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THE EMERGENCE OF CHAOS FROM CLASSICAL ECONOMIC GROWTH*

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This paper shows how fluctuations of an erratic and unstable nature can emerge from the classical, deterministic economic growth process. Implications of the analysis would appear to be at least two. First, pronounced changes in the way an economy behaves need not cause us to reject our understanding of how it works. Second, we need not seek in exogenous forces an explanation as to why behavioral patterns change and why it may be so difficult to anticipate future events from the profile of past experience.

I. INTRODUCTION

This paper is concerned with the emergence of erratic fluctuations in economic growth processes, fluctuations of a highly irregular and unstable nature termed "chaotic" in the mathematical literature, that emerge endogenously through the interplay of technology, preferences, and behavioral rules alone, with no exogenous interference from stochastic shocks. For the purpose of introducing this subject, we reconsider the familiar classical model of productivity and population growth. This theory provides a convenient and natural starting point for such an investigation: convenient because it provides the simplest possible setting in which to study chaos; natural because Malthus [1817, p. 91], who gave that theory its definitive form, emphasized the likelihood that oscillation must have been the common mode of population and income dynamics in early times. ("A faithful history ... would probably prove the existence of retrograde and progressive movements ... [and] ... the times of their vibrations must necessarily be rendered irregular.")

The formal theory of "chaos" in deterministic dynamic systems is of relatively recent origin, and although some early investigators in other fields (e.g., Ford [1977]) suggested its possible relevance for economics, our own work [Benhabib and Day, 1981, 1982a, 1982b] and that of Stutzer [1980] are apparently the first attempts to study the phenomenon in economic models. For this reason, we briefly review the basic properties and sufficient conditions for chaos before proceeding to an analysis of the classical model. After these general considerations, examples involving specific production and birth rate

* The simulations displayed in Figures II and III were computed by Ken Hanson, Department of Economics, University of Southern California.

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functions are given. The paper concludes with remarks on related applications and interpretations.

The classical growth theory that we use here is familiar terrain to most economists. Yet the reader should be prepared for a few surprises: primarily the nature of mathematical "chaos" itself, of course, but also the ability of a nonlinear model to evolve strikingly different patterns of behavior, endogenously evolving from one pattern to another with no corresponding change in underlying structure. Two numerical simulations illustrate this phenomenon, as well as Malthus' "irregular . . . retrograde and progressive population movements."

II. Chaos

The type of dynamic phenomenon of interest in the present discussion has a recent, but surprisingly varied, history and has arisen in several different scientific and mathematical contexts. See, for example, Lorenz [1964] (meteorology), Ulam [1963] and Ruelle and Takens [1971] (physics), and May [1974] (biology). In the present study classical economic assumptions about productivity, population growth, and income distribution are used to derive a single, real-valued difference equation in human population:

$$x_{t+1} = \theta(x_t). \tag{2.1}$$

A trajectory generated by the map $\theta{:}x \rightarrow \theta(x)$ is a sequence $\{x_t\}_0^\infty$ whose elements satisfy (2.1). Chaotic trajectories are defined precisely by Li and Yorke [1975]. Their formal definition is reproduced below in the Appendix to this paper. In words, the properties of chaos are four. First, periodic cycles of every order exist. Second, there exists a "scrambled set" of chaotic trajectories containing no periodic cycles. Trajectories in this set have the following character. First, every chaotic trajectory wanders away from any other chaotic trajectory. Second, all chaotic trajectories wander arbitrarily close to each other. Third, chaotic trajectories wander away from any periodic trajectory: they are aperiodic and fail to converge to a cycle of any order. Evidently, chaotic trajectories are highly unstable and possess features common to our experience with numerical economic data that fluctuate, usually in a highly irregular manner and with exasperating unpredictability.

It is both natural and relevant to inquire under what conditions chaotic behavior may emerge from a model incorporating specific economic structures. The basic tool that we use in this paper for this purpose is the following:

THEOREM (Li-Yorke). Let the function θ of (2.1) be a continuous map of an interval $J \to J \subset R$. Suppose that there exists a point $x \in J$ such that

$$\theta^3(x) \leqq x < \theta(x) < \theta^2(x). \tag{2.2}$$

Then

(i) for every $k = 1,2,3,\ldots$ there exists a k-periodic trajectory in J, and

(ii) θ is chaotic on an uncountable scrambled set $S \subset J$.

The interpretation of this theorem and exegesis of its proof are found together with a great deal of useful material on the associated bifurcation theory in a number of studies. See, for example, May [1976], May and Oster [1976], Gukenheimer, Oster, and Ipaktachi [1977] and Stutzer [1980]. Elsewhere we have used it to develop a theory of erratic rational behavior, of chaotic dynamics for pure exchange economies with overlapping generations, and of irregular investment cycles in models of capital accumulation. See Benhabib and Day [1980, 1981a, and 1981b] and Day [forthcoming].

Our concern is limited here to continuous difference equations whose generating map θ can, for certain parameter values, take on a "single-humped" form with $\theta(0) = 0$ and with $\theta(x)$ increasing with $\theta'(x) > 1$ for sufficiently small positive x. For such a map we can define a maximum attainable population x^m generated by a preimage or maximizing population x^* by

$$x^m = \theta(x^*) = \max_{x \geqq 0} \theta(x) > 0. \tag{2.3}$$

In Figure I(b), (c), and (d), such populations exist though not in Figure I(a). Because of θ's single-humped nature the population x^* may have two preimages, the smaller of which we denote x^c. Of course, $\theta(x^c) = x^*$. If $\theta(x^m) \geqq 0$, then any initial condition x must map into the interval $[0,x^m]$ so that a set J required by the Li-Yorke theorem exists. The sufficient condition for chaos (2.2) can now be reexpressed as

$$0 < \theta(x^m) \leqq x^c < x^* < x^m. \tag{2.4}$$

The procedure to be followed is now clear. First, establish parameters, if such can be found, so that x^m exists as defined by (2.3). Then find parameters such that the four inequalities of (2.4) can be satisfied. This is now accomplished for several versions of the classical economic growth model.

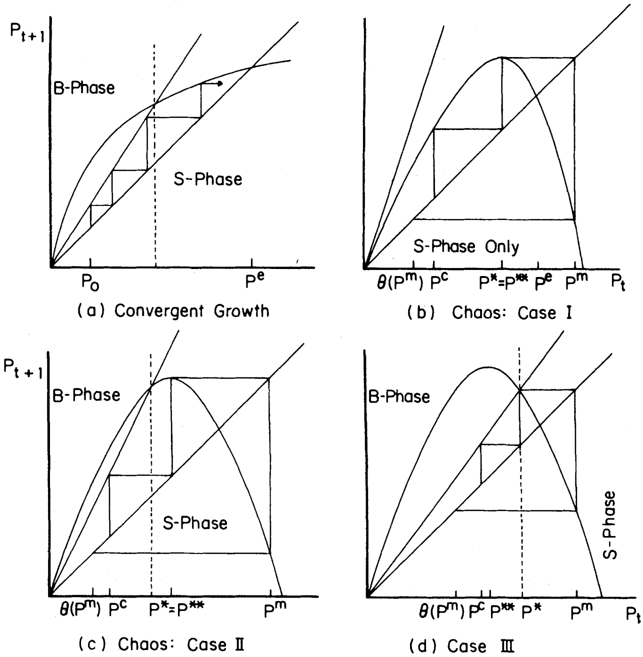

FIGURE I
The Classical Agrarian Economy
Convergent Growth and Three Types of Chaos

III. THE CLASSICAL AGRARIAN ECONOMY

In its simplest form the classical growth theory is based on three ingredients: an equation relating the net birth rate to income, a production function describing "the immediate product of labor," and a distribution function that defines the wages of labor. Malthus argued that when the necessities of life were in abundance, population tended to grow at a maximal biological or natural rate, say λ; when they were scarce, he assumed that net population birth rates were the maximum attainable under a culturally determined subsistence level, say σ. Accordingly, the population rate of growth in per capita terms is governed by the function,

$$\Delta P/P = \min\{\lambda,(w - \sigma)/\sigma\}, \tag{3.1}$$

in which $\Delta P/P$ is the net birth rate and w is the wage rate. Letting $\Delta P = P_{t+1} - P_t$, we see that the population growth equation becomes

$$P_{t+1} = \min\{(1 + \lambda)P_t,w_tP_t/\sigma\}. \tag{3.2}$$

Following Malthus, we note that the time unit is a "generation" of twenty-five years so that (3.2) describes the evolution of a sequence of generations.[1]

Consider an egalitarian, agrarian society in which aggregate output, as determined by the production function $Y = f(P)$ is distributed according to the average product (see Georgescu-Roegen [1960]) so that

$$w_t = f(P_t)/P_t. \tag{3.3}$$

It is assumed that $f(0) = 0$ and that f is continuous and "single-humped." Then (3.2) becomes

$$P_{t+1} = \theta(P_t) := \min\{(1 + \lambda)P_t, f(P_t)/\sigma\}, \tag{3.4}$$

which, though it is made up of two segments and is kinked, is continuous and single-humped. Several phase diagrams for equation (3.4) are shown in Figure I.

Assuming that the natural growth rate λ is not too big, we see that the history of the economy is governed by two regimes, one in which the rate of growth is constrained by the natural rate λ and one in which the means of subsistence govern population. The former may be referred to as the "biological- or B-phase"; the latter as the "subsistence- or S-phase." In Figure I(a) monotonic growth occurs in the biological phase for a time leading to a stationary population P^e, where $f(P^e) = \sigma P^e$, i.e., where output is just sufficient to sustain the population at the subsistence level. In Figure I(b)–(d) a population overshoot occurs and cycles emerge with periods of plenty alternating with periods of famine.

Given that f is single-humped then P^m exists, and

$$\begin{aligned} P^m &= \theta(P^*) = \min\{(1 + \lambda)P^*, f(P^*)/\sigma\} \\ &= \max_{P \geqq 0} \min\{(1 + \lambda)P, f(P)/\sigma\}. \end{aligned} \tag{3.5}$$

Defining P^c to be the preimage of P^*, i.e., so that $\theta(P^c) = P^*$, we see that the chaos theorem will be satisfied if

$$0 < \theta(P^m) \leqq P^c < P^* < P^m. \tag{3.6}$$

The second inequality of (3.6) implies that

$$f(P^m)/(P^m) \leqq (P^c)/(P^m)\sigma; \tag{3.7}$$

1. I have used as literal an interpretation of Malthus as possible. Earlier formal explorations of the classical model include Samuelson [1948, pp. 296–98] and Baumol [1970, pp. 266–68]. Explicit inclusion of Malthus' natural growth constraint simplifies the application of the sufficient conditions.

that is, at maximum population P^m the average product of labor falls below subsistence by the fraction P^c/P^m. Or alternatively, production falls below the level required to sustain the population P^c at a subsistence level; i.e.,

$$f(P^m) \leqq \sigma P^c. \tag{3.8}$$

Because of the two-phase, piecewise character of (3.2), several types of oscillation are possible. To identify these, let P^{**} be the preimage of the maximum population when the biological constraint is disregarded; i.e.,

$$\theta(P^{**}) = \max_{P>0} \frac{f(P)}{\sigma}. \tag{3.9}$$

(Of course, $P^* \geqq P^{**}$.) Three distinct cases are now seen to arise, depending on the natural growth rate and the parameters of the production function $f(\cdot)$. These are as follows:

Case I: P^c, P^*, and P^m all lie on the subsistence phase.

Case II: P^c lies on the biological phase; P^* and P^m lie on the subsistence phase.

Case III: P^c, P^*, and P^m are generated by the biological regime; P^m simultaneously lies on the subsistence phase.

In Case I, shown in Figure I(b), the natural rate is so large that the Li-Yorke conditions become

$$0 \leqq f(P^m)/\sigma \leqq P^c < P^{**} < P^m = f(P^{**})/\sigma, \tag{3.10}$$

where $f(P^c) = \sigma P^{**}$. In Case II, shown in Figure I(c), $P^* = P^{**}$, and $P^c = P^{**}/(1 + \lambda)$. Hence, the sufficient conditions become

$$0 \leqq \frac{1+\lambda}{\sigma} f\left[\frac{f(P^{**})}{\sigma}\right] \leqq P^{**} < \frac{f(P^{**})}{\sigma}. \tag{3.11}$$

In Case III, shown in Figure I(d), the natural growth rate is smaller, ceteris paribus, than in Cases I and II so that populations P^c, P^*, and $f(P^m)/\sigma$ are constrained to lie in the biological phase. Hence, $P^* > P^{**}$, and $P^m = (1 + \lambda)P^* = f(P^*)/\sigma$. From these facts and noting that $P^c = P^*/(1 + \lambda)$, we find that in Case III the sufficient conditions for chaos, given the existence of an overshoot, boil down to the inequalities,

$$0 \leqq f(P^m)/P^m \leqq \sigma/(1 + \lambda)^2. \tag{3.12}$$

That is, the average product of labor at the maximum population must be positive but less than the fraction $1/(1 + \lambda)^2$ of subsistence.

IV. EXAMPLES

The striking range of qualitative modes of behavior inherent in the classical model and the evolution of chaos when the sufficient conditions are satisfied can be illustrated when the production function is specified. A flexible function that may be used for this purpose is

$$f(P) = AP^{\beta}(1 - P)^{\gamma}, \tag{4.1}$$

in which the term AP^{β} is the usual power production function and the term $(1 - P)^{\gamma}$ represents a productivity-reducing factor caused by an excessively concentrated population. With this technology the difference equation (3.5) becomes

$$P_{t+1} = \min\{(1 + \lambda)P_t, AP_t^{\beta}(1 - P_t)^{\gamma}/\sigma\}. \tag{4.2}$$

Example 1. For simplicity, let $\beta = 1$, and let γ be quite small. This means that the productivity-inhibiting factor will not have much influence until P is large (i.e., close to one). In the absence of a natural growth rate constraint the maximum of (4.2) would be reached at

$$P^m = \frac{A/\sigma}{1 + \gamma}\left(\frac{\gamma}{1 + \gamma}\right)^{\gamma}, \tag{4.3}$$

which is attained at the preimage,

$$P^{**} = 1/(1 + \gamma). \tag{4.4}$$

Thus, the smaller γ is, the more dramatic will be the population overshoot, and the sharper the decline in population once P^{**} is exceeded. Now suppose that Case III occurs. This implies that

$$A\left[(1 + \lambda)\left(\frac{(1 - \lambda)}{A}\right)^{1/\gamma} - \lambda\right]^{\gamma} \leqq \frac{\sigma}{(1 + \lambda)^2} \tag{4.5}$$

is a sufficient condition for chaos.

Example 2. Equation (4.2) was simulated for parameter values that satisfy the Case III sufficient condition just derived (4.5).[2] The trajectory is shown in Figure II. A period of growth in the biological phase passes into a wandering oscillation with relatively small population changes interspersed with very large fluctuations at sporadic intervals. Notice that a stationary state is approximated twice in the simulation but that erratic cycles gradually emerge.

2. The parameter values are $\sigma = \beta = 1$, $\gamma = 0.25$, $\lambda = 0.75\gamma$, $A = (1 + \gamma)[(1 + \gamma)/\gamma]^{\gamma}$, $P_0 = 0.10$.

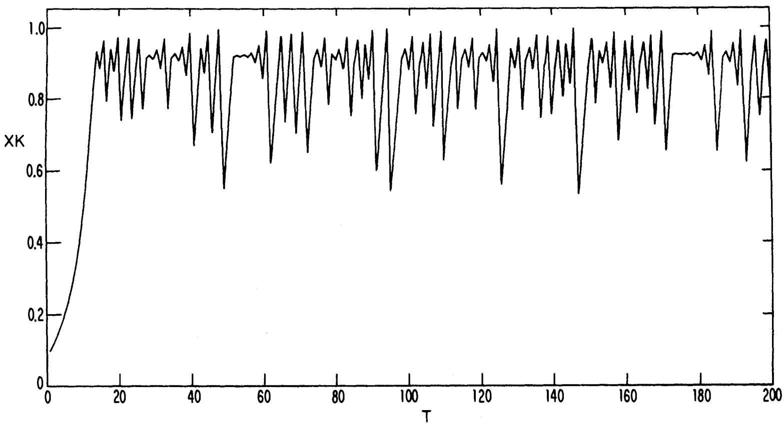

FIGURE II
The Emergence of Chaos from the Classical Population Model

Example 3. Suppose that $\beta = \lambda = 1$ in (4.1). Then the production function assumes the quadratic expression,

$$f(P) = AP(1 - P). \tag{4.6}$$

Suppose also that $\sigma = 1$ and that $\lambda = 1$, a natural growth rate that allows population to double every generation as Malthus mistakenly assumed. (The growth rate was about half of what he thought.) This gives Case I for which $P^* = P^{**} = 1/2$ and $P^m = A/4$. Hence, the sufficient condition for chaos is found to be

$$A^2(4 - A)/16 < P^c, \quad \text{where} \quad AP^c(1 - P^c) = 1/2. \tag{4.7}$$

For this Case I example the subsistence phase holds for all initial conditions $P_0 \geqq P^c$ so that in the interval $[P^c, A/4]$,

$$P_{t+1} = AP_t(1 - P_t). \tag{4.8}$$

This equation has been much studied, and it is known that the chaos threshold is about 3.57, somewhat less than that given by the Li-Yorke sufficient conditions. Consequently, for all A such that

$$3.57 \leqq A < 4, \tag{4.9}$$

chaotic trajectories exist. (See Hoppensteadt and Hyman [1977].)

V. HAAVELMO'S MALTHUSIAN MODEL

In the examples presented so far, the nonlinearity responsible for engendering chaos came from a downturn in the production function. We now eliminate this characteristic of the model by setting $\gamma = 0$ and show how a modified birth rate equation can be responsible when net births fall sharply enough as wages fall below subsistence. For this purpose we consider the equation,

$$\Delta P/P = \eta - \delta/w, \tag{5.1}$$

used by Haavelmo [1956], who expressed keen enthusiasm for Malthus' insights on economic growth. Combining this with Malthus' natural growth rate constraint gives

$$\Delta P/P = \min\{\lambda, \eta - \delta/w\}. \tag{5.2}$$

Now, the population equation,

$$P_{t+1} = \min\{(1+\lambda)P_t, (1+\eta)P_t - \delta/AP_t^{2-\beta}\}, \tag{5.3}$$

is obtained.[3]

Considering Case III, again for simplicity, we calculate that $P^* = [(\eta - \lambda)A/\delta]^{1/(1-\beta)}$. From (3.12) we get the inequality,

$$\frac{(1+\eta)P^m - (\delta/A)(P^m)^{2-\beta}}{P^m} \leqq \frac{\sigma}{(1+\lambda)^2}, \tag{5.4}$$

which reduces to

$$1 + \eta - (\delta/A)(P^m)^{1-\beta} \leqq \sigma/(1+\lambda)^2. \tag{5.5}$$

But for Case III, $P^m = (1+\lambda)P^*$. Substituting into the preceding inequality, we obtain the following sufficient condition for chaos:

$$0 \leqq 1 + \lambda - (1+\lambda)^{1-\beta}(\eta - \lambda) \leqq \sigma/(1+\lambda)^2. \tag{5.6}$$

Letting $(1+\lambda) = \sqrt{2}$, $\sigma = 1$, and $\beta = \frac{1}{2}$, we may boil this expression down to

$$\eta \geqq \frac{1 + 2^{7/4} - 2^{5/4}}{2^{5/4} - 2} \sim 5.235. \tag{5.7}$$

Hence, chaos can eventually emerge in the presence of a natural rate that would involve a doubling of population every two generations, a quite reasonable number given the historical record.

VI. Concluding Comments

For nonlinear systems behavior is related to the time period for which the dynamic structure is defined. In the present case the time

3. Stutzer [1980] studies the pure Haavelmo model that is obtained when (4.12) is substituted directly into (5.1). Our Malthusian form has the advantage of enabling us to obtain analytically the qualitative sufficient conditions for chaos. It also exhibits more realistic population fluctuations.

unit is interpreted as a "generation" of twenty-five years. This was the period that the classicals thought appropriate for the study of long-run dynamics.

The kind of phenomenon under study here can also be derived for continuous time models, that is, for differential equations. Such equations, to exhibit chaos, must be of the third order, which would require more advanced considerations as well as a greater dependence on numerical calculation. (See Yorke and Yorke [1979] for a review.) As the latter would have to be carried out in discrete time anyway, there is some advantage in considering the simplest discrete time theory first. The difference equation theory has been generalized by Diamond [1976] to n-dimensional systems and by ourselves [Benhabib and Day, 1982a] to the kind of set-valued dynamical systems that arise in general economic theory. What we have here then is merely an introduction to the application of the chaos concept to economics.

In the classical model humans are not represented as exhibiting farsighted behavior, which raises the question as to whether or not chaos may be due to myopia. In the papers already cited Benhabib and I show that this is indeed not the case, for there an overlapping-generations structure is used in which the young generation is assumed to plan for its old age and to exploit "rational expectations" or perfect foresight. Elsewhere, irregular investment cycles have been obtained in simulation models where agents make plans on the basis of a relatively long planning horizon (see Müller and Day [1978]).

Because the kind of trajectories under discussion here are highly unstable and unpredictable so that small perturbations in parameters or initial conditions lead to large divergencies in trajectories, it would seem appropriate to study the *distribution* of values generated by the model. This is the approach taken in ergodic theory and already applied to equation (4.3) by Lorenz and later by Hoppensteadt and Hyman in the references already cited. Lorenz found that the mean value of trajectories as a function of the parameter A in (4.3) was itself highly unstable in the interval [3.8+,4] exhibiting "wild oscillations" as A approached four. Hoppensteadt and Hyman computed the density functions for various values of A with similar results. And not only does the density function change drastically from one value of A to another, but for some values of A the density function is itself unbelievably complex, exhibiting extreme oscillations through its range.

The Li-Yorke proof uses a technique originated by Smale [1966] that involves studying iterates or higher powers of the generating function θ. It shows that the passage of time has the effect of stretching and folding the interval J within which the system's history

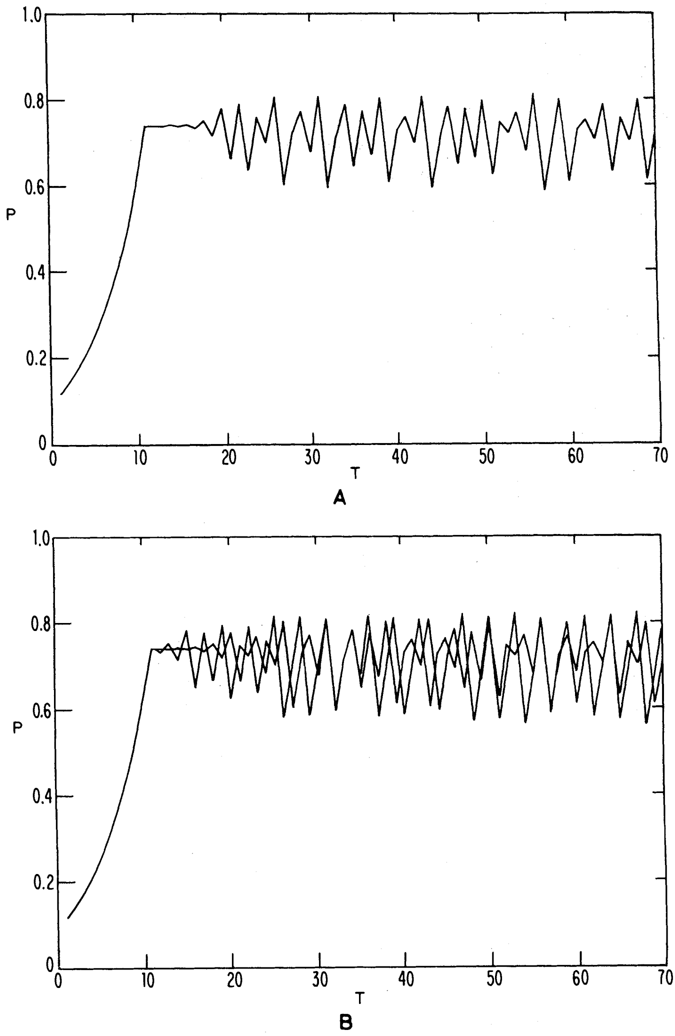

FIGURE III
The Instability of Chaotic Trajectories

is confined. The analogy has been suggested that this is somewhat like card shuffling, which provides an intuitive explanation of the more or less random fluctuation in the values of the cycles.

The instability of chaos is illustrated in Figure III.[4] Diagram III(A) illustrates a period of growth that appears to culminate in a

4. The parameter values for Figure III(A) are $\sigma = \beta = \gamma = 1$, $\lambda = 0.2$, $A = 3.8284$, $P_0 = 0.11929$; for Figure III(B) they are the same except $P_0 = 0.12$.

stationary state. Instead, a complex saw-tooth pattern develops. In III(B) a new trajectory is superimposed on the one of III(A). It was obtained by a 1 percent change in initial condition. Notice that once the fluctuating mode is reached, the time paths depart rapidly. Note, also, that they wander "close" several times over the seventy-period run only to depart rapidly each time!

APPENDIX

The following make precise the definitions used in the paper.

DEFINITION 1. An iterated map $x \to \theta^k(x)$ is defined recursively by $\theta^0(x) = x$ and $\theta^k(x) = \theta(\theta^{k-1}(x))$.

DEFINITION 2. A point x is k-periodic for θ if $\theta^k(x) = x$ and $\theta^i(x) \neq x$ for $0 < i < k$.

Li and Yorke [1975] give the following precise definition of chaos.

DEFINITION 3. Let J be an interval in R that is closed under θ, i.e., $\theta(x) \in J$ all $x \in J$. Suppose that there exists an uncountable set $S \subset J$ containing no periodic points such that X is closed under θ, and

(a) for all $x,y \in S$ such that $x \neq y$:

$$\lim_{t \to \infty} \sup |\theta^t(x) - \theta^t(y)| > 0,$$

$$\lim_{t \to \infty} \inf |\theta^t(x) - \theta^t(y)| = 0, \text{ and}$$

(b) for all periodic points $x \in J$ and all $y \in S$,

$$\lim_{t \to \infty} \sup |\theta^t(x) - \theta^t(y)| > 0,$$

then the map θ (difference equation (2.1)) is *chaotic on* S, and S is called a *scrambled set*.

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