COMP111: Artificial Intelligence

Section 7(b). KR&R: Propositional Logic and Review

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Propositional Logic

For many purposes, rules are not expressive enough. For example,

- one cannot express that something is not the case:

  not FrenchFootballClub(LiverpoolFC)

- one cannot connect sentences using 'or':

  Today, I will do the AI exercise or I will play football

Propositions

A proposition is a statement that can be true or false, but not both at the same time!

- Logic is easy;
- I eat toast;
- $2 + 3 = 5$;
- $2 \cdot 2 = 5$.

What about these?

- $4 + 5$;
- Which city is the capital of the UK?
- Logic is not easy;
- Logic is easy or I eat toast;

Reasoning about propositions

> *The meeting takes place if all members have been informed in advance, and it is quorate. It is quorate provided that there are at least 15 people present. Members will have been informed in advance if there is not a postal strike.*

Therefore,

> *If the meeting does not take place, we conclude that there were fewer than 15 members present, or there was a postal strike.*

Understanding compound true/false propositions

The meeting takes place if all members have been informed in advance, and it is quorate. It is quorate provided that there are at least 15 people present. Members will have been informed in advance if there is not a postal strike. Therefore, if the meeting does not take place, we conclude that there were fewer than 15 members present, or there was a postal strike.

Compound statements are built from atomic propositions — the 'simplest' statements that it is possible to make about the world

m: "the meeting takes place" a: "all members have been informed"
p: "there is a postal strike" q: "the meeting is quorate"
f: "there are at least 15 members present"

From

If a and q then m. If f then q. If not p then a.

conclude

If not m then not f or p.

Connectives

- Propositions may be combined with other propositions to form compound propositions. These in turn may be combined into further propositions.
- The connectives that may be used are

$\neg$	not	negation	$(\sim)$
$\wedge$	and	conjunction	(& or .)
$\vee$	or	disjunction	$(\mid$ or $+)$
$\Leftrightarrow$	if and only if	equivalence	$(\leftrightarrow)$
$\Rightarrow$	if ... then	implication	$(\rightarrow)$

- Some books use different notations. Some of these are given in parentheses.

Propositional Formulas

The set of propositional formulas is defined as follows:

- Every atomic proposition is a propositional formula. We denote atomic propositions by p, q, a, p_1, p_2, and so on.
- If P and Q are propositional formulas, then
  - $(P \wedge Q)$
  - $(P \vee Q)$
  - $(P \Rightarrow Q)$
  - $(P \Leftrightarrow Q)$

  are propositional formulas.
- If P is a propositional formula, then $\neg P$ is a propositional formula.

Example

The following are propositional formulas:

- p
- $\neg\neg p$
- $(p \vee q)$
- $(((p \Rightarrow q) \wedge \neg q) \Rightarrow \neg p)$

The following are not propositional formulas:

- $p \wedge q$
- (p)
- $(p \wedge q)\neg q$

Giving meaning to propositions: Truth values

An interpretation I assigns to every atomic proposition p a truth value

$$I(p) \in \{0, 1\}.$$

- If $I(p) = 1$, then p is called true under the interpretation I.
- If $I(p) = 0$, then p is called false under the interpretation I.

Given an assignment I we can compute the truth value of compound formulas step by step using truth tables.

Negation

The negation $\neg P$ of a formula P

It is not the case that P

Truth table:

P	$\neg P$
1	0
0	1

Examples

p: propositional logic is easy.
$\neg p$: propositional logic is not easy.

q: The exam is in January.
$\neg q$: It is not the case that the exam is in January

Conjunction

The conjunction $(P \wedge Q)$ of P and Q.

both P and Q are true

Truth table:

P	Q	$(P \wedge Q)$
1	1	1
1	0	0
0	1	0
0	0	0

Examples

p: Logic is easy.

q: I eat toast.

$(p \wedge q)$: Logic is easy and I eat toast

$(\neg p \wedge q)$: Logic is not easy and I eat toast

$(\neg p \wedge \neg q)$: Logic is not easy and I do not eat toast

Disjunction

The disjunction $(P \vee Q)$ of P and Q

at least one of P and Q is true

Truth table:

P	Q	$(P \vee Q)$
1	1	1
1	0	1
0	1	1
0	0	0

Examples

p: I'll have tea.
q: I'll have coffee.
$(p \vee q)$: I'll have tea or I'll have coffee

Equivalence

The equivalence $(P \Leftrightarrow Q)$ of P and Q

P and Q take the same truth value

Truth table:

P	Q	$(P \Leftrightarrow Q)$
1	1	1
1	0	0
0	1	0
0	0	1

Examples

p:	You can take a flight.
q:	You buy a ticket.
$(p \Leftrightarrow q)$:	You can take a flight if and only if you buy a ticket

Implication

The implication $(P \Rightarrow Q)$ of P and Q

if P *then* Q

Truth table:

P	Q	$(P \Rightarrow Q)$
1	1	1
1	0	0
0	1	1
0	0	1

Truth under an interpretation

So, given an interpretation I, we can compute the truth value of any formula P under I.

- If $I(P) = 1$, then P is called true under the interpretation I.
- If $I(P) = 0$, then P is called false under the interpretation I.

Example 1

List the interpretations I such that $P = ((p_1 \wedge \neg p_2) \Rightarrow (p_2 \wedge \neg p_1))$ is true under I.

p_1	p_2	$\neg p_2$	$\neg p_1$	$(p_1 \wedge \neg p_2)$	$(p_2 \wedge \neg p_1)$	P
1	1	0	0	0	0	1
1	0	1	0	1	0	0
0	1	0	1	0	1	1
0	0	1	1	0	0	1

Remember the negation truth table:

P	$\neg P$
1	0
0	1

Remember the conjunction truth table:

P	Q	$(P \wedge Q)$
1	1	1

Example 2

How many interpretations I of p_1, p_2 and p_3 are there such that $P = ((p_1 \vee p_2) \Leftrightarrow p_3)$ is true under I?

p_1	p_2	p_3	$(p_1 \vee p_2)$	P
1	1	1	1	1
1	1	0	1	0
1	0	1	1	1
1	0	0	1	0
0	1	1	1	1
0	1	0	1	0
0	0	1	0	0
0	0	0	0	1

Disjunction truth table:

P	Q	$(P \vee Q)$
1	1	1
1	0	1
0	1	1
0	0	0

Equivalence truth table:

P	Q	$(P \Leftrightarrow Q)$
1	1	1
1	0	0
0	1	0
0	0	1

Thus, there are 4 interpretations making P true.

Example 3

How many interpretations I of p_1, p_2 and p_3 are there such that $P = ((p_1 \vee \neg p_2) \wedge p_3)$ is true under I?

p_1	p_2	p_3	$\neg p_2$	$(p_1 \vee \neg p_2)$	P
1	1	1	0	1	1
1	1	0	0	1	0
1	0	1	1	1	1
1	0	0	1	1	0
0	1	1	0	0	0
0	1	0	0	0	0
0	0	1	1	1	1
0	0	0	1	1	0

Negation truth table:

P	$\neg P$
1	0
0	1

Disjunction truth table:

P	Q	$(P \vee Q)$
1	1	1
1	0	1
0	1	1
0	0	0

Conjunction truth table:

P	Q	$(P \wedge Q)$
1	1	1

Satisfiability

A propositional formula is satisfiable if there exists an interpretation under which it is true.

Example. The formula $(p \wedge \neg p)$ is not satisfiable, because

$$I(p \wedge \neg p) = 0$$

for all interpretations I:

p	$\neg p$	$(p \wedge \neg p)$
1	0	0
0	1	0

Example 1

$P = (p_1 \Rightarrow (p_2 \wedge \neg p_2))$ is satisfiable.

To see this, simply choose any interpretation I with $I(p_1) = 0$. Then $I(P) = 1$ and we have found an interpretation I that makes P true.

The same argument shows that every formula of the form $(p_1 \Rightarrow Q)$ is satisfiable.

Example 2

$P = (p_1 \Leftrightarrow ((p_2 \wedge \neg p_3) \wedge p_4))$ is satisfiable.

To see this, let I be any interpretation of p_2, p_3, p_4. Compute

$$I(((p_2 \wedge \neg p_3) \wedge p_4))$$

Then let $I(p_1) = I(((p_2 \wedge \neg p_3) \wedge p_4))$. I is an interpretation that makes P true.

The same argument shows that every formula of the form $(p_1 \Leftrightarrow Q)$ such that p_1 does not occur in Q is satisfiable.

Note that $(p_1 \Leftrightarrow \neg p_1)$ is not satisfiable. So the condition that p_1 does not occur in Q is needed.

Deciding satisfiability

- P is satisfiable if $I(P) = 1$ for some interpretation I.
- There are 2^n interpretations if P has n propositional atoms.
- Thus, to check whether P is satisfiable directly using truth tables, a table with 2^n rows is needed.
- This is not practical, even for small n (such as 100): combinatorial explosion again.
- There has been great progress in developing very fast satisfiability checking algorithms (called SAT solvers) that can deal with formulas with very large numbers of propositional atoms (using heuristics again).

A Knowledge Base

The meeting can take place if all members have been informed in advance, and it is quorate. It is quorate provided that there are at least 15 people present. Members will have been informed in advance if there is not a postal strike.
Consequence: Therefore, if the meeting was cancelled, we conclude that there were fewer than 15 members present, or there was a postal strike.

Let

m: "the meeting takes place" a: "all members have been informed"
p: "there is a postal strike" q: "the meeting is quorate"
f: "there are at least 15 members present"

Then the text can be formalised as the following knowledge base:

$$((a \wedge q) \Rightarrow m), \quad (f \Rightarrow q), \quad (\neg p \Rightarrow a)$$

Propositional Knowledge Bases and Reasoning

A propositional knowledge base X is a finite set of propositional formulas.

Suppose a propositional knowledge base X is given. Then a propositional formula P follows from X if the following holds for every interpretation I:

$$\text{If } I(Q) = 1 \text{ for all } Q \in X\text{, then } I(P) = 1.$$

This is denoted by

$$X \models P.$$

Example. We have seen that:

$$\{((a \wedge q) \Rightarrow m), (f \Rightarrow q), (\neg p \Rightarrow a)\} \models (\neg m \Rightarrow (\neg f \vee p))$$

Example 1

Show $\{(p_1 \wedge p_2)\} \models (p_1 \vee p_2)$.

p_1	p_2	$(p_1 \wedge p_2)$	$(p_1 \vee p_2)$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	0

Thus, from $I(p_1 \wedge p_2) = 1$ it follows that $I(p_1 \vee p_2) = 1$.

But $\{(p_1 \vee p_2)\} \not\models (p_1 \wedge p_2)$ since there exists an interpretation I with $I(p_1 \vee p_2) = 1$ and $I(p_1 \wedge p_2) = 0$. For example, $I(p_1) = 1$ and $I(p_2) = 0$.

Example 2

Show $\{p_1, p_1 \Rightarrow p_2\} \models p_2$.

p_1	p_2	$(p_1 \Rightarrow p_2)$
1	1	1
1	0	0
0	1	1
0	0	1

Thus, whenever $I(p_1) = 1$ and $I(p_1 \Rightarrow p_2) = 1$, then $I(p_2) = 1$.

But $\{p_1, p_2 \Rightarrow p_1\} \not\models p_2$. This is shown by the interpretation I with $I(p_1) = 1$ and $I(p_2) = 0$.

Reasoning

- $X \models P$ if $I(P) = 1$ for all interpretations I such that $I(Q) = 1$ for all $Q \in X$.
- There are 2^n different relevant interpretations if P and X together have n propositional atoms.
- Thus, to check $X \models P$ directly using truth tables, a table with 2^n rows is needed.
- $X \models P$ can be checked using a SAT solver: assume that X contains $P_1, \ldots, P_n$ and let

  $$Q = P_1 \wedge \cdots \wedge P_n \wedge \neg P$$

  Then

  $$X \models P \quad \text{if and only if} \quad Q \text{ is not satisfiable}$$

$\{P_1, \ldots, P_n\} \models P$ and satisfiability of $P_1 \wedge \ldots \wedge P_n \wedge \neg P$

Let $X = \{P_1, \ldots, P_n\}$ be a knowledge base.

P follows from X if for every interpretation I, if $I(P_i) = 1$ for all $P_i \in X$, then $I(P) = 1$.

P does NOT follow from X if there exists an interpretation I such that $I(P_i) = 1$ for all $P_i \in X$, but $I(P) = 0$ ($I(\neg P) = 1$).

P does NOT follow from X if $P_1 \wedge \ldots \wedge P_n \wedge \neg P$ is satisfiable.

P follows from X if $P_1 \wedge \ldots \wedge P_n \wedge \neg P$ is NOT satisfiable.

What is the point?
Different points of view can lead to different techniques

- different algorithms
- different heuristics

for example, checking all interpretations vs looking for one

Summary

- In KR&R knowledge is stored in a knowledge base. Reasoning is needed to make implicit knowledge explicit.
- We have considered rule-based and propositional knowledge bases and corresponding reasoning.
- In the rule-based case we have given reasoning algorithms.
- Databases only store atomic assertions. In contrast, knowledge bases store knowledge that goes beyond atomic assertions (such as rules and compound propositions).
- Knowledge stored in knowledge bases is mostly incomplete, whereas knowledge stored in databases is mostly complete.
- Many other KR&R languages with corresponding reasoning algorithms have been developed.

How to apply what we have learned?

When facing a new problem, ask yourself

- can I use any of the KR languages I know?
- Answer depends on whether any of the KR languages is sufficiently expressive to model the problem.

Modelling Example

The meeting can take place if all members have been informed in advance, and it is quorate. It is quorate provided that there are at least 15 people present. Members will have been informed in advance if there is not a postal strike. Therefore, if the meeting was cancelled, we conclude that there were fewer than 15 members present, or there was a postal strike.

m: "the meeting takes place" a: "all members have been informed"
p: "there is a postal strike" q: "the meeting is quorate"
f: "there are at least 15 members present"

The problem can be modelled as

$$\{((a \wedge q) \Rightarrow m), (f \Rightarrow q), (\neg p \Rightarrow a)\} \models (\neg m \Rightarrow (\neg f \vee p))$$

Recalling Examples: Example 1

Consider the following knowledge base:

- If I have an AI lecture today, then it is Tuesday or Friday.
- It is not Tuesday.
- I have an AI lecture today or I have no class today.
- If I have no class today, then I am sad.
- I am not sad.

Can you infer what day it is?

Can we model it using the rule-based approach?

- No, some concepts cannot be expressed. For example, "it is Tuesday or Friday", and "it is not Tuesday".

We can model it using propositional logic.

Recalling Examples: Example 2

Consider the following medical knowledge base:

- Pericardium is a tissue contained in the heart
- Pericarditis is an inflammation located in the pericardium
- Inflammation is a disease that acts on tissue
- A disease located in something contained in the heart is a heartdisease

Can you infer that pericarditis is a heartdisease?

Can we model it using propositional logic?

- No, some concepts cannot be expressed. For example, we cannot express "any inflammation is a disease".

We can model it using a rule-based approach.

Modelling Example 2

Representation in Rule-Based Language. Let K be the following knowledge base.

Tissue(Pericardium), ContinHeart(Pericardium)
Inflammation(Pericarditis)
LocatedIn(Pericarditis,Pericardium)
$\text{Inflammation}(x) \Rightarrow \text{Disease}(x)$
$\text{Disease}(x) \wedge \text{LocatedIn}(x, y) \wedge \text{contInHeart}(y) \Rightarrow \text{Heartdisease}(x)$

Then

$$K \models \text{Heartdisease(Pericarditis)}$$

Modelling Example 1

Consider the following knowledge base:

- If I have an AI lecture today, then it is Tuesday or Friday.
- It is not Tuesday.
- I have an AI lecture today or I have no class today.
- If I have no class today, then I am sad.
- I am not sad.

Can you infer what day it is?

Which means: is it Monday? Is it Tuesday? . . . Is it Friday?

Let us focus only on the question: is it Friday?

Modelling Example 1: propositions

Consider the following knowledge base:

- If I have an AI lecture today, then it is Tuesday or Friday.
- It is not Tuesday.
- I have an AI lecture today or I have no class today.
- If I have no class today, then I am sad.
- I am not sad.

What propositions do we need?

a: "I have an AI lecture today" c: "I have class today"
t: "it is Tuesday" f: "it is Friday"
s: "I am sad"

Modelling Example 1: rewriting the knowledge base

Consider the following knowledge base:

- If I have an AI lecture today, then it is Tuesday or Friday.
  - $(a \Rightarrow (t \vee f))$
- It is not Tuesday.
  - $\neg t$
- I have an AI lecture today or I have no class today.
  - $(a \vee \neg c)$
- If I have no class today, then I am sad.
  - $(\neg c \Rightarrow s)$
- I am not sad.
  - $\neg s$

Our propositions

a: "I have an AI lecture today" c: "I have class today"
t: "it is Tuesday" f: "it is Friday"
s: "I am sad"

Modelling Example 1: is it Friday?

Let us define some abbreviation:

- $P_1 = (a \Rightarrow (t \vee f))$
- $P_2 = \neg t$
- $P_3 = (a \vee \neg c)$
- $P_4 = (\neg c \Rightarrow s)$
- $P_5 = \neg s$

And $X = \{P_1, P_2, P_3, P_4, P_5\}$

Checking if it is Friday corresponds to check whether $X \models f$ holds.

Alternatively: is $P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge P_5 \wedge \neg f$ not satisfiable?.

Remember:

$X \models f$ if and only if $P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge P_5 \wedge \neg f$ is not satisfiable

Modelling Example 1: does $X \models f$ hold?

We have 5 propositions, meaning 2^5 rows using a truth table. Let us try to avoid a huge truth table.

We show using a proof by contradiction that $P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge P_5 \wedge \neg f$ is not satisfiable. Thus, assume that $P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge P_5 \wedge \neg f$ is satisfiable. Then there is an interpretation I such that

- $I(P_1) = 1$, $I(P_2) = 1$, $\ldots$, $I(P_5) = 1$, and $I(\neg f) = 1$;
- $I(\neg f) = 1$ means $I(f) = 0$;
- since $P_2 = \neg t$ and $P_5 = \neg s$, then $I(t) = 0$ and $I(s) = 0$;
- since $P_4 = (\neg c \Rightarrow s)$, then $I(P_4) = 1$ only if $I(c) = 1$;
- since $P_1 = (a \Rightarrow (t \vee f))$, then $I(P_1) = 1$ only if $I(a) = 0$;
- since $P_3 = (a \vee \neg c)$, then $I(P_3) = 0$!

We have derived a contraction as $I(P_3) = 1$ and $I(P_3) = 0$.

Thus, $X \models f$ holds.

Not always that easy

How would you model this using propositional logic?

			7			4	1	
		3		2				6
1		7	4			5	2	3
4		1	6				8	
	2	9		7		6	3	
	7				4	2		1
7	5	2			6	3		9
3				4		1		
	1	4			3			

What propositions?
Hint: number x is in row y and column z

What to model?

- at least one number per cell (pair of row and column)
- at most one number per cell (pair of row and column)
- no number can be repeated in a row
- no number can be repeated in a column
- no number can be repeated in a region