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# FACETS OF THE KNAPSACK POLYTOPE*

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A necessary and sufficient condition is given for an inequality with coefficients 0 or 1 to define a facet of the knapsack polytope, i.e., of the convex hull of 0–1 points satisfying a given linear inequality. A sufficient condition is also established for a larger class of inequalities (with coefficients not restricted to 0 and 1) to define a facet for the same polytope, and a procedure is given for generating all facets in the above two classes. The procedure can be viewed as a way of generating cutting planes for 0–1 programs.

## 1. Introduction: Canonical equivalents of a linear inequality in 0–1 variables

Consider the inequality

$$\sum_{j\in N} a_j x_j \leqslant a_0 , \tag{1}$$

where $a_0 > 0$, $a_j > 0$, and $x_j = 0$ or $1, j \in N = \{1, ..., n\}$. Let $N$ and all of its subsets to be considered below be ordered so that $a_j \geqslant a_{j+1}, j = 1, ..., n-1$.

A set $S \subset N$ will be called a *cover* or *covering set* for (1), if

(i) $$\sum_{j\in S} a_j > a_0 .$$

A cover for (1) will be called *minimal,* if

(ii) $$\sum_{j\in Q} a_j \leqslant a_0 \quad \text{for all proper subsets } Q \text{ of } S .$$

The set $E(S) = S \cup S'$, where

$$S' = \{j \in N - S\colon\ a_j \geqslant a_{j_1}\} ,$$

* This paper was first circulated under [1]. A brief note on it was published under [2].

and

$$a_{j_1} = \max_{j \in S} a_j\,, \tag{2}$$

will be called the *extension* of $S$ to $N$.

In [3, Theorem 2] (see also [4, 5]), Balas and Jeroslow have shown that if $\mathcal{S}$ is the family of all minimal covers $S$ for (1), then (1) is equivalent to the set of (canonical) inequalities

$$\sum_{j \in E(S)} x_j \leqslant |S| - 1 \qquad \text{for all } S \in \mathcal{S} \tag{3}$$

in the sense that $x \in \mathbf{R}^n$, $x_j = 0$ or $1, j \in N$, satisfies (1) if and only if it satisfies (3). Further (see Remark to Theorem 2 of [3]), it was shown that (1) is also equivalent to the set

$$\sum_{j \in S} x_j \leqslant |S| - 1, \qquad S \in \mathcal{S}\,. \tag{4}$$

Since $E(S) \supseteq S$ and more often than not $E(S) - S \neq 0$, the inequalities of (3) always dominate, and in most cases strictly dominate, the corresponding inequalities of (4).

The equivalence of (1) and (4) was independently established by Granot and Hammer [7], who also extended it to the case where (1) is nonlinear. More recently, further properties of canonical inequalities were studied by Glover [6]. Also, several authors (including the writer), have used canonical inequalities in enumerative or cutting plane algorithms, but here we are concerned with structural properties only.

If $S$ and $T$ are minimal covers such that $|S| = |T|$ and $E(S) \subset E(T)$, the inequality of (3) corresponding to $S$ is obviously redundant. Removing from (3) such obviously redundant inequalities amounts to restricting the family $\mathcal{S}$ to

$$\bar{\mathcal{S}} = \{S \in \mathcal{S} : E(S) \not\subset E(T) \quad \text{for all } T \in \mathcal{S} - S \text{ such that } |T| = |S|\}.$$

The members of $\bar{\mathcal{S}}$ will be termed *strongly minimal* (or *strong*) covers. Clearly, a minimal cover $S$ is strong if and only if either $E(S) = N$, or else no set of the form $(S - \{j_1\}) \cup \{i\}$, for some $i \in N - E(S)$, is a cover; and the latter condition is easily seen to hold if and only if the set $(S - \{j_1\}) \cup \{i_1\}$ is not a cover, where $i_1$ is the first index of $N - E(S)$.

Hence, a set $S \subseteq N$ is a strong (strongly minimal) cover for (1) if and only if it satisfies (i), (ii) and

(iii) if $E(S) \neq N$, then $\displaystyle\sum_{j \in (S - \{j_1\}) \cup \{i_1\}} a_j \leqslant a_0\,,$

where $j_1$ is defined by (2), and $i_1$ by

$$a_{i_1} = \max_{j \in N - E(S)} a_j .$$

Restricting the family $\mathcal{S}$ of minimal covers to the family $\bar{\mathcal{S}} \subset \mathcal{S}$ of strong covers yields the set of strong canonical inequalities

$$\sum_{j \in E(S)} x_j \leqslant |S| - 1, \qquad S \in \bar{\mathcal{S}} \tag{3'}$$

which of course is still equivalent to (1) in the sense of having the same 0–1 solution set as (1), since the only inequalities that were removed from (3) were the obviously redundant ones. Also, the family $\bar{\mathcal{S}}$ was shown by Glover [6] to be minimal, i.e., the equivalence ceases to hold if any member of $\bar{\mathcal{S}}$ is removed.

It was empirically observed that the strong canonical inequalities often define facets of the convex hull of 0–1 points satisfying (1), i.e., facets of the "knapsack polytope"

$$P = \operatorname{conv} \left\{ x \in \mathbf{R}^n \colon \sum_{j \in N} a_j x_j \leqslant a_0, \ x_j = 0 \text{ or } 1, j \in N \right\}.$$

In this paper we establish precisely when this is the case. The necessary and sufficient conditions, stated in Proposition 2 and Theorem 1, are remarkably easy to check. Furthermore, the reasoning used in the proof of sufficiency is conducive to a more general result, stated in Theorem 2, which characterizes a broader class of facets of $P$, whose coefficients are not restricted to 0 or 1. Finally, a procedure is given for generating the members of this class.

## 2. Canonical facets of the knapsack polytope

Let $d$ be the dimension of $P$. The inequality

$$\sum_{j \in M} x_j \leqslant k , \tag{5}$$

where $M \subseteq N$ and $k \geqslant 0$, is said to define a facet [$(d-1)$-dimensional face] of $P$, if and only if the halfspace defined by (5) contains $P$, and the hyperplane defined by

$$\sum_{j \in M} x_j = k \tag{5'}$$

contains exactly $d$ (affinely) independent points of $P$.

**Proposition 1.** $d = n - n'$, *where* $n' = |N'|$ *and*

$$N' = \{j \in N \colon a_j > a_0\}\ .$$

**Proof.** $d \geqslant n - n'$, since $P$ contains the $n - n'$ unit vectors $e_j$, $j \in N - N'$. Also, $d \leqslant n - n'$, since $x_j = 0$ for all $j \in N'$, for any $x \in P$.

If $d = 0$, $P$ has no facets. If $d > 0$, then since $d = n - n'$, (5) defines a facet of $P$ if and only if the inequality

$$\sum_{j \in M \sim N'} x_j \leqslant k$$

defines a facet of the polytope $P \cap R^{n-n'}$, where $R^{n-n'}$ is the ($d$-dimensional) subspace of the variables $x_j$, $j \in N - N'$.

Hence we can (and will) assume without loss of generality, that $P$ is full-dimensional, i.e., $a_j \leqslant a_0$ for all $j \in N$, and $d =$ n. Also, we will assume that $k \geqslant 1$, which is a necessary condition for (5) to define a facet for $P$ when $P$ is full-dimensional. Since $k \geqslant 1$ implies $x \neq 0$ for any $x$ satisfying (5′), the hyperplane (5′) contains $n$ affinely independent vertices of $P$ if and only if it contains $n$ linearly independent vertices of $P$.

If $|M| = 1$, i.e., if $M = \{j\}$ for some $j \in N$, the inequality (5) becomes $x_j \leqslant k$. For $k > 1$, no $x \in P$ satisfies $x_j = k$; hence $k = 1$ is a necessary condition for $x_j \leqslant k$ to define a facet of $P$.

On the other hand, unlike the canonical inequalities with $|M| \geqslant 2$, which are implied by (1) and the 0–1 condition, the inequalities $x_j \leqslant 1$ are implied by the 0–1 condition alone. This lends them a special status and we treat them accordingly.

**Proposition 2.** *The inequality* $x_j \leqslant 1$ *defines a facet of P if and only if*

$$a_{j_*} + a_j \leqslant a_0\ , \tag{6}$$

*where*

$$a_{j_*} = \max_{i \in N - \{j\}} a_i\ .$$

**Proof.** Let $e_i$ be the $i^{\text{th}}$ unit vector, $i \in N$. Then $e_j$ and the $n - 1$ vectors $e_i + e_j$, for all $i \in N - \{j\}$, constitute a set of $n$ linearly independent vectors which are contained in $P$ if (6) holds. Hence if (6) holds, $x_j \leq 1$ defines a facet of $P$.

On the other hand, if $x_j \leq 1$ defines a facet of $P$, then $x_j^i = 1$ holds for $n$ linearly independent vertices $x^i$ of $P$, $i = 1, ..., n$. Let $X$ be the $n \times n$ matrix whose rows are the vectors $x^i$. If (6) does not hold, then $x_{j*}^i = 0$, $i = 1, ..., n$, and $X$ is singular; thus (6) must hold.

**Example 1.** Let $P$ be the convex hull of 0–1 points satisfying

$$8x_1 + 8x_2 + 7x_3 + 5x_4 + 2x_5 + x_6 + x_7 \leq 11 .$$

Then $x_j \leq 1$ defines a facet of $P$ for $j = 5, 6, 7$, but not for $j = 1, ..., 4$, since (6) holds for 5, 6, 7 only.

We now turn to the inequalities (5) with $|M| \geq 2$.

**Lemma.** *If* $|M| \geq k + 1 \geq 2$ *and* (5) *defines a facet of* $P$, *then for each* $i \in N$, $P$ *has a vertex* $\bar{x}$ *satisfying* (5′) *and such that* $\bar{x}_i = 1$; *and for each* $i \in M$, $P$ *has a vertex* $\hat{x}$ *satisfying* (5′) *and such that* $\hat{x}_i = 0$.

**Proof.** We prove the Lemma by contradiction. Since (5) defines a facet of $P$, (5′) is satisfied by $n$ linearly independent vertices $x^h$, $h = 1, ..., n$, of $P$. If $X$ is the $n \times n$ matrix whose rows are these vertices $x^h$, then $x_i^h = 0$, $h = 1, ..., n$, for some $i \in N$, implies that $X$ is singular. Hence for each $i \in N$, $x_i^h = 1$ for some $h \in \{1, ..., n\}$.

Also, if $X_M$ is the submatrix of $X$ whose columns are indexed by $M$, then each row of $X_M$ has exactly $k$ entries equal to 1; and therefore if $x_i^h = 1$ for $h = 1, ..., n$ and some $i \in M$, then the $i^{\text{th}}$ column of $X_M$ is the sum of the remaining columns of $X_M$ divided by $k - 1$, hence the columns of $X$ are linearly dependent. Thus for each $i \in M$, $x_i^h = 0$ for some $h \in \{1, ..., n\}$.

The next theorem fully characterizes canonical facets (i.e., facets with coefficients 0 or 1) of the knapsack polytope. The same characterization was also established, in a somewhat different context and with a proof different from ours, by Hammer, Johnson and Peled [8], as well as by Wolsey [12]. Though the report [8] is dated October 1973, the Hammer–Johnson–Peled result was obtained earlier during the summer, i.e., prior to [1].

**Theorem 1.** *The inequality* (5), *where* $|M| \geqslant 2$, *defines a facet of* $P$ *if and only if* $M$ *is the extension of a strong cover* $S$ *for* (1), *such that* $|S| = k + 1$, *and*

$$\sum_{j \in T} a_j \leqslant a_0, \qquad T = (S - \{j_1, j_2\}) \cup \{1\}, \tag{7}$$

*with* $j_1$ *defined by* (2), *and* $j_2$ *by*

$$a_{j_2} = \max_{j \in S - \{j_1\}} a_j .$$

**Proof.** ($\alpha$) Assume (5) defines a facet of $P$. Then obviously $1 \leqslant k < |M| = m$. Further, we claim that the set $S$ of the last $k + 1$ elements of $M$, is a strong cover. Suppose it is not; then (i), (ii) or (iii) is false.

If (i) is false, then (5) cuts off $x \in P$, defined by $x_j = 1, j \in S$, $x_j = 0$, $j \in N - S$, contrary to the assumption that (5) defines a facet of $P$. If (i) is true but (ii) is false for some $Q = S - \{i\}$, then each $x \in P$ satisfies

$$\sum_{j \in M - \{i\}} x_j \leqslant k - 1 . \tag{8}$$

But a vertex $x$ of $P$ satisfying (5′) can only satisfy (8) if $x_i = 1$, which contradicts the Lemma.

Finally, if (i) and (ii) are true but (iii) is false, then $R = (S - \{j_1\}) \cup \{i_1\}$ is a minimal cover with $E(R) \supseteq M \cup \{i_1\}$, where $E$ is the extension of $R$ to $N$, and therefore each $x \in P$ satisfies

$$\sum_{j \in M \cup \{i_1\}} x_j \leqslant k . \tag{9}$$

But a vertex $x$ of $P$ satisfying (5′) can only satisfy (9) if $x_{i_1} = 0$, which again contradicts the Lemma. Hence (iii) is true.

We have shown that $S$ is a strong cover, with $|S| = k + 1$. Next we show that $M = E(S)$. If this is false, i.e., if there exists $h \in N - M$ such that $a_h \geqslant a_{j_1}$, then each $x \in P$ satisfies

$$\sum_{j \in M \cup \{h\}} x_j \leqslant k \tag{10}$$

and the same argument holds as for (9); i.e., each vertex $x$ of $P$ satisfying (5′) and (10) has $x_h = 0$, contrary to the Lemma. Hence $M = E(S)$.

It remains to be shown that (7) holds. If $j_1 = 1$, (7) follows from (ii).

If $j_1 \neq 1$ and (7) is false, then $x_1 = 0$ for every $x \in \operatorname{vert} P$ satisfying (5′), contrary to the lemma.

This proves the "only if" part of the theorem.

($\beta$) Assume that $M$ is the extension of a strong cover $S$ for (1), and $k = |S| - 1$. From property (i) of $S$, every vector $x \in \mathbf{R}^n$ which satisfies (1) also satisfies (5). To show that (5) defines a facet of $P$, we will prove constructively that the hyperplane defined by (5′) contains $n$ linearly independent vertices of $P$.

Consider the $n \times n$ matrix

$$X = \begin{bmatrix} I_{m-k-1} & B_1 & 0 \\ 0 & C & 0 \\ 0 & B_0 & I_{n-m} \end{bmatrix},$$

where $I_p$ is the identity matrix of order $p$; $B_1$ and $B_0$ are $(m-k-1) \times (k+1)$ and $(n-m) \times (k+1)$ respectively, each of them having identical rows of the form

$$b_1 = (0, 0, 1, 1, ..., 1), \qquad b_0 = (0, 1, 1, 1, ..., 1)$$

respectively (i.e., 1 for all entries except for the first two in the case of the rows of $B_1$, and for the first one in the case of those of $B_0$); the zeros in $X$ stand for zero matrices of appropriate dimension; and $C = (c_{ij})$ is the $(k+1) \times (k+1)$ matrix

$$C = \begin{bmatrix} 1 & 1 & \dots & 1 & 0 \\ 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 1 & 1 \\ 1 & 1 & \dots & 0 & 1 \end{bmatrix},$$

i.e.,

$$c_{ij} = \begin{cases} 0 & \text{for } i = 1, j = k+1; \text{ and } i \neq 1, j = i-1, \\ 1 & \text{for } i = 1, j \neq k+1; \text{ and } i \neq 1, j \neq i-1. \end{cases}$$

$C$ is nonsingular, with inverse $C^{-1} = (\gamma_{ij})$, where

$$\gamma_{ij} = \begin{cases} 1/k - 1 & \text{for } j = 1, i = k+1; \text{ and } j \neq 1, i = j-1, \\ 1/k & \text{for } j = 1, i \neq k+1; \text{ and } j \neq 1, i \neq j-1 \end{cases}$$

and since

$$\det X = \det I_{m-k-1} \times \det C \times \det I_{n-m} \neq 0,$$

it follows that the $n$ rows of $X$ are linearly independent. Furthermore, each of these row vectors $x^i$ has exactly $k$ among its first $m$ entries equal to 1, hence satisfies the equation (5′).

We claim that each of the $n$ row vectors $x^i$ satisfies the inequality

$$a^* x \leqslant a_0 , \tag{1*}$$

obtained from (1) by permuting the $n$ indices $j \in N$ so that the first $m = |M|$ components of $a^*$ are those coefficients $a_j$ of (1) such that $j \in M$, while the order defined on $N$ is preserved for $M$ as well as for $N - M$. To show that the $n$ row vectors $x^i$ satisfy (1*), we notice that for the rows of $X$ corresponding to $I_{m-k-1}$ (which exist only if $M - S \neq \emptyset$)

$$\begin{aligned} a^* x^i &= \sum_{j \in (S - \{j_1, j_2\}) \cup \{i\}} a_j \qquad \text{for some } i \in M - S \\ &\leqslant \sum_{j \in T} a_j \\ &\leqslant a_0 \qquad \text{[from (7)]}. \end{aligned}$$

On the other hand, for the rows corresponding to $C$ and $I_{n-m}$, $x^i$ satisfies (1*) since $x^i_j = 1$ for exactly $k = |S| - 1$ indices $j \in S$, and $S$ is a minimal cover.

This proves the "if" part of the theorem.

**Example 2.** Let $P$ be the convex hull of 0–1 points satisfying

$$5x_1 + 5x_2 + 4x_3 + 3x_4 + 2x_5 + 2x_6 + x_7 \leqslant 6 .$$

From Proposition 2, the only inequality of the form $x_j \leqslant 1$ to define a facet of $P$, is $x_7 \leqslant 1$.

From Theorem 1, the only inequalities of the form (5) with $|M| \geqslant 2$ which define facets of $P$, are

$$\begin{aligned} &x_1 + x_2 + x_3 + x_4 && \leqslant 1, \\ &x_1 + x_2 + x_5 && \leqslant 1, \\ &x_1 + x_2 + x_6 && \leqslant 1, \\ &x_1 + x_2 + x_3 + x_5 + x_7 && \leqslant 2, \\ &x_1 + x_2 + x_3 + x_6 + x_7 && \leqslant 2 . \end{aligned}$$

The canonical inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leqslant 2$$

which is satisfied by all 0–1 points satisfying (1) and is associated with the strong cover $S = \{4, 5, 6\}$, belongs to the set (3′); yet, it does not define a facet of $P$, since

$$T = (\{4, 5, 6\} - \{4, 5\}) \cup \{1\} = \{1, 6\},$$

and

$$a_1 + a_6 = 5 + 2 > 6.$$

Proposition 2 and the above theorem give a complete characterization of the class of facets of $P$ defined by inequalities with coefficients 0 or 1 and right-hand side $k \geqslant 1$. However, this class is not exhaustive, i.e., $P$ may have facets defined by inequalities having coefficients other than 0 or 1. For instance, the following inequality defines a facet of the knapsack polytope of Example 2:

$$2x_1 + 2x_2 + x_3 + x_4 + x_5 + x_6 \leqslant 2.$$

The following corollary to Theorem 1 shows that a strong cover, even when it does not yield an inequality defining a facet of $P$, does yield one defining a facet for some polytope $P_V = P \cap R^v$, where $V \subset N$, $v = |V|$, and $R^v$ is the subspace of the variables $x_j$, $j \in V$.

**Corollary 1.** *Let $S \subseteq N$ be a strong cover for* (1), $M = E(S)$, *and let*

$$M' = \left\{ i \in M\colon \sum_{j \in T(i)} a_j \leqslant a_0 \right\},$$

*where*

$$T(i) = (S - \{j_1, j_2\}) \cup \{i\}, \qquad i \in M,$$

*with $j_1$ and $j_2$ defined as in Theorem* 1.

*Then the inequality*

$$\sum_{j \in M'} x_j \leqslant k, \tag{11}$$

*where $k = |S| - 1$, defines a facet of $P_V = P \cap R^v$, i.e., of*

$$P_V = \operatorname{conv} \left\{ x \in R^v\colon \sum_{j \in V} a_j x_j \leqslant a_0,\ x_j = 0 \text{ or } 1, j \in V \right\},$$

*with*

$$V = (N - M) \cup M', \qquad v = |V|.$$

**Proof.** By definition, if $S$ is a strong cover for (1), then it is a strong cover also for

$$\sum_{j \in V} a_j x_j \leqslant a_0 \, .$$

Further, $M'$ is easily seen to be the extension of $S$ to $V$, and by the definition of $M'$, the inequality (7) of Theorem 1 holds when $T$ is replaced by $T(i_*)$, where

$$a_{i_*} = \max_{j \in V} a_j \, .$$

Hence, from Theorem 1, (11) defines a facet of $P_V$.

According to Corollary 1, every strong cover for (1) yields a facet for some polytope $P_V$, $V \subseteq N$. Furthermore, the corollary shows how to identify the largest such set $V$.

In Example 2, for instance, applying the test of the corollary to the strong cover $S = \{4, 5, 6\}$, we find

$$M = \{1, 2, 3, 4, 5, 6\},$$

$$T(i) = \{6\} \cup \{i\}, \qquad i = 1, \ldots, 6,$$

$$\sum_{j \in T(i)} a_j = 2 + a_i \leqslant 6 \quad \text{for } i = 3, 4, 5, 6, \text{ but } \textit{not} \text{ for } i = 1, 2 \, .$$

Thus $M' = \{3, 4, 5, 6\} \neq M$ and the strong cover $\{4, 5, 6\}$ does not yield a facet for $P$, but it does yield the inequality

$$x_3 + x_4 + x_5 + x_6 \leqslant 2$$

which defines a facet of $P_V$, where $V = \{3, 4, 5, 6, 7\}$, i.e., for

$$P_V = \operatorname{conv} \{x \in \mathbf{R}^5 \colon 4x_3 + 3x_4 + 2x_5 + 2x_6 + x_7 \leqslant 6, \; x_j = 0 \text{ or } 1 \text{ for all } j\} \, .$$

## 3. A class of facets with nonnegative integer coefficients

In Theorem 2 below we give a sufficient condition for a class of inequalities with nonnegative integer coefficients to define a facet of $P$. The class of facets characterized in Theorem 2, which subsumes as a subclass the canonical facets of Theorem 1, can be calculated at a low computational cost.

First, we mention that Corollary 1 can be used, in conjunction with a result of Padberg, to generate a more comprehensive class of facets, at a higher computational cost. Padberg's result, first established for the node-packing/set partitioning problem [10], then extended to 0–1 programs with nonnegative coefficients [11] (see also [9]), associates with every facet of the lower-dimensional polytope $P_V$ (for any $V \subset N$), one or several facets of the full-dimensional polytope $P$. In particular, from each inequality of the form

$$\sum_{j \in V} \pi_j x_j \leqslant \pi_0 \qquad (12_V)$$

with $\pi_0 > 0$, $\pi_j \geqslant 0$, which defines a facet of $P_V$, Padberg's procedure generates one or several inequalities of the form

$$\sum_{j \in N} \pi_j x_j \leqslant \pi_0 \,, \qquad (12_N)$$

each of which defines a facet of $P$. The coefficients $\pi_j$, $j \in V$, in $(12_N)$ are of course the same as in the inequality $(12_V)$, whereas the $\pi_j$, $j \in N - V$, are nonnegative integers, found by solving a sequence of integer programs (in our case, knapsack problems), one for each coefficient. The sequence in which the coefficients are calculated may affect the outcome, and whenever it does, more than one inequality $(12_N)$ is generated from the same starting inequality $(12_V)$.

Our next theorem is motivated by the desire to obtain (noncanonical) facets of the above type without solving a knapsack problem. The sufficient condition given in Theorem 2 below characterizes a class of facets associated with strong covers, that are easy to generate.

We recall that the set $N$ is ordered and so are all its subsets considered here.

**Theorem 2.** *The inequality*

$$\sum_{j \in N} \pi_j x_j \leqslant \pi_0 \,, \qquad (12)$$

*where $\pi_0$ is a positive integer, is satisfied by all $x \in P$, if $N$ can be partitioned into $q + 1$ subsets $N_h$, $h = 0, 1, ..., q$, $1 \leqslant q \leqslant \pi_0$, such that*

($\alpha$) $\quad \pi_j = h \quad$ *for all $j \in N_h$,* $\quad h = 0, 1, ..., q$;

($\beta$) $\quad M = \bigcup_{h=1}^{q} N_h$ *is the extension of some minimal cover $S$ for* (1), *such that $S \subseteq N_1$, and $|S| = \pi_0 + 1$;*

$$(\gamma) \qquad N_0 = N - M, \qquad N_1 = M - \mathbf{U}_{h=2}^{q} N_h \,,$$

$$N_h = \left\{ i \in N\colon \sum_{j \in S_h} a_j \leqslant a_i < \sum_{j \in S_{h+1}} a_j \right\}, \qquad h = 2, ..., q,$$

*where $S_h$ is the set of the first $h$ elements of $S$, $h = 2, ..., q+1$.*

*If, in addition to* ($\alpha$), ($\beta$), ($\gamma$), *one also has*

$$(\delta) \qquad \sum_{j \in S - S_{h+1}} a_j + a_i \leqslant a_0 \quad \text{for all } i \in N_h, \qquad h = 0, 1, ..., q\,,$$

*then* (12) *defines a facet of $P$.*

**Proof.** We first prove that (12) is satisfied by all $x \in P$ if conditions ($\alpha$), ($\beta$) and ($\gamma$) of the theorem hold, by showing that any 0–1 point which violates (12) in the presence of these conditions, also violates (1).

Suppose $\bar{x}$ violates (12). Let $J_k = \{j \in N\colon \bar{x}_j = k\}$, $k = 0, 1$. If

$$\sum_{j \in N_1 \cap J_1} a_j \geqslant \sum_{j \in S} a_j \,,$$

then from (i),

$$\sum_{j \in N_1 \cap J_1} a_j > a_0 \,,$$

i.e., $x$ violates (1). Now assume

$$\sum_{j \in N_1 \cap J_1} a_j < \sum_{j \in S} a_j \,, \tag{13}$$

which in view of ($\beta$) implies $|N_1 \cap J_1| \leqslant \pi_0$, and define the 0–1 vector $\hat{x}$ as follows:

(i) $\hat{x}_j = \bar{x}_j$ unless otherwise specified by (ii), (iii) below.

(ii) $\hat{x}_j = 1 - \bar{x}_j = 1$ for $j \in \overline{N}_1$, where $\overline{N}_1$ is any subset of $N_1 \cap J_0$ with cardinality

$$|\overline{N}_1| = \pi_0 - |N_1 \cap J_1| + 1\,.$$

The existence of such a subset follows from the fact that $S \subseteq N_1$ and $|S| = \pi_0 + 1$.

(iii) $\hat{x}_j = 1 - \bar{x}_j = 0$ for $j \in \widetilde{N}_h \subseteq (N_h \cap J_1)$, $h = 2, ..., q$, where the family of subsets $\overline{N}_h$ satisfies

$$\sum_{h=2}^{q} h\, |\widetilde{N}_h| = |\overline{N}_1|\,.$$

The existence of such a family of subsets follows from

$$\sum_{j\in N} \pi_j \bar{x}_j = \sum_{h=1}^{q} h\, |N_h \cap J_1| \qquad \text{[from } (\alpha)\text{]}$$

$$= \sum_{h=2}^{q} h\, |N_h \cap J_1| + |N_1 \cap J_1|$$

$$> \pi_0 \qquad \text{[since } \bar{x} \text{ violates (12)]}$$

which in turn implies

$$\sum_{h=2}^{q} h\, |N_h \cap J_1| \geqslant \pi_0 - |N_1 \cap J_1| + 1 = |\bar{N}_1| \,.$$

From the definition of $\hat{x}$, $\hat{x}_j = 1$ for exactly $\pi_0 + 1 = |S|$ indices $j \in N_1$. Therefore,

$$\sum_{j\in N} a_j \hat{x}_j \geqslant \sum_{j\in N_1} a_j \hat{x}_j$$

$$\geqslant \sum_{j\in S} a_j \qquad \text{[from the way } S \text{ is specified in } (\beta)\text{]}$$

$$> a_0 \qquad \text{[since } S \text{ is a cover]}\,.$$

This proves that $\hat{x}$ violates (1). We prove that $\bar{x}$ also violates (1), by showing that

$$\sum_{j\in N} a_j \bar{x}_j - \sum_{j\in N} a_j \hat{x}_j \geqslant 0\,. \tag{14}$$

We have

$$\sum_{j\in N} a_j(\bar{x}_j - \hat{x}_j) = \sum_{h=2}^{q} \sum_{j\in \tilde{N}_h} a_j - \sum_{j\in \bar{N}_1} a_j \quad \text{[from the definition of } \hat{x}\text{]}$$

$$= \sum_{h=2}^{q} \sum_{j\in \tilde{N}_h} (a_j - \sum_{i\in R_j} a_i)$$

where the sets $R_j \subseteq \bar{N}_1$ are disjoint and $|R_j| = h$ for all $j \in \tilde{N}_h$, $h = 2, ..., q$. But from (13) and ($\gamma$) we have

$$\sum_{i\in R_j} a_i \leqslant \sum_{i\in S_h} a_i \leqslant a_j \qquad \text{for all } j \in N_h, \qquad h = 2, ..., q\,,$$

for all subsets $R_j \subseteq N_1$ of cardinality $|R_j| = h$. This proves (14).

We have proved that (12) is satisfied by all $x \in P$ if ($\alpha$), ($\beta$) and ($\gamma$) holds. We will now assume that ($\delta$) also holds, and will show that (12) defines a facet of $P$, by exhibiting $n$ linearly independent vertices of $P$ which satisfy the equation

$$\sum_{j \in N} \pi_j x_j = \pi_0 . \tag{12'}$$

Consider the $n \times n$ matrix

$$X = \begin{pmatrix} I_{n_q} & \dots & 0 & 0 & B_q & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & I_{n_2} & 0 & B_2 & 0 \\ 0 & \dots & 0 & I_p & B_1 & 0 \\ 0 & \dots & 0 & 0 & C & 0 \\ 0 & \dots & 0 & 0 & B_0 & I_{n_0} \end{pmatrix},$$

where $I_{n_h}$ is the identity matrix of order $n_h = |N_h|$, $h = 0, ..., q$; $I_p$ is the identity matrix of order $p = n_1 - \pi_0 - 1$; the zeros are zero matrices of compatible dimensions; $B_h$, for $h = 0, 1, ..., q$, is a matrix with $\pi_0 + 1$ columns and identical rows of the form

$$b_h = (\underbrace{0, ..., 0}_{h+1}, \underbrace{1, ..., 1}_{\pi_0 - h}),$$

where the number of rows is $p$ for $h = 1$, $n_h$ for $h = 0$ and $h = 2, ..., q$; and, finally, $C = (c_{ij})$ is the $(\pi_0 + 1) \times (\pi_0 + 1)$ matrix used in the proof of Theorem 1 (where $\pi_0$ was called $k$).

Since $C$ is nonsingular and

$$\det X = \det I_{n_q} \times ... \times \det I_p \times \det C \times \det I_{n_0} \neq 0 ,$$

it follows that the $n$ rows $x^i$ of $X$ are linearly independent. Furthermore,

$$\sum_{j \in N} \pi_j x_j^i = h + (\pi_0 - h) \qquad \text{[from } (\alpha)]$$

$$= \pi_0$$

for the row vectors $x^i$ corresponding to $I_p$ and $I_{n_h}$, $h = 2, ..., q$; and the vectors $x^i$ satisfy (12′) for the remaining rows too, since both $C$ and $B_0$ have exactly $\pi_0$ entries equal to 1 in each row, while $\pi_j = 1$ for all $j \in N_1$, and $\pi_j = 0$ for all $j \in N_0$. Hence (12′) holds for each of the $n$ linearly independent rows $x^i$ of $X$.

Further, we claim that each of the $n$ row vectors $x^i$ satisfies the inequality (1*), obtained from (1) by permuting the indices $j \in N$ so that the first $n_q$ components of $a^*$ are those coefficients $a_j$ of (1) for which $j \in N_q$, the next $n_{q-1}$ components are those $a_j$ for which $j \in N_{q-1}$, etc., and the last $n_0$ components are those $a_j$ with $j \in N_0$; where the sets $N_h$, $h = 0, \ldots, q$, are those defined (uniquely) by ($\gamma$), and the order defined on $N$ is preserved within each set $N_h$.

To show that each $x^i$ satisfies (1*), we notice that for the rows of $X$ corresponding to $I_{n_q}, \ldots, I_{n_2}, I_p$ and $I_0$,

$$a^* x^i = a_{k(i)} + \sum_{j \in S - S_{h+1}} a_j \qquad \text{for some } k(i) \in N_h$$

$$\leqslant a_0 \qquad [\text{from } (\delta)]$$

as one can easily see from the structure of the matrices $B_q, \ldots, B_0$. Also, for the rows $x^i$ corresponding to $C$, (1*) holds since $x^i_j = 1$ for exactly $\pi_0 = |S| - 1$ indices $j \in S$, and $S$ is a minimal cover.

Since each $x^i$ satisfies (1*), each vector $y^i$ obtained from the corresponding $x^i$ by the inverse of the permutation mapping $(a_1, \ldots, a_n)$ into $a^*$, satisfies (1). These $n$ vectors $y^i$ satisfy (12′) and are linearly independent, since so are the vectors $x^i$.

**Remark.** Condition ($\delta$) implies that $S$ is a strong cover.

**Corollary 2.** *Theorem 2 remains true if* ($\gamma$) *is replaced by the* (*weaker*) *condition*

$$(\gamma') \qquad N_0 = N - M, \qquad N_1 = M - \bigcup_{h=2}^{q} N_h\,,$$

$$N_h = \left\{ i \in N\colon \sum_{j \in S_h} a_j \leqslant a_i \right\}, \qquad h = 2, \ldots, q\,.$$

*However, the unique inequality* (12) *defined by* ($\alpha$), ($\beta$) *and* ($\gamma$) *for a given minimal cover S, dominates all inequalities of the form* (12) *whose coefficients satisfy* ($\alpha$), ($\beta$) *and* ($\gamma'$) *for the same cover S.*

**Proof.** Let $N_h$ and $N'_h$ be the sets defined by ($\gamma$) and ($\gamma'$) respectively, and let $\pi_i$, $\pi'_i$ be the corresponding coefficients of (12). From the definitions,

$$i \in N'_h,\ i \notin N_h \Rightarrow i \in N_k$$

for some $k > h$; hence $\pi_i \geqslant \pi'_i$ for all $i \in N$.

Theorem 2 lays the groundwork for generating a family $\mathcal{F}$ of valid cutting planes, i.e., inequalities satisfied by all 0–1 points satisfying (1), most of which are facets of $P$. This family, which has exactly one member for each strong cover of (1), can be generated by the following procedure.

(1) Find a strong cover $S \subseteq N$ not yet considered; i.e., a set $S \subseteq N$ such that

(i) $\quad \sum_{j\in S} a_j > a_0\,,$

(ii) $\quad \sum_{j\in Q} a_j \leqslant a_0 \quad$ for all $Q \subset S, \qquad |Q| < |S|$,

(iii) $\quad$ if $E(S) \neq N, \quad$ then $\sum_{j\in(S-\{j_1\})\cup\{i_1\}} a_j \leqslant a_0\,,$

where

$$a_{j_1} = \max_{j\in S} a_j, \qquad a_{i_1} = \max_{j\in N-E(S)} a_j\,,$$

$$E(S) = S \cup \{j \in N - S\colon a_j \geqslant a_{j_1}\}\,.$$

If there is none, stop: all cuts in the family $\mathcal{F}$ have been generated. Otherwise go to 2.

(2) Let $\pi_0 = |S| - 1$, and define the coefficients $\pi_j$ and the index sets $N_h$ by ($\alpha$) and ($\gamma$) respectively, i.e., let

$$\pi_j = h \quad \text{for all } j \in N_h, \qquad h = 0, 1, ..., q\,;$$

$$N_0 = N - E(S), \qquad N_1 = E(S) - \textstyle\bigcup_{h=2}^{q} N_h\,,$$

$$N_h = \left\{ i \in N\colon \sum_{j\in S_h} a_j \leqslant a_i < \sum_{j\in S_{h+1}} a_j \right\}, \qquad h = 2, ..., q,$$

where $S_h$ is the set of the first $h$ elements of $S$, for $h = 2, ..., q+1$.

Then the inequality

$$\sum_{j\in N} \pi_j x_j \leqslant \pi_0 \tag{12}$$

is a valid cut (i.e., is satisfied by all $x \in \operatorname{vert} P$); and if

$$\sum_{j\in S-S_{h+1}} a_j + a_i \leqslant a_0$$

holds for all $i \in N_h$, $h = 1, ..., q$, then (12) defines a facet of $P$.

Go to 1.

**Example 3.** Let $P$ be the convex hull of 0–1 points satisfying

$$9x_1 + 7x_2 + 6x_3 + 4x_4 + 3x_5 + 3x_6 + 2x_7 + 2x_8 + x_9 \leqslant 10 .$$

Table 1 lists the family of cutting planes characterized by Theorem 2, along with the strong covers from which they are generated. Of the 26 members of the family, 20 are facets of $P$.

The table is to be read as follows. Take line 1: $S = \{2, 3\}$ is a cover, since $a_2 + a_3 = 7 + 6 > a_0 = 10$; a minimal cover, since $6 \leqslant 10$; a strong cover, since $a_3 + a_4 = 6 + 4 \leqslant 10$.

The extension of $S$ is $E(S) = \{1, 2, 3\}$, and $N_0 = \{4, 5, 6, 7, 8, 9\}$; Further, $N_h = 0$, $h > 1$, and $N_1 = E(S) = \{1, 2, 3\}$. $\pi_j = 0, j \in N_0$, $\pi_j = 1$, $j \in N_1$; also, $\pi_0 = |S| - 1 = 1$.

Table 1

| Strong cover $S$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ | $\pi_7$ | $\pi_8$ | $\pi_9$ | $\pi_0$ | Facet (–), or condition ($\delta$) violated |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2, 3 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | – |
| 2, 4 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | – |
| 1, 5 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | – |
| 1, 6 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | – |
| 1, 7 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | – |
| 1, 8 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | – |
| 3, 4, 5 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 2 | $3+9 \nleqslant 10$ |
| 3, 4, 6 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 2 | $3+9 \nleqslant 10$ |
| 3, 5, 6 | 2 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 2 | – |
| 3, 4, 7 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 2 | $2+9 \nleqslant 10$ |
| 3, 5, 7 | 2 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 2 | – |
| 3, 6, 7 | 2 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 2 | – |
| 3, 4, 8 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 2 | $2+9 \nleqslant 10$ |
| 3, 5, 8 | 2 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 2 | – |
| 3, 6, 8 | 2 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 2 | – |
| 2, 7, 8 | 2 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 2 | – |
| 3, 4, 9 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 2 | – |
| 2, 5, 9 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 2 | – |
| 2, 6, 9 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 2 | – |
| 4, 5, 6, 7 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 3 | $3+2+6 \nleqslant 10$ |
| 4, 5, 6, 8 | 2 | 2 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 3 | $3+2+6 \nleqslant 10$ |
| 4, 5, 7, 8 | 3 | 2 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 3 | – |
| 4, 6, 7, 8 | 3 | 2 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 3 | – |
| 4, 5, 6, 9 | 2 | 2 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 3 | – |
| 3, 7, 8, 9 | 2 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 3 | – |
| 5, 6, 7, 8, 9 | 3 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 4 | – |

Hence the first cut in the family $\mathcal{F}$ is

$$x_1 + x_2 + x_3 \leqslant 1 ,$$

which defines a facet of $P$, since $S - S_2 = \emptyset$, and $a_1 = 9 \leqslant a_0 = 10$.

Now take, for instance, line 5 from the bottom. $S = \{4, 5, 6, 8\}$ is a cover, since $4 + 3 + 3 + 2 > 10$; a minimal cover, since $(4 + 3 + 3 + 2) - 2 \leqslant 10$; a strong cover, since $3 + 3 + 2 + 2 \leqslant 10$.

Further, $E(S) = \{1, 2, 3, 4, 5, 6, 8\}$, and $N_0 = \{7, 9\}$. $N_2 = \{1, 2\}$, since $a_4 + a_5 \leqslant a_i < a_4 + a_5 + a_6$ for $i = 1, 2$, but $a_3 < a_4 + a_5$.

$N_h = 0, h > 2; N_1 = E(S) - N_2 = \{3, 4, 5, 6, 8\}$. $\pi_j = h, j \in N_h$, $h = 0, 1, 2$, $\pi_0 = |S| - 1 = 3$, and the cut associated with $S$ is

$$2x_1 + 2x_2 + x_3 + x_4 + x_5 + x_6 + x_8 \leqslant 3 ,$$

which does not define a facet of $P$, since

$$a_3 + a_6 + a_8 = 6 + 3 + 2 \not\leqslant a_0 = 10 .$$

Finally, the last cut of the family is generated from the strong cover $S = \{5, 6, 7, 8, 9\}$; with $E(S) = N$, $N_0 = 0$, $N_2 = \{2, 3\}$; $N_3 = \{1\}$; $N_1 = \{4, 5, 6, 7, 8, 9\}$; and the cut itself is

$$3x_1 + 2x_2 + 2x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 \leqslant 4 .$$

This inequality again defines a facet of $P$, as one can easily check.

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