

## MASS QUADRUPOLE AS A SOURCE OF NAKED SINGULARITIES

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We investigate the gravitational field of a static mass with quadrupole moment in empty space. It is shown that in general this configuration is characterized by the presence of curvature singularities without a surrounding event horizon. These naked singularities generate an effective field of repulsive gravity which, in turn, drastically changes the behavior of test particles. As a possible consequence, the accretion disk around a naked singularity presents a particular discontinuous structure.

*Keywords:* Quadrupole moment; naked singularities; accretion disks.

### 1. Introduction

According to the black hole uniqueness theorems, the most general black hole solution in empty space is described by the Kerr metric<sup>1</sup>

$$ds^2 = \frac{r^2 - 2Mr + a^2}{r^2 + a^2 \cos^2 \theta} (dt - a \sin^2 \theta d\varphi)^2 - \frac{\sin^2 \theta}{r^2 + a^2 \cos^2 \theta} [(r^2 + a^2) d\varphi - a dt]^2 \\ - \frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2Mr + a^2} dr^2 - (r^2 + a^2 \cos^2 \theta) d\theta^2,$$

which represents the exterior gravitational field of a rotating mass  $M$  with specific angular momentum  $a = J/M$ . A true curvature singularity is determined by the equation  $r^2 + a^2 \cos^2 \theta = 0$  and is interpreted as a ring singularity situated on the equatorial plane  $\theta = \pi/2$ . The ring singularity is isolated from the exterior space by an event horizon situated on a sphere of radius  $r_h = M + \sqrt{M^2 - a^2}$ . In the case  $a^2 > M^2$  no event horizon exists and the ring singularity becomes naked. Different studies<sup>2–4</sup> show, however, that, in realistic situations where astrophysical objects are surrounded by accreting matter, a Kerr naked singularity is an unstable

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configuration that rapidly decays into a black hole. Moreover, it now seems established that a gravitational collapse cannot in generic situations lead to the formation of a final configuration resembling the Kerr solution with a naked singularity. These results indicate that rotating naked singularities cannot be very common objects in nature.

The above results seem to support the validity of the cosmic censorship hypothesis<sup>5</sup> according to which a physically realistic gravitational collapse, which evolves from a regular initial state, can never lead to the formation of a naked singularity; that is, all singularities formed as the result of such a collapse should always be enclosed within an event horizon and hence invisible to outside observers. Many attempts have been made to prove this conjecture with the same mathematical rigor used to show the inevitability of singularities in general relativity.<sup>6</sup> So far, no general proof has been formulated. Instead, particular scenarios of gravitation collapse have been investigated some of which indeed corroborate the correctness of the conjecture. Other studies, however, indicate that under certain circumstances naked singularities can appear as a result of a realistic gravitational collapse. Indeed, it turns out that in an inhomogeneous collapse, there exists a critical degree of inhomogeneity below which black holes form. Naked singularities appear if the degree of inhomogeneity is higher than the critical value. The speed of the collapse and the shape of the collapsing object are also factors that play an important role in the determination of the final state of the collapse. Naked singularities form more frequently if the collapse occurs very rapidly and the object is not exactly spherically symmetric.

In view of this situation it seems reasonable to investigate the effects of naked singularities on the surrounding spacetime. This is the main aim of the present work. We investigate the simplest generalization of the Schwarzschild metric that contains a naked singularity. In fact, we show that, starting from the Schwarzschild metric, the Zipoy–Voorhees<sup>8,9</sup> transformation can be used to generate a static axisymmetric spacetime which describes the field of a mass with a particular quadrupole moment. For any values of the quadrupole, the spacetime is characterized by the presence of naked singularities situated at a finite distance from the origin of coordinates. We argue that the effects of the quadrupole can be described by means of an effective potential with an effective mass that, for certain values of the quadrupole parameter and coordinates, can become negative, giving rise to phenomena related to repulsive gravity. This is valid for any gravitational configuration with quadrupole moment. An example is shown where an accretion disk becomes discontinuous because of the presence of a naked singularity in its center.

## 2. The Simplest Static Spacetime with Quadrupole Moment

Static axisymmetric gravitational fields in empty space can be described by the Weyl line element<sup>10</sup>

$$ds^2 = e^{2\psi} dt^2 - e^{-2\psi} [e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\varphi^2], \quad (1)$$

where  $\psi = \psi(\rho, z)$  and  $\gamma = \gamma(\rho, z)$ . The field equations  $\psi_{\rho\rho} + \psi_{zz} + \psi_{\rho}/\rho = 0$ ,  $\gamma_{\rho} = \rho(\psi_{\rho}^2 - \psi_z^2)$ , and  $\gamma_z = 2\rho\psi_{\rho}\psi_z$  are invariant with respect to the Zipoy–Voorhees<sup>8,9</sup> (ZV) transformation  $\psi \rightarrow \delta\psi$  and  $\gamma \rightarrow \delta^2\gamma$ , where  $\delta$  is a real constant parameter. For any particular solution, the ZV transformation generates a family of solutions that is parametrized by  $\delta$ .

In terms of multipole moments, the simplest static solution contained in the Weyl class is the Schwarzschild metric which is the only one that possesses a mass monopole moment only. From a physical point of view, the next interesting solution must describe the exterior field of a mass with quadrupole moment. In this case, it is possible to find a large number of exact solutions with the same quadrupole<sup>11</sup> that differ only in the set of higher multipoles. A common characteristic of solutions with quadrupole is that their explicit form is rather cumbersome, making them difficult to be handled analytically.<sup>12</sup> To meliorate this situation, we derive here an exact solution with quadrupole which can be written in a compact and simple form. To this end, we consider the Schwarzschild solution and apply a ZV transformation with  $\delta = 1 + q$ . The resulting metric can be written in spherical coordinates as

$$ds^2 = \left(1 - \frac{2m}{r}\right)^{1+q} dt^2 - \left(1 - \frac{2m}{r}\right)^{-q} \left[ \left(1 + \frac{m^2 \sin^2 \theta}{r^2 - 2mr}\right)^{-q(2+q)} \left(\frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2\right) + r^2 \sin^2 \theta d\varphi^2 \right]. \quad (2)$$

This solution is axially symmetric and reduces to the spherically symmetric Schwarzschild metric only for  $q \rightarrow 0$ . It is asymptotically flat for any finite values of the parameters  $m$  and  $q$ . Moreover, in the limiting case  $m \rightarrow 0$  it can be shown that, independently of the value of  $q$ , there exists a coordinate transformation that transforms the resulting metric into the Minkowski solution. This last property is important from a physical point of view because it means that the parameter  $q$  is related to a genuine mass distribution. To see this explicitly, we calculate the multipole moments of the solution by using the invariant definition proposed by Geroch.<sup>13,14</sup> The lowest mass multipole moments  $M_n$ ,  $n = 0, 1, \dots$  are given by

$$M_0 = (1 + q)m, \quad M_2 = -\frac{m^3}{3}q(1 + q)(2 + q), \quad (3)$$

whereas higher moments are proportional to  $mq$  and can be completely rewritten in terms of  $M_0$  and  $M_2$ . This means that the arbitrary parameters  $m$  and  $q$  determine the mass and quadrupole which are the only independent multipole moments of the solution. In the limiting case  $q = 0$  only the monopole  $M_0 = m$  survives, as in the Schwarzschild spacetime. In the limit  $m = 0$ , with  $q \neq 0$ , all moments vanish identically, implying that no mass distribution is present and the spacetime must be flat. This is in accordance with the result mentioned above for the metric (2). Furthermore, notice that all odd multipole moments are zero because the solution possesses an additional reflection symmetry with respect to the equatorial plane.

We conclude that the above metric describes the exterior gravitational field of a static deformed mass. The deformation is described by the quadrupole moment  $M_2$  which is positive for a prolate mass distribution and negative for an oblate one. Notice that the condition  $q > -1$  must be satisfied in order to avoid the appearance of a negative total mass  $M_0$ .

To investigate the structure of possible curvature singularities, we consider the Kretschmann scalar  $K = R_{\mu\nu\lambda\tau}R^{\mu\nu\lambda\tau}$ . A straightforward computation leads to

$$K = \frac{16m^2(1+q)^2}{r^{4(2+2q+q^2)}} \frac{(r^2 - 2mr + m^2 \sin^2 \theta)^{2(2q+q^2)-1}}{(1 - 2m/r)^{2(q^2+q+1)}} L(r, \theta), \quad (4)$$

with

$$L(r, \theta) = 3(r - 2m - qm)^2(r^2 - 2mr + m^2 \sin^2 \theta) + q(2 + q) \sin^2 \theta [q(2 + q) + 3(r - m)(r - 2m - qm)]. \quad (5)$$

In the limiting case  $q = 0$ , we obtain the Schwarzschild value  $K = 48m^2/r^6$  with the only singularity situated at the origin of coordinates  $r \rightarrow 0$ . In general, one can show that the singularity at the origin is present for any values of  $q$ . Moreover, an additional singularity appears at the radius  $r = 2m$  which, according to the metric (2), is also a horizon in the sense that the norm of the timelike Killing tensor vanishes at that radius. Outside the hypersurface  $r = 2m$  no additional horizon exists, indicating that the singularities situated at the origin and at  $r = 2m$  are naked. A more detailed analysis of the Kretschmann scalar (4) shows that in the interval  $q \in (-1, -1 + \sqrt{3/2}] \setminus \{0\}$  two further singularities appear inside the singular hypersurface  $r = 2m$ . This configuration of naked singularities is schematically illustrated in Fig. 1. We conclude that the presence of a quadrupole moment

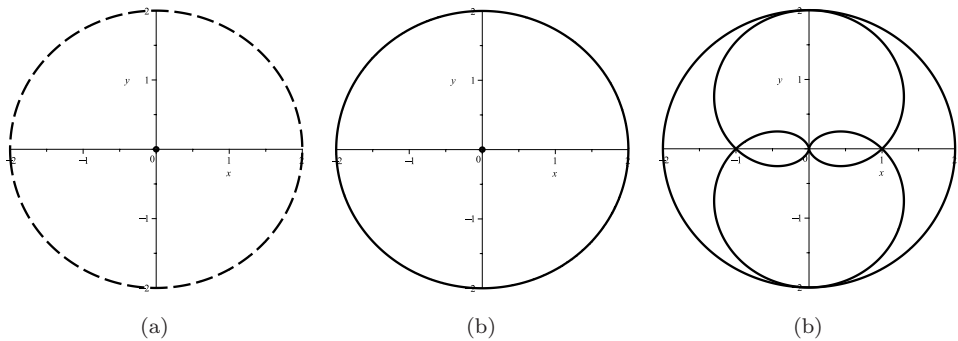


Fig. 1. Structure of naked singularities of a spacetime with quadrupole. Plot (a) represents the limiting case of a Schwarzschild spacetime ( $q = 0$ ) with a singularity at the origin surrounded by the horizon (dashed curve) situated at  $r = 2m$ . Once the quadrupole parameter  $q$  is included, the horizon transforms into a naked singularity (solid curve) and the central singularity becomes naked as well. This case is illustrated in plot (b). For values of the quadrupole parameter within the interval  $q \in (-1, -1 + \sqrt{3/2}] \setminus \{0\}$ , two additional naked singularities appear as depicted in plot (c).

completely changes the structure of spacetime. In particular, naked singularities are always present, regardless of the value of the quadrupole parameter.

### 3. Geodesic Motion

The study of the motion of test particles in the gravitational field of naked singularities has shown that certain effects appear that can be explained by assuming the existence of an effective gravitational potential that generates repulsive gravity under specific circumstances. Currently, there is no invariant definition of repulsive gravity in the context of general relativity, although some attempts have been made by using invariant quantities constructed with the curvature of spacetime.<sup>15,16</sup> Nevertheless, it is possible to consider an intuitive approach by using the fact that the motion of test particles in stationary axisymmetric gravitational fields reduces to the motion in an effective potential. This is a consequence of the fact that the geodesic equations possess two first integrals associated with stationarity and axial symmetry. The explicit form of the effective potential depends also on the type of motion under consideration. But in general one can find certain similarities between the effective potential for geodesic motion and the effective Newtonian potential which follows from the metric as  $g_{tt} \approx 1 - 2V_N = 1 - 2M_{\text{eff}}/r$ . For a mass  $M_0$  with quadrupole moment  $M_2$ , the Newtonian potential reads  $V_N = M_0/r + (1 - 3\cos^2\theta)M_2/(4r^3)$  so that in the particular case of the metric (2), we obtain the effective mass

$$M_{\text{eff}} = m(1 + q) \left[ 1 - \frac{m^2}{12r^2} q(2 + q)(1 - 3\cos^2\theta) \right]. \quad (6)$$

Clearly, this expression can become negative for certain values of the coordinates and parameters. This behavior is illustrated in Fig. 2.

One can then intuitively expect that in the region of negative effective mass the motion of test particles will experience effects that can be interpreted as due to the presence of repulsive gravity. In fact, it is even possible to find situations in which repulsive gravity compensates the attractive gravitational force and test particles can remain at rest with respect to static observers situated at infinity. We have performed an analysis of the motion of test particles moving along circular orbits around naked singularities of different types: Reissner–Nordström naked singularity with mass  $M$  and electric charge  $Q$  with  $M < |Q|$ , Kerr naked singularity with mass  $M$  and specific angular momentum  $a$  with  $M < |a|$ , Kerr–Newman naked singularity with  $M^2 < a^2 + Q^2$ , ZV naked singularity with mass parameter  $m$  and arbitrary quadrupole parameter  $q$ , and rotating ZV naked singularity with an additional arbitrary parameter  $j$  associated to the angular momentum. In all these cases we found that it is possible to introduce an effective mass that can become negative in a certain region of spacetime and resembles the behavior illustrated in Fig. 2.

A detailed study of the motion of test particles in the gravitational field of naked singularities is beyond the scope of the present work. In general it is necessary to

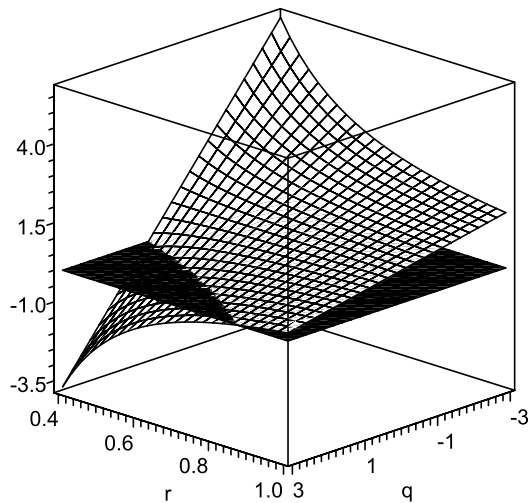


Fig. 2. Behavior of the effective mass for the metric (2). The effective mass (6) has been evaluated on the equatorial plane  $\theta = \pi/2$  and is plotted as a function of the parameter  $q$  and the radial coordinate  $r$ . The plane  $M_{\text{eff}} = 0$  has been plotted to distinguish the region where the effective mass becomes negative.

perform numerical computations to solve the geodesic equations. Nevertheless, the case of circular motion around a Reissner–Nordström naked singularity can be studied analytically<sup>17</sup> and the results reproduce in general terms those of other naked configurations. For this reason we present here only the results for the Reissner–Nordström case. Imagine a test particle of mass  $\mu$  on a circular trajectory around a naked singularity with mass  $M$  and charge  $Q$ . Then, the geodesic motion is equivalent to the motion in the effective potential

$$V_{\text{eff}} \equiv \sqrt{\left(1 + \frac{L^2}{\mu^2 r^2}\right) \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)}, \quad (7)$$

where  $r$  is the radial coordinate and  $L$  the angular momentum of the test particle. It is then possible to find the radius of the last stable circular orbit  $r_{\text{lsc}}$ . The result is plotted in Fig. 3. The most interesting feature is that below the value  $Q/M = \sqrt{5}/2$  there are two regions of stability. Let us suppose that an accretion disk around such a singularity is made off test particles on circular motion. For the particular value  $Q/M = 1.1$ , particles with radius in the intervals  $(r_*, r_{c-})$  and  $(r_{c+}, +\infty)$  are stable. If a particle happens to be within the region of instability  $(r_{c-}, r_{c+})$ , it necessarily will move toward one of the regions of stability (see Fig. 3). Then, the accretion disk will become discontinuous. In a realistic situation one would expect that the interior ring would be situated outside the classical radius  $r_*$  and very close to the last stable radius  $r_{c-}$ . The exterior disk would then be situated around the radius  $r_{c+}$ . This structure is schematically illustrated in Fig. 4. A more detailed

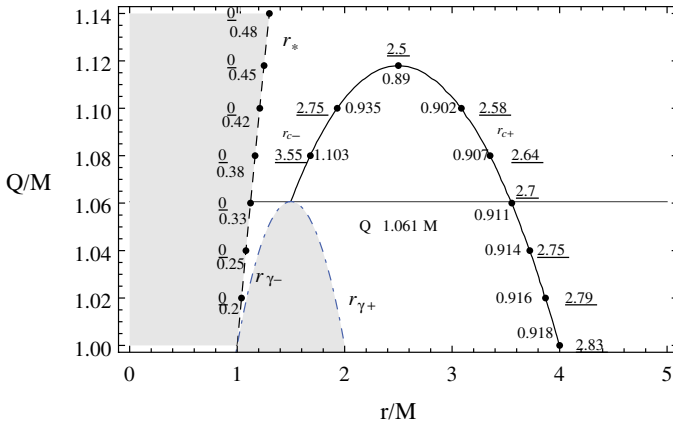


Fig. 3. The minimum radius  $r_{\text{lSCO}}/M$  (solid curve) is plotted as a function of the ratio  $Q/M$ . Shaded regions are forbidden (only spacelike geodesics are allowed). The dashed curve represents the classical radius  $r_* = Q^2/M$  where  $L = 0$  and the particle stays at “rest”. The dotted curve  $r_{\gamma-}$  and the dot-dashed curve  $r_{\gamma+}$  represent lightlike orbits. The region inside the curves  $r_{c+}$ ,  $r_{c-}$  and  $r_{\gamma+}$  corresponds to the region of instability. Outside this region (and outside the forbidden zone) any point represents a stable circular orbit. Numbers close to the selected points represent the value of the energy  $E/\mu$  and the angular momentum  $L/(\mu M)$  (underlined numbers) of the last stable circular orbits.

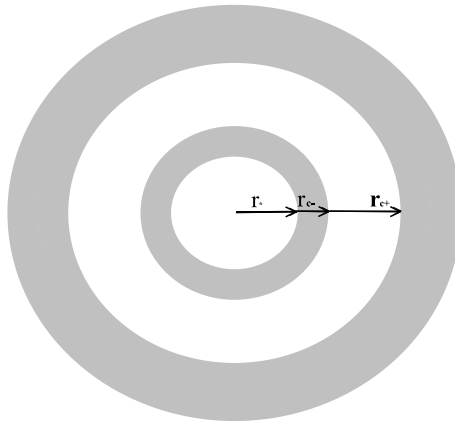


Fig. 4. Structure of an accretion disk around a naked singularity. The interior ring is situated within the interior region of stability.

study of discontinuous accretion disks around naked singularities will be presented elsewhere.

#### 4. Conclusions

In this work we presented an exact solution of Einstein’s field equations in empty space which represents the exterior gravitational field of a static naked singularity.

It was obtained by applying a particular ZV transformation on the Schwarzschild metric. To our knowledge, this is the simplest solution which contains a mass parameter and a quadrupole parameter as well. The quadrupole moment of the source is identified as the source of the naked singularity. We briefly discuss the motion of test particles around naked singularities and argue that an effective mass can be introduced that in certain regions of spacetime can become negative and give rise to repulsive gravity. The analysis of circular motion around naked singularities indicates that there exist discontinuous regions of stability. Consequently, a hypothetical accretion disk made off test particles on circular motion present a discontinuous structure. It would be interesting to further analyze this type of accretion disks to determine if they could lead to observational consequences.

In the context of singularities the question about stability is essential. It is interesting to notice that only in the case of ZV naked singularities the parameters that enter the metric are arbitrary. This could be interpreted as an indication of the stability of naked singularities generated by a quadrupole. In fact, all naked singularities which have a black hole counterpart must satisfy certain conditions among the parameters entering the corresponding metric. The analysis of stability indicates that perturbations modify these conditions in such a way that the resulting configuration must rapidly decay into a black hole. In the case of the static ZV naked singularity studied above a perturbation of the parameters  $m \rightarrow m + \delta m$  and  $q \rightarrow q + \delta q$  does not lead to any additional condition among  $m$  and  $q$ . A more exhaustive analysis will be necessary to determine if this heuristic argument is indeed an indication of stability.

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