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Stable, Cyclic and Chaotic Growth: The Dynamics of a Discrete-Time Version of Goodwin's Growth Cycle Model

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1. Introduction

The stability properties of Dr. R. M. Goodwin's (1967) growth cycle model have attracted the attention of some economists, apparently because the model whose solutions are periodic for all initial states is structurally unstable. This fact makes it interesting to explore what particular modifications of the underlying economic hypotheses result in a qualitatively different behaviour of the solutions. The model has been studied under alternative assumptions about production technology [Akerlof and Stiglitz (1969), Desai (1973)] and wage bargaining [Akerlof and Stiglitz (1969), Desai (1973), Pohjola (1979), Velupillai (1979)]. The purpose of this paper is to add a new aspect to the discussion by analyzing a discrete-time version of the system.

The structure of the model is also slightly modified in another respect: Goodwin's real wage bargaining equation — a Phillips-curve — is replaced by Kuh's (1967) alternative to it which makes the level of wages — as opposed to the relative change — to depend positively on the employment rate. This modification results in a non-linear first-order difference equation which possesses a rich spectrum of dynamic behaviour that goes, as the parameters of the model are tuned, from a stable equilibrium point into stable cycles and, finally, into a regime that is called chaos in the literature on the mathematical properties of non-linear difference systems [see

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the review by May (1976)]. Chaotic solutions look irregular and are indistinguishable from the sample function of a stochastic process although they are generated by a completely deterministic system. On the contrary, the continuous-time version of this equation is rather uninteresting in the sense that it cannot produce any oscillations at all, as was shown by Akerlof and Stiglitz (1969).

The plan of the paper is as follows: In the next section the model is set up. Section 3 describes the dynamic behaviour of the resulting difference equation. In concluding the paper we discuss the relevance and the implications of chaos to the business cycle theory and to economic modelling in general.

2. The Model

We shall consider a closed economy where a single homogeneous good is produced that can be either consumed or invested. There are two factors of production, labour and capital, and two classes of people, workers and capitalists. Labour supply, N_t, is taken to grow at a constant rate, n:

$$N_{t+1} = (1+n)\, N_t; \quad n \geq 0; \tag{1}$$

where t denotes the time-period. Labour and capital are combined in fixed proportions to produce the output under constant returns to scale. We shall here be concerned with situations of less than full-employment so the assumption that labour is never the limiting factor in production is made. Then, if technical progress is labour-augmenting at a constant rate, σ:

$$Y_{t+1}/L_{t+1} = (1+\sigma)\, Y_t/L_t; \quad \sigma \geq 0 \tag{2}$$

where Y_t is real output and L_t labour employed, production possibilities can be characterized by a constant capital-output ratio, μ:

$$K_t = \mu Y_t; \quad \mu > 0, \tag{3}$$

where K_t is the capital stock at the beginning of period t. Finally, the savings function is assumed to be of the classical type: all wages are consumed, all profits saved and automatically invested. The product market equilibrium can then be expressed as

$$K_{t+1} - K_t = (1 - A_t)\, Y_t, \tag{4}$$

A_t being workers' share of total output, or since they do not save, the wage share.

Define the employment rate as $E_t = L_t/N_t$ and consider its relative change which is obtained from (1)—(4) as

$$E_{t+1}/E_t = 1 + (1 - \mu g - A_t)/\mu\ (1+g), \tag{5}$$

where g is the natural rate of growth, $g = n + \sigma + n\sigma$. This specifies the dynamics of the employment rate for any given income distribution A_t. The equilibrium wage share is then $A^* = 1 - \mu g$. If income distribution stays constant, then the employment rate grows or declines at a constant rate when the wage share is below or above its equilibrium value, respectively.

So far we have followed Goodwin's (1967) hypotheses. In postulating how income distribution reacts to the employment situation we, however, deviate from his specification and follow Kuh (1967) who assumes that wages are marked up on average labour productivity, the mark-up factor depending on the demand conditions in the labour market. Making the strong assumption that workers do not suffer from money illusion and approximating the demand conditions by the employment rate, the bargaining equation can be expressed as

$$W_t = h\ (E_t)\ Y_t/L_t \tag{6}$$

where W_t denotes the real wage rate and $h\ (E_t)$ is a positive, increasing function of E_t for $0 < E_t < 1$. It also has to grow fast enough when full-employment is approached to ensure that labour never becomes the limiting factor in production, as was assumed earlier. This corresponds to the property of standard Phillips-curves that the rate of growth of wages goes to infinity as full-employment is approached. Kuh (1967) ended up with a formulation similar to (6) in terms of money wages from the observation that in many estimated Phillips-curves the rate of change of employment turned out to be more significant than the level of employment in explaining money wage changes.

By observing that $A_t = W_t L_t/Y_t$ and approximating $h(E)$ by a linear function $-\alpha + \beta E$, $\alpha \geq 0$, $\beta > 0$, we obtain the equation

$$A_t = -\alpha + \beta E_t \tag{7}$$

that closes our model. Substituting it into (5), the model is reduced to

$$E_{t+1} = E_t\ [1 + r\ (1 - E_t/E^*)], \tag{8}$$

where $r = (1 - \mu g + \alpha)/\mu\ (1+g)$ and $E^* = (1 - \mu g + \alpha)/\beta$. It can be interpreted as a logistic difference equation with r as the intrinsic rate of growth and E^* as its equilibrium.

The approximation (7) plays a central part in the analysis that will follow and is the source of the unrealistic feature of the model that small enough employment makes the wage share negative, as Fig. 1 illustrates. We have, however, wanted to gain analytical simplicity at the expense of economic realism since our aim is not to construct an empirically realistic model but to show that even the simplest non-linear models can have complicated solutions. This

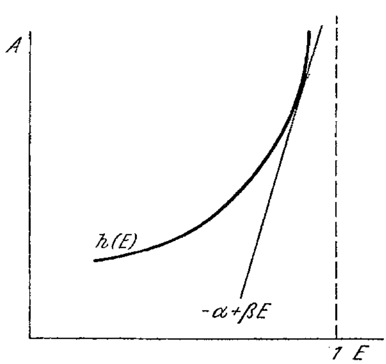

Fig. 1. The relation between the wage share (A) and the employment rate (E)

modelling deficiency could easily be avoided by approximating $h(E)$ by a non-linear function such as γE^{δ}, $\gamma > 0$, $\delta > 0$; the nature of the results that will be presented soon would not change, only the numbers would be different.

3. The Dynamics: Stable, Cyclic and Chaotic Growth

To study the dynamics of our model, let $x_t = rE_t/(1+r)E^*$ and express (8) as

$$x_{t+1} = (1+r)\, x_t\, (1-x_t) = F\,(x_t; r) \tag{9}$$

which is a simple non-linear first-order difference equation whose dynamics can be described in terms of one parameter, r. To keep the solutions of the model economically meaningful let us impose the following additional restrictions on the parameter values. First, assume that $1-\mu g+\alpha > 0$ to keep r positive. Next make the assumption that $-\alpha+\beta > 1+\mu$ to ensure that $E_t < 1$ for all t. This means that at full employment workers' real income claim is higher than the obtainable maximum which is current output plus the existing capital stock and ensures that labour is never the limiting factor in production. Finally, we take an upper limit for r, $r<3$, which together with $r>0$ guarantees that $E_t>0$ for all t. For these parameter values we have $0<E_t<1$ and $-\alpha<A_t<1+\mu$ in all periods, i. e., the solutions are bounded and, except for the non-positive lower limit imposed on the wage share, within economically reasonable bounds. The corresponding equilibrium values are $E^*=(1-\mu g+\alpha)/\beta<1$ and $A^*=1-\mu g\leq 1$. From the definition of x_t we can see that these restrictions also guarantee the boundedness of the solution trajectories of (9): $0<x_t<1$ for all t. The corresponding equilibrium is $x^*=r/(1+r)$. In this sense Eqs. (8) and (9) are globally stable.

The dynamics of Eq. (8), or (9), is well documented in the ecological and mathematical literature [see the survey by May (1976)]. We shall summarize it here briefly. Let us consider (9). First note that besides the trivial unstable fixed point at zero it has also an equilibrium at $x^*=r/(1+r)$. This is locally attracting if the slope of $F(.)$ at x^* is less than one in the absolute value, i. e., if $|\lambda(r)|=|F'(x^*;r)|=|1-r|<1$. The stability condition is then $0<r<2$. The equilibrium is approached monotonically for $0<r<1$ and in an oscillatory fashion for $1<r<2$. But what happens if $r>2$? The fixed point is now repelling but, on the other hand, we know that the solution trajectories are bounded. To find out where they go, consider $x_{t+2}=F(F(x_t;r))=F^{(2)}(x_t;r)$. Solve $x^{(2)*}=F^{(2)}(x^{(2)*};r)$ to obtain the period-two fixed points. It turns out that for $0<r<2$ we have only two real solutions, namely the period-one fixed points zero and x^*. But once $r>2$ it has two new real solutions $x_1{}^{(2)*}$ and $x_2{}^{(2)*}$. The stability of these new fixed points is again given by the slope of $F^{(2)}(.)$ at $x_1{}^{(2)*}$ and $x_2{}^{(2)*}$. But applying the chain rule we can see that these slopes are equal: $\lambda^{(2)}(r)=F^{(2)\prime}(x_1{}^{(2)*};r)=F^{(2)\prime}(x_2{}^{(2)*};r)=F'(x_1{}^{(2)*};r)\,F'(x_2{}^{(2)*};r)$. The stability of this two-point cycle can now be expressed in terms of r alone. Performing the actual calculations we obtain $x_{1,2}^{(2)*}=(2+r\pm\sqrt{r^2-4})/2\,(1+r)$ and $\lambda^{(2)}(r)=5-r^2$. The two-period cycle is then attracting if $|\lambda^{(2)}(r)|<1$, i. e., if $2<r<\sqrt{6}=2.449$.

As r is increased beyond 2.449 the initially stable two-period cycle becomes repelling. To find out where the trajectories then go, consider $x_{t+4}=F^{(4)}(x_t;r)$. For $r>2.449$ this has four different fixed points in addition to the period-one and period-two points. The stability of this four-period cycle is once again given by the eigenvalue $\lambda^{(4)}=\prod_{i=1}^{4}F'(x_i{}^{(4)*};r)$. Numerical calculations show that $|\lambda^{(4)}(r)|<1$ for $2.449<r<2.544$. If r is increased beyond 2.544 we obtain an initially stable 8-period cycle, then a 16-period cycle and so on. In this way the initially attracting equilibrium point x^* bifurcates into cycles of period 2^k where k is a positive integer.

The bifurcation process is, however, convergent: r is bounded from above by the critical value $r_c = 2.570$ at which Eq. (9) has an infinite number of periodic points. As r is increased beyond r_c, all these infinitely many cycles have even periodicities until the first odd period cycle is created at $r = 2.679$. Its period is very long but increasing r produces odd period cycles with diminishing periodicities and finally at $r = 2.828$ the three-period cycle is created. Thus for $r > 2.828$ we can obtain cycles of every integer period. Fortunately, Guckenheimer, Oster and Ipaktchi (1977) have been

Table 1

The nature of dynamic solutions of (8) and (9) for $r \leq 3$. Note that for $r > 3$ the model becomes economically infeasible. For those values of r that separate different regions from each other there are no attracting cycles, i. e., the solution are chaotic. (For the calculation of these values, see May (1976) and the references given there)

Dynamic behaviour	Value of r
Stable equilibrium point	$0 < r < 2$
— monotonic convergence	$0 < r < 1$
— oscillations	$1 < r < 2$
Stable cycles of period 2^k	$2 < r < 2.570$
— 2-period cycle	$2 < r < 2.449$
— 4-period cycle	$2.449 < r < 2.544$
— 8-period cycle	$2.544 < r < 2.564$
— 16, 32, 64, ...	$2.564 < r < 2.570$
Chaotic behaviour	$2.570 < r \leq 3$
— first odd period cycle	$r = 2.679$
— 3-period cycle	$r = 2.828$

able to show that, for each value of r, Eq. (9) has at most one stable cycle. Therefore, even if for $r > 2.828$ the equation is capable of generating cycles of every integer period, only one of these can be attracting. Their proof holds for all first-order difference equations that are analytic but it does not necessarily apply to more general forms of $F(.)$ or to higher dimensional systems.

But if Eq. (9) is capable of generating the three-period cycle then, by Li and Yorke (1975), it can also have totally aperiodic trajectories. The measure of the uncountable number of initial conditions x_0 whose trajectories are aperiodic is, however, zero on the unit interval so that these solutions are very rare. From the practical point of view it does not really matter whether solutions are aperiodic or periodic with very long periodicities since it is likely that, in the latter case, the transients associated with initial conditions take very many periods to wear out; in both cases the solution trajectories look like the sample function of a random process. It has been proposed that in these cases the model should be analyzed by statistical methods even though it is completely deterministic [May (1976), Guckenheimer, Oster and Ipaktchi (1977)].

The case $r > 2.570$ has been termed chaos, since now Eq. (9) can generate almost any kind of behaviour, depending on the initial condition x_0. It can produce cycles of every integer period but there are also initial points whose trajectories do not converge to any finite cycle. Table 1 summarizes the dynamics.

Let us next examine the dynamics of our model in terms of the original parameters α, β, μ and g. First note that β affects the equilibrium employment rate but not the nature of the dynamic solutions if α and β can be chosen independently. But if E^* is regarded as a parameter, we then have $A - A^* = \beta\,(E - E^*)$, i. e., $\alpha = \beta E^* - A^*$. If this interpretation is preferred, then the dynamics depends also on β. We first reduce the number of parameters by assuming $g = 0$ and later examine how a positive natural growth rate modifies the results. Then $r = (1 + \alpha)/\mu$ and $E^* = (1 + \alpha)/\beta$. The following figure illustrates the regimes of solutions of Eq. (8) in the parameter space:

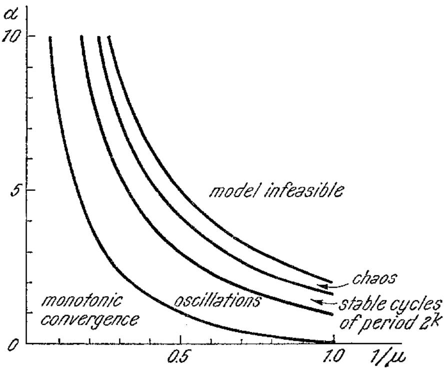

Fig. 2. The regimes of solutions of (8) in the parameter space when $g = 0$

In drawing the figure it has been assumed that $\mu > 1$ which we regard as realistic if the length of the period implicit in the analysis is less than a year. This makes the output-capital ratio, $1/\mu$, smaller than one. α can be interpreted to reflect capitalists' bargaining power since, given β, an increase in α shifts the mark-up function in Fig. 1 to the right which means that a smaller wage share corresponds to a given employment rate than previously.

Assume first that the parameters are such that the equilibrium (E^*, A^*) is approached monotonically. Then if either capitalists' bargaining power, α, or the output-capital ratio, $1/\mu$, increases enough, the economy starts oscillating. As the parameters further increase, it goes through a regime of stable cycles into the chaotic state before collapsing. Fig. 3 picture some of the solution trajectories of the model.

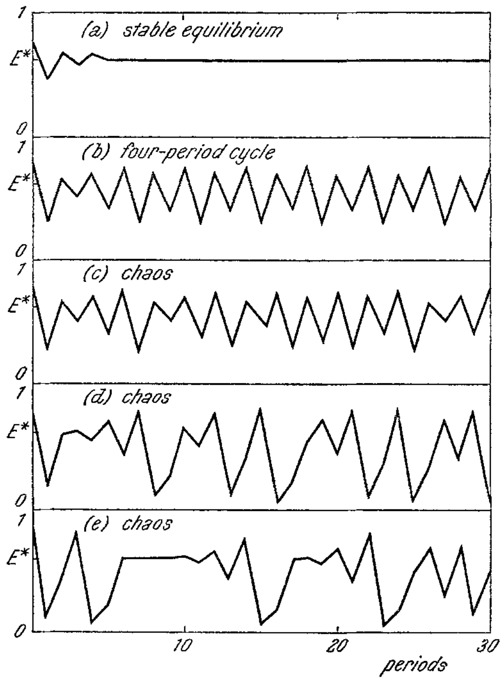

Fig. 3. Simulations of Eq. (8) with $g=0$, $E^*=0.60$ and (a) $r=1.67$, (b) $r=2.50$, (c) $r=2.63$, (d) $r=2.94$, (e) $r=2.94$. The initial condition is $E_0=0.75$ in (a)—(d) and $E_0=0.77$ in (e)

If the natural growth rate is positive, $g>0$, the whole family of the curves in Fig. 2 shifts to the right. Thus, oscillations, stable cycles and chaos become less likely and the economy can sustain more powerful capitalists and higher output-capital ratios then when $g=0$. The relaxation of the assumptions that there is no saving out of wages and no consumption out of profits would have the same effect, since for these cases we obtain the intrinsic growth rate in (8) as

$$r = [s_p - \mu g + (s_p - s_w)\, \alpha]/\mu\, (1+g),$$

where s_p, s_w denote the savings propensities out of profits and wages, respectively, and $s_w < s_p$.

The results we have obtained in this section concern the employment rate. They can, however, be converted to apply to the rate of economic growth since from Eqs. (3) and (4) it follows that $Y_{t+1}/Y_t = 1 + (1 - A_t)/\mu$ and since $A_t = -\alpha + \beta E_t$. Thus, the rate of growth of real output is a linear function of the employment rate. The qualitative conclusions obtained about the dynamics of the employment rate apply directly to the growth rate.

4. Concluding Remarks

We have constructed a simple discrete-time model that is capable of generating chaotic growth paths. It can be interpreted as a discrete-time version of Goodwin's (1967) growth cycle model with the exception that his real wage formation equation — a Phillips-curve — is replaced by the bargaining Eq. (6). This equation was linearized to make the model simpler to analyze. In so doing, we abstracted from such factors as the subsistence real wage and the minimum profit rate that would have kept the cyclic and chaotic solutions of the model within a narrower band. It turned out that the model produced stable growth at the natural rate for the parameter values such that $0 < r < 2$ where $r = (1 - \mu g + \alpha)/\mu\, (1+g)$. However, for $r > 2$ the model first gave cyclic solutions having periodicities 2^k where k is a positive integer until finally for $r > 2.570$ the solutions proved to be irregular or chaotic. The nature of the growth paths in this chaotic regime were seen to depend on the initial conditions and they looked noisy, like the sample function of a stochastic process.

Bearing in mind that Goodwin's model is based on assumptions that are quite standard in prototype macro-economic models and that the effects of their generalizations on economic growth have been examined elsewhere [Akerlof and Stiglitz (1969), Desai (1973), Pohjola (1979), Velupillai (1979)], we concentrate on the modifications to the original model that we have made in this paper in trying to judge the economic relevance of the results. Firstly, Kuh's (1967) wage formation Eq. (6) can be regarded as an alternative to the Phillips-curve on both theoretical and empirical grounds, and, as was already mentioned, the linear approximation that was used is in no way necessary for the qualitative results obtained. Besides, it is known that the likelihood of chaotic solutions increases in higher dimensional models [Guckenheimer, Oster and Ipaktchi (1977)]. Then, we should expect our model to produce cyclic and chaotic solutions for lower values of r if we replaced our bargaining equation with a Phillips-curve where the rate of the real wage increase is taken to depend on employment, since we need two variables to describe growth paths. The analysis of higher dimensional models is, however, much more difficult although Marotto (1978) has already proven a general existence theorem for chaotic solutions.

Secondly, we should show that discrete-time modelling is the relevant way to characterize lags in economic behaviour. However, since economic agents do not all have to revise their behaviour at the same time and since there are many agents, discrete-time models might be regarded as inappropriate. But there are many factors, such as wage bargaining and price controls, that fix the levels of some variables for a certain period even at the macro-economic level. What is essential in our case is that difference models, as opposed to first-order differential equations, make the overshooting of the equilibrium possible. As far as the income distribution is concerned, we personally regard this possibility as a realistic feature of capitalist economies. In a more general context the existence of discrete lags is not, however, necessary for chaos to occur since it is known that even differential models of a dimensionality higher than two can possess irregular solutions [May (1976)]. This means that chaos might be a more likely state of affairs in real and model economies than what our simple system implies.

As for the empirical realism of the numerical results obtained, we recall that the model was not constructed to picture reality but to demonstrate how economic hypotheses that we regard as realistic can have cyclic and chaotic consequences. If we view E^* as a parameter and assume that $s_p=1$, $s_w=0$, $\mu=3.7$, $g=0.05$ and $E^*=0.97$, implying that $A^*=0.82$, the model produces stable cycles of period 2^k for $8.00<\beta<10.28$ and chaos for $10.28<\beta<12.00$. Changing the savings propensities to $s_p=0.5$, $s_w=0.1$ and keeping the other parameters as above, 2^k-period cycles are created when $20.00<\beta<25.70$ and chaos follows if $25.70<\beta<30.00$. These β-values may be too high to occur in practice.

The possibility of chaos has, however, some interesting implications for the business cycle theory and for economic modelling in general. Firstly, since Slutzky's (1937) seminal paper, random factors have been used in business cycle models to explain the observed irregularities in the cyclic behaviour of economies. Chaotic solutions of deterministic difference systems offer an alternative explanation: non-linearity combined with discrete adjustment mechanisms can generate the irregularities. In fact, non-linearity of economic relations can alone create the observed behaviour in models of a dimensionality higher than two.

Secondly, chaos has some disconcerting implications for policy analysis that is based on simulation experiments alone as can be seen from Fig. 3. (c)—(e) picture solutions in the chaotic regime. They all look noisy and depend crucially on the initial condition as (d) and (e) illustrate. This fact may make the simulation analysis of policy changes difficult and statistical methods should be employed. Also, as the trajectory in (e) shows, the model looks deceitfully stable after a few iterations, the trajectory being "locked in" near the equilibrium. Thus, a careful analysis of the simulation results is needed.

Finally, chaos has interesting implications for the rational expectations literature. If the economy happens to be in the chaotic regime, then, even if economic agents know perfectly how the economy functions, they are unable to predict its behaviour, except probabilistically. As far as we know, this possibility has not been examined in the literature.

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