

# Modeling Power of Mixed Integer Convex Optimization Problems And Their Effective Solution with Julia and JuMP

Juan Pablo Vielma

Massachusetts Institute of Technology

Mathematics of Data and Decisions at Davis,  
Department of Mathematics, UC Davis,  
Davis, CA, April, 2019.

Funded by NSF OAC-1835443, ONR N00014-18-1-2079 and NSF CMMI-1351619

# Mixed Integer Convex Optimization (MICP)

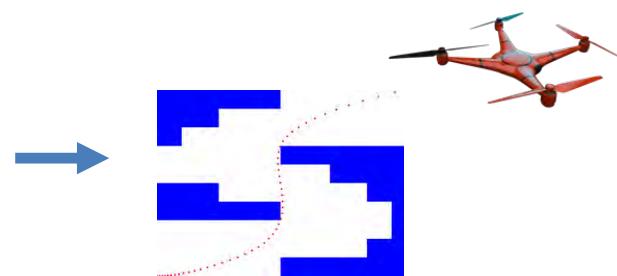
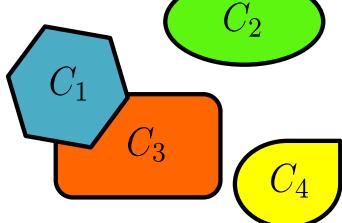
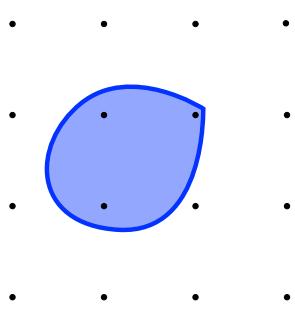
$$\min f(x)$$

s.t.

$$x \in C$$

$$x_i \in \mathbb{Z} \quad i \in I$$

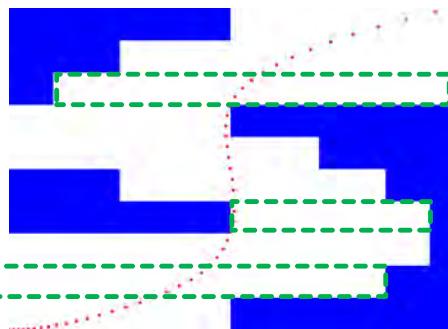
convex  $f$  and  $C$ .



<http://www.gurobi.com/company/example-customers>

# A Mixed-Integer Infinite Dimensional Example

- Obstacle avoiding trajectory:



$$(x(t), y(t))_{t \in [0,1]}$$

- Step 1: discretize time into intervals  
 $0 = T_1 < T_2 < \dots < T_N$  s.t.  
 $(x(t), y(t)) = p_i(t) \quad t \in [T_i, T_{i+1}]$

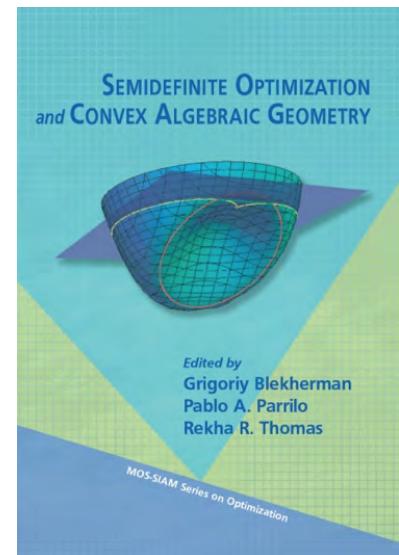
- Step 2: “safe polyhedrons”  
 $P^r = \{x \in \mathbb{R}^2 : A^r x \leq b^r\}$  s.t.  
 $\forall i \exists r \text{ s.t. } p_i(t) \in P^r \quad t \in [T_i, T_{i+1}]$

- $p_i(t) \in P^r \rightarrow q_{i,r}(t) \geq 0 \forall t$

SOS:

$$q_{i,r}(t) = \sum_j r_j^2(t)$$

- Bound degree of polynomials:  
Semidefinite Programming (SDP)



- MI-SDP solver:

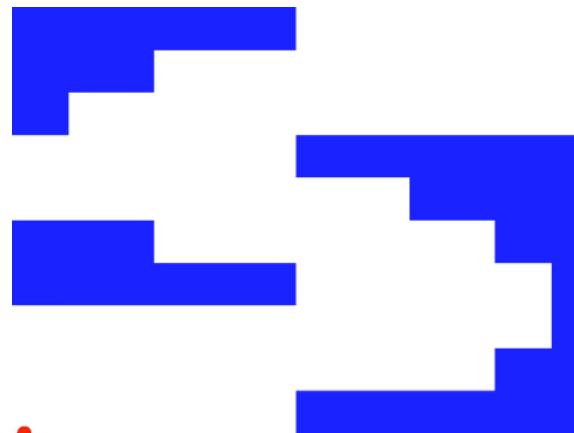


## Solving MI-SDP to Global Optimality?

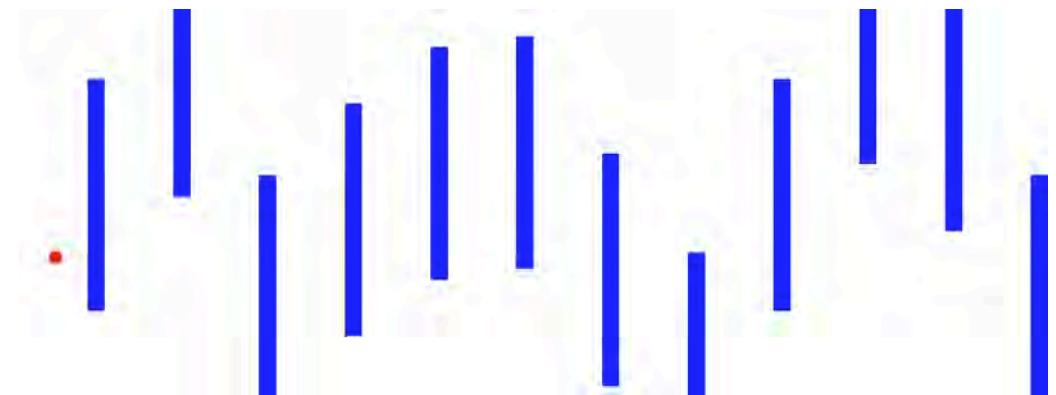
---



Optimal “Smoothness” in 651 seconds



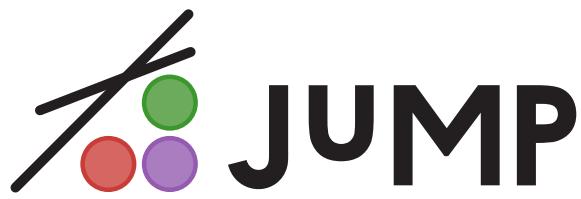
Optimal “clicks” in 80 seconds



## Outline

---

- **MICP Representability:** Characterize what can we model with MICP?
  - Joint work with M. Lubin and I. Zadik
- **Computational Solution of MICP:** Methods and Solvers based on



- MICP solvers is joint work with R. Bent, C. Coey, I. Dunning, J. Huchette, L. Kapelevich, M. Lubin, E. Yamangil
- JuMP is joint work with and independent work by M. Lubin, I. Dunning, J. Huchette, B. Legat, O. Dowson, C. Coey, C. Coffrin, J. Dias Garcia, T. Koolen, V. Nesello, F. Pacaud, R. Schwarz, I. Tahiri, U. Worsøe, ....

What can we model with MICP?

## MICP /MICONV Formulations and Representability

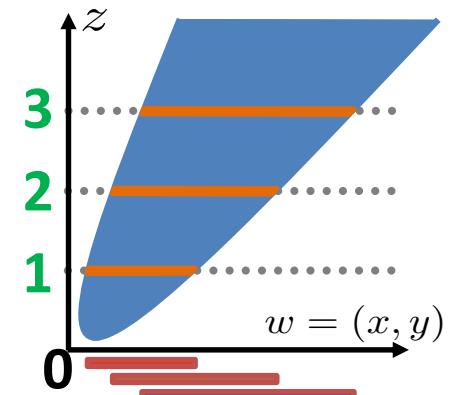
---

- $S \subseteq \mathbb{R}^n$  is MICP representable (MICP-R) iff it has an **MICP formulation**:
  - A closed convex set  $M \subseteq \mathbb{R}^{n+p+d}$
  - auxiliary continuous variables  $y \in \mathbb{R}^p$
  - auxiliary integer variables  $z \in \mathbb{Z}^d$

$$x \in S \iff \begin{aligned} & \exists(y, z) \in \mathbb{R}^p \times \mathbb{Z}^d \text{ s.t.} \\ & (x, y, z) \in M \end{aligned}$$

or equivalently

$$S = \text{proj}_x (M \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^d))$$



## MICP-R $\Rightarrow$ Countable Union of Projections of Closed Convex Sets

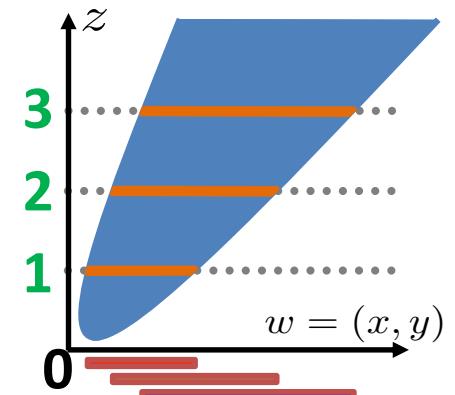
---

- $S \subseteq \mathbb{R}^n$  is MICP representable (MICP-R) iff it has an **MICP formulation**:
  - A closed convex set  $M \subseteq \mathbb{R}^{n+p+d}$
  - auxiliary continuous variables  $y \in \mathbb{R}^p$
  - auxiliary integer variables  $z \in \mathbb{Z}^d$

$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \text{proj}_x(B_z)$$

$I = \text{proj}_z(M)$  convex

$B_z = M \cap (\mathbb{R}^{n+p} \times \{z\})$   
closed and convex



- Simple Proposition:
  - Complement of any convex body is a **countable union** of projections of **closed convex sets**

## What Countable Unions are MICP-R? Jeroslow and Lowe Regularity

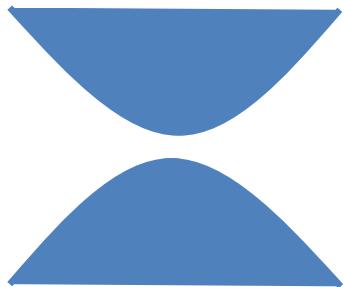
---

$$\exists \{r(i)\}_{i=1}^t \subseteq \mathbb{Q}^n \text{ s.t. } S = \bigcup_{i=1}^k P_i + \left\{ \sum_{i=1}^t \lambda_i r(i) : \lambda \in \mathbb{Z}_+^t \right\}$$

- Regularity Conditions:
  - $M$  rational polyhedron  $\Rightarrow P_i$  = rational polytopes (Jeroslow and Lowe '84):
  - $M = \{x \in \mathbb{Z}^2 : x_1 \cdot x_2 \geq \alpha\}$   $\Rightarrow P_i$  = points (Dey & Moran '13)
  - $M$  = Rational Polyhedron  $\cap$  “Rational” Ellipsoidal Cylinder  $\Rightarrow P_i$  = Rational Ellipsoid  $\cap$  Polytope (Del Pia & Poskin '16)
  - $M$  = Compact Convex + Rational Polyhedron Cone  $\Rightarrow P_i$  = Compact Convex (Lubin, Zadik & V. 17')

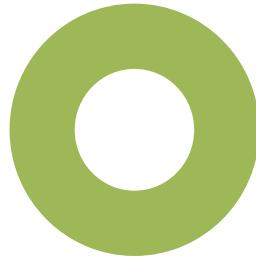
## What Other Countable Unions are MICP-R?

Two sheet hyperbola?



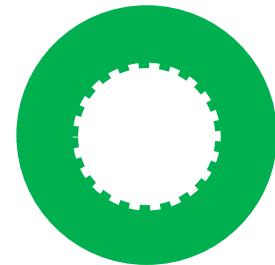
$$\{x \in \mathbb{R}^2 : 1 + x_1^2 \leq x_2^2\}$$

Spherical shell?



$$\{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\}$$

“Clopen” Spherical shell?



$$\{x \in \mathbb{R}^2 : 1 < \|x\| \leq 2\}$$

- Integer points in parabola  $\{(x, x^2) : x \in \mathbb{Z}\}$ ?
- The set of  $n \times n$  matrices with rank  $\leq k$ ?
- Dense discrete set?  $\{\sqrt{2}x - \lfloor \sqrt{2}x \rfloor : x \in \mathbb{N}\} \subseteq [0, 1]$
- Set of prime numbers?

"God made the integers, all else is the work of man"

- Leopold Kronecker

## 0-1 MICP = Finite Union of (Closed) Convex Sets

- $T_1, \dots, T_k$  be **closed convex** set. Formulation of  $x \in \bigcup_{i=1}^k T_i$ :

$$(x^i, z_i) \in \overline{\text{cone}}(T_i \times \{1\}) \quad \forall i \in [k]$$

$$\|x^i\|_2^2 \leq z_i t_i. \quad \forall i \in [k]$$

$$\sum_{i=1}^k x^i = x,$$

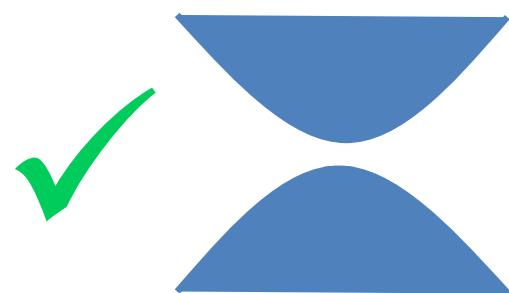
$$\sum_{i=1}^k z_i = 1,$$

$$z \in \{0,1\}^k,$$

$$t \in \mathbb{R}_+^k,$$

$$x^i \in \mathbb{R}^n \quad \forall i \in [k]$$

Two sheet hyperbola?



$$\{x \in \mathbb{R}^2 : 1 + x_1^2 \leq x_2^2\}$$

## A Simple Lemma for non-MICP Representability

---

- Obstruction for MICP representability of  $S$  :

$$\text{infinite } R \subseteq S \quad \text{s.t.} \quad \frac{u+v}{2} \notin S \quad \forall u, v \in R, u \neq v$$

- Proof: Assume for contradiction there exists  $M$  such that:

$$S = \text{proj}_x (M \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^d))$$

$$(u, y_u, z_u) \in M \quad \Rightarrow \quad \frac{z_u + z_v}{2} \notin \mathbb{Z}^d$$
$$(v, y_v, z_v) \in M$$

$$z_u \equiv z_v \pmod{2} \text{ component-wise} \quad \Rightarrow \quad \frac{z_u + z_v}{2} \in \mathbb{Z}^d$$

component-wise parity classes =  $2^d < |R| = \infty$        $\not\equiv$

## A Simple Lemma for non-MICP Representability

- Obstruction for MICP representability of  $S$  :

infinite  $R \subseteq S$     s.t.     $\frac{u+v}{2} \notin S \quad \forall u, v \in R, u \neq v$

Spherical shell?



$$\{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\}$$

- ✗ Integer points in parabola  $\{(x, x^2) : x \in \mathbb{Z}\}$ ?
- ✗ The set of  $n \times n$  matrices with rank  $\leq k$ ?
- ✗ Set of prime numbers?

## A Simple Lemma for non-MICP Representability

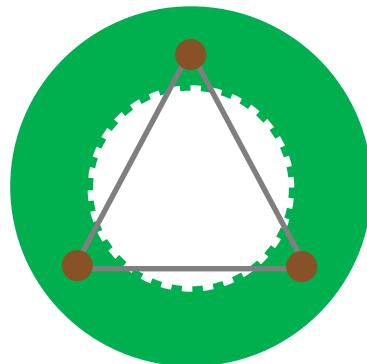
---

$$R \subseteq S \quad \text{s.t.} \quad \frac{u+v}{2} \notin S \quad \forall u, v \in R, u \neq v \quad |R| = k$$

- MICP formulation of  $S$  needs  $\log_2 k$  variables :

“Clopen” Spherical shell?

X



Arbitrarily large  $R$

$$\{x \in \mathbb{R}^2 : 1 < \|x\| \leq 2\}$$

## What About Irrationality?

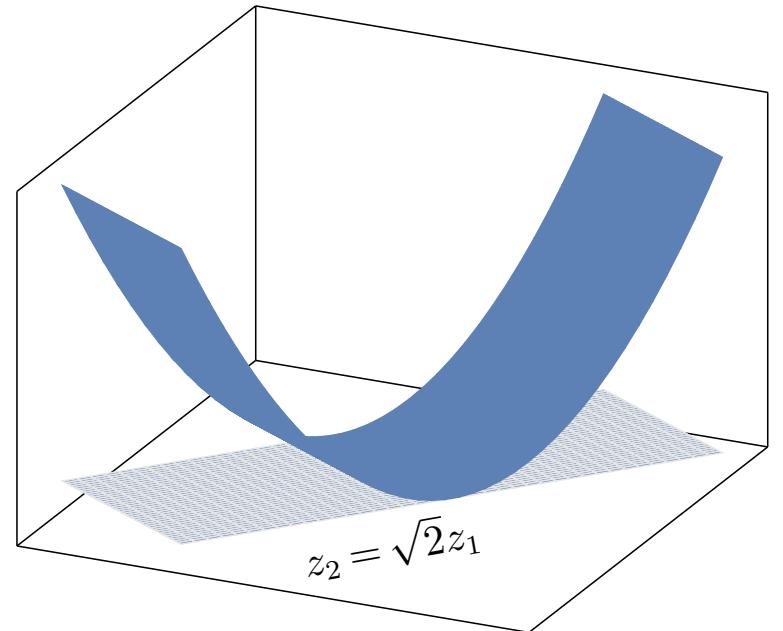
---

- A **set  $S$**  is **periodic** if and only if:  
$$\exists r \in \mathbb{R}^n \setminus \{0\} \quad \text{s.t.} \quad x + \lambda r \in S \quad \forall \lambda \in \mathbb{Z}_+, \quad x \in S$$
- Non-periodic MICP-R sets
  - Dense discrete set  $\{\sqrt{2}x - \lfloor \sqrt{2}x \rfloor : x \in \mathbb{N}\} \subseteq [0, 1]$   
$$\|(z_1, z_1)\|_2 \leq z_2 + 1, \quad \|(z_2, z_2)\|_2 \leq 2z_1, \quad x_1 = y_1 - z_2,$$
  
$$\|(z_1, z_1)\|_2 \leq y_1, \quad \|(y_1, y_1)\|_2 \leq 2z_1, \quad z \in \mathbb{Z}^2$$
  - Set of naturals  $\{x \in \mathbb{N} : \sqrt{2}x - \lfloor \sqrt{2}x \rfloor \notin (\varepsilon, 1 - \sqrt{2}\varepsilon)\}$   
$$\|(x_1, x_1)\|_2 \leq x_2 + \varepsilon, \quad \|(x_2, x_2)\|_2 \leq 2x_1 + 2\varepsilon, \quad x \in \mathbb{Z}_+^2$$

## A Definition of Rational MICP-R

$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \text{proj}_x(B_z) \quad S = \text{proj}_x(\mathcal{M} \cap (\mathbb{R}^{n+p} \times \mathbb{Z}^d))$$
$$I = \text{proj}_z(\mathcal{M})$$

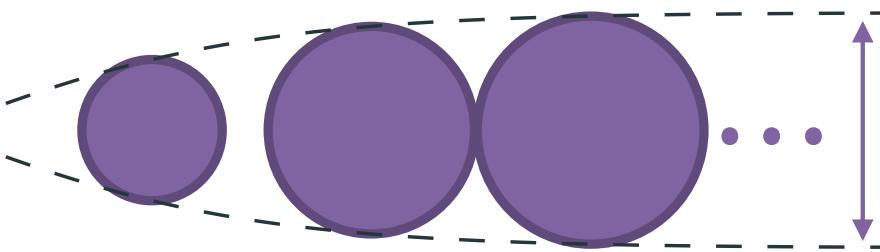
- Any rational affine mapping of index set  $I$ :
  - Is bounded, or
  - Has an integer (rational) recession direction
- Irrational directions can hide!



## Properties of Rational MICP-R Sets

---

- A **rational MICP-R** set  $S$  is a **finite union** of **convex** or **periodic** sets if
  - $S$  is **closed** and its **convex subsets** have **upper bounded diameter**
  - $S$  is a **not necessarily closed discrete** set
    - **Dense discrete** and **non-periodic naturals NOT R-MICPR**



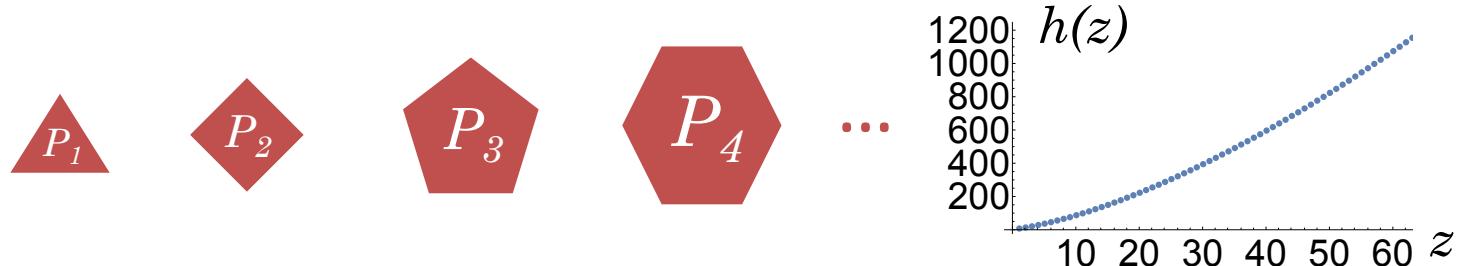
## Properties of Rational MICP-R Sets

---

- Other consequences:
  - **Compact rational MICP-R :**
    - finite unions of compact convex sets
  - **Rational MICP-R** subsets of the **naturals** =
    - Union of **finite** points and **one periodic set**
    - Union of **finite** points and **one MILP-R** set
- **Rational MICP Representability:**
  - **Closed** under: **Finite Union, Cartesian Product, rational affine transformations and Minkowski sum**
  - **NOT Closed** under **Intersection.**

## Rational MICP-R does Not Imply Finite Shapes

---

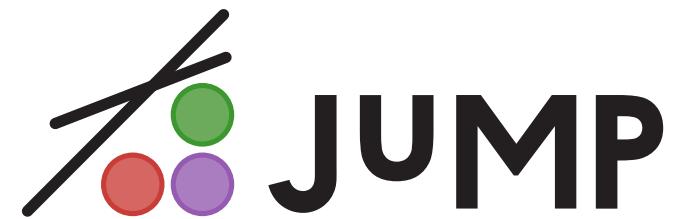


- There exists an increasing function  $h$  such that:
  - $P_z \subseteq \mathbb{R}^2$  regular  $h(z)$ -gon centered at  $(z, 0)$
  - $P_z \cap P_{z'} = \emptyset, z \neq z'$
  - $S = \bigcup_{z=1}^{\infty} P_z$  is R-MICPR and periodic
- Equal volume  $\Rightarrow$  Finite # of Shapes

# MICP with

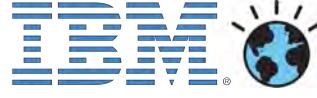


&



## 50+ Years of MIP = Significant Solver Speedups

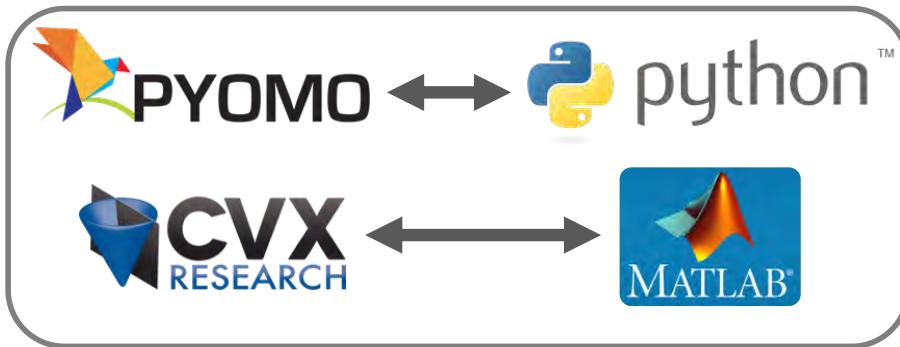
---

- Algorithmic Improvements (**Machine Independent**):
  - **CPLEX** →  → 
    - v1.2 (1991) – v11 (2007): **29,000 x** speedup
  - 
    - v1 (2009) – v6.5 (2015): **48.7 x** speedup
- Also MICP:
  - 
    - v6.0 (2014) – v6.5 (2015) quadratic: **4.43 x**  
(V., Dunning, Huchette, Lubin, 2015)

## Accessing MIP Solvers = Modelling Languages

---

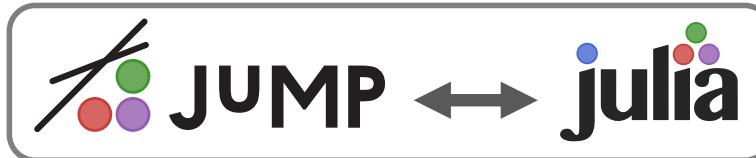
- User-friendly algebraic modelling languages (AML):



Standalone and Fast

Based on General Language and Versatile

- Fast and Versatile, but complicated (and possibly proprietary)
  - Low-level C/C++ solver or Coin-OR interfaces & frameworks
- Best of all worlds?



# 21st Century Programming/Modelling Languages

---



- Open-source and free!
- “Floats like python/matlab, strings like C/Fortran”
- Petaflop scaling: C/C++, Fortran and Julia!
- Powerful **compiler** and **meta-programming** features



- Open-source and free!
- Modelling language, interface and software ecosystem for optimization (~20 solvers)
- **Easy to use, fast** and advanced
- Integrated into Julia

Large Software Stack and Vibrant Community



**juliacon** 2018

University College London



## JuliaCon is coming to Baltimore

**Monday 22nd to Friday 26th of July, 2019**  
at the University of Maryland Baltimore (UMB),  
**Baltimore, MD, USA**



# Large Software Stack and Vibrant Community



**JuMP Developers Workshop**

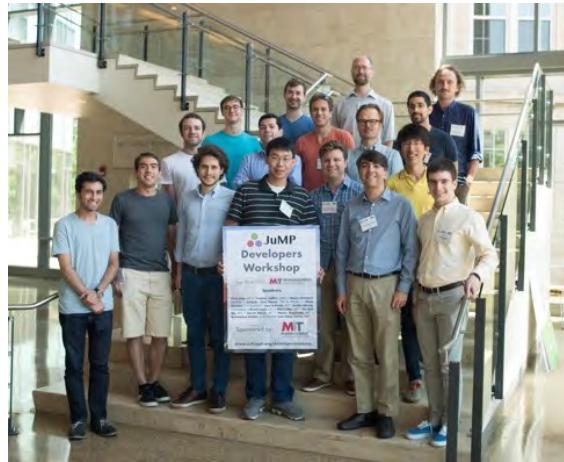
June 12-16, 2017. **MIT MANAGEMENT SLOAN SCHOOL**

**Speakers**

Chris Coey, MIT • Carlton Coffrin, LANL • Steven Diamond, Stanford • Joaquim Dias Garcia, PSR & PUC-Rio • Oscar Dowson, U. of Auckland • Joey Huchette, MIT • Jordan Jolivng, UW-Madison • Benoît Legat, UCL • Miles Lubin, MIT • Yee Sian Ng, MIT • Jarrett Revels, MIT • Nestor Sepulveda, MIT • Bartolomeo Stellato, U. of Oxford • Juan Pablo Vielma, MIT

**Sponsored by:** **MIT MANAGEMENT LATIN AMERICA OFFICE**

[www.juliaopt.org/developersmeetup](http://www.juliaopt.org/developersmeetup)



**3RD ANNUAL JUMLP-dev WORKSHOP**

ESCUELA DE INGENIERÍA  
PONTIFICIA UNIVERSIDAD  
CATÓLICA DE CHILE

Instituto Chileno  
**Norteamericano**

March 12-14, 2019

Chris Coey MIT	Lea Kapelevich PSR	David Sanders Universidad Autónoma de México
Joaquim Dias Garcia PSR & PUC-Rio	Stefan Karpinski Julia Computing	Thuenir Silva PUC-Rio
Oscar Dowson Northwestern U.	Tomas Lagos González U. de Chile	Alessandro Soares PSR
Marcelo Fortes UdeiaR	José Daniel Lara UC Berkeley & NREL	Mario Souto PUC-Rio
Michael Garstka U. of Oxford	Benoit Legat UCLouvain	Mathieu Tanneau Polytechnique Montréal
Alvaro González Skoltech	Miles Lubin Google	Tillmann Weisser Los Alamos N. L.
Joey Huchette Google & Rice U.	Harsha Nagarajan Los Alamos N. L.	Andrew David Werner Rosemberg
Jordan Jolivng U. of Wisconsin-Madison	Vitor Nevello U. of Bordeaux	PUC-Rio

**Sponsored by:**

NSF → MISTI INSTITUTO DE INGENIERÍA  
NUMFOCUS MATEMÁTICA Y COMPUTACIONAL  
OPEN CODE - BETTER SCIENCE MIT DEPARTAMENTO DE INGENIERÍA  
JULIA OPERATIONS RESEARCH SLOAN SCHOOL INDUSTRIAL Y DE SISTEMAS

[www.juliaopt.org/meetings/santiago2019](http://www.juliaopt.org/meetings/santiago2019)



**THE SECOND ANNUAL JUMLP-dev WORKSHOP**

June 27-29, 2018. **Institut de Mathématiques de Bordeaux**

**Speakers**

Martin Dostál, KTH • Oscar Dowson, U. of Auckland • Joaquim Dias Garcia, PSR & PUC-Rio • Hassan Hijazi, LANL • Jean-Hubert Hours, Arélys • Oliver Huber, UVA-Madison • Joey Huchette, MIT • Ole Kroger, Uni Heidelberg • Benoît Legat, UCLouvain • Miles Lubin, Google • Guillaume Marques, U. de Bordeaux • Harsha Nagarajan, LANL • François Pacaud, CERMICS, ENPC • Abel Soares Siqueira, Federal University of Paraná • Julie Silivrić, RTE • Mohamed Tarek, UNSW Canberra • Matthew Wilhelm, U. of Connecticut • Ulf Worsøe, Mosk

**Sponsored by:**

INITIATIVE EXCELLENCE MIT INSTITUTO DE INGENIERÍA  
MANAGEMENT SLOAN SCHOOL MATEMÁTICA Y COMPUTACIONAL  
JULIA OPERATIONS RESEARCH A Private Consulting Organization  
[www.juliaopt.org/meetings/bordeaux2018](http://www.juliaopt.org/meetings/bordeaux2018)



# State-of-the-~~JU~~ JUMP

- NumFOCUS Sponsored project since 2018
- NSF funding for annual meeting until 2023 (OAC-1835443)
- Hoping to have the first 3 GSoC students under NumFOCUS umbrella in 2019
- Towards v1.0 (see Miles talk at JuMP-dev):
  - v0.19 released on February 2019:
    - ~2 year and ~30,000 lines of code
  - To-do: Documentation, usability and regressions from v0.18



Google  
Summer of Code

TOS: M. Lubin, I. Dunning & J. Huchette  
Next-Gen: B. Legat & O. Dowson  
Contributors: C. Coey, C. Coffrin, J. Dias Garcia, T. Koolen, V. Nesello, F. Pacaud, R. Schwarz, I. Tahiri, J. P. Vielma, U. Worsøe

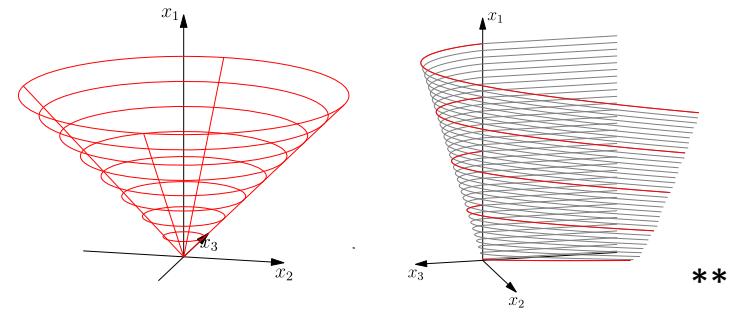
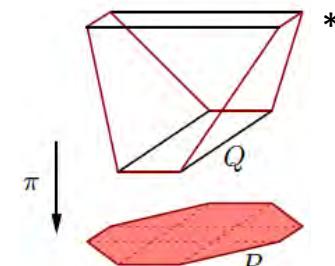
# Polyhedral Outer-Approximation for MICP Solvers

- Dynamically approximate **convex constraints** with **polyhedral** to combine **MILP** and **convex solvers**
  - Basis for most commercial & open source solvers
- Performance keys:
  1. Use power of **projection** to build **extended** or **lifted polyhedral relaxations**
  2. Exploit **geometry** and **duality** from **Conic Programming** (SOCP, SDP, etc.):

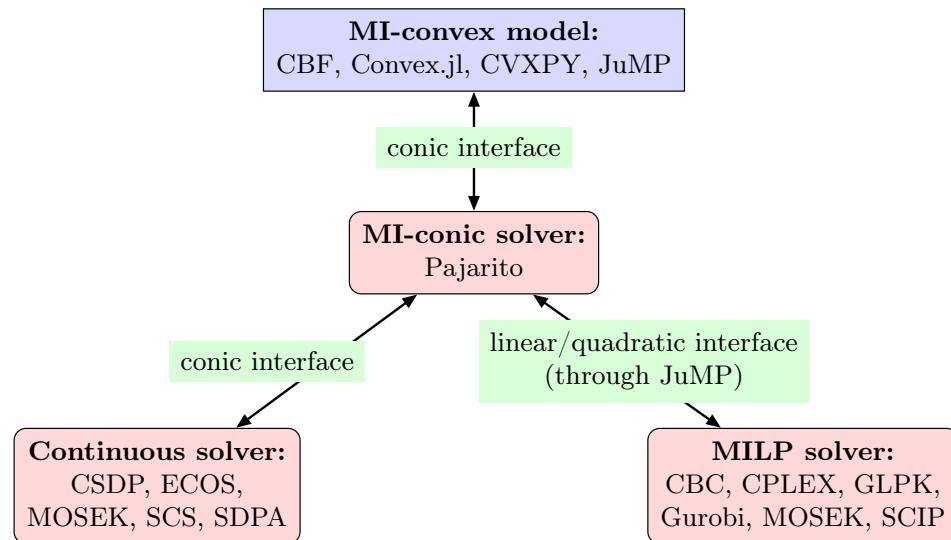
$$\min_{\mathbf{x} \in \mathbb{R}^N} \langle \mathbf{c}, \mathbf{x} \rangle : \mathbf{b}_k - \mathbf{A}_k \mathbf{x} \in \mathcal{C}_k, \quad x_i \in \mathbb{Z}$$

For closed convex cones  $\mathcal{C}_k$

\* <https://rjlipton.wordpress.com>, \*\* MOSEK Modelling Cookbook



# Pajarito: A Julia-based MI<sup>CONIC</sup> Solver



- Solved gams01, tls5 and tls6 (MINLPLIB2)
- Fastest “open-source” MISOCP solver:
  - faster than Bonmin nearly matches SCIP
- Improves performance & reliability of CPLEX

## Stability of **CONIC Interior Point** Algorithms is KEY!

---

- Why? Avoid non-differentiability issues? Stronger theory?
- Industry change in 2018:
  - **KNITRO**® version 11.0 adds support for SOCP constraints
  - **mosek** version 9.0 deprecates nonlinear formulations

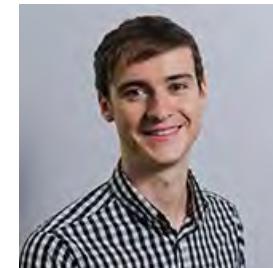
$$\begin{array}{ll}\min & f(x) \\ \text{subject to} & g(x) \leq 0,\end{array}$$

and focuses on pure conic (linear, SOCP, rotated SOCP, SDP, exp & power)

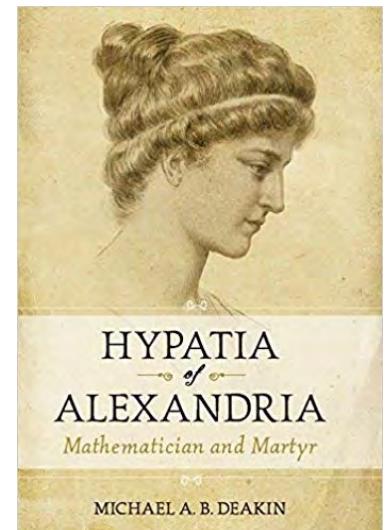
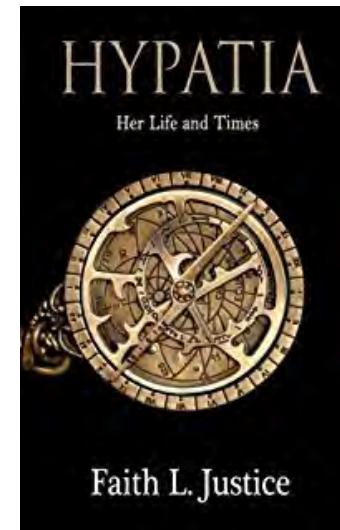
# Hypatia: Pure Julia-based IPM Beyond “Standard” Cones

---

- A homogeneous interior-point solver for nonsymmetric cones (based on Skajaa and Ye ‘15, Papp and Yıldız ‘17, Andersen, Dahl, and Vandenberge ‘04-18)
- Cones:
  - LP, SOCP, RSOCP, 3-dim exponential cone, PSD,  $L_\infty$ , n-dim power cone, spectral norm, log-Det cone,...
  - Sum-of-Squares, “Matrix” Sum-of-Squares, SOCP Sum-of-Squares, ...
- Customizable: “Bring your own barrier”



Chris Coey   Lea Kapelevich



## Summary

---

- MICP can model many problems (but not all)
- How to solve MICP? Don't solve MICP, solve MICONIC
- Easy access to optimization modeling and solvers with
- Advanced solver development with Julia &

