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The arising of cooperation in Cournot duopoly games

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ABSTRACT

In literature, several studies have been made to study conflict among duopolistic Cournot game with incomplete information. This paper attempts to investigate equilibrium stability of a nonlinear Cournot duopoly game on which cooperation may be existed among firms. Discrete time scales under the assumption of logarithmic price function and linear cost are used to build our models in the proposed games. We study here an adjustment dynamic strategy based on the so-called tit-for-tat strategy. For each model, the stability analysis of the fixed point is analyzed. Numerical simulations are carried out to show the complex behavior of the proposed models and to point out the impact of the models' parameters on the cooperation.

1. Introduction

Game theory is characterized by its ability to consider interactions among firms. It is one of the most important theories that is used to describe and study such competition among competitors statically and dynamically. The dynamic case in which the equilibrium point (Nash equilibrium) is sought and its complex dynamic characteristics are of main interest have been studied in literature [1–14]. In such games there are often several duopolistic firms in economic market where competition among them is controlled by the amount of commodities they produce, the demand scheme they adopt and the profit each firm wants to maximize. In competition, firms produce the same or homogenous goods and they must focus not only on the market size, but also on the actions their competitors do.

The Cournot duopolistic game has been studied intensively in literature. Agiza and Elsadany [14] have modeled a Cournot duopolistic game on which one of the competitors is heterogeneous. They have studied the proposed game in details and particularly when the game's fixed point becomes unstable due to bifurcation occurred. In [28], a heterogeneous duopoly game with rational and adaptive competitors was examined. The authors have studied the impact of quadratic cost function on the complex behavior of the game and came up with the conclusion that introducing nonlinearity in cost generalizes the implications of heterogeneous Cournot duopoly with adjusting strategies. Other interesting works related to Cournot duopolistic games can be in literature [29–31].

In this paper, we focus and study the cooperation between firms in repeated Cournot duopoly games with a logarithmic price function. In Cournot duopoly games, Nash equilibrium or Cournot equilibrium is the basic solution in such games and reflects the rationality of the firms within games. Since, firm rationality contradicts with Pareto optimality (in cooperation case), then Nash equilibrium in duopoly game is not Pareto optimal. In other words, Pareto optimality in such games cannot be achieved by firm interest's maximization. As reported in [15,16], theoretical and experimental studies have leaded up to several ways by

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which the cooperative solution can be obtained. For instance, in the well-known short game of prisoner's dilemma, the Nash equilibrium point is Pareto optimal as cooperation is obtainable. But for the repeated games, emergence of cooperation among competitors (firms) may be possible to achieve and then cooperation in iterated prisoner's dilemma can be explained [15,17]. In [18], it has been shown that the conditional cooperative strategy such as the so-called "tit-for-tat" may be used to achieve cooperation among firms in repeated games.

The emergence of cooperation has attracted much of interest for a long time and it would look like even pleonastic to report some of the recent and important papers in this field. In [19], setups based on discrete, continuous and mixed strategy have been proposed in the social dilemma games and their performance on networks populations has been shown. A useful source of information on the evolutionary games on multilayer networks and particularly in the evolution of cooperation is reported in details in [20]. An evolutionary dictator game model is introduced in [21] by which the evolution of altruism and fairness of populations has been studied. In this study, the influence of assignation on heterogeneous populations has been investigated. An important review of the universality of scaling for the dilemma strength in evolutionary games has been reported in [22]. The review has shown that social viscosity or spatial structure causes the existing scaling parameters to fail. In addition to the review has developed new parameters to resolve the paradox of cooperation benefits. Two-layer scale-free networks has been introduced in [23] to show evolution of cooperation. In [24], the authors have demonstrated that the influence of simple strategy-independent form may expand the scope of cooperation on structured populations. For more related works, readers are advised to have a look on some important papers [25,26] and a more informational report [27].

The current paper is motivated by the work done by Ding and Shi [15]. We introduce a duopoly game based on a logarithmic price function. An adjustment dynamic strategy is introduced and studied to detect the cooperation condition that may be occurred based on this strategy. Under the proposed function, the tit-for-tat is used and a two dimension discrete map is introduced. The complex dynamic characteristics of this map are studied and the stability of the Nash equilibrium is investigated. The qualitative study of bifurcation is studied analytically and numerically. We conclude our study with a tit-for-tat strategy with team profit.

The structure of the paper is as follows: In Section 2 a description of a Cournot duopoly game based on a generalized inverse demand function is presented. In Section 3, the two-dimensional map whose iteration gives the time evolution based on a proposed dynamic adjustment is defined. The steady state point of the map, which is Nash equilibrium, is computed. Then the stability of this point is investigated and its complex dynamics is detected. Section 4 introduces a tit-for-tat Cournot duopoly game using the same function. As in Section 3, the dynamic characteristics of the game are investigated. In Section 5, the system studied in Section 4 is improved by adding a control strategy in this system and some discussion is illustrated. Finally, we end the paper with some conclusions to show the significance of our results.

2. Cooperative duopoly model

Suppose a market with two firms producing the same product or homogenous product. The decisions in this market are the quantities both firms sell in the market and are taken at discrete time scale, $t = 0, 1, 2, \ldots$. The produced quantity by each firm at time t is denoted by $q_{i,t}$ $(i = 1, 2)$. We assume that the cost of production is linear, $C_{i,t} = c_i q_{i,t}$, where $c \geq 0$ is a marginal cost. Further we assume an inverse demand function as follows:

$$p_t = a - b \log Q \tag{1}$$

It is well-known that p indicates commodity price, while $Q = \sum_{i=1}^{2} q_i$ is the total quantity produced by the two firms. In addition, a and b are positive constants. Non-negative price is obtained if $Q < e^{\frac{a}{b}}$. One can see that the direct demand function is exponential and therefore it is a log-concave function in the price. Log-concavity of function is achieved if and only if its logarithm is concave. Many standard demand functions are log-concave, including those that are strictly concave such as linear and exponential functions. A log-concave function means that it is no longer convex than an exponential function [6]. The profit of each firm is now given by:

$$\pi_{i,t} = [a - c - b \log Q] q_{i,t} \tag{2}$$

In [6], the case of non-cooperative duopolistic game based on incomplete information and the price function given in (1) has been studied. In the current paper, we study the cooperation situation under the incomplete information. We assume that both firms' profits are used in the cooperative profit that is denoted by π_c[1]. The first-order derivative of the total profits gives the cooperative output $q_c = \frac{1}{2} e^{\left(\frac{a-b-c}{b}\right)}$ and then $\pi_c = \frac{b}{2} e^{\left(\frac{a-b-c}{b}\right)}$.

3. Dynamic tit-for-tat behavior

For achieving the cooperation between the two firms, the tit-for-tat strategy is used. With this strategy, every firm is doing what its opponent has done in the previous move. This is an incomplete information scenario however the only things each firm knows are the output and the profit. In this situation each firm compares its profit $\pi_{i,t}$ with the cooperative profit π_c that is Pareto optimal. If $\pi_{i,t} > \pi_c$ this means that each firm will probably reduce its output to keep the cooperation between each

[1] Cooperative profit is a maximization problem that is given by: $\max \pi = \pi_1 + \pi_2$.

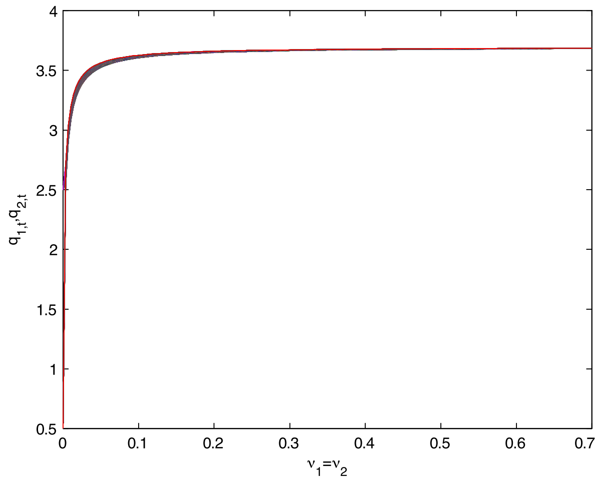

Fig. 1. Behavior of $q_{1,\,t}$ and $q_{2,\,t}$ at the parameters: $a = 5, c = 2, b = 1, q_{0,1} = 2.5, q_{0,2} = 0.5$.

other. On the other hand, cooperation cannot be realized if $\pi_{i,\,t} < \pi_c$ because that indicates one of the firm is non-cooperative. Based on these thoughts, the following dynamic map is built:

$$q_{i,t+1} = q_{i,t} + \nu_i(\pi_c - \pi_{i,t}) \tag{4}$$

where $\nu_i > 0, i = 1, 2$ is an adjusting parameter. This map can be rewritten in the following form:

$$\begin{aligned} q_{1,t+1} &= q_{1,t} + \nu_1[\pi_c - (a - c - b\log Q)q_{1,t}] \\ q_{2,t+1} &= q_{2,t} + \nu_2[\pi_c - (a - c - b\log Q)q_{2,t}] \end{aligned} \tag{5}$$

The system (5) has a unique positive fixed point which is $q_1 = q_2 = q_c$. The Jacobian at this point is:

$$J(q_c, q_c) = \begin{pmatrix} 1 - \dfrac{b}{2}\nu_1 & \dfrac{b}{2}\nu_1 \\ \dfrac{b}{2}\nu_2 & 1 - \dfrac{b}{2}\nu_2 \end{pmatrix}$$

whose eigenvalues are:

$$\begin{aligned} \lambda_1 &= 1, \\ \lambda_2 &= 1 - \frac{b}{2}(\nu_1 + \nu_2) \end{aligned}$$

where $|\lambda_2| < 1$ if $0 < (\nu_1 + \nu_2) < \frac{4}{b}$ but $\lambda_1 = 1$ is a critical condition by which we cannot detect whether the system (5) is stable or not. Therefore, we perform some numerical simulation to investigate the characteristics of this system. For the system (5), we take $a = 5, c = 2, b = 1, q_{0,1} = 2.5, q_{0,2} = 0.5$. With these values, Fig. 1 shows that the equilibrium point of the system (5) becomes asymptotically stable. It is observed that the outputs of both firms are stable and Pareto optimality may be achieved. On the other hand, with parameters $a = 5, c = 2, b = 4, q_{0,1} = 2.5, q_{0,2} = 0.5$, Fig. 2 shows that if $b > c$ the equilibrium point becomes unstable and Pareto optimality cannot be achieved. In addition, if $c > b$ and $c > a$ the equilibrium point becomes unstable and Pareto optimality cannot be achieved and this is shown in Fig. 3.

4. Dynamic tit-for-tat with control

A feedback control is added in the map (4) to improve it. This gives the following new map:

$$q_{i,t+1} = q_{i,t} + \nu_i(\pi_c - \pi_{i,t}) - \nu(q_{i,t} - q_c), i = 1, 2 \tag{6}$$

where $\nu_i > 0, i = 1, 2$ is an adjusting parameter, and $\nu > 0$ and $-\nu(q_{i,t} - q_c), i = 1, 2$ is the feedback control of the system. This feedback control shows that whether the firms cooperate or not. It is used to show that if $q_{i,\,t} > q_c$ $(i = 1, 2)$ the firm is going to reduce its output in the next period of time since, the current is over the cooperative output. On the other hand, if $q_{i,\,t} < q_c$ $(i = 1, 2)$ the firm has to properly increase its output in the next period of time. Because of this feedback control, both firms has the intention to cooperate. The system (6) is an improvement of system (4). It shows that firms are willing to have a tit-for-tat thoughts and a cooperative intention at the same time. This cooperation can be last by all players for a long time. Indeed, the proposed system (6) depends on incomplete information about the players and we can say that it gives a plausible way by which

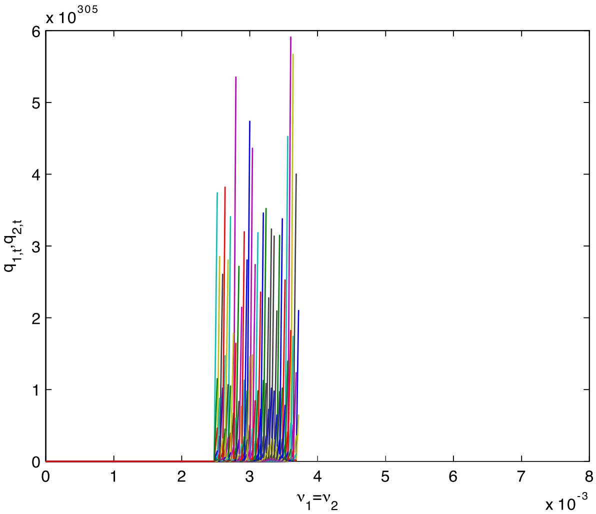

Fig. 2. Behavior of $q_{1,t}$ and $q_{2,t}$ at the parameters: $a = 5, c = 2, b = 4, q_{0,1} = 2.5, q_{0,2} = 0.5$.

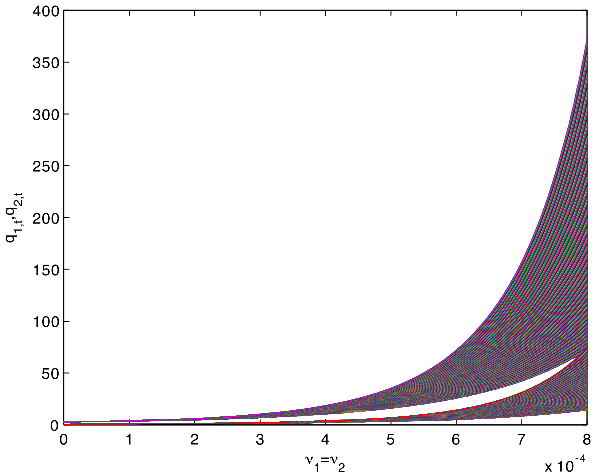

Fig. 3. Behavior of $q_{1,t}$ and $q_{2,t}$ at the parameters: $a = 5, c = 8, b = 1, q_{0,1} = 2.5, q_{0,2} = 0.5$.

firms may be able to achieve Pareto optimal cooperation in conflicting economic markets. Now, the system (6) can be rewritten in the form:

$$\begin{aligned} q_{1,t+1} &= q_{1,t} + \nu_1[\pi_c - (a - c - b\log Q)q_{1,t}] - \nu(q_{1,t} - q_{c)} \\ q_{2,t+1} &= q_{2,t} + \nu_2[\pi_c - (a - c - b\log Q)q_{2,t}] - \nu(q_{2,t} - q_{c)} \end{aligned} \tag{7}$$

The system (7) has a unique positive fixed point which is $q_1 = q_2 = q_c$. The Jacobian at this point is:

$$J(q_c, q_c) = \begin{pmatrix} 1 - \nu - \frac{b}{2}\nu_1 & \frac{b}{2}\nu_1 \\ \frac{b}{2}\nu_2 & 1 - \nu - \frac{b}{2}\nu_2 \end{pmatrix}$$

whose eigenvalues are:

$$\lambda_1 = 1 - \nu,$$
$$\lambda_2 = 1 - \nu - \frac{b}{2}(\nu_1 + \nu_2)$$

where $|\lambda_1| < 1$ if $0 < \nu < 2$ and $|\lambda_2| < 1$ if $0 < \nu + \frac{b}{2}(\nu_1 + \nu_2) < 2$ and therefore the fixed point is locally asymptotically stable except at $\nu = 1$, the fixed point is stable. When $\nu = 1$ this means that $\lambda_1 = 0$ and $\lambda_2 = -\frac{b}{2}(\nu_1 + \nu_2) < 0$ that lead to $Tr(J) = \lambda_1 + \lambda_2 = -\frac{b}{2}(\nu_1 + \nu_2) < 0$, $Det(J) = \lambda_1\lambda_2 = 0$, $\Delta = (Tr(J))^2 - 4Det(J) > 0$ and therefore the fixed point is stable. We perform some numerical simulation to investigate the characteristics of this system.

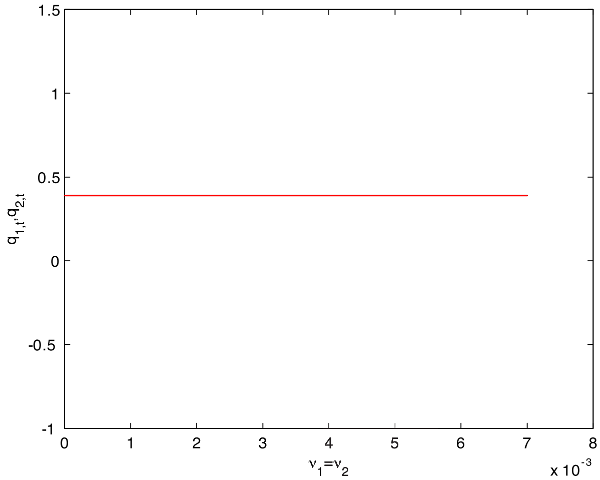

Fig. 4. Behavior of $q_{1,\,t}$ and $q_{2,\,t}$ at the parameters: $a = 5, c = 2, b = 4, q_{0,1} = 2.5, q_{0,2} = 0.5$ and $\nu = 0.6$.

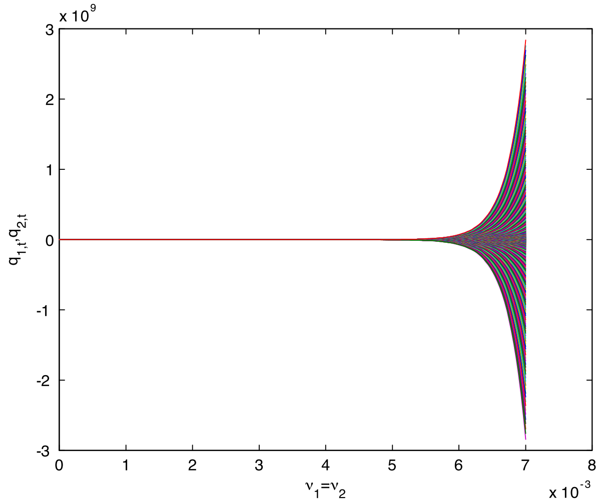

Fig. 5. Behavior of $q_{1,\,t}$ and $q_{2,\,t}$ at the parameters: $a = 5, c = 2, b = 4, q_{0,1} = 2.5, q_{0,2} = 0.5$ and $\nu = 2$.

We reconsider the unstable situation ($a = 5, c = 2, b = 4, q_{0,1} = 2.5, q_{0,2} = 0.5$ and $\nu = 0.6$) in the previous section. Let $\nu = 0.6$, the equilibrium point is asymptotically stable, as shown in Fig. 4. In addition, the numerical simulation has shown that as long as $\nu \in (0, 2)$ with the other conditions of the eigenvalues hold, the equilibrium point will be stable and the Pareto optimality holds. Fig. 5 shows that Pareto optimality cannot be achieved when $\nu = 2$ while Fig. 6 shows the stability of the fixed point when $\lambda_1 = 0$ and $\lambda_2 = -\frac{b}{2}(\nu_1 + \nu_2)$.

5. Dynamic tit-for-tat team profit

Here, we assume that both firms construct a team with a team profit given in the following form:

$$\Pi_{team} = \omega[(a - c - b\log Q)q_{1,t}] + (1 - \omega)[(a - c - b\log Q)q_{2,t}] \tag{8}$$

where, ω is the weight parameter of the firm's profit. The first-order derivative of (8) gives the cooperative output $q_c^* = \frac{1}{2}e^{\left(\frac{2\omega(a-c)-b}{2\omega b}\right)}$ and then $\Pi_c = \frac{b}{4\omega}e^{\left(\frac{2\omega(a-c)-b}{2\omega b}\right)}$. Similar to the discussion in Section 2, the following system is obtained.

$$\begin{aligned} q_{1,t+1} &= q_{1,t} + \nu_1[\Pi_c - (a - c - b\log Q)q_{1,t}] \\ q_{2,t+1} &= q_{2,t} + \nu_2[\Pi_c - (a - c - b\log Q)q_{2,t}] \end{aligned} \tag{9}$$

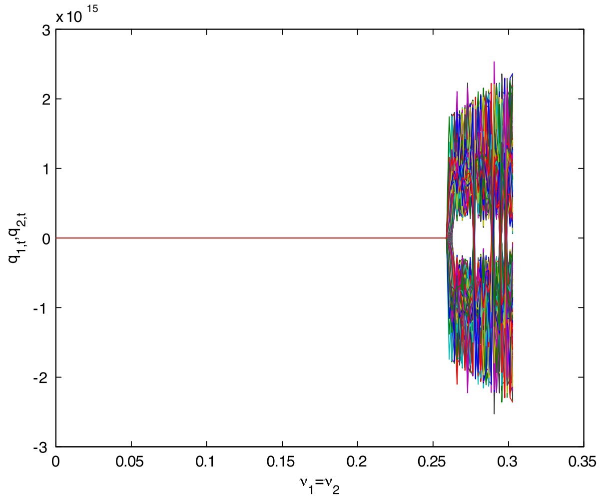

Fig. 6. Behavior of $q_{1,t}$ and $q_{2,t}$ at the parameters: $a = 5, c = 2, b = 4, q_{0,1} = 2.5, q_{0,2} = 0.5$ and $\nu = 1$.

The system (9) has a unique positive fixed point which is $q_1 = q_2 = q_c^*$. The Jacobian at this point is:

$$J(q_c, q_c) = \begin{pmatrix} 1 - \dfrac{b\nu_1(1-\omega)}{2\omega} & \dfrac{b}{2}\nu_1 \\ \dfrac{b}{2}\nu_2 & 1 - \dfrac{b\nu_2(1-\omega)}{2\omega} \end{pmatrix}$$

whose trace and determinant are respectively:

$$\begin{aligned} Tr &= 2 - \frac{b(1-\omega)(\nu_1+\nu_2)}{2\omega} \\ Det &= 1 - \frac{1}{4\omega^2}\left[2\omega b(1-\omega)(\nu_1+\nu_2) - b^2\nu_1\nu_2(1-2\omega)\right] \end{aligned} \tag{10}$$

To discuss the stability of the fixed point, the following conditions which are known with Schur's conditions are used:

$$\begin{aligned} &1 + Tr + Det > 0, \\ &1 - Tr + Det > 0, \\ &1 - Det > 0 \end{aligned} \tag{11}$$

In general, the violation of any single inequality in (11), with the other two inequalities being simultaneously fulfilled leads to a bifurcation. It leads to a flip bifurcation (real eigenvalue that passes through -1) once $1 + Tr + Det = 0$, while $1 - Tr + Det = 0$

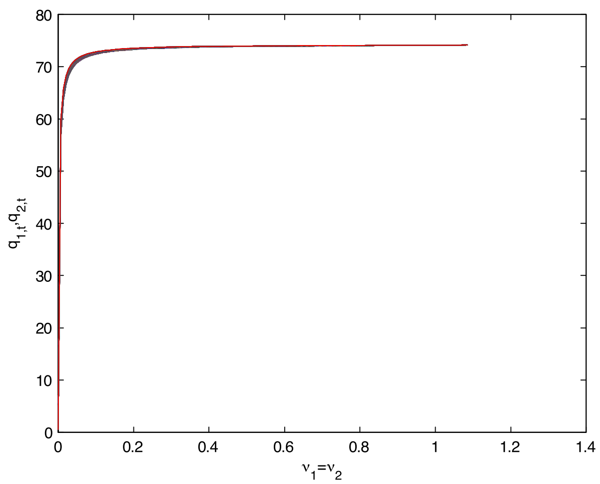

Fig. 7. Behavior of $q_{1,t}$ and $q_{2,t}$ at the parameters: $a = 8, c = 2, b = 1, q_{0,1} = 2.5, q_{0,2} = 0.5$ and $\omega = 0.5$.

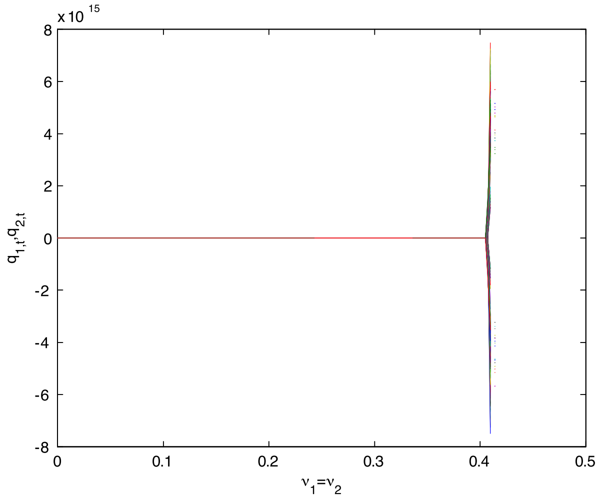

Fig. 8. Behavior of $q_{1,\,t}$ and $q_{2,\,t}$ at the parameters: $a = 8, c = 2, b = 1, q_{0,1} = 2.5, q_{0,2} = 0.5$ and $\omega = 0.1$.

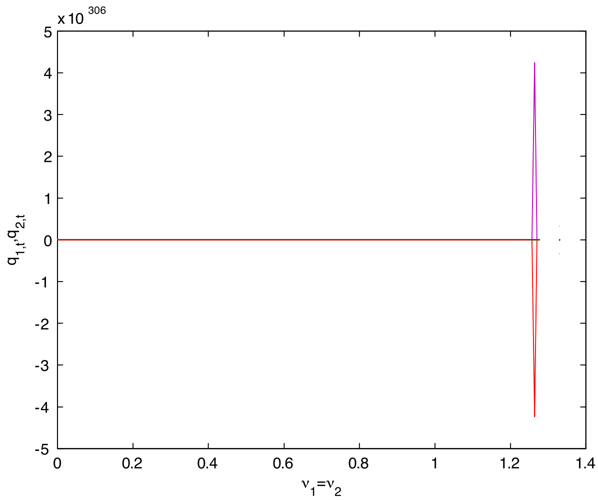

Fig. 9. Behavior of $q_{1,\,t}$ and $q_{2,\,t}$ at the parameters: $a = 8, c = 2, b = 1, q_{0,1} = 2.5, q_{0,2} = 0.5$ and $\omega = 0.9$.

leads to a fold or transcritical bifurcation (real eigenvalue that passes through +1). In the case of $Det = 1$, a Neimark–Sacker bifurcation (the modulus of complex eigenvalue that passes through +1) is found. Using (10), the Schur's stability conditions can be rewritten in the following form:

$$\begin{aligned}
&4 - b(1-\omega)(\nu_1+\nu_2) + \frac{b^2\nu_1\nu_2(1-2\omega)}{4\omega} > 0,\\
&b^2\nu_1\nu_2(1-2\omega) > 0,\\
&2\omega(1-\omega)(\nu_1+\nu_2) - b\nu_1\nu_2(1-2\omega) > 0
\end{aligned} \tag{12}$$

From (12), it is not clear whether all the conditions are fulfilled or not. For example, if we take $a = 8, b = 1, c = 2, \nu_1 = \nu_2 = 1.1$ and $\omega = 0.5$ all the conditions are satisfied and fixed point is stable. This is illustrated in Fig. 7. Using the same parameters values but for values $\omega < 0.5$ the third condition gets violated and the other two conditions are fulfilled. Therefore, the fixed point can lose stability through neither a Neimark–Sacker bifurcation. This can be seen in Fig. 8. Yet if we take $\omega > 0.5$, with the same values of parameters, the second condition gets violated and transcritical bifurcation occurred. This is shown in Fig. 9.

6. Conclusion

In this paper, we have studied the cooperation that may be obtained among duopolistic firms in the economic market. Based on a logarithmic price function, three duopolistic Cournot models have been investigated. For each model, the fixed point has been computed and complete analytical and numerical studies of the stability conditions for the fixed point have been obtained.

The analysis has shown that under the dynamic adjustment strategy and the tit-for-tat strategy, the cooperation may be achieved, but the stability in those systems is sensitive to the parameters, and the Pareto optimality can be achieved, and is stable within the parameters' certain field. So the cooperation can be the result of such a strategy under certain condition.

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