Mixed Integer Programming (MIP) for Daily Fantasy Sports, Statistics and Marketing

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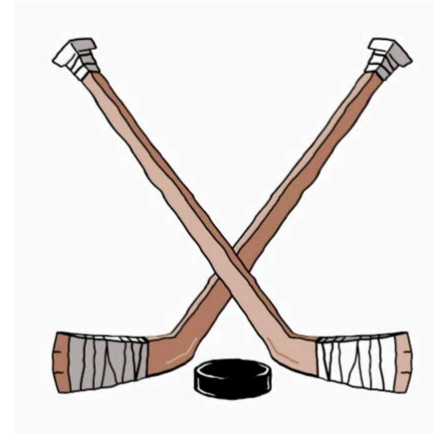
MIP & Daily Fantasy Sports

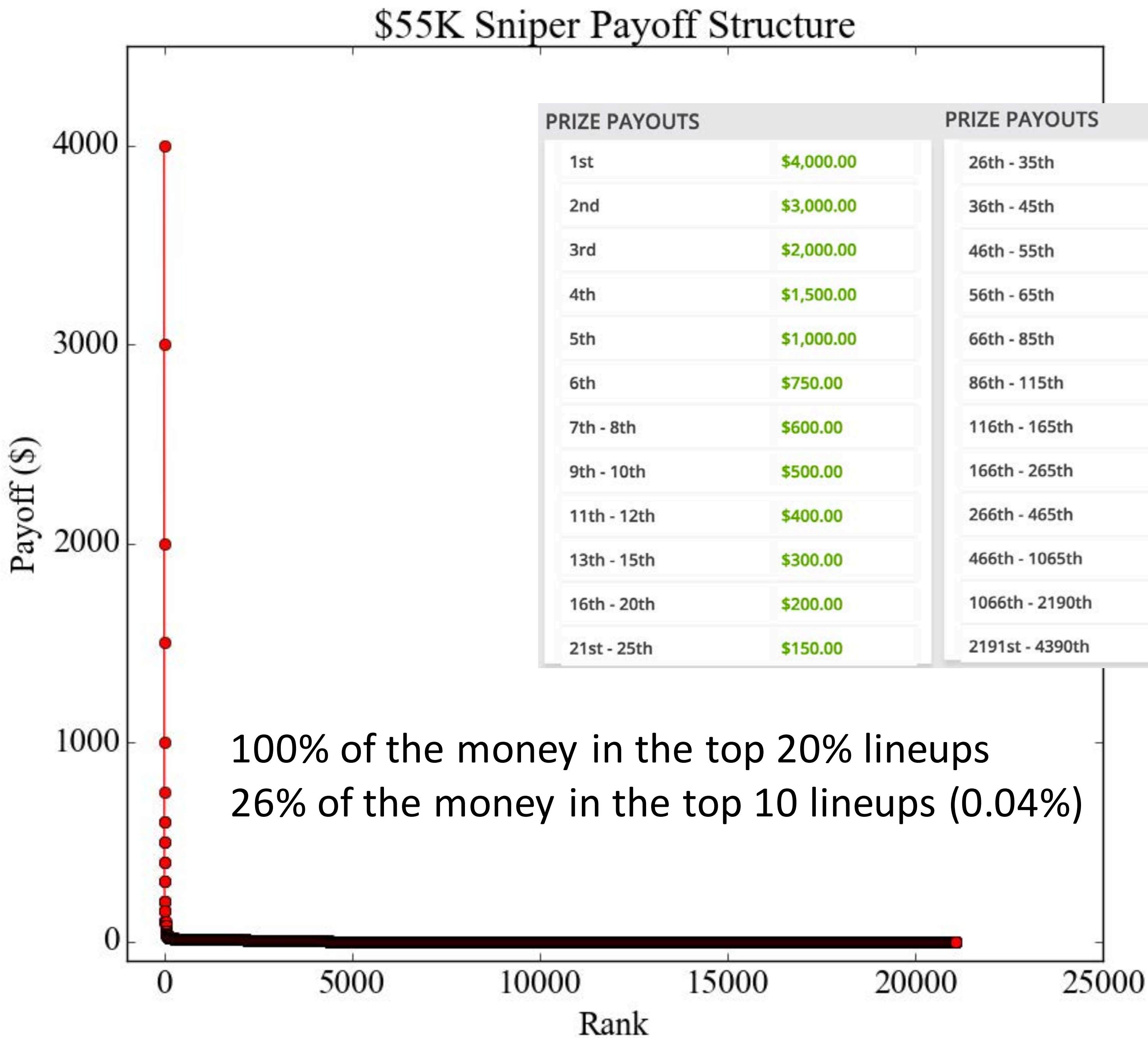
Example Entry



LINEUP

Avg. Rem. / Player: $0
Rem. Salary: $0

POS	PLAYER	OPP	FPPG	SALARY
C	Jussi Jokinen	Fla@Anh	3.1	$5,300
C	Brandon Sutter	Pit@Van	3.0	$4,400
W	Nikolaj Ehlers	Wpg@Tor	3.9	$4,800
W	Daniel Sedin	Pit@Van	3.8	$6,400
W	Radim Vrbata	Pit@Van	3.4	$5,800
D	Brian Campbell	Fla@Anh	2.6	$4,100
D	Morgan Rielly	Wpg@Tor	3.5	$4,200
G	Corey Crawford P	StL@Chi	6.3	$7,800
UTIL	Blake Wheeler	Wpg@Tor	4.8	$7,200



PRIZE PAYOUTS	
1st	$4,000.00
2nd	$3,000.00
3rd	$2,000.00
4th	$1,500.00
5th	$1,000.00
6th	$750.00
7th - 8th	$600.00
9th - 10th	$500.00
11th - 12th	$400.00
13th - 15th	$300.00
16th - 20th	$200.00
21st - 25th	$150.00

PRIZE PAYOUTS	
26th - 35th	$100.00
36th - 45th	$75.00
46th - 55th	$50.00
56th - 65th	$40.00
66th - 85th	$30.00
86th - 115th	$25.00
116th - 165th	$20.00
166th - 265th	$15.00
266th - 465th	$12.00
466th - 1065th	$10.00
1066th - 2190th	$8.00
2191st - 4390th	$6.00

100% of the money in the top 20% lineups

26% of the money in the top 10 lineups (0.04%)

Building a Lineup

Integer Programming Formulation

- We will make a bunch of lineups consisting of 9 players each
- Use an integer programming approach to find these lineups

Decision variables

$$x_{pl} = \begin{cases} 1, & \text{if player } p \text{ in lineup } l \\ 0, & \text{otherwise} \end{cases}$$

Basic Feasibility

- 9 different players
- Salary less than $50,000

Basic constraints

$$\sum_{p=1}^{N} c_p x_{pl} \leq \$50,000, \quad \text{(budget constraint)}$$

$$\sum_{p=1}^{N} x_{pl} = 9, \quad \text{(lineup size constraint)}$$

$$x_{pl} \in \{0, 1\}, \quad 1 \leq p \leq N.$$

Position Feasibility

- Between 2 and 3 centers
- Between 3 and 4 wingers
- Between 2 and 3 defensemen
- 1 goalie

Position constraints

$$2 \leq \sum_{p \in C} x_{pl} \leq 3, , \quad \text{(center constraint)}$$

$$3 \leq \sum_{u \in W} x_{pl} \leq 4, \quad \text{(winger constraint)}$$

$$2 \leq \sum_{u \in D} x_{pl} \leq 3, \quad \text{(defensemen constraint)}$$

$$\sum_{u \in G} x_{pl} = 1 \quad \text{(goalie constraint)}$$

Team Feasibility

- At least 3 different NHL teams

Team constraints

$$t_i \leq \sum_{p \in T_i} x_{pl}, \quad \forall\, i \in \{1, \ldots, N_T\}$$

$$\sum_{i=1}^{N_T} t_i \geq 3,$$

$$t_i \in \{0, 1\}, \quad \forall\, i \in \{1, \ldots, N_T\}.$$

Maximize Points

- Forecasted points for player p: f_p

RotoGrinders
The Daily Fantasy Games Authority

dfnerd

Score type	Points
Goal	3
Assist	2
Shot on Goal	0.5
Blocked Shot	0.5
Short Handed Point Bonus (Goal/Assist)	1
Shootout Goal	0.2
Hat Trick Bonus	1.5
Win (goalie only)	3
Save (goalie only)	0.2
Goal allowed (goalie only)	-1
Shutout Bonus (goalie only)	2

Table 1 Points system for NHL contests in DraftKings.

Points Objective Function

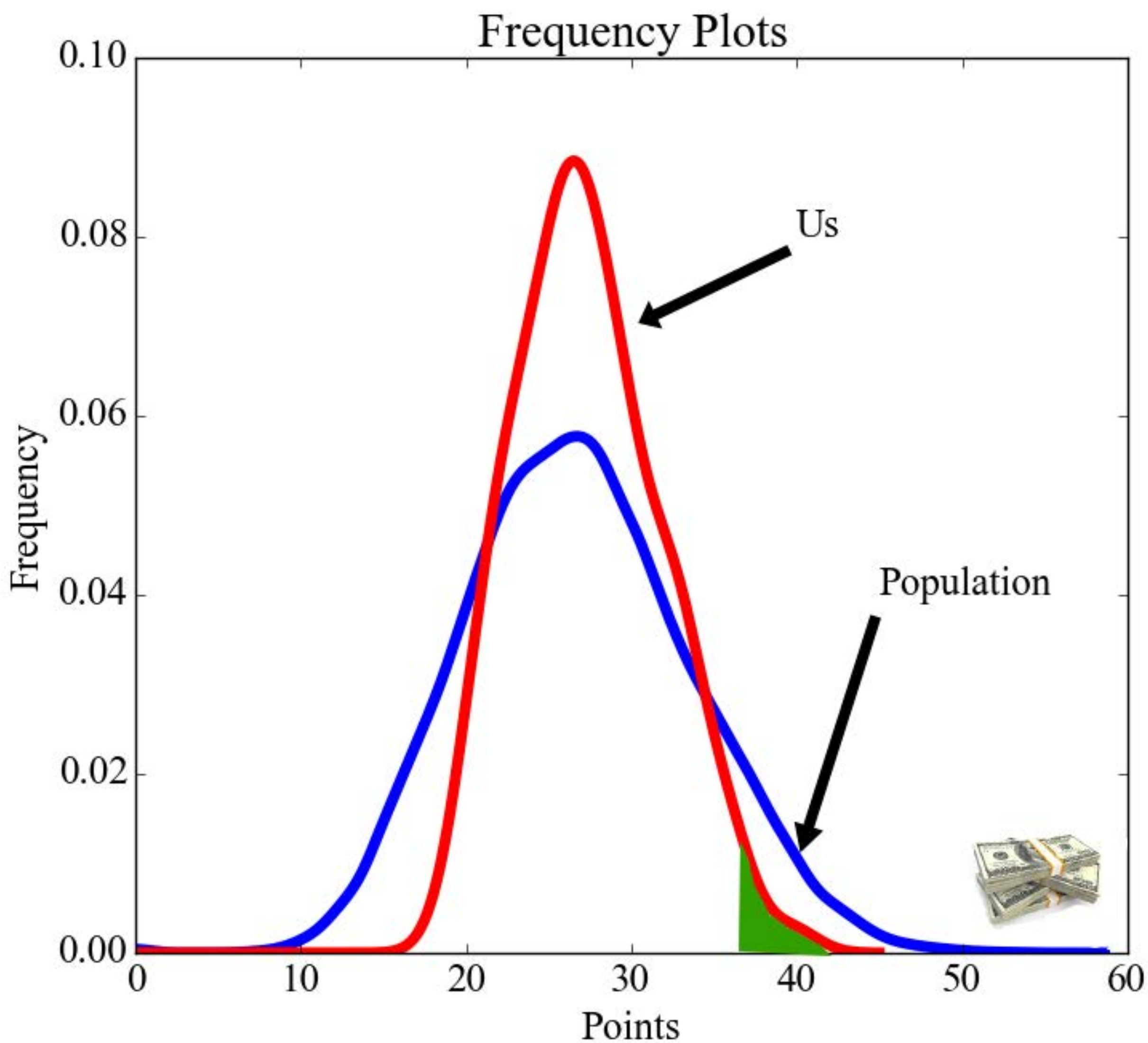
$$\sum_{p=1}^{N} f_p x_{pl}$$

Lineup

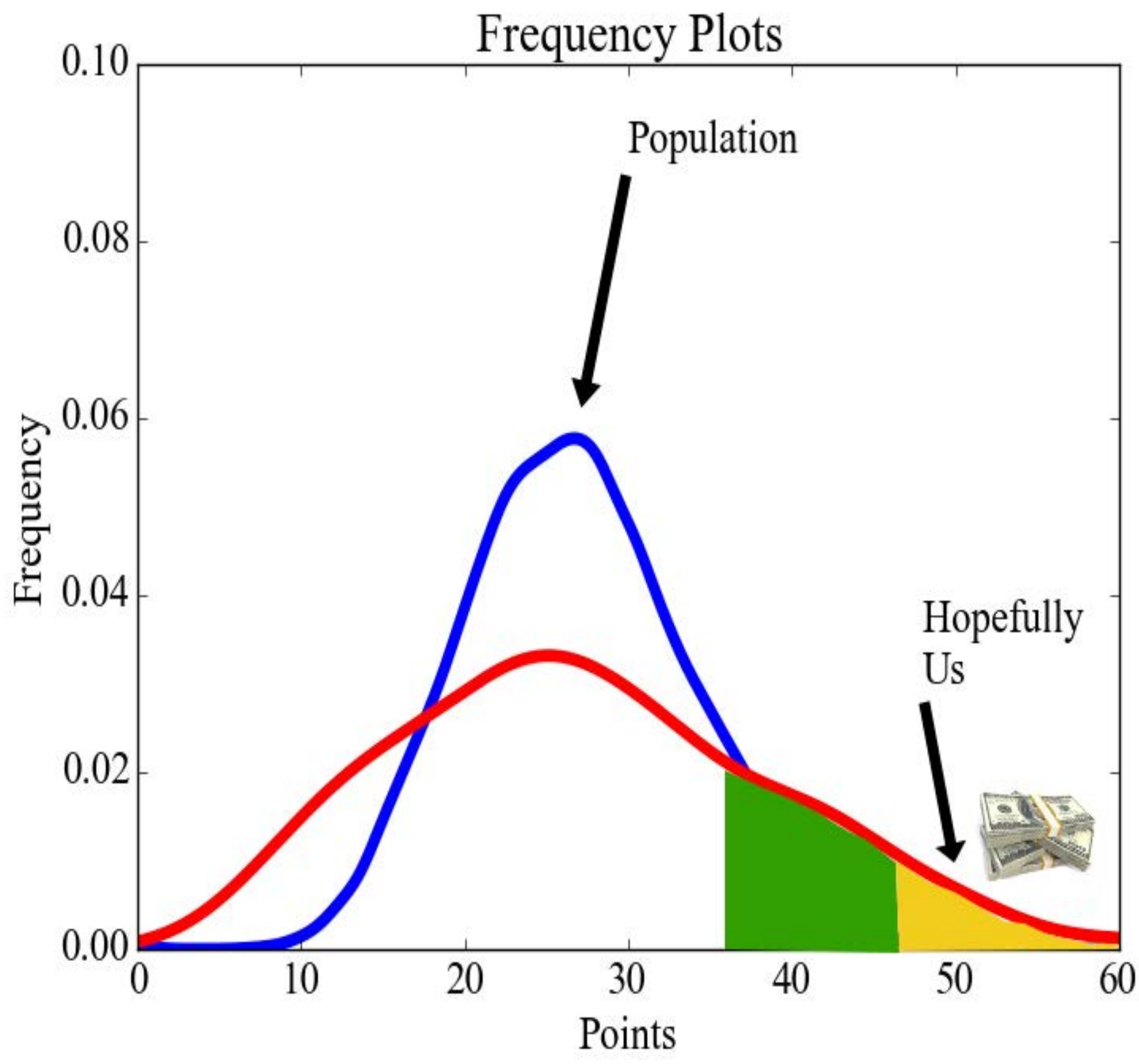
Projections:	5.4	2.5	3.4	3.0	3.2	4.2	3.5	3.4	5.7
	$9500	$2700	$4600	$3800	$4600	$6400	$5200	$5100	$8000
	W	UTIL	D	D	C	C	W	W	G

23 points on average

Need > 38 points for a chance to win



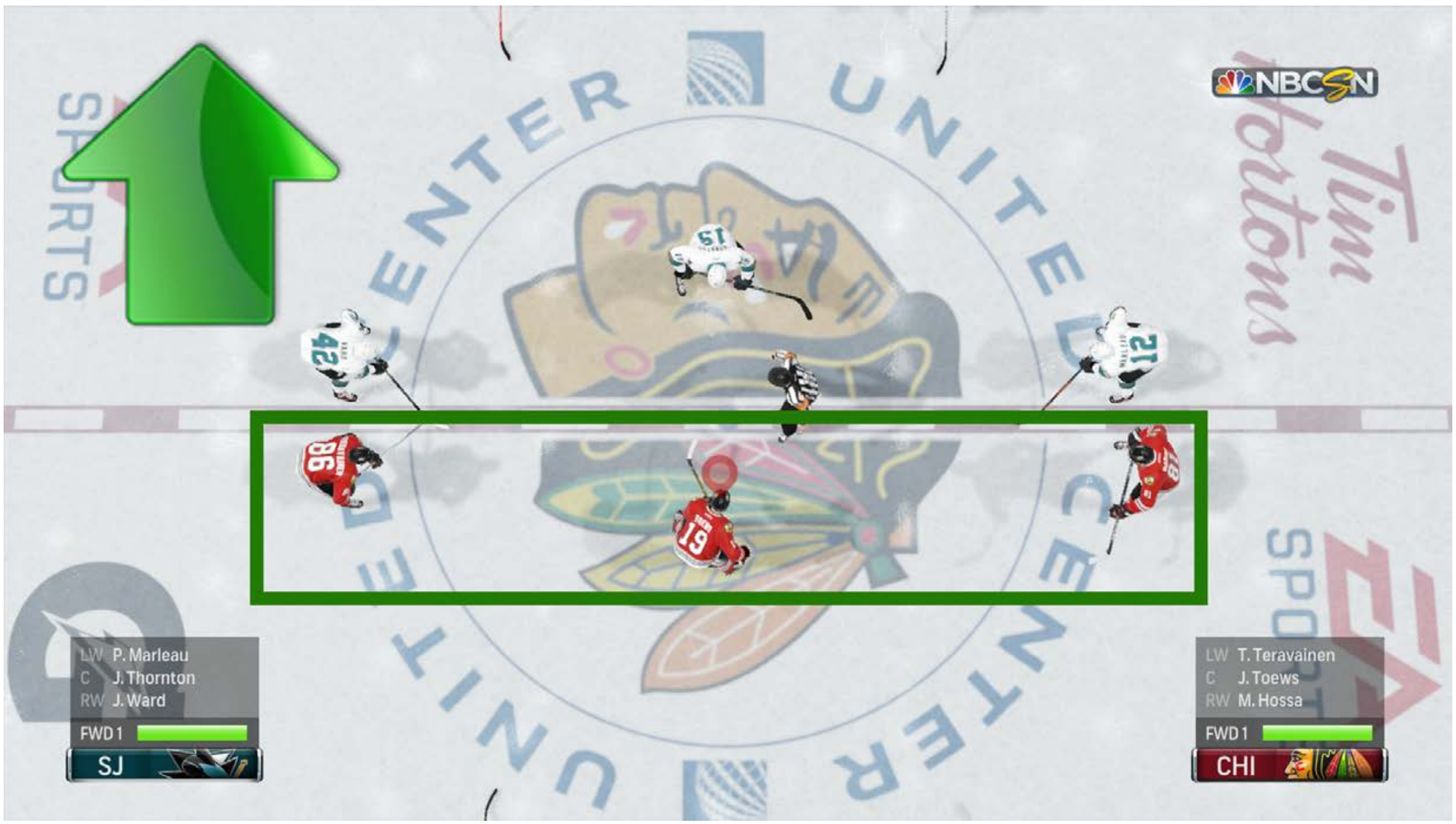
Increase variance to have a chance



Structural Correlations - Teams

Structural Correlations - Lines

- Goal = 3 pt, assist = 2 pt



Structural Correlations – Lines = Stacking

- At least 1 complete line (3 players per line)
- At least 2 partial lines (at least 2 players per line)

1 complete line constraint

$$3v_i \leq \sum_{p \in L_i} x_{pl}, \qquad \forall i \in \{1, \ldots, N_L\}$$

$$\sum_{i=1}^{N_L} v_i \geq 1$$

$$v_i \in \{0, 1\}, \qquad \forall i \in \{1, \ldots, N_L\}.$$

2 partial lines constraint

$$2w_i \leq \sum_{p \in L_i} x_{pl}, \qquad \forall i \in \{1, \ldots, N_L\}$$

$$\sum_{i=1}^{N_L} w_i \geq 2$$

$$w_i \in \{0, 1\}, \qquad \forall i \in \{1, \ldots, N_L\}.$$

Structural Correlations – Goalie Against Opposing Players

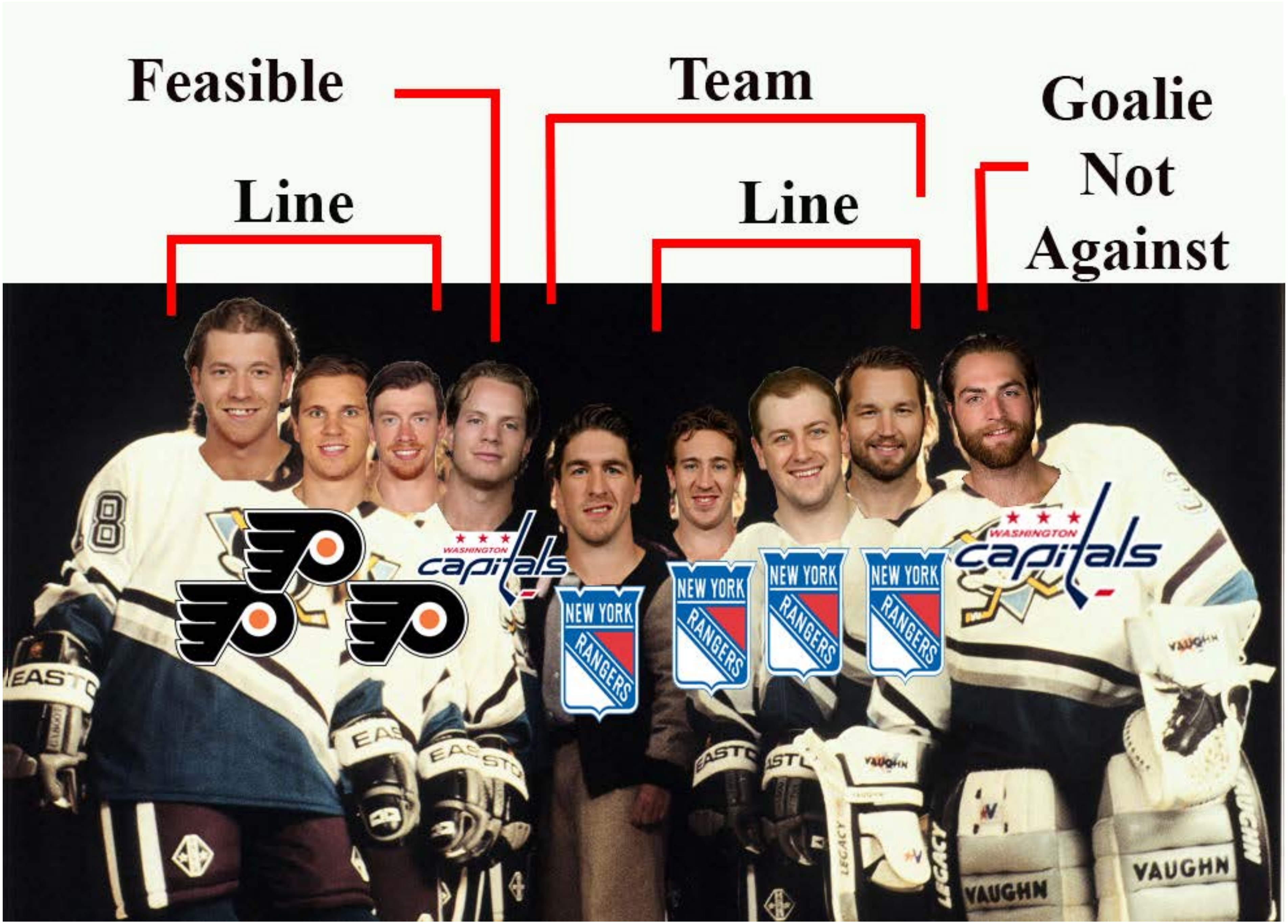
Structural Correlations – Goalie Against Skaters

- No skater against goalie

No skater against goalie constraint

$$6x_{pl} + \sum_{q \in Opponents_p} x_{ql} \leq 6, \quad \forall p \in G$$

Good, but not great chance



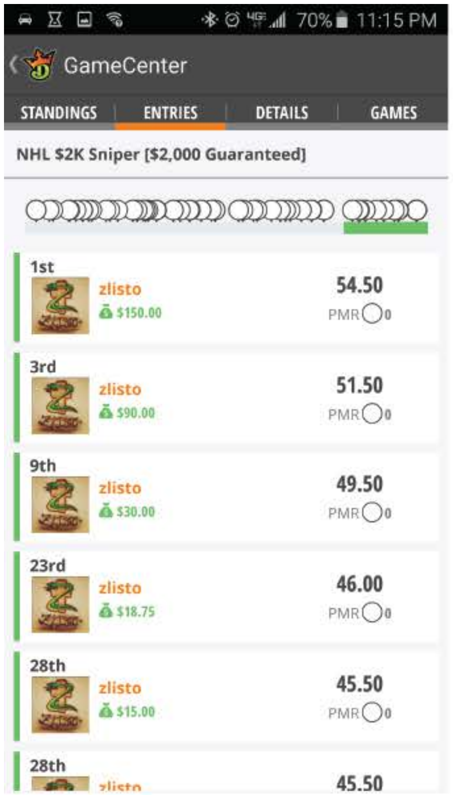
Play many diverse Lineups

- Make sure lineup l has no more than γ players in common with lineups 1 to l-1

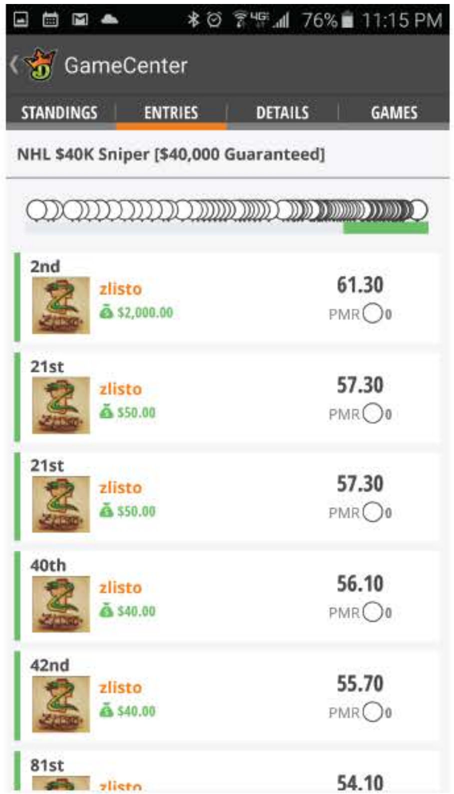
Diversity constraint

$$\sum_{p=1}^{N} x^*_{pk} x_{pl} \leq \gamma, k = 1, \ldots, l-1$$

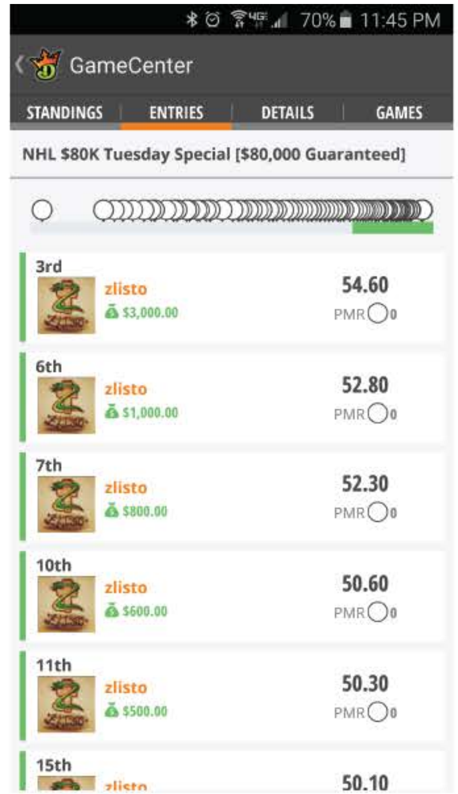
Were we able to do it?



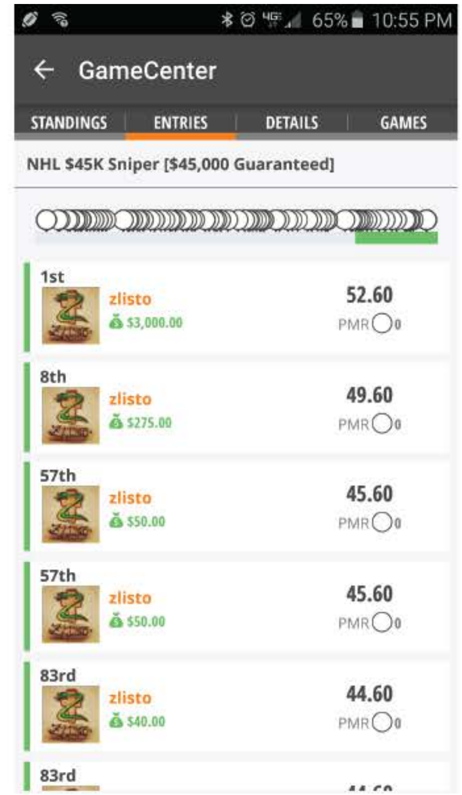
November 15, 2015



November 16, 2015



November 17, 2015



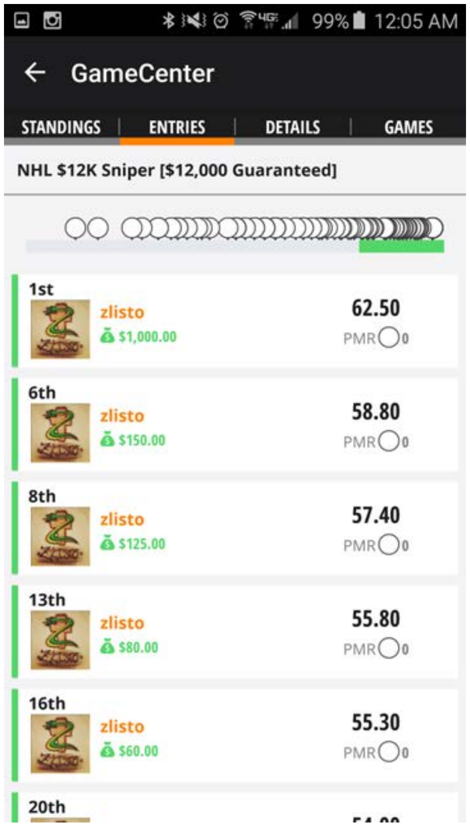
November 23, 2015

200 lineups

Policy Change

200 lineups -> 100 lineups

Were we able to continue it?



December 12, 2015

The Greater Boston FOOD BANK

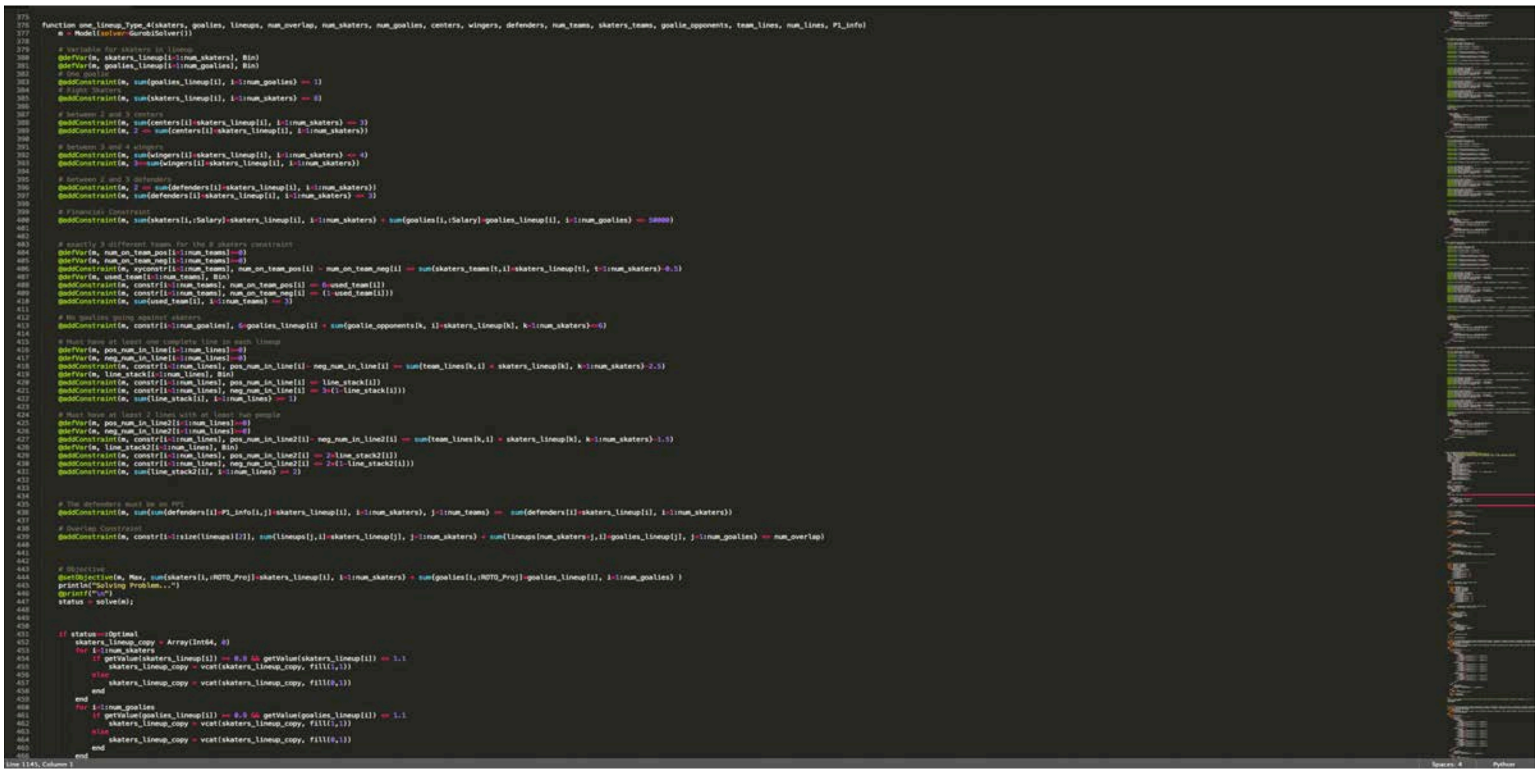
> $15K

100 lineups

How can you do it?

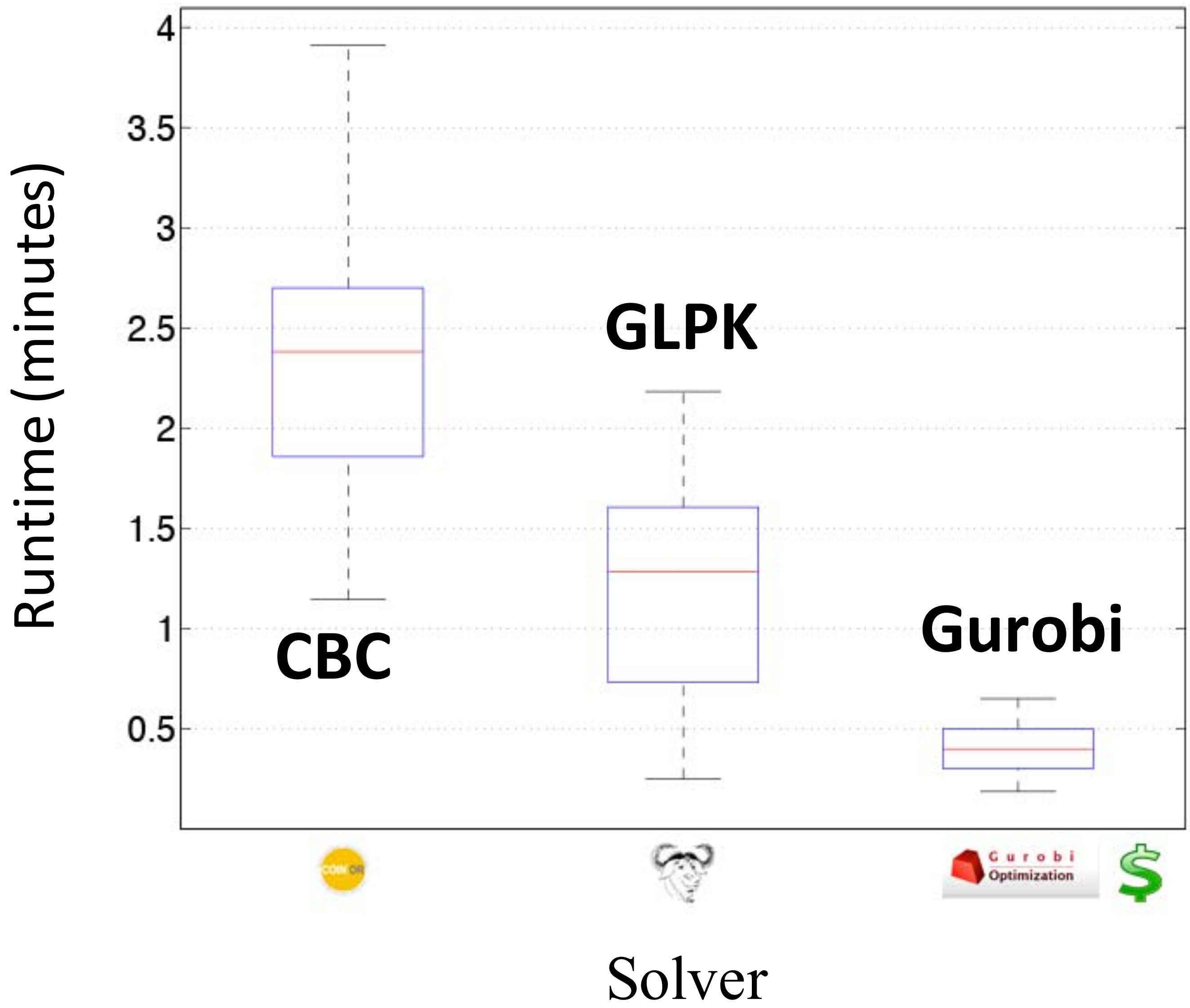
Download Code from Github:

https://github.com/dscotthunter/Fantasy-Hockey-IP-Code



http://arxiv.org/pdf/1604.01455v1.pdf

Performance Time
< 30 Minutes



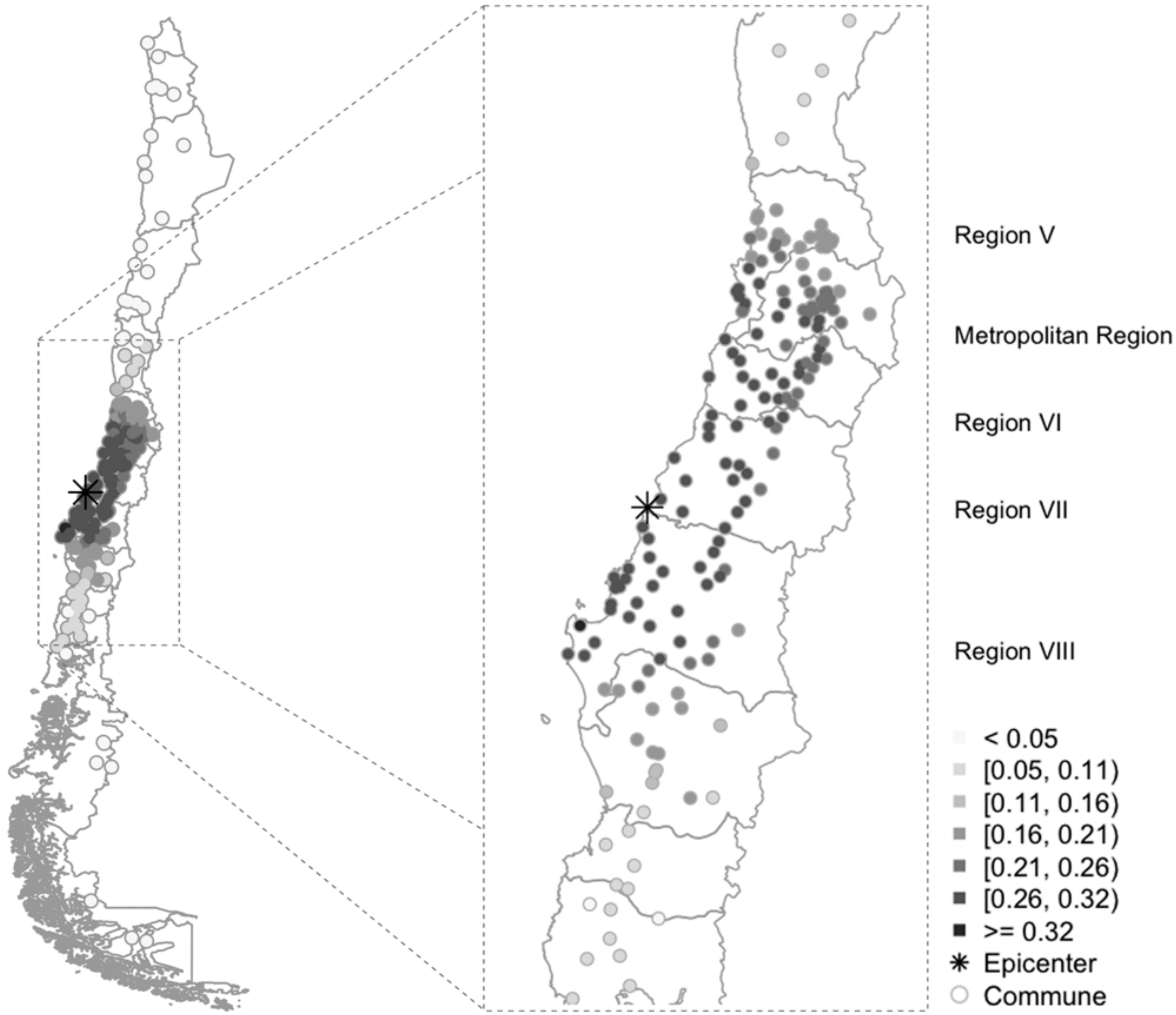
MIP and Statistics: Inference for the Chilean Earthquake

The 2010 Chilean Earthquake

6th Strongest in Recorded History (8.8)

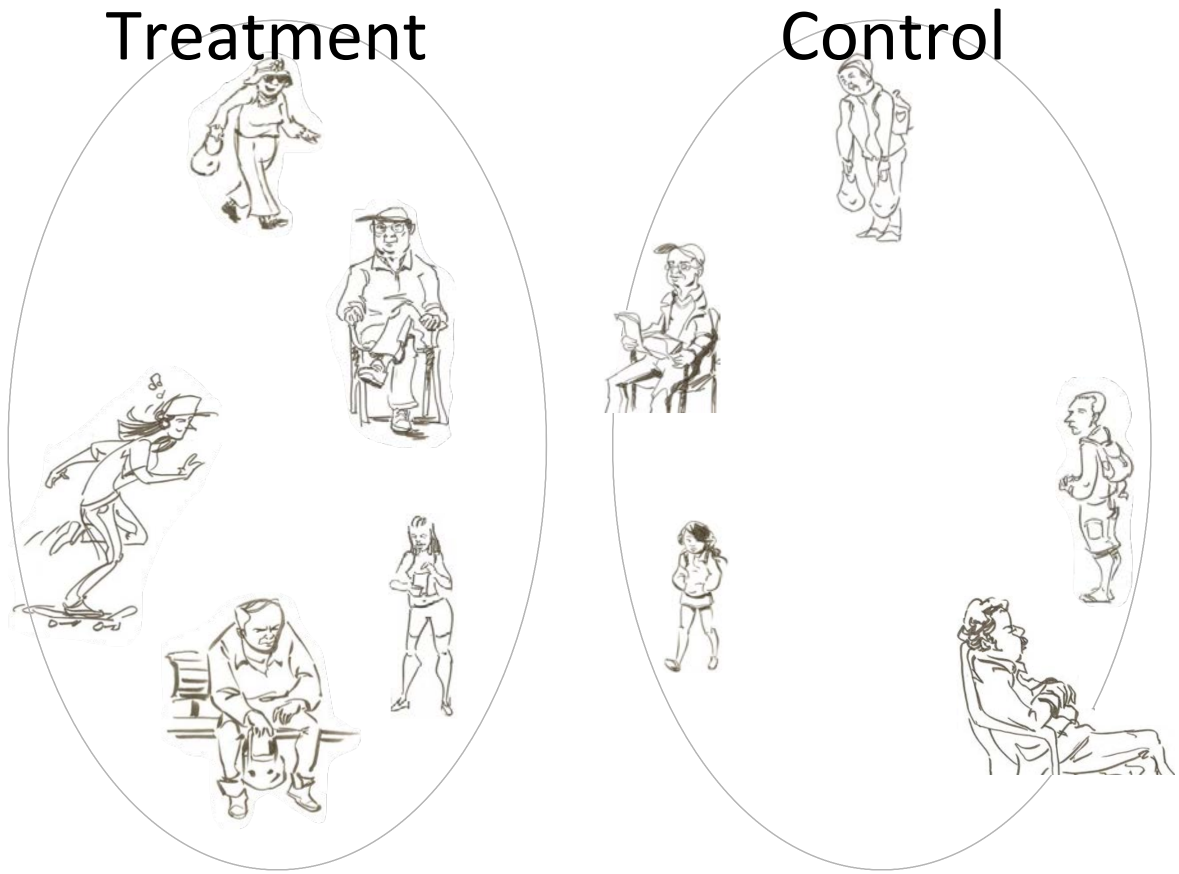
Impact on Educational Achievement? PSU = SAT

Earthquake Intensity + Great Demographic Info



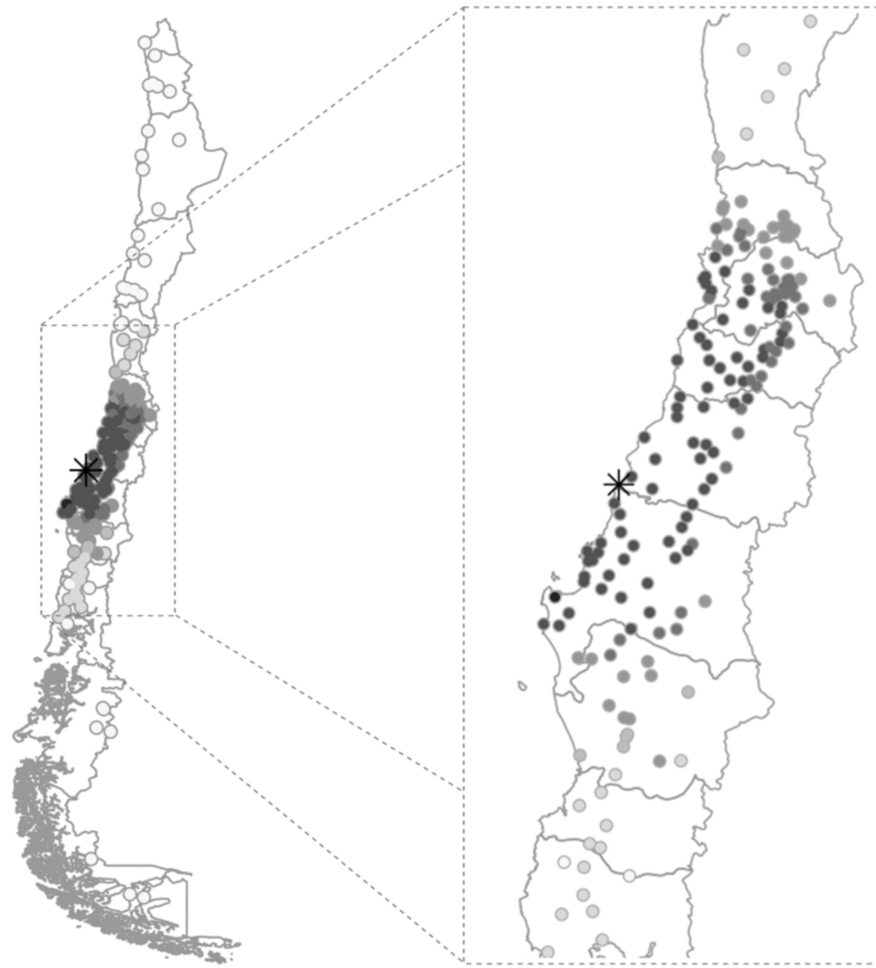
Randomized experiment

- Treatment / control have similar characteristics (covariates).



Covariate Balance Important for Inference

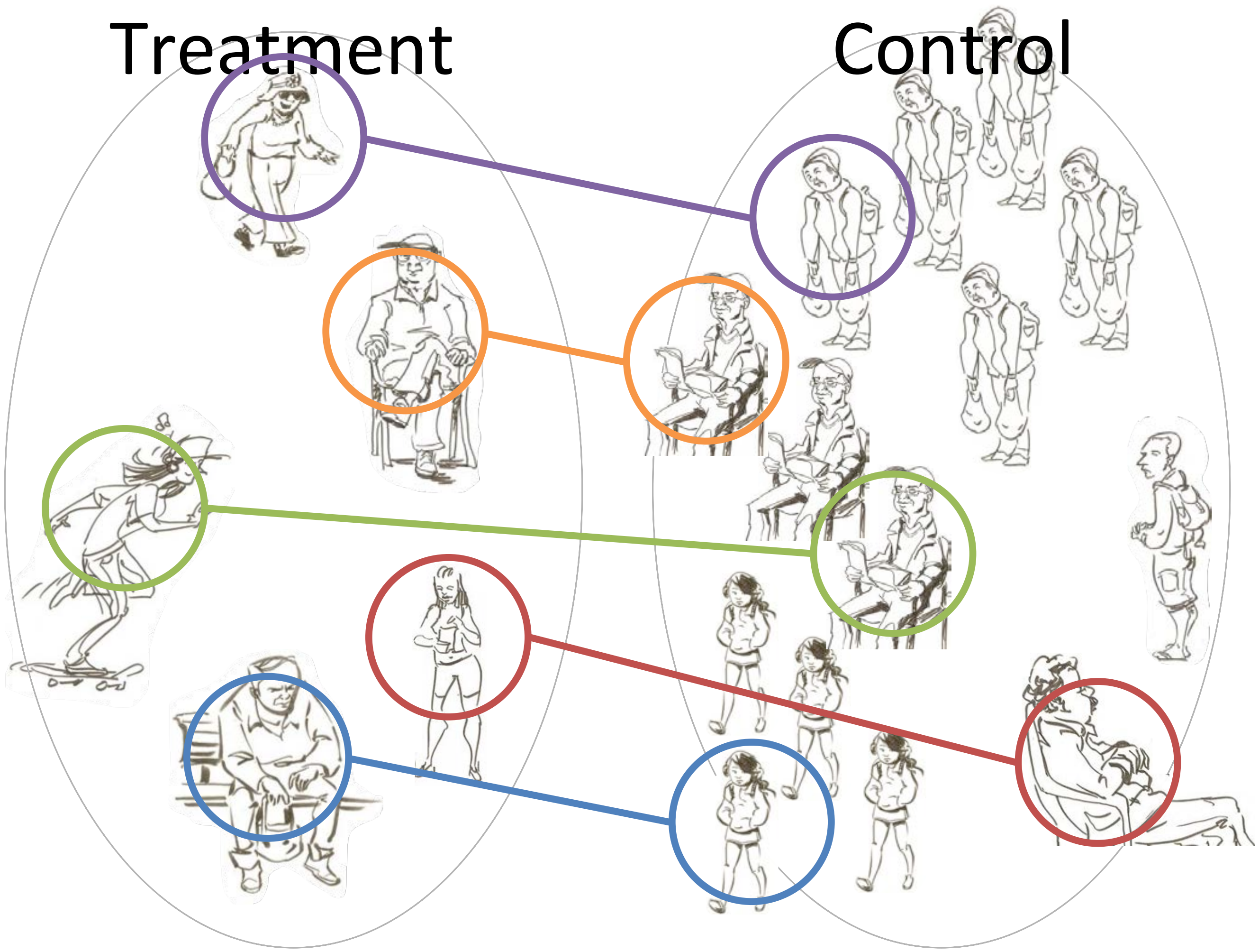
Covariate	Dose 1	Dose 2
Gender		
Male	462	462
Female	538	538
School SES		
Low	75	75
Mid-low	327	327
Medium	294	294
Mid-high	189	189
High	115	115
Mother's education		
Primary	335	335
Secondary	426	426
Technical	114	114
College	114	114
Missing	11	11
⋮		



Observational Study: e.g. After Earthquake

- Treatment / control can have different characteristics.

Solution
=
Matching?



Matching

Treated Units: $\mathcal{T} = \{t_1, \ldots, t_T\}$

Control Units: $\mathcal{C} = \{c_1, \ldots, c_C\}$

Observed Covariates: $\mathcal{P} = \{p_1, \ldots, p_P\}$

Covariate Values:
$$\begin{aligned} \mathbf{x}^t &= \left(x_p^t\right)_{p \in \mathcal{P}}, \quad & t \in \mathcal{T} \\ \mathbf{x}^c &= \left(x_p^c\right)_{p \in \mathcal{P}}, \quad & c \in \mathcal{C} \end{aligned}$$

Nearest Neighbor Matching

$$\underset{\boldsymbol{m}}{\text{minimize}} \quad \sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} \delta_{t,c} m_{t,c}$$

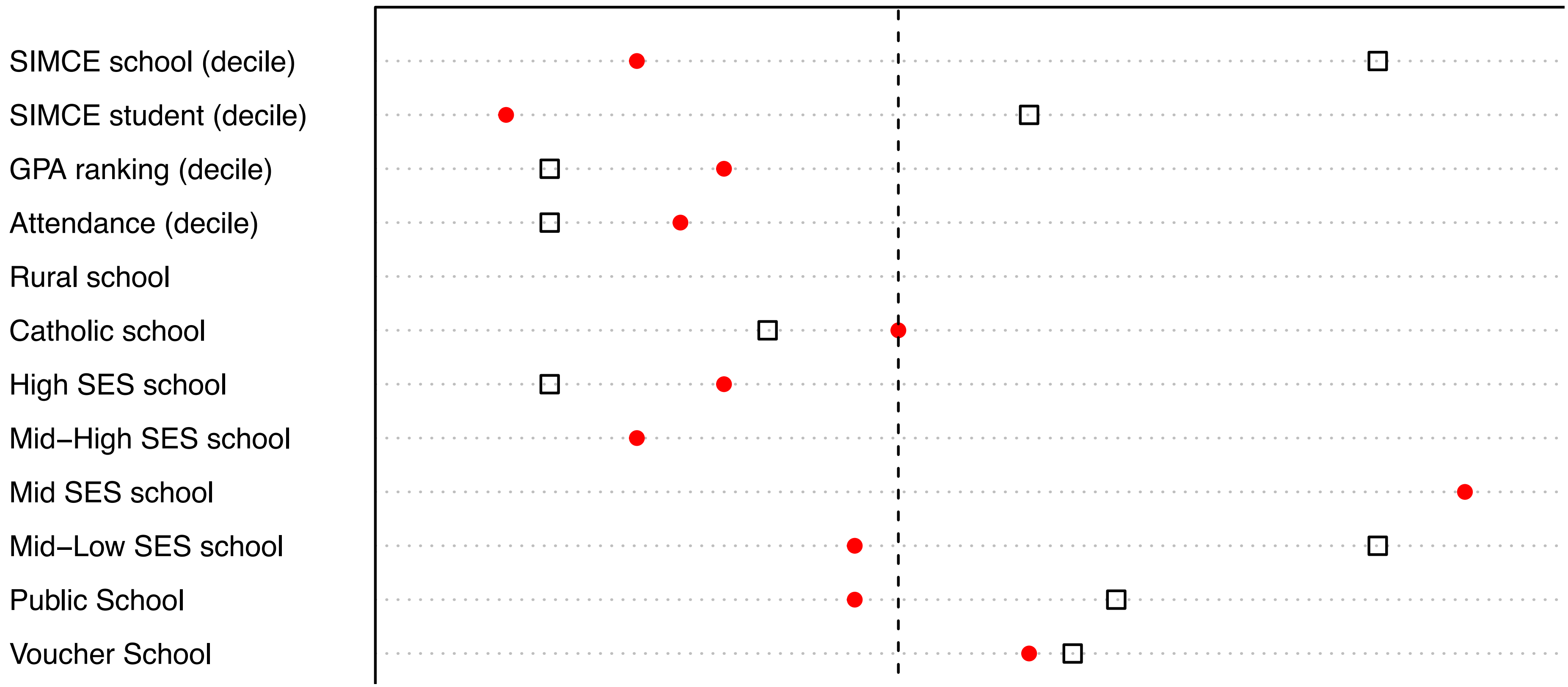
$$\text{subject to} \quad \sum_{c \in \mathcal{C}} m_{t,c} = 1, \; t \in \mathcal{T}$$

$$\sum_{t \in \mathcal{T}} m_{t,c} \leq 1, \; c \in \mathcal{C}$$

$$0 \leq m_{t,c} \leq 1 \quad \cancel{m_{t,c}} \in \{0, 1\}, \; t \in \mathcal{T}, c \in \mathcal{C}$$

- e.g. $\delta_{t,c} = \left\| \mathbf{x}^t - \mathbf{x}^c \right\|_2$
- Easy to solve

Balance Before After Matching



Maximum Cardinality Matching

$$\mathcal{K}(p) = \{\mathbf{x}_p^c\}_{c\in\mathcal{P}} \cup \{\mathbf{x}_p^t\}_{t\in\mathcal{T}}$$

$$\mathcal{C}_{p,k} = \{c \in \mathcal{C} \;:\; \mathbf{x}_p^c = k\}$$

$$\mathcal{T}_{p,k} = \{t \in \mathcal{T} \;:\; \mathbf{x}_p^t = k\}$$

$$\begin{aligned}
\max \quad & \sum_{t\in\mathcal{T}}\sum_{c\in\mathcal{C}} m_{t,c} \\
s.t. \quad & \sum_{t\in\mathcal{T}} m_{t,c} \leq 1, && \forall c \in \mathcal{C} \\
& \sum_{c\in\mathcal{C}} m_{t,c} \leq 1, && \forall t \in \mathcal{T} \\
& \sum_{t\in\mathcal{T}_{p,k}}\sum_{c\notin\mathcal{C}_{p,k}} m_{t,c} = \sum_{t\notin\mathcal{T}_{p,k}}\sum_{c\in\mathcal{C}_{p,k}} m_{t,c} && \forall p \in \mathcal{P}, k \in \mathcal{K}(p) \\
& m_{t,c} \in \{0,1\} && \forall t \in \mathcal{T}, \quad c \in \mathcal{C}.
\end{aligned}$$

- Very hard to solve (and very hard to understand!)

Advanced Maximum Cardinality Matching

$$\max \quad \sum_{t \in \mathcal{T}} x_t$$

$$\mathcal{K}(p) = \{\mathbf{x}_p^c\}_{c \in \mathcal{P}} \cup \{\mathbf{x}_p^t\}_{t \in \mathcal{T}}$$

$$\mathcal{C}_{p,k} = \{c \in \mathcal{C} \;:\; \mathbf{x}_p^c = k\}$$

$$\mathcal{T}_{p,k} = \{t \in \mathcal{T} \;:\; \mathbf{x}_p^t = k\}$$

$s.t.$

$$\sum_{t \in \mathcal{T}} x_t = \sum_{c \in \mathcal{C}} y_c,$$

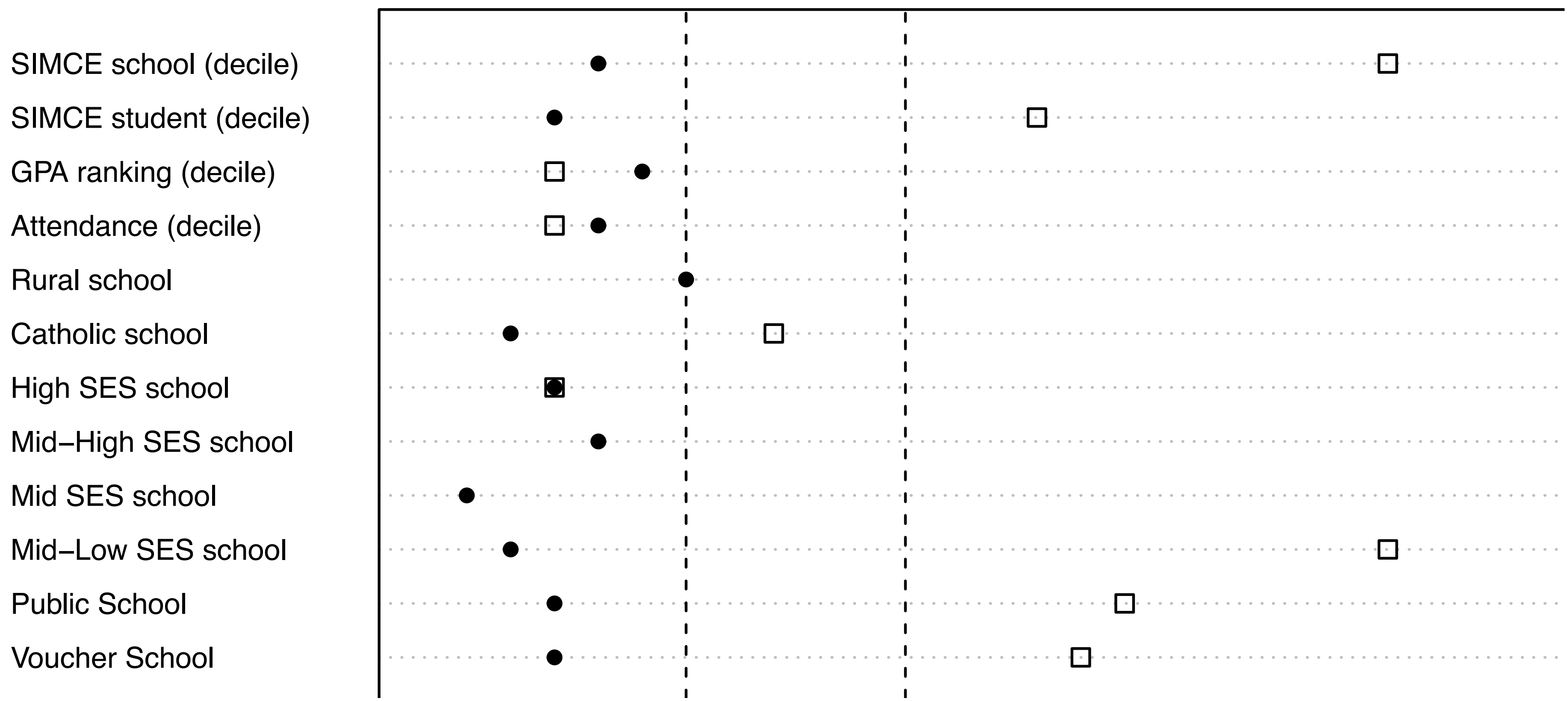
$$\sum_{t \in \mathcal{T}_{p,k}} x_t = \sum_{c \in \mathcal{C}_{p,k}} y_c, \quad \forall p \in \mathcal{P}, \quad k \in \mathcal{K}(p)$$

$$x_t \in \{0,1\} \qquad \forall t \in \mathcal{T}$$

$$y_c \in \{0,1\} \qquad \forall c \in \mathcal{C}.$$

- Matching without matching variables
- Easy to solve: Small, but inherits matching properties

Balance Before After Cardinality Matching

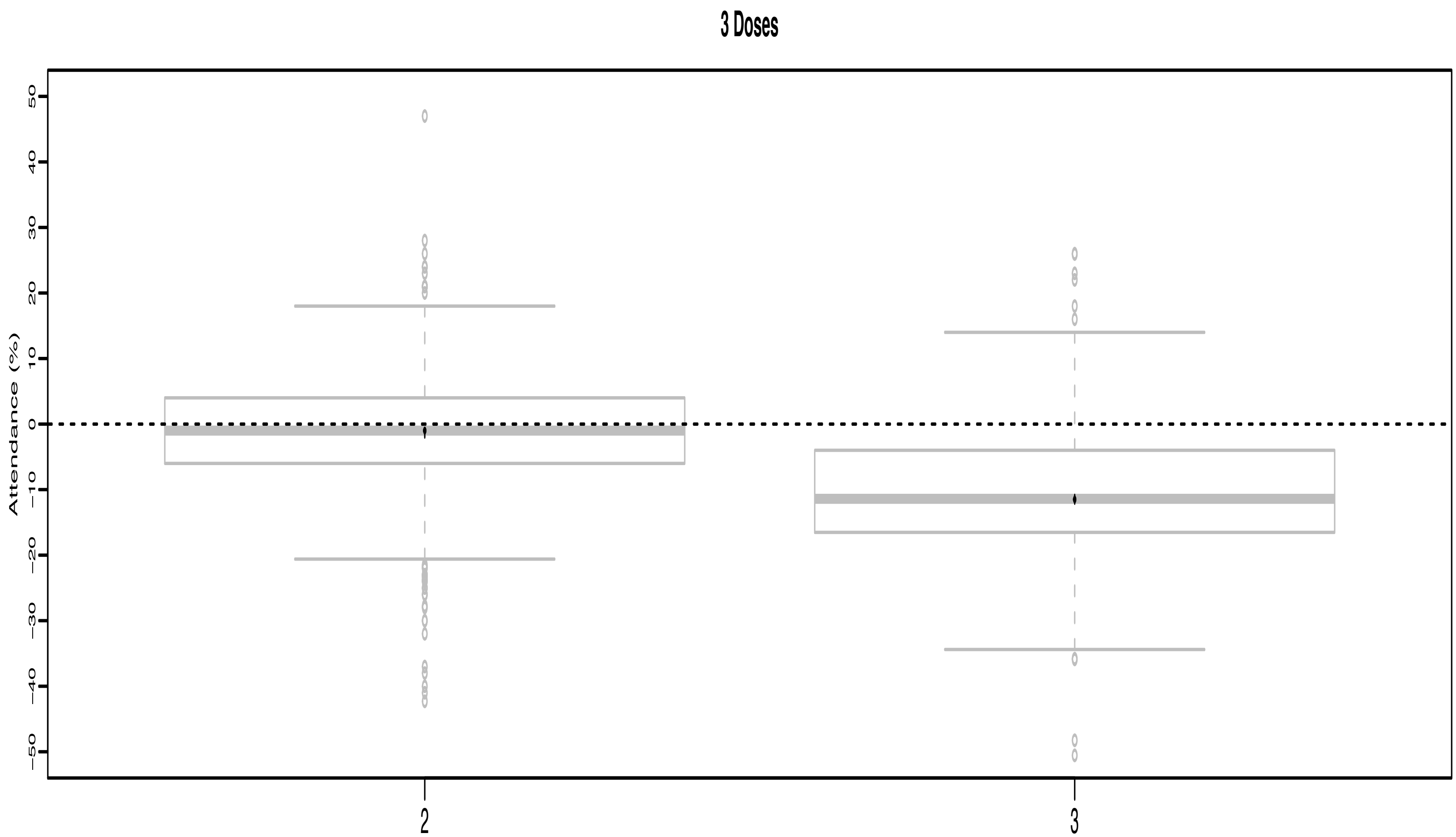


Can Also do Multiple Doses

- Dose
  1. No quake
  2. Medium quake
  3. Bad quake

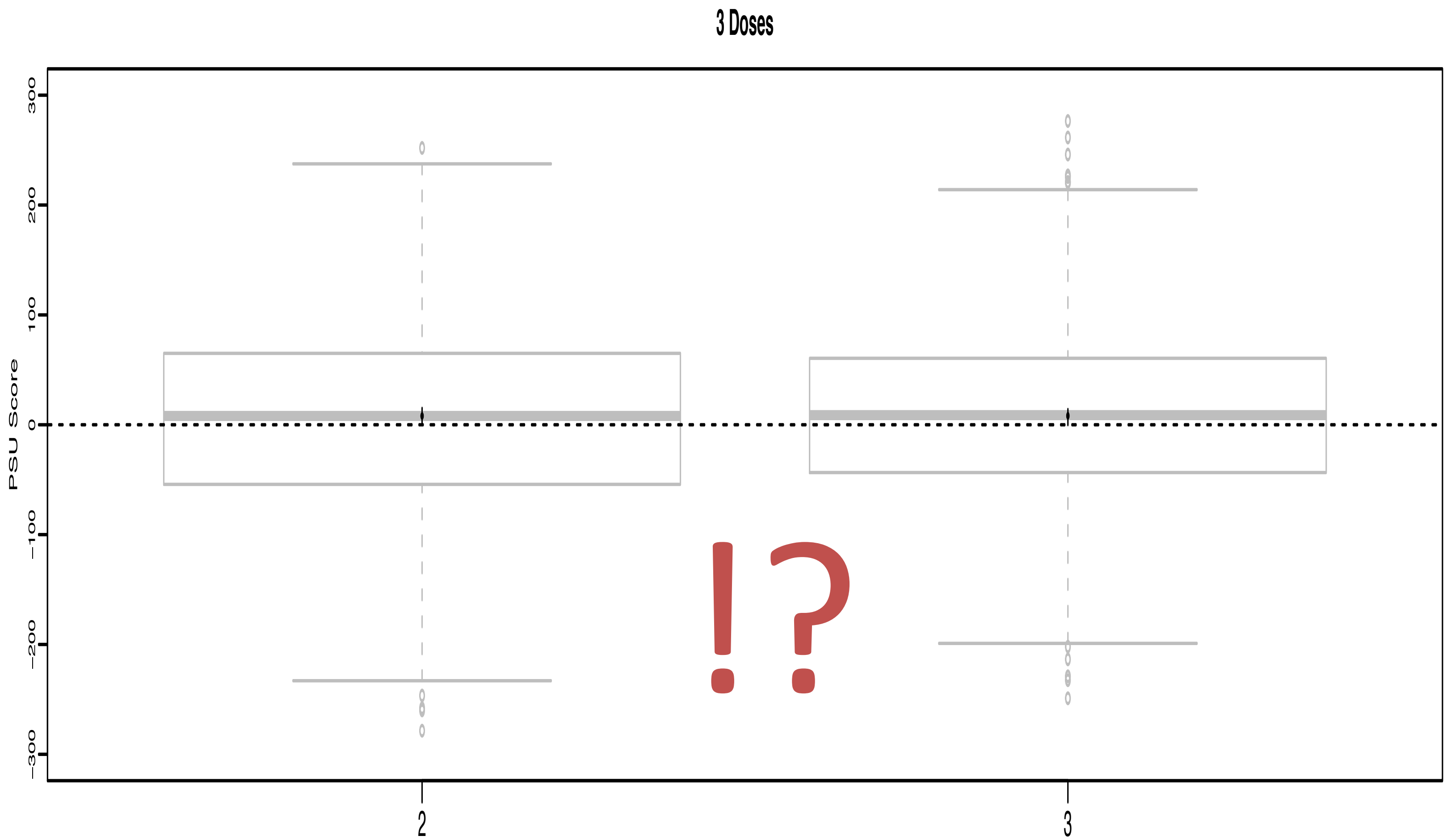
	Dose		
Covariate	1	2	3
Gender			
Male	462	462	462
Female	538	538	538
School SES			
Low	75	75	75
Mid-low	327	327	327
Medium	294	294	294
Mid-high	189	189	189
High	115	115	115
Mother's education			
Primary	335	335	335
Secondary	426	426	426
Technical	114	114	114
College	114	114	114
Missing	11	11	11
⋮			

Relative (To no Quake) Attendance Impact



Medium quake　　　　Bad quake

Relative (To no Quake) PSU Score Impact



Medium quake

Bad quake

MIP and Marketing: Chewbacca or BB-8?

Adaptive Preference Questionnaires

Feature	SX530	RX100
Zoom	50x	3.6x
Prize	$249.99	$399.99
Weight	15.68 ounces	7.5 ounces
Prefer	☑	☐

Feature	TG-4	G9
Waterproof	Yes	No
Prize	$249.99	$399.99
Weight	7.36 lb	7.5 lb
Prefer	☐	☑

Feature	TG-4	Galaxy 2
Waterproof	Yes	No
Prize	$249.99	$399.99
Viewfinder	Electronic	Optical
Prefer	☑	☐

We recommend:

Choice-based Conjoint Analysis (CBCA)

Feature	Chewbacca	BB-8
Wookiee	Yes	No
Droid	No	Yes
Blaster	Yes	No
I would buy toy	☑	☐

Product Profile	x^1	x^2

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = x^2$$

Preference Model and Geometric Interpretation

- Utilities for 2 products, d features, logit model

$$U_1 = \beta \cdot x^1 + \cancel{\epsilon_1} = \sum_{i=1}^{d} \beta_i x_i^1 + \cancel{\epsilon_1}$$

$$U_2 = \beta \cdot x^2 + \cancel{\epsilon_2} = \sum_{i=1}^{d} \beta_i x_i^2 + \cancel{\epsilon_2}$$

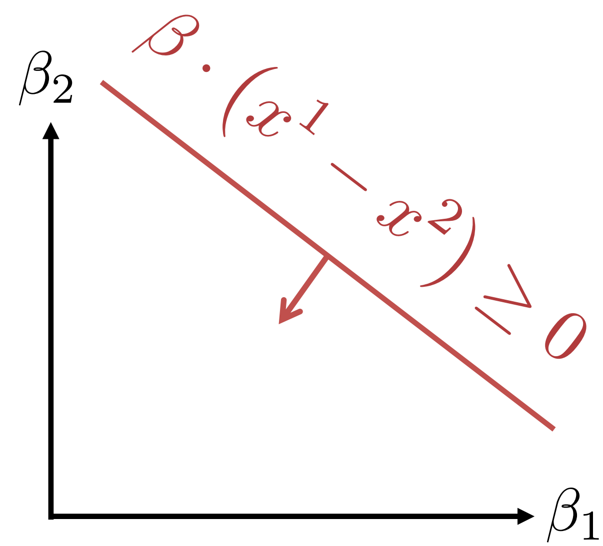
part-worths

product profile

noise (gumbel)

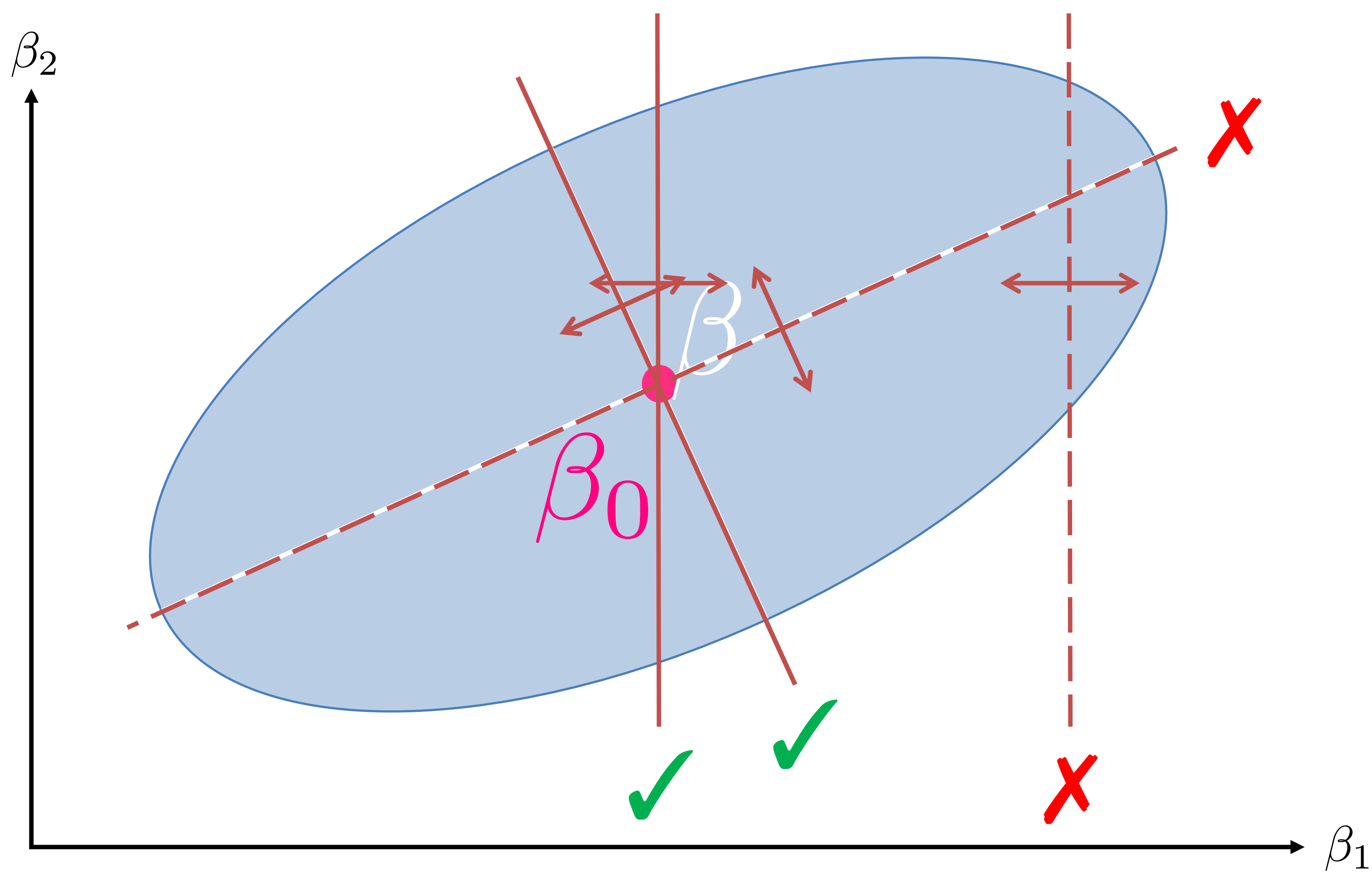
- Utility maximizing customer
  - Geometric interpretation of preference for product 1 without error

$$x^1 \succeq x^2 \Leftrightarrow U_1 \geq U_2$$

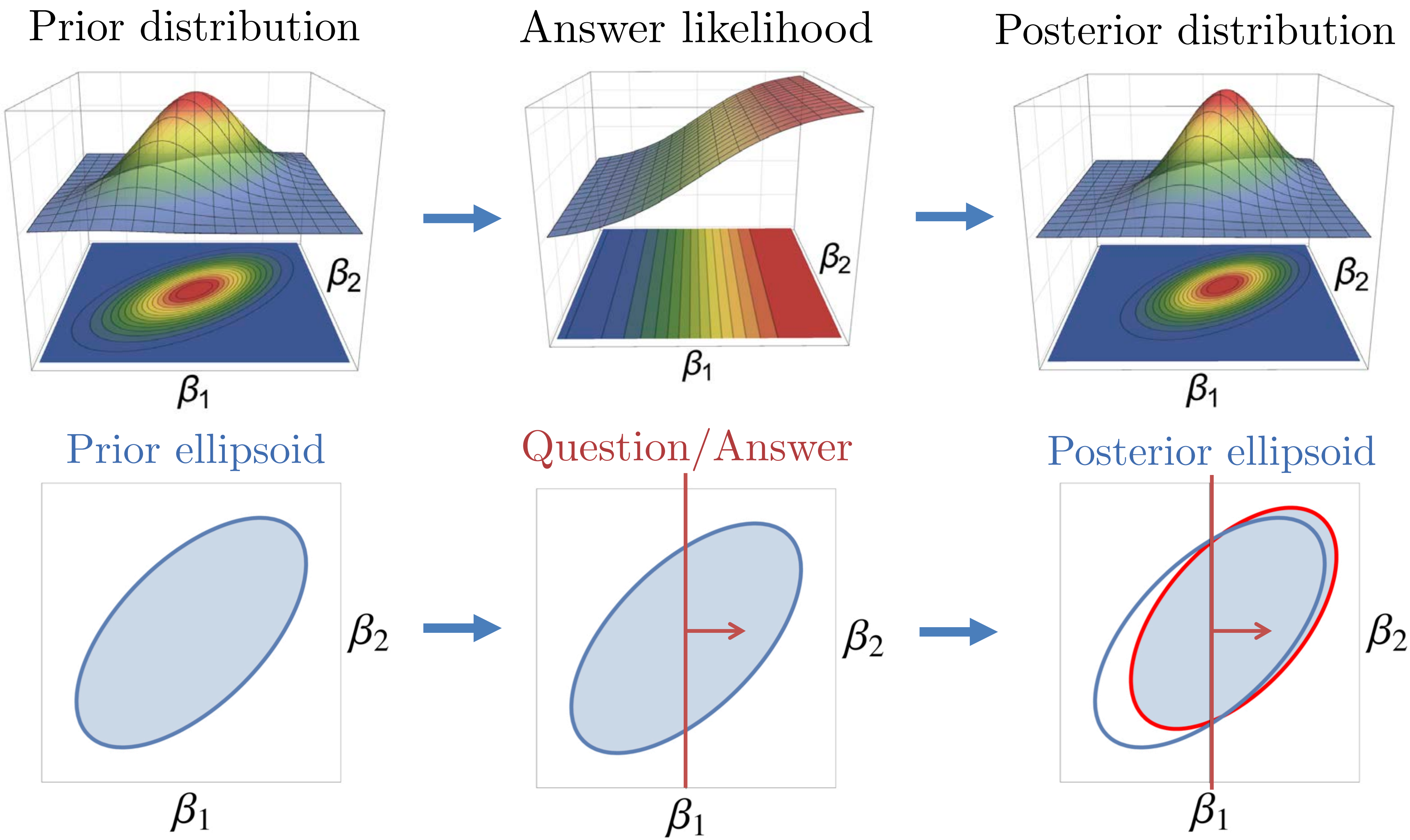


Next Question = Minimize (Expected) Volume

Good estimator for β?



With Error = Volume of Ellipsoid $f\left(x^1, x^2\right)$

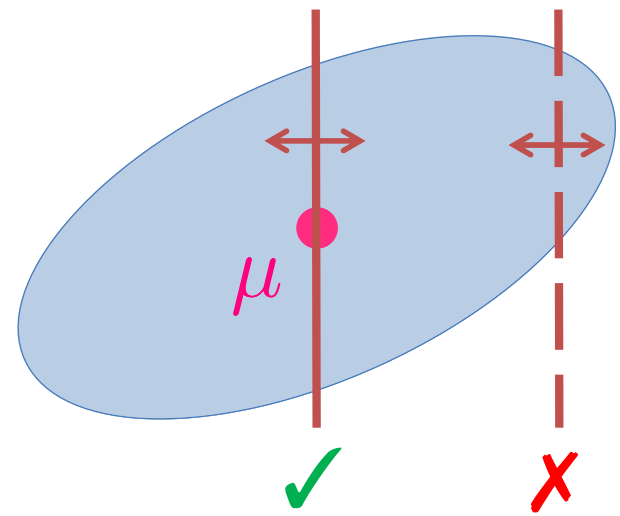


Rules of Thumb Still Good For Ellipsoid Volume

$$(\beta-\mu)' \cdot \Sigma^{-1} \cdot (\beta-\mu) \le r$$

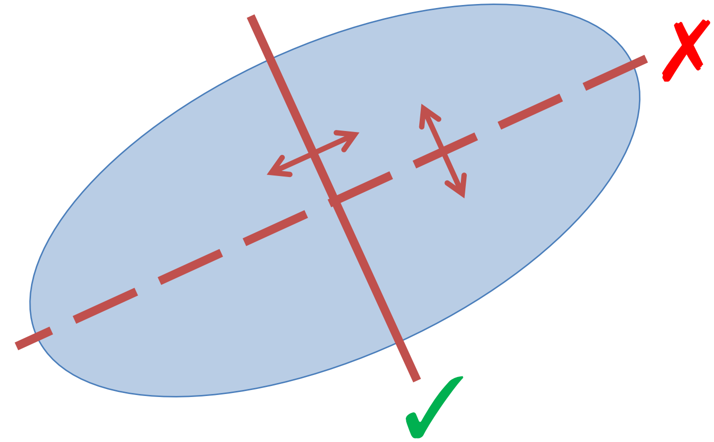
- Choice balance:
  - Minimize distance to center

$$\mu \cdot \left(x^1 - x^2\right)$$



- Postchoice symmetry:
  - Maximize variance of question

$$\left(x^1 - x^2\right)' \cdot \sum \cdot \left(x^1 - x^2\right)$$

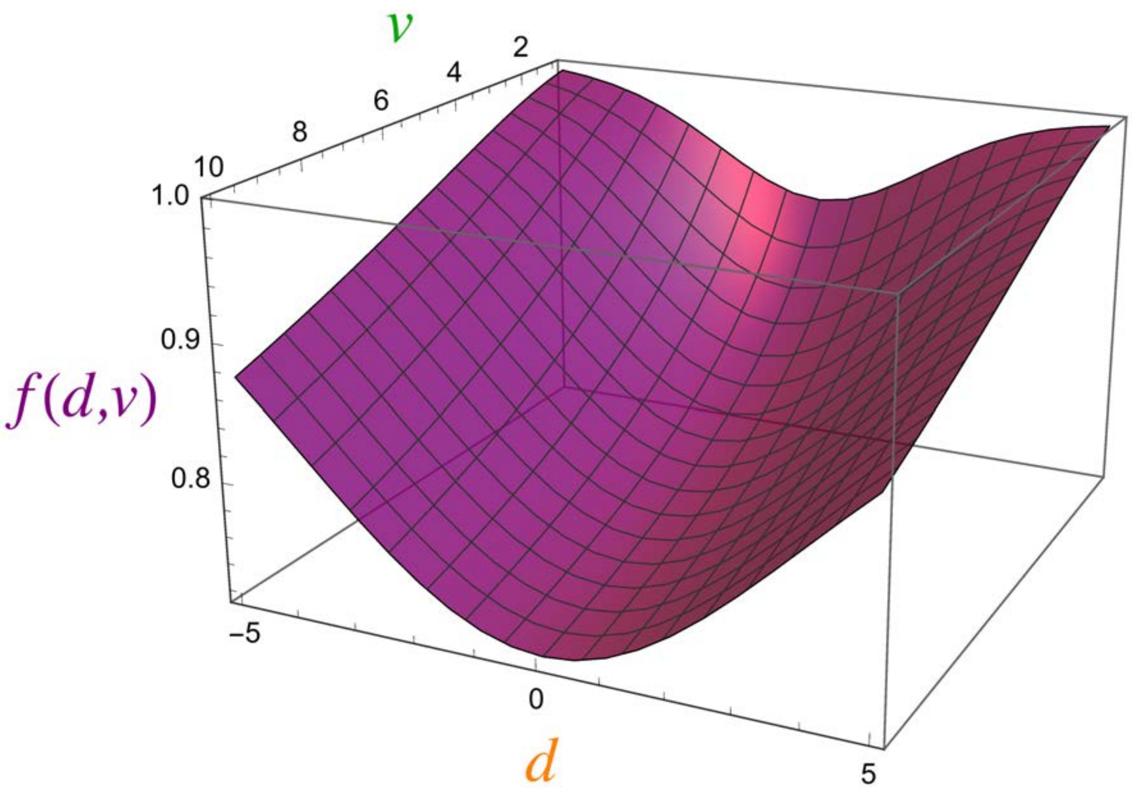


“Simple” Formula for Expected Volume

- Expected Volume = Non-convex function $f(d, v)$ of

distance: $d := \mu \cdot (x^1 - x^2)$

variance: $v := (x^1 - x^2)' \cdot \sum \cdot (x^1 - x^2)$



Can evaluate $f(d, v)$ with 1-dim integral 🙂

Optimization Model

$$\min \qquad f(d, v) \qquad ✗$$

$$s.t.$$

$$\mu \cdot (x^1 - x^2) = d \qquad ✓$$

$$(x^1 - x^2)' \cdot \Sigma \cdot (x^1 - x^2) = v \qquad ✗✓$$

$$A^1 x^1 + A^2 x^2 \leq b \qquad ✓$$

Formulation trick:

$$\text{linearize } x_i^k \cdot x_j^l$$

$$x^1 \neq x^2 \qquad ✗✓$$

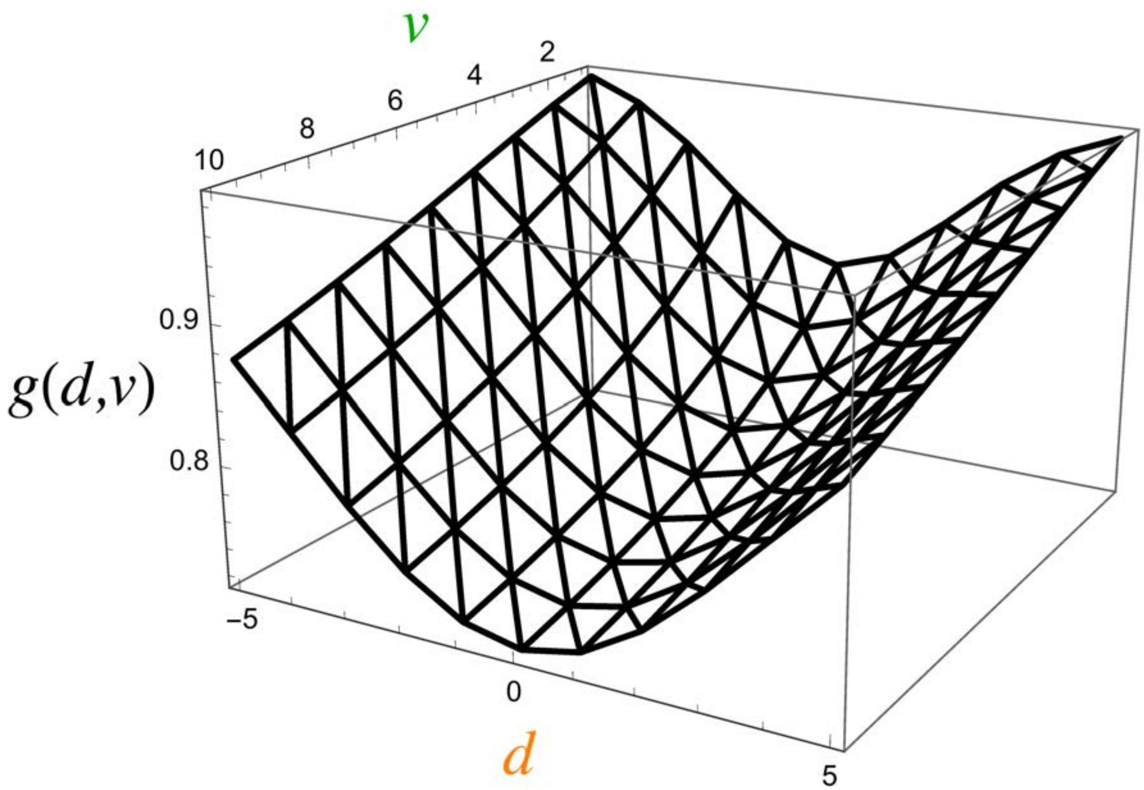
$$x^1, x^2 \in \{0,1\}^n$$

Technique 2: Piecewise Linear Functions

- D-efficiency = Non-convex function $f(d, v)$ of

distance: $d := \mu \cdot (x^1 - x^2)$

variance: $v := (x^1 - x^2)' \cdot \sum \cdot (x^1 - x^2)$



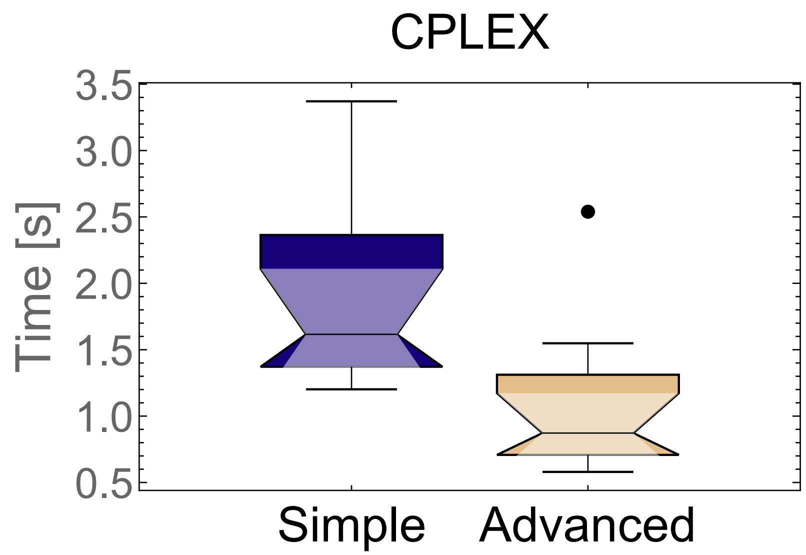
Can evaluate $f(d, v)$ with 1-dim integral 🙂

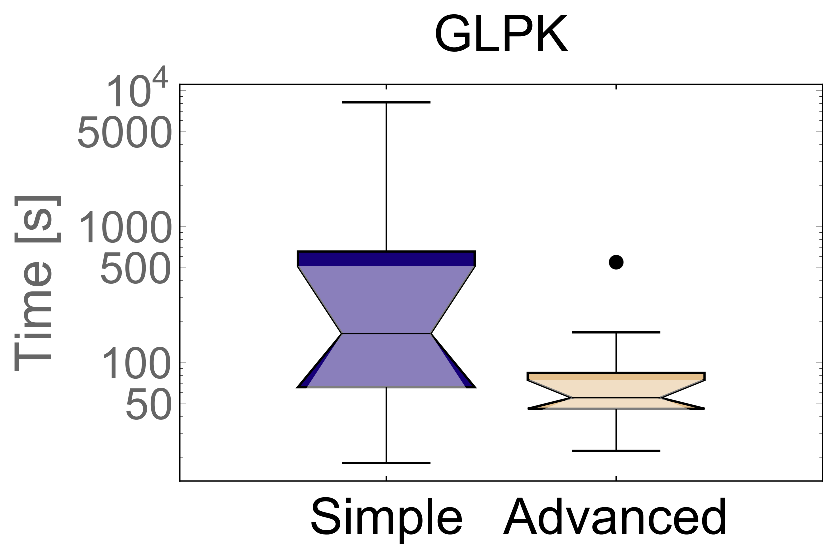
Piecewise Linear Interpolation

MIP formulation

Computational Performance

- Advanced formulations provide an computational advantage
- Advantage is significantly more important for free solvers
- State of the art commercial solvers can be significantly better that free solvers
- Still, free is free!





Summary and Main Messages

- Always choose Chewbacca!

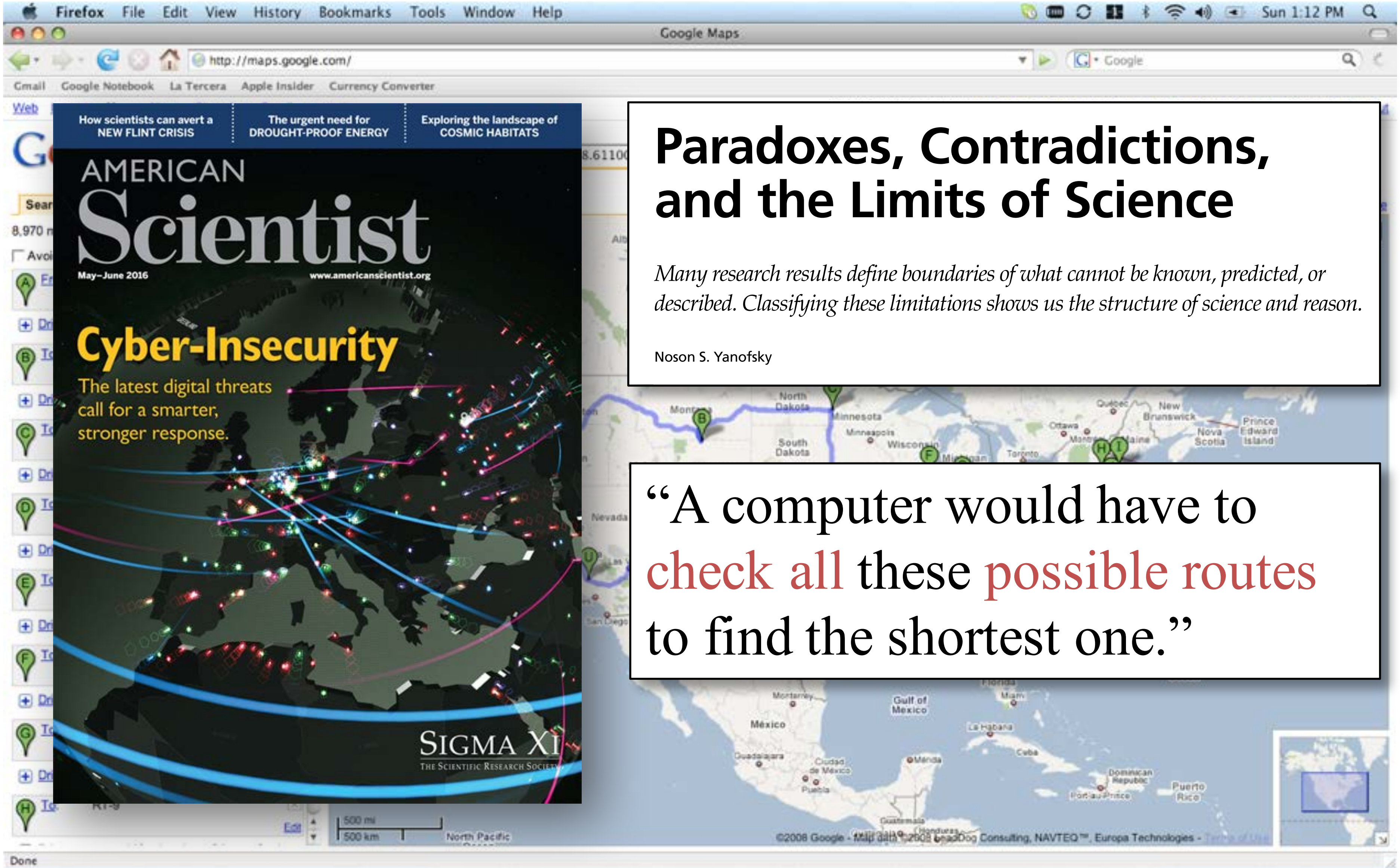
- How to YOU use MIP / Optimization / OR / Analytics?
  - Study for the 2nd midterm!
  - Use JuMP and Julia Opt.
  - How about grad school down the river?
    - Masters of Business Analytics / OR
    - Ph.D. in Operations Research

OPERATIONS
RESEARCH
CENTER

https://orc.mit.edu

How Hard is MIP?

How hard is MIP: Traveling Salesman Problem ?



Paradoxes, Contradictions, and the Limits of Science

Many research results define boundaries of what cannot be known, predicted, or described. Classifying these limitations shows us the structure of science and reason.

Noson S. Yanofsky

"A computer would have to check all these possible routes to find the shortest one."

MIP = Avoid Enumeration

- Number of tours for 49 cities $= 48!/2 \approx 10^{60}$
- Fastest supercomputer $\approx 10^{17}$flops
- Assuming one floating point operation per tour:
  $> 10^{35}$ years $\approx 10^{25}$ times the age of the universe!
- How long does it take on an iphone?
  - Less than a second!
  - 4 iterations of **cutting plane** method!
  - Dantzig, Fulkerson and Johnson 1954 did it by hand!
  - For more info see tutorial in ConcordeTSP app
  - **Cutting planes** are the key for effectively solving (even NP-hard) MIP problems in practice.

50+ Years of MIP = Significant Solver Speedups

- Algorithmic Improvements (Machine Independent):
  - **CPLEX** v1.2 (1991) – v11 (2007): 29,000x speedup
  - Gurobi v1 (2009) – v6.5 (2015): 48.7x speedup
  - Commercial, but free for academic use
- (Reasonably) effective free / open source solvers:
  - GLPK, **COIN-OR (CBC)** and SCIP (only for non-commercial)
- Easy to use, fast and versatile modeling languages
  - Julia based JuMP modelling language
  - http://julialang.org
  - http://www.juliaopt.org

Technique 1: Binary Quadratic $x^1, x^2 \in \{0,1\}^n$

$$(x^1 - x^2)' \cdot \Sigma \cdot (x^1 - x^2) = v$$

$$X^l_{i,j} = x^l_i \cdot x^l_j \quad (l \in \{1,2\}, \quad i,j \in \{1,\dots,n\}):$$

$$X^l_{i,j} \le x^l_i, \quad X^l_{i,j} \le x^l_j, \quad X^l_{i,j} \ge x^l_i + x^l_j - 1, \quad X^l_{i,j} \ge 0$$

$$W_{i,j} = x^1_i \cdot x^2_j :$$

$$W_{i,j} \le x^1_i, \quad W_{i,j} \le x^2_j, \quad W_{i,j} \ge x^1_i + x^2_j - 1, \quad W_{i,j} \ge 0$$

$$\sum_{i,j=1}^{n} \left(X^1_{i,j} + X^2_{i,j} - W_{i,j} - W_{j,i}\right) \Sigma_{i,j} = v$$

Technique 1: Binary Quadratic $x^1, x^2 \in \{0,1\}^n$

$$x^1 \neq x^2 \quad \Leftrightarrow \quad \|x^1 - x^2\|_2^2 \geq 1$$

$$X^l_{i,j} = x^l_i \cdot x^l_j \quad (l \in \{1,2\}, \quad i,j \in \{1,\dots,n\}):$$

$$X^l_{i,j} \leq x^l_i, \quad X^l_{i,j} \leq x^l_j, \quad X^l_{i,j} \geq x^l_i + x^l_j - 1, \quad X^l_{i,j} \geq 0$$

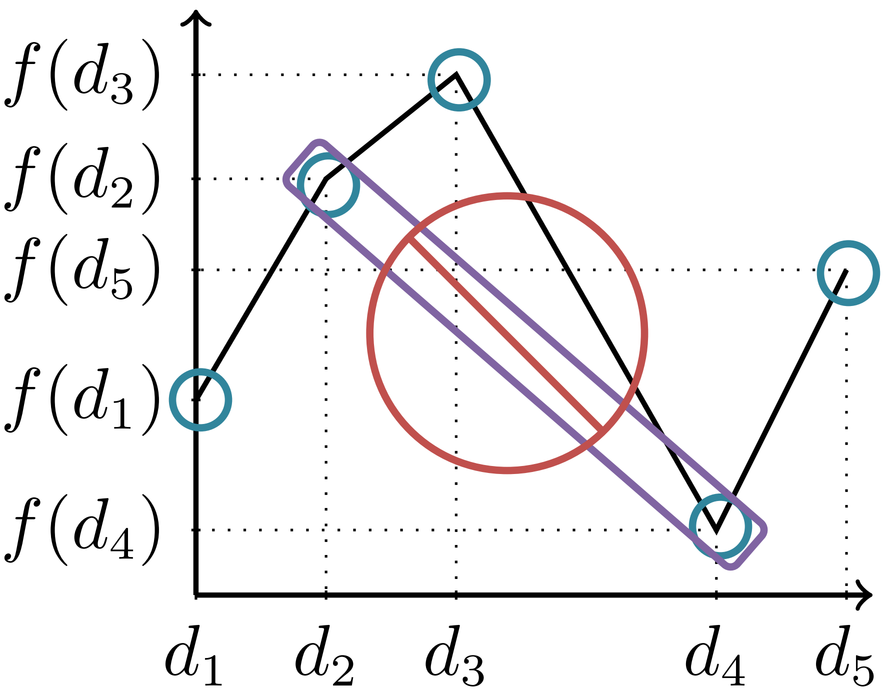
$$W_{i,j} = x^1_i \cdot x^2_j :$$

$$W_{i,j} \leq x^1_i, \quad W_{i,j} \leq x^2_j, \quad W_{i,j} \geq x^1_i + x^2_j - 1, \quad W_{i,j} \geq 0$$

$$\sum_{i,j=1}^{n} \left(X^1_{i,j} + X^2_{i,j} - W_{i,j} - W_{j,i}\right) \geq 1$$

Simple Formulation for Univariate Functions

$$z = f(x)$$



$$\text{Size} = O(\#\text{ of segments})$$

Non-Ideal: Fractional Extreme Points

$$\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{5} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_j$$

$$1 = \sum_{j=1}^{5} \lambda_j, \quad \lambda_j \geq 0$$

$$y \in \{0,1\}^4, \quad \sum_{i=1}^{4} y_i = 1$$

$$0 \leq \lambda_1 \leq y_1$$

$$0 \leq \lambda_2 \leq y_1 + y_2$$

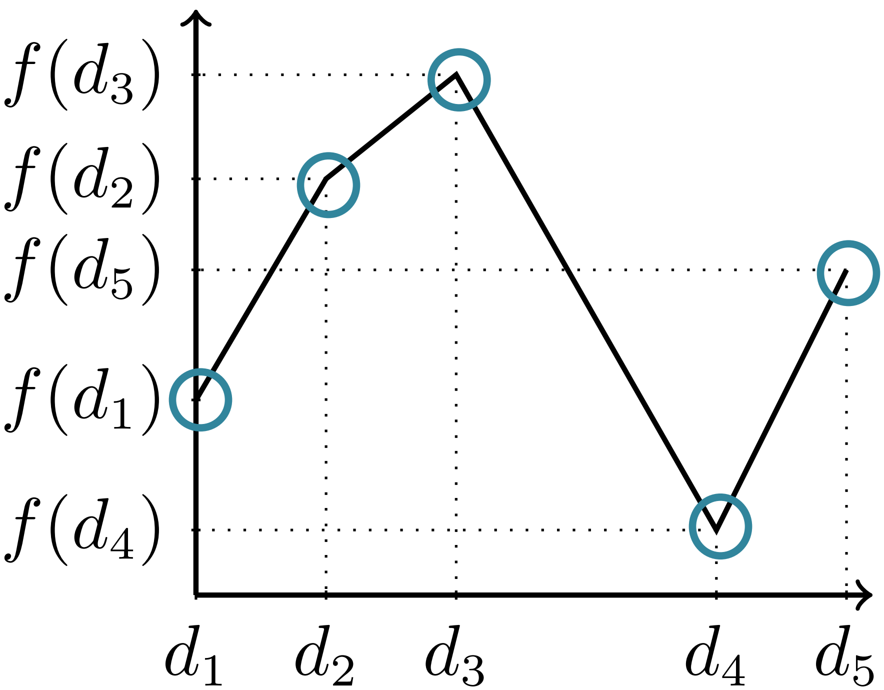
$$0 \leq \lambda_3 \leq y_2 + y_3$$

$$0 \leq \lambda_4 \leq y_3 + y_4$$

$$0 \leq \lambda_5 \leq y_4$$

Advanced Formulation for Univariate Functions

$$z = f(x)$$



$$\text{Size} = O\left(\log_2 \#\text{ of segments}\right)$$

Ideal: Integral Extreme Points

$$\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{5} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_j$$

$$1 = \sum_{j=1}^{5} \lambda_j, \quad \lambda_j \geq 0$$

$$y \in \{0,1\}^2$$

$$0 \leq \lambda_1 + \lambda_5 \leq 1 - y_1$$

$$0 \leq \lambda_3 \leq y_1$$

$$0 \leq \lambda_4 + \lambda_5 \leq 1 - y_2$$

$$0 \leq \lambda_1 + \lambda_2 \leq y_2$$