Regularity in mixed-integer convex representability

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Characterizations of the sets with mixed integer programming (MIP) formulations using only rational linear inequalities (rational MILP representable) and those with formulations that use arbitrary closed convex constraints (MICP representable) were given by Jeroslow and Lowe (1984), and Lubin, Zadik and Vielma (2017). The latter also showed that even MICP representable subsets of the natural numbers can be more irregular than rational MILP representable ones, unless certain rationality is imposed on the formulation. In this work we show that for MICP representable subsets of the natural numbers, a cleaner version of the rationality condition from Lubin, Zadik and Vielma (2017) still results in the same periodical behavior appearing in rational MILP representable sets after a finite number of points are excluded. We further establish corresponding results for compact convex sets, the epigraphs of certain functions with compact domain and the graphs of certain piecewise linear functions with unbounded domains. We then show that MICP representable sets that are unions of an infinite family of convex sets with the same volume are unions of translations of a finite sub-family. Finally, we conjecture that all MICP representable sets are (possibly infinite) unions of homothetic copies of a finite number of convex sets.

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1. Introduction Mixed-integer programming representability considers the study and classification of sets that have mixed-integer programming (MIP) formulations. That is, sets that can be represented by the feasible region of a mathematical programming problem that includes integrality constraints on some of its variables. An iconic example of this research area is the 1985 result by Jerowslow and Lowe [15] that resolved the decade-long quest of characterizing what sets can be modeled using *rational* mixed integer linear programming (MILP) formulations that include integrality requirements and linear inequalities with rational coefficients (e.g. [24, Section 11]). Recently MIP representability has received renewed interest through alternative algebraic characterizations of rational MILP representability [2] and extensions to mixed-integer convex representability by considering MIP formulations that include integrality requirements and arbitrary closed convex constraints [4, 5, 6, 7, 9, 19, 25]. The later results have been partially motivated by recent algorithmic developments for the solution of mixed integer convex programming (MICP) [17, 18]. In this paper we continue the study of MICP representable sets by further analyzing and refining a characterization for such sets introduced in [19].

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The characterization from [19] shows that a set is MICP representable precisely when it is the union of an infinite family of convex sets that obey some structural relation between them and are indexed by integer vectors in a convex set. The classical characterization of rational MILP representability by Jerowslow and Lowe [15] has an analog description that requires the representable set to be an infinite union of rational polytopes (bounded polyhedra) indexed by integer vectors in rational polyhedra (possibly unbounded). However, the rational MILP characterization additionally imposes the regularity condition on the infinite family of polytopes that they must be translations of a finite number of polytopes. Furthermore, the associated translation vectors must be elements of a finitely generated integral monoid (or equivalently must be integer points in a rational polyhedral cone). We would obtain a nice convex, but non-polyhedral version of Jerowslow and Lowe's result if the infinite family of convex sets from the MICP representability characterization were also only translations of a finite number of convex sets (even possibly unbounded convex sets), and the translation vectors had a regularity similar to that for the rational MILP case. However, because the class of MICP representable sets is much richer that this nicely behaved sub-classs, we have the ufortunate side effect that MICP representable sets can be quite irregular. Some minor irregularities of these examples include the fact that the convex sets in the infinite family do not need to be closed (they are only projections of closed convex sets). However, MICP representable sets can also have major irregularities such as the sets not being translations of a finite number of sets or the translations being pathologically irregular when the sets do have exactly the same shape (e.g. they are all a single point). In contrast, [19] presented a simple combinatorial sufficient condition for being non-MICP representable that was used to show that the extremely irregular infinite union of singletons[1] corresponding to the set of prime numbers in the real line is not MICP representable even though it can be described by a finite number of polynomial equations and integrality constraints [14].

One of the main goals of this paper is to understand and classify the possible irregularities present in MICP representable sets. For instance, a regularily property that is not satisfied by the prime numbers, but is always satisfied by the more restricted class of rational MILP representable sets is having a periodic structure (e.g. for subsets of the natural numbers, containing infinite arithmetic progressions). In the context of subsets of the natural numbers, [19] showed that the periodicity of rational MILP representable sets translates precisely to such sets being finite unions of infinite arithmetic progressions with the same step size. In contrast, [19] presented an infinite MICP representable subset of the natural numbers that fails to contain infinite arithmetic progressions because of irrational unbounded directions in its description. This difficulty coming from the irrationality we just described inspired a definition of a mild *rational* restriction of MICP representability that aligns with rational MILP representability and whose representable subsets of the natural numbers are precisely unions of a finite number of points and a rational MILP representable set. In this paper we further study and develop such regularity conditions on MICP representable results through the following major contributions:

1. **Review and refinement of the results from [19]:** We review the results from [19] and include all proofs that were omitted from its conference proceedings version. However, we also provide more detailed explanations of the results that were omitted because of space constraints and provide some refinements of the results. In particular, we provide an updated analysis of Proposition 1 in [19], which showed that any finite union of convex sets that are projections of closed convex sets has a MICP formulation with only binary variables. This provides an extension of classical results that require the convex sets to have the same recession cone. Our updated analysis explains how this extension is achieved and introduces a variant of the formulation (Proposition 1) that satisfies the desirable strength property of being *integral* or *ideal*

[1] i.e. convex sets with a single point

(cf. Definition 5). Furthermore, in Lemma 2 we refine the non-MICP representability sufficient condition (Lemma 2 in [19]) to allow restricting the number of integer variables in the MICP representation. We include refinements and simplified proofs of the representability results for subsets of the natural numbers from [19] that incorporate our updated version of rational MICP representability and our categorization of desirable regularity conditions described below.

2. **Redefinition of rational MICP representability:** We introduce an updated rationality requirement for MICP representability. This new notion of rational MICP representability is significantly cleaner than the original version in [19], but it is apparently more restrictive. However, in Theorem 6 we show that when applied to subsets of the natural numbers it still coincides with being the union of a finite set and a rational MILP representable set and hence is equivalent to the original definition in this context. We also show that the updated notion of rational MICP representability is closed under unions, which allows for simplified proofs of existing and new results.
3. **Categorization and study of regularity of conditions for MICP representability:** The most significant contribution of the paper is the distillation of the regularity conditions associated to rational MILP representable sets and an analysis of their presence in MICP representable sets. These conditions consider several versions of the finite-number-of-shapes (i.e., translations of a finite number of sets) and periodicity properties discussed above, and are presented in Corollary 3. Our first analysis shows how these conditions can fail for MICP representable sets. We then show how our new rationality requirement for MICP representability can help satisfy the periodicity conditions. In particular, we show that compact rational MICP representable set are finite unions of convex sets (Theorem 4) and the same holds for graphs of certain functions over compact domains (Theorem 5). We then extend the results from [19] to show that for subsets of the natural numbers, periodic sets are precisely the rational MILP representable sets (Lemma 6), that such sets have the simplest possible periodicity structure (Lemma 6) and that rational MICP representable subsets of the naturals are precisely the union of a finite set and a periodic set (Theorem 6). We also show analog results for the graphs of certain piecewise linear functions (Lemma 8 and Theorem 7). We then use the Brunn-Minkowski inequality to show that if all convex sets from the infinite family defining an MICP representation of a set have the same volume then, they satisfy the finite number of shapes condition, and that this holds even without any rationality condition (Theorem 8). Finally, we propose a variant of the equal shapes condition that we conjecture holds for all MICP representable sets.

The remainder of the paper is organized as follows. In Section 2 we review the general concept of MICP representability and the results from [19]. We also introduce and motivate the various regularity conditions, and define the new version of rational MICP representability. Then in Section 3 we state our main results, which are proven in Section 3. Other omitted proofs are included in Section 5. With regards to notation we let $\mathbb{N}$ be the set of nonnegative integers $\{0,1,2,\dots\}$ and otherwise mostly follow the standard notation from [12]. We will often work with projections of a set $M \subseteq \mathbb{R}^{n+p+d}$ for some $n,p,d \in \mathbb{N}$. We identify the variables in $\mathbb{R}^n$, $\mathbb{R}^p$ and $\mathbb{R}^d$ of this set as $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ and we let

$$\operatorname{proj}_x(M) = \left\{\boldsymbol{x} \in \mathbb{R}^n : \exists\,(\boldsymbol{y},\boldsymbol{z}) \in \mathbb{R}^{p+d} \text{ s.t. } (\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \in M\right\}.$$

We similarly define $\operatorname{proj}_y(M)$ and $\operatorname{proj}_z(M)$.

2. Mixed-Integer Convex Representability

A mixed-integer convex representation or formulation of a set is a collection of convex constraints, auxiliary variables and integrality constraints that precisely describe the set. The following definition formalizes this and further qualifies the case in which the representation only uses linear inequalities and/or only uses bounded integer variables.

Definition 1. Let $M \subseteq \mathbb{R}^{n+p+d}$ be a closed, convex set and $S \subseteq \mathbb{R}^n$. We say M induces an MICP formulation of S if and only if

$$S = \operatorname{proj}_x\left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d\right)\right). \tag{1}$$

A set $S \subseteq \mathbb{R}^n$ is **MICP representable** if there exists a closed convex set $M \subseteq \mathbb{R}^{n+p+d}$ which induces an MICP formulation of S. If such formulation exists for a (rational) polyhedron M then we say S is (rational) MILP representable.

A set S is **bounded MICP (MILP)** representable if there exists a closed convex set $M \subseteq \mathbb{R}^{n+p+d}$ which induces an MICP formulation of S and satisfies $\left|\operatorname{proj}_z\left(M \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^d)\right)\right| < \infty$. That is, there are only finitely many feasible assignments of the integer variables z. □

A characterization of general MICP representable sets was given in [19]. The characterization shows that such sets are precisely countable unions of projections of the elements of a specially structured family of convex sets.

Definition 2. Let $C \subseteq \mathbb{R}^d$ be a convex set and $(A_z)_{z\in C}$ be a family of convex sets in $\mathbb{R}^n$. We say that the family of sets is **convex** if for all $\boldsymbol{z}, \boldsymbol{z}' \in C$ and $\lambda \in [0,1]$ it holds

$$\lambda A_{\boldsymbol{z}} + (1-\lambda) A_{\boldsymbol{z}'} \subseteq A_{\lambda \boldsymbol{z} + (1-\lambda)\boldsymbol{z}'}.$$

We further say that the family is **closed** if $A_{\boldsymbol{z}}$ is closed for all $\boldsymbol{z} \in C$ and for any convergent sequences $\{\boldsymbol{z}_m\}_{m\in\mathbb{N}}, \{\boldsymbol{x}_m\}_{m\in\mathbb{N}}$ with $\boldsymbol{z}_m \in C$ and $\boldsymbol{x}_m \in A_{\boldsymbol{z}_m}$ we have $\lim_{m\to\infty} \boldsymbol{x}_m \in A_{\lim_{m\to\infty} \boldsymbol{z}_m}$. □

Theorem 1 (**[19]**). *If $M \subseteq \mathbb{R}^{n+p+d}$ induces an MICP-formulation of $S \subseteq \mathbb{R}^n$ then*

$$S = \bigcup_{\boldsymbol{z} \in C \cap \mathbb{Z}^d} \operatorname{proj}_x(B_{\boldsymbol{z}}), \tag{2}$$

where $C = \operatorname{proj}_z(M)$ and $B_{\boldsymbol{z}} = M \cap (\mathbb{R}^{n+p} \times \{\boldsymbol{z}\})$ for any $\boldsymbol{z} \in C$. In addition, $(B_{\boldsymbol{z}})_{\boldsymbol{z}\in C}$ is a closed convex family.

Conversely, if (2) holds for a convex set $C \subseteq \mathbb{R}^d$ and closed convex family $(B_{\boldsymbol{z}})_{\boldsymbol{z}\in C}$, then S is MICP representable.

Theorem 1 states that MICP representable sets are of the form $\bigcup_{\boldsymbol{z}\in C\cap\mathbb{Z}^d} A_{\boldsymbol{z}}$ where both C and $A_{\boldsymbol{z}}$ are projections of closed convex sets. As described by Theorem 1, the fact that these sets are projections of certain affine sections (obtained by fixing the $\boldsymbol{z}$ variables) of a common convex set M imposes a particular structure to them. As discussed in the introduction, the objective of this paper is to understand the regularity properties of this underlying structure. For "brevity" we will often use the following nomenclature associated to a MICP formulation.

Definition 3. Let $M \subseteq \mathbb{R}^{n+p+d}$ be a closed, convex set that induces a MICP formulation of $S \subseteq \mathbb{R}^n$. We refer to $\operatorname{proj}_z(M) \cap \mathbb{Z}^d$ as the *index set* of the MICP representation, to $C = \operatorname{proj}_z(M)$ as the *relaxation* of the index set or the *relaxed index set* and to $A_{\boldsymbol{z}} = \operatorname{proj}_x(M \cap (\mathbb{R}^{n+p} \times \{\boldsymbol{z}\}))$ as the $\boldsymbol{z}$-projected sets.

Finally, we let d be the MICP dimension of the MICP formulation induced by M, and for an MICP representable set S we let its MICP dimension be the smallest dimension among all its MICP formulations. □

2.1. Simple properties and limits to MICP representability We begin with a lemma stating some operations under which the family of MICP representable sets is closed. We include a proof of this result in Section 5.

Lemma 1. *A finite intersection, Cartesian product or Minkowski sum of MICP (MILP) representable sets is MICP (MILP) representable.*

A basic, albeit very useful, property of MICP representable sets can be obtained by considering the limitations of MICP formulations. One such limit for MICP representability was introduced in [19] and referred to as the *midpoint lemma* as it is based on the following notion of nonconvexity which is based on the midpoints of a set.

DEFINITION 4. Let $w \in \mathbb{N} \cup \{+\infty\}$. We say that a set $S \subseteq \mathbb{R}^n$ is **w-strongly nonconvex**, if there exists a subset $R \subseteq S$ with $|R| = w$ such that for all pairs $\boldsymbol{x}, \boldsymbol{y} \in R$,

$$\frac{\boldsymbol{x}+\boldsymbol{y}}{2} \notin S, \tag{3}$$

that is, an subset of points in S of cardinality w such that the midpoint between any pair is not in S. $\square$

The midpoint lemma from [19] is useful to recognize sets that are not MICP representable no matter how many integer variables are used. However, as the following lemma shows, it can also be extended to recognize sets that are not MICP representable with a given fixed number of integer variables. The proof of this lemma is similar to that of the original midpoint lemma, but for completeness we include a proof in Section 5.

LEMMA 2. *Let $S \subseteq \mathbb{R}^n$. If S is w-strongly nonconvex, then S cannot be MICP representable with MICP-dimension less than $\lceil \log_2(w) \rceil$.*

It is interesting to recognize the corner cases $w = 2$ and $w = +\infty$. The former implies that S is nonconvex and hence we need at least one integer variable. The later corresponds to sets that are not MICP representable and corresponds to the original midpoint lemma from [19].

With regards to finite w we obtain an interesting case when S is a subset of the binary hypercube $\{0,1\}^n$. It is clear that a formulation with binary integer variables requires at least $\lceil \log_2 |S| \rceil$ binary integer variables (e.g. [13, Proposition 1]). However, by taking $R = S$ in Lemma 2 we have that the same lower bound holds if we use unbounded integer variables. That is, using general integer variables instead of binary variables does not provide any advantage in this case. In particular, we can use Lemma 2 to contrast the number of integer variables needed to construct a formulation for a subset of the binary hypercube and those needed to construct a formulation of its convex hull as studied in [1, 11]. For instance let $S = \{x \in \{0,1\}^n : \|x\|_1 \text{ is even}\}$. As noted in [1, 11] the convex hull of S can be described with a single integer variable by noting that

$$\operatorname{conv}(S) = \operatorname{conv}\left(\operatorname{proj}_x\left(M \cap \left(\mathbb{R}^n \times \mathbb{Z}^d\right)\right)\right) \tag{4}$$

for $d = 1$ and $M = \left\{(\boldsymbol{x}, \boldsymbol{z}) \in \mathbb{R}^{n+d} : \sum_{i=1}^n x_i = 2z_1\right\}$. Hence, for any linear function $l : \mathbb{R}^n \to \mathbb{R}$ we have

$$\min\left\{l(\boldsymbol{x}) : \boldsymbol{x} \in S\right\} = \min\left\{l(\boldsymbol{x}) : (\boldsymbol{x}, \boldsymbol{z}) \in M \cap \left(\mathbb{R}^n \times \mathbb{Z}^d\right)\right\}. \tag{5}$$

However, to guarantee (5) holds for an arbitrary function l we need

$$S = \operatorname{proj}_x\left(M \cap \left(\mathbb{R}^n \times \mathbb{Z}^d\right)\right) \tag{6}$$

instead of (4), and using Lemma 2 we can show that (6) can hold only for $d = \Theta(n)$.

Finally, as noted in the introduction, midpoint lemma with $w = +\infty$ was used in [19] to show that the set of prime numbers is not MICP representable, even though it can be represented by a nonconvex polynomial mixed integer programming formulation [14]. Furthermore, this result gives an example of a countable union of MICP representable sets (each composed of a single non-negative integer) that is not MICP representable.

2.2. Bounded MICP representability We can use standard formulations to see that a finite union of compact convex sets has an MICP representation with only binary integer variables (e.g. [3, Theorem], [25, Theorem 1] or [19, Proposition 1]). It is also not hard to see that the requirement of a finite index set for bounded MICP representability is equivalent to allowing only binary variables and hence a bounded MICP representable set is equal to a finite union of convex sets. Classical bounded MIP representability results (e.g. [4, 15] and the references in [25, Section 2]) focus on finite unions of polyhedra or closed convex sets and show that unions of unbounded sets can be represented as long as they share a common recession cone. The general bounded MICP representability clearly also includes certain unions of non-polyhedral convex sets and through the projection $\operatorname{proj}_x(B_{\boldsymbol{z}})$ of convex set $B_{\boldsymbol{z}}$ it also includes certain unions of non-closed convex sets. However, Proposition 1 in [19] shows that the projection of convex sets also allows for a significant extension of the classical results by removing the common recession cone condition. One illustrative way to understand this result is by comparing it to Theorem 1 from [25] which subsumes all formulations for unions of convex sets predating [19]. This theorem is essentially equivalent to the following restricted version of Proposition 1 of [19].

THEOREM 2. *Let $\{C^i\}_{i=1}^k$ be a family of non-empty closed convex sets with recession cones $\{C^i_\infty\}_{i=1}^k$ such that for all $i,j \in [\![k]\!]$ we have $C^i_\infty = C^j_\infty$. Furthermore, for each $i \in [\![k]\!]$ let*

$$\hat{C}^i = \operatorname{cl}(\{(\boldsymbol{x}, z) : \boldsymbol{x}/z \in C^i, z > 0\}) = \overline{\operatorname{cone}}\left(C^i \times \{1\}\right)$$

be the closed conic hull of C^i. Then a formulation for $\bigcup_{i=1}^k C^i$ is given by

$$\boldsymbol{x} = \sum_{i \in [\![k]\!]} \boldsymbol{x}^i, \quad \left(\boldsymbol{x}^i, z_i\right) \in \hat{C}^i \quad \forall i \in [\![k]\!], \quad \sum_{i \in [\![k]\!]} z_i = 1, \; \boldsymbol{z} \in \{0,1\}^k. \tag{7}$$

The way Proposition 1 in [19] bypasses the common recession cone condition is by noting that any union of projections of closed convex sets can be represented as the union of projections of closed convex sets with the same recession cones. This can be formalized in the following straightforward lemma.

LEMMA 3. *Let $S = \bigcup_{i=1}^k S^i \subseteq \mathbb{R}^n$ where for each $i \in [\![k]\!]$ there exists a closed convex set $T^i \subseteq \mathbb{R}^{n+p}$ such that $S^i = \operatorname{proj}_x(T^i)$. Then $S = \bigcup_{i=1}^k \operatorname{proj}_x(C^i)$ where for each $i \in [\![k]\!]$ we have*

$$C^i = \left\{ (\boldsymbol{x}, \boldsymbol{y}, t) \in \mathbb{R}^{n+p+1} : (\boldsymbol{x}, \boldsymbol{y}) \in T^i, \quad \|(\boldsymbol{x}, \boldsymbol{y})\|_2^2 \le t, \quad t \ge 0 \right\}.$$

Furthermore, for all $i,j \in [\![k]\!]$ we have that C^i is a closed convex set and $C^i_\infty = C^j_\infty$. Finally, for all $i \in [\![k]\!]$ we have

$$\hat{C}^i = \left\{ (\boldsymbol{x}, \boldsymbol{y}, t, z) \in \mathbb{R}^{n+p+1} : (\boldsymbol{x}, \boldsymbol{y}, z) \in \hat{T}^i, \quad \|(\boldsymbol{x}, \boldsymbol{y})\|_2^2 \le t \cdot z, \quad t, z \ge 0 \right\}.$$

Using the fact that projection and union commute as operators of sets we obtain the following variant of Proposition 1 in [19] as a direct corollary of Theorem 2 and Lemma 3.

COROLLARY 1. *$S \subseteq \mathbb{R}^n$ is bounded MICP representable if and only if there exist nonempty, closed, convex sets $T_1, T_2, \ldots, T_k \subset \mathbb{R}^{n+p}$ for some $p, k \in \mathbb{N}$ such that $S = \bigcup_{i \in [\![k]\!]} \operatorname{proj}_x T_i$. A formulation of such an S is given by*

$$\boldsymbol{x} = \sum_{i \in [\![k]\!]} \boldsymbol{x}^i, \quad (\boldsymbol{x}^i, \boldsymbol{y}^i, z_i) \in \hat{T}_i \; \forall i \in [\![k]\!], \quad \sum_{i \in [\![k]\!]} z_i = 1, \; \boldsymbol{z} \in \{0,1\}^k, \tag{8a}$$

$$\|\left(\boldsymbol{x}^i, \boldsymbol{y}^i\right)\|_2^2 \le z_i t_i, \quad \forall i \in [\![k]\!], \boldsymbol{t} \ge 0 \tag{8b}$$

where $\hat{T}_i$ is the closed conic hull of T_i, i.e., $\operatorname{cl}(\{(\boldsymbol{x}, \boldsymbol{y}, z) : (\boldsymbol{x}, \boldsymbol{y})/z \in T_i, z > 0\})$.

We obtain Proposition 1 from [19] by noting that (8) remains a valid formulation when (8b) is replaced by

$$||\boldsymbol{x}^i||_2^2 \leq z_i t, \quad \forall i \in [\![k]\!], t \geq 0 \tag{9}$$

and we use a single common t variable. Now, removing variables and simplifying constraints can prevent a common desirable property of MIP formulations that requires the extreme points of its continuous relaxation to satisfy the integrality conditions of the formulation (A condition satisfied by many classical formulations [24]). Formulations satisfying this property are often denoted *ideal* and it can be formalized for general MICP formulations with continuous relaxations possibly containing lines as follows (e.g. see [25]).

Definition 5. A MIP formulation induced by a closed convex set $M \subseteq \mathbb{R}^{n+p+d}$ is *ideal* if and only if for any minimal face F of M we have $\boldsymbol{z} \in \mathbb{Z}^d$ for all $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in F$. □

The following proposition, which we prove in Section 5, shows that a variant of the formulation from Proposition 1 in [19] that only partially simplifies (8b) is ideal.

Proposition 1. *Let $\{T_i\}_{i=1}^k$ be a family of non-empty closed convex sets in $\mathbb{R}^{n+p}$. Then an ideal formulation for $S = \bigcup_{i \in [\![k]\!]} \operatorname{proj}_x T_i$ is given by*

$$\boldsymbol{x} = \sum_{i \in [\![k]\!]} \boldsymbol{x}^i, \quad (\boldsymbol{x}^i, \boldsymbol{y}^i, z_i) \in \hat{T}_i \; \forall i \in [\![k]\!], \quad \sum_{i \in [\![k]\!]} z_i = 1, \; \boldsymbol{z} \in \{0,1\}^k, \tag{10a}$$

$$||\boldsymbol{x}^i||_2^2 \leq z_i t_i, \quad \forall i \in [\![k]\!], \boldsymbol{t} \geq 0. \tag{10b}$$

Unfortunately, we also note that the use of the squared norm in the formulations of this section can result in computational complications in practice (e.g. [10, 18]).

2.3. Closure under finite union Using any of the formulations, such as formulation (10), we can use as $\{T_i\}_{i=1}^k$ an arbitrary family of MICP representable sets to immediately get an MICP representation for a finite union of MICP representable sets. In particular the following is established.

Corollary 2. *A finite union of MICP representable sets is MICP representable.*

Corollary 2 will play a central role in our main results.

2.4. Unbounded MICP representability, first results Classical and recent results on MICP representability with general unbounded integer variables often use the following mathematical notion to represent discrete unboundedness directions.

Definition 6. For a set of integral vectors $Z \subset \mathbb{Z}^d$ we denote the integral cone generated by Z as

$$\operatorname{intcone}(Z) = \left\{ \sum_{i=1}^t \lambda_i \boldsymbol{z}^i : \lambda \in \mathbb{N}^t, \quad \left\{\boldsymbol{z}^i\right\}_{i=1}^t \subseteq Z, \quad t \in \mathbb{N}_{>0} \right\}.$$

We say a set is an integral cone if it is the integral cone generated by some $Z \subset \mathbb{Z}^d$, and if $|Z| < \infty$ we say the integral cone is finitely generated. □

For general MILP representability we have the following result by Jerowslow and Lowe [15].

Theorem 3 (**[15]**). *A set $S \subseteq \mathbb{R}^n$ is rational MILP representable if and only if there exist $t, k \in \mathbb{N}$, integer vectors $\{\boldsymbol{z}^i\}_{i=1}^t \subseteq \mathbb{Z}^n$ and rational polytopes $S_1, S_2, \ldots, S_k \subseteq \mathbb{R}^n$ such that*

$$S = \bigcup_{i \in [\![k]\!]} S_i + \operatorname{intcone}\left(\left\{\boldsymbol{z}^i\right\}_{i=1}^t\right). \tag{11}$$

Characterization (11) does not hold in general for non-polyhedral M. For instance, integer points in a convex cone are integral cones, but they are finitely generated if and only if they are rational polyhedral cones [9]. However, [19] reviews two cases in which (11) holds for non-polyhedral cases. The first one is the following example derived from results in [7].

Example 1. Theorem 6 in [7] can be used to show that for any $\alpha > 0$, $P_\alpha := \{\boldsymbol{x} \in \mathbb{Z}^2 : x_1 x_2 \geq \alpha\}$ satisfies a representation of the form (11) with each polyhedron S_i containing a single integer vector for each $i \in [\![k]\!]$. $\square$

The second one is Theorem 2 from [5] that states that (11) holds when M is the intersection of a rational polyhedron with an ellipsoidal cylinder having a rational recession cone. This last result was extended in [19] to the following proposition. For completeness we include a proof in Section 5 as the proof was not included in the conference proceeding version of [19].

Proposition 2 (**[19, Proposition 2]**). *If M induces an MICP-formulation of S and $M = B + K$ where B is a compact convex set and K is a rational polyhedral cone, then for some $k, t \in \mathbb{N}$ there exist compact convex sets $S_1, S_2, \ldots, S_k$ and integer vectors $\boldsymbol{r}^1, \boldsymbol{r}^2, \ldots, \boldsymbol{r}^t \subseteq \mathbb{Z}^n$ such that*

$$S = \bigcup_{i \in [\![k]\!]} S_i + \operatorname{intcone}(\boldsymbol{r}^1, \boldsymbol{r}^2, \ldots, \boldsymbol{r}^t). \tag{12}$$

The countable unions of convex sets whose MICP representability is considered in Proposition 2 have several regularity properties that are implied by characterization (12). We enumerate some of these properties in the following corollary.

Corollary 3. *If M induces an MICP-formulation of $S \subseteq \mathbb{R}^n$ and $M = B + K$ where B is a compact convex set and K is a rational polyhedral cone, then $S = \bigcup_{\boldsymbol{z} \in C \cap \mathbb{Z}^d} A_{\boldsymbol{z}}$ where C and the elements of $\{A_{\boldsymbol{z}}\}_{\boldsymbol{z} \in C \cap \mathbb{Z}^d}$ are convex sets satisfying the following properties.*

(i) **Finite Number of Shapes:** *There exist a finite $C_0 \subseteq C \cap \mathbb{Z}^d$ such that every set in $\{A_{\boldsymbol{z}}\}_{\boldsymbol{z} \in C \cap \mathbb{Z}^d}$ is a translation of a set in $\{A_{\boldsymbol{z}}\}_{\boldsymbol{z} \in C_0}$.*

(ii) **Locally Finite:** *For every compact set $K \subset \mathbb{R}^n$ we have that $\left|\left\{\boldsymbol{z} \in C \cap \mathbb{Z}^d : A_{\boldsymbol{z}} \cap K \neq \emptyset\right\}\right| < \infty$.*

(iii) **Periodic:** *Either S is bounded or there exist $\boldsymbol{r} \in \mathbb{Z}^n$ such that for any $\boldsymbol{x} \in S$ and $\lambda \in \mathbb{N}$ we have $\boldsymbol{x} + \lambda \boldsymbol{r} \in S$. We say $\boldsymbol{r}$ is a period of S and we let $R(S)$ be the set of all periods of S (Note that $R(S)$ is an integral cone, but is not necessarily finitely generated).*

(iv) **Periodically Generated:** *There exists a finite family of bounded convex sets $\{S_i\}_{i=1}^k$ such that $S = \bigcup_{i=1}^k S_i + R(S)$.*

(v) **Finitely Generated Periodicity:** *$R(S)$ is a finitely generated integral cone.*

Because MICP representability is closed under finite unions (Corollary 2), a conjecture based on the analogy with the MILP results would be that MICP representable sets are equal to finite unions of sets satisfying the properties of Corollary 3. However, the following series of examples show how MICP representable sets may fail each one of these properties even when allowing such finite unions.

Example 2 (Finite Shapes). As expected MICP sets do not have to satisfy the finite shapes condition. A simple example that shows that this can happen even for periodic sets is $S : \{\boldsymbol{x} \in \mathbb{R}^n : \exists z \in \mathbb{Z}_+ \text{ s.t. } \|\boldsymbol{x} - z\boldsymbol{v}\|_2 \leq f(z)\}$ where $\boldsymbol{v} \in \mathbb{Z}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a non-decreasing concave function. We can check that S is MICP-representable, as the constraint $\|\boldsymbol{x} - z\boldsymbol{v}\|_2 \leq f(z)$ is convex in $(\boldsymbol{x}, z) \in \mathbb{R}^{n+1}$, and that $\boldsymbol{v} \in R(S)$. We note though that interestingly S satisfies a restricted version of the finite shape condition that allows scalings of a finite number of shapes. $\square$

Example 3 (Locally Finite). The following example shows how even the locally finite condition can fail for an MICP-representable infinite union of points. Let

$$S := \left\{ \boldsymbol{x} \in \mathbb{R} : \exists (\boldsymbol{y}, \boldsymbol{z}) \in \mathbb{R}^3 \times \mathbb{Z}^2 \text{ s.t. } \begin{array}{rr} \|(z_1, z_1)\|_2 \leq z_2 + 1, & \|(z_2, z_2)\|_2 \leq 2z_1, \\ \|(z_1, z_1)\|_2 \leq y_1, & \|(y_1, y_1)\|_2 \leq 2z_1 \\ & x_1 = y_1 - z_2 \end{array} \right\}$$

$$= \left\{ \boldsymbol{x} \in \mathbb{R} : \exists \boldsymbol{z} \in \mathbb{Z} \text{ s.t. } x_1 = \sqrt{2}z - \left\lfloor \sqrt{2}z \right\rfloor \right\},$$

By Kroneckers Approximation Theorem S is dense in $[0,1]$ and hence does not satisfy the locally finite condition. □

Example 4 (Periodic Generation). Periodic generation implies finite number of shapes. However, the following examples adapted from [19] show that periodic generation can fail for periodic sets given by translations of a single shape. For this let $S := \{\boldsymbol{x} \in \mathbb{N} \times \mathbb{R} : x_1 x_2 \geq 1\}$. For each $z \in \mathbb{N}, z \neq 0$ let $A_z := \{x \in \mathbb{R}^2 : x_1 = z,\, x_2 \geq 1/z\}$ so that $S = \bigcup_{z=1}^{\infty} A_z$. Suppose for contradiction that S is union of m periodically generated sets S^j and let $\{S_{i,j}\}_{i \in [\![k]\!],\, j \in [\![m]\!]}$ be the associated bounded convex sets from these periodically generated set. By convexity of $S_{i,j}$ and there exists $z_0 \in \mathbb{Z}$ such that $\bigcup_{i \in [\![k]\!],\, j \in [\![m]\!]} S_{i,j} \subset \bigcup_{z \in [\![z_0 - 1]\!]} A_z$. Because $\min_{\boldsymbol{x} \in A_{z_0}} x_2 < \min_{\boldsymbol{x} \in A_z} x_2$ for all $z \in [\![z_0 - 1]\!]$ we have that there exists $j \in [\![m]\!]$ and $\boldsymbol{r} \in R(S^j)$ such that the second component of $\boldsymbol{r}$ is strictly negative. However, this implies that there exists $\boldsymbol{x} \in S$ such that $x_2 < 0$ which is a contradiction with the definition of S. We can check that $(1,0) \in R(S)$. We note that S satisfies the finite shape condition only through unboundedness as it contains the set $S' := \{\boldsymbol{x} : x_2 \leq 2\}$ which does not satisfy the finite shape condition. □

Example 5 (Periodicity). The previous example may suggest that unboundedness was the reason that an MICP representable set could satisfy the equal shapes condition through unboundedness but not necessarily be periodic or periodically generated. The following example from [19] shows that periodicity can also fail for unions of integer points, which satisfy the equal shapes condition without the use of unboundedness. For $x \in \mathbb{R}$ let $f(x) = x - \lfloor x \rfloor$. For $\varepsilon > 0$ consider the set

$$K_\varepsilon = \{\boldsymbol{x} \in \mathbb{R}^2 : \|(x_1, x_1)\|_2 \leq x_2 + \varepsilon, \quad \|(x_2, x_2)\|_2 \leq 2x_1 + 2\varepsilon, \quad x_1, x_2 \geq 0\} \tag{13}$$

$$= \{\boldsymbol{x} \in \mathbb{R}^2 : \sqrt{2}x_1 - \varepsilon \leq x_2 \leq \sqrt{2}x_1 + \sqrt{2}\varepsilon, \quad x_1, x_2 \geq 0\} \tag{14}$$

and $S_\varepsilon = \{x_1 \in \mathbb{R} : \exists x_2 \text{ s.t. } (x_1, x_2) \in K_\varepsilon \cap \mathbb{Z}^2\} = \{x \in \mathbb{N} : f(\sqrt{2}x) \notin (\varepsilon, 1 - \sqrt{2}\varepsilon)\}$. Let $\varepsilon_0 < 1/(1+\sqrt{2})$ be rational (e.g. $\varepsilon = 0.4$). Suppose that for some $a, b \in \mathbb{N}, a \geq 1$ it holds $ak + b \in S_{\varepsilon_0}$ for all $k \in \mathbb{N}$. $\emptyset \neq (\varepsilon_0, 1 - \sqrt{2}\varepsilon_0) \subseteq (0,1)$, so by Kronecker's Approximation Theorem we have that there exist $k_0 \in \mathbb{N}$ such that $f(\sqrt{2}(ak_0 + b)) \in (\varepsilon_0, 1 - \sqrt{2}\varepsilon_0)$ which is a contradiction. Therefore the set S_{ε_0} is not periodic. □

Example 6 (Finitely Generated Periodicity). Finally, a simple example of a set that satisfies all properties, but finitely generated periodicity is $S = \{(t, \boldsymbol{x}) \in \mathbb{Z}^{n+1} : \|\boldsymbol{x}\|_2 \leq t\}$. □

2.5. Rational MICP It is apparent that the reason why Examples 3 and 5 do not satisfy the periodicity and locally finite properties is the inclusion of irrational numbers, and more precisely irrational directions of unboundedness, in their definitions. However, this irrationality appears indirectly from the non-polyhedral constraints as the sets can be described without any irrational data. More precisely, as described in [19] for Example 5, we can describe the convex set M that induces a MICP formulation for this set through conic conditions of the form $\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \in K$ where $\boldsymbol{A}$ and $\boldsymbol{b}$ are an appropriately sized rational matrix and rational vector, and K is a specially structured convex cone (in this case a product of Lorentz cones defined as $\mathcal{L}_n := \{(t, \boldsymbol{x}) \in \mathbb{R}^n : ||\boldsymbol{x}||_2 \leq t\}$). Alternatively, we may describe M through polynomial inequalities with rational coefficients.

To resolve these difficulties, [19] introduced a notion of *rational* MICP representability that restricts irrational unbounded directions and ensures that the properties of Corollary 3 are satisfied (up to a union with a finite set) when S is a subset of the natural numbers. This notion concerns unbounded directions of the index set of a MICP representation and [19] showed that, in the context of natural numbers, it is equivalent to S being a finite union of periodic sets. Unfortunately the definition required some awkward technical conditions to deal with the fact that the relaxation of the index set may not be a closed set and hence the directions of unboundedness of the index set could be dependent of the starting integer point (a common issue in geometric studies of MICP; e.g. [7, Section 2.2]). Fortunately, Theorem 6 in Section 3.3 shows the following cleaner, but apparently more restrictive version of rational MICP representability is equivalent to that in [19] for subsets of the natural numbers. Furthermore, the periodicity properties proven in [19] for rational MICP representable subsets of the natural numbers also extend to more complex sets that satisfy the new rational MICP representable definition (e.g. Theorem 7).

Definition 7. We say that an unbounded convex set $C \subseteq \mathbb{R}^d$ is **rationally unbounded** if the image C' of any rational affine mapping of C, is either bounded or there exists $\boldsymbol{r} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ such that $\boldsymbol{x} + \lambda \boldsymbol{r} \in C'$ from any $\boldsymbol{x} \in C'$ and $\lambda \geq 0$. (i.e., $\boldsymbol{r}$ is a recession direction.)

We say that a set S is **rational MICP representable** if it has a MICP representation induced by the set M and with relaxed index set $C = \operatorname{proj}_z(M)$ that is either bounded or rationally unbounded. □

We can check that the relaxed index sets of the formulations used to describe the sets in Examples 3 and 5 are not rationally unbounded. However, it is not clear if there is an alternative MICP representation of either of these sets that does have a rationally unbounded relaxed index set. In Corollary 5 we show that the set from Example 5 indeed fails to be rational MICP representable.

3. Main Results

We now present the statements of our main results. All omitted proofs are included in Section 4.

3.1. Basic properties of rational MICP representable sets

Rational MICP representability is trivially preserved for the basic operations of Cartesian product and Minkowski summation but, it is not clear if they are preserved under unions and intersections. We obtain the result for unions through the following variant of the bounded MICP formulations from Section 2.4.

Lemma 4. *Let $S_1, S_2 \subseteq \mathbb{R}^n$, $p_1, p_2, d_1, d_2 \in \mathbb{N}$, $M_1 \subseteq \mathbb{R}^{n+p_1+d_1}$ and $M_2 \subseteq \mathbb{R}^{n+p_2+d_2}$ be such that M_i induces a formulation of S_i for each $i \in [\![2]\!]$. Then a formulation of $S_1 \cup S_2$ is given by*

$$(\boldsymbol{x}^1, \boldsymbol{y}^1, \boldsymbol{z}^1) \in M_1, \tag{15a}$$
$$(\boldsymbol{x}^2, \boldsymbol{y}^2, \boldsymbol{z}^2) \in M_2, \tag{15b}$$
$$||\boldsymbol{x} - \boldsymbol{x}^1||_2^2 \leq t z', \tag{15c}$$
$$||\boldsymbol{x} - \boldsymbol{x}^2||_2^2 \leq t(1 - z'), \tag{15d}$$
$$t \geq 0, \tag{15e}$$
$$(\boldsymbol{z}^1, \boldsymbol{z}^1, z') \in \{0,1\}^{d_1+d_2+1}. \tag{15f}$$

Furthermore, if M_1 and M_2 have rationally unbounded relaxed index sets, then $S_1 \cup S_2$ is rational MICP representable.

Corollary 4. *The finite union, Cartesian product or Minkowski sum of non-empty rational MICP representable sets is rational MICP representable. In particular, the finite union of non-empty rational MILP representable sets is rational MICP representable.*

We can use Example 5 to see that the intersection of rational MICP representable sets is not necessarily rational MICP representable. We give a short proof of this in Corollary 5 of Section 3.3.

3.2. Representability of compact sets

The issues with Examples 3 and 5 stem from the irregularities in the index set and not from the $\boldsymbol{z}$-projected sets. The following lemma shows that for closed MICP representable sets we can assume that the $\boldsymbol{z}$-projected sets of its representation inherit the regularity of being closed.

Lemma 5. *Suppose $S \subseteq \mathbb{R}^n$ is a closed set with MICP representation induced by M. Let $A_{\boldsymbol{z}} = \operatorname{proj}_x(M \cap (\mathbb{R}^{n+p} \times \{\boldsymbol{z}\}))$ and $C = \operatorname{proj}_z(M)$. Then $S = \bigcup_{\boldsymbol{z} \in (C \cap \mathbb{Z}^d)} \operatorname{cl}(A_{\boldsymbol{z}})$.*

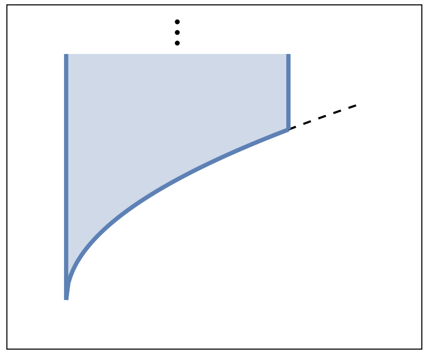

FIGURE 1. The set $\{(x,y) : y \geq \sqrt{x}, 0 \leq x \leq 3\}$ is not rational MICP representable (with an additional technical condition) because it is not a finite union of convex sets; see Theorem 5.

Proof By definition, $S = \bigcup_{\boldsymbol{z}\in(C\cap\mathbb{Z}^d)} A_{\boldsymbol{z}}$, so $S \subseteq \bigcup_{\boldsymbol{z}\in(C\cap\mathbb{Z}^d)} \mathrm{cl}(A_z)$. Fix $\boldsymbol{z} \in C\cap\mathbb{Z}^d$. Since $A_{\boldsymbol{z}} \subseteq S$ and S is closed, it follows that $\mathrm{cl}(A_{\boldsymbol{z}}) \subseteq S$, hence the desired statement holds. $\square$

As illustrated in Examples 3, MICP representable sets with closed $\boldsymbol{z}$-projected sets can still be quite irregular. However, closed sets that are rational MICP representable are indeed quite regular.

THEOREM 4. *Suppose $S \subseteq \mathbb{R}^n$ is a compact set which has a rational MICP representation. Then S is a finite union of compact convex sets.*

While this result is not unexpected, it does provide a simpler alternative to the midpoint lemma to prove that sets are not *rational* MICP representable. For instance, the theorem trivially implies that the set $\{1/n : n = 1, 2, \ldots\} \cup \{0\}$, the annulus $\{\boldsymbol{x} \in \mathbb{R}^n : 1 \leq \|\boldsymbol{x}\|_2 \leq 2\}$, and the set of rank 1 contained in some compact domain are not rational MICP representable[2]. We note that Theorem 4 cannot be used to prove that the set from Examples 3 is not rational MICP representable as it is not closed.

Another direct corollary of Theorem 4 is that if the graph of a continuous function over a compact domain is rational MICP representable, then the function is piecewise linear with finitely many affine pieces. The following theorem shows that this result can also be extended to the epigraph of lower semi-continuous functions that have rational MICP representations with an additional natural regularity condition.

THEOREM 5. *Let R be a compact set and $f : R \to \mathbb{R}$ such that the epigraph $S = \{(x', \boldsymbol{x}) \in \mathbb{R}\times R : x' \geq f(\boldsymbol{x})\}$ is closed and rational MICP representable where there exits an upper bound u on f such that whenever a $\boldsymbol{z}$-projected set contains a point $(x', \boldsymbol{x})$ it also contains $(u, \boldsymbol{x})$. Then S is a finite union of closed convex sets.*

One useful corollary of Theorem 5 is that epigraphs of strictly nonconvex lower semi-continuous functions over a compact domain cannot have rational MICP representations with this additional regularity condition (e.g. see Figure 1). In fact, it follows from Theorem 5 that the epigraph of a lower semi-continuous functions over a compact domain has such regular rational MICP representation if and only if the function is piecewise convex with finitely many pieces.

3.3. Representability of subsets of the natural numbers One of the results in [19] showed that for subsets of the natural numbers, rational MICP representability under a broader definition than that used in Definition 7 is equivalent to rational MILP representability plus finite

[2] As shown in [19], the midpoint lemma can also be used to prove that the last two sets are not MICP representable even without the rationality restriction. The midpoint lemma also can be used for the first set by noting that for any $n, m \in \mathbb{N}$ with $n \neq m$ there is no $k \in \mathbb{N}$ such that $(2^{-n} + 2^{-m})/2 = 1/k$.

unions of points. As noted in Section 2.5, this variant required a technical condition to account for the possibility of the relaxed index set failing to be closed. This condition basically allowed for the exclusion of a finite number of indices from the definition of rationally unbounded. The fact that the equivalence still holds for the simplified version from Definition 7 will follow from the closure of rational MICP representable sets under finite unions (Lemma 4), a property discovered after [19]. The result still uses the fact that rational MILP representable subsets of the natural numbers are precisely those that are periodic and all such periodic sets have a unique period. We formalize this in the follow lemma, and for completeness we include a proof in Section 4 as the proof did not appear in the conference proceeding version of [19].

Lemma 6. *An infinite subset S of the natural numbers is periodic iff it is rational MILP representable. Furthermore, if S is periodic then $R(S)$ is finitely generated by a single point.*

Theorem 6. *Let $S \subseteq \mathbb{N}$ with $|S| = \infty$. Then the following are equivalent:*
(a) S is rational MICP representable.
(b) There exists a finite set S_0 and an infinite periodic set S_1 such that $S = S_0 \cup S_1$.
(c) There exists a finite set S_0 and a rational-MILP-representable set S_1 such that $S = S_0 \cup S_1$.

We do not know if the way we define rational MICP representability is necessary for the finite union property in Theorems 4 and 5. However, the set from Example 5 fails to be periodic even after removing a finite number of elements and hence shows that rational MICP representability in Theorem 6 is necessary for the periodicity property. Finally, Theorem 6 and Example 5 can be used to provide the following simple proof of the non-closure of MICP representability under finite intersections.

Corollary 5. *The intersection of two rational MICP representable sets is not in general rational MICP representable.*

Proof Let $K_1 = \{\boldsymbol{x} \in \mathbb{Z}^2 : \sqrt{2}x_1 - 0.4 \leq x_2, \quad x_1, x_2 \geq 0\}$ and $K_1 = \{\boldsymbol{x} \in \mathbb{Z}^2 : x_2 \leq \sqrt{2}x_1 + 0.4\sqrt{2}, \quad x_1, x_2 \geq 0\}$. We can check that K_1 and K_2 are rational MICP representable. However, from Example 5 and Theorem 6 we have that $\operatorname{proj}_{x_1}(K_1 \cap K_2)$ is not rational MICP representable. The result follows because orthogonal projections preserve rational MICP representability. □

3.4. Representability of piecewise linear functions Theorem 6 can also be extended to the following class of piecewise linear functions.

Definition 8. The continuous function $\mathcal{P} : \mathbb{R}_+ \to \mathbb{R}$ is a $\mathcal{PWL}$-function if it is piecewise linear on with breakpoints only at integer values. If, furthermore, $\mathcal{P}(i) \in \mathbb{Q} \, \forall i \in \mathbb{N}$ then we say $\mathcal{P}(i)$ is a rational $\mathcal{PWL}$-function. □

$\mathcal{PWL}$-functions are uniquely defined by their values at the integers $\{\mathcal{P}(i)\}_{i\in\mathbb{N}}$. We parameterize unit-step segments $conv(\{(i, \mathcal{P}(i)), (i+1, \mathcal{P}(i+1))\})$ of the graph of $\mathcal{P}$ by their starting point $(i, \mathcal{P}(i))$ and slope $\mathcal{P}(i+1) - \mathcal{P}(i)$ by letting $P_{(i,x,c)} := \operatorname{conv}(\{(i,x),(i+1,x+c)\})$. Then the graph of $\mathcal{P}$ is $\bigcup_{i\in\mathbb{N}} P_{(i,\mathcal{P}(i),\mathcal{P}(i+1)-\mathcal{P}(i))}$ and while this object is 2-dimensional its periodic structure is unidimensional and can be described by periodicity of the slopes of the segments.

Lemma 7. *Let S be the graph of a $\mathcal{PWL}$-function $\mathcal{P}$. Then S is periodic if and only if there exists $t \in \mathbb{N} \setminus \{0\}$ such that $\mathcal{P}(\lambda t + i + 1) - \mathcal{P}(\lambda t + i) = \mathcal{P}(i+1) - \mathcal{P}(i)$ for all $i \in [\![t-1]\!]$ and $\lambda \in \mathbb{N}$.*

Similar to subsets of the naturals we have that periodic $\mathcal{PWL}$-functions are precisely those that are rational MILP representable.

Lemma 8. *Let S be the graph of a rational $\mathcal{PWL}$-function $\mathcal{P}$. Then S is periodic iff S is rational MILP representable. Furthermore, we can write down a rational MILP formulation of S where each $\boldsymbol{z}$-projected set is a full segment.*

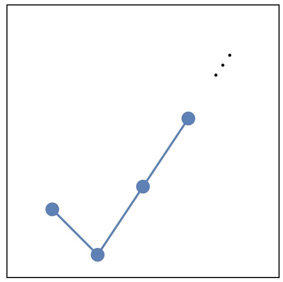

FIGURE 2. The graph of the piecewise linear function depicted above taking values $1, 0, 1.5, 3, 4.5, \ldots$ at $i = 0, 1, 2, 3, 4, \ldots$ respectively is rational MICP representable but not rational MILP representable. The graph is representable if the first segment on the left is excluded.

Lemma 8 is sufficient to separate rational MILP from rational MICP for the graphs of rational $\mathcal{PWL}$-functions. The graph depicted in Figure 2 is not MILP representable because the corresponding function is not periodic. It is rational MICP representable because it is the union of the segment on the left with the remaining segments, which are rational MILP representable. Such unions of rational MILP representable sets correspond precisely to the graphs of rational MICP representable functions under the mild regularity condition that the $\boldsymbol{z}$-projected sets of the representation correspond to complete unit-step line segments. To formally state this result we use the following definition

DEFINITION 9. We say that an infinite subset S of segments is periodic if $\exists t \in \mathbb{N}, t > 0$ such that if for some i, x, c we have $P_{(i,x,c)} \subset S$ then $\forall r \in \mathbb{N}$ $P_{(rt+i,x',c)} \subset S$ for some x'. We say t is a period of S. $\square$

A finite union of non-overlapping infinite periodic subsets of segments is also periodic, as we can consider the product of the periods as the period for the finite union. Furthermore, if a periodic infinite subset S of segments is the graph of some continuous $\mathcal{PWL}$-function $\mathcal{P}$ then $\mathcal{P}$ is periodic.

THEOREM 7. *Let $\mathcal{P}$ be a rational $\mathcal{PWL}$-function. Let S be the graph of $\mathcal{P}$. Then the following are equivalent:*

(a) S is rational MICP representable where each $\boldsymbol{z}$-projected set is a full unit-step segment.
(b) There exists a finite subset of segments S_0 and an infinite periodic subset of segments S_1 such that $S = S_0 \cup S_1$.
(c) There exists a finite subset of segments T_0 and a rational-MILP representable subset of segments T_1 such that $S = T_0 \cup T_1$, where each $\boldsymbol{z}$-projected set in the MILP formulation is a full unit-step segment.

3.5. A simple condition for finite shapes We end the paper with the following simple result, which shows that equal shape regularity holds for certain unions of convex sets with equal volume.

THEOREM 8. *If S has an MICP representation such that all $\boldsymbol{z}$-projected sets have the same volume, then there exists a finite family of convex sets $\{T_i\}_{i=1}^m$ such that all $\boldsymbol{z}$-projected sets are translations of sets in this family.*

The translation-of-finite-family-of-convex-sets property of Theorem 8 also holds if the set of volumes of the $\boldsymbol{z}$-projected sets is finite. Furthermore, for an infinite set of volumes this equal-shape regularity seems to fail only by allowing scalings of the sets. For this reason we conjecture the following variant of the equal shape condition holds for all MICP representable sets.

CONJECTURE 1 **(Finite Similar Shapes)**. *Every MICP representable set $S \subseteq \mathbb{R}^n$ is of the form $S = \bigcup_{\boldsymbol{z} \in C \cap \mathbb{Z}^d} A_{\boldsymbol{z}}$ where $C \subset \mathbb{R}^d$ and $A_{\boldsymbol{z}} \subset \mathbb{R}^d$ for each $\boldsymbol{z} \in C \cap \mathbb{Z}^d$ are projections of closed convex sets and there exist a finite $C_0 \subseteq C \cap \mathbb{Z}^d$ such that for all $\boldsymbol{z} \in C \cap \mathbb{Z}^d$ there exist $\boldsymbol{z}_0 \in C_0$ such that $A_{\boldsymbol{z}}$ is homothetic to (i.e. is a translation and scaling of) $A_{\boldsymbol{z}_0}$.*

4. Proofs of main results

4.1. Basic properties To prove Lemma 4 we use the following straightforward result.

LEMMA 9. *If $C_1 \subseteq \mathbb{R}^{n_1}$ and $C_2 \subseteq \mathbb{R}^{n_2}$ are bounded or rationally unbounded sets, then $C_1 \times C_2$ is bounded or rationally unbounded.*

LEMMA 4. *Let $S_1, S_2 \subseteq \mathbb{R}^n$, $p_1, p_2, d_1, d_2 \in \mathbb{N}$, $M_1 \subseteq \mathbb{R}^{n+p_1+d_1}$ and $M_2 \subseteq \mathbb{R}^{n+p_2+d_2}$ be such that M_i induces a formulation of S_i for each $i \in [\![2]\!]$. Then a formulation of $S_1 \cup S_2$ is given by*

$$(\boldsymbol{x}^1, \boldsymbol{y}^1, \boldsymbol{z}^1) \in M_1, \tag{15a}$$
$$(\boldsymbol{x}^2, \boldsymbol{y}^2, \boldsymbol{z}^2) \in M_2, \tag{15b}$$
$$||\boldsymbol{x} - \boldsymbol{x}^1||_2^2 \le tz', \tag{15c}$$
$$||\boldsymbol{x} - \boldsymbol{x}^2||_2^2 \le t(1 - z'), \tag{15d}$$
$$t \ge 0, \tag{15e}$$
$$(\boldsymbol{z}^1, \boldsymbol{z}^1, z') \in \{0,1\}^{d_1+d_2+1}. \tag{15f}$$

Furthermore, if M_1 and M_2 have rationally unbounded relaxed index sets, then $S_1 \cup S_2$ is rational MICP representable.

Proof Validity of the formulation follows from noting that $z' = 0$ implies $\boldsymbol{x} \in S_1$ and $z' = 1$ implies $\boldsymbol{x} \in S_2$ by validity of the formulations M_i. Rationality follows from Lemma 9 by noting that if $M \subseteq \mathbb{R}^{3n+p_1+p_2+d_1+d_2+2}$ is the continuous relaxation of (15), $C_1 = \operatorname{proj}_z(M_1)$, $C_2 = \operatorname{proj}_z(M_2)$ and $C = \operatorname{proj}_z(M)$ then $C = C_1 \times C_2 \times [0,1]$. $\square$

We now establish a number of properties of rational MICP that we use in our main proofs. In particular, we will follow the same approach we used in [19] of iteratively decomposing a MICP representable set into unions of simple sets and MICP representable sets with strictly smaller MICP dimension. As in [19] we obtain such smaller dimensional sets by relaxing integrality of the MICP representation over an integer unbounded direction obtained from the rational MICP requirement. However, to simplify the proofs and extend their applicability we apply a unimodular transformation of the integer variables that translates the relaxation over the integer unbounded direction into the relaxation of a single integer variable. The following lemmas prove the existence of such transformation and show that both the transformation and the relaxation preserve MICP representability.

LEMMA 10. *Let $\boldsymbol{r} \in \mathbb{Z}^d$ nonzero with $\gcd(r_1, \ldots, r_d) = 1$. Then there exists a $d \times d$ unimodular matrix $\boldsymbol{U} \in \mathbb{Z}^{d\times d}$ with $\boldsymbol{r}$ as the last column.*

Proof Recall [20, p. 189]: A square invertible, integer matrix $\boldsymbol{H} \in \mathbb{Z}^{d\times d}$ is said to be in Hermite normal form if it is 1) lower triangular, 2) has positive entries on the diagonal, and 3) has nonpositive entries off the diagonal with magnitude smaller than the element on the diagonal for the same row. Then if $\boldsymbol{A}$ is a square invertible matrix, there exists a unimodular matrix $\boldsymbol{U}$ such that $\boldsymbol{AU} = \boldsymbol{H}$ for some $\boldsymbol{H}$ in an Hermite normal form. Since $\boldsymbol{A}$ is invertible, we have $\boldsymbol{U} = \boldsymbol{A}^{-1}\boldsymbol{H}$. Since $\boldsymbol{H}$ is lower triangular, the last column of $\boldsymbol{U}$ is a positive integer multiple (H_{dd}) of the last column of $\boldsymbol{A}^{-1}$. We'll use this property to prove the claim.

Now, let $\boldsymbol{B}$ be a rational invertible matrix with $\boldsymbol{r}$ on the last column (which is always possible because we can complete $\boldsymbol{r}$ into a rational basis of $\mathbb{R}^d$). The matrix $\boldsymbol{B}$ is rational so $\boldsymbol{B}^{-1}$ is as well. Let $q \in \mathbb{N}$ be a positive number such that $q\boldsymbol{B}^{-1}$ has all integer entries and consider the decomposition such that $(q\boldsymbol{B}^{-1})\boldsymbol{U} = \boldsymbol{H}$, i.e., $\boldsymbol{U} = \frac{1}{q}\boldsymbol{BH}$.

We see via this decomposition that there exists a unimodular matrix $\boldsymbol{U}$ with last column equal to the vector $(H_{dd}/q)r$. The unique solution to the system $\boldsymbol{Ux} = \boldsymbol{r}$ is the vector $(q/H_{dd})\boldsymbol{e}(d)$, which must be integral since $\boldsymbol{U}$ is unimodular. Therefore (q/H_{dd}) is a positive integer. Entries of unimodular matrices must be integral, so we require $(H_{dd}/q)\boldsymbol{r} \in \mathbb{Z}^d$. Since $\gcd(r_1, \ldots, r_d) = 1$ by assumption, we must have that H_{dd}/q is also a positive integer. It follows that $H_{dd}/q = 1$. $\square$

Lemma 11. *Suppose $S \subseteq \mathbb{R}^n$ has a rational MICP representation induced by M with MICP dimension d, and let $\mathcal{R}: \mathbb{R}^d \to \mathbb{R}^d$ be a rational invertible affine transformation. Then the set $S' \subseteq \mathbb{R}^n$ defined by*

$$\boldsymbol{x} \in S' \text{ iff } \exists \boldsymbol{y} \in \mathbb{R}^p, \boldsymbol{z} \in \mathbb{Z}^d \text{ such that } (\boldsymbol{x}, \boldsymbol{y}, \mathcal{R}(\boldsymbol{z})) \in M, \tag{16}$$

is rational MICP representable with MICP dimension at most d. Furthermore, if $\mathcal{R}$ maps integers to integers, i.e., $\mathcal{R}(\mathbb{Z}^d) \subseteq \mathbb{Z}^d$, then $S' \subseteq S$, and if the inverse transformation maps integers to integers, i.e., $\mathcal{R}^{-1}(\mathbb{Z}^d) \subseteq \mathbb{Z}^d$ then $S \subseteq S'$.

Proof Define the extended affine transformation $\mathcal{R}_{\text{ext}}: \mathbb{R}^{n+p+d} \to \mathbb{R}^{n+p+d}$ by $\mathcal{R}_{\text{ext}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (\boldsymbol{x}, \boldsymbol{y}, \mathcal{R}(\boldsymbol{z}))$. Let $M' = \mathcal{R}_{\text{ext}}^{-1}(M)$ be the image of M under $\mathcal{R}_{\text{ext}}^{-1}$. We claim that M' induces a rational MICP formulation of S'. Note $(\boldsymbol{x}, \boldsymbol{y}, \mathcal{R}(\boldsymbol{z})) \in M$ iff $\mathcal{R}_{\text{ext}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in M$ iff $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in M'$, so

$$\boldsymbol{x} \in S' \text{ iff } \exists \boldsymbol{y} \in \mathbb{R}^p, \boldsymbol{z} \in \mathbb{Z}^d \text{ such that } (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in M'. \tag{17}$$

Also, M' is convex since it is the image of a convex set under an affine transformation. Finally, $\text{proj}_z(M')$ is the image under the rational affine mapping $\mathcal{R}^{-1}$ of $\text{proj}_z(M)$, so $\text{proj}_z(M')$ is either bounded or rationally unbounded. The set M' is closed because M is closed and $\mathcal{R}$ is invertible.

For the inclusion statements, suppose $\boldsymbol{x} \in S'$ with corresponding $\boldsymbol{y} \in \mathbb{R}^p, \boldsymbol{z} \in \mathbb{Z}^d$ such that $(\boldsymbol{x}, \boldsymbol{y}, \mathcal{R}(\boldsymbol{z})) \in M$. If $\mathcal{R}(\boldsymbol{z}) \in \mathbb{Z}^d$ then $\boldsymbol{x} \in S$. Suppose now $\boldsymbol{x} \in S$ with corresponding $\boldsymbol{y} \in \mathbb{R}^p, \boldsymbol{z} \in \mathbb{Z}^d$ such that $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in M$. If $\mathcal{R}^{-1}(\boldsymbol{z}) \in \mathbb{Z}^d$ then $\boldsymbol{x} \in S'$. $\square$

Lemma 12. *Suppose $S \subseteq \mathbb{R}^n$ has a rational MICP representation induced by M with MICP dimension d. Then the set $S' \subseteq \mathbb{R}^n$ defined by*

$$\boldsymbol{x} \in S' \text{ iff } \exists \boldsymbol{y} \in \mathbb{R}^p, z_0 \in \mathbb{R}, \bar{\boldsymbol{z}} \in \mathbb{Z}^{d-1} \text{ such that } (\boldsymbol{x}, \boldsymbol{y}, z_0, \bar{\boldsymbol{z}}) \in M, \tag{18}$$

is rational MICP representable with MICP dimension at most $d-1$.

Proof The S' can be called a relaxation of S since it is derived by relaxing the integrality restriction on the variable z_0, so we note that $S \subseteq S'$. Let C be the projection of M onto the last d variables and C' the projection of M onto the last $d-1$ variables. We need only show that C' is either bounded or rationally unbounded. This follows since C' is the image of C under the rational linear mapping that discards the first dimension and acts as an identity on the remaining dimensions. Note that the set M does not change in the definition of S'. $\square$

To understand the effect of the integer variables or integer unbounded directions we establish some basic properties of the support function (cf. [12, Chapter C]) of the z-projected sets $A_{\boldsymbol{z}}$ as a function of the index $\boldsymbol{z}$. For that we will use the following simple convex analysis result in which following Rockafellar [22] we denote by $\text{aff}(\cdot)$, $\text{int}(\cdot)$ and $\text{relint}(\cdot)$ the affine hull, the interior and relative interior of a set, by $\text{dom}(\cdot)$ and $\partial\cdot$ the domain and subdifferential of a function.

Lemma 13. *Let $C \subseteq \mathbb{R}^d$ be a convex set, $h: C \to \mathbb{R}$ a nonpositive convex function and $(\boldsymbol{x}^i)_{i \in [\![k]\!]} \subset C$ such that $h(\boldsymbol{x}^1) = 0$ and $x^1 \in \text{relint}\left(\text{aff}\left((\boldsymbol{x}^i)_{i \in [\![k]\!]}\right) \cap C\right)$. Then $h(\boldsymbol{x}) = 0$ for all $x \in \text{aff}\left((\boldsymbol{x}^i)_{i \in [\![k]\!]}\right) \cap C$.*

Proof After an affine transformation we may assume without loss of generality that $\text{aff}\left((\boldsymbol{x}^i)_{i \in [\![k]\!]}\right) = \mathbb{R}^d$. Let $\bar{h}: \mathbb{R}^d \to \mathbb{R} \cap \{\infty\}$ so that $\bar{h}(\boldsymbol{x}) = h(\boldsymbol{x})$ for all $\boldsymbol{x} \in C$ and $\bar{h}(\boldsymbol{x}) = \infty$ otherwise. We have that $\boldsymbol{x}^1 \in \text{int}\left(\text{dom}\left(\bar{h}\right)\right) = \text{int}\left(C\right)$ and hence $\partial \bar{h}\left(\boldsymbol{x}^1\right)$ is nonempty and bounded [22]. If there exist $\boldsymbol{u} \in \partial \bar{h}\left(x^1\right) \setminus \{0\}$ then for sufficiently small $\varepsilon > 0$ we have $\boldsymbol{x}^1 + \varepsilon \boldsymbol{u} \in \text{int}\left(C\right)$ and $0 \geq \bar{h}\left(\boldsymbol{x}^1 + \varepsilon \boldsymbol{u}\right) \geq \bar{h}\left(\boldsymbol{x}^1\right) + \varepsilon ||\boldsymbol{u}||_2 > 0$, which is a contradiction. Hence $\partial \bar{h}\left(\boldsymbol{x}^1\right) = \{\boldsymbol{0}\}$ so $h\left(\boldsymbol{x}\right) = \bar{h}\left(\boldsymbol{x}\right) \geq \bar{h}\left(\boldsymbol{x}^1\right) = 0$ for all $\boldsymbol{x} \in C$. $\square$

LEMMA 14. *Let M be an MICP formulation for $S \subseteq \mathbb{R}^n$, $C = \operatorname{proj}_z(M)$ and $A_{\boldsymbol{z}} = \operatorname{proj}_x(M \cap (\mathbb{R}^{n+p} \times \{\boldsymbol{z}\}))$. Then for any $\boldsymbol{c} \in \mathbb{R}^n$, the function $g_{\boldsymbol{c}}(\boldsymbol{z}) : C \to \mathbb{R} \cup \{\infty\}$ defined by $g_{\boldsymbol{c}}(\boldsymbol{z}) = \sup\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{x} \in A_{\boldsymbol{z}}\}$ is concave and the function $f_{\boldsymbol{c}}(\boldsymbol{z}) : C \to \mathbb{R} \cup \{\infty\}$ defined by $f_{\boldsymbol{c}}(\boldsymbol{z}) = \inf\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{x} \in A_{\boldsymbol{z}}\}$ is convex*

Proof Note $A_{\boldsymbol{z}}$ is nonempty for $\boldsymbol{z} \in C$, so $g_{\boldsymbol{c}}(\boldsymbol{z})$ is well defined. Choose any two $\boldsymbol{z}^1, \boldsymbol{z}^2 \in C$ and $\lambda \in [0,1]$. We will show $g_{\boldsymbol{c}}(\lambda \boldsymbol{z}^1 + (1-\lambda)\boldsymbol{z}^2) \geq \lambda g_{\boldsymbol{c}}(\boldsymbol{z}^1) + (1-\lambda) g_{\boldsymbol{c}}(\boldsymbol{z}^2)$. Let $\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3, \cdots$ and $\boldsymbol{y}^1, \boldsymbol{y}^2, \boldsymbol{y}^3, \cdots$ be sequences contained in $A_{\boldsymbol{z}^1}$ and $A_{\boldsymbol{z}^2}$ respectively such that $\lim_{i\to\infty} \boldsymbol{c}^T\boldsymbol{x}^i = g_{\boldsymbol{c}}(\boldsymbol{z}^1)$ and $\lim_{i\to\infty} \boldsymbol{c}^T\boldsymbol{y}^i = g_{\boldsymbol{c}}(\boldsymbol{z}^2)$. Since M is convex, it follows that for each i, $\lambda\boldsymbol{x}^i + (1-\lambda)\boldsymbol{x}^i \in A_{\lambda\boldsymbol{z}^1 + (1-\lambda)\boldsymbol{z}^2}$, so

$$g_{\boldsymbol{c}}(\lambda \boldsymbol{z}^1 + (1-\lambda)\boldsymbol{z}^2) \geq \lim_{i\to\infty} \boldsymbol{c}^T(\lambda\boldsymbol{x}^i + (1-\lambda)\boldsymbol{y}^i) = \lambda g_{\boldsymbol{c}}(\boldsymbol{z}^1) + (1-\lambda) g_{\boldsymbol{c}}(\boldsymbol{z}^2). \tag{19}$$

The result for f is analogous. □

As a first example of the use of Lemma 14 we can combine it with the following simple convex analysis result Lemma 15 to show Lemma 16 according to which all unbounded integer directions in the index set of a rational MICP representation of a bounded set can be relaxed.

LEMMA 15. *Let $l \in \mathbb{R}$ and let $f : [l, \infty) \to \mathbb{R} \cup \{-\infty\}$ be an extended-value concave function. If $\exists x > x' \in [l, \infty)$ such that $f(x) < f(x')$ then $\lim_{x\to\infty} f(x) = -\infty$.*

Proof Look at the set of supergradients at x, which is in the relative interior of the domain. A supergradient with zero or positive slope would contradict $f(x) < f(x')$, so there has to be a supergradient with negative slope which forces $f(x)$ to $-\infty$ as $x \to \infty$. □

LEMMA 16. *Let $S \subseteq \mathbb{R}^n$ be a set which has an MICP representation induced by M. Let $C = \operatorname{proj}_z(M)$ and $A_{\boldsymbol{z}} = \operatorname{proj}_x(M \cap (\mathbb{R}^{n+p} \times \{\boldsymbol{z}\}))$. Let $\boldsymbol{r} \in \mathbb{Z}^d$ and $\boldsymbol{z} \in C \cap \mathbb{Z}^d$. If the set $\ell_{\boldsymbol{z}} = \{\boldsymbol{z} + \lambda\boldsymbol{r} : \lambda \in \mathbb{R}\} \cap C$ is unbounded and $\exists$ a bounded set $T \subset \mathbb{R}^n$ and λ' such that $\forall \lambda \geq \lambda'$ with $\lambda \in \mathbb{Z}$, we have that $A_{\boldsymbol{z}+\lambda\boldsymbol{r}} \subseteq T$, i.e., the $\boldsymbol{z}$-projected sets along $\ell_{\boldsymbol{z}}$ at integer $\boldsymbol{z}$ points are eventually contained in a bounded region, then $\bigcup_{\boldsymbol{z}' \in \ell_{\boldsymbol{z}}} \operatorname{cl}(A_{\boldsymbol{z}'}) = \bigcup_{\boldsymbol{z}' \in \ell_{\boldsymbol{z}} \cap \mathbb{Z}^d} \operatorname{cl}(A_{\boldsymbol{z}'})$, i.e., integrality can be relaxed along this ray modulo closure.*

Proof We know that $\boldsymbol{z} + \lambda\boldsymbol{r} \in C$ for all $\lambda \geq 0$. Fix $\boldsymbol{c} \in \mathbb{R}^n$ and let $g_{\boldsymbol{c}}(\boldsymbol{z}) = \sup\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{x} \in A_{\boldsymbol{z}}\}$. Recall from Lemma 14 that $g_{\boldsymbol{c}}$ is concave over its domain C. Note that since T is bounded, there exist finite bounds α and β (depending on $\boldsymbol{c}$ but not λ, $\boldsymbol{z}$, or $\boldsymbol{r}$) such that $\alpha \leq g_{\boldsymbol{c}}(\boldsymbol{z} + \lambda\boldsymbol{r}) \leq \beta$ whenever $A_{\boldsymbol{z}} \subseteq T$, i.e., whenever $\lambda \geq \lambda'$ and $\lambda \in \mathbb{Z}$. We now claim that $g_{\boldsymbol{c}}(\boldsymbol{z} + \lambda\boldsymbol{r})$ is nondecreasing as a function of λ. If it strictly decreases anywhere, then $\lim_{\lambda\to\infty} g_{\boldsymbol{c}}(\boldsymbol{z} + \lambda\boldsymbol{r}) = -\infty$ by Lemma 15, which leads to a contradiction with the lower bound.

Since the choice of $\boldsymbol{c}$ was arbitrary, it follows that for any $\lambda_1 \leq \lambda_2$ such that $\boldsymbol{z} + \lambda_1\boldsymbol{r}, \boldsymbol{z} + \lambda_2\boldsymbol{r} \in \ell_{\boldsymbol{z}}, g_{\boldsymbol{c}}(\boldsymbol{z} + \lambda_1\boldsymbol{r}) \leq g_{\boldsymbol{c}}(\boldsymbol{z} + \lambda_2\boldsymbol{r}) \forall \boldsymbol{c} \in \mathbb{R}^n$. Seen as a function of $\boldsymbol{c}$, $g_{\boldsymbol{c}}(\boldsymbol{z})$ is the support function of $A_{\boldsymbol{z}}$, so it follows that $\operatorname{cl}(A_{\boldsymbol{z}+\lambda_1\boldsymbol{r}}) \subseteq \operatorname{cl}(A_{\boldsymbol{z}+\lambda_2\boldsymbol{r}})$ [12, p. 225]. The desired claim (in the nontrivial $\subseteq$ direction) follows by noting that for any $\boldsymbol{z}' \in \ell_{\boldsymbol{z}}$, there exists $\boldsymbol{z}'' \in \ell_{\boldsymbol{z}} \cap \mathbb{Z}^d$ such that $\operatorname{cl}(A_{\boldsymbol{z}'}) \subseteq \operatorname{cl}(A_{\boldsymbol{z}''})$. □

COROLLARY 6. *Suppose $S \subseteq \mathbb{R}^n$ is a bounded set which has an MICP representation induced by M. Let $C = \operatorname{proj}_z(M)$ and $A_{\boldsymbol{z}} = \operatorname{proj}_x(M \cap (\mathbb{R}^{n+p} \times \{\boldsymbol{z}\}))$. Let $\boldsymbol{r} \in \mathbb{Z}^d$ and $\boldsymbol{z} \in C \cap \mathbb{Z}^d$. If the set $\ell_{\boldsymbol{z}} = \{\boldsymbol{z} + \lambda\boldsymbol{r} : \lambda \in \mathbb{R}\} \cap C$ is unbounded then $\bigcup_{\boldsymbol{z}' \in \ell_{\boldsymbol{z}}} \operatorname{cl}(A_{\boldsymbol{z}'}) = \bigcup_{\boldsymbol{z}' \in \ell_{\boldsymbol{z}} \cap \mathbb{Z}^d} \operatorname{cl}(A_{\boldsymbol{z}'})$, i.e., integrality can be relaxed along this ray modulo closure.*

Proof Take $T = S$ in Lemma 16, since $A_{\boldsymbol{z}} \subseteq S, \forall \boldsymbol{z} \in C \cap \mathbb{Z}^d$. □

4.2. Representability of compact sets

THEOREM 4. *Suppose $S \subseteq \mathbb{R}^n$ is a compact set which has a rational MICP representation. Then S is a finite union of compact convex sets.*

Proof Let $M \subseteq \mathbb{R}^{n+p+d}$ be a convex set that induces a rational MICP formulation of S. Let $C = \mathrm{proj}_z(M)$ be the convex set which is bounded or rationally unbounded by assumption. Let $A_{\boldsymbol{z}} = \mathrm{proj}_x(M \cap (\mathbb{R}^{n+p} \times \{\boldsymbol{z}\}))$, so that, by Lemma 5, $S = \bigcup_{\boldsymbol{z} \in (C \cap \mathbb{Z}^d)} \mathrm{cl}(A_{\boldsymbol{z}})$. If C is bounded then S is precisely a finite union of compact convex sets. Suppose then that C is unbounded. Since C is rationally unbounded, let $\boldsymbol{r} \in \mathbb{Z}^d$ such that the ray $\boldsymbol{z} + \lambda \boldsymbol{r} \in C, \forall \boldsymbol{z} \in C, \lambda \geq 0$.

We prove now that without loss of generality, we may assume $\boldsymbol{r} = \boldsymbol{e}(1)$. First, rescale $\boldsymbol{r}$ if necessary so that $\gcd(\boldsymbol{r}) = 1$. Then there exists a $d \times d$ unimodular matrix $\boldsymbol{U}$ with $\boldsymbol{r}$ as the first column via Lemma 10 (we can freely permute columns of a unimodular matrix). The columns of the unimodular matrix $\boldsymbol{U}$ form a lattice basis of $\mathbb{Z}^d$, so the matrix can be thought of as mapping from integers expressed in the nonstandard basis (with $\boldsymbol{r}$) to the standard basis. We have that $\boldsymbol{x} \in S$ iff $\exists \boldsymbol{y} \in \mathbb{R}^p, \boldsymbol{z} \in \mathbb{Z}^d$ such that $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{U}\boldsymbol{z}) \in M$. Following Lemma 11, we can redefine M to be M' such that $\boldsymbol{x} \in S$ iff $\exists \boldsymbol{y} \in \mathbb{R}^p, \boldsymbol{z} \in \mathbb{Z}^d$ such that $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in M'$ and $\boldsymbol{e}(1)$ is a ray of $C' := \mathrm{proj}_z(M')$.

Consider the set Q defined by

$$\boldsymbol{x} \in Q \text{ iff } \exists\, \boldsymbol{y} \in \mathbb{R}^p, z_0 \in \mathbb{R}, \bar{\boldsymbol{z}} \in \mathbb{Z}^{d-1} \text{ such that } (\boldsymbol{x}, \boldsymbol{y}, z_0, \bar{\boldsymbol{z}}) \in M. \tag{20}$$

Note that formulation (20) is obtained solely by relaxing the integrality constraint on the first component of the d integer-constrained variables, so clearly $S \subseteq Q$ and Q is rational MICP representable with one fewer integer dimension via Lemma 12.

Corollary 6 together with the trivial observation that for any $z \in C$, $A_z \subseteq \mathrm{cl}(A_{\boldsymbol{z}})$ imply that no new points are created in the relaxation, i.e., $Q = S$. We now have a formulation for S with one fewer dimension and can repeat our argument until C is bounded (it is trivially bounded when $d = 0$). $\square$

THEOREM 5. *Let R be a compact set and $f : R \to \mathbb{R}$ such that the epigraph $S = \{(x', \boldsymbol{x}) \in \mathbb{R} \times R : x' \geq f(\boldsymbol{x})\}$ is closed and rational MICP representable where there exits an upper bound u on f such that whenever a $\boldsymbol{z}$-projected set contains a point $(x', \boldsymbol{x})$ it also contains $(u, \boldsymbol{x})$. Then S is a finite union of closed convex sets.*

Proof Let $M \subseteq \mathbb{R}^{n+p+d}$ be a convex set that induces a rational MICP formulation of S. Let $C = \mathrm{proj}_z(M)$ be the convex set which is bounded or rationally unbounded by assumption. Let $A_{\boldsymbol{z}} = \mathrm{proj}_x(M \cap (\mathbb{R}^{n+p} \times \{\boldsymbol{z}\}))$. So that, by Lemma 5, $S = \bigcup_{\boldsymbol{z} \in (C \cap \mathbb{Z}^d)} \mathrm{cl}(A_{\boldsymbol{z}})$. If C is bounded then S is precisely a finite union of closed convex sets. Suppose then that C is unbounded. Since C is rationally unbounded, let $\boldsymbol{r} \in \mathbb{Z}^d$ such that the ray $\boldsymbol{z} + \lambda \boldsymbol{r} \in C, \forall \boldsymbol{z} \in C, \lambda \geq 0$. By the same argument as in Theorem 4, we may assume $\boldsymbol{r} = \boldsymbol{e}(1)$ without loss of generality, so the ray $\ell_{\boldsymbol{z}} := \{\boldsymbol{z} + \lambda \boldsymbol{e}(1) : \lambda \geq 0\}$ is contained in C for all $\boldsymbol{z} \in C$.

Consider the set Q defined by

$$(x', \boldsymbol{x}) \in Q \text{ iff } \exists\, \boldsymbol{y} \in \mathbb{R}^p, z_0 \in \mathbb{R}, \bar{\boldsymbol{z}} \in \mathbb{Z}^{d-1} \text{ such that } (x', \boldsymbol{x}, \boldsymbol{y}, z_0, \bar{\boldsymbol{z}}) \in M. \tag{21}$$

Note that formulation (21) is obtained solely by relaxing the integrality constraint on the first component of the d integer-constrained variables, so clearly $S \subseteq Q$ and Q is rational MICP representable with one fewer integer dimension via Lemma 12.

Let $A'_{\boldsymbol{z}}$ be the projection of $A_{\boldsymbol{z}}$ onto the last $n-1$ variables, i.e., onto the domain R of f. Let $(x', \boldsymbol{x}) \in Q$ with corresponding $\boldsymbol{y} \in \mathbb{R}^p, z_0 \in \mathbb{R}, \bar{\boldsymbol{z}} \in \mathbb{Z}^{d-1}$ such that $(x', \boldsymbol{x}, \boldsymbol{y}, z_0, \bar{\boldsymbol{z}}) \in M$. As in Lemma 16, we can argue because R is compact that along a ray we have the containment $\mathrm{cl}(A'_{(z_0, \bar{\boldsymbol{z}})}) \subseteq \mathrm{cl}(A'_{(z_0+\lambda, \bar{\boldsymbol{z}})})$ for any $\lambda \geq 0$. So in particular there exists $x'' \in \mathbb{R}$ such that $(x'', \boldsymbol{x}) \in S$, i.e., $\boldsymbol{x} \in R$, because $\mathrm{cl}(A_{(\lceil z_0 \rceil, \bar{\boldsymbol{z}})}) \subseteq S$. In terms of having points in Q that are not in S, we only need to worry about the case $x' < x''$ because since S is an epigraph, $x' > x''$ implies $(x', \boldsymbol{x}) \in S$. If $x' < x''$ then the function $h_{\boldsymbol{x}}(\boldsymbol{z}) = \inf\{\beta' : (\beta', \boldsymbol{\beta}) \in A_{\boldsymbol{z}}, \boldsymbol{\beta} = \boldsymbol{x}\}$ increases at some point along the ray $\ell_{(z_0, \bar{\boldsymbol{z}})}$. By an extension of Lemma 14 we see that for any $\boldsymbol{x}$ fixed, $h_{\boldsymbol{x}}(\boldsymbol{z})$ is convex in $\boldsymbol{z} \in C$. Convexity combined with increasing at some point in $\ell_{(z_0, \bar{\boldsymbol{z}})}$ implies that $\lim_{\lambda \to \infty} h_{\boldsymbol{x}}(z_0 + \lambda, \bar{\boldsymbol{z}}) = \infty$ by Lemma 15. However, this contradicts our assumption on the $\boldsymbol{z}$-projected set of the formulation which implies $h_{\boldsymbol{x}}(z_0 + \lambda, \bar{\boldsymbol{z}}) \leq u \, \forall \lambda \geq 0$ with $z_0 + \lambda \in \mathbb{Z}$. From this discussion we conclude that no new points are created in the relaxation, i.e., $Q = S$. We now have a formulation for S with one fewer dimension and can repeat our argument until C is bounded (it is trivially bounded when $d = 0$). $\square$

4.3. Representability of subsets of the natural numbers

Lemma 6. *An infinite subset S of the natural numbers is periodic iff it is rational MILP representable. Furthermore, if S is periodic then $R(S)$ is finitely generated by a single point.*

Proof ($\Rightarrow$): Suppose S is periodic with period t. We will show that there is a finite set of integers S_0 such that $S = S_0 + \mathrm{intcone}(t)$, because the right-hand side is rational MILP representable. For every $i \in [\![t-1]\!]$ either there exists a unique minimal $r_i \in \mathbb{N}$ such that $i + r_i t \in S$ or $i + rt \notin S \, \forall r \in \mathbb{N}$. Supposing r_i exists, then $i + (r_i + r')t \in S \, \forall r' \in \mathbb{N}$ because S is periodic, and all integers in S with remainder i modulo t are generated in this manner. Define S_0 be the collection of all such $i + r_i t$ for $i \in [\![t-1]\!]$. Finally, note that by because of representation $S = S_0 + \mathrm{intcone}(t)$ we have that $R(S)$

($\Leftarrow$): Suppose S is rational MICP representable. Then $S = S_0 + \mathrm{intcone}(R)$ for some finite non-empty sets $S_0 \subset \mathbb{N}$ and $R \subset \mathbb{N}$. We have that $\mathrm{intcone}(R) = g \cdot \mathrm{intcone}\,(R')$ where $g := \gcd(R)$ and R' is a finite set of coprime numbers. Furthermore, by Schur's upper bound on Frobenius number (e.g. [21]) there exist a finite set $R_0 \subseteq \mathrm{intcone}\,(R')$ such that

$$\mathrm{intcone}\,(R') = R_0 \cup \{x \in \mathbb{N} : z \geq \alpha_0\} \tag{22}$$

where $\alpha_0 = \max\,(R_0)$. We claim that $\mathrm{intcone}\,(R') = J + \mathrm{intcone}(\beta)$ where $\beta = \prod_{\alpha \in R_0} \alpha \in \mathrm{intcone}\,(R')$ and $J = \{x \in \mathrm{intcone}\,(R') : x \leq 2\beta\} \subseteq \mathrm{intcone}\,(R')$. The right to left containment follows from closure of $\mathrm{intcone}\,(R')$ under addition. For the reverse containment let $x \in \mathrm{intcone}\,(R')$ and let $k \in \mathbb{N}$ be the largest integer such that $x - \beta \cdot k \in \mathrm{intcone}\,(R')$. If $x - \beta \cdot k \notin J$, then $x - \beta \cdot (k+1) > \beta > \alpha_0$ and hence by (22) we have $x - \beta \cdot k \in \mathrm{intcone}\,(R')$, which contradicts the maximality of k. Hence $x - \beta \cdot k \notin J$ and $x \in J + \mathrm{intcone}(\beta)$, which proves the claim. Finally, we have $S = S_0 + g\,(J + \mathrm{intcone}(\beta)) = (S_0 + gJ) + \mathrm{intcone}(g \cdot \beta)$ and hence S is periodic with $R(S)$ generated by $g \cdot \beta$. $\square$

Lemma 17. *Suppose $S \subseteq \mathbb{N}$ is rational MICP representable with MICP dimension d. Then either S is a finite set or there exists $k \in \mathbb{N}$ such that $S = S_0 \cup \bigcup_{i \in [\![k]\!]} S_i$ where S_0 an infinite periodic subset of $\mathbb{N}$, and for each $i \in [\![k]\!]$, S_i is rational MICP representable with MICP-dimension at most $d-1$.*

Proof Let $M \subseteq \mathbb{R}^{1+p+d}$ be a convex set that induces an MICP formulation of S. If S is bounded then we're done. Suppose S is unbounded. Let $C := \mathrm{proj}_z(M)$ be the convex set which is rationally unbounded by assumption. Let $A_{\boldsymbol{z}} = \mathrm{proj}_x(M \cap (\mathbb{R}^{1+p} \times \{\boldsymbol{z}\}))$ which for any $\boldsymbol{z} \in C \cap \mathbb{Z}^d$ by assumption is equal to some element of $\mathbb{N}$. Since C is rationally unbounded, let $\boldsymbol{r} \in \mathbb{Z}^d$ such that the ray $\boldsymbol{z} + \lambda \boldsymbol{r} \in C \, \forall \boldsymbol{z} \in C, \lambda \geq 0$. By the same argument as in Theorem 4, we may assume $\boldsymbol{r} = \boldsymbol{e}(1)$ without loss of generality, so the ray $\ell_{\boldsymbol{z}} := \{\boldsymbol{z} + \lambda \boldsymbol{e}(1) : \lambda \geq 0\}$ is contained in C for all $\boldsymbol{z} \in C$.

Let $\{T_i\}_{i \in [\![2^d]\!]}$ be such that $C \cap \mathbb{Z}^d = \bigcup_{i \in [\![2^d]\!]} T_i$ and $z_j \equiv z'_j \mod 2$ for all $j \in [\![d]\!]$, $i \in [\![2^d]\!]$ and $\boldsymbol{z}, \boldsymbol{z}' \in T_i$. Let $S_i = \bigcup_{\boldsymbol{z} \in T_i} A_{\boldsymbol{z}}$ be the subset of $\mathbb{N}$ generated by T_i. We claim that $\mathbf{S_i}$ **is either an infinite periodic subset of** $\mathbb{N}$ **or is rational MICP representable with MICP dimension**

at most $d-1$**.** This would prove the desired result because there are finitely many S_i sets and finite unions of periodic sets are periodic, so we can take S_0 to be the union of the S_i sets that are periodic.

Define $f(\boldsymbol{z}) := \inf\{x : x \in A_{\boldsymbol{z}}\}$ and $g(\boldsymbol{z}) := \sup\{x : x \in A_{\boldsymbol{z}}\}$ Define $h : C \to \mathbb{R}$ as $h(\boldsymbol{z}) := f(\boldsymbol{z}) - g(\boldsymbol{z})$. The function h is concave, nonnegative, and takes values 0 for $\boldsymbol{z} \in C \cap \mathbb{Z}^d$. For fixed $i \in [\![2^d]\!]$ we have $\frac{\boldsymbol{z}+\boldsymbol{z}'}{2} \in C \cap \mathbb{Z}^d$ and $\ell_{\frac{\boldsymbol{z}+\boldsymbol{z}'}{2}} \subset C$ for any $\boldsymbol{z}, \boldsymbol{z}' \in T_i$. Then $L := \operatorname{conv}\left(\left\{\ell_{\boldsymbol{z}}, \ell_{\frac{\boldsymbol{z}+\boldsymbol{z}'}{2}}, \ell_{\boldsymbol{z}'}\right\}\right)$ is a subset of C which importantly contains integer points in its relative interior. Let $\tilde{\boldsymbol{z}}$ be one such integer point. Then $h(\tilde{\boldsymbol{z}}) = 0$ which implies h must be entirely zero over all L by Lemma 13. This implies $f(\boldsymbol{z}) = g(\boldsymbol{z})$ is both convex and concave on L and so is affine. Then, since the choice of $\boldsymbol{z}, \boldsymbol{z}' \in T_i$ was arbitrary there exist $\boldsymbol{\alpha}^i \in \mathbb{R}^d, \beta_i \in \mathbb{R}$ such that $f(\tilde{\boldsymbol{z}}) = \tilde{\boldsymbol{z}}^T \boldsymbol{\alpha}^i + \beta_i \forall \tilde{\boldsymbol{z}} \in \ell_{\boldsymbol{z}}$ for any $\boldsymbol{z} \in T_i$. Let $s_i = \boldsymbol{e}(1)^T \boldsymbol{\alpha}^i$. We know that f takes nonnegative integer values at integer points along a ray in direction $\boldsymbol{e}(1)$ so s_i in particular must be a nonnegative integer.

For the cases where $\mathbf{s_i \neq 0}$, we claim that the set of numbers S_i is periodic with period s_i. For any $x \in S_i$, $\exists \boldsymbol{z} \in C \cap \mathbb{Z}^d$ such that $A_{\boldsymbol{z}} = \{x\} = \{f(\boldsymbol{z})\}$. Then $\boldsymbol{z} + \boldsymbol{e}(1) \in C \cap \mathbb{Z}^d$ and $A_{\boldsymbol{z}} = \{f(\boldsymbol{z} + \boldsymbol{e}(1))\} = \{x + s_i\}$ so $x + s_i \in S_i$.

For the cases where $\mathbf{s_i = 0}$, we will show that S_i is rational MICP representable with MICP dimension at most $d-1$. Note that S_i is rational MICP representable because it can be obtained by performing an invertible affine transformation on C. That is, let $\tilde{\boldsymbol{z}} \in T_i$, then

$$\boldsymbol{x} \in S_i \text{ iff } \exists \boldsymbol{y} \in \mathbb{R}^p, \boldsymbol{z} \in \mathbb{Z}^d \text{ such that } (\boldsymbol{x}, \boldsymbol{y}, \tilde{\boldsymbol{z}} + 2\boldsymbol{z}) \in M. \tag{23}$$

Let M_i be the convex set that induces the rational MICP representation of S_i via Lemma 11. Consider the set Q_i defined by

$$\boldsymbol{x} \in Q_i \text{ iff } \exists \boldsymbol{y} \in \mathbb{R}^p, z_0 \in \mathbb{R}, \bar{\boldsymbol{z}} \in \mathbb{Z}^{d-1} \text{ such that } (\boldsymbol{x}, \boldsymbol{y}, z_0, \bar{\boldsymbol{z}}) \in M_i. \tag{24}$$

Note that formulation (24) is obtained solely by relaxing the integrality constraint on the first component of the d integer-constrained variables, so clearly $S_i \subseteq Q_i$ and Q_i is rational MICP representable with MICP dimension at most $d-1$ via Lemma 12. We now claim that $S_i = Q_i$. Let $x \in Q_i$ with corresponding $\boldsymbol{y} \in \mathbb{R}^p, z_0 \in \mathbb{R}, \bar{\boldsymbol{z}} \in \mathbb{Z}^{d-1}$ such that $(x, \boldsymbol{y}, z_0, \bar{\boldsymbol{z}}) \in M_i$. Let $C_i = \operatorname{proj}_z(M_i)$ be the projection of M_i onto the last d variables. Note that $\boldsymbol{e}(1)$ remains a recession direction of C_i. Therefore $(z_0 + \lambda, \bar{\boldsymbol{z}}) \in C_i \forall \lambda \geq 0$, and in particular $\exists z_0' \in \mathbb{Z}$ such that $z_0' > z_0$ and $(z_0', \bar{\boldsymbol{z}}) \in C_i$. We know that $f(\boldsymbol{z}) = \boldsymbol{z}^T \boldsymbol{\alpha}^i + \beta_i$ on $\ell_{(z_0', \bar{\boldsymbol{z}})}$ and since $s_i = 0$, the ray starting at $(z_0', \bar{\boldsymbol{z}})$ projects to a single natural number, i.e., $A_{\boldsymbol{z}} = \{\sum_{j \in [\![d-1]\!]} \alpha_{j+1}^i \bar{z}_j + \beta_i\} \forall \boldsymbol{z} \in \ell_{(z_0', \bar{\boldsymbol{z}})}$. Since the $\boldsymbol{z}$-projected sets are eventually bounded on the ray, we may apply Lemma 16 to conclude that $x \in S_i$. $\square$

THEOREM 6. *Let $S \subseteq \mathbb{N}$ with $|S| = \infty$. Then the following are equivalent:*
(a) S is rational MICP representable.
(b) There exists a finite set S_0 and an infinite periodic set S_1 such that $S = S_0 \cup S_1$.
(c) There exists a finite set S_0 and a rational-MILP-representable set S_1 such that $S = S_0 \cup S_1$.

Proof (a) $\Rightarrow$ (b): Repeatedly apply Lemma 17.
(b) $\Rightarrow$ (c): Use Lemma 8 to obtain a rational MILP formulation of S_1.
(c) $\Rightarrow$ (a): Corollary 4. $\square$

4.4. Representability of piecewise linear functions To show Lemmas 7 and 8 we use the following auxiliary lemma.

LEMMA 18. *If $\mathcal{P}$ is a $\mathcal{PWL}$-function, $t \in \mathbb{N} \setminus \{0\}$ and $\mathcal{P}(\lambda t + i + 1) - \mathcal{P}(\lambda t + i) = \mathcal{P}(i+1) - \mathcal{P}(i)$ for all $i \in [\![t-1]\!]$ and $\lambda \in \mathbb{N}$, then $\mathcal{P}(i + \lambda t) = \lambda(\mathcal{P}(t) - \mathcal{P}(0)) + \mathcal{P}(i)$ for all $i \in [\![t-1]\!]$ and $\lambda \in \mathbb{N}$.*

Proof

$$\mathcal{P}(i+\lambda t) = \sum_{k=0}^{i+\lambda t-1} (\mathcal{P}(k+1)-\mathcal{P}(k)) + \mathcal{P}(0) \tag{25}$$
$$= \sum_{k=0}^{\lambda t-1} (\mathcal{P}(k+1)-\mathcal{P}(k)) + \sum_{k=\lambda t}^{i+\lambda t-1} (\mathcal{P}(k+1)-\mathcal{P}(k)) + \mathcal{P}(0) \tag{26}$$
$$= \lambda(\mathcal{P}(t)-\mathcal{P}(0)) + \sum_{k=0}^{i-1} (\mathcal{P}(k+1)-\mathcal{P}(k)) + \mathcal{P}(0) \tag{27}$$
$$= \lambda(\mathcal{P}(t)-\mathcal{P}(0)) + \mathcal{P}(i). \tag{28}$$

□

LEMMA 7. *Let S be the graph of a $\mathcal{PWL}$-function $\mathcal{P}$. Then S is periodic if and only if there exists $t \in \mathbb{N}\setminus\{0\}$ such that $\mathcal{P}(\lambda t+i+1)-\mathcal{P}(\lambda t+i) = \mathcal{P}(i+1)-\mathcal{P}(i)$ for all $i \in [\![t-1]\!]$ and $\lambda \in \mathbb{N}$.*

Proof If $\boldsymbol{r} \in \mathbb{Z}^2$ is a period of $\mathcal{P}$, then

$$P_{(i+\lambda r_1, \mathcal{P}(i)+\lambda r_2, \mathcal{P}(i+1)-\mathcal{P}(i))} = P_{(i+\lambda r_1, \mathcal{P}(i+\lambda r_1), \mathcal{P}(i+\lambda r_1+1)-\mathcal{P}(i+\lambda r_1))}$$

for all $i \in [\![r_1-1]\!]$ and $\lambda \in \mathbb{N}$ and hence the first implication follows by letting $t = r_1$.

For the reverse implication let $\boldsymbol{r} = (t, \mathcal{P}(t)-\mathcal{P}(0))$. Then, for any $j \in \mathbb{N}$ we have that there exist $q \in \mathbb{N}$ and $i \in [\![t-1]\!]$ such that $j = i+qt$. Then for any $\lambda \in \mathbb{N}$ we have

$$\begin{aligned} P_{(j+\lambda r_1, \mathcal{P}(j)+\lambda r_2, \mathcal{P}(j+1)-\mathcal{P}(j))} &= P_{(i+(\lambda+q)t, \mathcal{P}(i+qt)+\lambda(\mathcal{P}(t)-\mathcal{P}(0)), \mathcal{P}(i+qt+1)-\mathcal{P}(i+qt))} \\ &= P_{(i+(\lambda+q)t, \mathcal{P}(i+(\lambda+q)t), \mathcal{P}(i+qt+1)-\mathcal{P}(i+qt))} \\ &= P_{(i+(\lambda+q)t, \mathcal{P}(i+(\lambda+q)t), \mathcal{P}(i+(\lambda+q)t+1)-\mathcal{P}(i+(\lambda+q)t))} \end{aligned}$$

where the second equality follows from Lemma 18 and the third equality follows from the assumption. Then $\boldsymbol{r}$ is a period of $\mathcal{P}$. □

LEMMA 8. *Let S be the graph of a rational $\mathcal{PWL}$-function $\mathcal{P}$. Then S is periodic iff S is rational MILP representable. Furthermore, we can write down a rational MILP formulation of S where each $\boldsymbol{z}$-projected set is a full segment.*

Proof (⇒): We will show that the segments in S are finitely generated. Let t be a period of $\mathcal{P}$. Let $T_0 = \bigcup_{i\in[\![t-1]\!]} P_{(i,\mathcal{P}(i),\mathcal{P}(i+1)-\mathcal{P}(i))}$ be the graph of $\mathcal{P}$ over the interval $[0,t]$. T_0 is a finite union of bounded rational polyhedra, because the endpoints of the segments are rational numbers. Let $\boldsymbol{r} = (t, \mathcal{P}(t)-\mathcal{P}(0))$. Then we claim $S = T_0 + \operatorname{intcone}(\boldsymbol{r})$, the right-hand side being rational MILP representable where each $A_{\boldsymbol{z}}$ is a full segment. To prove the $\supseteq$ direction consider $\tilde{P} := P_{(i,\mathcal{P}(i),s)} + \lambda \boldsymbol{r}$ for some $i \in [\![t-1]\!]$ and $\lambda \in \mathbb{N}$ where $s = \mathcal{P}(i+1)-\mathcal{P}(i)$. $\tilde{P}$ is the segment $P_{(i+\lambda t, \mathcal{P}(i)+\lambda(\mathcal{P}(t)-\mathcal{P}(0)), s)}$. Since $\mathcal{P}$ is periodic, the slope of $\tilde{P}$ matches the slope of $\mathcal{P}$ between $i+\lambda t$ and $i+\lambda t+1$ by definition, so it remains to show that $\mathcal{P}(i)+\lambda(\mathcal{P}(t)-\mathcal{P}(0)) = \mathcal{P}(i+\lambda t)$, which follows from Lemma 18. The $\subseteq$ direction follows by reversing the same argument.

(⇐): Suppose S is rational MILP representable. Then there exist $\boldsymbol{r}^1, \boldsymbol{r}^2, \ldots, \boldsymbol{r}^t \subseteq \mathbb{Z}^n$ and rational polytopes $S_1, S_2, \ldots, S_k$ such that

$$S = \bigcup_{i\in[\![k]\!]} S_i + \operatorname{intcone}(\boldsymbol{r}^1, \boldsymbol{r}^2, \ldots, \boldsymbol{r}^t). \tag{29}$$

In particular, for any subset $T \subset S$, we have $T + \operatorname{intcone}(\boldsymbol{r}^1, \boldsymbol{r}^2, \ldots, \boldsymbol{r}^t) \subset S$. Since S is the graph of some piecewise linear function $\mathcal{P}$, one of the integer rays must have a positive first component, say $r_1^1 > 0$. Then r_1^1 must be a period of $\mathcal{P}$ because any segment in S translated by $\boldsymbol{r}^1$ retains the same slope and belongs to the graph of $\mathcal{P}$. □

Lemma 19. *Suppose S is a subset of segments of the graph of a rational $\mathcal{PWL}$-function $\mathcal{P}$ which is rational MICP or MICP representable with MICP dimension d where each $\boldsymbol{z}$-projected set is a full unit-step segment. Then either S is bounded or there exists a $k \in \mathbb{N}$ such that $S = S_0 \cup \bigcup_{i \in [\![k]\!]} S_k$ where S_0 a periodic infinite subset of segments and for each i, S_i is a rational MICP representable subset of segments included in S with MICP dimension at most $d-1$.*

Proof Let $M \subseteq \mathbb{R}^{2+p+d}$ be a convex set that induces an MICP formulation of S. If S is bounded then we're done. Suppose S is unbounded. Let $C := \operatorname{proj}_z(M)$ be the convex set which is rationally unbounded by assumption. Let $A_{\boldsymbol{z}} = \operatorname{proj}_x(M \cap (\mathbb{R}^{1+p} \times \{\boldsymbol{z}\}))$ which for any $\boldsymbol{z} \in C \cap \mathbb{Z}^d$ by assumption is equal to the segment P_i for some i. Since C is rationally unbounded, let $\boldsymbol{r} \in \mathbb{Z}^d$ such that the ray $\boldsymbol{z} + \lambda \boldsymbol{r} \in C \,\forall \boldsymbol{z} \in C, \lambda \geq 0$. By the same argument as in Theorem 4, we may assume $\boldsymbol{r} = \boldsymbol{e}(1)$ without loss of generality, so the ray $\ell_{\boldsymbol{z}} := \{\boldsymbol{z} + \lambda \boldsymbol{e}(1) : \lambda \geq 0\}$ is contained in C for all $\boldsymbol{z} \in C$.

Let $\{T_i\}_{i \in [\![2^d]\!]}$ be such that $C \cap \mathbb{Z}^d = \bigcup_{i \in [\![2^d]\!]} T_i$ and $z_j \equiv z'_j \mod 2$ for all $j \in [\![d]\!]$, $i \in [\![2^d]\!]$ and $\boldsymbol{z}, \boldsymbol{z}' \in T_i$. Let $S_i = \bigcup_{\boldsymbol{z} \in T_i} A_{\boldsymbol{z}}$ be the subset of segments generated by T_i.

We claim that **all segments in S_i have the same slope**. For fixed $i \in [\![2^d]\!]$ we have $\frac{\boldsymbol{z}+\boldsymbol{z}'}{2} \in C \cap \mathbb{Z}^d$ and $\ell_{\frac{\boldsymbol{z}+\boldsymbol{z}'}{2}} \subset C$ for any $\boldsymbol{z}, \boldsymbol{z}' \in T_i$. Then $L := \operatorname{conv}\left(\left\{\ell_{\boldsymbol{z}}, \ell_{\frac{\boldsymbol{z}+\boldsymbol{z}'}{2}}, \ell_{\boldsymbol{z}'}\right\}\right)$ is a subset of C which importantly contains integer points in its relative interior. Let $\tilde{\boldsymbol{z}}$ be one such integer point. Then $A_{\tilde{\boldsymbol{z}}} = P_j$ for some j with slope $c = \mathcal{P}(j+1) - \mathcal{P}(j)$. Define $h_c(\boldsymbol{z}) = \sup\{x_1 + (-1/c)x_2 : \boldsymbol{x} \in A_{\boldsymbol{z}}\} - \inf\{x_1 + (-1/c)x_2 : \boldsymbol{x} \in A_{\boldsymbol{z}}\}$. We can see that h_c is concave (as a sum of two functions which are concave by Lemma 14), nonnegative, and takes value zero at $\tilde{\boldsymbol{z}}$ and so must be entirely zero over all L by Lemma 13. But $h_c(\boldsymbol{z}) = 0$ implies that $A_{\boldsymbol{z}}$ falls in an affine subspace of the form $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = cx_1 + \gamma_{\boldsymbol{z}}\}$ for some $\gamma_{\boldsymbol{z}} \in \mathbb{R}$ [12, p. 209], so all segments in S_i must have the same slope since the choice of $\boldsymbol{z}, \boldsymbol{z}'$ was arbitrary.

We will now characterize the starting points for the segments contained in S_i. Define $f_1(\boldsymbol{z}) = \inf\{x_1 : \boldsymbol{x} \in A_{\boldsymbol{z}}\}$ and $g_1(\boldsymbol{z}) = \sup\{x_1 : \boldsymbol{x} \in A_{\boldsymbol{z}}\}$. By Lemma 14, f is convex and g is concave. Define $h' : C \to \mathbb{R}$ as $h'(\boldsymbol{z}) = g_1(\boldsymbol{z}) - f_1(\boldsymbol{z})$. Analogously to h_c, h' is concave and nonnegative. It also satisfies $h'(\boldsymbol{z}) = 1 \,\forall \boldsymbol{z} \in C \cap \mathbb{Z}^d$ because by assumption $A_{\boldsymbol{z}}$ is a segment that spans a unit step in the first coordinate. Consider L as before for some choice of $\boldsymbol{z}, \boldsymbol{z}' \in T_i$. Then, by Lemma 13, $h'(\tilde{\boldsymbol{z}}) = 1$ for all $\tilde{\boldsymbol{z}} \in L$ because L has an integer point in its relative interior. It follows that $f_1(\tilde{\boldsymbol{z}}) = g_1(\tilde{\boldsymbol{z}}) - 1 \,\forall \tilde{\boldsymbol{z}} \in L$, so in particular f is both concave and convex and is therefore affine. Then, since the choice of $\boldsymbol{z}, \boldsymbol{z}' \in T_i$ was arbitrary there exist $\boldsymbol{\alpha}^i \in \mathbb{R}^d, \beta_i \in \mathbb{R}$ such that $f_1(\tilde{\boldsymbol{z}}) = \tilde{\boldsymbol{z}}^T \boldsymbol{\alpha}^i + \beta_i \,\forall \tilde{\boldsymbol{z}} \in \ell_{\boldsymbol{z}}$ for any $\boldsymbol{z} \in T_i$. Let $s_i = \boldsymbol{e}(1)^T \boldsymbol{\alpha}^i$. We know that f_1 takes nonnegative integer values at integer points along a ray in direction $\boldsymbol{e}(1)$ so s_i in particular must be a nonnegative integer.

For the cases where $\mathbf{s_i \neq 0}$, we claim that the set of segments S_i is periodic. Note that the set of segments is unbounded because $f(\tilde{\boldsymbol{z}})$ generates an infinite arithmetic progression along any ray $\ell_{\boldsymbol{z}}$ for $\boldsymbol{z} \in T_i$. The latter combined with the observation that all segments in S_i have the same slope implies that if $P_{(j, \mathcal{P}(j), c)}$ is a segment in S_i then $P_{(j+rs_i, \mathcal{P}(j+rs_i), c)} \subset S_i \,\forall r \in \mathbb{N}$. Now, take the union of all S_i for i with $s_i \neq 0$ to obtain S_0 in the statement of the lemma. There are at most 2^d such i so this is a finite union.

For the cases where $\mathbf{s_i = 0}$, we claim, to complete the proof, that the set of segments S_i is rational MICP representable with MICP dimension at most $d-1$. Note that S_i is rational MICP representable because it can be obtained by performing an invertible affine transformation on C. That is, let $\tilde{\boldsymbol{z}} \in T_i$, then

$$\boldsymbol{x} \in S_i \text{ iff } \exists \boldsymbol{y} \in \mathbb{R}^p, \boldsymbol{z} \in \mathbb{Z}^d \text{ such that } (\boldsymbol{x}, \boldsymbol{y}, \tilde{\boldsymbol{z}} + 2\boldsymbol{z}) \in M. \tag{30}$$

Let M_i be the convex set that induces the rational MICP representation of S_i via Lemma 11. Consider the set Q_i defined by

$$\boldsymbol{x} \in Q_i \text{ iff } \exists \boldsymbol{y} \in \mathbb{R}^p, z_0 \in \mathbb{R}, \bar{\boldsymbol{z}} \in \mathbb{Z}^{d-1} \text{ such that } (\boldsymbol{x}, \boldsymbol{y}, z_0, \bar{\boldsymbol{z}}) \in M_i. \tag{31}$$

Note that formulation (31) is obtained solely by relaxing the integrality constraint on the first component of the d integer-constrained variables, so clearly $S_i \subseteq Q_i$ and Q_i is rational MICP representable with MICP dimension at most $d-1$ via Lemma 12. We now claim that $S_i = Q_i$.

Let $\boldsymbol{x} \in Q_i$ with corresponding $\boldsymbol{y} \in \mathbb{R}^p, z_0 \in \mathbb{R}, \bar{\boldsymbol{z}} \in \mathbb{Z}^{d-1}$ such that $(\boldsymbol{x}, \boldsymbol{y}, z_0, \bar{\boldsymbol{z}}) \in M_i$. Let $C_i = \operatorname{proj}_z(M_i)$ be the projection of M_i onto the last d variables. Note that $\boldsymbol{e}(1)$ remains a recession direction of C_i, and the ray $\ell_{(z_0,\bar{\boldsymbol{z}})}$ contains the ray $\ell_{(\lceil z_0 \rceil,\bar{\boldsymbol{z}})}$, and both are contained in C_i. We claim that $\mathrm{cl}(A_{\boldsymbol{z}})$ is a constant segment $\forall \boldsymbol{z} \in \ell_{(z_0,\bar{\boldsymbol{z}})} \cap \mathbb{Z}^d$. Since $s_i = 0$, we have that $f_1(\boldsymbol{z})$ (properly redefined on C_i) is constant $\forall \boldsymbol{z} \in \ell_{(\lceil z_0 \rceil,\bar{\boldsymbol{z}})}$. The value $f_1(\boldsymbol{z})$ is the first component of the starting point of the segment $\mathrm{cl}(A_{\boldsymbol{z}}) \subseteq S_i$ when $\boldsymbol{z} \in \mathbb{Z}^d \cap C_i$, and since segments in S_i form part of the graph of a $\mathcal{PWL}$-function, there can be at most one segment in the range $[f_1(\boldsymbol{z}), f_1(\boldsymbol{z})]$, hence $\mathrm{cl}(A_{\boldsymbol{z}})$ must be a unique constant segment $\forall \boldsymbol{z} \in \ell_{(z_0,\bar{\boldsymbol{z}})} \cap \mathbb{Z}^d$. Now we may apply Lemma 16 to conclude that $\boldsymbol{x} \in S_i$.

□

THEOREM 7. *Let $\mathcal{P}$ be a rational $\mathcal{PWL}$-function. Let S be the graph of $\mathcal{P}$. Then the following are equivalent:*

(a) S is rational MICP representable where each $\boldsymbol{z}$-projected set is a full unit-step segment.

(b) There exists a finite subset of segments S_0 and an infinite periodic subset of segments S_1 such that $S = S_0 \cup S_1$.

(c) There exists a finite subset of segments T_0 and a rational-MILP representable subset of segments T_1 such that $S = T_0 \cup T_1$, where each $\boldsymbol{z}$-projected set in the MILP formulation is a full unit-step segment.

Proof (a) ⇒ (b): Repeatedly apply Lemma 19.

(b) ⇒ (c): It is possible that the infinite periodic subset of segments S_1 contains gaps, i.e., does not define the graph of a piecewise linear function. However, since S_0 is finite we can define T_1 to be the infinite set of segments that begins after all segments in S_0. Define $T_0 = S \setminus T_1$. T_1 remains periodic and defines the graph of a periodic piecewise linear function, so we can apply Lemma 8 to obtain a rational MILP formulation of T_1 (it does not matter that the function does not start at 0).

(c) ⇒ (a): Corollary 4. The $\boldsymbol{z}$-projected sets are preserved from the MILP to the MICP formulation. □

4.5. A simple condition for finite shapes To prove Theorem 8 we use the following consequence of the Brunn-Minkowski inequality [23].

LEMMA 20. *Let C be the relaxed index set of a MICP representation of a set $S \subseteq \mathbb{R}^n$ and let $\{A_{\boldsymbol{z}}\}_{\boldsymbol{z}\in C}$ be the $\boldsymbol{z}$-projected sets of this representation. Then $h : C \to \mathbb{R}$ defined by $h(\boldsymbol{z}) = (\mathrm{Vol}(A_{\boldsymbol{z}}))^{\frac{1}{n}}$ is a concave function. Furthermore, for any $\boldsymbol{z}, \boldsymbol{w} \in C$ and $\lambda \in [0,1]$ it holds:*

$$h(\lambda \boldsymbol{z} + (1-\lambda)\boldsymbol{w}) \geq \mathrm{Vol}(\lambda A_{\boldsymbol{z}} + (1-\lambda) A_{\boldsymbol{w}})^{\frac{1}{n}} \geq \lambda h(\boldsymbol{z}) + (1-\lambda) h(\boldsymbol{w}),$$

Proof Indeed for any $\boldsymbol{z}, \boldsymbol{w} \in C$ and $\lambda \in [0,1]$ by Theorem 1 we have that $\lambda A_{\boldsymbol{z}} + (1-\lambda) A_{\boldsymbol{w}} \subseteq A_{\lambda \boldsymbol{z} + (1-\lambda)\boldsymbol{w}}$, and hence

$$h(\lambda \boldsymbol{z} + (1-\lambda)\boldsymbol{w}) = (\mathrm{Vol}(A_{\lambda \boldsymbol{z} + (1-\lambda)\boldsymbol{w}}))^{\frac{1}{n}} \geq \mathrm{Vol}(\lambda A_{\boldsymbol{z}} + (1-\lambda) A_{\boldsymbol{w}})^{\frac{1}{n}}.$$

But now by Brunn-Minkowski inequality

$$\mathrm{Vol}(\lambda A_{\boldsymbol{z}} + (1-\lambda) A_{\boldsymbol{w}})^{\frac{1}{n}} \geq \lambda \mathrm{Vol}(\lambda A_{\boldsymbol{z}})^{\frac{1}{n}} + (1-\lambda) \mathrm{Vol}(A_{\boldsymbol{w}})^{\frac{1}{n}}.$$

The above two inequalities and the definition of h imply

$$h(\lambda \boldsymbol{z} + (1-\lambda)\boldsymbol{w}) \geq \mathrm{Vol}(\lambda A_{\boldsymbol{z}} + (1-\lambda) A_{\boldsymbol{w}})^{\frac{1}{n}} \geq \lambda h(\boldsymbol{z}) + (1-\lambda) h(\boldsymbol{w}),$$

as needed. □

THEOREM 8. *If S has an MICP representation such that all $\boldsymbol{z}$-projected sets have the same volume, then there exists a finite family of convex sets $\{T_i\}_{i=1}^m$ such that all $\boldsymbol{z}$-projected sets are translations of sets in this family.*

Proof Let C be the relaxed index set of a MICP representation of a set $S \subseteq \mathbb{R}^n$ and let $\{A_{\boldsymbol{z}}\}_{\boldsymbol{z}\in C}$ be the $\boldsymbol{z}$-projected sets of this representation. For h defined in Lemma 20 we have that there exists $\alpha > 0$ such that $h(\boldsymbol{z}) = \alpha$ for all $\boldsymbol{z} \in C \cap \mathbb{Z}^d$. We claim for take any two $\boldsymbol{z}, \boldsymbol{w} \in C \cap \mathbb{Z}^d$ with with the same modulo 2 pattern in their coordinates (i.e. $(\boldsymbol{z}+\boldsymbol{w})/2 \in \mathbb{Z}^d$) $A_{\boldsymbol{z}}$ is a translation of $A_{\boldsymbol{w}}$. Indeed, we have $h(\boldsymbol{z}) = h(\boldsymbol{w}) = h((\boldsymbol{z}+\boldsymbol{w})/2) = \alpha$, which implies

$$\frac{1}{2}h(\boldsymbol{z}) + \frac{1}{2}h(\boldsymbol{w}) = h\left(\frac{\boldsymbol{z}+\boldsymbol{w}}{2}\right) = \alpha.$$

Together with Lemma 20 this implies

$$\operatorname{Vol}\left(\frac{1}{2}A_{\boldsymbol{z}} + \frac{1}{2}A_{\boldsymbol{w}}\right)^{\frac{1}{n}} = \frac{1}{2}\operatorname{Vol}(A_{\boldsymbol{z}})^{\frac{1}{n}} + \frac{1}{2}\operatorname{Vol}(A_{\boldsymbol{w}})^{\frac{1}{n}} = \alpha.$$

But this implies equality in the Brunn-Minkowski inequality for the convex sets $A_{\boldsymbol{z}}, A_{\boldsymbol{w}}$ which implies that they are homothetic. Since they also have the same volume, this implies our translation claim.

Finally, the result follows by letting $\{T_i\}_{i=1}^m$ include one representative $A_{\boldsymbol{z}}$ for each of the finite number of modulo 2 patterns that appear for some $\boldsymbol{z} \in \mathbb{Z}^d \cap C$. $\square$

5. Other omitted proofs

LEMMA 1. *A finite intersection, Cartesian product or Minkowski sum of MICP (MILP) representable sets is MICP (MILP) representable.*

Proof

For each $i \in [\![m]\!]$, let $M_i \subseteq \mathbb{R}^{n_i+p_i+d_i}$ be a closed, convex set that induces a MICP representation of $S_i \subseteq \mathbb{R}^{n_i}$. A MICP formulation for $\tilde{\boldsymbol{x}} \in \prod_{i=1}^m M_i \subset \mathbb{R}^{\sum_{i=1}^m n_i}$ is given by

$$\left(\boldsymbol{x}^i, \boldsymbol{y}^i, \boldsymbol{z}^i\right) \in M \cap \left(\mathbb{R}^{n_i+p_i} \times \mathbb{Z}^{d_i}\right) \quad \forall i \in [\![m]\!], \quad \tilde{\boldsymbol{x}} = \prod_{i=1}^m \boldsymbol{x}^i.$$

If $n_i = n_j$ for all $i, j \in [\![m]\!]$ a formulation for $\tilde{\boldsymbol{x}} \in \sum_{i=1}^m M_i$ is given by

$$\left(\boldsymbol{x}^i, \boldsymbol{y}^i, \boldsymbol{z}^i\right) \in M \cap \left(\mathbb{R}^{n_i+p_i} \times \mathbb{Z}^{d_i}\right) \quad \forall i \in [\![m]\!], \quad \tilde{\boldsymbol{x}} = \sum_{i=1}^m \boldsymbol{x}^i,$$

and a formulation for $\tilde{\boldsymbol{x}} \in \bigcap_{i=1}^m M_i$ is given by

$$\left(\tilde{\boldsymbol{x}}, \boldsymbol{y}^i, \boldsymbol{z}^i\right) \in M \cap \left(\mathbb{R}^{n_i+p_i} \times \mathbb{Z}^{d_i}\right) \quad \forall i \in [\![m]\!].$$

$\square$

LEMMA 2. *Let $S \subseteq \mathbb{R}^n$. If S is w-strongly nonconvex, then S cannot be MICP representable with MICP-dimension less than $\lceil \log_2(w) \rceil$.*

Proof Suppose we have R as in the statement above and there exists an MICP formulation of S, that is, a closed convex set $M \subset \mathbb{R}^{n+p+d}$ such that $\boldsymbol{x} \in S$ iff $\exists \boldsymbol{z} \in \mathbb{Z}^d, y \in \mathbb{R}^p$ such that $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in M$. Then for each point $\boldsymbol{x} \in R$ we associate at least one integer point $\boldsymbol{z}^{\boldsymbol{x}} \in \mathbb{Z}^d$ and a $\boldsymbol{y}^{\boldsymbol{x}} \in \mathbb{R}^p$ such that $(\boldsymbol{x},\boldsymbol{y}^{\boldsymbol{x}},\boldsymbol{z}^{\boldsymbol{x}})\in M$. If there are multiple such pairs of points $\boldsymbol{z}^{\boldsymbol{x}},\boldsymbol{y}^{\boldsymbol{x}}$ then for the purposes of the argument we may choose one arbitrarily.

Suppose $d<\lceil\log_2(w)\rceil$. We will derive a contradiction by proving that there exist two points $\boldsymbol{x},\boldsymbol{x}'\in R$ such that the associated integer points $\boldsymbol{z}^{\boldsymbol{x}},\boldsymbol{z}^{\boldsymbol{x}'}$ satisfy

$$\frac{\boldsymbol{z}^{\boldsymbol{x}}+\boldsymbol{z}^{\boldsymbol{x}'}}{2}\in\mathbb{Z}^d. \tag{32}$$

Indeed, this property combined with convexity of M, i.e., $\left(\frac{\boldsymbol{x}+\boldsymbol{x}'}{2},\frac{\boldsymbol{y}^x+\boldsymbol{y}^{\boldsymbol{x}'}}{2},\frac{\boldsymbol{z}^{\boldsymbol{x}}+\boldsymbol{z}^{\boldsymbol{x}'}}{2}\right)\in M$ would imply that $\frac{\boldsymbol{x}+\boldsymbol{x}'}{2}\in S$, which contradicts the definition of R.

Recall a basic property of integers that if $i,j\in\mathbb{Z}$ and $i\equiv j \pmod 2$, i.e., i and j are both even or odd, then $\frac{i+j}{2}\in\mathbb{Z}$. We say that two integer vectors $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{Z}^d$ have the same *parity* if α_i and β_i are both even or odd for each component $i=1,\dots,d$. Trivially, if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have the same parity, then $\frac{\boldsymbol{\alpha}+\boldsymbol{\beta}}{2}\in\mathbb{Z}^d$. Given that we can categorize any integer vector according to the 2^d possible choices for whether its components are even or odd, and we notice that from any collection of integer vectors of size greater than 2^d+1 we must have at least one pair that has the same parity. Therefore since $|R|=w\geq 2^d+1$ we can find a pair $\boldsymbol{x},\boldsymbol{x}'\in R$ such that their associated integer points $\boldsymbol{z}^{\boldsymbol{x}},\boldsymbol{z}^{\boldsymbol{x}'}$ have the same parity and thus satisfy (32). □

Proposition 1. *Let $\{T_i\}_{i=1}^k$ be a family of non-empty closed convex sets in $\mathbb{R}^{n+p}$. Then an ideal formulation for $S=\bigcup_{i\in[\![k]\!]}\operatorname{proj}_x T_i$ is given by*

$$\boldsymbol{x}=\sum_{i\in[\![k]\!]}\boldsymbol{x}^i,\quad (\boldsymbol{x}^i,\boldsymbol{y}^i,z_i)\in\hat{T}_i\ \forall i\in[\![k]\!],\quad \sum_{i\in[\![k]\!]}z_i=1,\ \boldsymbol{z}\in\{0,1\}^k, \tag{10a}$$

$$||\boldsymbol{x}^i||_2^2\leq z_i t_i,\qquad \forall i\in[\![k]\!],\boldsymbol{t}\geq 0. \tag{10b}$$

Proof Note that the constraints define a convex set because the conic hull of a convex set is convex, and $||\boldsymbol{x}^i||_2^2\leq z_it_i$ is a form of the rotated second-order cone, which is also convex. Any feasible assignment of the integer vector $\boldsymbol{z}$ has at most one nonzero component. Without loss of generality we may take this to be the first component, so $z_1=1$. Since t_i is unrestricted in the positive direction, the constraint $||\boldsymbol{x}^1||_2^2\leq t_i$ imposes no restrictions on the vector $\boldsymbol{x}^1$ and $\boldsymbol{x}^1\in\operatorname{proj}_x T_1$ iff there exists $\boldsymbol{y}^1\in\mathbb{R}^p$ such that $(\boldsymbol{x}^1,\boldsymbol{y}^1,1)\in\hat{T}_1$. For $i>1$, the constraint $||\boldsymbol{x}^i||_2^2\leq 0$ implies $\boldsymbol{x}^i=0$, and this is feasible because $(\mathbf{0},\mathbf{0},0)\in\hat{T}_i$ given $\hat{T}_i$ is nonempty by assumption.

For simplicity of exposition suppose $k=2$. Let F be a minimal face of M and $\boldsymbol{\beta}=(\boldsymbol{x},\boldsymbol{x}^1,\boldsymbol{y}^1,z_1,t_1,\boldsymbol{x}^2,\boldsymbol{y}^2,z_2,t_2)\in F$ with $0<z_1,z_2<1$. Define

$$\boldsymbol{\beta}^1=(\boldsymbol{x}^1/z_1,\boldsymbol{x}^1/z_1,\boldsymbol{y}^1/z_1,1,t_1/z_1,\mathbf{0},\mathbf{0},0,0) \tag{33}$$

and

$$\boldsymbol{\beta}^2=(\boldsymbol{x}^2/z_2,\mathbf{0},\mathbf{0},0,0,\boldsymbol{x}^2/z_2,\boldsymbol{y}^2/z_2,1,t_2/z_2). \tag{34}$$

Then by construction $\boldsymbol{\beta}=z_1\boldsymbol{\beta}^1+z_2\boldsymbol{\beta}^2$ $(z_1+z_2=1)$. One may verify as well that $\boldsymbol{\beta}^1$ and $\boldsymbol{\beta}^2$ satisfy conditions (10) and hence $\boldsymbol{\beta}^1,\boldsymbol{\beta}^2\in F$. However, then $\{(\boldsymbol{x},\boldsymbol{x}^1,\boldsymbol{y}^1,z_1,t_1,\boldsymbol{x}^2,\boldsymbol{y}^2,z_2,t_2)\in F: z_1=0\}\subsetneq F$, which contradicts the minimality of F. □

Proposition 2 **([19, Proposition 2])**. *If M induces an MICP-formulation of S and $M=B+K$ where B is a compact convex set and K is a rational polyhedral cone, then for some $k,t\in\mathbb{N}$ there exist compact convex sets $S_1,S_2,\dots,S_k$ and integer vectors $\boldsymbol{r}^1,\boldsymbol{r}^2,\dots,\boldsymbol{r}^t\subseteq\mathbb{Z}^n$ such that*

$$S=\bigcup_{i\in[\![k]\!]}S_i+\operatorname{intcone}(\boldsymbol{r}^1,\boldsymbol{r}^2,\dots,\boldsymbol{r}^t). \tag{12}$$

Proof This argument is an extension of Theorem 11.6 of [24]. Suppose $M = C + K$ where C is a compact convex set and K is a polyhedral cone generated by rational rays $(\boldsymbol{r}_x^1, \boldsymbol{r}_y^1, \boldsymbol{r}_z^1), (\boldsymbol{r}_x^2, \boldsymbol{r}_y^2, \boldsymbol{r}_z^2), \dots, (\boldsymbol{r}_x^t, \boldsymbol{r}_y^t, \boldsymbol{r}_z^t)$. (We may assume without loss of generality that these rays furthermore have integer components). We will prove that there exist sets $S_1, \dots, S_k$ such that

$$S = \text{proj}_x \left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right) = \bigcup_{i \in [\![k]\!]} S_i + \text{intcone}(\boldsymbol{r}_x^1, \boldsymbol{r}_x^2, \dots, \boldsymbol{r}_x^t), \tag{35}$$

where each S_i is a projection of a closed convex set.

For any $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*) \in M$ there exist a finite (via Carathéodory) set of extreme points $(\boldsymbol{x}^1, \boldsymbol{y}^1, \boldsymbol{z}^1), (\boldsymbol{x}^2, \boldsymbol{y}^2, \boldsymbol{z}^2), \dots, (\boldsymbol{x}^w, \boldsymbol{y}^w, \boldsymbol{z}^w)$ of C and nonnegative multipliers $\boldsymbol{\lambda}$, $\boldsymbol{\gamma}$ with $\sum_{i \in [\![w]\!]} \lambda_i = 1$ such that

$$(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*) = \sum_{i \in [\![w]\!]} \lambda_i (\boldsymbol{x}^i, \boldsymbol{y}^i, \boldsymbol{z}^i) + \sum_{j \in [\![t]\!]} \gamma_j (\boldsymbol{r}_x^j, \boldsymbol{r}_y^j, \boldsymbol{r}_z^j). \tag{36}$$

Define

$$(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}) = \sum_{i \in [\![w]\!]} \lambda_i (\boldsymbol{x}^i, \boldsymbol{y}^i, \boldsymbol{z}^i) + \sum_{j \in [\![t]\!]} (\gamma_j - \lfloor \gamma_j \rfloor)(\boldsymbol{r}_x^j, \boldsymbol{r}_y^j, \boldsymbol{r}_z^j) \tag{37}$$

and

$$(\boldsymbol{x}^\infty, \boldsymbol{y}^\infty, \boldsymbol{z}^\infty) = \sum_{j \in [\![t]\!]} \lfloor \gamma_j \rfloor (\boldsymbol{r}_x^j, \boldsymbol{r}_y^j, \boldsymbol{r}_z^j) \tag{38}$$

so that $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*) = (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}) + (\boldsymbol{x}^\infty, \boldsymbol{y}^\infty, \boldsymbol{z}^\infty)$.

Note that $(\boldsymbol{x}^\infty, \boldsymbol{y}^\infty, \boldsymbol{z}^\infty) \in \text{intcone}((\boldsymbol{r}_x^1, \boldsymbol{r}_y^1, \boldsymbol{r}_z^1), \dots, (\boldsymbol{r}_x^t, \boldsymbol{r}_y^t, \boldsymbol{r}_z^t)) =: M^\infty$ and $\hat{\boldsymbol{z}} = \boldsymbol{z}^* - \boldsymbol{z}^\infty \in \mathbb{Z}^d$, so $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ belongs to a bounded set

$$\hat{M} = (C + B) \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^d), \tag{39}$$

where $B = \{\sum_{j \in [\![t]\!]} \gamma_j (\boldsymbol{r}_{\boldsymbol{x}}^j, \boldsymbol{r}_{\boldsymbol{y}}^j, \boldsymbol{r}_{\boldsymbol{z}}^j) : \boldsymbol{0} \le \boldsymbol{\gamma} \le \boldsymbol{1}\}$. Since this decomposition holds for any points $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*) \in M$, it follows that $M \subseteq \hat{M} + M^\infty$. Since $\hat{M}$ is bounded, $\hat{M}$ is a finite union of bounded convex sets. Also $\hat{M} + M^\infty \subseteq M$ is easy to show, so we've demonstrated $M = \hat{M} + M^\infty$. The statement (35) follows from projection of M onto the x variables. $\square$

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