Traveling Salesman Problem

Approximation Algorithms

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Quotes of the day

“Problem solving is hunting. It is savage pleasure and we are born to it.”

Thomas Harris

“Algorithms must be seen to be believed.”

Donald Knuth

Outline

- Metric TSP
  - Double-Tree Algorithm
  - Christofides Algorithm
- Euclidean TSP
  - Arora’s Algorithm
- Real-world application

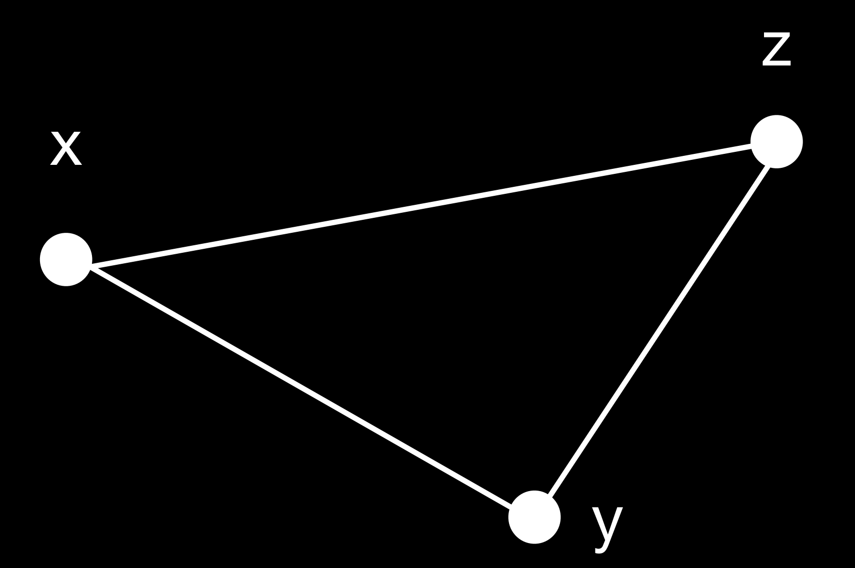
Goals of this talk

- Understand presented approximation algorithms
- Implement presented algorithms to solve real world problems

Metric TSP

$d(\cdot,\cdot) : X \times X \to \mathbb{R}, d(x,y) \mapsto \|x-y\|$

- $d(x,y) \geq 0$, for $d(x,y)=0 \Rightarrow x=y$
- $d(x,y) = d(y,x)$
- $d(x,z) \leq d(x,y) + d(y,z)$



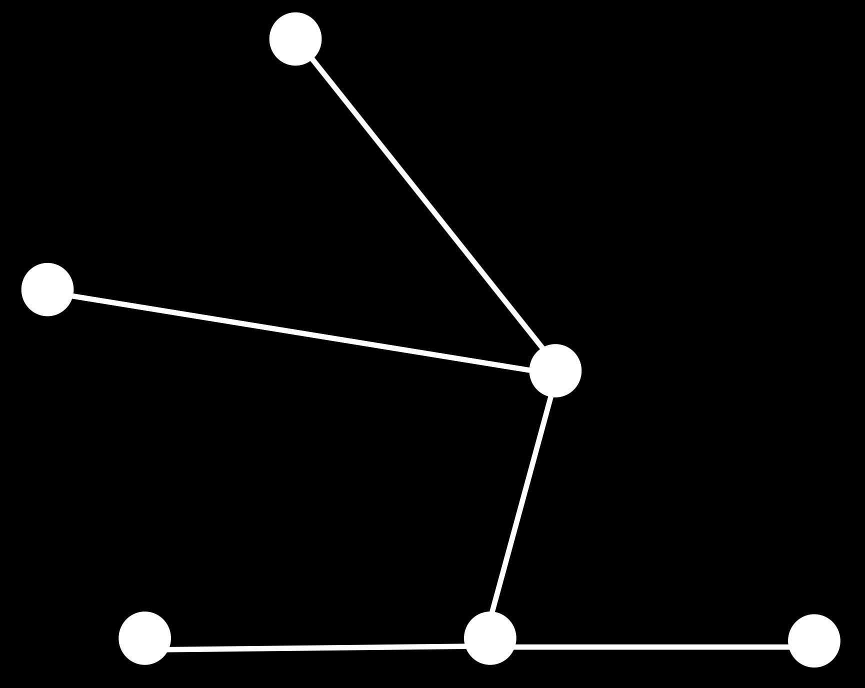
<u>Instance:</u> A complete graph G with weights $c : E(G) \to \mathbb{R}_+$ such that metric properties hold

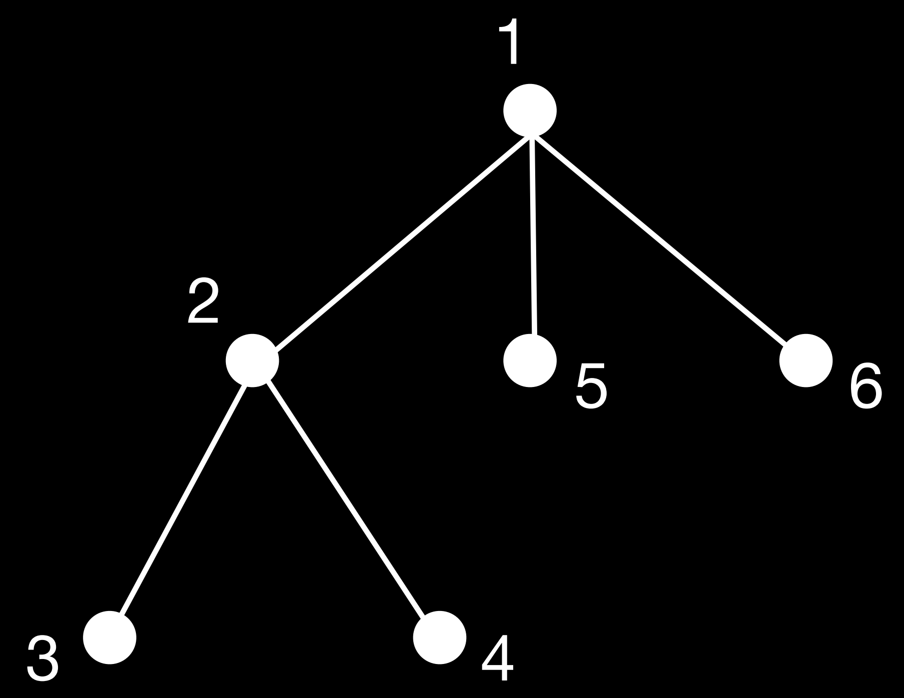
<u>Goal:</u> Find Hamiltonian cycle in G of minimum weight

Theorem 1. The Metric TSP is NP-hard

Approximation algorithms

Double-Tree Algorithm



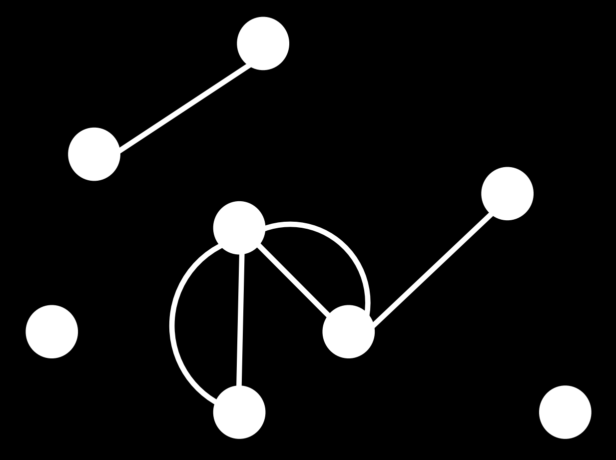


DFS traversal: 1, 2, 3, 2, 4, 2, 1, 5, 1, 6, 1

Instance: An instance (G, c) of the Metric TSP

Goal: A cycle

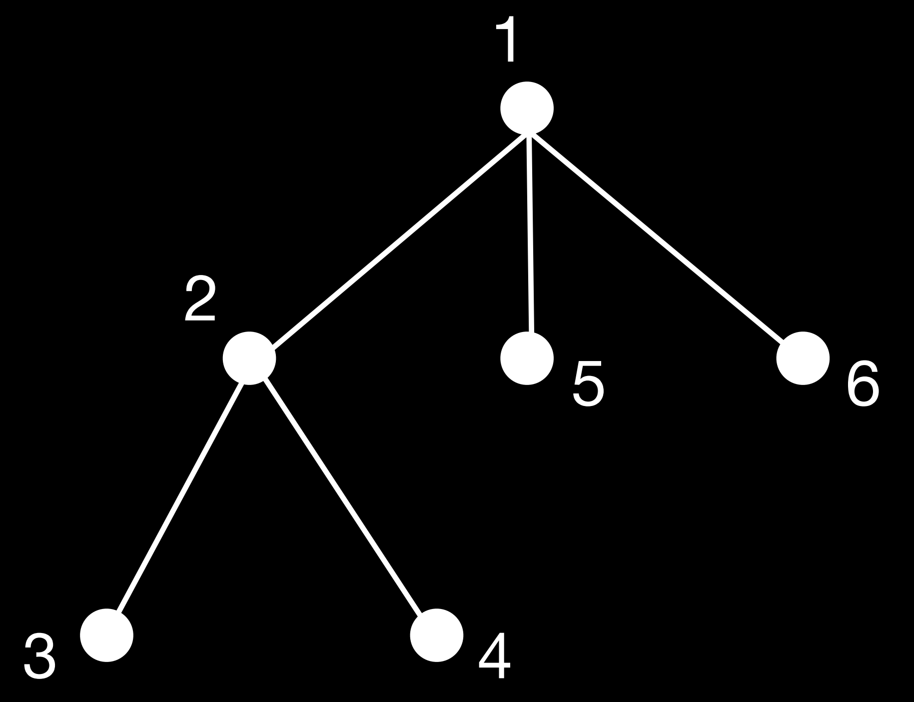
Theorem 2. The Double-Tree Algorithm is a 2-factor approximation algorithm for the Metric TSP



S: (Multi)-set of edges
c(S): Sum of cost of all edges in S
$c(H_G^*)$: cost of optimal Hamiltonian cycle

Approximation algorithms

Double-Tree Algorithm



DFS traversal: 1, 2, 3, 2, 4, 2, 1, 5, 1, 6, 1

DFS traversal: 1, 2, 3, ~~2,~~ 4, ~~2,~~ ~~1,~~ 5, ~~1,~~ 6, ~~1~~

$$T \longrightarrow G \longrightarrow G'$$

$2 \cdot c(T) = c(G)$ $\quad$ $c(G') \leq c(G)$

(1) $\Rightarrow c(G') \leq 2 \cdot c(T)$

<u>Instance:</u> An instance (G, c) of the Metric TSP

Goal: A cycle

Theorem 2. The Double-Tree Algorithm is a 2-factor approximation algorithm for the Metric TSP

$1 \rightarrow 2 \rightarrow 3 \rightarrow 2 \rightarrow 4 \rightarrow \ldots$

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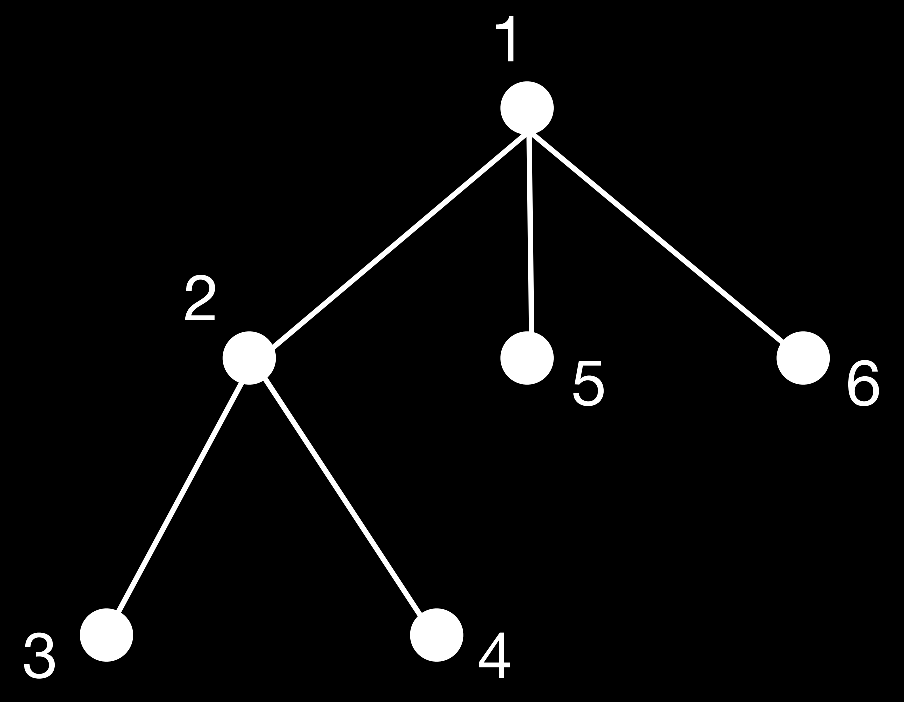
- $d(x,z) \leq d(x,y) + d(y,z)$

Approximation algorithms

Double-Tree Algorithm



DFS traversal: 1, 2, 3, 2, 4, 2, 1, 5, 1, 6, 1

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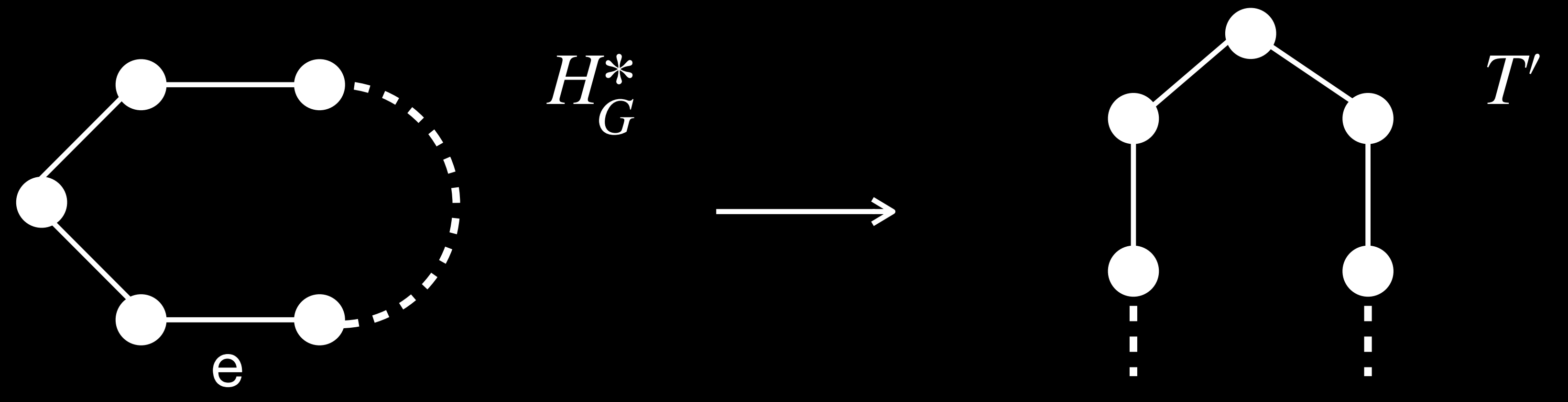
Theorem 2. The Double-Tree Algorithm is a 2-factor approximation algorithm for the Metric TSP



(2) $c(H_G^*) \geq c(H_G^* - e) \geq c(T)$

$\Longrightarrow$ From (1) and (2) $c(G') \leq 2 \cdot c(H_G^*)$

Approximation algorithms

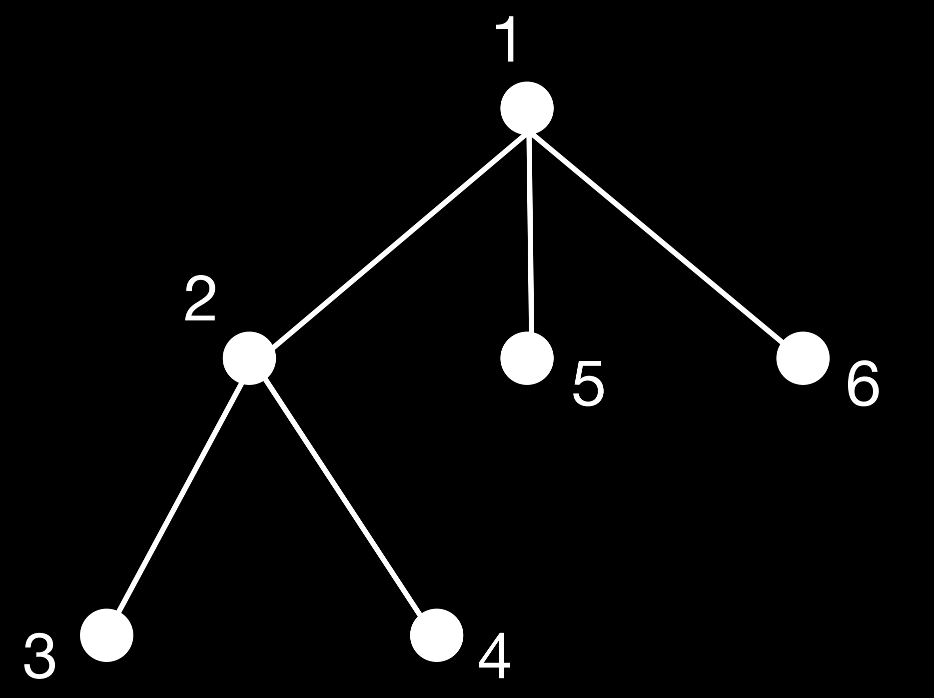
Double-Tree Algorithm - Recap

<u>Instance:</u> An instance (G, c) of the Metric TSP

Goal: A cycle

Theorem 2. The Double-Tree Algorithm is a 2-factor approximation algorithm for the Metric TSP

1.



DFS traversal: 1, 2, 3, 2, 4, 2, 1, 5, 1, 6, 1

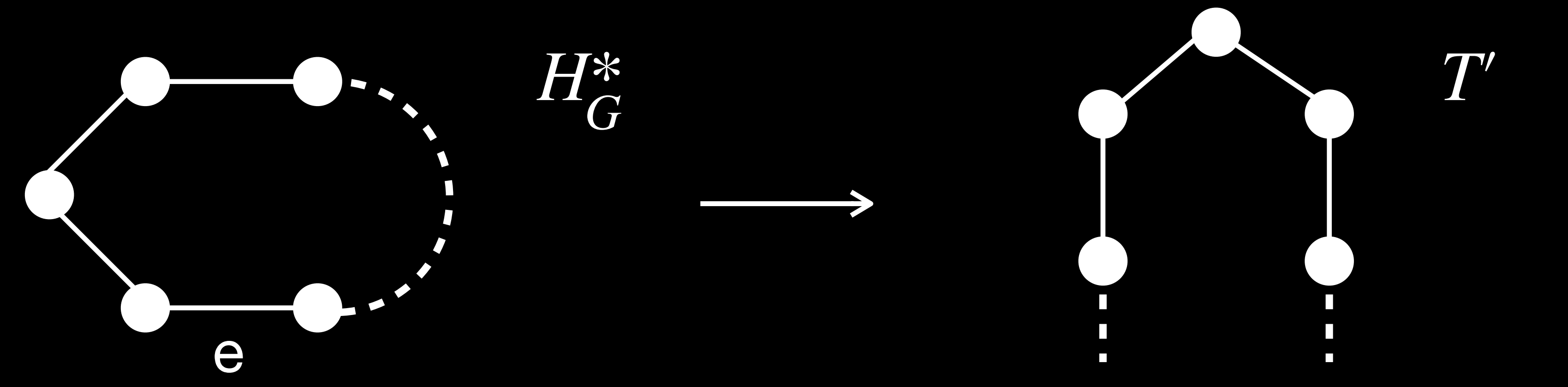
2. DFS traversal: 1, 2, 3, ~~2,~~ 4, ~~2,~~ ~~1,~~ 5, ~~1,~~ 6, ~~1~~

3. $$T \longrightarrow G \longrightarrow G'$$

$2 \cdot c(T) = c(G)$ $\quad c(G') \leq c(G)$

(1) $\Rightarrow c(G') \leq 2 \cdot c(T)$

4.

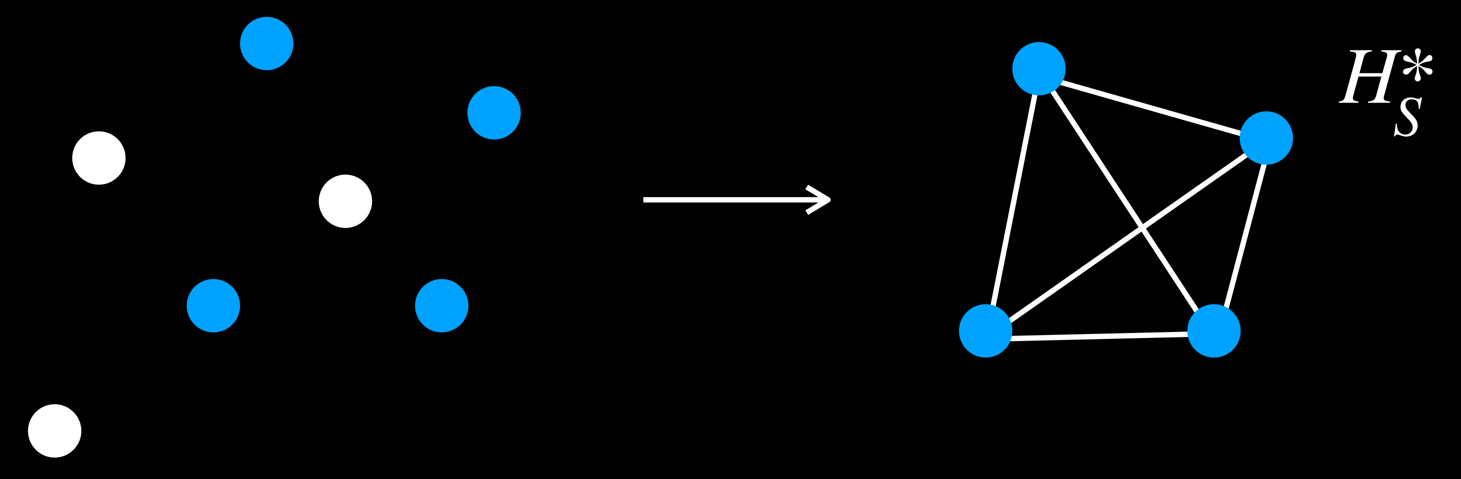


(2) $c(H_G^*) \geq c(H_G^* - e) \geq c(T)$

$\Longrightarrow$ From (1) and (2) $c(G') \leq 2 \cdot c(H_G^*)$

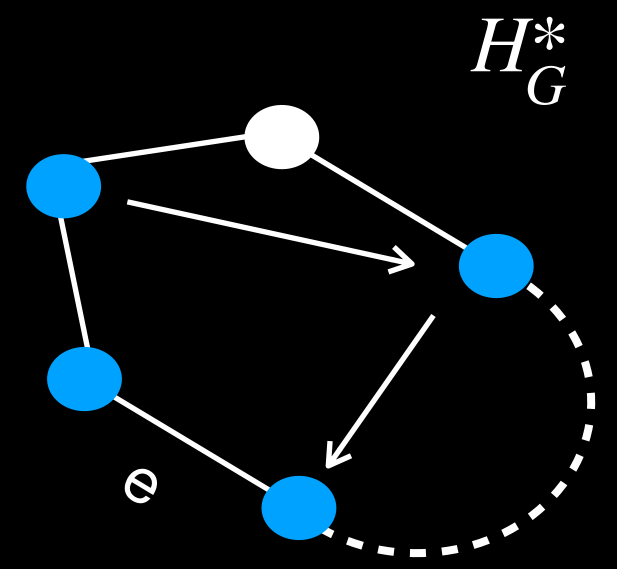
Some useful concepts and lemmas

$S \subseteq V$

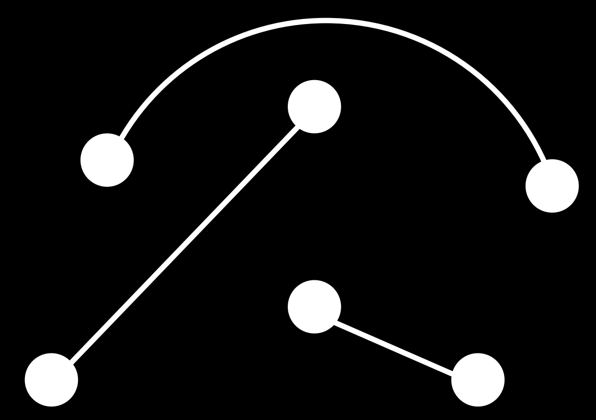


Claim: $c(H_S^*) \leq c(H_G^*)$

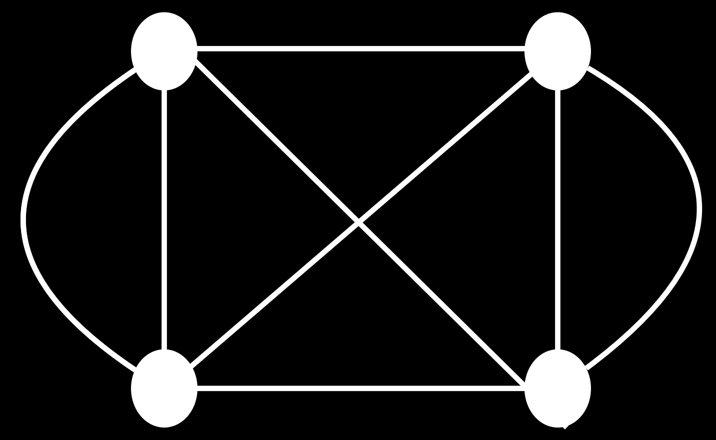
Say $c(H_S^*) > c(H_G^*)$



Perfect Matchings

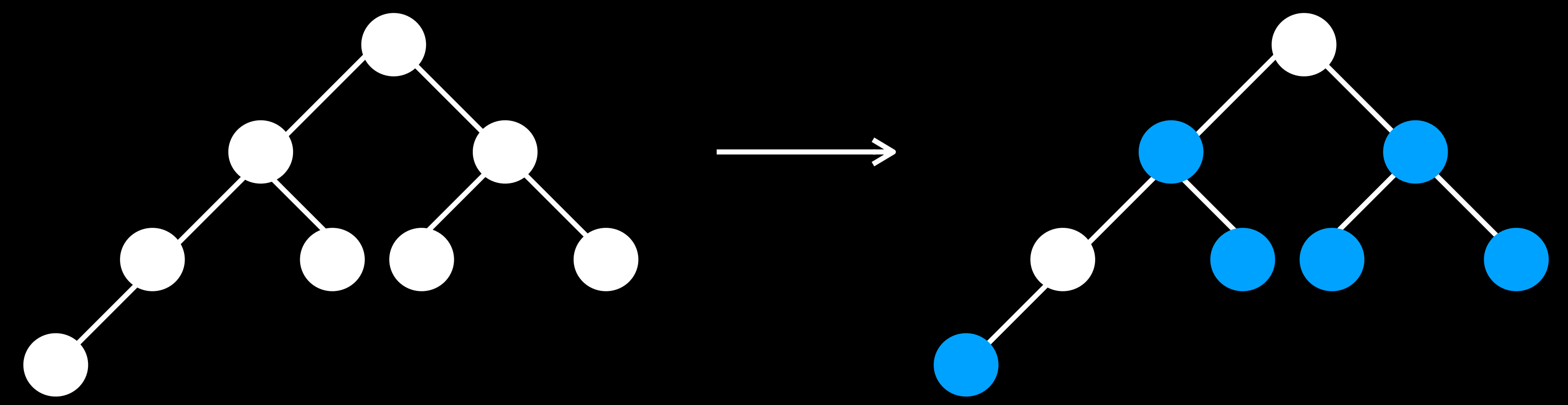


Eulerian Graph

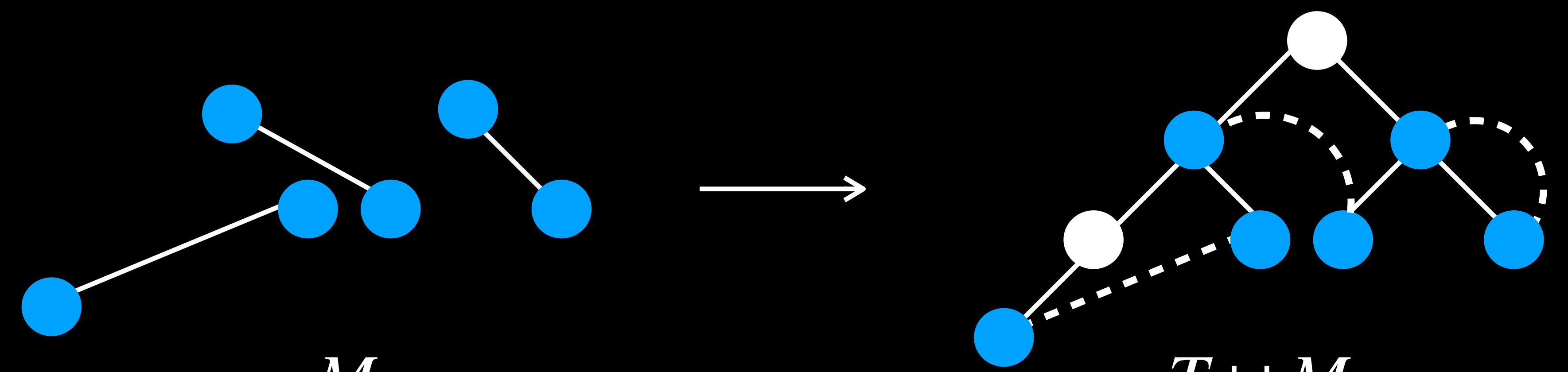


Approximation algorithms

Christofides' Algorithm



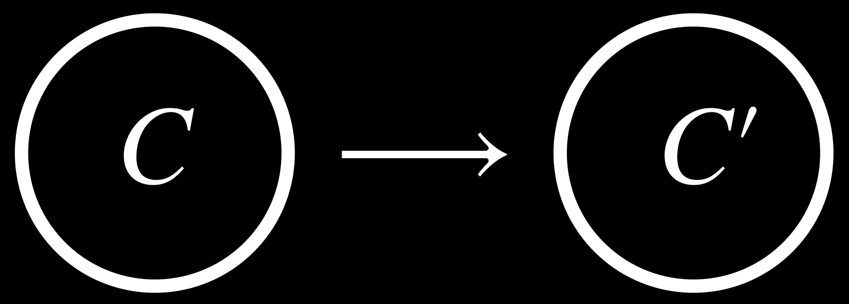
T



M $\quad$ $T \cup M$

$c(C) = c(T) + c(M)$

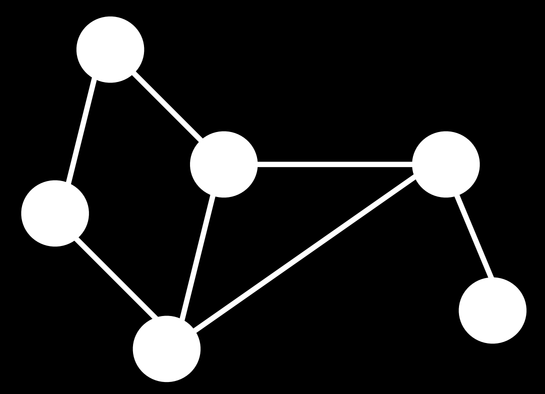
Duplication argument:



<u>Instance:</u> An instance (G, c) of the Metric TSP

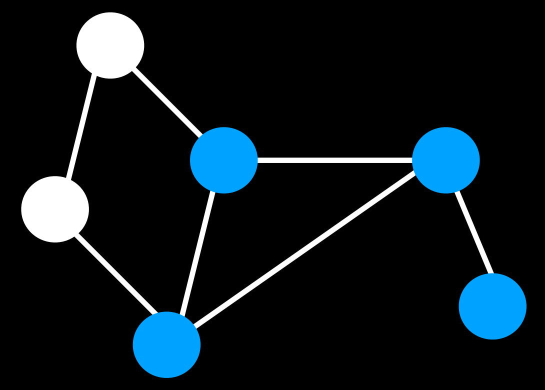
Goal: A cycle

Theorem 3. Christofides' Algorithm is a 3/2-factor approximation algorithm for the Metric TSP



$$\sum_{v_i \in V} d_i = 2 \cdot |E|$$

$\sum_{v_i \in V}^{r} d_i$ is even, $\forall i = 1,\ldots,r : v_i$ has even degree



$$\sum_{v_i \in V} d_i = \sum_{v_i \in V:\ v_i \text{ even}} d_i + \sum_{v_i \in V:\ v_i \text{ odd}} d_i$$

$\Longrightarrow$ Number of vertices with odd degree (number of terms in $\sum_{v_i \in V:\ v_i \text{ odd}} d_i$) must be even

Approximation algorithms

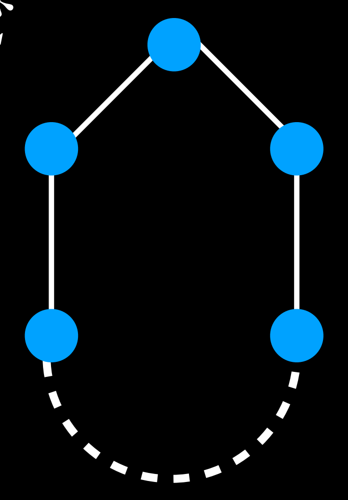
Christofides' Algorithm

$c(C) = c(T) + c(M)$

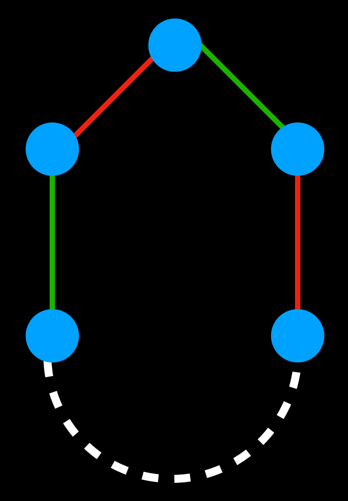
Duplication argument: $C \longrightarrow C'$

From Double-Tree algorithm: $c(T) \leq c(H_G^*)$

H_S^*



From previous lemma: $c(H_S^*) \leq c(H_G^*)$



M_1 $\quad c(M) \leq c(M_1)$

M_2 $\quad c(M) \leq c(M_2)$

<u>Instance:</u> An instance (G, c) of the Metric TSP

Goal: A cycle

Theorem 3. Christofides' Algorithm is a 3/2-factor approximation algorithm for the Metric TSP

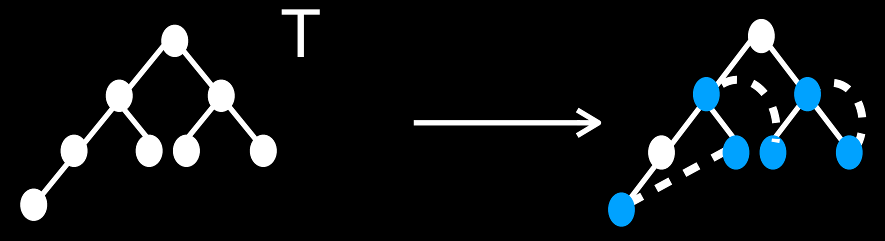
$$\begin{aligned} \Rightarrow c(M) &\leq \frac{1}{2}(c(M_1) + c(M_2)) \\ &= \frac{1}{2}c(H_S^*) \\ &\leq \frac{1}{2}c(H_G^*) \end{aligned}$$

$$\begin{aligned} c(C) &= c(T) + c(M) \\ &\leq c(H_G^*) + \frac{1}{2}c(H_G^*) \end{aligned}$$

$$\Longrightarrow c(C') \leq \frac{3}{2}c(H_G^*)$$

Approximation algorithms

Christofides' Algorithm - Recap

1. 

2. $C \longrightarrow C'$ Duplication argument

3. Claim: $c(C) = c(T) + c(M)$

4. M bounded by H_S^* $\to M_1$, $\to M_2$

<u>Instance:</u> An instance (G, c) of the Metric TSP

Goal: A cycle

Theorem 3. Christofides' Algorithm is a 3/2-factor approximation algorithm for the Metric TSP

5.

$$\begin{aligned} c(M) &\leq \frac{1}{2}(c(M_1) + c(M_2)) \\ &= \frac{1}{2}c(H_S^*) \\ &\leq c(H_G^*) \end{aligned}$$

$$\begin{aligned} c(C) &= c(T) + c(M) \\ &\leq c(H_G^*) + \frac{1}{2}c(H_G^*) \end{aligned}$$

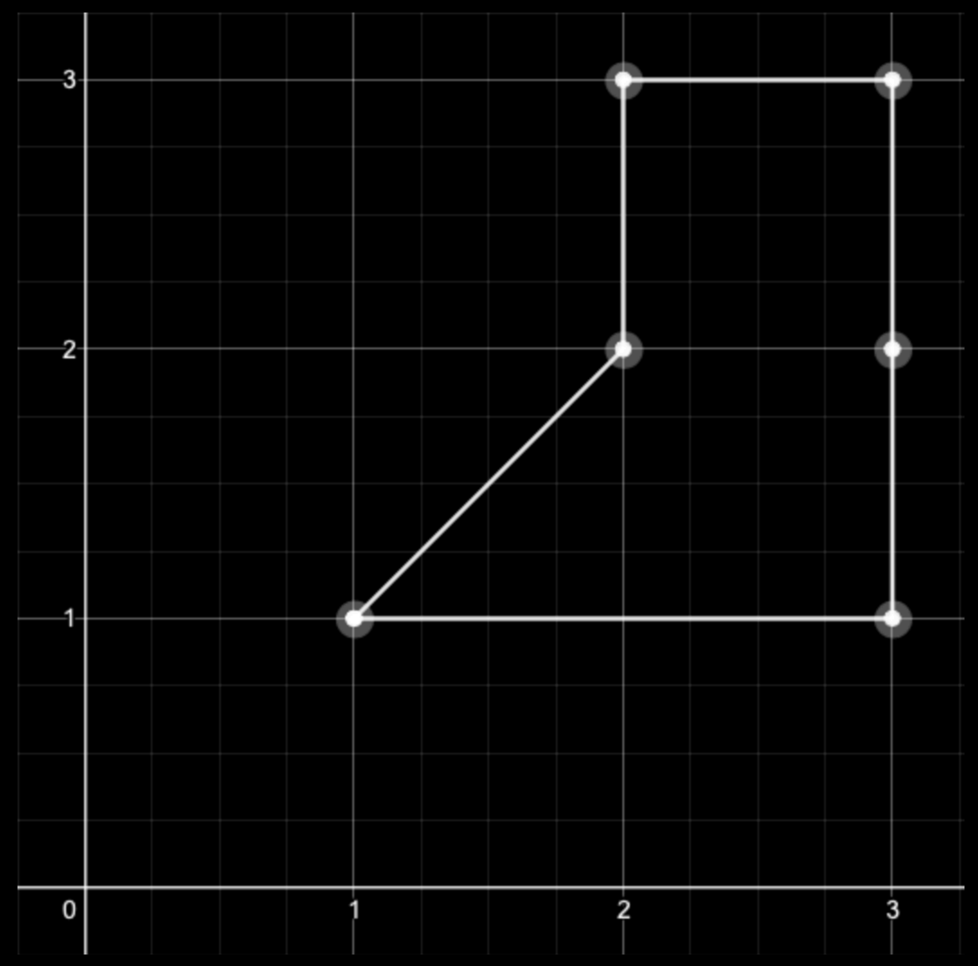
$$\implies c(C') \leq \frac{3}{2}c(H_G^*)$$

Up to this point

Euclidean TSP

$d(\cdot,\cdot): X \times X \to \mathbb{R}, d(x,y) \mapsto \|x-y\|$

- $d(x,y) \geq 0$, for $d(x,y) = 0 \Rightarrow x = y$
- $d(x,y) = d(y,x)$
- $d(x,z) \leq d(x,y) + d(y,z)$



<u>Instance:</u> Finite set $V \subseteq \mathbb{R}^2, |V| > 3$ with euclidean distances, i.e., $d(x,y) = \|x-y\|$

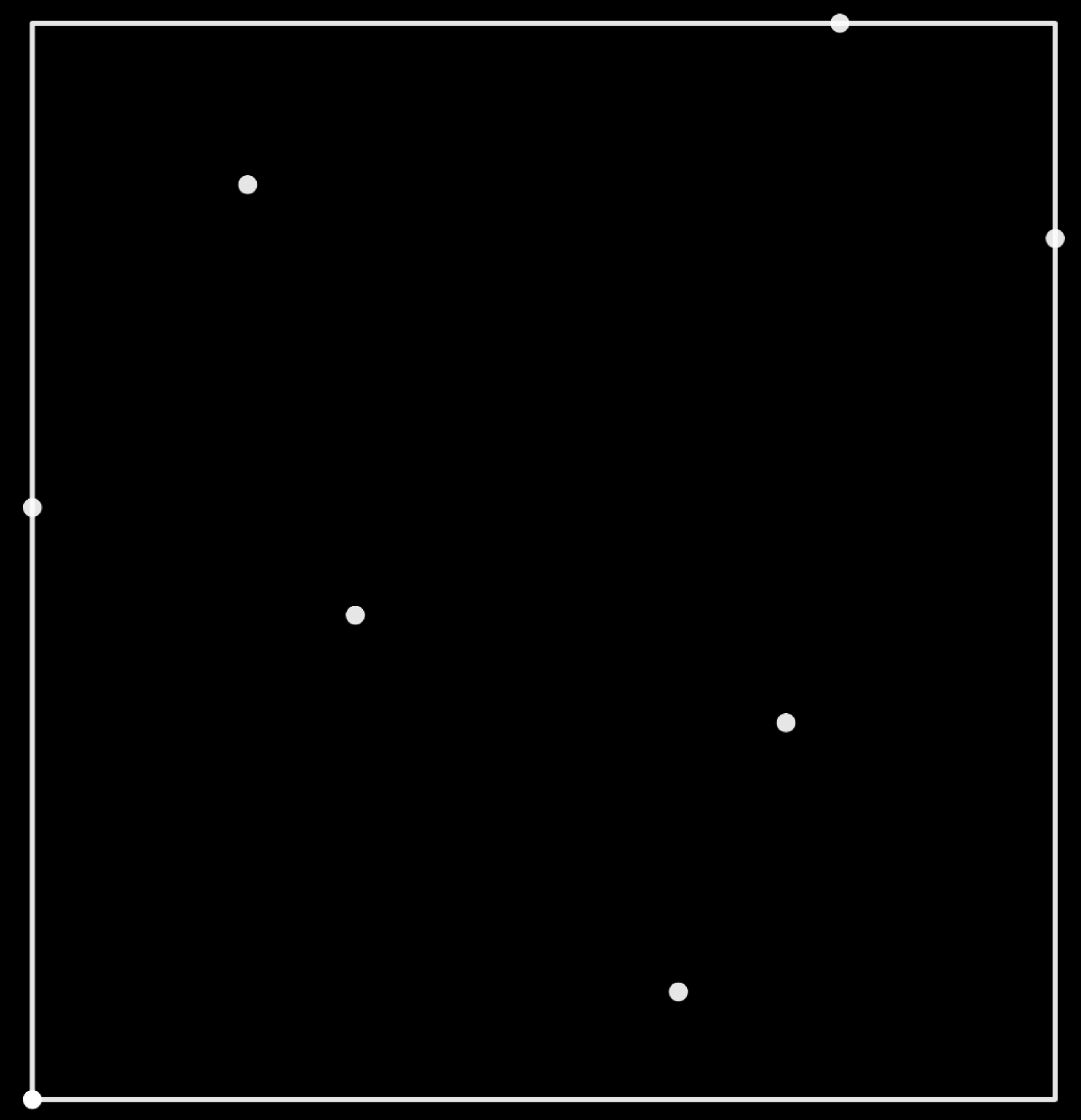
<u>Goal:</u> Find Hamiltonian cycle in G of minimum weight

Theorem 4. The Euclidean TSP is NP-hard

Some useful definitions and lemmas

Definition. (ϵ-nice instances): An instance of euclidean TSP is called ϵ-nice if the following conditions hold:

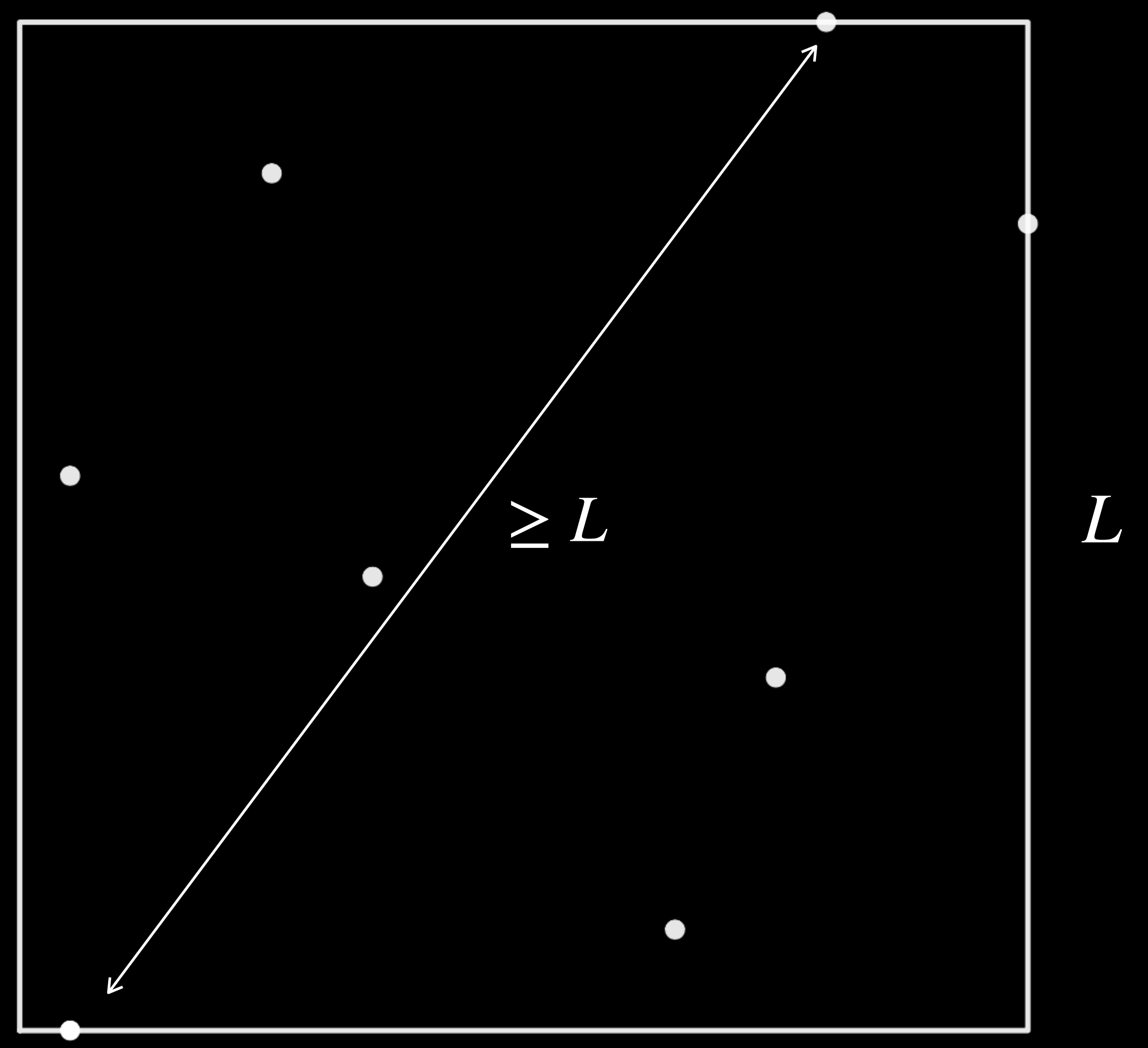
- Every point has integral coordinates in the interval $\left[0,\ O\left(\frac{n}{\epsilon}\right)\right]^2$
- Any two different points have distances at least 4.



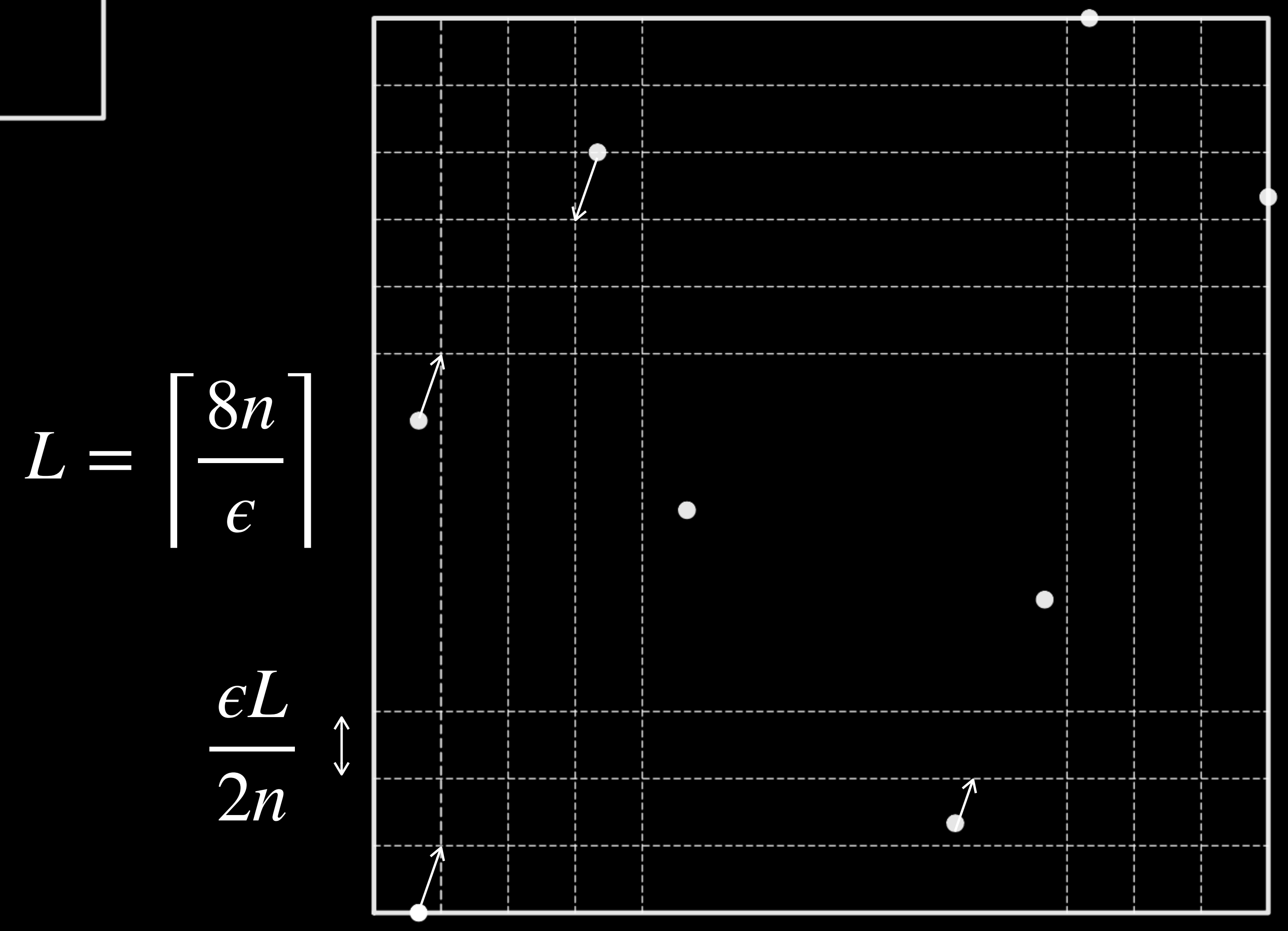
$$L \longrightarrow L = \left\lceil \frac{8n}{\epsilon} \right\rceil$$

Some useful definitions and lemmas

Arora's Algorithm



$OPT_I \geq 2L \geq L$



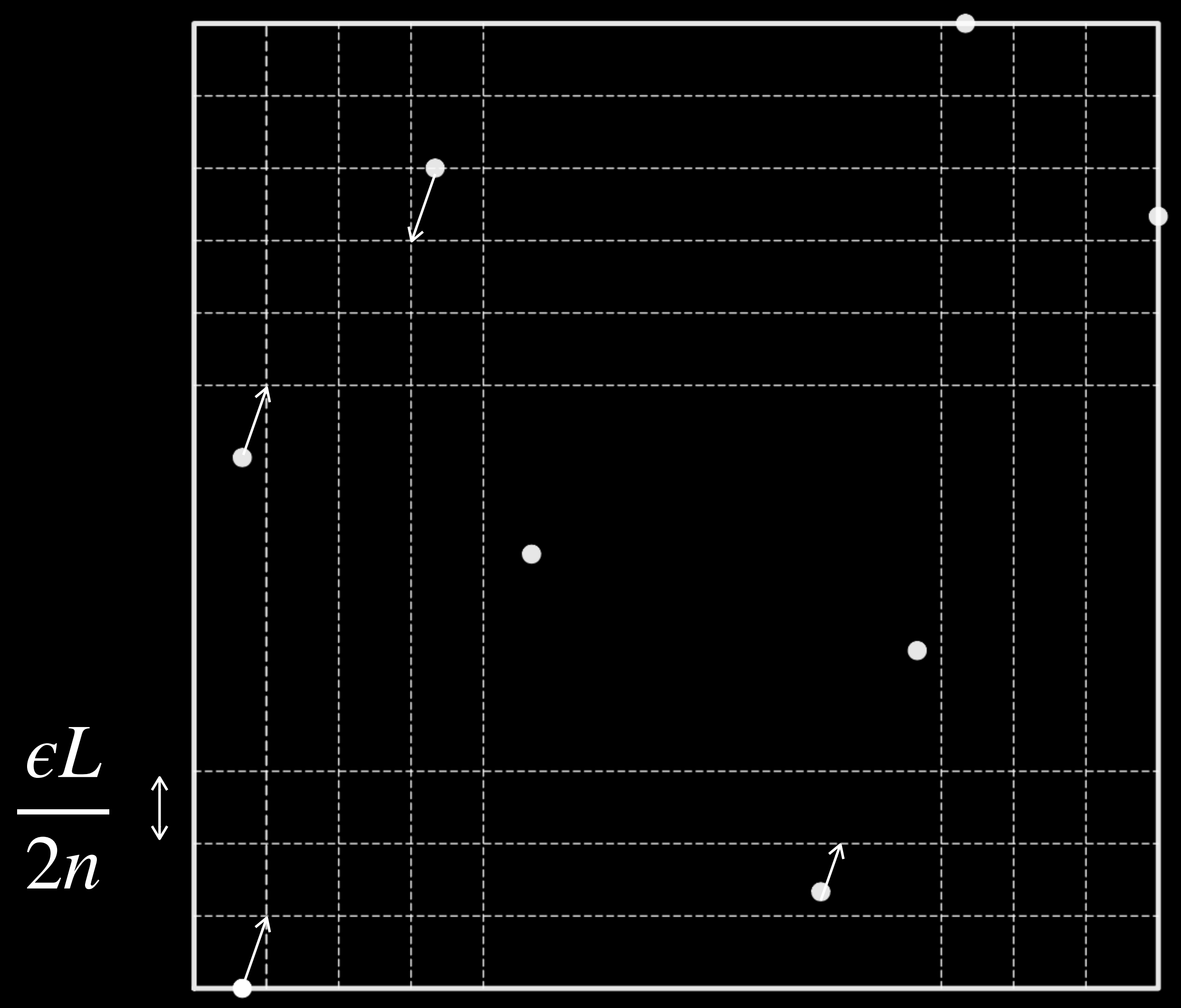
Lemma. Let I be an arbitrary instance to euclidean TSP. Let OPT_I denote the length of the optimum tour in I. We can transform I into an ϵ-nice instance I' such that $OPT_{I'} \leq (1+\epsilon)OPT_I$

- Every point has integral coordinates in the respective range since $L = \lceil 8n/\epsilon \rceil \in O(n/\epsilon)$
- Grid spacing is $\frac{\epsilon L}{2n} \geq \frac{\epsilon}{2n}\frac{8n}{\epsilon} \Rightarrow d(x,y) \geq 4$

$\Longrightarrow I'$ is ϵ-nice

Some useful definitions and lemmas

Arora's Algorithm



$$L = \left\lceil \frac{8n}{\epsilon} \right\rceil$$

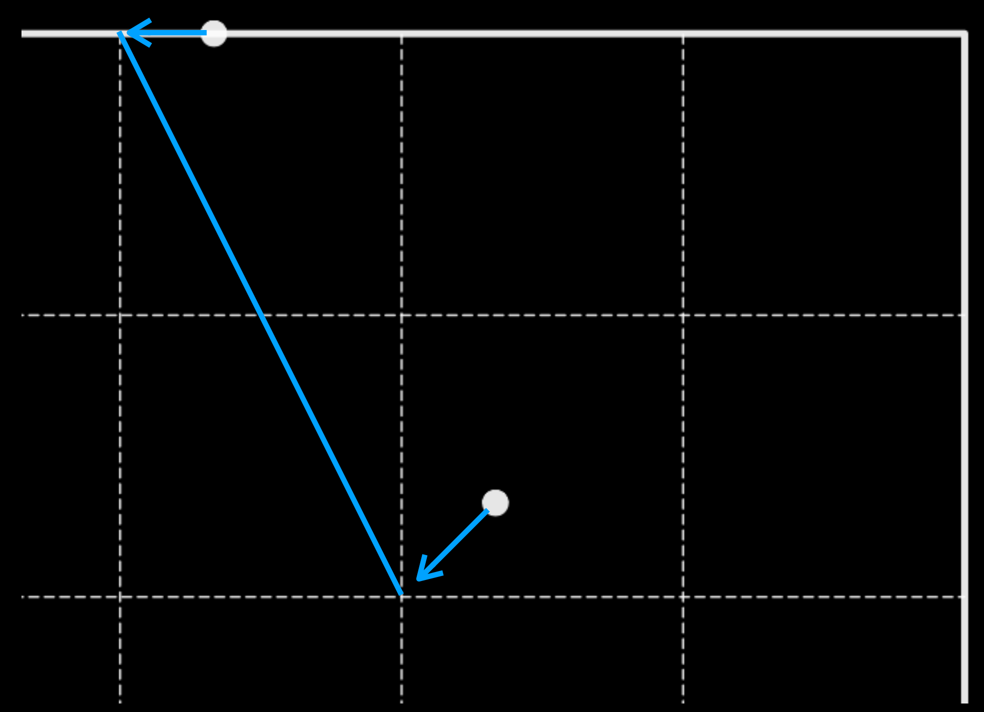
Mapping points from I to I': distance $\leq \frac{\epsilon L}{2n}$

$\Rightarrow$ Mapping edges from I to I': length edge $\leq \frac{\epsilon L}{n}$

Lemma. Let I be an arbitrary instance to euclidean TSP. Let OPT_I denote the length of the optimum tour in I. We can transform I into an ϵ-nice instance I' such that $OPT_{I'} \leq (1+\epsilon)OPT_I$

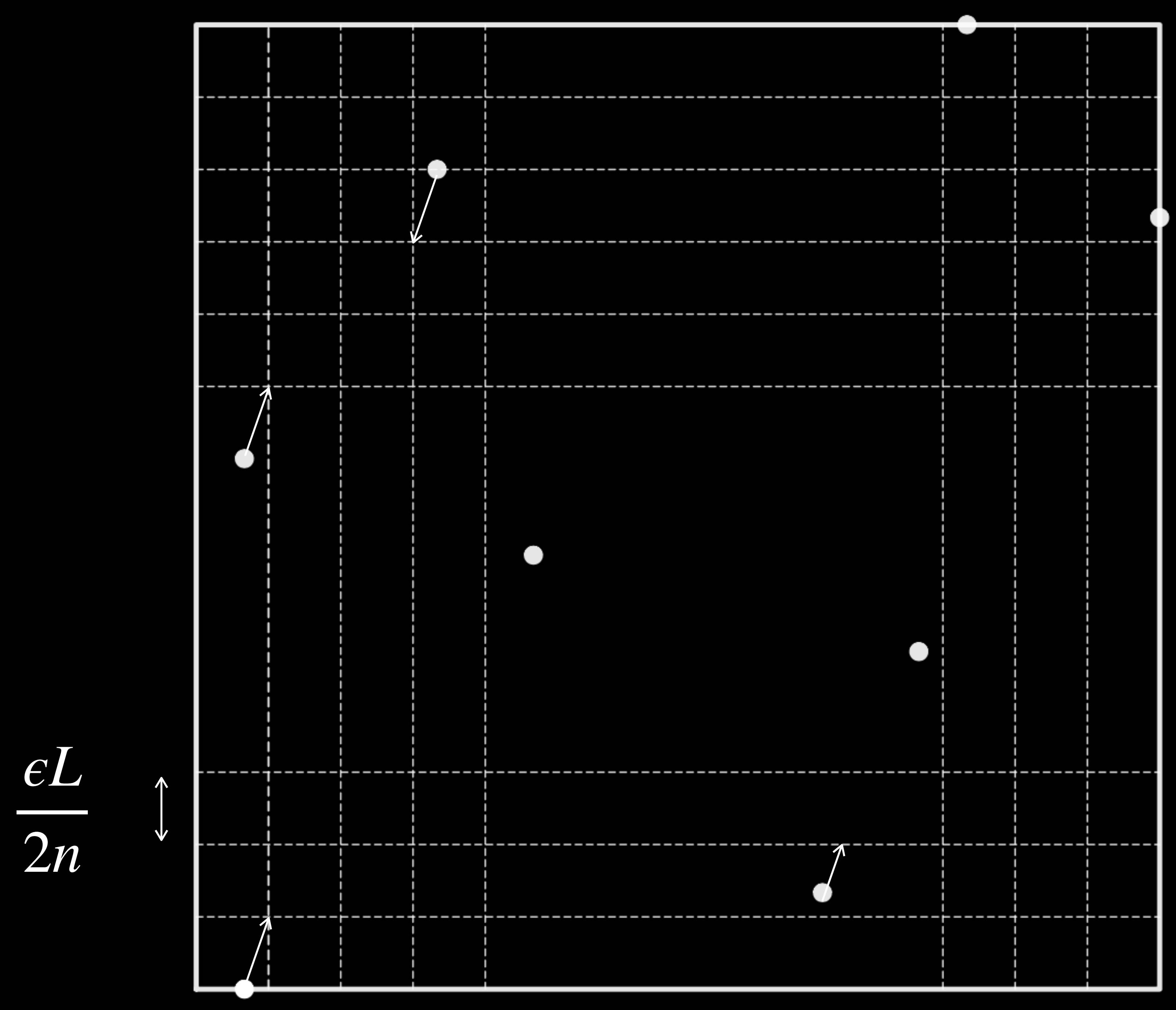
$$c^2 = \left(\frac{\epsilon L}{2n}\right)^2 + 2\left(\frac{\epsilon L}{2n}\right)^2$$

$$\Leftrightarrow c = \sqrt{3}\left(\frac{\epsilon L}{2n}\right)^2 \leq \frac{\epsilon L}{n}$$



Some useful definitions and lemmas

Arora's Algorithm



$$L = \left\lceil \frac{8n}{\epsilon} \right\rceil$$

Lemma. Let I be an arbitrary instance to euclidean TSP. Let OPT_I denote the length of the optimum tour in I. We can transform I into an ϵ-nice instance I' such that $OPT_{I'} \leq (1+\epsilon)OPT_I$

Mapping points from I to I': distance $\leq \frac{\epsilon L}{2n}$

$\Rightarrow$ Mapping edges from I to I': length edge $\leq \frac{\epsilon L}{n}$

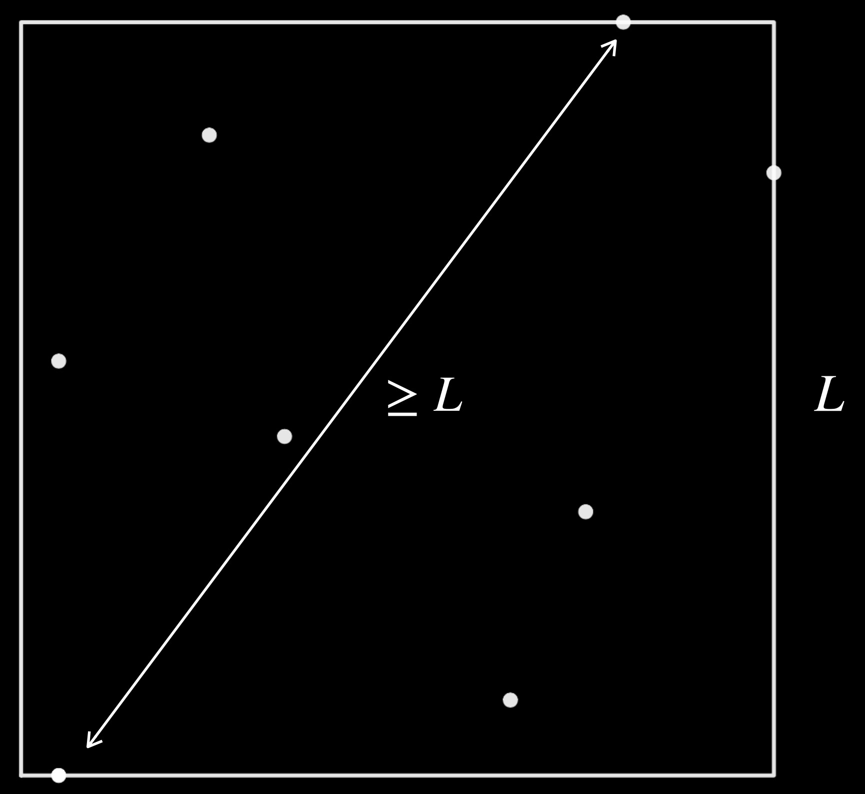
$\Rightarrow OPT_{I'} \leq OPT_I + \epsilon L$

$\implies OPT_{I'} \leq OPT_I + \epsilon L \leq (1+\epsilon)\ OPT_I$

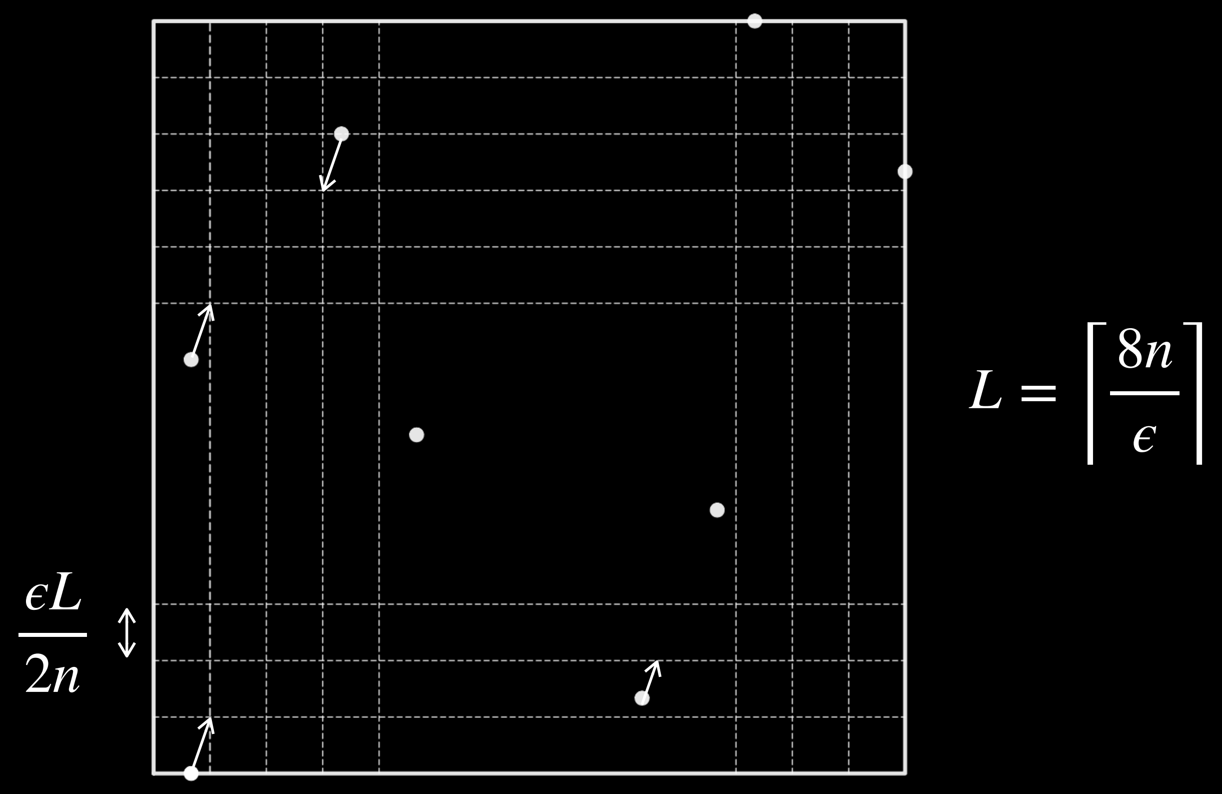
Euclidean TSP

Arora's Algorithm - Recap

1.



2.



Mapping points from I to I': distance $\leq \dfrac{\epsilon L}{2n}$

$\Rightarrow$ Mapping edges from I to I': length edge $\leq \dfrac{\epsilon L}{n}$

$\Rightarrow OPT_{I'} \leq OPT_I + \epsilon L$

$\Longrightarrow OPT_{I'} \leq OPT_I + \epsilon L \leq (1+\epsilon)\ OPT_I$

Real world application

Thank you for your attention!

References

- Demaine, E., Devadas, S., & Lynch, N. (2015). Approximation Algorithms: Traveling Salesman Problem. Retrieved from https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-046j-design-and-analysis-of-algorithms-spring-2015/recitation-notes/MIT6_046JS15_Recitation9.pdf
- Svensson Ola (2013). Approxiimation algorithms: Euclidean TSP. Retrieved from https://theory.epfl.ch/osven/courses/Approx13/Notes/lecture4-5.pdf
- Korte, B. H., & Vygen, J. (2010). Combinatorial optimization: Theory and algorithms. Berlin: Springer.Korte, B. H., & Vygen, J. (2010). Combinatorial optimization: Theory and algorithms. Berlin: Springer.

Github: https://github.com/juan190199/TravelingSalesmanProblem