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Adaptive learning and roads to chaos

The case of the cobweb

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We investigate the dynamics of the cobweb model with adaptive expectations, a linear demand curve, and a nonlinear, S-shaped, increasing supply curve. Both stable periodic and chaotic price behaviour can occur. We investigate, how the dynamics of the model depend on the parameters. Both infinitely many period doubling and period halving bifurcations can occur, when the demand curve is shifted upwards. The same result holds with respect to the expectations weight factor.

1. Introduction

In economics there is a growing interest in the use of non-linear deterministic models. The main reason is that a non-linear deterministic model may exhibit both stable periodic and chaotic behaviour, and hence may provide an endogenous explanation of the periodicity and irregularity observed in economic time series. [For a survey on non-linear economic models exhibiting chaos, see e.g. Lorenz (1989).]

In this paper, we investigate the dynamics of one of the simplest nonlinear economic models: the cobweb model with adaptive expectations. The demand curve is linearly decreasing, while the supply curve is non-linear, S-shaped and increasing. The dynamics of the expected prices in the model is described by a one-dimensional nonlinear difference equation $x_{n+1} = f(x_n)$. Chiarella (1988) approximated this model by the well known logistic map $x_{n+1} = \mu x_n(1 - x_n)$. Unfortunately, since the map f is either increasing or has two critical points, the quadratic map (which has one critical point) is not a good approximation of the map f. In another related paper, Finkenstädt and Kuhbier (1990) present numerical evidence of the occurrence of chaos, in the case of linear supply and a non-linear, decreasing demand curve.

In particular, we investigate how the dynamics of the model depend on the height of the demand curve and the expectations weight factor. In this paper we present numerical results, and explain the validity of these results by means of theoretical results. The proofs of these theoretical results will be presented in a forthcoming long paper [Hommes (1991)].

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2. The cobweb model with adaptive expectations

The well known cobweb model is one of the simplest economic models. The model described the price behaviour in a single market. We write p_t for the price, $\hat{p}_t$ for the expected price, q_t^d for the demand for goods and q_t^s for the supply for goods, all at time t. The *cobweb model* is given by the following three equations:

$$q_t^d = D(p_t), \tag{1}$$

$$q_t^s = S(\hat{p}_t), \tag{2}$$

$$q_t^d = q_t^s. \tag{3}$$

In the traditional version of the cobweb model, the expected price equals the previous actual price, that is $\hat{p}_t = p_{t-1}$. It is well known that if in the traditional cobweb model both the supply and demand curves are monotonic, then basically three types of price dynamics occur: Convergence to an equilibrium price, convergence to period two price oscillations or unbounded, exploding price oscillations. Recently, it was shown by Artstein (1983) and Jensen and Urban (1984) that chaotic price behaviour can occur if at least one of the supply and demand curves is non-monotonic, see also Lichtenberg and Ujihara (1989).

Nerlove (1958) introduced adaptive price expectations into the cobweb model, in the case of linear supply and demand curves. Adaptive expectations is described by the following equation

$$\hat{p}_t = \hat{p}_{t-1} + w(p_{t-1} - \hat{p}_{t-1}), \quad 0 \leq w \leq 1. \tag{4}$$

The parameter w is called the *expectations weight factor*, and for $w = 1$ the model reduces to the traditional cobweb model. In the case of linear supply and demand curves, the introduction of adaptive expectations in the cobweb model has a stabilizing effect on the price dynamics, see Nerlove (1958). However, the equilibrium price may still be unstable.

The question we address is: '*What can be said about the price behaviour in the case of non-linear, monotonic supply and demand curves?*'

For simplicity we assume that the demand curve is linearly decreasing, and is given by

$$D(p_t) = a - bp_t, \quad b > 0. \tag{5}$$

Concerning the supply, we start off with the following two Economic Considerations:

(*EC1*) If prices are low then supply increases slowly, because of start-up costs and fixed production costs.

(*EC2*) If prices are high then supply increases slowly, because of supply and capacity constraints.

Based on these considerations we choose a non-linear, increasing supply curve.

The simplest smooth curve satisfying (EC1) and (EC2) is an S-shaped curve S with the property that S has a unique inflection point $\bar{p}$, such that (1) the slope S' of S is maximal in $\bar{p}$, (2) S$'$ is increasing for $p < \bar{p}$ and (3) S' is decreasing for $p > \bar{p}$. We change coordinates and choose the inflection point of the supply curve to be the new origin. Note that with respect to this new origin both 'prices' and 'quantities' can be negative. As an example of an S-shaped supply curve satisfying the above assumptions we choose

$$S_\lambda(x) = \arctan(\lambda x), \quad \lambda > 0. \tag{6}$$

Observe that the parameter λ tunes the 'steepness' of the S-shape.

Equations (1) through (5) yield a difference equation $x_{t+1} = f(x_t)$ describing the expected price dynamics, with f given by

$$f_{a,b,w,\lambda}(x) = -wS_\lambda(x)/b + (1-w)x + aw/b. \tag{7}$$

We would like to point out that the price dynamics and the quantity dynamics are equivalent to the dynamics of the expected prices.

The map $f_{a,b,w,\lambda}$ has a unique fixed point, which is the equilibrium price corresponding to the intersection point of the supply and demand curves. An important question is: 'What can be said about the global dynamics of the model, when the equilibrium price is unstable?'

3. Roads to chaos

In this section we investigate how the dynamics of the model depends on the height of the demand curve (parameter a) and the expectations weight factor w. Figure 1 shows a bifurcation diagram with respect to the parameter a, with the other parameters fixed at $b = 0.25$, $w = 0.3$ and $\lambda = 4.8$. In fact a

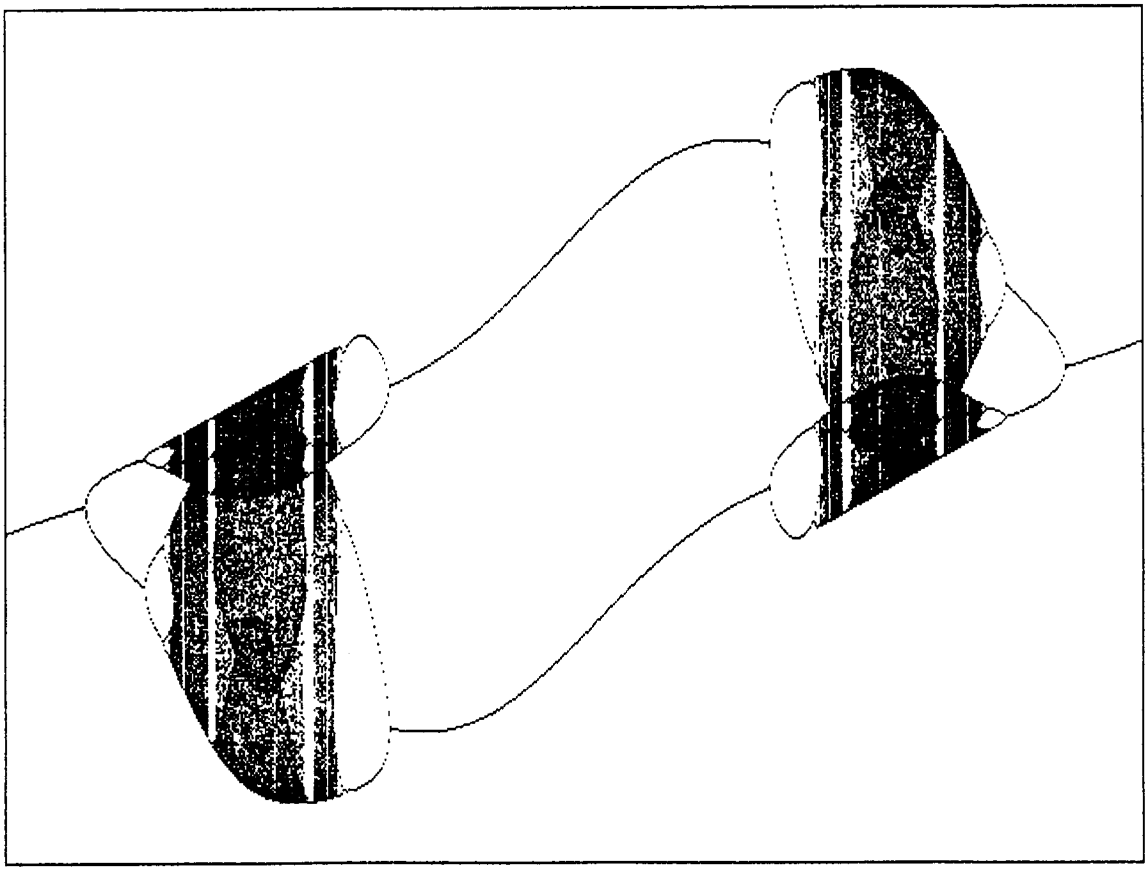

Fig. 1. Shifting the demand curve upwards, stable periodic and chaotic price behaviour may interchange several times.

bifurcation diagram of a one-dimensional model shows an attractor of the model as a (multi-valued) function of one parameter.

Figure 1 suggests the following bifurcation scenario. If a is small then there exists a stable equilibrium. If a is increased, then the equilibrium becomes unstable and period doubling bifurcations occur. After infinitely many period doubling bifurcations the price behaviour becomes chaotic, as a is increased. Next, after infinitely many period halving bifurcations the price behaviour becomes more regular again. A stable period 2 orbit occurs for an interval of a-values, containing $a = 0$. When a is further increased, once more, after infinitely many period doubling bifurcations chaotic behaviour arises. Finally, after infinitely many period halving bifurcations, we have a stable equilibrium again, when a is sufficiently large.

Concerning the dynamics of the model, for the particular supply curve in (6), the results in Hommes (1991) imply the following (we write f for $f_{a,b,w,\lambda}$).

Given b > 0 and $0 < w < 1$, if λ is sufficiently large, then there exist $a_1 < a_2 < 0 < a_3 < a_4$ such that:

(1) f has a globally stable fixed point, if $a < a_1$.
(2) f has a period 3 orbit, for $a = a_2$.
(3) f has an unstable fixed point, a stable period 2 orbit, and no other periodic points, for $a = 0$.
(4) f has a period 3 orbit, for $a = a_3$.
(5) f has a globally stable fixed point, fi $a > a_4$.

From a well known result by Li and Yorke (1975) it follows that in the cases (2) and (4) the map $f_{a,b,w,\lambda}$ is topological chaotic, that is: (i) there exist infinitely many periodic points with different period, and (ii) there exists an uncountable set of aperiodic points, for which there is sensitive dependence on initial conditions. Concerning the bifurcations scenario with respect to the parameter a, properties (1)–(5) imply the following:

(a) Infinitely many period doubling bifurcations occur in the parameter intervals (a_1, a_2) and $(0, a_3)$.
(b) Infinitely many period halving bifurcations occur in the parameter intervals $(a_2, 0)$ and (a_3, a_4).

Nusse and Yorke [1988] present a nice example $x_{n+1} = \mu F(x_n)$ (where F is a one-hump map with negative Schwarzian derivative and μ is a parameter) for which they showed that both infinitely many period doubling and period halving bifurcations do occur as μ is increased.

Recall that increasing the parameter a is just shifting the demand curve vertically upwards. Hence our theoretical results imply that, if the demand curve is shifted vertically upwards, then both infinitely many period doubling and period halving bifurcations occur and periodic and chaotic behaviour interchange several times.

A bifurcation diagram with respect to the expectations weight factor w, with the other parameters fixed at $a = 0.8$, $b = 0.25$ and $\lambda = 4$, is shown in figure 2. The diagram suggests that a stable equilibrium occurs for w close to 0. After infinitely many period doubling bifurcations the price dynamics becomes chaotic, as w is increased. Next, after infinitely many period halving bifurcations, the price behaviour becomes more regular again, until a stable period 2 cycle occurs, for w close to 1.

The results in [H] imply the following. Given $b > 0$, if λ is sufficiently large and for a suitable choice of the parameter a we have:

(1) f has a globally stable fixed point for w close to 0.
(2) f has a period 3 orbit for intermediate values of w.
(3) f has a stable period 2 orbit for w close to 1.

This result implies that both infinitely many period doubling and period halving bifurcations occur as w is increased from 0 to 1.

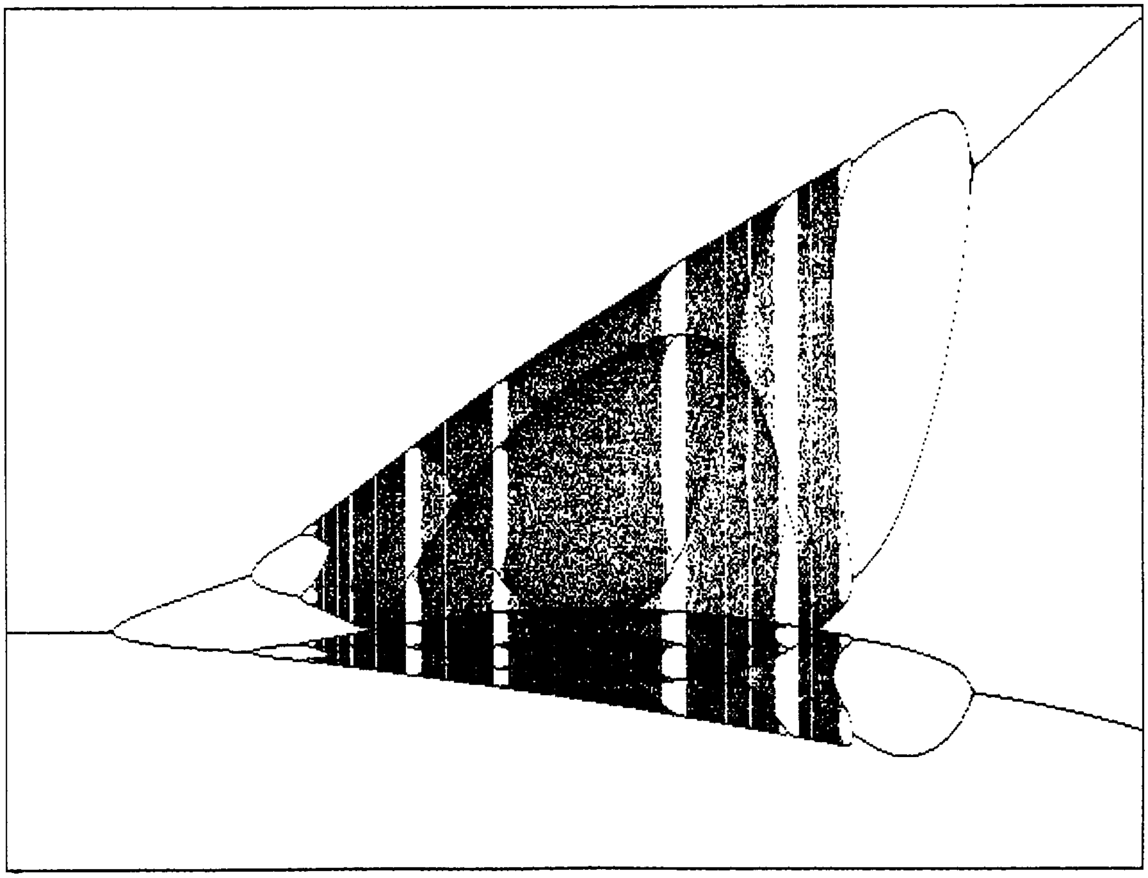

Fig. 2. For w close to 0 a stable equilibrium occurs, while for w close to 1 stable period 2 oscillations with large amplitude occur; for intermediate values of w chaotic price oscillations with moderate amplitude arise.

In the case of linear supply and demand curves, the introduction of adaptive expectations into the cobweb model has a stabilizing effect on the price dynamics, see Nerlove (1958). We present a corresponding result, in the case of non-linear, monotonic supply and/or demand curves. Recall that $w = 1$ corresponds to the traditional cobweb model. As w is decreased from 1 to 0, then the amplitude of the price oscillations becomes smaller, see fig. 2. In the case of non-linear, monotonic supply and/or demand curves the introduction of adaptive expectations into the cobweb model dampens the amplitude of the price cycles. Meanwhile a price cycle may become unstable and chaotic price oscillations may arise. Hence, from a quantitative point of view adaptive expectations have a stabilizing effect, but from a qualitative point of view adaptive expectations may have a destabilizing effect upon the price behaviour.

4. Discussion

Expectations and learning behaviour play an important role in economics. One of the fundamental differences between the physical and the social sciences is that in a physical system the laws of the system are often fixed, while in a social system individuals learn from he past and influence the laws of the system. It has been argued, that, because of this fundamental difference, deterministic chaos, which plays an important role in the physical sciences nowadays, is of less interest to the economic and social sciences.

In the traditional cobweb model, there is no learning behaviour: the suppliers believe that today's price will also hold tomorrow. Recently, Holmes and Manning (1988) showed that in a cobweb model with nonlinear, monotonic supply and demand, when the suppliers learn by arithmetic mean, that is, if they employ the mean of all past prices as their expectation of tomorrow's price, then prices always converge to a stable equilibrium. Holmes and Manning conclude that if agents use their memories, then chaos can not occur; only when agents are forgetful, chaos may occur. Stated according to their aphorism: '*Those who do not learn from history are condemned to never repeat it*', see Holmes and Manning (1988, p. 7).

Our results show that the conclusions of Holmes and Manning are not true in general. Whether or not chaotic behaviour is possible when agents learn from the past, depends on both the model and the type of learning behaviour. We have seen that in a cobweb model with adaptive learning and non-linear, monotonic supply and demand, chaotic price behaviour is possible, even in the long run. It is well known that adaptive price expectations means that the expected price is a weighted average, with geometrically declining weights, of all past observed prices. Hence, adaptive learning seems to be much more realistic than learning by arithmetic mean, since adaptive learning puts higher weights to the most recent prices. Unlike the aphorism of Holmes and Manning, one might say: *Even those who learn from the past, may never repeat it.*

In Hommes (1991) we will analyse the dynamics of the model, for a general class of S-shaped supply curves, in more detail. Moreover, we will present a geometric explanation of the occurrence of chaotic price behaviour in the cobweb model with adaptive expectations, for a large class of non-linear, monotonic supply and demand curves.

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