Irregular Growth Cycles

By RICHARD H. DAY*

This paper uses the familiar, neoclassical theory of capital accumulation to show how complex behavior can emerge from quite simple economic structures. Indeed, when sufficient nonlinearities and a production lag are present, the interaction alone of the propensity to save and the productivity of capital can lead to growth cycles that exhibit a wandering, sawtooth pattern not unlike those observed in reality. These fluctuations need not converge to a cycle of any regular periodicity so they are not quasi periodic.

Because such "chaotic" trajectories are unstable, errors of estimation in parameters or initial conditions, however tiny, will accumulate rapidly into substantial errors of forecast. Moreover, periods of erratic cycling can be interspersed with periods of more or less stable growth. Evidently, the "future" behavior of a model solution cannot be anticipated from its patterns in the "past," a situation that seems to mimic experience. Apparent structural change and unpredictability is explained in the present theory by a deterministic, single equation model. Random shocks play no role.

So the reader can visualize just what it is we are are talking about, a noteworthy simulation is presented in Figure 1 for *GNP* in a growth model that is described below in Section III. A period of relatively rapid growth is followed by a period of cycles. Then, remarkably, for a considerable time (about twenty "periods") apparently "steady-state" growth occurs. Wandering cycles, however, emerge. Another brief period close to the steady state appears again toward the end of the series.

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I establish conditions of savings and productivity that lead to results of this kind. This analysis makes use of the mathematical theory of "chaos" which, in the form exploited here, originated in the work of Edward Lorenz. A formal definition of chaos and sufficient conditions for chaotic trajectories were provided in a seminal paper by T-Y Li and James Yorke. A survey of these related contributions is found in Yorke and Evelyn Yorke. This theory was introduced into economics by Jess Benhabib and myself (1981), where we showed that sequences of rational choices can be erratic when preferences depend on experience in a certain way; by Michael Stutzer, who provides a detailed analysis of Trygve Haavelmo's growth model; by Benhabib and myself (1980) in an application of the overlapping generations model; and in my forthcoming study of the classical growth model.

I. Chaotic Cycles

A. *The Neoclassical Model in Discrete Time*

When the standard neoclassical growth model (Robert Solow, 1956) is modified by introducing a production lag, it can be expressed as a difference equation in the capital-labor ratio $k_t \equiv K_t/L_t$,

$$(1) \qquad k_{t+1} = s(k_t)\cdot f(k_t)/(1+\lambda),$$

in which $s(\cdot)$ is the savings ratio, $f(\cdot)$ the per capita production function, and λ the "natural" rate of population growth. Aggregate output is, of course, given by the aggregate production function $Y_t = F(K_t, L_t) \equiv f(k_t)L_t$ where the aggregate capital stock is k_tL_t and the supply of labor is $L_t = (1+\lambda)^t L_0$. The savings ratio may depend on income, wealth, and the interest rate, but using $y = f(k)$ and the real interest rate $r = f'(k)$, it reduces to a dependence on k alone.

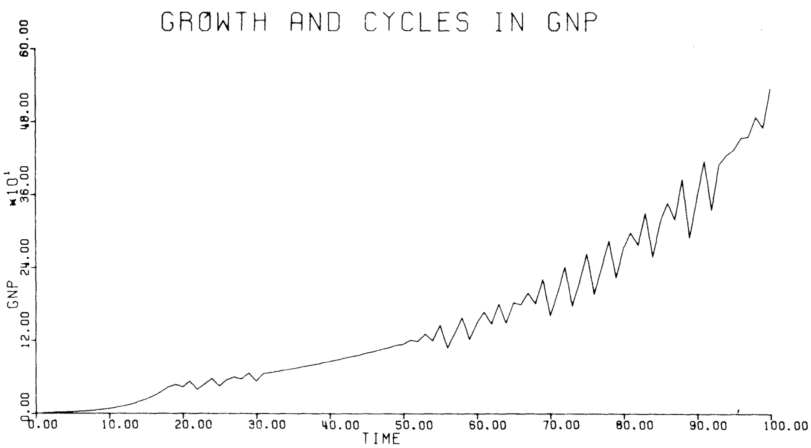


FIGURE 1. IRREGULAR GROWTH CYCLES
(A Simulation of Equation (19))

B. *Qualitative Dynamics*

Balanced growth paths of capital accumulation and population growth are associated with stationary states of (1), that is, capital-labor ratios k satisfying $k = s(k)\cdot f(k)/(1+\lambda)$. Whether these are stable or not depends on the stability of (1) in the neighborhood of these stationary states. Thus, depending on savings behavior and on the productivity of capital, growth can converge to a steady state, or oscillate about one in fluctuations that may converge to a cycle of some order, results that can be illustrated in the usual phase diagram in the (k_{t+1}, k_t) plane as shown in Figure 2.

The possibility that fluctuations might not converge to a cycle of *any* order was discovered for the special difference equation

$$x_{t+1} = Ax_t(1-x_t) \tag{2}$$

by Lorenz, a definitive analysis of which was subsequently provided by F. Hoppensteadt and J. Hyman. As the parameter A is increased from zero, the qualitative behavior of trajectories of x is seen to change at specific values of A, called *bifurcation points*. Thus, confining attention to initial conditions in the interval $(0,1)$ we have the following:

$0 < A \leqslant 1$, monotonic contraction to $x = 0$,
$1 < A \leqslant 2$, monotonic growth converging to $x = (A-1)/A$,
$2 < A \leqslant 3$, oscillations converging to $(A-1)/A$,
$3 < A \leqslant 4$, continued oscillations.

Lorenz uncovered extremely complex behavior for (2) as A increased in the interval $[3,4]$; in particular, trajectories in which no segment of past history was repeated. Hoppensteadt and Hyman showed that a threshold existed at $A = 3.57$ after which cycles of all orders existed at most one of which could be stable.

Similar behavior was subsequently generated by Robert May, and May and George Oster, in a series of investigations of specific difference equations arising in population biology. General difference equations that exhibit complex behavior began to receive attention, for example, in M. Metropolis et al., Li and Yorke, and John Guckenheimer et al. It is the Li-Yorke contribution that I apply in the remainder of this paper.

C. *Sufficient Conditions for Chaos*

Consider a real-valued, continuous function θ that depends on a vector of parameters, say π, that maps an interval J into itself. Such a function generates a difference equation

$$x_{t+1} = \theta(x_t;\pi) \equiv \theta(x_t), \tag{3}$$

whose behavior depends on the parameters of π. If for some value of these parameters

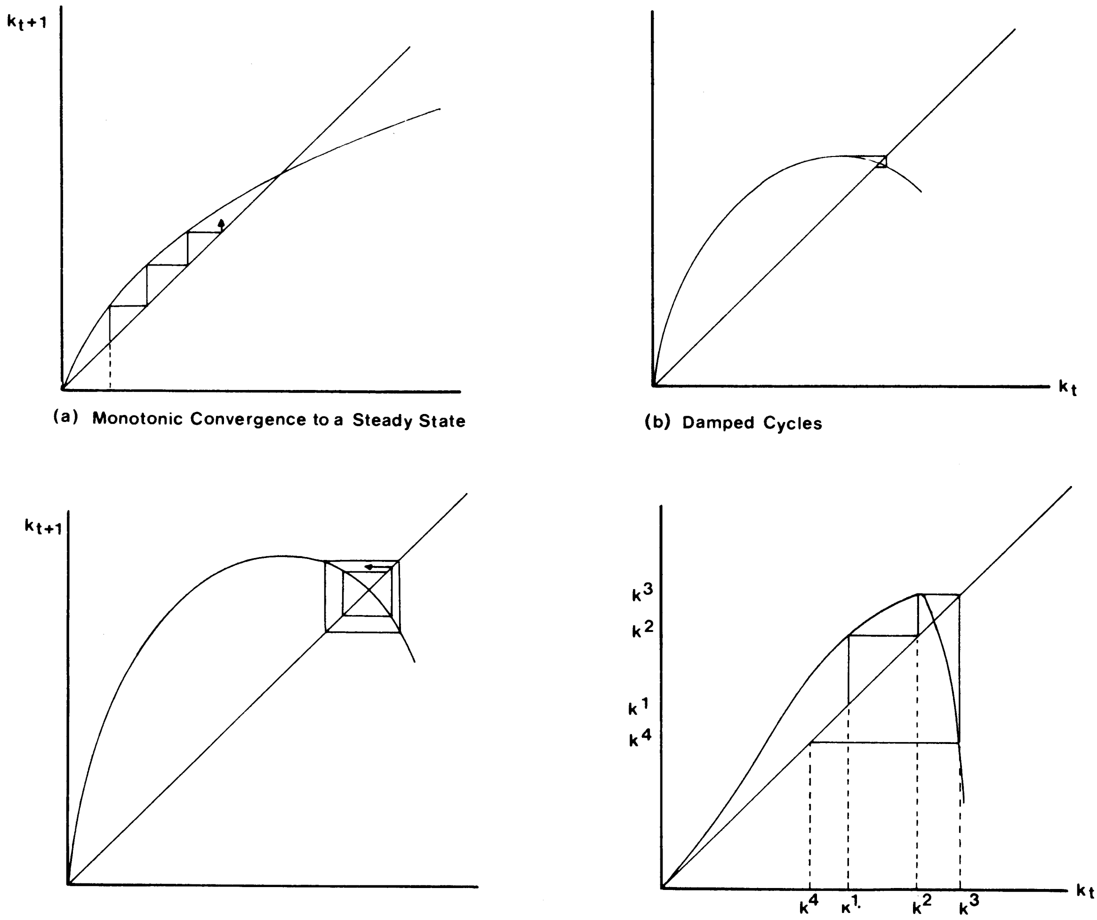


FIGURE 2. DYNAMICS OF THE CAPITAL-LABOR RATIO: ALTERNATIVE POSSIBILITIES
(Case (d) Illustrates the Chaos Inequalities)

there exists a point in J, say x^c, such that

$$\theta^3(x^c) \leqslant x^c < \theta(x^c) < \theta^2(x^c), \tag{4}$$

where $\theta^1(x) \equiv \theta(x)$, $\theta^2(x) \equiv \theta(\theta(x))$, and $\theta^{n+1}(x) \equiv \theta(\theta^n(x))$, then Li and Yorke showed that

A. there exist cycles of every order in J. (That is, for every n, $n = 1, 2, 3, \ldots$, there exist points in J satisfying $x = \theta^n(x)$.)

B. there exists an uncountable set S in J such that all trajectories with initial conditions in S remain in S and

B1. every trajectory in S wanders arbitrarily close to every other one,

B2. no matter how close two distinct trajectories in S may come to each other, they must eventually wander away,

B3. every trajectory in S wanders away from a cycle of any order in J, however close it may approximate one for a time.

The sufficient conditions for chaos given by (4) are illustrated in Figure 2(d).

The Li-Yorke theorem is proved, following the general approach of Steve Smale, by considering the iterates θ^n of the map θ, which in effect means studying the properties of the general (symbolic) solution of (3),

$$x(t) = \theta^t(x), x = x(0), \theta^0(x) = x. \tag{5}$$

These iterates are themselves maps of J into J and, when the sufficient conditions (4) are satisfied, fold the interval in a way that contains more and more "wrinkles." The process, it has been suggested, is somewhat like shuffling cards, so that the "passage of time" through the nonlinear dynamic equation acts more or less like a random number generator!

In the remainder of this paper, I consider examples of the neoclassical model (1) and show the existence of parameter values for the savings and production function for which the inequalities (4) can be satisfied. My approach is to consider cases where a maximum attainable capital labor ratio $k^m = s(k^*)\cdot f(k^*)/(1+\lambda)$ exists, where k^* maximizes $s(k)\cdot f(k)$, assuming that $k^m > k^*$. I then take the smallest root of $s(k)\cdot f(k)/(1+\lambda) = k^*$; call it k^c. The sufficient condition (4) now becomes

$$(4')\quad s(k^m)f(k^m)/(1+\lambda)\leqslant k^c < k^* < k^m.$$

It should be noted that the neoclassical theory assumes continuous substitution of labor and capital, assumes full employment, and precludes disequilibrium on the capital market with the consequence that condition (4′) implies rather abrupt and historically unprecedented changes in the capital-labor ratio. The business cycle probably involves alternative relationships, but explicit consideration of them would carry us beyond the realm of the present discussion. For our current heuristic purpose, it may be most appropriate to think of the model as representing a sequence of generations so that saving is made on the basis of a fairly long horizon, but with an imperfect ability to foretell the consequences a generation hence of actions taken in the present.

II. Examples

A. *Example* 1: *Constant Savings Ratio with a Neoclassical Production Function*

As a benchmark for comparison, let us consider a constant savings ratio $s(k)\equiv\sigma$ with a Cobb-Douglas production function

$$(6)\qquad f(k) = Bk^{\beta}.$$

The neoclassical model (1) becomes

$$(7)\qquad k_{t+1} = \sigma Bk_t^{\beta}/(1+\lambda).$$

For $0<\beta$ the right-hand side of (7) is monotonically, strictly increasing so that all trajectories converge to the balanced growth path determined by the steady-state capital-labor ratio $k^s = [(\sigma B)/(1+\lambda)]^{1/(1-\beta)}$. This case is illustrated in Figure 2(a) above.

B. *Example* 2: *A Productivity Inhibiting Effect*

Suppose now that productivity is reduced by a "pollution effect" caused by increasing concentrations of capital according to a multiplicative term $(m-k)^{\gamma}$. When γ is close to zero, this term remains close to unity until k gets close to m when it falls rapidly. Thus, the production function

$$(8)\qquad f(k) = Bk^{\beta}(m-k)^{\gamma}$$

behaves like the conventional power function (6) over much of its range.

Retaining the constant savings ratio σ, the difference equation is obtained;

$$(9)\qquad k_{t+1} = \sigma Bk_t^{\beta}(m-k_t)^{\gamma}/(1+\lambda).$$

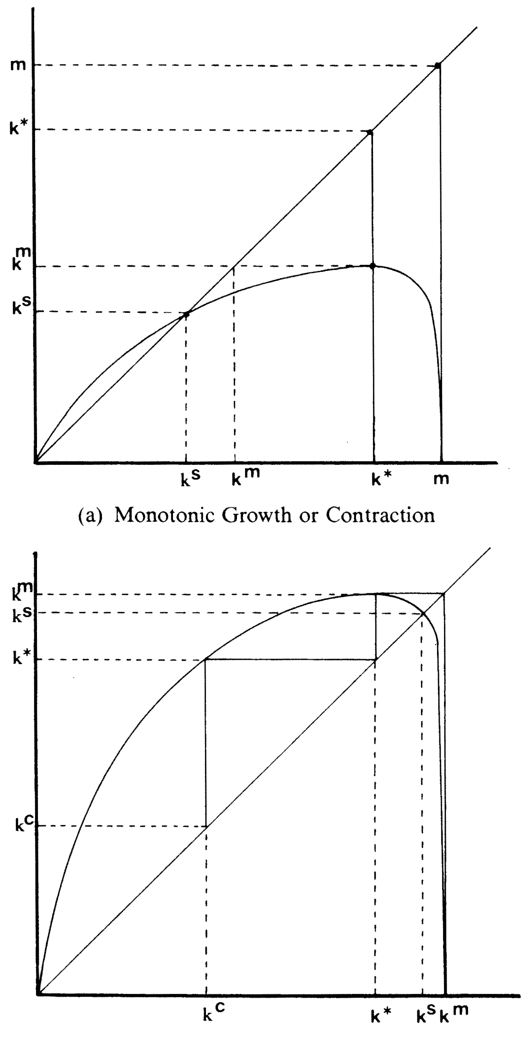
For positive values of β and γ, this function has a single humped concave shape with a unique maximum capital-labor ratio $k^m = f(k^*)$ where

$$(10)\qquad k^* = (\beta/\beta+\gamma)m,$$

so that

$$(11)\quad k^m = \frac{B\sigma}{(1+\lambda)}\beta^{\beta}\gamma^{\gamma}\left(\frac{m}{\beta+\gamma}\right)^{\beta+\gamma}.$$

Notice that this latter value depends on B, but that the capital-ratio k^* generating it



(b) Sufficient Condition for Chaos ($k^m = m$)

FIGURE 3. STRETCHING A FUNCTION TO GENERATE CHAOS

does not. This means that the graph of (9) can be stretched as B is increased continuously upward without changing k^*. Notice, also, that as k approaches zero, the slope of the production function grows indefinitely large. Consequently, for sufficiently small initial conditions, growth must be positive for any positive value of B.

For small enough values of B the situation is like that shown in Figure 3(a) with a positive stationary state approached monotonically either from above or below. As the production parameter B increased, the steady-state capital stock increases until eventually $k^m = k^* = k^s$. This is a bifurcation point at which oscillations in the capital-labor ratio must emerge for higher values of B. To prevent negative values in k, we must have $k^m \leqslant m$. Hence, for values of our parameters satisfying

(12)
$$\frac{\beta}{\beta+\gamma} m < \frac{\sigma B}{1+\lambda} \beta^\beta \gamma^\gamma \left(\frac{m}{\beta+\gamma} \right)^{\beta+\gamma} \leqslant m,$$

the capital-labor ratio eventually exhibits bounded oscillations, perhaps after a period of growth. Relations (12) are thus sufficient conditions for growth cycles in model (9).

To see if these cycles can be chaotic we need to see if parameter values exist that satisfy the sufficient conditions (4′). For the model under consideration this can always be done! Choose the value of B, say B'', such that the right-hand side of (12) is an equality, that is, such that

$$(13) \quad k^m = \frac{\sigma B''}{1+\lambda} \beta^\beta \gamma^\gamma \left(\frac{m}{\beta+\gamma} \right)^{\beta+\gamma} = m.$$

From (10) clearly $k^* < k^m$. The situation is shown in Figure 3(b). Clearly there exist two positive roots of the equation

$$(14) \qquad \frac{\sigma B}{1+\lambda} k^\beta (m-k)^\gamma = k^*.$$

Let k^c be the smaller of these two roots. We now have

$$(15) \qquad 0 < k^c < k^* < k^m,$$

where k^c generates k^*, k^* generates k^m and k^m generates 0. Moreover, our function maps the interval $[0, m]$ into itself so all the conditions of the Li-Yorke theorem described above are satisfied. Since our phase diagram can be stretched continuously, there exists a smaller value of B, say B', such that, for any B in the interval $[B', B'']$ there exists a point k^c (depending on B) satisfying (14) and satisfying

(16)

$$0 \leqslant \frac{\sigma B}{(1+\lambda)} k^{m\beta}(m-k^m)^{\gamma} \leqslant k^c < k^* < k^m,$$

so that for all such B irregular oscillations occur.

For the special case in which $\beta=\gamma=m=1$ the difference equation (9) reduces to (2) with $A=\sigma B/(1+\lambda)$. Hence for parameter combinations such that

$$3.57 \leqslant \frac{\sigma B}{(1+\lambda)} \leqslant 4$$

irregular investment cycles exist. (See Yorke and Yorke, pp. 16–20.)

C. *Example* 3: *A Variable Savings Ratio*

Return now to the standard production function (6) of Example 1, but consider a variable savings ratio $s(k)$ which depends on income, wealth, and the real rate of interest r. Without going into a derivation based on an explicit utility function, let us suppose that per capita savings are proportional to wealth (k) and increase with the rate of interest. Let us suppose further that when the latter is small enough, saving is insufficient to maintain capital stock and wealth is consumed. A function with this character is

$$s(k)y = a\left(1-\frac{b}{r}\right)k, \tag{17}$$

which, using (6) and the fact that $r=\beta y/k$ gives the difference equation

(18)

$$k_{t+1}=[a/(1+\lambda)]k_t[1-(b/\beta B)k_t^{1-\beta}].$$

This equation ranges in the interval $[0,(\beta B/\ b)^{1/(1-\beta)}]$ and has the concave single humped shape with which we are already familiar, and which is illustrated in Figure 3. We can, therefore, stretch the graph of (18) in a similar manner by varying the savings parameter "a" continuously, yielding a series of bifurcation points at which qualitative behavior changes with higher and higher order cycles emerging. Likewise, we can find an interval $[a', a'']$ with chaos points satisfying the sufficient conditions (4′) for any savings parameter in this interval.

D. *Example* 4: *A Behavioral Hypothesis*: *Restrained Growth*

So far I have shown that parameters of technology or savings behavior exist that may lead to erratic accumulation cycles. Calculating actual parameter values that satisfy the sufficient conditions (4) can usually be accomplished only by using numerical procedures.[1] However, if the substitution of capital for labor is constrained by a maximal potential growth rate, say ρ, then analysis of the dynamics is greatly facilitated. Such a constraint would arise if (*i*) for purely psychological reasons, agents had an absolute preference for limiting the rate of change in the technique of production, (*ii*) because of a cost of adjustment (Lucas), or (*iii*) as a tactic for avoiding the uncertainties inherent in a changing way of life (my 1979 paper). In this case, the transition equation for the economy becomes, instead of (1),

(19)

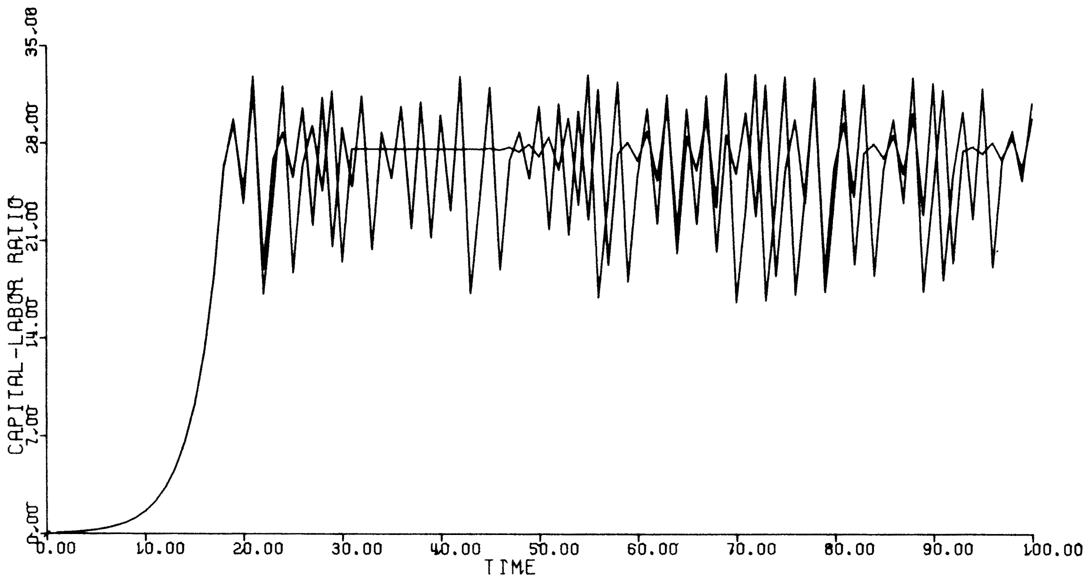
$$k_{t+1}=min\{(1+\rho)k_t, s(k_t)f(k_t)\}/(1+\lambda),$$

where $s(k_t)$ is the savings ratio that would hold in the absence of the precautionary restraint. The simplest situation, and the only one I shall go into here, occurs when the capital-labor ratio k^{**} giving the maximum of (19) is at least as great as that which maximizes (1). In this case, k^{**} is the largest solution of the equation

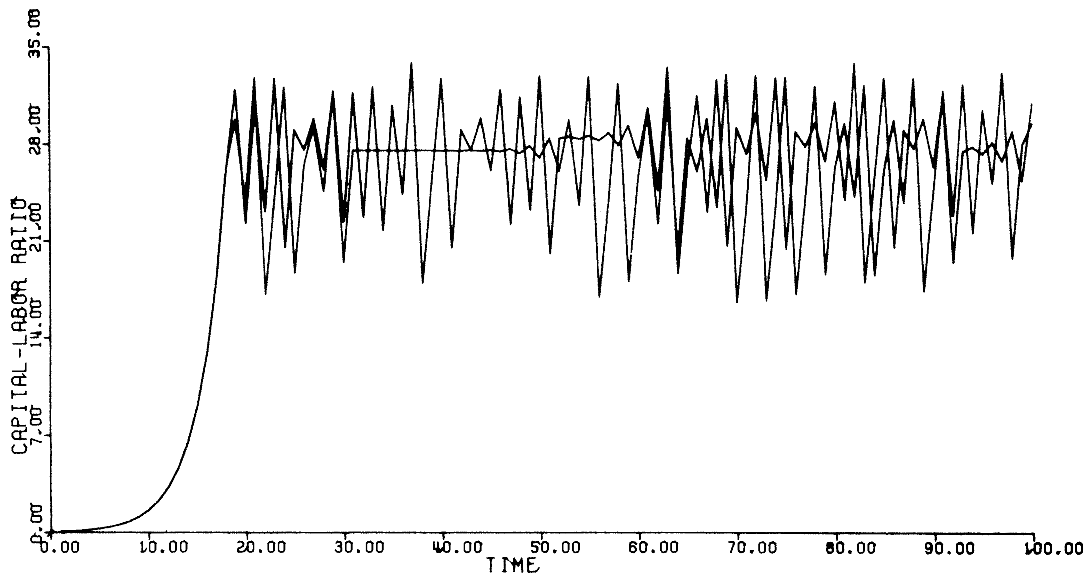
$$(1+\rho)k = s(k)\cdot f(k), \tag{20}$$

(assuming a solution exists).

[1]This could be done, for example, using Newton's method for solving roots of nonlinear equations. My initial experiments so far suggest difficulties even here especially with the functional forms used below.



(a) A Small Perturbation in Initial Condition



(b) A Small Perturbation in Savings Rate

FIGURE 4. INSTABILITY OF CHAOTIC TRAJECTORIES
(Note that the trajectories wander away, wander close, and wander away from each other as time progresses.)

As $k^m=(1+\rho/1+\lambda)k^{**}$ and as $k^{**}=(1+\rho/1+\lambda)k^c$, the inequalities $k^c<k^{**}<k^m$ will hold as long as $\rho>\lambda$. Assuming this restriction, the sufficient conditions for chaos (4′) reduce to the two inequalities

$$(21)\qquad 0\leqslant s(k^m)\cdot f(k^m)/(1+\lambda) \leqslant (1+\lambda/1+\rho)^2 k^m = k^c,$$

where $k^m=[(1+\rho)/(1+\lambda)]k^{**}$.

If we assume the savings function (17) and the usual production function (6), we obtain for inequality (21) the expression

$$(22)\qquad 0\leqslant a\left[B(k^m)^{\beta-1}-b/\beta\right]/(1+\lambda) \leqslant \left(\frac{1+\lambda}{1+\rho}\right)^2$$

where

$$(23)\qquad k^m=\frac{1+\rho}{1+\lambda}\,\frac{(1+\rho)(1+\lambda)}{aB}+\frac{b}{\beta B}1/(\beta-1).$$

The simulation shown in Figure 1 is for this model.[2]

III. Instability and Predictability

By now we are familiar with the kinds of nonlinearity in production or savings functions in the neoclassical one-sector model of capital accumulation that can induce irregular, aperiodic trajectories in the capital-labor ratio and *GNP*. I conclude by illustrating the unstable nature of these chaotic trajectories. Two examples are shown in Figure 4. In panel (a), two trajectories for the self-restrained version of the variable savings ratio model are shown (equation (19)) using the production function (6) and savings function (17). These trajectories have *identical* parameter values differing only in their initial starting points by one tenth of a percent.[3] In panel (b), two trajectories are shown for the same model that emanate from identical initial conditions and which differ in the parameter b by one tenth of a percent.[4]

Evidently, in models with the nonlinearities under investigation here, extremely accurate approximations of model parameters or initial conditions do not make possible accurate predictions for very many periods into the future. Indeed, numerical round-off error, which is inevitable in digital computation, is sufficient to cause rapid divergence of a computed model solution from its "true" path.

If such results were mere artifacts of a particular nonlinear difference equation, such as (2), these implications would give us little pause. That they hold for wide classes of difference equations is noteworthy. Moreover, according to evidence cited by J. R. Beddington et al., and May, chaotically unstable trajectories of this type are more likely to occur with "less" nonlinearity for multi-equation models.

I may note, finally, that in the phenomenon of "strange attractors," various investigators are finding similar dynamic behavior for continuous time, nonlinear differential equations. A survey of this work is found in the paper by Yorke and Yorke cited earlier. Consequently, the insights of the present analysis are not limited to discrete-time modelling. In spite of the simplicity of the basic model, therefore, we have here an introduction to a type of dynamics that seems likely to arise in a wide variety of economic contexts where nonlinearities are present. Although their relevance and full scope can only be revealed after much more detailed research, the implications for how we think about economic theory and policy may be profound.

[2]For this simulation $\rho=.4143$, $a=5.25$, $b/B=2$, $\beta=.5$, and $k(0)=.072068$.

[3]The initial condition $k(0)=.072788$ for this run.

[4]The parameter $\beta=.505$ for this run.

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