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# $C^3$ matching for asymptotically flat spacetimes

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**Abstract**
We propose a criterion for finding the minimum distance at which an interior solution of Einstein's equations can be matched with an exterior asymptotically flat solution. The location of the matching hypersurface is thus constrained by a criterion defined in terms of the eigenvalues of the Riemann curvature tensor by using repulsive gravity effects. To determine the location of the matching hypersurface, we use the first derivatives of the curvature eigenvalues, implying $C^3$ differentiability conditions. The matching itself is performed by demanding continuity of the curvature eigenvalues across the matching surface. We apply the $C^3$ matching approach to spherically symmetric perfect fluid spacetimes and obtain the physically meaningful condition that density and pressure should vanish on the matching surface. Several perfect fluid solutions in Newton and Einstein gravity are tested.

Keywords: asymptotically flat spacetimes, matching conditions, curvature eigenvalues

(Some figures may appear in colour only in the online journal)

  

## 1. Introduction

General relativity is a theory of the gravitational interaction and, in particular, should describe the gravitational field of relativistic compact objects. In this case, the spacetime can be split into two different parts, namely, the interior region described by an exact solution $g^{-}_{\mu\nu}$ of Einstein's equations with a physically reasonable energy–momentum tensor and the exterior region, which corresponds to an exact vacuum solution $g_{\mu\nu}$. This implies that the spacetime $M$ can be considered as split into two regions $(M^{-}, g^{-}_{\mu\nu})$ and $(M^{+}, g^{+}_{\mu\nu})$ with a special hypersurface $\Sigma$ at which the two regions should be matched. In the case of compact objects, $\Sigma$ should be identified with the surface of the object, i.e. it is a time-like hypersurface. It then follows that at $\Sigma$ certain matching conditions should be imposed in order for the spacetime to be well defined.

Two sets of matching conditions are commonly used in the literature. The Darmois conditions [1] demand that the first and second fundamental forms (the intrinsic metric and the extrinsic curvature) be continuous across $\Sigma$. The Lichnerowicz conditions state that the metric and all its first derivatives must be continuous across $\Sigma$ in 'admissible' coordinates that traverse $\Sigma$. The Darmois conditions are expressed in terms of tensorial quantities and hence they can be considered as a covariant formulation of the matching problem. In the case of the Lichnerowicz conditions, the term 'admissible' coordinates is used, indicating that the choice of a coordinate system is essential. The equivalence between Darmois and Lichnerowicz conditions can be proved by using Gaussian normal coordinates. This proof allows one also to precise the concept of 'admissible' coordinates which are then defined as coordinates related to Gaussian normal coordinates by means of $(C^2_\Sigma, C^4)$ transformations [2]. However, as pointed out by Israel in [3], the explicit form of the matching conditions are of limited utility, since 'admissible' coordinates usually are not the most convenient for handling the matching problem in practice.

An alternative approach in which the extrinsic curvature is not necessarily continuous across $\Sigma$ was proposed by Israel [3]. An effective energy–momentum tensor which determines a thin shell is defined in terms of the difference of the extrinsic curvature evaluated inside and outside the hypersurface $\Sigma$. This means that $\Sigma$ can now be interpreted as a thin shell that separates $M^{+}$ and $M^{-}$, is part of the entire spacetime $M$ and as such plays an important role in the determination of the spacetime dynamics.

In all the $C^2$ matching conditions described above, it is important to know *a priori* the location of $\Sigma$. Although in the case of compact objects, we can identify $\Sigma$ with the surface of the body, in general, it is not easy to find the equation that determines the surface and, if possible, it often is not given in the 'admissible' coordinates that are essential for treating the matching problem. This is probably the reason why the matching conditions have been applied so far only in cases characterized by very high symmetries.

In this work, we propose to use a $C^3$ criterion to find information about the location of the hypersurface $\Sigma$. It is defined in terms of the eigenvalues of the Riemann curvature tensor which are invariant quantities. The idea is simple. Since the curvature tensor is a measure of the gravitational interaction, the curvature eigenvalues provides us with an invariant measure of the gravitational interaction. This invariance should be understood as independence with respect to the choice of coordinates and reference frames. This is due to the fact that the curvature eigenvalues can be treated as scalars under coordinate transformations. Whereas single components of the curvature tensor are not scalars in general and, consequently, cannot be used to define gravitational interaction in an invariant way, the eigenvalues with their scalar property represent the gravitational interaction independently of the choice of coordinates. An example of the application of the curvature eigenvalues is in the formulation of the invariant Petrov classification of the curvature tensor and, correspondingly, of the gravitational fields [4].

Since for a compact object, one expects the spacetime to be asymptotically flat, the curvature eigenvalues should vanish at spatial infinite and the behavior of the eigenvalues approaching the gravitational source could give some information about its boundaries. Here, we implement this simple idea in an invariant manner and show its applicability in the case of several exact solutions of Einstein's equations. The original idea of using curvature eigenvalues to formulate $C^3$ matching conditions was first presented by one of us in [5]. It has been used also to formulate an invariant definition of repulsive gravity [6, 7] and to construct cosmological models [8].

This paper is organized as follows. In section 2, we use Cartan's formalism to investigate the general form of a curvature tensor that satisfies Einstein equations with a perfect-fluid source. Moreover, we find the general form of the curvature eigenvalues and derive some identities relating them. In section 3, we review the definition of repulsive gravity in terms of the curvature eigenvalues. This definition is then used in section 4 to propose the $C^3$ matching approach, whose objective is to perform the matching in such a way that the effects of repulsive gravity cannot be detected. We also apply the method to spherically symmetric solutions in Newton gravity and obtain the general result that the mass density should vanish at the matching surface. In Einstein gravity, in addition, the pressure should also vanish. Then, in the appendix, we present all the metrics of the Tolman class of interior solutions and calculate the corresponding curvature eigenvalues to show that none of them satisfies the $C^3$ matching conditions. Finally, in section 6, we discuss our results and propose some tasks for future investigation.

## 2. Curvature eigenvalues and Einstein equations

Our approach is based upon the analysis of the behavior of the curvature eigenvalues. There are different ways to determine these eigenvalues [4]. Our strategy is to use local tetrads and differential forms. From the physical point of view, a local orthonormal tetrad is the simplest and most natural choice for an observer in order to perform local measurements of time, space, and gravity. Moreover, once a local orthonormal tetrad is chosen, all the quantities related to this frame are invariant with respect to coordinate transformations. The only freedom remaining in the choice of this local frame is a Lorentz transformation. So, let us choose the orthonormal tetrad as

$$ds^2 = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \eta_{ab}\vartheta^a \otimes \vartheta^b, \tag{1}$$

with $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$, and $\vartheta^a = \mathrm{e}^a{}_\mu \mathrm{d}x^\mu$. The first

$$\mathrm{d}\vartheta^a = -\omega^a{}_b \wedge \vartheta^b, \tag{2}$$

and second Cartan equations

$$\Omega^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2}R^a{}_{bcd}\vartheta^c \wedge \vartheta^d \tag{3}$$

allow us to compute the components of the Riemann curvature tensor in the local orthonormal frame.

It is possible to represent the curvature tensor as a $(6 \times 6)$-matrix by introducing the bivector indices $A, B, ...$ which encode the information of two different tetrad indices, i.e. $ab \to A$.

We follow the convention proposed in [9] which establishes the following correspondence between tetrad and bivector indices

$$01 \to 1, \quad 02 \to 2, \quad 03 \to 3, \quad 23 \to 4, \quad 31 \to 5, \quad 12 \to 6\,. \tag{4}$$

Then, the Riemann tensor can be represented by the symmetric matrix $\mathbf{R}_{AB} = \mathbf{R}_{BA}$ with 21 components. The first Bianchi identity $R_{a[bcd]} = 0$, which in bivector representation reads

$$\mathbf{R}_{14} + \mathbf{R}_{25} + \mathbf{R}_{36} = 0, \tag{5}$$

reduces the number of independent components to 20.

Einstein's equations with cosmological constant

$$R_{ab} - \frac{1}{2} R\eta_{ab} + \Lambda\eta_{ab} = \kappa T_{ab}, \quad R_{ab} = R^{c}{}_{acb}, \tag{6}$$

can be written explicitly in terms of the components of the curvature tensor in the bivector representation, resulting in a set of ten algebraic equations that relate the components $\mathbf{R}_{AB}$. This means that only ten components $\mathbf{R}_{AB}$ are algebraic independent which can be arranged in the $6 \times 6$ curvature matrix

$$\mathbf{R}_{AB} = \begin{pmatrix} \mathbf{M}_1 & \mathbf{L} \\ \mathbf{L} & \mathbf{M}_2 \end{pmatrix}, \tag{7}$$

where

$$\mathbf{L} = \begin{pmatrix} \mathbf{R}_{14} & \mathbf{R}_{15} & \mathbf{R}_{16} \\ \mathbf{R}_{15} - \kappa T_{03} & \mathbf{R}_{25} & \mathbf{R}_{26} \\ \mathbf{R}_{16} + \kappa T_{02} & \mathbf{R}_{26} - \kappa T_{01} & -\mathbf{R}_{14} - \mathbf{R}_{25} \end{pmatrix},$$

and $\mathbf{M}_1$ and $\mathbf{M}_2$ are $3 \times 3$ symmetric matrices

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & \mathbf{R}_{13} \\ \mathbf{R}_{12} & \mathbf{R}_{22} & \mathbf{R}_{23} \\ \mathbf{R}_{13} & \mathbf{R}_{23} & -\mathbf{R}_{11} - \mathbf{R}_{22} - \Lambda + \kappa\left(\frac{T}{2} + T_{00}\right) \end{pmatrix},$$

$$\mathbf{M}_2 = \begin{pmatrix} -\mathbf{R}_{11} + \kappa\left(\frac{T}{2} + T_{00} - T_{11}\right) & -\mathbf{R}_{12} - \kappa T_{12} & -\mathbf{R}_{13} - \kappa T_{13} \\ -\mathbf{R}_{12} - \kappa T_{12} & -\mathbf{R}_{22} + \kappa\left(\frac{T}{2} + T_{00} - T_{22}\right) & -\mathbf{R}_{23} - \kappa T_{23} \\ -\mathbf{R}_{13} - \kappa T_{13} & -\mathbf{R}_{23} - \kappa T_{23} & \mathbf{R}_{11} + \mathbf{R}_{22} + \Lambda - \kappa T_{33} \end{pmatrix},$$

with $T = \eta^{ab} T_{ab}$. This is the most general form of a curvature tensor that satisfies Einstein's equations with cosmological constant and arbitrary energy–momentum tensor. The traces of the matrices entering the final form of the curvature turn out to be of particular importance. First, the matrix $\mathbf{L}$ is traceless by virtue of the Bianchi identities, as shown above. Moreover, for the remaining matrices we obtain

$$\mathrm{Tr}(\mathbf{M_1}) = -\Lambda + \kappa\left(\frac{T}{2} + T_{00}\right), \quad \mathrm{Tr}(\mathbf{M_2}) = +\Lambda + \kappa T_{00}, \tag{8}$$

so that

$$\mathrm{Tr}(\mathbf{R_{AB}}) = \kappa\left(\frac{T}{2} + 2T_{00}\right)\,. \tag{9}$$

We see that these traces depend on the components of the energy–momentum tensor only.

### *2.1. Vacuum spacetimes*

In the particular case of vanishing cosmological constant ($\Lambda = 0$) and vacuum fields ($R_{ab} = 0$), the curvature matrix reduces to

$$\mathbf{R}_{AB} = \begin{pmatrix} \mathbf{M} & \mathbf{L} \\ \mathbf{L} & -\mathbf{M} \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & \mathbf{R}_{13} \\ \mathbf{R}_{12} & \mathbf{R}_{22} & \mathbf{R}_{23} \\ \mathbf{R}_{13} & \mathbf{R}_{23} & \mathbf{R}_{33} \end{pmatrix}, \tag{10}$$

and the $3 \times 3$ matrices $L$ and $M$ are symmetric and trace free,

$$\mathrm{Tr}(\mathbf{L}) = 0, \qquad \mathrm{Tr}(\mathbf{M}) = 0\,. \tag{11}$$

Notice that the relation $\mathrm{Tr}(\mathbf{L}) = 0$ is valid in general as a consequence of the Bianchi identities, whereas $\mathrm{Tr}(\mathbf{M}) = 0$ holds only in the limiting vacuum case.

### *2.2. Conformally flat spacetimes*

In the case of a conformally flat spacetime

$$g_{ab} = \mathrm{e}^{\Psi} \eta_{ab} \tag{12}$$

where $\Psi$ is an arbitrary $C^2$ function, we have that

$$R_{abcd} = \frac{1}{6} R(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}) + \frac{1}{2}(\eta_{ac}R_{bd} - \eta_{ad}R_{bc} - \eta_{bc}R_{ad} + \eta_{bd}R_{ac}), \tag{13}$$

where

$$R_{ab} = \Psi_{,ab} - \frac{1}{2}\Psi_{,a}\,\Psi_{,b} + \frac{1}{2}\eta_{ab}(\Box\Psi + \chi)\,, \tag{14}$$

and

$$R = 3\mathrm{e}^{-\psi}\left(\Box\Psi + \frac{1}{2}\chi\right)\,, \tag{15}$$

with $\Box\Psi \equiv \eta^{ab}\Psi_{,ab}$ and $\chi \equiv \eta^{ab}\Psi_{,a}\Psi_{,b}$.

Then, taking into account Einstein equations, the curvature matrix reduces to

$$\mathbf{R}_{AB} = \begin{pmatrix} \mathbf{M}_1 & \mathbf{L} \\ \mathbf{L} & \mathbf{M}_2 \end{pmatrix}, \tag{16}$$

with

$$\mathbf{L} = \frac{k}{2}\begin{pmatrix} 0 & T_{03} & -T_{02} \\ -T_{03} & 0 & T_{01} \\ T_{02} & -T_{01} & 0 \end{pmatrix}, \tag{17}$$

$$\mathbf{M_1} = \begin{pmatrix} -\frac{1}{3}\Lambda + \frac{1}{3}kT + \frac{1}{2}k(T_{00} - T_{11}) & -\frac{1}{2}kT_{12} & -\frac{1}{2}kT_{13} \\ -\frac{1}{2}kT_{12} & -\frac{1}{3}\Lambda + \frac{1}{3}kT + \frac{1}{2}k(T_{00} - T_{22}) & -\frac{1}{2}kT_{23} \\ -\frac{1}{2}kT_{13} & -\frac{1}{2}kT_{23} & -\frac{1}{3}\Lambda + \frac{1}{3}kT + \frac{1}{2}k(T_{00} - T_{33}) \end{pmatrix} \tag{18}$$

and

$$\mathbf{M_2} = \begin{pmatrix} \frac{1}{3}\Lambda + \frac{1}{6}kT + \frac{1}{2}k(T_{00} - T_{11}) & -\frac{1}{3}kT_{12} & -\frac{1}{2}kT_{13} \\ -\frac{1}{2}kT_{12} & \frac{1}{3}\Lambda + \frac{1}{6}kT + \frac{1}{2}k(T_{00} - T_{22}) & -\frac{1}{2}kT_{23} \\ -\frac{1}{2}kT_{13} & -\frac{1}{2}kT_{23} & \frac{1}{3}\Lambda + \frac{1}{6}kT + \frac{1}{2}k(T_{00} - T_{33}) \end{pmatrix}. \tag{19}$$

As before, the matrix $\mathbf{L}$ is traceless by virtue of the Bianchi identities. Moreover, for the remaining matrices we obtain

$$\mathrm{Tr}(\mathbf{M_1}) = -\Lambda + \kappa\left(\frac{T}{2} + T_{00}\right), \quad \mathrm{Tr}(\mathbf{M_2}) = +\Lambda + \kappa T_{00}, \tag{20}$$

so that

$$\mathrm{Tr}(\mathbf{R_{AB}}) = \kappa\left(\frac{T}{2} + 2T_{00}\right) . \tag{21}$$

An additional reduction is obtained if conformal invariance with $T = 0$ is demanded:

$$\mathbf{R}_{AB} = \begin{pmatrix} \mathbf{M}_1 & \mathbf{L} \\ \mathbf{L} & \mathbf{M}_2 \end{pmatrix}, \tag{22}$$

with

$$\mathbf{L} = \frac{k}{2}\begin{pmatrix} 0 & T_{03} & -T_{02} \\ -T_{03} & 0 & T_{01} \\ T_{02} & -T_{01} & 0 \end{pmatrix}, \tag{23}$$

$$\mathbf{M_1} = \begin{pmatrix} -\frac{1}{3}\Lambda + \frac{1}{2}k(T_{00} - T_{11}) & -\frac{1}{2}kT_{12} & -\frac{1}{2}kT_{13} \\ -\frac{1}{2}kT_{12} & -\frac{1}{3}\Lambda + \frac{1}{2}k(T_{00} - T_{22}) & -\frac{1}{2}kT_{23} \\ -\frac{1}{2}kT_{13} & -\frac{1}{2}kT_{23} & -\frac{1}{3}\Lambda + \frac{1}{2}k(T_{00} - T_{33}) \end{pmatrix} \tag{24}$$

and

$$\mathbf{M_2} = \begin{pmatrix} \frac{1}{3}\Lambda + \frac{1}{2}k(T_{00}-T_{11}) & -\frac{1}{3}kT_{12} & -\frac{1}{2}kT_{13} \\ -\frac{1}{2}kT_{12} & \frac{1}{3}\Lambda + \frac{1}{2}k(T_{00}-T_{22}) & -\frac{1}{2}kT_{23} \\ -\frac{1}{2}kT_{13} & -\frac{1}{2}kT_{23} & \frac{1}{3}\Lambda + \frac{1}{2}k(T_{00}-T_{33}) \end{pmatrix}. \tag{25}$$

Again, the matrix $\mathbf{L}$ is traceless by virtue of the Bianchi identities. Moreover, for the remaining matrices we obtain

$$\mathrm{Tr}(\mathbf{M_1}) = -\Lambda + \kappa T_{00}, \quad \mathrm{Tr}(\mathbf{M_2}) = +\Lambda + \kappa T_{00}, \tag{26}$$

so that

$$\mathrm{Tr}(\mathbf{R_{AB}}) = 2\kappa T_{00}\,. \tag{27}$$

### *2.3. Perfect fluid spacetimes*

For later use, we also consider the case of a perfect fluid energy–momentum tensor with density $\rho$ and pressure $p$

$$T_{ab} = (\rho + p)u_a u_b + p\eta_{ab}, \tag{28}$$

where $u_a$ is the four-velocity of the fluid which for simplicity can always be chosen as the comoving velocity $u^a = (-1,0,0,0)$. Then,

$$T_{ab} = \mathrm{diag}(\rho, p, p, p)\,. \tag{29}$$

The curvature matrix for a perfect fluid is then given by equation (7) with

$$\mathbf{L} = \begin{pmatrix} \mathbf{R}_{14} & \mathbf{R}_{15} & \mathbf{R}_{16} \\ \mathbf{R}_{15} & \mathbf{R}_{25} & \mathbf{R}_{26} \\ \mathbf{R}_{16} & \mathbf{R}_{26} & -\mathbf{R}_{14} - \mathbf{R}_{25} \end{pmatrix},$$

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & \mathbf{R}_{13} \\ \mathbf{R}_{12} & \mathbf{R}_{22} & \mathbf{R}_{23} \\ \mathbf{R}_{13} & \mathbf{R}_{23} & -\mathbf{R}_{11} - \mathbf{R}_{22} - \Lambda + \frac{\kappa}{2}\,(3p+\rho) \end{pmatrix},$$

$$\mathbf{M}_2 = \begin{pmatrix} -\mathbf{R}_{11} + \frac{\kappa}{2}(\rho+p) & -\mathbf{R}_{12} & -\mathbf{R}_{13} \\ -\mathbf{R}_{12} & -\mathbf{R}_{22} + \frac{\kappa}{2}(\rho+p) & -\mathbf{R}_{23} \\ -\mathbf{R}_{13} & -\mathbf{R}_{23} & \mathbf{R}_{11} + \mathbf{R}_{22} + \Lambda - \kappa p \end{pmatrix}.$$

The eigenvalues of the curvature tensor correspond to the eigenvalues of the matrix $\mathbf{R}_{AB}$. In general, they are functions $\lambda_i$, with $i = 1, 2, ..., 6$, which depend on the parameters and coordinates entering the tetrads $\vartheta^a$. As shown above, in the case of a vacuum solution the curvature matrix is traceless and hence the eigenvalues must satisfy the condition

$$\sum_{i=1}^{6} \lambda_i = 0 \,. \tag{30}$$

In the case of a perfect fluid solution, the curvature eigenvalues are related by

$$\sum_{i=1}^{6} \lambda_i = \frac{3\kappa}{2}(\rho + p) \,. \tag{31}$$

These identities are a consequence of applying Einstein's equations to the general form of the curvature matrix $\mathbf{R_{AB}}$ and, consequently, they should contain information about the behavior of the gravitational field. We will verify these statements in the examples to be presented below.

## 3. Repulsive gravity

The effects of repulsive gravity have been found in several contexts and different gravitational fields (see, for instance, [10–20] and the references cited therein). In particular, repulsive effects have been identified in the gravitational field of naked singularities and near black holes [21]. In the literature, there are several intuitive definitions of repulsive gravity, but only recently an invariant definition was proposed in [7] by using the eigenvalues of the curvature tensor. The idea consists in using the eigenvalues to detect the regions of the gravitational field of compact objects, where repulsive effects are of importance. Indeed, since the gravitational field of compact objects is asymptotically flat, the eigenvalues should vanish at infinity. When approaching the object, the eigenvalues can either increase exponentially until they diverge at the singularity or they change their sign at some point, indicating the character of gravity has changed. This behavior is schematically illustrated in figure 1. The dotted curve corresponds to an eigenvalue with no change in the character of gravity whereas the solid curve shows a maximum at a distance $r = r_{\rm rep}$, indicating the presence of repulsive gravity. Within the region $r < r_{\rm rep}$, repulsive gravity exists and even becomes dominant once the eigenvalue changes its sign.

The above intuitive description of repulsive gravity can be formalized as follows. Let $\{\lambda_i^+\}$ $(i = 1, ..., 6)$ represents the set of eigenvalues of an exterior spacetime. As explained above, the presence of an extremum in an eigenvalue is an indication of the existence of repulsive gravity. Then, let $\{r_l\}, l = 1,2,...$ with $0 < r_l < \infty$ represents the set of solutions of the equation

$$\frac{\partial \lambda_i}{\partial r}\bigg|_{r=r_l} = 0, \quad \text{with} \quad r_{\rm rep} = \max\{r_l\}, \tag{32}$$

i.e. $r_{\rm rep}$ is the largest extremum of the eigenvalues and is called repulsion radius. Notice that at $r_{\rm rep} = \max\{r_l\}$ the corresponding curvature eigenvalue should show a true change in the behavior of the gravitational interaction, i.e. the set up of repulsive gravity should be clear. For this reason, one should also impose the condition that the second derivative of the eigenvalue at the extremum be different from zero to avoid the presence of saddle points. In all the examples we will consider below, this condition is satisfied.

This definition is obviously related to the existence of eigenvalue extrema. It could happen that no extrema exist at all. This would correspond to the case illustrated in figure 1 with a dotted curve, i.e. the non-existence of extrema implies that no repulsion radius exists and, consequently, no repulsive effects can be observed in such a gravitational field. We will see

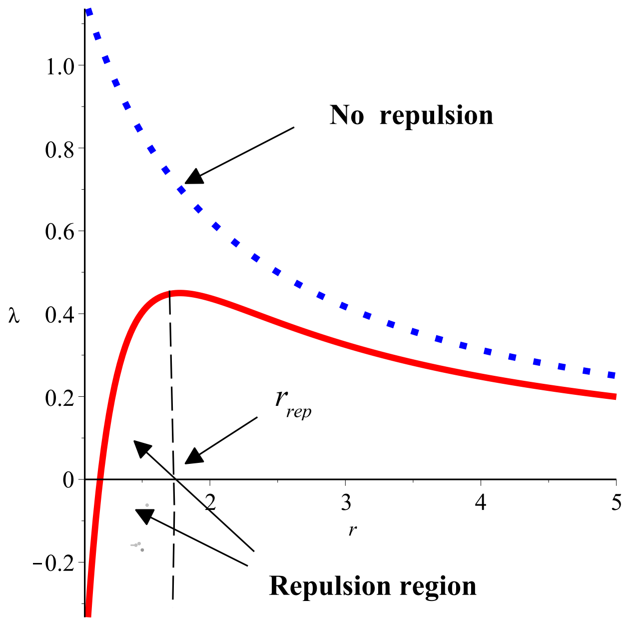


**Figure 1.** Schematic representation of the behavior of the curvature eigenvalue. Here, $r$ represents the distance from the source. The repulsive region $r < r_{\rm rep}$ includes positive values of the eigenvalue, where repulsive effects can be detected, and negative values, where repulsion becomes dominant.

below that this is the case of the Schwarzschild spacetime. If there is only one extremum, the repulsion radius coincides with the value of the radial coordinate at the extremum. On the other hand, if several extrema exist, the region contained between two extrema would show the effects corresponding to a transition between, say, a local maximum attraction and a local maximum repulsion. We will show in the next section that in the Kerr, Reissner–Nordström, and Kerr–Newman spacetimes there are several extrema. This seems to indicate that a general condition for the existence of a repulsion radius is related to the existence of multipole moments other than the mass monopole.

Since the curvature eigenvalues characterize in an invariant way the gravitational interaction, the above definition represents a invariant method to derive the repulsion region of asymptotically flat spacetimes, which describe the gravitational field of compact objects.

### *3.1. Repulsive gravity around black holes*

In this work, we will use the above definition of repulsive radius to establish the minimum radius at which an exterior spacetime can be matched with an interior spacetime. In particular, we are interested in matching asymptotically flat exterior spacetimes with interior spacetimes that are free of singularities inside a certain particular region. It is therefore important to determine the location of the repulsion radius of known spacetimes that describe the gravitational field of compact objects. Let us thus consider the Kerr–Newman black hole spacetime [4]:

$$
\begin{aligned}
\mathrm{d}s^2 = &-\frac{r^2-2Mr+a^2+Q^2}{r^2+a^2\cos^2\theta}(\mathrm{d}t-a\sin^2\theta\mathrm{d}\varphi)^2 \\
&+\frac{\sin^2\theta}{r^2+a^2\cos^2\theta}[(r^2+a^2)\mathrm{d}\varphi-a\mathrm{d}t]^2 \\
&+\frac{r^2+a^2\cos^2\theta}{r^2-2Mr+a^2+Q^2}\mathrm{d}r^2+(r^2+a^2\cos^2\theta)\mathrm{d}\theta^2,
\end{aligned}
\tag{33}
$$

where $M$ is the mass of the rotating central object, $a = J/M$ is the specific angular momentum, and $Q$ represents the total electric charge. The orthonormal tetrad can be chosen as

$$
\begin{aligned}
\vartheta^0 &= \left(\frac{r^2-2Mr+a^2+Q^2}{r^2+a^2\cos^2\theta}\right)^{1/2}(\mathrm{d}t-a\sin^2\theta\mathrm{d}\varphi), \\
\vartheta^1 &= \frac{\sin\theta}{(r^2+a^2\cos^2\theta)^{1/2}}[(r^2+a^2)\mathrm{d}\varphi-a\mathrm{d}t], \\
\vartheta^2 &= (r^2+a^2\cos^2\theta)^{1/2}\mathrm{d}\theta, \\
\vartheta^3 &= \left(\frac{r^2+a^2\cos^2\theta}{r^2-2Mr+a^2+Q^2}\right)^{1/2}\mathrm{d}r\ .
\end{aligned}
\tag{34}
$$

Following the approach presented in the last section, lengthy computations lead to the following eigenvalues [7]

$$\lambda_1^+ + \mathrm{i}\lambda_4^+ = \lambda_2^+ + \mathrm{i}\lambda_5^+ = l, \quad \lambda_3^+ + \mathrm{i}\lambda_6^+ = -2l + k, \tag{35}$$

where

$$l = -\left[M - \frac{Q^2\,(r+\mathrm{i}a\cos\theta)}{r^2+a^2\cos^2\theta^2}\right]\left(\frac{r-\mathrm{i}a\cos\theta}{r^2+a^2\cos^2\theta}\right)^3, \tag{36}$$

$$k = -\frac{Q^2}{(r^2+a^2\cos^2\theta)^2}\ . \tag{37}$$

A straightforward analysis shows that all the eigenvalues have extrema located at different values of $r$, but the largest one is associated with the equation $\frac{\partial\lambda_3^+}{\partial r} = 0$, the roots of which are determined by the equation

$$r^3(Mr-2Q^2) + a^2\cos^2\theta[2Q^2r + M(a^2\cos^2\theta - 6r^2)] = 0\ . \tag{38}$$

The analytical solution to this quartic equation can be represented as

$$r_{\text{rep}}^{\text{KN}} = \frac{1}{2M}\left[Q^2 + A\left(1+\sqrt{2+\frac{2Q^2}{A}}\right)\right], \quad A = \sqrt{Q^4+4a^2M^2\cos^2\theta}, \tag{39}$$

which in the limiting case $a = 0$ reduces to the Reissner–Nordström repulsion radius

$$r_{\text{rep}}^{\text{RN}} = \frac{2Q^2}{M}, \tag{40}$$

for $Q = 0$ leads to repulsion radius of the Kerr source

$$r_{\text{rep}}^{\text{K}} = (1+\sqrt{2})a\cos\theta, \tag{41}$$

and in the Schwarzschild limiting case yields no repulsion radius. We see that the repulsion region is located very closed to the source. In the case of the Reissner–Nordström spacetime, the repulsion radius is twice the classical radius of a particle with mass $M$ and charge $Q$, implying that repulsion is a pure classical effect. In the case of elementary particles, like the electron, the value of the classical radius would imply that the surrounding Reissner–Nordström spacetime is a naked singularity. If we also take into account the rotational properties of the particle and use the Kerr–Newman metric, the resulting spacetime corresponds to a naked singularity as well (see [22] and the references cited therein). In the Kerr spacetime, the repulsion radius depends on the polar angle, vanishing on the equatorial plane and reaching its maximum value at the poles. This result is frame independent in the sense that it is derived from a curvature eigenvalue, which can be considered as a scalar. The angular dependence of the Kerr repulsion radius is also in agreement with the fact that on the equator the curvature of the Kerr metric coincides with that of the Schwarzschild metric, which has no repulsion radius. In figure 2, we illustrate the shape of the repulsion radii of black holes and compare them with the location of the horizon and ergosphere. Indeed, we see that the repulsion region is located in the region of spacetime occupied by the horizon and ergosphere, which is closed to the gravitational source. This agrees with the regions where repulsive effects have been identified by using a completely different approach based on the analysis of the motion of test particles along circular orbits around black holes and naked singularities [21, 23, 24].

## 4. $C^3$ matching

The exterior field of compact objects is usually described by vacuum exact solutions characterized by singularities in a region closed to the source of gravity. To describe the entire spacetime and to avoid the presence of singularities, we usually say that the exterior spacetime must be matched with an interior spacetime which 'covers' the region with singularities. To do this in concrete examples, we usually apply physical intuition to determine the matching surface and anyone of the $C^2$ methods mentioned in section 1 to carry out the matching. The goal of the $C^3$ matching approach is to determine in an invariant manner where and how to 'cover' the singular spacetime region.

To be more specific, consider an exterior spacetime $(M^+, g^+_{\mu\nu})$ and an interior spacetime $(M^-, g^-_{\mu\nu})$, which are characterized by the curvature eigenvalues $\{\lambda_i^+\}$ and $\{\lambda_i^-\}$, respectively. Then, the $C^3$ matching procedure consists in $(i)$ establishing the matching surface $\Sigma$ as determined by the matching radius [5]

$$r_{\rm match} \in [r_{\rm rep}, \infty), \quad \text{with} \quad r_{\rm rep} = \max\{r_l\}, \quad \left.\frac{\partial \lambda_i^+}{\partial r}\right|_{r=r_l} = 0, \tag{42}$$

and $(ii)$ performing the matching of the spacetimes $(M^+, g^+_{\mu\nu})$ and $(M^-, g^-_{\mu\nu})$ at $\Sigma$ by imposing the conditions

$$\lambda_i^+\Big|_\Sigma = \lambda_i^-\Big|_\Sigma \quad \forall i\,. \tag{43}$$

The idea of defining the matching surface in terms of the matching radius was first proposed in [5]. Continuing with the formalization of the $C^3$ matching procedure, in the present work, we introduce the matching conditions (43) that are $C^2$ conditions similar to those assumed in other matching formalisms.

Thus, the $C^3$ matching demands that the curvature eigenvalues be continuous across the matching surface $\Sigma$ which should be located at any radius between the repulsion radius and infinity. Notice that the repulsion radius is determined by the eigenvalues of the exterior curvature tensor. Accordingly, the minimum matching radius coincides with the repulsion radius. Physically, this means that the $C^3$ matching is intended to avoid the presence of repulsive gravity in the case of gravitational compact objects. The motivation for this requirement is that so far no repulsive gravity has been detected in the gravitational field of realistic compact objects like white dwarfs, neutron stars and other kinds of stars and planets. We can, therefore, consider repulsive gravity as an unphysical phenomenon at the level of compact objects and we propose to use the $C^3$ matching procedure to avoid such unphysical situation by covering the repulsive region by an physical interior solution.

### *4.1. Newtonian gravity*

It is well known that Newtonian gravity is contained in the Einstein gravity theory as a special case. In some sense, we could also say that Newtonian gravity is the simplest non-trivial special case of Einstein gravity. It is, therefore, reasonable to test the $C^3$ matching in this simple special case. If it turns out that this procedure does not lead to physically meaningful results in the Newtonian limiting case, it would imply a failure of the method. Then, the possibility of being successful in the general Einstein theory would be very small.

Let us consider the line element for the nearly Newtonian metric in spherical coordinates $x^\alpha = (t, r, \theta, \varphi)$ (see [9], p 470)

$$\mathrm{d}s^2 = -(1+2\Phi)\mathrm{d}t^2 + (1-2\Phi)(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2) \tag{44}$$

where $\Phi \ll 1$ is the Newtonian potential. In this work, we limit ourselves to the study of spherically symmetric gravitational configurations and so we assume that $\Phi$ depends on $r$ only. The components of the orthonormal tetrad are then

$$\vartheta^0 = \sqrt{1+2\Phi}\mathrm{d}t, \quad \vartheta^1 = \sqrt{1-2\Phi}\mathrm{d}r, \tag{45}$$

$$\vartheta^2 = \sqrt{1-2\Phi}\, r\,\mathrm{d}\theta, \quad \vartheta^3 = \sqrt{1-2\Phi}\, r\sin\theta\,\mathrm{d}\varphi, \tag{46}$$

which in the first-order approximation lead to the connection 1-form

$$\omega^0{}_1 = \Phi_r\vartheta^0, \quad \omega^2{}_3 = -\frac{1}{r}(1+\Phi)\cot\theta\vartheta^3, \tag{47}$$

$$\omega^1{}_2 = -\frac{1}{r}(1+\Phi - r\Phi_r)\vartheta^2, \quad \omega^1{}_3 = -\frac{1}{r}(1+\Phi - r\Phi_r)\vartheta^3, \tag{48}$$

where $\Phi_r$ denotes the radial derivative of $\Phi$. Moreover, the only non-vanishing components of the curvature 2-form can be expressed up to the first order in $\Phi$ as

$$\Omega^0{}_1 = -\Phi_{rr}\,\vartheta^0\wedge\vartheta^1, \;\; \Omega^0{}_2 = -\frac{1}{r}\Phi_r\,\vartheta^0\wedge\vartheta^2, \tag{49}$$

$$\Omega^0{}_3 = -\frac{1}{r}\Phi_r\,\vartheta^0\wedge\vartheta^3, \;\; \Omega^2{}_3 = \frac{2}{r}\Phi_r\vartheta^2\wedge\vartheta^3, \tag{50}$$

$$\Omega^3{}_1 = (\Phi_{rr} + \frac{1}{r}\Phi_r)\vartheta^2\wedge\vartheta^3, \;\; \Omega^1{}_2 = (\Phi_{rr} + \frac{1}{r}\Phi_r)\vartheta^1\wedge\vartheta^2\,. \tag{51}$$

It then follows that the only non-zero components of the curvature tensor are

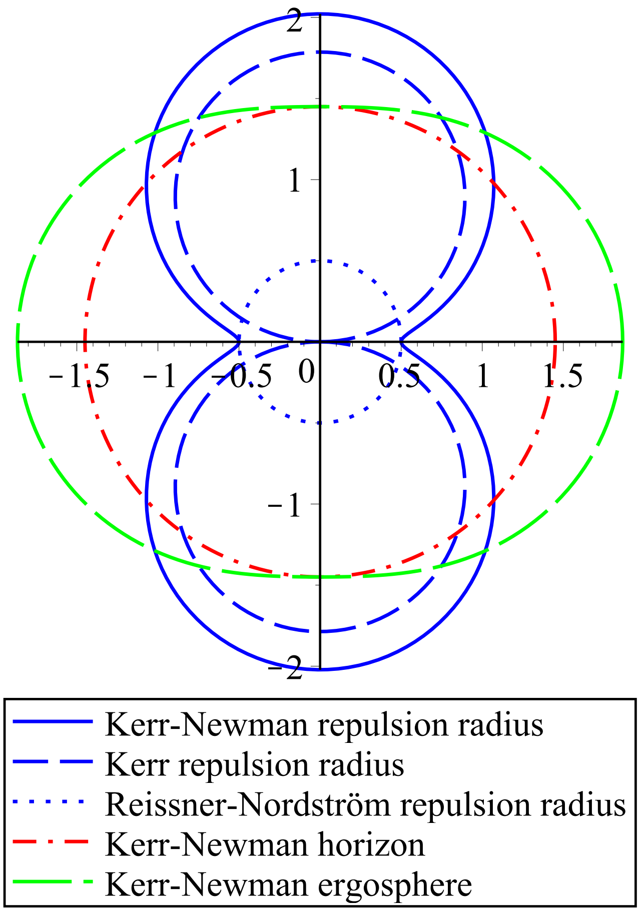


**Figure 2.** Repulsion radius of black holes with $M = 1$, $Q = 0.5$ and $a = 0.74$. For comparison, the horizon and ergosphere of the Kerr–Newman black hole are also plotted.

$$R_{0101} = \mathbf{R}_{11} = \Phi_{rr}\,, \quad R_{0202} = \mathbf{R}_{22} = R_{0303} = \mathbf{R}_{33} = \frac{1}{r}\Phi_r\,, \tag{52}$$

$$R_{2323} = \mathbf{R}_{44} = \frac{2}{r}\Phi_r\,, \quad R_{3131} = \mathbf{R}_{55} = R_{1212} = \mathbf{R}_{66} = \Phi_{rr} + \frac{1}{r}\Phi_r\,. \tag{53}$$

Consequently, the curvature matrix $\mathbf{R_{AB}}$ (7) is diagonal with eigenvalues

$$\lambda_1 = \Phi_{rr}, \quad \lambda_2 = \lambda_3 = \frac{1}{r}\Phi_r, \tag{54}$$

$$\lambda_4 = \frac{2}{r}\Phi_r, \quad \lambda_5 = \lambda_6 = \Phi_{rr} + \frac{1}{r}\Phi_r, \tag{55}$$

which satisfy the relationship

$$\sum_{i=1}^{6} \lambda_i = 3\left(\Phi_{rr} + \frac{2}{r}\Phi_r\right) = 3\,\nabla^2\Phi\,. \tag{56}$$

We then conclude that in Newtonian gravity the eigenvalue identity (31) is equivalent to the Poisson equation, i.e.

$$\sum_{i=1}^{6} \lambda_i = \frac{3\kappa}{2}\rho \quad \Leftrightarrow \quad \nabla^2\Phi = \frac{\kappa}{2}\rho\,. \tag{57}$$

### *4.2. $C^3$ matching in Newtonian gravity*

To illustrate the $C^3$ matching approach in Newtonian gravity, let us consider a spherically symmetric solution. Then, the corresponding exterior field should correspond to that of a sphere described by a solution of the Laplace equation with

$$\rho_{\text{ext}} = 0, \qquad \Phi_{\text{ext}} = -\frac{M}{r}, \tag{58}$$

where $M$ is a constant. It is then straightforward to calculate the curvature eigenvalues which turn out to be

$$\lambda_1^+ = -\lambda_4^+ = -\frac{2M}{r^3}, \quad \lambda_2^+ = \lambda_3^+ = -\lambda_5^+ = -\lambda_6^+ = \frac{M}{r^3}. \tag{59}$$

The condition

$$\left.\frac{\mathrm{d}\lambda_i^+}{\mathrm{d}r}\right|_{r_{\text{match}}} = 0 \tag{60}$$

does not offer any positive finite root for the minimum matching radius, indicating that the matching can be performed at any radius $r_{\text{match}} \in (0, \infty)$. We proceed now to the second step and demand the equality of the eigenvalues across the matching surface, i.e.

$$\left.\lambda_i^-\right|_{r_{\text{match}}} = \left.\lambda_i^+\right|_{r_{\text{match}}}. \tag{61}$$

Using the expressions (54) for the eigenvalues of the interior solution, we obtain that at the matching radius the following conditions must be satisfied

$$\lambda_1^- = \frac{\kappa}{2}\rho - \frac{2}{r}\Phi_r = -\frac{2M}{r^3}, \quad \lambda_2^- = \frac{1}{r}\Phi_r = \frac{M}{r^3}, \quad \lambda_3^- = \frac{1}{r}\Phi_r = \frac{M}{r^3}, \tag{62}$$

$$\lambda_4^- = \frac{2}{r}\Phi_r = \frac{2M}{r^3}, \quad \lambda_5^- = \frac{\kappa}{2}\rho - \frac{1}{r}\Phi_r = -\frac{M}{r^3}, \quad \lambda_6^- = \frac{\kappa}{2}\rho - \frac{1}{r}\Phi_r = -\frac{M}{r^3}, \tag{63}$$

where we have used Poisson's equation (57) to replace the second derivative of $\Phi$. The above equations represent a system of three independent algebraic equations from which we obtain that the only compatible solution is

$$\rho = 0 \quad \text{at} \quad r = r_{\text{match}}\,. \tag{64}$$

This condition is in agreement with our physical intuition as we expect that the density vanishes at the matching surface. Notice that to obtain this result, we used, on the one hand, Poisson's equation for the internal potential, without specifying any particular equation of state for the matter density $\rho$, i.e. no particular solution of Poisson's equation was involved in the proof. Consequently, this result is independent of the equation of state of the spherical mass distribution. On the other hand, for the matching with the external metric, we used the Newtonian potential of a sphere, which is described by a unique solution (58) of the Laplace equation. We conclude, therefore, that the $C^3$ matching condition (64) is valid, in general, for any spherically symmetric field in Newtonian gravity.

### *4.3. Particular solutions in Newtonian gravity*

We now consider some solutions of the Poisson equation which determine interior Newtonian fields. Remember that a solution of the Poisson equation $\nabla^2\Phi = \frac{\kappa}{2}\rho$ depends on the specific form of the density function $\rho$. In general, the density is given *a priori* and the potential $\Phi$ is determined from the Poisson differential equation. Several particular solutions are known. For concreteness, let us consider the following solutions [25]

$$\rho_{\mathrm{HS}} = \rho_0, \qquad \Phi_{\mathrm{HS}} = -\frac{k\rho_0}{4}\left(a^2 - \frac{r^2}{3}\right), \tag{65}$$

$$\rho_{\mathrm{P}} = \frac{6Mb^2}{k(r^2+b^2)^{5/2}}, \quad \Phi_{\mathrm{P}} = -\frac{M}{(r^2+b^2)^{1/2}}, \tag{66}$$

$$\rho_{\mathrm{IP}} = \frac{2Mb\left[3b^2 + 3b(b^2+r^2)^{1/2} + 2r^2\right]}{k(b^2+r^2)^{3/2}\left[b + (b^2+r^2)^{1/2}\right]^3}, \quad \Phi_{\mathrm{IP}} = -\frac{M}{b+(b^2+r^2)^{1/2}}, \tag{67}$$

where $\rho_0$, $a$ and $b$ are constants. These solutions are known as the homogeneous sphere, Plummer model and isochrone potential, respectively. As we can see, in general, none of these solutions satisfies the matching condition $\rho = 0$. This means that strictly speaking none of them can be matched with the exterior solution of a sphere (58). However, to illustrate the validity of the matching procedure, consider, for instance, the eigenvalues of the Plummer model, which can be written as

$$\lambda_1^- = \frac{M(b^2-2r^2)}{(b^2+r^2)^{5/2}}, \tag{68}$$

$$\lambda_2^- = \lambda_3^- = \lambda_4^-/2 = \frac{M}{(b^2+r^2)^{3/2}},$$
$$\lambda_5^- = \lambda_6^- = \frac{M(2b^2-r^2)}{(b^2+r^2)^{5/2}}\ . \tag{69}$$

A straightforward computation of the conditions $\lambda_i^+ = \lambda_i^-$ shows that the only possible solution is $b = 0$, which coincides with the matching condition $\rho_P = 0$. Moreover, from the expressions for the eigenvalues we see that

$$\lim_{r\to\infty} \lambda_i^- = \lambda_{Ei}^+, \tag{70}$$

indicating that the matching can be performed only at infinity. An analysis of the interior solutions for the homogeneous sphere and the isochrone potential leads to similar results. This corroborates the validity of the $C^3$ matching conditions in Newtonian gravity.

## 5. Spherically symmetric relativistic fields

For the investigation of relativistic fields, we consider the general spherically symmetric line element

$$ds^2 = -e^{\nu}dt^2 + e^{\phi}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{71}$$

where $\nu$ and $\phi$ depend on $r$ only. It then follows that the corresponding orthonormal tetrad can be chosen as

$$\vartheta^0 = e^{\nu/2}dt, \quad \vartheta^1 = e^{\phi/2}dr, \quad \vartheta^2 = rd\theta, \quad \vartheta^3 = r\sin\theta d\varphi\, . \tag{72}$$

It is straightforward to compute the connection 1-form

$$\begin{aligned} &\omega^1{}_2 = -\frac{1}{r}e^{-\phi/2}\vartheta^2, \quad \omega^1{}_3 = -\frac{1}{r}e^{-\phi/2}\vartheta^3, \\ &\omega^2{}_3 = -\frac{1}{r}\cot\theta\vartheta^3, \quad \omega^1{}_0 = -\frac{\nu_r}{2r}e^{-\phi/2}\vartheta^4, \end{aligned} \tag{73}$$

and from here the curvature 2-form and the components of the curvature tensor, which can be expressed as

$$\begin{aligned} &R_{0101} = \mathbf{R}_{11} = -\frac{1}{4}(\phi_r\nu_r - \nu_r^2 - 2\nu_{rr})e^{-\phi}, \quad R_{0202} = \mathbf{R}_{22} = \frac{1}{2r}\nu_r e^{-\phi}, \\ &R_{0303} = \mathbf{R}_{33} = \frac{1}{2r}\nu_r e^{-\phi}, \quad R_{2323} = \mathbf{R}_{44} = \frac{1}{r^2}(1 - e^{-\phi}), \\ &R_{3131} = \mathbf{R}_{55} = \frac{1}{2r}\phi_r e^{-\phi}, \quad R_{1212} = \mathbf{R}_{66} = \frac{1}{2r}\phi_r e^{-\phi}\, . \end{aligned} \tag{74}$$

We then obtain the following eigenvalues for the curvature tensor of an interior perfect fluid solution

$$\lambda_1^- = \mathbf{R}_{11},\ \lambda_2^- = \mathbf{R}_{22},\ \lambda_3^- = \mathbf{R}_{33}, \tag{75}$$

$$\lambda_4^- = -\lambda_1^- + \frac{k(\rho+p)}{2},\ \lambda_5^- = -\lambda_2^- + \frac{k(\rho+p)}{2},\ \lambda_6^- = -\lambda_3^- + \frac{k(\rho+p)}{2}. \tag{76}$$

Moreover, Einstein's equations can be expressed as

$$\nu_{rr} + \frac{1}{2}\nu_r^2 - \frac{\nu_r}{2r}(2 + r\phi_r) - \frac{\phi_r}{r} - \frac{2}{r^2}(1 - e^{\phi}) = 0, \tag{77}$$

$$\kappa\rho = \frac{1}{r^2}[1 + e^{-\phi}(r\phi_r - 1)],\ \ \kappa p = -\frac{1}{r^2}[1 - e^{-\phi}(1 + r\nu_r)]. \tag{78}$$

### 5.1. $C^3$ matching in general relativity

To proceed with the matching, we consider the Schwarzschild solution

$$\rho_S = p_S = 0,\ \ \phi_S = -\nu_S = -\ln(1 - 2M/r) \tag{79}$$

as the only available spherically symmetric vacuum solution. The eigenvalues are as follows:

$$\lambda_1^+ = -\lambda_4^+ = -2M/r^3,\ \lambda_2^+ = \lambda_3^+ = -\lambda_5^+ = -\lambda_6^+ = M/r^3. \tag{80}$$

The $C^3$ matching condition $\mathrm{d}\lambda_i^+/\mathrm{d}r = 0$ does not lead to any repulsion radius and so the matching can be carried out within the interval $r_{\mathrm{match}} \in (0, \infty)$, resembling the situation in the case of Newtonian gravity. If, in addition, we demand that the exterior (79) and interior eigenvalues (75) coincide on the matching surface, we obtain that the only solution is

$$\rho = 0,\ p = 0 \quad \text{at} \quad r = r_{\mathrm{match}}\ . \tag{81}$$

Again, this result is very consistent and corroborates in an invariant way our physical expectation of vanishing pressure and density on the matching surface. In the Newtonian limit, which at the level of the energy–momentum tensor is obtained by neglecting the pressure, we corroborate the result obtained above for Newtonian gravity that the density should vanish at the matching surface.

Notice that these conditions should be imposed on the equation of state, which determines the internal structure of the spherically symmetric mass distribution. This means that they are physical conditions to be satisfied by the interior solution. Usually, it is difficult to find analytic solutions to the Einstein equations. If, in addition, we demand that the solutions be physical, the number of known solutions reduces dramatically. This seems to be the case of spherically symmetric solutions. Indeed, the physical conditions (81), which follow from the $C^3$ matching procedure, seem to discard a large number of known analytic solutions. In the appendix, we include a series of spherically symmetric interior solutions which are known as Tolman *I–VIII*. For all the solutions of this class we computed the interior eigenvalues, which are also included in the appendix. It is straightforward to show that none of these solutions can be matched with the exterior Schwarzschild metric.

## 6. Conclusions

In this work, we propose a new method for matching two spacetimes in general relativity. We demand that the curvature eigenvalues of the interior and exterior solutions be continuous across the matching surface. To determine the matching surface, we assume that the exterior spacetime is asymptotically flat and consider the behavior of the corresponding eigenvalues as the source is approached. A monotonous growth of the eigenvalues is interpreted as corresponding to the presence of attractive gravitational interaction throughout the entire space. On the contrary, if an eigenvalue shows local extrema and even changes its sign as the source is approached, we interpret this behavior as due to the presence of repulsive gravity. The repulsion radius is defined by the location of the first extremum ($C^3$ condition), which appears as the source is approached from spatial infinity. In turn, the repulsion radius is defined as the minimum radius, where the matching can be carried out, i.e. the matching surface can be located anywhere between the repulsion radius and infinity. This means that the goal of fixing a minimum radius for the matching surface is to avoid the presence of repulsive gravity because it has not been detected at least in the gravitational field of compact objects.

We tested the $C^3$ matching procedure in the case of spherically symmetric perfect fluid spacetimes in Newtonian gravity and in general relativity. Remarkably, our method leads to completely general results, independently of any particular solution of the field equations. In the case of Newtonian gravity, we obtain that the matter density $\rho$ of the gravitational source should vanish on the matching surface. This result is valid in general because to obtain it, we used, on the one hand, Poisson's equation for the internal potential, without specifying any particular form for the matter density $\rho$. Consequently, this result is independent of the equation of state of the spherical mass distribution. On the other hand, for the matching with the external metric, we used the Newtonian potential of a sphere, which is described by a unique solution of the Laplace equation. This means that the $C^3$ matching condition $\rho = 0$ on the matching surface is valid for any spherically symmetric field in Newtonian gravity. In the case of general relativity, we applied a similar procedure, in which only the interior Einstein field equations and the exterior Schwarzschild metric are involved in the analysis of the $C^3$ matching conditions, and obtained that the matter density and the pressure should vanish on the matching surface. Again, this result is independent of the equation of state, which determines the internal structure of the spherical mass distribution. These conditions are very plausible from a physical point of view and, therefore, establish the validity of the $C^3$ matching. We analyzed several particular examples of well-known spherically symmetric perfect fluid solutions and found out that, in general, it is not possible to satisfy the $C^3$ matching conditions. This result indicates that to obtain physically meaningful interior solutions, it would be convenient to start from metrics, satisfying the matching conditions *a priori* and containing arbitrary functions that are then determined by the field equations.

In this work, we limited ourselves to the study of spherically symmetric solutions so that the matching surface is easily identified as a sphere. However, it is possible to apply the $C^3$ matching method to the case of axially symmetric spacetimes, which are more realistic as models for describing the gravitational field of astrophysical compact objects. Preliminary results show that in the case of metrics with quadrupolar moment, the repulsion radius depends on the angular coordinate so that the matching surface is different from an ideal sphere. In this case, the $C^3$ matching implies a detailed numerical analysis of the curvature eigenvalues. This work is in progress and will be presented elsewhere.

As presented here, the $C^3$ matching method has been specially adapted for the study of asymptotically flat spacetimes, which can be used to describe the gravitational field of astrophysical compact objects. However, it is also possible to consider other physical situations in which, for instance, cosmological models or collapsing shells are to be matched. We expect to investigate this type of configurations in future works.

The generality of the results obtained in this work points out to the possibility of applying the $C^3$ matching procedure in other theories and scenarios. For instance, matching conditions are important in the context of local strings, domail walls, braneworld scenarios, etc. We will analyze these problems in other works.

The matching problem in general relativity and other theories can also be investigated by using the distributional calculus techniques. In particular, Israel's matching conditions of general relativity can be formulated in a very elegant way by using these techniques (see, for instance, [29]). Israel's conditions can be understood as a $C^2$ matching procedure because they involve only second order derivatives of the metric. The disadvantage of this method is that it can be applied only once the matching surface is known. On the other hand, the $C^3$ method we propose here can be used to find the matching surface by imposing the condition that no repulsive gravity exists in the gravitational field of compact objects. However, our approach is different and cannot be formulated with the elegant method of distributional calculus. Indeed, we impose that the curvature eigenvalues of the interior and exterior metrics coincide on the matching surface and, therefore, in the case of no coincidence our method does not apply. This is a disadvantage of our method. However, in the case of no coincidence it seems reasonable to try to apply distributional calculus to handle correctly the jump in the eigenvalues. We expect to consider this problem in future works.

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## Appendix. Spherically symmetric interior solutions

In 1939, Tolman [26] carried out an exhaustive analysis of the Einstein field equations in the presence of a perfect fluid with spherical symmetry. The method consisted in giving *a priori* one of the unknowns, the density $\rho$ for instance, and finding the remaining unknowns from the field equations. As a result, Tolman derived eight different solutions, which are known as the Tolman class. This class of exact analytic solutions has many features that make it interesting to describing physical cosmology and for modeling high density astronomical objects. In particular, the solutions IV, VI and VII have received great attention and have been generalized in different contexts of the general relativity (see [27, 28]).

The Tolman solutions can be represented by the metric functions $\nu(r)$ and $\phi(r)$ of the line element

$$\mathrm{d}s^2 = -\mathrm{e}^{\nu}\mathrm{d}t^2 + \mathrm{e}^{\phi}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) \tag{A.1}$$

and by the density $\rho$ and pressure $p$ of the perfect fluid. In this appendix, we present the explicit form of all the eight interior solutions and calculate the corresponding curvature eigenvalues $\lambda_i$. The solutions and eigenvalues are denoted by a subscript in front of all the relevant quantities.

**Tolman I: The Einstein universe**
**Solution:**

$$\begin{aligned} &\phi_{\mathrm{EU}} = -\ln\left(1 - r^2/\kappa^2\right), \quad \nu_{\mathrm{EU}} = 2\ln(c), \\ &\rho_{\mathrm{EU}} = \frac{3}{k\kappa^2}, \quad p_{\mathrm{EU}} = -\frac{\rho_{\mathrm{EU}}}{3}, \end{aligned} \tag{A.2}$$

where $c$, $k$ and $\kappa$ are constants.

**Eigenvalues:**

$$\begin{aligned} &\lambda_{\mathrm{EU}_1} = \lambda_{\mathrm{EU}_2} = \lambda_{\mathrm{EU}_3} = 0, \\ &\lambda_{\mathrm{EU}_4} = \lambda_{\mathrm{EU}_5} = \lambda_{\mathrm{EU}_6} = 1/\kappa^2. \end{aligned} \tag{A.3}$$

**Tolman II: The Schwarzschild–de Sitter solution**
**Solution:**

$$
\begin{aligned}
&\phi_{\mathrm{SdS}} = -\ln\left(1 - 2M/r - r^2/\kappa^2\right),\\
&\nu_{\mathrm{SdS}} = \ln\left[c^2\left(1 - 2M/r - r^2/\kappa^2\right)\right],\\
&\rho_{\mathrm{SdS}} = -p_{\mathrm{SdS}} = \frac{3}{k\kappa^2},
\end{aligned}
\tag{A.4}
$$

where $c$, $k$ and $\kappa$ are constants.

**Eigenvalues:**

$$
\lambda_{\mathrm{SdS}_1} = -\lambda_{\mathrm{SdS}_4} = -\frac{2\kappa^2 M + r^3}{\kappa^2 r^3},
\tag{A.5}
$$

$$
\lambda_{\mathrm{SdS}_2} = \lambda_{\mathrm{SdS}_3} = -\lambda_{\mathrm{SdS}_5} = -\lambda_{\mathrm{SdS}_6} = \frac{\kappa^2 M - r^3}{\kappa^2 r^3}.
\tag{A.6}
$$

**Tolman III: Schwarzschild interior solution**
**Solution:**

$$
\begin{aligned}
&\phi_{\mathrm{SI}} = -\ln\left[1 - 2Mr^2/\kappa^3\right],\\
&\nu_{\mathrm{SI}} = 2\ln\left[\frac{3(1-2M/\kappa)^{1/2}}{2} - \frac{\left(1 - 2Mr^2/\kappa^3\right)^{1/2}}{2}\right],\\
&\rho_{\mathrm{SI}} = \frac{6M}{k\kappa^3}, \quad p_{\mathrm{SI}} = \frac{6M\left[(1-2Mr^2/\kappa^3)^{1/2} - (1-2M/\kappa)^{1/2}\right]}{k\kappa^3\left[3(1-2M/\kappa)^{1/2} - (1-2Mr^2/\kappa^3)^{1/2}\right]},
\end{aligned}
\tag{A.7}
$$

where $k$, $\kappa$ and $M$ are constants.

**Eigenvalues:**

$$
\begin{aligned}
&\lambda_{\mathrm{SI}_1} = \lambda_{\mathrm{SI}_2} = \lambda_{\mathrm{SI}_3} = -\frac{2M(2Mr^2 - k^3)^{1/2}}{\kappa^3\left[(2Mr^2 - k^3)^{1/2} - 3\kappa(2M-\kappa)^{1/2}\right]},\\
&\lambda_{\mathrm{SI}_4} = \lambda_{\mathrm{SI}_5} = \lambda_{\mathrm{SI}_6} = 2M/\kappa^3.
\end{aligned}
\tag{A.8}
$$

**Tolman IV**
**Solution:**

$$
\begin{aligned}
&\phi_{IV} = \ln\left[\frac{1 + 2r^2/A^2}{(1 - r^2/\kappa^2)(1 + r^2/A^2)}\right], \ \ \nu_{IV} = \ln\left[B^2(1 + r^2/A^2)\right],\\
&\rho_{IV} = \frac{3A^2(A^2+\kappa^2) + (7A^2 + 2\kappa^2)r^2 + 6r^4}{k\kappa^2(A^2+2r^2)^2}, \ \ p_{IV} = \frac{\kappa^2 - A^2 - 3r^2}{k\kappa^2(A^2+2r^2)}\ .
\end{aligned}
\tag{A.9}
$$

**Eigenvalues:**

$$
\begin{aligned}
\lambda_{IV_1} &= \frac{(\kappa^2-2r^2)A^2-2r^4}{\kappa^2(A^2+2r^2)^2},\\
\lambda_{IV_2} = \lambda_{IV_3} &= \frac{\kappa^2-r^2}{\kappa^2(A^2+2r^2)},\\
\lambda_{IV_4} &= \frac{\kappa^2+A^2+r^2}{\kappa^2(A^2+2r^2)},\\
\lambda_{IV_5} = \lambda_{IV_6} &= \frac{A^4+(\kappa^2+2r^2)A^2+2r^4}{\kappa^2(A^2+2r^2)^2}\ . \qquad \text{(A.10)}
\end{aligned}
$$

**Tolman V**
**Solution:**

$$
\begin{aligned}
\phi_V &= \ln\left[\frac{1+2K-K^2}{1-(1+2K-K^2)(r/\kappa)^N}\right], \quad \nu_V = \ln(B^2 r^{2K}),\\
\rho_V &= \frac{2K-K^2}{k(1+2K-K^2)r^2} + \frac{3+5K-2K^2}{k(1+K)\kappa^2}\left(\frac{r}{\kappa}\right)^M,\\
p_V &= \frac{K^2}{k(1+2K-K^2)r^2} - \frac{1+2K}{k\kappa^2}\left(\frac{r}{\kappa}\right)^M, \qquad \text{(A.11)}
\end{aligned}
$$

where $N = 2(1+2K-K^2)/(1+K)$ and $M = 2K(1-K)/(1+K)$.

**Eigenvalues:**

$$
\begin{aligned}
\lambda_{V_1} &= -\frac{K\left[2K(K^2-2K-1)(r/\kappa)^N + K^2-1\right]}{(K+1)(K^2-2K-1)r^2},\\
\lambda_{V_2} = \lambda_{V_3} &= -\frac{K\left[1+(K^2-2K-1)(r/\kappa)^N\right]}{(K^2-2K-1)r^2},\\
\lambda_{V_4} &= \frac{(K^2-2K-1)(r/\kappa)^N + K^2-2K}{(K^2-2K-1)r^2},\\
\lambda_{V_5} = \lambda_{V_6} &= \frac{N}{2r^2}\left(\frac{r}{\kappa}\right)^N. \qquad \text{(A.12)}
\end{aligned}
$$

**Tolman VI**
**Solution:**

$$
\begin{aligned}
\phi_{VI} &= \ln\left(2-K^2\right), \quad \nu_{VI} = 2\ln\left(Ar^{1-K} - Br^{1+K}\right),\\
\rho_{VI} &= \frac{1-K^2}{8\pi(2-K^2)r^2},\\
p_{VI} &= \frac{(1-K)^2A-(1+K)^2Br^{2K}}{8\pi(2-K^2)(A-Br^{2K})r^2} - \frac{1+2K}{8\pi\kappa^2}\left(\frac{r}{\kappa}\right)^M, \qquad \text{(A.13)}
\end{aligned}
$$

where $A, B$ and $K$ are arbitrary constants.

**Eigenvalues:**

$$
\begin{aligned}
\lambda_{VI_1} &= \frac{K\left[A(K-1)-B(K+1)r^{2K}\right]}{(K^2-2)(A-Br^{2K})r^2},\\
\lambda_{VI_2} &= \lambda_{VI_3} = \frac{-A(K-1)-B(K+1)r^{2K}}{(K^2-2)(A-Br^{2K})r^2},\\
\lambda_{VI_4} &= \frac{K^2-1}{(K^2-1)r^2},\\
\lambda_{VI_5} &= \lambda_{VI_6} = 0\,.
\end{aligned}
\tag{A.14}
$$

**Tolman VII**
**Solution:**

$$
\phi_{VII} = -\ln\left(1-r^2/\kappa^2+4r^4/A^4\right), \nu_{VII} = 2\ln\left[B\sin\left(\ln T^{1/2}\right)\right], \tag{A.15}
$$

$$
\begin{aligned}
\rho_{VII} &= \frac{3A^2-20\kappa^2r^2}{8\pi\kappa^2A^2}\\
p_{VII} &= \frac{B\cot\left(\ln T^{1/2}\right)-A^2(A^4-4\kappa^2r^2)\left(1-r^2/\kappa^2+4r^4/A^4\right)^{1/2}}{8\pi\kappa^2A^6\left(1-r^2/\kappa^2+4r^4/A^4\right)^{1/2}}
\end{aligned}
\tag{A.16}
$$

where $B\equiv\left[4\kappa^2A^4-4r^2(A^4-4\kappa^2r^2)\right]$, $T$ is given by

$$
c\,T\equiv\left(1-r^2/\kappa^2+4r^4/A^4\right)^{1/2}+2r^2/A^2-A^2/(4\kappa^2)
$$

and $c$, $\kappa$ and $A$ are arbitrary constants.

**Eigenvalues:**

$$
\begin{aligned}
4\lambda_{VII_1} &= (1-r^2/\kappa^2+4r^4/A^4)\\
&\quad\times\left[\frac{\cot\left(\ln T^{1/2}\right)}{T}(FT_r+2T_{rr})-\frac{T_r^2}{T^2}\left(2\cot\left(\ln T^{1/2}\right)+1\right)\right],\\
F &\equiv \frac{-2r/\kappa^2+16r^3/A^4}{1-r^2/\kappa^2+4r^4/A^4},\\
\lambda_{VII_2} &= \lambda_{VII_3} = -(1-r^2/\kappa^2+4r^4/A^4)\frac{T_r\cot\left(\ln T^{1/2}\right)}{2rT},\\
\lambda_{VII_4} &= 1/\kappa^2-4r^2/A^4,\\
\lambda_{VII_5} &= \lambda_{VII_6} = 1/\kappa^2-8r^2/A^4\,.
\end{aligned}
\tag{A.17}
$$

Here $T_r$ indicates derivative with respect to $r$.

**Tolman VIII**
**Solution:**

$$\begin{aligned}
\phi_{VIII} &= -\ln\left[\frac{2}{(a-b)(a+2b-1)} - \left(\frac{2M}{r}\right)^{a+2b-1} - \left(\frac{r}{\kappa}\right)^{a-b}\right],\\
\nu_{VIII} &= \ln\left(B^2 r^{2b}\right) - \phi_{VIII},\\
8\pi r^2 \rho_{VIII} &= 1 - \frac{2}{(a-b)(a+2b-1)} - (a+2b-2)\left(\frac{2M}{r}\right)^{a+2b-1}\\
&\quad + (a-b+1)\left(\frac{r}{\kappa}\right)^{a-b},
\end{aligned} \tag{A.18}$$

$$\begin{aligned}
p_{VIII} = &\Big[(a-2)(a+2b-1)(a-b)\left(\frac{2M}{r}\right)^{a+2b-1}\\
&- (a+b+1)(a+2b-1)(a-b)\left(\frac{r}{\kappa}\right)^{a-b}\\
&- a^2 + (1-b)a + 2b^2 + 3b + 2\Big/\Big[8\pi(a-b)(a+2b-1)r^2\Big],
\end{aligned}$$

where $a$, $b$ and $M$ are constants and $b \equiv (a^2 - a - 2)/(3 - a)$

**Eigenvalues:**

$$\begin{aligned}
\lambda_{VIII_1}/D + 4b^2 + 4b &= (a+2b-1)(a^2-b^2)(a-1)\\
&\quad \times \left[\left(\frac{2M}{r}\right)^{a+2b-1} + \left(\frac{r}{\kappa}\right)^{a-b}\right],\\
\lambda_{VIII_2}/D + 4b &= (a+2b-1)(a-b)\\
&\quad \times \left[(1-a)\left(\frac{2M}{r}\right)^{a+2b-1} + (a+b)\left(\frac{r}{\kappa}\right)^{a-b}\right],\\
\lambda_{VIII_3} &= \lambda_{VIII_2},\\
\lambda_{VIII_4}/2D + 2 &= (a+2b-1)(a-b)\\
&\quad \times \left[\left(\frac{2M}{r}\right)^{a+2b-1}\left(\frac{r}{\kappa}\right)^{a-b} + 1\right],\\
\lambda_{VIII_5/D} &= \lambda_{VIII_6} = (b-a)\left(\frac{2M}{r}\right)^{a+2b-1}\\
&\quad + (a+2b-1)\left(\frac{r}{\kappa}\right)^{a-b},
\end{aligned} \tag{A.19}$$

where

$$D \equiv \frac{1}{2(a-b)(a+2b-1)r^2}\ .$$

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