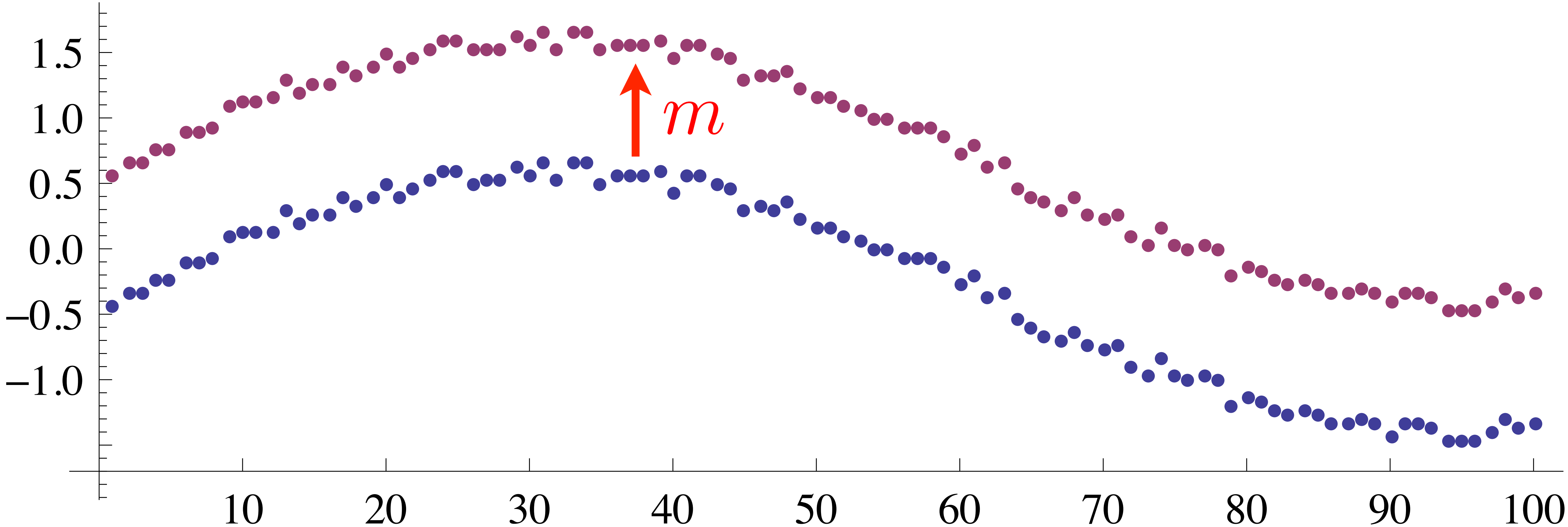
Conclusions

- Robust Constraints = Risk Measures
  - Add risk control
  - Easy to solve conservative approximation of hard to solve probabilistic constraints
  - Very easy to implement for scenario based
  - Do not need a priori scenario probabilities

References

- Everything in this talk:
  - D. Bertsimas and D. B. Brown, 2010. Constructing Uncertainty Sets for Robust Linear Optimization, Operations Research, 58, 1220-1234.
- Books:
  - A. Ben-Tal, L. El Ghaoui and A. Nemirovski, 2010. Robust Optimization, Princeton University Press.
  - A. Shapiro, D. Dentcheva and A. Ruszczyski, 2010. Lectures on Stochastic Programming: Modeling and Theory, SIAM.

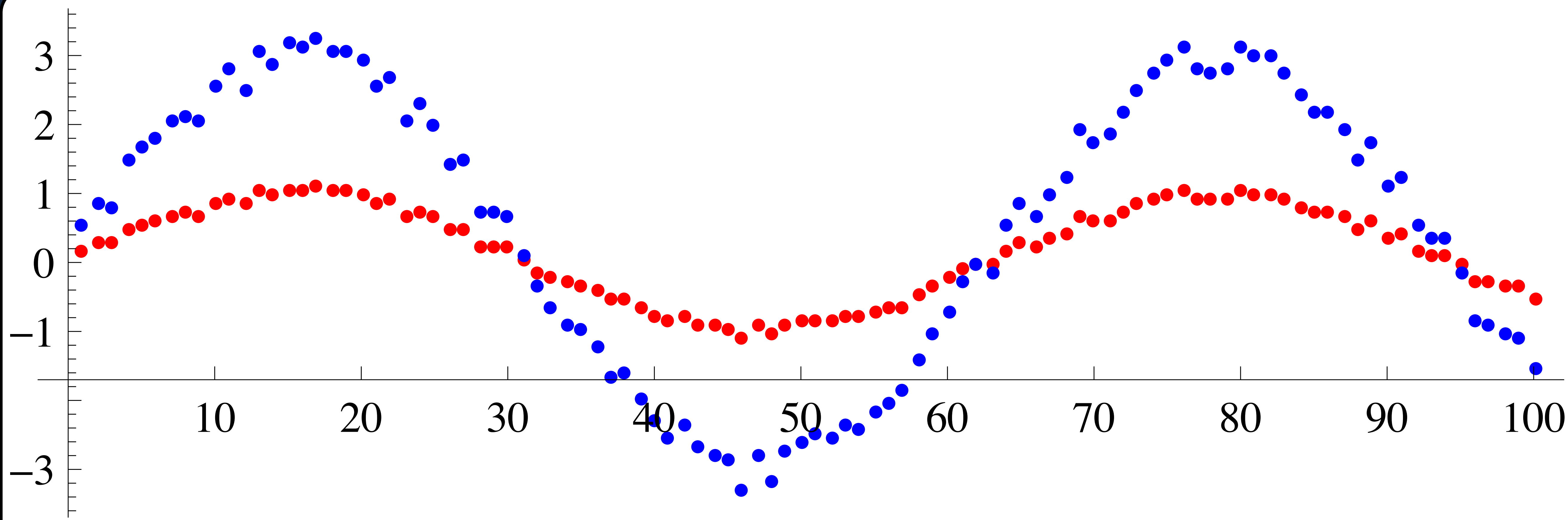
Translation Invariance



$$\mathbb{A}_k\left(L^x + m\right) = \mathbb{A}_k\left(L^x\right) + m$$

$$\mathbb{A}_k\left(L^x - m\right) = \mathbb{A}_k\left(L^x\right) - m$$

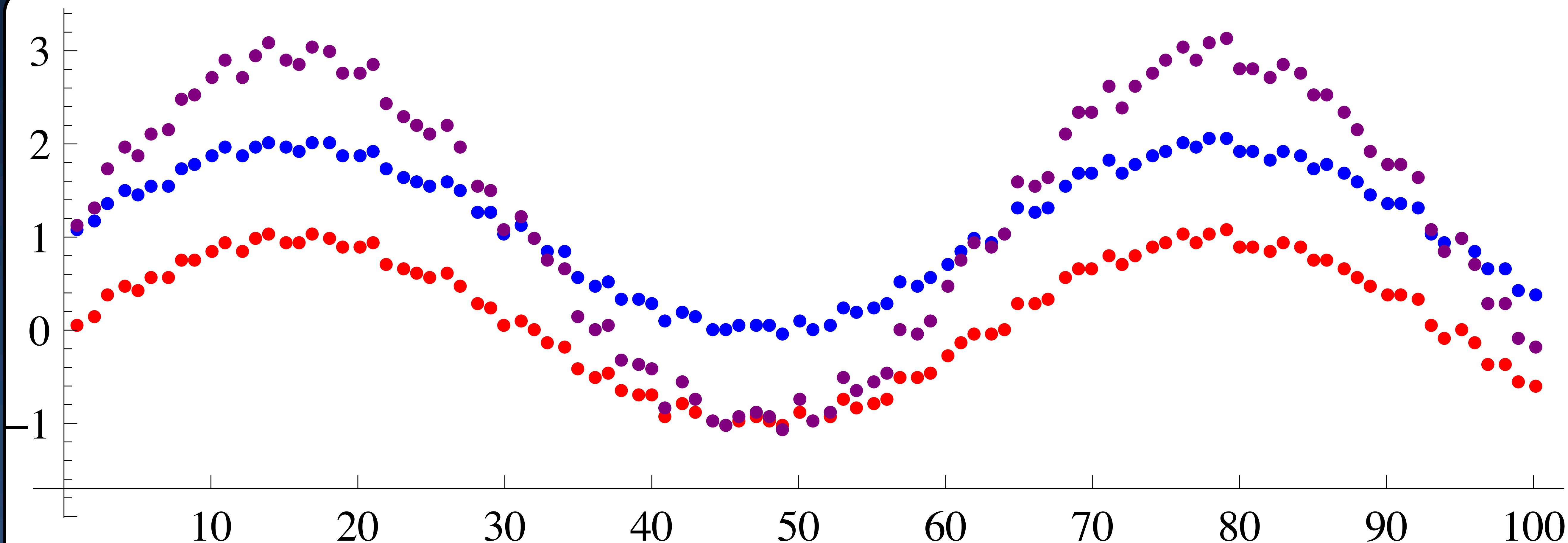
Positive Homogeneous



$$\mathbb{A}_k\left(\lambda L^x\right) = \lambda \mathbb{A}_k\left(L^x\right)$$

$$\forall \lambda > 0$$

Co-Monotonicity

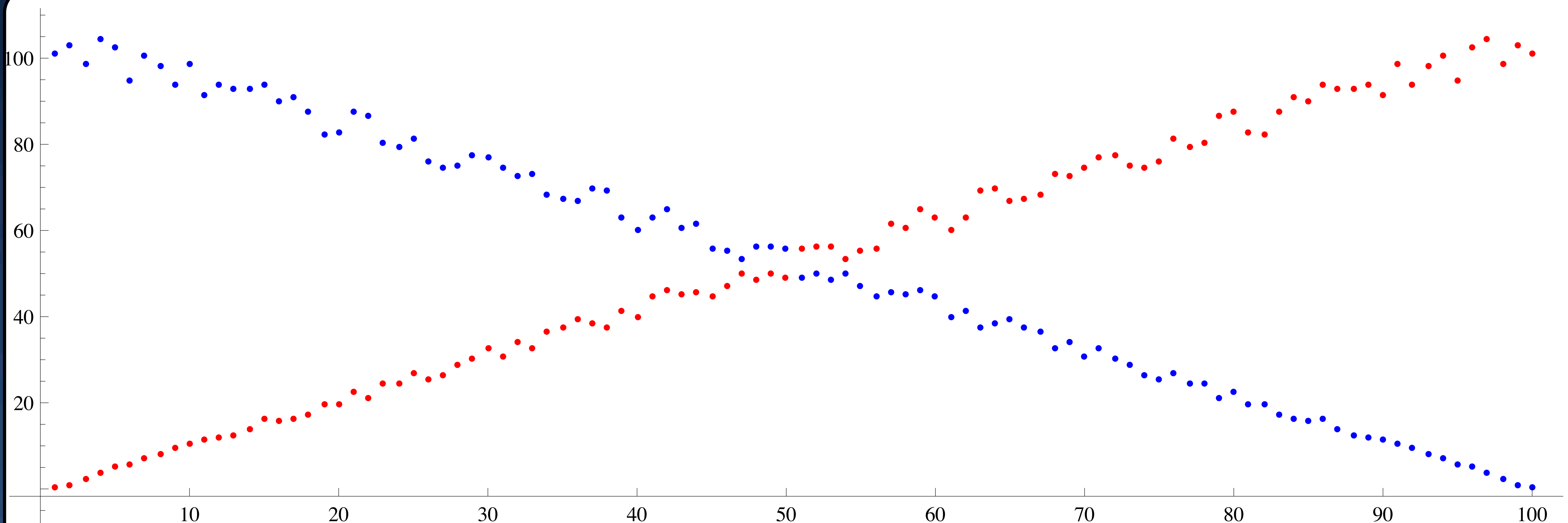


$$\left(L^{x^1}(\omega) - L^{x^1}(\omega')\right)\left(L^{x^2}(\omega) - L^{x^2}(\omega')\right) \geq 0 \quad \forall \omega,\, \omega'$$

$$\Downarrow$$

$$\mathbb{A}_k\left(L^{x^1} + L^{x^2}\right) = \mathbb{A}_k\left(L^{x^1}\right) + \mathbb{A}_k\left(L^{x^2}\right)$$

Law Invariance (for $\mathbb{P}\left(\omega = \omega^i\right) = \mathbb{P}\left(\omega = \omega^j\right)$)



$$L^{x_1}\left(\omega^i\right) = L^{x_2}\left(\omega^{\pi(i)}\right) \qquad \pi \text{ permutation}$$

$$\Downarrow$$

$$\mathbb{A}_k\left(L^{x_1}\right) = \mathbb{A}_k\left(L^{x_2}\right)$$