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Complex dynamics analysis for a duopoly Stackelberg game model with bounded rationality

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ABSTRACT

In view of the effect of differences between plan products and actual products, a duopoly Stackelberg model of competition on output is formulated. The firms announce plan products sequentially in planning phase and act simultaneously in production phase. Backward induction is used to solve subgame Nash equilibrium. The equilibrium outputs and equilibrium profits are affected by cost coefficients. For the duopoly Stackelberg model, a nonlinear dynamical system which describes the time evolution with bounded rationality is analyzed. The equilibria of the corresponding discrete dynamical systems are investigated. The local stability analysis has been carried out. The stability of Nash equilibrium gives rise to complex dynamics as some parameters of the model are varied. Numerical simulations were used to show bifurcation diagram, stability region and chaos. It is also shown that the state variables feedback and parameter variation method can be used to keep the system from instability and chaos.

1. Introduction

An oligopoly is a market form between monopoly and perfect competition, in which a market has a dominant influence on a small number of firms (oligopolists) [1]. The dynamic of an oligopoly game is more complex because firms must consider not only the behaviors of the consumers, but also the reactions of the competitors. The first formal theory of oligopoly was introduced by Cournot, in 1838 [2]. Significant additions to the theory were made exactly one hundred years later by H. von Stackelberg [3]. In the repeated oligopoly game all players maximize their profits. Recently, the dynamics of the duopoly game has been studied [4–16]. The general formula of the oligopoly model with a form of bounded rationality has been investigated [6]. The results show that the dynamics of the game can lead to complex behaviors such as cycles and chaos. The complex dynamics of a bounded rationality duopoly game with a nonlinear demand function has been studied [10]. Depending on the strategy that the firms use and the expectations of the output the firms have to maximize, the modification of the duopoly game has been discussed [11,12]. With bounded rationality, Ref. [13] examined the dynamical behavior of Bowley's model. Furthermore, Ref. [14] used the Jury condition to discuss the stability of a modification of Puu's model. The development of complex oligopoly dynamics theory has been reviewed in Ref. [16]. Other studies on the dynamics of oligopoly models with more firms and other modifications have been studied in Ref. [17–21]. Based on bounded rationality, a linear dynamic system for the duopoly game of renewable resource extraction was proposed [22]. Also in the past decade, there has been a great deal of interest in chaos control of duopoly games because of its complexity in Ref. [23] and Ref. [24] and their references.

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Most of the previous works are based on the Cournot model and the modifications to discuss the complex dynamics. There is little literature dealing with Stackelberg model and the modifications in studying dynamical behaviors with bounded rationality. The Stackelberg games are natural models for many important applications that involve human interaction. Existing algorithms for Stackelberg games find optimal solutions (leader strategy) efficiently, but they critically assume that the follower plays optimally. Unfortunately, in many applications, players facing human followers (adversaries) may deviate from their expected responses to the game theoretic optimal choice, because of their bounded rationality and limited observation [25]. Thus, a human adversary may cause an unacceptable degradation in the leader's reward [26].

Expectation plays an important role in modelling economic phenomena. A firm can choose its expectation rules of many available techniques to adjust its strategy. The present work aims to formulate a duopoly Stackelberg game with bounded rationality and study the dynamical behaviors. The leader firm chooses strategic variable first, then the follower firm chooses strategic variable. In the subsequent stages, two firms update their strategies in order to maximize their profits in the market. Each firm adjusts its strategy according to the expected marginal profit, therefore the decision of each firm depends on local information about its output. In this Stackelberg game each firm tries to maximize its profit according to local information of its strategy.

This paper is organized as follows. In Section 2, the Stackelberg game and the duopoly game with bounded rationality are briefly described. Some properties about the equilibrium output and equilibrium profit are investigated. The dynamics for a duopoly Stackelberg game model with bounded rationality are analyzed. The local stability analysis has been carried out. In Section 3, we present the numerical simulations to verify our theoretical results. In Section 4, we exerted control on the duopoly Stackelberg game model. Finally, some remarks are presented in Section 5.

2. The model

2.1. The Stackelberg game and duopoly Stackelberg model

The original Stackelberg model is a sequential quantity choice game in a homogeneous product market. In Stackelberg games, one player, the leader, commits to a strategy publicly before the remaining players, the followers, make their decisions. The followers selfishly optimize their rewards, considering the action chosen by the leader. The leader knows ex ante that the followers observe his action. The decision and actual performance of the followers will influence the cost and benefit of the leader, so the followers must make decision to predict the decision of the leader, that is to say, the decision of the leader and followers is influenced mutually. A Stackelberg equilibrium is a subgame perfect equilibrium of the above game.

The classic Stackelberg game is divided into two stages. In stage 1, the*planning phase*, each player chooses strategies, and concludes forward contracts for output. In stage 2, the *production phase*, they choose the quantities to be produced. The players act sequentially in planning phase, act simultaneously in production phase, and the choices made in stage 1 are common knowledge in stage 2. There are no costs of production besides the costs of capacity. The forward sales are priced competitively in that the eventual resulting market price is anticipated correctly in equilibrium.

We consider two firms, labelled by $i = 1, 2,$ producing the same goods for sale in the market. Firm 1 is the Stackelberg leader and firm 2 is the follower. Production decisions of both firms are made at discrete periods $t = 0, 1, 2, \ldots$. Let $q_i(t) > 0$ represents the output of firm i during period t, with a production cost function $C_i(q_i)$. The price prevailing in period t is determined by the total supply $q(t) = q_1(t) + q_2(t)$ through a demand function $p = f(q)$. In this model the demand function is assumed linear, which has the form $f(q) = a - bq,$ where a and b are positive constants.

Let Q_i be the announced plan products of the firm $i, i = 1, 2$ respectively. There is a difference between the announced product and the actual output of firm i during period t. In this work we assume that the firms use different production method and the cost function is proposed in the nonlinear form

$$C_i(q_i) = c_i(q_i - Q_i)^2, \quad i = 1, 2$$

where the parameters c_i are positive shift parameters to the cost function of the firm i, $i = 1, 2$ respectively. In fact, there are many factors affecting the difference between ideal product and actual product. For the sake of analysis, all of these factors are summarized as the cost coefficient c_i. With these assumptions, the single-period profit of the firm i is given by

$$\Pi_i(q_1, q_2) = q_i(a - bq) - c_i(q_i - Q_i)^2, \quad i = 1, 2$$

The empirical estimate of marginal profit for the firm i at the point (q_1, q_2) is given by

$$\frac{\partial \Pi_i}{\partial q_i} = a + 2c_iQ_i - 2(b + c_i)q_i - bq_j, \quad i = 1, 2, \ i \neq j \tag{1}$$

In order to maximize profit for the firm i, let the partial derivative of Π_i respect to q_i equal to zero.

$$\frac{\partial \Pi_i}{\partial q_i} = a + 2c_iQ_i - 2(b + c_i)q_i - bq_j = 0, \quad i = 1, 2, \ i \neq j$$

From the equation group, we obtain the equilibrium solution for the firms.

$$\begin{cases} q_1 = \dfrac{a(b+2c_2)+4(b+c_2)c_1Q_1-2bc_2Q_2}{3b^2+4b(c_1+c_2)+4c_1c_2} \\ q_2 = \dfrac{a(b+2c_1)+4(b+c_1)c_2Q_2-2bc_1Q_1}{3b^2+4b(c_1+c_2)+4c_1c_2} \end{cases} \tag{2}$$

The planning products $Q_i, i=1,2$ can be solved to find the subgame perfect Nash equilibrium(SPNE). The model is solved by backward induction. The leader considers what the best response of the follower is, i.e. how it will respond once it has observed the quantity of the leader. Then the leader picks a quantity that maximizes its payoff, and anticipates the predicted response of the follower. The follower actually observes this and picks the expected quantity as a response in equilibrium.

To calculate the SPNE, the best response function of the follower must first be calculated. The firm 1 (the leader) announces the plan production Q_1 first, and then the firm 2 (the follower) selfishly chooses the plan output Q_2 for the biggest profit, considering the action chosen by the leader. Take Eq. (2) into Π_2, the profit of firm 2. Hence, the maximum of Π_2 with respect to Q_2 will be found. We calculate a derivative of Π_2 with respect to Q_2 and set it to zero for the optimal action of firm 2. This optimization problem has unique solution in the form

$$Q_2 = \frac{4(b+c_1)(b+c_2)[a(b+2c_1)-2bc_1Q_1)]}{b[b(3b+4c_1)^2+8(b+c_1)(b+2c_1)c_2]} \tag{3}$$

The leader will anticipate the Q_2, predicted response of the follower. Take Eqs. (2) and (3) into Π_1, differentiate Π_1 with respect to Q_1. Equating the partial derivative to zero, we can obtain the optimal action of firm 1 in the form

$$Q_1 = \frac{4a(b+c_1)(\Delta_1-bc_2)(\Delta_1-2bc_2)}{b\Delta} \tag{4}$$

where $\Delta_1 = 3b^2+4b(c_1+c_2)+4c_1c_2$, and $\Delta = (9b+8c_1)\Delta_1^2-24b(b+c_1)c_2\Delta_1+16b^2(b+c_1)c_2^2$. Take Eq. (4) into Eq. (3) and the simplified form is

$$Q_2 = \frac{a(b^2+\Delta_1)[(3b+2c_1)\Delta_1-4b(b+c_1)c_2]}{b\Delta} \tag{5}$$

The planning products $Q_i, i=1,2$ have been found out. They are functions of the parameters c_1 and c_2. Substituting Eqs. (4) and (5) into Eq. (2), we obtain the equilibrium solution for the firms in the Stackelberg game.

$$\begin{cases} q_1^* = \dfrac{a[(3b+4c_1)\Delta_1-4b(b+c_1)c_2](\Delta_1-2bc_2)}{b\Delta} \\ q_2^* = \dfrac{a[(3b+2c_1)\Delta_1-4b(b+c_1)c_2]\Delta_1}{b\Delta} \end{cases} \tag{6}$$

Because of

$$Q_1 - q_1^* = \frac{a\Delta_1(\Delta_1-2bc_2)}{\Delta}$$

$$Q_2 - q_2^* = \frac{ab[(3b+2c_1)\Delta_1-4b(b+c_1)c_2]}{\Delta}$$

We know that $Q_1-q_1^*>0$ and $Q_2-q_2^*>0$, it will bring about $Q_1>q_1^*$ and $Q_2>q_2^*$. We have the following results.

Proposition 1. *The equilibrium production for each firm is not more than it's announced plan product.*

Substituting q_1^*, q_2^*, Q_1 and Q_2 into Π_i, the single-period profit of the firm i, simple calculations show that

$$\Pi_1 = \frac{a^2(b+c_1)(\Delta_1-2bc_2)^2}{b\Delta}$$

$$\Pi_2 = \frac{a^2(b+c_2)[(3b+2c_1)\Delta_1-4b(b+c_1)c_2]^2[(3b+4c_1)\Delta_1-4b(b+c_1)c_2]}{b\Delta^2}$$

Since a and b are positive constants, we know that q_1^*, q_2^*, Π_1 and Π_2 are the functions of the parameters c_1 and c_2. In order to analyze the effects of the cost coefficients on the equilibrium solution and the single-period profit of the firm i, further analysis results show that

$$\frac{\partial q_1^*}{\partial c_1}>0,\quad \frac{\partial q_2^*}{\partial c_2}>0,\quad \frac{\partial \Pi_1}{\partial c_1}>0,\quad \frac{\partial \Pi_1}{\partial c_2}<0,\quad \frac{\partial \Pi_2}{\partial c_1}<0,\quad \frac{\partial \Pi_2}{\partial c_2}>0$$

Clearly the following results are true.

Proposition 2. *The equilibrium solution q_1^* and q_2^* for the firms will increase as c_1 and c_2 increase respectively.*

Proposition 3. *With the increase of c_1 , Π_1 will increase and Π_2 will decrease respectively. With the increase of c_2 , Π_2 will increase and Π_1 will decrease respectively.*

2.2. Duopoly Stackelberg game model with bounded rationality

The duopoly method with bounded rational players [6] for our assumptions has the following two dimensional nonlinear map $T(q_1, q_2) \to (q'_1, q'_2)$ which is defined by

$$T: \ q'_i = q_i + \alpha_i(q_i)\frac{\partial \Pi_i}{\partial q_i}, \quad i = 1, 2$$

where $\alpha_i(q_i)$ is a positive function which gives the extend of the production variation of the firm i according to its computed profit signal $\partial \Pi_i/\partial q_i$, prime ($'$) denotes the unit-time advancement operator. The right-hand side variables are productions of period t, and the left-hand side variables are productions of period $(t+1)$. We take the function $\alpha_i(q_i)$ in the linear form $\alpha_i(q_i) = v_i q_i$. Form Eq. (1) the two-dimensional nonlinear map $T(q_1, q_2) \to (q'_1, q'_2)$ is defined by the following two nonlinear difference equations

$$\begin{cases} q_1(t+1) = q_1(t)\{1 + v_1[a + 2c_1Q_1 - 2(b + c_1)q_1(t) - bq_2(t)]\} \\ q_2(t+1) = q_2(t)\{1 + v_2[a + 2c_2Q_2 - 2(b + c_2)q_2(t) - bq_1(t)]\} \end{cases} \tag{7}$$

where $v_i, i = 1, 2$ is a positive constant which is called the speed of adjustment of firm i [6,19]. This means that, if the marginal profit of firm i is positive/negative it increases /decreases its production q_i in the next output period.

2.3. Equilibrium points and local stability

In order to study the qualitative behavior of the solutions of the nonlinear difference Eq. (7), we define the equilibrium points of the dynamic duopoly game as a nonnegative fixed points of the map (7), i.e. the solution of the nonlinear algebraic system

$$\begin{cases} q_1(t)[a + 2c_1Q_1 - 2(b + c_1)q_1(t) - bq_2(t)] = 0 \\ q_2(t)[a + 2c_2Q_2 - 2(b + c_2)q_2(t) - bq_1(t)] = 0 \end{cases} \tag{8}$$

which is obtained by setting $q_i(t+1) = q_i(t), \ i = 1, 2$ in (7). It is clear that the algebraic system (8) has four equilibria:

$$E_0 = (0, 0) \quad E_1 = \left(\frac{a + 2c_1Q_1}{2(b + c_1)}, 0\right) \quad E_2 = \left(0, \frac{a + 2c_2Q_2}{2(b + c_2)}\right) \quad E_3 = (q_1^*, q_2^*)$$

Obviously, E_0, E_1 and E_2 are boundary equilibrium points. Q_1 and Q_2 are given by (4) and (5) respectively. E_3 is the unique Nash equilibrium point with components in Eq. (6).

The study of the local stability of the fixed points of the two-dimensional system (7) depends on the eigenvalues of the Jacobian matrix of (7). The Jacobian matrix $J(q_1, q_2)$ for the system (7) at any point (q_1, q_2) takes the form

$$\begin{pmatrix} 1 + v_1[a + 2c_1Q_1 - 4(b + c_1)q_1 - bq_2] & -bv_1q_1 \\ -bv_2q_2 & 1 + v_2[a + 2c_2Q_2 - 4(b + c_2)q_2 - bq_1] \end{pmatrix}$$

Proposition 4. *The boundary equilibria E_0, E_1 and E_2 of the system (7) are unstable equilibrium points.*

In order to prove this result, we find the eigenvalues of the Jacobian matrix $J(q_1, q_2)$ at each boundary equilibria E_0, E_1 and E_2. At E_0 the Jacobian matrix $J(q_1, q_2)$ is the diagonal matrix

$$J(E_0) = \begin{pmatrix} 1 + v_1(a + 2c_1Q_1) & 0 \\ 0 & 1 + v_2(a + 2c_2Q_2) \end{pmatrix}$$

The eigenvalues of $J(E_0)$ are given by $\lambda_1 = 1 + v_1(a + 2c_1Q_1)$ and $\lambda_2 = 1 + v_2(a + 2c_2Q_2)$. Since the parameters are positive constants then $\lambda_i > 1, \ i = 1, 2$. Hence the equilibrium point E_0 is unstable node.

At E_1 the Jacobian matrix $J(q_1, q_2)$ becomes a triangular matrix

$$J(E_1) = \begin{pmatrix} 1 - v_1(a + 2c_1Q_1) & -\dfrac{v_1b(a + 2c_1Q_1)}{2(b + c_1)} \\ 0 & 1 + \dfrac{v_2\Delta_1q_2^*}{2(b + c_1)} \end{pmatrix}$$

whose eigenvalues are given by the diagonal entries. They are $\lambda_1 = 1 - v_1(a + 2c_1Q_1)$ and $\lambda_2 = 1 + \frac{v_2\Delta_1q_2^*}{2(b+c_1)}$. Since a, b, c_1, c_2, v_1 and v_2 are positive parameters, and $q_2^* > 0$ in Eq. (6), then the eigenvalue λ_2 is greater than 1 and λ_1 is less than 1. Therefore E_1 is a saddle point (unstable). Similarly we can prove that E_2 is also a saddle point.

2.4. Local stability of Nash equilibrium point

In order to study the local stability of the Nash equilibrium $E_3 = (q_1^*, q_2^*)$ of the system (7), we estimate the Jacobian matrix $J(q_1, q_2)$ at E_3, which is

$$J(E_3) = \begin{pmatrix} 1 - 2v_1(b + c_1)q_1^* & -bv_1q_1^* \\ -bv_2q_2^* & 1 - 2v_2(b + c_2)q_2^* \end{pmatrix}$$

The characteristic equation is

$$\lambda^2 - Tr(J)\lambda + Det(J) = 0$$

where $Tr(J)$ is the trace and $Det(J)$ is the determinant of the Jacobian matrix $J(E_3)$.

$$\begin{aligned} Tr(J) &= 2 - 2v_1(b+c_1)q_1^* - 2v_2(b+c_2)q_2^* \\ Det(J) &= 1 - 2v_1(b+c_1)q_1^* - 2v_2(b+c_2)q_2^* + v_1 v_2 \Delta_1 q_1^* q_2^* \end{aligned}$$

Since

$$(Tr(J))^2 - 4Det(J) = [2v_1(b+c_1)q_1^* - 2v_2(b+c_2)q_2^*]^2 + 4b^2 v_1 v_2 q_1^* q_2^* \geq 0$$

This means that the eigenvalues of Nash equilibrium are real.

Following a standard stability analysis a sufficient and necessary conditions for the local stability of Nash equilibrium E_3 is that the eigenvalues of the Jacobian matrix $J(E_3)$ are inside the unit circle of the complex plane, this is true if and only if the following Jury's conditions [11,14] hold.

(i) $Det(J) < 1$.
This condition becomes

$$2v_1(b+c_1)q_1^* + 2v_2(b+c_2)q_2^* - v_1 v_2 \Delta_1 q_1^* q_2^* > 0$$

(ii) $1 - Tr(J) + Det(J) = v_1 v_2 \Delta_1 q_1^* q_2^* > 0$

(iii) $1 + Tr(J) + Det(J) = 4 - 4v_1(b+c_1)q_1^* - 4v_2(b+c_2)q_2^* + v_1 v_2 \Delta_1 q_1^* q_2^* > 0$
This condition becomes

$$4v_1(b+c_1)q_1^* + 4v_2(b+c_2)q_2^* - v_1 v_2 \Delta_1 q_1^* q_2^* - 4 < 0 \tag{9}$$

In fact, the inequality (ii) is always satisfied.

Proposition 5. *When the inequality (iii) is true, the inequality (i) will be true.*

Putting $v_2 = 0$ in the inequality (9), we get $v_1 < \frac{1}{(b+c_1)q_1^*}$. When $0 < v_1 \leq \frac{2(b+c_2)}{\Delta_1 q_1^*}$, we know that

$$2v_1(b+c_1)q_1^* + 2v_2(b+c_2)q_2^* - v_1 v_2 \Delta_1 q_1^* q_2^* = 2v_1(b+c_1)q_1^* + v_2 q_2^*[2(b+c_2) - v_1 \Delta_1 q_1^*] > 0$$

We can easily verify that

$$(b+c_1)\Delta_1 q_1^2 v_1^2 - 2\Delta_1 q_1 v_1 + 4(b+c_2) > 0 \tag{10}$$

for all v_1. Since $\frac{1}{(b+c_1)q_1^*} < \frac{4(b+c_2)}{\Delta_1 q_1^*}$, when $\frac{2(b+c_2)}{\Delta_1 q_1^*} < v_1 < \frac{1}{(b+c_1)q_1^*}$, from the inequality (9) and (10), we obtain

$$\begin{aligned} v_2 &< \frac{4 - 4v_1(b+c_1)q_1^*}{4(b+c_2)q_2^* - v_1 \Delta_1 q_1^* q_2^*} \\ &< \frac{2v_1(b+c_1)q_1^*}{v_1 \Delta_1 q_1^* q_2^* - 2(b+c_2)q_2^*} \end{aligned}$$

Therefore, we can deduce that the inequality (i) is satisfied in this case. In a word, when the inequality (iii) is true, the inequality (i) will also be true.

The inequalities (i)–(iii) define a region of stability in the plane of speeds of adjustment (v_1, v_2), this stability region is bounded by the portion of hyperbola with positive v_1 and v_2, whose equation is given by the vanishing of the left-hand side of inequality (9) i.e.

$$4v_1(b+c_1)q_1^* + 4v_2(b+c_2)q_2^* - v_1 v_2 \Delta_1 q_1^* q_2^* - 4 = 0$$

For the values of (v_1, v_2) inside the stability region (see Fig. 1), the Nash equilibrium E_3 is stable node and looses its stability through a period doubling bifurcation. This bifurcation curve intersects the axes v_1 and v_2 in the points A_1 and A_2 respectively, whose coordinates are given by

$$A_1 = \left(\frac{1}{(b+c_1)q_1^*}, 0\right), \quad A_2 = \left(0, \frac{1}{(b+c_2)q_2^*}\right)$$

The stability of E_3 depends on the system parameters. From these results, we obtain information on the effects of the model parameters on the local stability of Nash equilibrium E_3. For example, the increase of the speeds of adjustment has a destabilizing effect with the other parameters held fixed. In fact, an increase of v_1 and v_2 starting from the local stability of Nash equilibrium, can bring the point (v_1, v_2) out of the region of stability, crossing the flip bifurcation curve. Similar arguments apply if the parameters v_1, v_2, b, c_1 and c_2 are fixed and the parameter a, which represents the maximum price of the good produced, is increased. In this case the region of stability becomes small, and this can cause a loss of stability of E_3. Indeed, we know that an increase of the cost parameter c_1 and c_2 held fixed will cause a displacement of the point A_1 to the left and of A_2 upwards. Similarly, an increase of c_2 and c_1 held fixed, causes a displacement of the point A_2 downwards and of A_1 almost a fixed point.

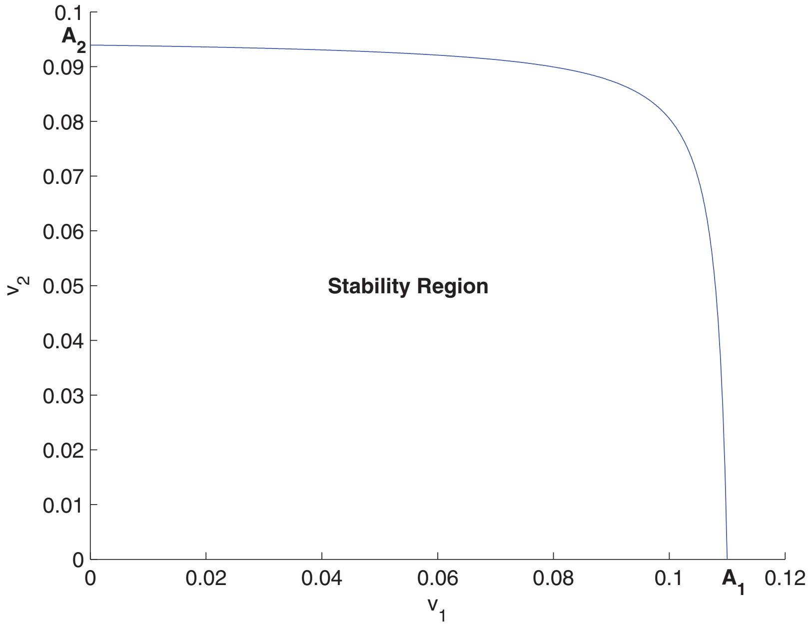

Fig. 1. The region of stability of the Nash equilibrium of duopoly game (7) in v_1v_2-plane. The values of the parameters are $a = 7, b = 0.5, c_1 = 1$ and $c_2 = 2$.

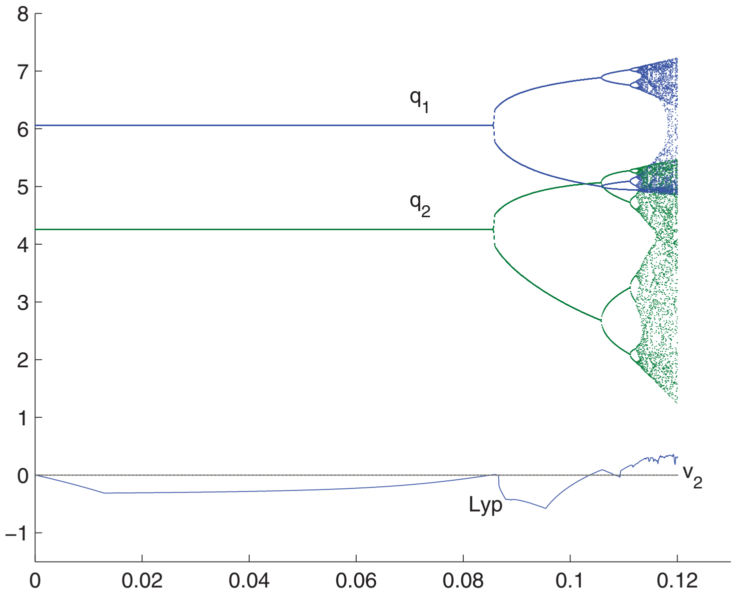

Fig. 2. The bifurcation diagram of the solutions q_1 and q_2 of the system (7) vs v_2 and down the maximum Lyapunov exponent Lyp vs v_2 ($a = 7, b = 0.5, v_1 = 0.095, c_1 = 1$ and $c_2 = 2$).

3. Simulation

The main purpose of this section is to show that the qualitative behavior of the solutions of the nonlinear duopoly game. Numerical experiments are simulated to show the stability and period doubling bifurcation route to chaos for the two-dimensional system (7). In the simulation here, we take $a = 7,\ b = 0.5,\ c_1 = 1$ and $c_2 = 2$. If we fix the system parameters and vary one for example v_2, bifurcations and chaos occur, which is detected by using Lyapunov exponents.

Fig. 2 presents the bifurcation diagrams with respect to the adjustment speed v_2. From Fig. 2 we see that a low adjustment speed can make the system stable and its increase will make the equilibrium become unstable even chaotic. Fig. 2 shows that the system is always stable when v_2 is located in a large interval. Bifurcation occurs only when v_2 is numerically shown close about to 0.083, and the system turns into chaos when v_2 takes its large value around 0.12. We know that Fig. 1 shows the region

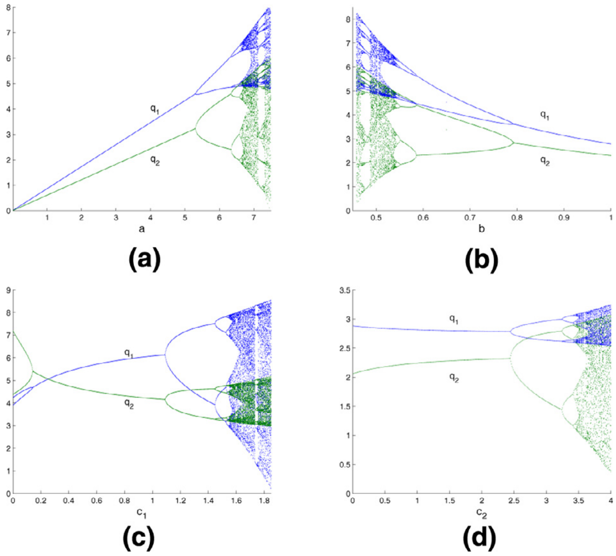

Fig. 3. The bifurcation diagram of the system (7) with respect to other parameters. The values of the parameters are $\nu_1 = 0.095$ and $\nu_2 = 0.12$ for each case. (a) shows the bifurcation diagram with respected to a. The values of the parameters are $b = 0.5, c_1 = 1$ and $c_2 = 2$. (b) shows the bifurcation diagram with respected to b. The values of the parameters are $a = 7, c_1 = 1$ and $c_2 = 2$. (c) shows the bifurcation diagram with respected to c_1. The values of the parameters are $a = 7, b = 0.5$ and $c_2 = 1$. (d) shows the bifurcation diagram with respected to c_2. The values of the parameters are $a = 7, b = 1$ and $c_1 = 1$.

of stability of the Nash equilibrium for the system (7). Crossing the bifurcation curve we obtain period doubling bifurcation and final chaotic behavior. Also the maximum Lyapunov exponent is plotted in Fig. 2 to show bifurcations and chaos, where positive values show the chaotic behaviors. At bifurcation point the maximum Lyapunov exponents is zero.

Fig. 3 shows the bifurcation diagram with respect to the parameters in system (7). From Fig. 3(a) and (d), we can see that the Nash equilibrium E_3 is locally stable for small values of a or c_2. As a or c_2 increases, the Nash equilibrium point becomes unstable and complex dynamic behavior occurs, including higher-order cycles and chaos. Fig. 3(b) gives the bifurcation diagram respect to b. We can see that the system dynamics is chaotic if b is small. As b increases, there exist period-halving bifurcations. The system (7) experiences chaos and period-halving bifurcation. From Fig. 3(c) we observe that the Nash equilibrium E_3 is locally stable for some properly values of c_1. From the Nash equilibrium point, there exist period-halving bifurcations as c_1 decreases. As c_1 increases from the Nash equilibrium point, the system dynamics is chaotic.

The strange attractor is another characteristic of chaos of the system, and it reflects the inherent regularity of the complex phenomena in a chaotic state. Thus players can forecast the market output in a short term according to inherent regularity while the system is in a chaotic state. Fig. 4(a), (b) and (c) shows the change situation of strange attractor of the system (7) for $\nu_1 = 0.09$ and $\nu_2 = 0.115, 0.120$ and 0.125, respectively. Fig. 4(d), (e) and (f) shows the change situation of strange attractor of the system (7) for $\nu_1 = 0.095$ and $\nu_2 = 0.115, 0.120$ and 0.125, respectively. Fig. 4(g), (h) and (i) shows the change situation of strange attractor of the system (7) for $\nu_1 = 0.1$ and $\nu_2 = 0.115, 0.120$ and 0.125, respectively. From Fig. 4, we can see the bifurcation processes, where invariant curves take place after the equilibrium bifurcates properly and chaos occurs when ν_1 and ν_2 take their value big enough. The system trajectories are plotted in a two-dimensional phase plane of the variables q_1 and q_2.

Fig. 5 demonstrates the sensitivity of system (7) to initial conditions. For each state variable Fig. 5 plots two orbits initially from the slightly deviated points $(q_1^1(0), q_2^1(0))$ and $(q_1^2(0), q_2^2(0)) = (q_1^1(0) + 10^{-4}, q_2^1(0))$, respectively. The red curves(labeled by superscript 1) start from (6.0608, 4.2583) and the blue curves (labeled by 2) start form (6.0609, 4.2583). It suffices to show that although the two orbits of each variable are indistinguishable at the beginning, the difference between them is obviously built after a series of iterations.

4. Chaos control

From the numerical simulations above, we observe that the adjustment speed ν_1 and ν_2 tremendously influence the stability of system (7). If the parameters fail to locate in the stable region, the dynamic behaviors of the system will be much complicated and even chaotic. The appearance of chaos in the economic system is not expected and even is harmful. We hope that the system chaos can be avoided or controlled on some level such that the dynamic system could perform better. A wide variety of methods have been proposed for controlling chaos in oligopoly models. Ott *et al.* presented the **OGY** method depends on varying the parameters in such a way as to project the system onto the stable manifold [27]. The **OGY** method had been applied to control chaos in the Kopel duopoly game model [9]. Chaos control with modified straight-line stabilization method in an output duopoly competing evolution model had been studied [23]. Chaos control with time-delayed feedback method in an economical model had been studied [28]. The dynamics and adaptive control of a duopoly advertising model based on heterogeneous expectations

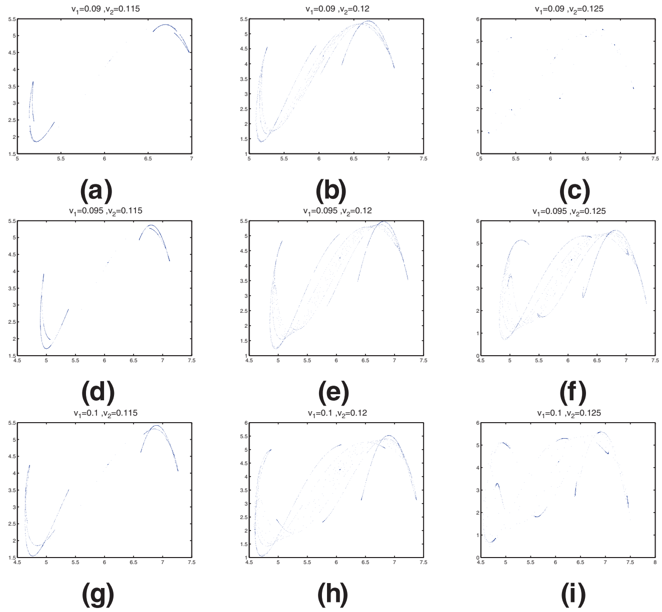

Fig. 4. The strange attractors for the two-dimensional system (7). The values of parameters are $a = 7, b = 0.5, c_1 = 1$ and $c_2 = 2$. With various values of adjustment speed v_1 and v_2, this figure plots the system's trajectories after full iteration in a two-dimensional plane, where the x-axis describes the product q_1 for player 1 and the y-axis writes the one for player 2.

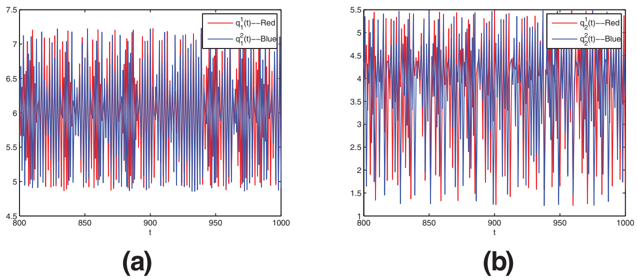

Fig. 5. Sensitive dependence of system (7) on initial conditions. The numerical simulations are done by setting $v_1 = 0.095$ and $v_2 = 0.12$. The system orbits in the time periods [800, 1000] are plotted. The red curves are associated with the initial state $q_1(0) = 6.0608$, and $q_2(0) = 4.2583$. The blue ones are initially associated with the initial state $q_1(0) = 6.0609$, and $q_2(0) = 4.2583$.

was presented [29], and so on. Elabbasy *et al.* considered a feedback method to control the chaotic behavior of the game with heterogeneous players [30,31]. Feedback control had been applied on multi-team Bertrand model [32], and on a Cournot investment game with heterogeneous players [33].

Feedback and parameter variation are two methods for the chaos control. Recently, a new control method had been proposed. It is called as control strategy of the state variables feedback and parameter variation [34]. This method had been considered in the four oligopolist model [35]. The same method had been used to control the chaos [36]. In this section, we use the method to control the chaos of system (7). We change two-dimensional discrete dynamic system (7) into the following format:

$$\begin{cases} q_1(t+1) = (1-\mu)q_1(t)\{1 + v_1[a + 2c_1Q_1 - 2(b+c_1)q_1(t) - bq_2(t)]\} + \mu q_1(t) \\ q_2(t+1) = (1-\mu)q_2(t)\{1 + v_2[a + 2c_2Q_2 - 2(b+c_2)q_2(t) - bq_1(t)]\} + \mu q_2(t) \end{cases} \tag{11}$$

System (7) will fall into instability region and chaos with the change of product modification speed for player 1 and player 2. The chaotic state of system (7) with the change of speed adjustment v_1 for player 1 and v_2 for player 2 are controlled. Fig. 6 is the bifurcation diagram of controlled system (11) withe the change of control parameter μ after adding control to the chaotic state ($a = 7, b = 0.5, c_1 = 1, c_2 = 2, v_1 = 0.095$ and $v_2 = 0.12$). It can be seen form Fig. 6 that the period-doubling bifurcation disappears, and the system stabilizes at the Nash equilibrium point when $\mu < 0.24$ as shown numerically. The system gradually gets out of chaos and becomes stable when the controlling parameter μ is properly large. Thus chaos control is successful. The method presented here can be applied to many chaotic dynamical system.

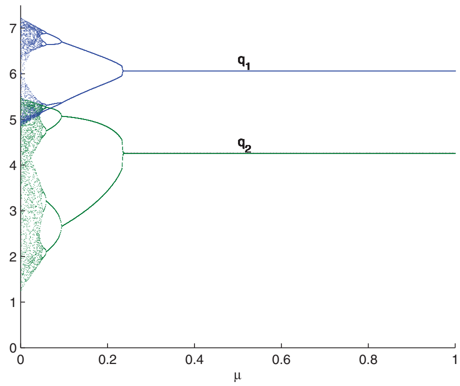

Fig. 6. The bifurcation diagram with respect to control parameter μ. The two bifurcation diagrams show that the system chaos has being gradually controlled with the parameter μ increasing and the system will be led to stability when μ is large enough.

5. Conclusion

In this paper, a duopoly Stackelberg model has been proposed, where players use different production methods. The equilibrium outputs and equilibrium profits are affected by cost coefficients. The dynamic of duopoly Stackelberg model with bounded rationality has been analyzed. The local stability of four equilibrium points is investigated in this game. Basic properties of the game have been analyzed by means of stability region, bifurcation diagram and strange attractor. The study shows that the stability of Nash equilibrium, as some parameters of the model are varied, gives rise to complex dynamics such as cycles and chaos. The model is quickly arrived at the Nash equilibrium point when a suitable controlling parameter is chosen. Similar methods can be used to study the dynamics of the duopoly Stackelberg game with delayed bounded rationality. Some modifications of the Stackelberg game can also be studied.

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References

[1] R. Gibbons, A Primer in Game Theory, Simon & Schuster, New York, 1992.
[2] A.A. Cournot, Researches into the Principles of the Theory of Wealth, in: Irwin Paper Back Classics in Economics, 1963.
[3] H. von Stackelberg, Probleme der unvollkommenen konkurrenz, Weltwirtschaftliches Archiv 48 (1938) 95–141.
[4] T. Puu, Chaos in duopoly pricing, Chaos Solit. Fract. 1 (1991) 573–581.
[5] T. Puu, The chaotic duopolists revisited, J. Econ. Behav. Organ. 37 (1998) 385–394.
[6] G.I. Bischi, A. Naimzada, Global Analysis of a Dynamic Duopoly Game with Bounded Rationality, in: Advanced in Dynamic Games and Application, 5, Birkhouser, 1999. Chapter 20.
[7] G.I. Bischi, A.K. Naimzada, L. Sbragia, Oligopoly games with Local Monopolistic Approximation, J. Econ. Behav. Organ. 62 (2007) 371–388.
[8] G.I. Bischi, L. Stefanini, L. Gardini, Synchronization intermittency and critical curves in a duopoly game, Math. Comput. Simul. 44 (1998) 559–585.
[9] H.N. Agiza, On the stability, bifurcations, chaos and chaos control of Kopel map, Chaos Solit. Fract. 11 (1999) 1909–1916.
[10] H.N. Agiza, A.S. Hegazi, A.A. Elsadany, Complex dynamics and synchronization of duopoly game with bounded rationality, Math. Comput. Simul. 58 (2002) 133–146.
[11] H.N. Agiza, A.A. Elsadany, Chaotic dynamics in nonlinear duopoly game with heterogeneous players, Appl. Math. Comput. 149 (2004) 843–860.
[12] H.N. Agiza, A.A. Elsadany, Nonlinear dynamics in the Cournot duopoly game with heterogeneous players, Phys. A 320 (2003) 512–524.
[13] H.N. Agiza, A.S. Hegazi, A.A. Elsadany, The dynamics of Bowley's model with bounded rationality, Chaos Solit. Fract. 12 (2001) 1705–1717.
[14] E. Ahmed, H.N. Agiza, S.Z Hassan, On modifications of Puu's dynamical duopoly, Chaos Solit. Fract. 7 (2000) 1025–1028.
[15] A.A. Elsadany, Dynamics of a delayed duopoly game with bounded rationality, Mathemat. Comput. Model. 52 (2010) 1479–1489.
[16] J.B. Rosser, The development of complex oligopoly dynamics theory, in: Text Book Oligopoly Dynamics : Models and Tools, Springer-Verlag, 2002.
[17] A. Naimzada, L. Sbragia, Oligopoly games with nonlinear demand and cost functions: Two boundedly rational adjustment processes, Chaos Solit. Fract. 2 (2006) 707–722.
[18] H.N. Yao, F. Xu, Complex dynamics analysis for a duopoly advertising model with nonlinear cost, Appl. Math. Comput. 180 (2006) 134–145.
[19] M.T. Yassen, H.N. Agiza, Analysis of a duopoly game with delayed baounded rationality, Appl. Math. Comput. 138 (2003) 387–402.
[20] J. Zhang, Q. Da, Y. Wang, Analysis of nonlinear duopoly game with hetergeneous players, Econ. Model. 24 (2007) 138–148.
[21] J. Du, T. Huang, New results on stable region of Nash equilibrium point of output game model, Appl. Math. Comput. 192 (2007) 12–19.
[22] Z. Ding, Q. Li, D. Ge, S. Jiang, Research on dynamics in a resource extraction game with bounded rationality, Appl. Math. Comput. 236 (2014) 628–634.
[23] J. Du, T. Huang, Z. Sheng, Anaylsis of decision-making in economic chaos control, Nonlinear Anal.: Real World Appl. 10 (2009) 2493–2501.
[24] Z. Sheng, T. Huang, J. Du, Q. Mei, H. Haung, Study on self-adaptive proportional control method for a class of output models, Discrete Contin. Dyn. Syst. Ser. B 11 (2009) 459–477.
[25] H. Simon, Rational choice and the structure of the environment, Psychol. Rev. 63 (1956) 129–138.
[26] James Pita, Manish Jain, Milind Tambe, Fernando Ordóñez, Sarit Kraus, Robust solutions to Stackelberg games: Addressing bounded rationality and limited observations in human cognition, Artif. Intel. 174 (2010) 1142–1171.
[27] E. Ott, C. Grebogi, J.A. Yorke, Controlling chaos, Phys. Rev. Lett. 64 (1990) 1196–1199.

[28] J. Holyst, K. Urbanowicz, Chaos control in economical model by time-delayed feedback method, Phys. A 287 (2000) 587–598.
[29] J. Ding, Q. Mei, H. Yao, Dynamics and adaptive control of a duopoly advertising model based on heterogeneous expectations, Nonlinear Dyn. 67 (1) (2012) 129–138.
[30] E.M. Elabbasy, H.N. Agiza, A.A. Elsadany, Analysis of nonlinear triopoly game with heterogeneous players, Comput. Math. Appl. 57 (2009) 488–499.
[31] A.A. Elabbasy, H.N. Agiza, E.M. Elsadany, Complex dynamics and chaos control of heterogeneous quadropoly game, Comput. Math. Appl. 219 (2013) 11110–11118.
[32] Z. Ding, Q. Hang, H. Yang, Analysis of the dynamics of multi- team Bertrand game with heterogeneous players, Int. J. Syst. Sci. 42 (2010) 1047–1056.
[33] Z. Ding, Q. Li, S. Jiang, X. Wang, Dynamics in a Cournot investment game with heterogeneous players, Comput. Math. Appl. 256 (2015) 939–950.
[34] X.S. Luo, G.R. Chen, B.H. Wang, J.Q. Fang, Y.L. Zou, H.J. Quan, Control of period-doubling bifurcation and chaos in a discrete nonlinear system by the feedback of states and parameter adjustment, Acta. Phys. Sin. Ch Ed. 52 (2003) 790–794.
[35] X. Pu, J. Ma, Complex dynamics and chaos control in nonlinear four-oligopolist game with different expectations, Int. J. Bifurcat. Chaos 23 (3) (2013) 1350053.
[36] Z. Ding, Q. Li, S. Jiang, X. Wang, Dynamics in a Cournot investment game with heterogeneous players, Comput. Math. Appl. 256 (2015) 939–950.