Economic Modelling 29 (2012) 1197–1204

Contents lists available at SciVerse ScienceDirect

Economic Modelling

journal homepage: www.elsevier.com/locate/ecmod

Population dynamics and utilitarian criteria in the Lucas–Uzawa Model

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ARTICLE INFO

Article history:
Accepted 22 January 2012

JEL Classification:
O40
O41
J13

Keywords:
Population change
Utilitarian criteria
Uzawa–Lucas model
Transitional dynamics
Stochastic shocks
Closed-form solution

ABSTRACT

This paper introduces population growth in the Uzawa–Lucas model, analyzing the implications of the choice of the welfare criterion on the model's outcome. Traditional growth theory assumes population growth to be exponential, but this is not a realistic assumption (see Brida and Accinelli, 2007). We model exogenous population change by a generic function of population size. We show that a unique non-trivial equilibrium exists and the economy converges towards it along a saddle path, independently of population dynamics. What is affected by the type of population dynamics is the dimension of the stable manifold, which can be one or two, and when the equilibrium is reached, which can happen in finite time or asymptotically. Moreover, we show that the choice of the utilitarian criterion will be irrelevant on the equilibrium of the model, if the steady state growth rate of population is null, as in the case of logistic population growth. Then, we show that a closed-form solution for the transitional dynamics of the economy (both in the case population dynamics is deterministic and stochastic) can be found for a certain parameter restriction.

1. Introduction

In standard economic growth theory, population is assumed to grow at an exogenous and exponential rate. This assumption has been firstly introduced by the Solow–Swan model (1956) and it has been applied also to following models with optimizing behavior, as the single-sector Ramsey–Cass–Koopmans (1965) model and the two-sector Lucas–Uzawa (1988) model. Such an assumption however is not without consequences for the analysis of growing economies. In fact, exponential population growth models imply unconstrained growth of population size. However, most populations are constrained by limitations on resources, at least in the short run, and none is unconstrained forever. For this reason, firstly Malthus (1798) discusses about the inevitable dire consequences of exponential growth of the human population of the earth. Recently, Brida and Accinelli clearly state: *"The simple exponential growth model can provide an adequate approximation to such growth only for the initial period because, growing exponentially, as $t\rightarrow\infty$, labor force will approach infinity, which is clearly unrealistic. As labor force becomes large enough, crowding, food shortage and environmental effects come into play, so that population growth is naturally bounded. This limit for the population size is usually called the carrying capacity of the environment".*

Some decades ago, Maynard Smith (1974) concluded that the growth of natural populations is more accurately depicted by a logistic law. This result has been recently used to claim that such a dynamics can probably better describe also human population growth. In fact, several studies support the idea that human population growth is decreasing and tending towards zero[1] (as Day, 1996). Even the Belgian mathematician Verhulst in the XIX century studies this idea; using data from the first five U.S. censuses, he makes a prediction in 1840 of the U.S. population in 1940 and was off by less than 1%. Moreover, based on the same idea, he predicts the upper limit of Belgian population; more than a century later, but for the effect of immigration, his prediction looks good (Verhulst, 1838). More recently, several studies try to understand which function fits better human population dynamics, showing that the exponential growth is reductive. For example, population dynamics can be described through a non-autonomous differential equation as $N_t = N_t g((N_t)$, where $g(N) = \sum_{i=0}^{m} g_i N^i$. The estimation of the parameters g_i can be done by using, for instance, fractal-based methods and penalization methods[2] as proposed and well-illustrated in Kunze et al. (2007a, 2007b, 2009a, 2009b, 2010) and Iacus and La Torre (2005a, 2005b). Table 1 provides the results to eight decimal digits by using data in six continents (Africa, Asia, Australia, Europe, South America, and North

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[1] According to up-to-date demographic forecasts of the United Nations, the world population annual growth rate is expected to fall gradually from 1.8% (1950–2000) to 0.9% (2000–2050), before reaching a value of 0.2% between the years 2050 and 2100.

[2] See also La Torre (2003), La Torre and Rocca (2003,2005); La Torre and Vrscay (2009) for further details on the fractal-based methods.

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doi:10.1016/j.econmod.2012.01.016

Table 1
Parameters' estimations of population growth over the period 1870–2008.

	g_0	g_1	g_2	g_3
Africa	−0.00763537	0.00000018	−0.00000000	0.00000000
Asia	−0.02926752	0.00000005	−0.00000000	0.00000000
Australia	0.03003342	−0.00000383	0.00000000	−0.00000000
Europe	0.12968633	−0.00000070	0.00000000	−0.00000000
South America	0.00203530	0.00000025	−0.00000000	0.00000000
North America	0.03432962	−0.00000030	0.00000000	−0.00000000

America) over the period 1870–2008. A good fitting curve for Australia, Europe, and North America for this data is the logistic one while South America shows an exponential behavior (g_0, $g_1>0$). Africa and Asia show a negative coefficient g_0 which can be justified in terms of migration effects.

Accinelli and Brida (2005) firstly introduce non-exponential population growth in a growth model, assuming that population dynamics is described by a logistic function. After this work a growing literature studying how different demographic change functions modify standard growth models arises. For example, the Solow model has been extensively analyzed assuming different demographic dynamics. Guerrini (2006) and Brida and Pereyra (2008) introduce respectively bounded population growth (which represents a generalization of the logistic case) and a decreasing population growth in the Solow–Swan model; Bucci and Guerrini (2009) instead study its transitional dynamics in the case of AK technology and logistic population. Also the Ramsey model has been recently extended to encompass several types of population change functions. Brida and Accinelli (2007) study the case of logistic population growth while Guerrini (2009 and references therein) analyzes the case where population growth is given by a bounded function, both in the neoclassical and endogenous framework.

However, all these papers also relax an important standard assumption of optimal growth theory, namely the social welfare function is founded on the Benthamite criterion (total utilitarianism). This criterion says that total welfare is the sum of per-capita welfare across population (the product between population size and average welfare if no heterogeneity among agents is present). These papers[3] instead assume the social welfare function is based on the Millian criterion (average utilitarianism): total welfare equals average welfare or per-capita utility (see Marsiglio, 2010, for a discussion of the implications of both criteria). Such a criterion has been used in order to limit population size and in an optimal theory of growth seems to be somehow reductive. In fact, the main difference in the model's outcome is the effect of population growth on the per-capita consumption dynamics: the Benthamite criterion implies that consumption growth is independent of population dynamics, while the opposite is true for the Millian criterion.

Some papers in the literature discuss how the choice of total rather than average utilitarianism affects the outcome of the model. Such an issue has always been studied in a context of exponential population change, where the general conclusion is that the Benthamite and Millian criteria lead to different effects of population growth on economic performance. This issue is quite popular in the framework of endogenous fertility, in which the steady state outcome is represented by exponential population growth. For example, Nerlove et al. (1982, 1985) and Barro and Becker (1989) analyze a neoclassical setup while Palivos and Yip (1993) an endogenous growth context. Barro and Becker (1989) show that according to the degree of altruism towards future generations, the social welfare function results to be a mix of the Benthamite and Millian criteria. Palivos and Yip (1993) show instead that the Benthamite principle leads to a higher economic growth and a smaller population size. Few papers tackle the issue when population change is exogenous, namely Strulik (2005) and Bucci (2008). They both study the effect of exogenous population growth on the economic growth rate in an endogenous growth model driven by R&D activity, as the degree of agents' altruism towards future generation changes. They both show that the impact of demographic change on the economy varies as the magnitude of the altruism parameter does so. All these works assume population growth is exponential (at least in steady state) and suggest that different utilitarian criteria affect the economic growth rate.

The aim of this paper is studying the introduction of not exponential population change in endogenous growth models, and analyzing the effect of different utilitarian criteria on the model's outcome. We formalize demographic growth as a generic function of population size, discussing how different shapes affect the model. We focus our analysis on a two-sector model of endogenous-growth, á-la Uzawa (1965)–Lucas (1988), since, it has never been analyzed in a framework of non-exponential population growth and, as claimed in Boucekkine and Ruiz-Tamarit (2008), it is one of the most studied and interesting endogenous growth models. In Section 2 we introduce the model in its general form, namely we assume population change depends on a generic function of population size and the social welfare function results to be of the Benthamite or Millian type according to the value of a parameter (representing the degree of altruism). Section 3 performs steady state analysis, which is characterized by a balanced growth path or an asymptotic balanced growth path, according to the features of the population growth function. However, we show that independently on the shape of such a function, the economy converges towards its equilibrium along a saddle path. What is affected by its shape is the dimension of the stable manifold, which can be one or two. We also show the utilitarian criterion adopted is irrelevant for the economic growth rate if in steady state population growth is null, as in the case in which population growth is logistic. In Section 4, instead, we show different examples of population growth function which represent particular cases of our general model. In Section 5 we characterize the global dynamics of the model under a particular parametric restriction concerning the altruism parameter, namely in the case it equals both the capital share and the inverse of the intertemporal elasticity of substitution; in Section 6 we show that under the same condition it is possible to find a closed-form solution for the case in which population dynamics is subject to random shocks and show that uncertainty increases on average the stock of (per-capita) physical and human capital. Section 6 as usual concludes.

2. The model

The model is a Uzawa–Lucas model of optimal growth where the representative agent seeks to maximize his welfare subject to the capital and demographic constraints, choosing consumption, c_t, and the rate of investment in physical capital, u_t. The welfare is the infinite discounted sum of the product of the instantaneous utility function (assumed to be iso-elastic, $u(c_t)=\frac{c_t^{1-\sigma}}{1-\sigma}$, where $\sigma>0$) and the population size weighted by the agent's degree of altruism, $N_t^{1-\varepsilon}$, where $\varepsilon\in[0,1]$. The final good is produced combining physical capital, K_t, and the share of human capital allocated to final production, u_tH_t, according to a Cobb–Douglas technology: $Y_t=K_t^{\alpha}(u_tH_t)^{1-\alpha}$, where $0<\alpha<1$ and $u_t\in(0,1)$. Physical capital, K_t, accumulation is given by the difference between production of the final good and consumption activity: $\dot{K}_t=AK_t^{\alpha}(u_tH_t)^{1-\alpha}-c_tN_t$. The law of motion of human capital, H_t, is instead given by production of new human capital: $\dot{H}_t=B(1-u_t)H_t$. We assume for simplicity that physical and human capital do not depreciate over time. Demographic growth instead is given by a generic function of population size: $\dot{N}_t=N_tg(N_t)$. The shape of such a function, as we shall later show, results to be irrelevant for the equilibrium of the model; the transitional dynamics instead is differently affected by the fact that $g(\cdot)$ shows or not one or more zeros.

The social planner maximizes the social welfare function, that is it has to choose c_t and u_t, in order to maximize agents lifetime utility

[3] An exception is represented by La Torre and Marsiglio (2010). They introduce logistic population growth in a three sectors Uzawa–Lucas (1988) type growth model, in which the welfare function is defined according to the Benthamite criterion. However, since their goal is to focus on endogenous technical progress, they do not study population dynamics (because population size in steady state is constant, under the logistic assumption).

function subject to physical and human capital accumulation constraints, the demographic dynamics and the initial conditions:

$$\max_{c_t,u_t} \quad \int_0^\infty \frac{c_t^{1-\sigma}}{1-\sigma} N_t^{1-\varepsilon} e^{-\rho t} dt \tag{1}$$

$$s.t. \quad \dot{K}_t = AK_t^\alpha (u_t H_t)^{1-\alpha} - N_t c_t \tag{2}$$

$$\dot{H}_t = B(1-u_t)H_t \tag{3}$$

$$\dot{N}_t = N_t g(N_t) \tag{4}$$

$$K_0, H_0, N_0 \text{given.} \tag{5}$$

The term $1-\varepsilon$, $\varepsilon \in [0,1]$ represents the degree of intertemporal altruism. Notice that the degree of altruism towards future generations, given by the term $1-\varepsilon$, determines the type of social welfare function we are adopting. In fact, if $\varepsilon = 0$ ($\varepsilon = 1$), the social welfare is defined according to the Benthamite (Millian) criterion. In the former (latter) case, we are adopting total (average) utilitarianism.

2.1. Optimal paths

Notice that the dynamic equation for population change is an auxiliary equation (not a state or a control variable), since it is completely exogenous. Therefore, from the maximization problem we can set the Hamiltonian function (considering population dynamics as auxiliary):

$$\mathcal{H}_t(\cdot) \quad = \frac{c_t^{1-\sigma}}{1-\sigma} N_t^{1-\varepsilon} e^{-\rho t} + \lambda_t \left[AK_t^\alpha (u_t H_t)^{1-\alpha} - N_t c_t \right] + \nu_t B(1-u_t)H_t, \tag{6}$$

and derive the FOCs:

$$c_t^{-\sigma} N_t^{1-\varepsilon} e^{-\rho t} = \lambda_t N_t \tag{7}$$

$$(1-\alpha)AK_t^\alpha (u_t H_t)^{-\alpha} H_t \lambda_t = BH_t \nu_t \tag{8}$$

$$\lambda_t A\alpha K_t^{\alpha-1} (u_t H_t)^{1-\alpha} = -\dot{\lambda}_t \tag{9}$$

$$(1-\alpha)AK_t^\alpha (u_t H_t)^{-\alpha} u_t \lambda_t + \nu_t B(1-u_t) = -\dot{\nu}_t. \tag{10}$$

together with the initial conditions K_0 and H_0, the dynamic constraints:

$$\dot{K}_t = AK_t^\alpha (u_t H_t)^{1-\alpha} - N_t c_t \tag{11}$$

$$\dot{H}_t = B(1-u_t)H_t \tag{12}$$

and the transversality conditions:

$$\lim_{t\to\infty} \quad \lambda_t K_t = 0 \tag{13}$$

$$\lim_{t\to\infty} \quad \nu_t H_t = 0. \tag{14}$$

Solving the resulting system we obtain the Euler equations for per-capita consumption and share of human capital to be allocated to physical production:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma} \left[A\alpha K_t^{\alpha-1} (u_t H_t)^{1-\alpha} - \rho - \varepsilon g(N_t) \right] \tag{15}$$

$$\frac{\dot{u}_t}{u_t} = \frac{B(1-\alpha)}{\alpha} + Bu_t - \frac{N_t c_t}{K_t}. \tag{16}$$

Eqs. (15) and (16) are standard, unless for the presence of the term $-\varepsilon g(N_t)$ in the Euler equation of per-capita consumption. Population growth does not affect the path of the share of human capital allocated in the production sector while whether it does or not per-capita consumption growth depending on the adopted utilitarian criterion. If $\varepsilon = 0$, we are adopting a classical or total utilitarianism approach and population change is completely irrelevant for the dynamics of consumption, as in the standard Ramsey model. If instead $\varepsilon = 1$, welfare is based on average utilitarianism and population change has a negative impact on the dynamics of consumption, as for example in Brida and Accinelli (2007); the same is true for impure altruism values, that is $\varepsilon \in (0,1)$.

3. Steady state analysis

The dynamic behavior of the economy is summarized by Eqs. (2), (3), (4), (15) and (16). We now analyze the steady state of such an economy. We can study the dynamics of a simplified system, by introducing the intensive variables $\chi_t = \frac{N_t c_t}{K_t}$ and $\psi_t = \left(u_t \frac{H_t}{K_t}\right)^{1-\alpha}$, representing respectively the consumption–capital ratio and the average product of capital:

$$\frac{\dot{\psi}_t}{\psi_t} = \frac{B(1-\alpha)}{\alpha} - (1-\alpha)A\psi_t \tag{17}$$

$$\frac{\dot{\chi}_t}{\chi_t} = \frac{\alpha-\sigma}{\sigma} A\psi_t - \frac{\rho}{\sigma} + \chi_t + \frac{\sigma-\varepsilon}{\sigma} g(N_t) \tag{18}$$

$$\frac{\dot{u}_t}{u_t} = \frac{B(1-\alpha)}{\alpha} + Bu_t - \chi_t \tag{19}$$

$$\frac{\dot{N}_t}{N_t} = g(N_t). \tag{20}$$

Depending on the type of demographic dynamics, the equilibrium of the economy derives from a three-dimensional or a four-dimensional system. In fact, if the $g(\cdot)$ function does not show any zeros, we have a three-dimensional system, since a stationary population size does not exist and Eq. (20) does not imply any equilibrium value. A benchmark for such a case is represented by constant and exponential population growth. In the following discussion and analysis we focus on constant population growth, for a matter of tractability and since it is probably the most relevant case of growth function not showing any zeros. In such a case, the system of differential equations reduces to:

$$\frac{\dot{\psi}_t}{\psi_t} = \frac{B(1-\alpha)}{\alpha} - (1-\alpha)A\psi_t \tag{21}$$

$$\frac{\dot{\chi}_t}{\chi_t} = \frac{\alpha-\sigma}{\sigma} A\psi_t - \frac{\rho}{\sigma} + \chi_t + \frac{\sigma-\varepsilon}{\sigma} g_N \tag{22}$$

$$\frac{\dot{u}_t}{u_t} = \frac{B(1-\alpha)}{\alpha} + Bu_t - \chi_t. \tag{23}$$

The equilibrium point of such a system is characterized by strictly positive values for all the variables if $\rho > B(1-\sigma) + (\sigma-\varepsilon)g_N$, where g_N represents the constant and non-negative growth rate of population; in order to ensure that the share of human capital allocated to physical production is less than one we also need that $\rho < B + (\sigma-\varepsilon)g_N$. Notice that such a kind of conditions is almost standard in the literature and it is always satisfies for realistic parameter values if the growth rate of population is not too large. Such a system converges to its steady state equilibrium through a saddle path, along which the stable arm has dimension one. Moreover, we can study the implications of the utilitarian criteria on economic growth, simply analyzing the steady state values of the variables: the steady state of the consumption–capital ratio and the share of human capital allocated to physical production are affected by ε, while the average product of capital is not. Therefore, we can conclude that total utilitarianism leads to higher economic growth than average utilitarianism. We can summarize the main results in the following proposition:

Proposition 1. *Assume $B(1-\sigma)+(\sigma-\varepsilon)g_N<\rho<B+(\sigma-\varepsilon)g_N$; if the population growth function is constant, then the following results hold:*

- **(i)** *the economy converges to its steady state equilibrium, along a saddle path, and the stable arm is a one-dimensional locus;*
- **(ii)** *total utilitarianism ($\varepsilon=0$) implies a higher economic growth rate than average utilitarianism ($\varepsilon=1$) if $g_N>0$.*

Proof. From the system (21)–(23), the steady state levels of ψ_t, χ_t u_t are respectively $\psi^*=\frac{B}{\alpha A}$, $\chi^*=\frac{\rho\alpha-B(\alpha-\sigma)-\alpha(\sigma-\varepsilon)g_N}{\alpha\sigma}$ and $u^*=\frac{\rho-B(1-\sigma)-(\sigma-\varepsilon)g_N}{B\sigma}$. Appendix A.1 proves part (i). To prove part (ii) notice that the growth rate of per-capita consumption, from Eq. (15), in steady state is $\gamma=\frac{1}{\sigma}[A\alpha\psi^*-\rho-\varepsilon g_N]$; it is straightforward to see that its derivative respect to ε is negative: $\frac{\partial\gamma}{\partial\varepsilon}=-\frac{g_N}{\sigma}$. Therefore, the Benthamite criterion implies a higher economic growth than the Millian one. □

If instead the $g(\cdot)$ function shows any zeros, we have a four dimensional system since a stationary population size exists and therefore Eq. (20) implies an equilibrium value. The system of differential equations is the following:

$$\frac{\dot{\psi}_t}{\psi_t}=\frac{B(1-\alpha)}{\alpha}-(1-\alpha)A\psi_t \tag{24}$$

$$\frac{\dot{\chi}_t}{\chi_t}=\frac{\alpha-\sigma}{\sigma}A\psi_t-\frac{\rho}{\sigma}+\chi_t+\frac{\sigma-\varepsilon}{\sigma}g(N_t) \tag{25}$$

$$\frac{\dot{u}_t}{u_t}=\frac{B(1-\alpha)}{\alpha}+Bu_t-\chi_t \tag{26}$$

$$\frac{\dot{N}_t}{N_t}=g(N_t). \tag{27}$$

The equilibrium point of such a system is characterized by strictly positive values for all the variables if $\rho>B(1-\sigma)$, while u is less than 1 if $\rho<B$. As before such conditions are satisfied for realistic parameter values. By Eq. (27), the existence of a stationary population size is ensured if $g(N_t)=0$. Moreover, the equilibrium is saddle point stable in a generalized form, since the stable manifold has dimension one (two) and the unstable one has dimension three (two) if $\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*}>0(<0)$. As before, we can study the implications of average and total utilitarianism on the outcome of the model, simply analyzing the steady state values of the variables: the steady state values of all the economic variables are independent of ε. Therefore, the adopted utilitarian criterion affects only the transitional dynamics of the economy, but in steady state all the differences vanish. We can summarize this result in the following proposition:

Proposition 2. *Assume $B(1-\sigma)<\rho<B$; if the population growth function shows some zeros, then:*

- **(i)** *the economy converges to its steady state equilibrium, along a saddle path. The stable arm is a two-dimensional locus if $\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*}<0$, while it has dimension one if $\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*}>0$*
- **(ii)** *Whether the social welfare function is built on the Benthamite ($\varepsilon=0$) or the Millian ($\varepsilon=1$) criterion, the steady state growth rate of the economy does not change.*

Proof. From the system (24)–(27), the steady state levels of N_t, ψ_t, χ_t u_t are respectively N^* such that $g(N^*)=0$, $\psi^*=\frac{B}{\alpha A}$, $\chi^*=\frac{\rho\alpha-B(\alpha-\sigma)}{\alpha\sigma}$ and $u^*=\frac{\rho-B(1-\sigma)}{B\sigma}$. Appendix A.2 proves part (i). To prove part (ii) notice that in such a framework, since population growth is null in equilibrium, the growth rate of per-capita consumption, from Eq. (15), in steady state is $\gamma=\frac{1}{\sigma}[A\alpha\psi^*-\rho]$. It is straightforward to see that its derivative respect to ε is null and therefore the Benthamite and the Millian criteria do not imply any difference for the economic growth rate. □

We can notice that the equilibrium of such a model is only marginally affected by the shape of the population growth function. In fact, the economic variables (χ, ψ, u) converge to their equilibrium independently of the behavior of the demographic variable (N). The features of population dynamics affect mainly the timing of convergence, which can happen in finite time or asymptotically, characterizing the equilibrium respectively as a balanced growth path (BGP), as in the growth models with constant exponential population growth, or as an asymptotically balanced growth path (ABGP), as in the case of logistic population growth. If the population growth function shows any zeros, then its shape determines the dimension of the stable arm. In fact, if $\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*}<0$ the stable arm has dimension two while if $\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*}>0$ it has dimension one (implying uniqueness of the convergence path). We have just proved:

Proposition 3. *The economy described by (1)–(5) converges towards its unique (non-trivial) equilibrium independently of population dynamics. Demographic dynamics just determines the timing of convergence and the dimension of the stable arm.*

These results derive from the assumption that population growth is exogenous, and therefore it just represents an auxiliary variable in the optimal control problem (1)–(5). Under such an assumption, how we model this dynamics does not affect the main outcome of the model (clearly the equilibrium values of χ and u can change as we introduce a different law of motion for demography). What can change according to such a choice is the timing when the equilibrium is reached (finite or infinite) and the dimension of the stable arm (one or two) according to the features of the $g(\cdot)$ function.

4. Some examples of demographic change

In this section we discuss some examples of population dynamics introduced in the previous literature, showing how they are just particular cases of our general formulation. We consider the cases in which population is exponential, logistic and follows a von Bertalanffy law.

4.1. Exponential population

The standard assumption of growth theory on demography is that population growth is exponential and constant over time (see Solow, 1956). This in our general formulation represents the case in which $g(\cdot)$ is simply a constant:

$$g(N_t)=n, \tag{28}$$

where n can be positive, negative or null. If it is negative, population size constantly decreases and asymptotically will completely disappear; if it is null, population size is constant and it does not show any dynamics over time; if it is positive instead population constantly increases and it will asymptotically approach infinity. This last case gives birth to the critique to exponential demographic change, since it implies the absence of any natural and environmental limits to population growth. This specification implies that demographic dynamics is monotonic and Proposition 1 holds: in such a case the stable arm is a one-dimensional locus and the Benthamite criterion leads to a higher economic growth rate than the Millian one.

4.2. Logistic population

A first attempt to avoid the implications of exponential population growth has been the introduction of logistic demography (see Brida and Accinelli, 2007). This represents the case in which $g(\cdot)$ is:

$$g(N_t) = n - dN_t, \tag{29}$$

where n and d are both positive (if $d=0$, we are driven back to exponential growth), and n represents the trend of population growth. Notice that population dynamics is given by a Bernoulli-type differential equation which can be explicitly solved obtaining:

$$N_t = \frac{n}{d + \left(\frac{n}{N_0} - d\right)e^{-nt}}. \tag{30}$$

Population size therefore is increasing over time and it reaches a stationary level only when $t \to \infty$; in fact, $lim_{t\to\infty} N_t = \frac{n}{d}$. This formulation implies that population growth is null in steady state and therefore Proposition 2 holds: the stable arm is two-dimensional locus since $g'(\cdot)<0$ and the economic growth rate is independent of the adopted utilitarian criterion. If population growth is logistic, the equilibrium is only asymptotically approached since population size converges to its steady state value in the very long-run.

4.3. von Bertalanffy population

The von Bertalanffy population growth has been introduced by Accinelli and Brida (2007) to describe a population strictly increasing and bounded whose growth rate is strictly decreasing to zero. This function corresponds to:

$$g(N_t) = \frac{n(N_\infty - N_t)}{N_t}, \tag{31}$$

where N_∞ is the theoretical maximum population size and n determines the speed at which demography reaches its maximal level. The equation of population dynamics can be explicitly solved obtaining:

$$N_t = N_\infty - (N_\infty - N_0)e^{-nt}. \tag{32}$$

Population size therefore is increasing over time and it reaches a stationary level only when $t\to\infty$; in fact, $lim_{t\to\infty} N_t = N_\infty$. Also in this case Proposition 2 holds: the stable arm has dimension two since $g'(\cdot)<0$, and the utilitarian criteria do not imply any difference for the economic growth rate since the growth rate of population is null in steady state.

5. A closed-form solution

We now study the transitional dynamics of the economy. Since population growth is exogenous its entire dynamics is driven by the differential equation:

$$\dot{N}_t = g(N_t)N_t.$$

Notice that all the cases discussed in the previous section show a closed-form solution for their dynamic path, given by $N_t = N_0 e^{nt}$ for the constant growth case, Eq. (30) for the logistic case and Eq. (32) for the von-Bertalanffy one.

The dynamics of χ, ψ and u are instead given by:

$$\psi_t = \frac{e^{\frac{B(1-\alpha)}{\alpha}t}}{\psi_0^{-1} + \frac{\alpha A}{B}\left(e^{\frac{B(1-\alpha)}{\alpha}t} - 1\right)}$$

$$\chi_t = \frac{e^{\int_0^t \left[\frac{\alpha-\sigma}{\sigma}A\psi_s - \frac{\rho}{\sigma} + \frac{\sigma-\varepsilon}{\sigma}g(N_s)\right]ds}}{\chi_0^{-1} - \int_0^t e^{\int_0^s \left[\frac{\alpha-\sigma}{\sigma}A\psi_v - \frac{\rho}{\sigma} + \frac{\sigma-\varepsilon}{\sigma}g(N_v)\right]dv} ds}$$

$$u_t = \frac{e^{\int_0^t \left[\frac{B(1-\alpha)}{\alpha} - \chi_s\right]ds}}{u_0^{-1} - B\int_0^t e^{\int_0^s \left[\frac{B(1-\alpha)}{\alpha} - \chi_v\right]dv} ds}.$$

We now look for parameter restrictions allowing us to uncouple the equations of the system (17)–(20), as in Smith (2006), or equivalently to solve the integrals in the previous equations. This can be easily done when $\sigma=\varepsilon=\alpha$. In fact, in such a case, the evolution of u_t and χ_t can be rewritten as:

$$u_t = \frac{e^{\frac{B(1-\alpha)}{\alpha}t}\left(1 - \frac{\alpha\chi_0}{\rho}\right) + \frac{\alpha\chi_0}{\rho}e^{\frac{B(1-\alpha)-\rho}{\alpha}t}}{u_0^{-1} - \frac{\alpha}{1-\alpha}\left(1 - \frac{\alpha\chi_0}{\rho}\right)\left(e^{\frac{B(1-\alpha)}{\alpha}t} - 1\right) - \frac{\alpha^2 B\chi_0}{\rho[B(1-\alpha)-\rho]}\left(e^{\frac{B(1-\alpha)-\rho}{\alpha}t} - 1\right)}$$

$$\chi_t = \frac{e^{-\frac{\rho}{\sigma}t}}{\chi_0^{-1} + \frac{\sigma}{\rho}\left(e^{-\frac{\rho}{\sigma}t} - 1\right)}.$$

Notice that in order to have convergence to the nontrivial equilibrium, the initial conditions for the consumption–capital ratio and the rate of investment in physical capital need to be determined as follows: $\chi_0 = \frac{\rho}{\alpha}$ and $u_0 = \frac{\rho - B(1-\alpha)}{B\alpha}$. This allows us to find a full closed-form solution for the transitional dynamics of the control, c_t, u_t, and state variables $k_t \equiv \frac{K_t}{N_t}$, $h_t \equiv \frac{H_t}{N_t}$. The result is summarized in the following Proposition:

Proposition 4. *Assume $\sigma=\varepsilon=\alpha$. Then, the optimal paths of the control, c_t and u_t, and state, k_t and h_t, variables in the problem (1)–(5) are given by the following expressions:*

$$u_t = u = \frac{\rho - B(1-\alpha)}{B\alpha} \tag{33}$$

$$c_t = \frac{\rho}{\alpha}k_t \tag{34}$$

$$k_t = e^{-\frac{\rho}{\alpha}t - \int_0^t g(N_s)ds}\left[k_0^{1-\alpha} + \frac{Au^{1-\alpha}h_0^{1-\alpha}}{\frac{\rho}{\alpha} + B(1-u)}\left(e^{(1-\alpha)\left[\frac{\rho}{\alpha} + B(1-u)\right]t} - 1\right)\right]^{\frac{1}{1-\alpha}} \tag{35}$$

$$h_t = h_0 e^{B(1-u)t - \int_0^t g(N_s)ds}. \tag{36}$$

Proposition 4 tells us that the share of human capital employed in physical production and the consumption to capital ratio are constant and equal to their equilibrium level since time 0 (notice that they coincide with χ^* and u^*, under the condition $\sigma=\varepsilon=\alpha$). If the inverse of the intertemporal elasticity of substitution is impure and equals both the capital share and the altruism parameter, we can evaluate the dynamics of the control and state variables for all t. Notice that this kind of restriction is consistent with other previous works. For example, Smith (2006) studies the transitional dynamics of the Ramsey model under the assumption $\sigma=\alpha$; the same assumption has also been used by Chilarescu (2008) in the Uzawa–Lucas model with no population growth. In order to obtain the same result in a context of population growth, we need a further restriction: $\sigma=\varepsilon$.

6. The case of stochastic population

Demographic shocks consist of changes in population growth rates or immigration policies, and can have important macroeconomic effects.

Therefore, in this section we consider the case in which population dynamics is stochastic, namely we assume for simplicity that population is subject to random shocks and suppose that it follows an exogenous stochastic differential equation driven by a Brownian motion.

We therefore replace Eq. (4) in our model by a Brownian motion, as in Smith (2007), as follows:

$$dN_t = \mu N_t dt + N_t \theta dW_t; \tag{37}$$

where μ is the drift of the process driving population dynamics, while $\theta \geq 0$ is the constant variance parameter and dW_t is the increment of a Wiener process such that $E[dW_t]=0$ and $var[dW_t]=dt$. Since the presence of this random term, the objective function (1) has to be re-written as an expected term:

$$U = E\left[\int_0^\infty \frac{c_t^{1-\sigma}}{1-\sigma} N_t^{1-\varepsilon} e^{-\rho t} dt\right]. \tag{38}$$

Notice first of all that the maximization problem is totally equivalent to the following:

$$\max_{C_t, u_t} U = E\left[\int_0^\infty \frac{C_t^{1-\sigma}}{1-\sigma} N_t^{\sigma-\varepsilon} e^{-\rho t} dt\right] \tag{39}$$

$$s.t. \dot{K}_t = AK_t^\alpha (u_t H_t)^{1-\alpha} - C_t \tag{40}$$

$$\dot{H}_t = B(1-u_t)H_t \tag{41}$$

$$dN_t = \mu N_t dt + N_t \theta dW_t \tag{42}$$

$$K_0, H_0, N_0 \text{given}, \tag{43}$$

where $C_t = c_t N_t$ represents aggregate consumption.

Define $J(H_t, K_t, N_t)$ as the maximum expected value associated with the stochastic optimization problem described above. The Hamilton–Jacobi–Bellman (HJB) equation is:

$$\rho J = \max_{C_t, u_t} \left\{\frac{C_t^{1-\sigma}}{1-\sigma} N_t^{\sigma-\varepsilon} + J_K \dot{K}_t + J_H \dot{H}_t + J_N \mu N_t + \frac{J_{NN}\theta^2 N_t^2}{2}\right\}, \tag{44}$$

where the differential equations for K_t and H_t are defined in Eqs. (40) and (41) and subscripts denote partial derivatives of J with respect to the relevant variables of interest. Notice that if $\sigma=\varepsilon$, the first term on the RHS of Eq. (44) becomes $\frac{C_t^{1-\sigma}}{1-\sigma}$ since the population term vanishes (from now onwards we continue as such an assumption holds). Dropping the ts for clarity, differentiating Eq. (44) with respect to the control variables gives:

$$C = J_K^{-\frac{1}{\sigma}}, \tag{45}$$

$$u = \frac{K}{H}\left[\frac{(1-\alpha)AJ_K}{BJ_H}\right]^{\frac{1}{\alpha}}, \tag{46}$$

which substituted back into Eq. (44) yield:

$$0 = \left(\frac{\sigma}{1-\sigma}\right) J_K^{\frac{\sigma-1}{\sigma}} - \rho J + J_K\left[AK\left[\frac{(1-\alpha)AJ_K}{BJ_H}\right]^{\frac{1}{\alpha}}\left[\frac{BJ_H}{(1-\alpha)AJ_K}\right]\right] +$$

$$+ J_H\left[BH - BK\left[\frac{(1-\alpha)AJ_K}{BJ_H}\right]^{\frac{1}{\alpha}}\right] + J_N \mu N + \frac{J_{NN}\theta^2 N^2}{2}. \tag{47}$$

Using the guess and verify method, it is possible to show that a closed-form solution to the problem exists under a particular combination of parameter values.[4]

Proposition 5. *Assume that $\sigma=\varepsilon=\alpha$; then the HJB equation, as in Eq. (44), has a solution given by:*

$$J(H,K) = T_H H^{1-\alpha} + T_K K^{1-\alpha}, \tag{48}$$

where:

$$T_K = \frac{\alpha^\alpha}{(1-\alpha)\rho^\alpha} \quad \text{and} \quad T_H = \frac{A\alpha^{2\alpha}}{B^{1-\alpha}\rho^\alpha[\rho-(1-\alpha)B]^\alpha}. \tag{49}$$

The optimal rules for consumption and share of human capital allocated to physical production are respectively given by:

$$C_t = \frac{\rho}{\alpha}K_t, \quad u_t = u = \frac{\rho - B(1-\alpha)}{B\alpha}. \tag{50}$$

while the optimal paths of human and physical capital are the following:

$$H_t = H_0 e^{B(1-u)t}, \quad K_t = e^{-\frac{\rho}{\alpha}t}\left[K_0^{1-\alpha} + \frac{Au^{1-\alpha}H_0^{1-\alpha}}{\frac{\rho}{\alpha}+B(1-u)}\left(e^{(1-\alpha)\left[\frac{\rho}{\alpha}+B(1-u)\right]t} - 1\right)\right]^{\frac{1}{1-\alpha}}. \tag{51}$$

Proof. See Appendix B. □

Eq. (50) says that, along the optimal paths, the consumption–capital ratio and the share of human capital employed in final production are constant. Notice that such result coincides with what we showed for the deterministic case.

In order to understand the role of population shocks, we need to take expectations of per-capital physical and human capital. Using the fact that $E[X(t)] = X(0)e^{(\theta^2-\mu)t}$, where $X(t) = \frac{1}{N(t)}$, and since $E[k(t)] = E\left[\frac{K(t)}{N(t)}\right] = E[K(t)X(t)]$ it is straightforward finding the expected value of $k(t)$:

$$E[k_t] = e^{\left(\theta^2-\mu-\frac{\rho}{\alpha}\right)t}\left[k_0^{1-\alpha} + \frac{Au^{1-\alpha}h_0^{-\alpha}\left(e^{(1-\alpha)\left[\frac{\rho}{\alpha}+B(1-u)\right]t} - 1\right)}{\frac{\rho}{\alpha}+B(1-u)}\right]^{\frac{1}{1-\alpha}}; \tag{52}$$

the same reasoning applies fot $h(t)$:

$$E[h(t)] = h_0 e^{\left[B(1-u)+\theta^2-\mu\right]t}. \tag{53}$$

By setting $\theta=0$ in Eqs. (52) and (53) yields the levels of the state per-capita variable in the deterministic version of the model. By comparing the level of physical and human capital in the deterministic and stochastic version, it is straightforward to see that uncertainty increases on average the levels of both per-capita physical and human capital.

[4] Bucci et al. (2011) is the unique work to our knowledge to deal with a stochastic version of the Lucas–Uzawa model. They study the case in which technological progress follows a geometric Brownian motion, and they show that a closed-form solution can be found under two conditions, namely that the capital share is equal to the inverse of the intertemporal elasticity of substitution and that the rate of time preference is equal to the human capital net (of depreciation) productivity (in our notation $\sigma=\alpha$ and $B=\rho$, since we abstract from depreciation). This former condition is standard in this literature (see Smith (2007)) while the latter is new. Even if empirically supported, as the authors show, it implies that one of the coefficients of the value function depends on the initial value of a state variable, and this is probably a limitation of their result. Notice that, with respect to theirs, our solution is based on a single condition $\sigma=\alpha=\varepsilon$ (the same used to determine the whole transitional dynamics in the deterministic version of the model) and the coefficients of the value function are independent of the initial level of the state variables.

7. Conclusion

A standard assumption in growth theory is that population change is constant and exponential. Recently, the idea that such a specification is unrealistic has been arisen. This is due to an implication of such a hypothesis: population size goes to infinity as time goes to infinity. This is clearly unrealistic, since it would deny the presence of an environmental and economic carrying capacity (Brida and Accinelli, 2007). As a result, several papers study the introduction of different population growth functions in canonical growth models.

In this paper we introduce a generic population change function in a two-sector endogenous model of growth, á-la Lucas–Uzawa and we show that the outcome of the model does not depend on the choice of such a function. In fact, a unique non-trivial equilibrium exists and the economy converges towards it along a saddle path, independently of the shape of the population change function. What can be affected by its shape is the dimension of the stable transitional path, which can be one (if $\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*} > 0$) or two (if $\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*}<0$), and the timing of convergence, which can happen in finite (the steady state is characterized by a BGP) or infinite (by an ABGP) time. Moreover, with respect to other works dealing with non exponential population change we do not relax one of the standard assumptions in economic growth theory, that is the social welfare function is founded on the Benthamite criterion. In fact, we consider a generic welfare function which results to be based on the Benthamite or the Millian principle according to the value of the altruism parameter. We show that if population growth is null in steady state (as in the case of logistic population growth), choosing one or the other criterion is irrelevant for the outcome of the model. Instead, if population growth is constant in equilibrium, the Benthamite criterion leads to higher economic growth than the Millian criterion.

Then, we formalize the population growth function by different functions which represent alternative demographic dynamics studied in the previous literature and show how our model is able to encompass all of them as particular cases. We look for a closed-form solution of the model, showing that this can be fully characterized under a certain condition on the altruism parameter, namely when it coincides with the inverse of the intertemporal elasticity of substitution and the capital share. We also look for a closed-form solution of the model when population dynamics is subject to random shocks driven by a Brownian motion; we show that under the same condition on the altruism parameter such a closed-form solution can be found and uncertainty increases the levels of (per-capita) physical and human capital.

For further research, we suggest to study the dynamics of a two-sector economic growth model when population change is endogenous. Really few papers introduce endogenous population change in optimal models of growth and moreover they just analyze the case of a single sector economy. It can be interesting to see whether endogenizing fertility can play a crucial role in determining the transitional dynamics of multi-sector growth models and whether the degree of intertemporal altruism affects the economic equilibrium. In such a framework it can also be interesting to study the implications of endogenous population growth on sustainable development in a setting with renewable and non-renewable resources (a first attempt to tackle the issue is Marsiglio (2011) who analyzes a single-sector model showing that the development path followed by the economy can be sustainable or not according to steady state population growth rate).

Acknowledgments

We thank the participants of the workshop held at Monash University (Melbourne Graduate Student Conference in Economics 2010, July 2010) for the helpful comments and suggestions.

Appendix A. Transitional dynamics

A.1. Exponential population growth

The steady state of the quasi-linear three dimensional system (21)–(23) is given by $\psi^*=\frac{B}{\alpha A}$, $\chi^*=\frac{\rho\alpha-B(\alpha-\sigma)-\alpha(\sigma-\varepsilon)g_N}{\alpha\sigma}$ and $u^*=\frac{\rho-B(1-\sigma)-(\sigma-\varepsilon)g_N}{B\sigma}$. The steady state value of all variables is strictly positive if $\rho>B(1-\sigma)+(\sigma-\varepsilon)g_N$. In order to ensure $u^*<1$, we also need that $\rho<B+(\sigma-\varepsilon)g_N$. Notice that, if g_N is not too large nor too small, such conditions generally hold for reasonable values of σ; in fact several studies find that the inverse of the intertemporal elasticity of substitution is higher than one(see Mehra and Prescott, 1985; and more recently Obstfeld, 1994) and in this case both conditions are automatically satisfied.

We can study the stability of the system by linearizing around the steady state. The Jacobian matrix evaluated at steady state, $J(\chi^*,\psi^*,u^*)$, is:

$$\begin{bmatrix} \chi^* & \frac{\alpha-\sigma}{\sigma}A\chi^* & 0 \\ 0 & -(1-\alpha)\psi^* & 0 \\ -u^* & 0 & Bu^* \end{bmatrix}.$$

The eigenvalues results to be the elements on the main diagonal: therefore, we have two positive and one negative eigenvalues: the equilibrium is saddle-point stable. The system therefore converges to its steady state equilibrium through a saddle path, along which the stable arm is a one-dimensional locus while the unstable manifold has dimension two.

A.2. Non-Exponential Population Growth

The steady state of the quasi-linear four dimensional system (24)–(27) is given by N^* such that $g(N^*)=0$, $\psi^*=\frac{B}{\alpha A}$, $\chi^*=\frac{\rho\alpha-B(\alpha-\sigma)}{\alpha\sigma}$, $u^*=\frac{\rho-B(1-\sigma)}{B\sigma}$. The steady state value of all variables is strictly positive if $\rho>B(1-\sigma)$, while $u^*<1$ if $\rho<B$.

The Jacobian matrix evaluated at steady state, $J(\chi^*,\psi^*,u^*,N^*)$, is:

$$\begin{bmatrix} \chi^* & \frac{\alpha-\sigma}{\sigma}A\chi^* & 0 & \frac{\sigma-\varepsilon}{\sigma}\frac{\partial g(\cdot)}{\partial N_t}|N_t=N^*\chi^* \\ 0 & -(1-\alpha)\psi^* & 0 & 0 \\ -u^* & 0 & Bu^* & 0 \\ 0 & 0 & 0 & \frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*}N^* \end{bmatrix}.$$

Also in this case, the eigenvalues results to be the elements on the main diagonal: therefore, we have two positive and one negative eigenvalues, independent of the $g(\cdot)$ function, while the last one crucially depends on it. However, the equilibrium is saddle-point stable and the system therefore converges to its steady state equilibrium through a saddle path. The shape of the $g(\cdot)$ function affects only the dimension of the stable and unstable transitional paths. In fact, if $\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*}>0$ the stable arm has dimension one, while if $\frac{\partial g(\cdot)}{\partial N_t}|_{N_t=N^*}<0$ it has dimension two.

Appendix B. Stochastic population

As in Bucci et al. (2011), we postulate a value function separable in the state variables of the problem:

$$J(H,K)=T_H H^{\lambda_1}+T_K K^{\lambda_2},$$

where T_H and T_K are constant parameters. From this, we have:

$$J_H=\lambda_1 T_H H^{\lambda_1-1}, \quad J_K=\lambda_2 T_K K^{\lambda_2-1}, \quad J_N=J_{NN}=0,$$

which substituted into Eq. (44), yield:

$$0=\left(\frac{\sigma}{1-\sigma}\right)[\lambda_2 T_K]^{\frac{\sigma-1}{\sigma}}K^{\frac{(\lambda_2-1)(\sigma-1)}{\sigma}}-\rho\left(T_H H^{\lambda_1}+T_K K^{\lambda_2}\right)+\lambda_1 T_H B H^{\lambda_1}+$$

$$+\left(\frac{\alpha}{1-\alpha}\right)B\lambda_1 T_H\left[\frac{(1-\alpha)A\lambda_2 T_K}{\lambda_1 B T_H}\right]^{\frac{1}{\alpha}}K^{\frac{\lambda_2+\alpha-1}{\alpha}}H^{\frac{(1-\alpha)(1-\lambda_1)}{\alpha}}$$

Let $\lambda_1=1-\alpha$ and $\lambda_2=1-\alpha$. Then we get: $0=\left(\frac{\sigma}{1-\sigma}\right)[(1-\alpha)T_K]^{\frac{\sigma-1}{\sigma}}K^{\frac{-\alpha(\sigma-1)}{\sigma}}-\rho\left(T_H H^{1-\alpha}+T_K K^{1-\alpha}\right)+$

$$+\left(\frac{\alpha}{1-\alpha}\right)B(1-\alpha)T_H\left[\frac{(1-\alpha)AT_K}{BT_H}\right]^{\frac{1}{\alpha}}H^{1-\alpha}+(1-\alpha)T_H B H^{1-\alpha}.$$

If $\sigma=\alpha$, then

$$0=\left[\left(\frac{\alpha}{1-\alpha}\right)(1-\alpha)^{\frac{-(1-\alpha)}{\alpha}}T_K^{-\frac{1}{\alpha}}-\rho\right]T_K K^{1-\alpha}$$
$$+\left[(1-\alpha)B+\alpha B\left[\frac{(1-\alpha)AT_K}{BT_H}\right]^{\frac{1}{\alpha}}-\rho\right]T_H H^{1-\alpha}.$$

Since this equation has to be satisfied for all values of K and H, the square brackets have to be zero. This implies the values of T_K and T_H given by (49). Given the expression for J, as defined in Eq. (48), from Eqs. (45) and (46) we obtain the optimal policy rules for C and u, as given in Eq. (50); by plugging these into the state equations, we get the optimal dynamics of K_t and H_t, as shown in Eq. (51).

The verification theorem requires that the transversality condition is satisfied in order to have an optimal solution. The TVC implies that $lim_{t\to\infty}E[e^{-\rho t}J(H,\ K)]=lim_{t\to\infty}E[e^{-\rho t}(T_H H^{1-\alpha}+T_K K^{1-\alpha})]=0$. Both the first and the second term automatically converge to zero, under the assumption ensuring that $0<u^*<1$, namely that $B(1-\alpha)<\rho<B$.

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