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The Complex Dynamics of Economic Interaction

Essays in Economics and Econophysics

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Preface

This volume contains a selection of contributions presented at the WEHIA02 (Workshop on Economies with Heterogeneous Interacting Agents), which was held at the Abdus Salam International Centre for Physics of Trieste, Italy, on May 29- 31, under the auspices of Exystence, Complex System network of excellence. WEHIA02 is the 7th edition of a workshop, which was held for the first time at the University of Ancona, Italy, on 1996 (1), as a one-week series of seminars for the Italian PhD program. Ancona hosted the event in the years 1997 and 1998 (2), before it moved to Genoa 1999 (3), Marseille 2000 (4), Maastricht 2001 (5), and finally to Trieste 2002. The 2003 workshop was held in Kiel, Germany, while the next editions will be in Kyoto, Japan (2004) and Essex, UK (2005). The full list of participants, titles of the invited lectures, papers and posters of WEHIA can be found in http://econ.dea.it/wehia2.

The workshop has been always very informal aiming to put together scholars of different disciplines working on interaction of heterogeneous "something" (economists, physicists, biologists). To facilitate participation there has not been conference fee. It aimed to be a locus for communicate and exchange results and experiences. Several collaborations started with the Workshop and it is not without pride that the organizers claim such a result.

The WEHIA has had two *pars*: a *pars destruens* and a *pars adestruens*. The representative agent framework has been under heavy criticism since the first editions of the workshop. Its shortcuts and its inadequate methodology has been pointed out several times (e.g. Kirman, 1992 and Lippi and Forni, 1998), even though this point has not been recognized by mainstream economics. The rebuttal of any representative agent framework naturally evolved toward models of interactions in which non-linearity plays a central role. It is not surprising that WEHIA moved toward complexity and econophysics which now constitute almost half of the contributions (compared to 10% of the first 3 editions).

While the pars destruens is rooted in economic science, the pars adstruens draws from many sources and is of a more interdisciplinary nature, as we shall discuss below.

Most economists are interested in aggregate or macro problems. They are not interested in individual behaviour as such, but rather in how individual behaviour may be used to explain aggregate economic phenomena. They seek adequate micro-foundations for their analysis. In so doing they typically resort to the idea that the behaviour of each sector of the economy as a whole can be interpreted as being that of a representative individual of that sector. This seems a very curious approach to a modern physicist or biologist, and indeed the point of view conveyed by the contributions in this volume is very different to this. In fact, the characteristic of economic systems that is of particular interest to the contributors to this volume is that of the essential difference between individual and aggregate behaviour. From the point of view of the contributions here this difference is due to the interaction between individuals who differ from one another and who function according to rather simple rules. This interaction organises itself and gives rise to emergent properties of aggregate behaviour, which do not correspond to those of an individual. Such a view is one very much associated with that of statistical physics.

Perhaps the easiest description of this view is to think of the economy as a complex system where aggregate behaviour is determined by the complicated interaction between individuals at the micro level and whose aggregate behaviour presents systematic characteristics which do not reflect average or typical individual behaviour. The analogies with physical, chemical and biological systems are obvious. The point here is not; however, to develop a methodological position along these lines, but to suggest that there are important lessons about the way in which economies work which we can learn from the research in this area, which is reflected in the chapters of this book. There are economic phenomena or structures, which emerge from these interactive models, which are not readily obtained with more standard analysis. Perhaps most importantly such models do not suggest that markets are not important but rather, describe and incorporate features of markets and market behaviour which are absent from standard micro and macro models of the economy. In particular, they offer a useful way of looking at how market structure and organization emerge.

The typical approach in macroeconomics is to obtain structure at the aggregate level by imposing it at the individual level. If, then, one observes a certain consistency of aggregate behaviour then one ascribes this to the consistency of the behaviour of the individuals in the economy. The argument is simply that observed aggregate regularities reflect individual regularity. This is one side of the problem of aggregation. The other is that even if individuals did happen to satisfy certain properties it is by no means necessary that these properties carry over to the aggregate level (see e.g. Sonnenschein [1972] Mantel (1974) and Debreu [1974]). The two taken together mean that there is no direct connection between micro and macro behavior. This basic difficulty in the testing of aggregate models has been insisted upon in the past, (see Kirman [1992], Carroll and Summers [1991] and Lewbel [1989]) when discussing representative individual macro models but as Lewbel observes, this has not, and is unlikely to, stop the profession from testing individually derived hypotheses at the aggregate level. To take a simple example, as Summers has suggested, if we test and reject some theory such as the permanent income hypothesis at the macro level, we may not be rejecting the fact that individuals behave like this but, rather, we may be rejecting the assumption that the consumption sector can be represent as an individual. On the other hand, although some empirical properties of the aggregate relationships between prices charged and quantities purchased can particular markets may hold in the aggregate, this does not imply that the same properties hold for individuals nor that such behaviour was derived from simple maximizing and price taking.

These are just examples, but suggest an important idea. Aggregation may, as Werner Hildenbrand (1994) has suggested, generate structure and regularity. The individuals involved may have a very limited view of some part of the economy but in the large their activities may self-coordinate into well-defined aggregate behaviour. Without making invidious comparisons one has only to think of the way in which the activities of individual ants are coordinated into the organisation of the colony.

While the models developed here bring out the importance of interaction in general, models with local interaction are even more interesting, for they give much more concrete form to the idea that since agents are limited to a set of neighbours with whom they interact they do not have an overall view of the economy and, in addition, changes will not affect all agents simultaneously but rather diffuse across the economy.

Of course, if we are to talk about local interaction we have to specify who interacts with whom. Typically agents in models of local interaction are thought of as being placed on a lattice or on a line and interacting with their neighbours (see Benabou [1992], Blume [1993] and Brock and Durlauf (2001)). In this case one is interested to know whether aggregate behaviour will be characterised by uniformity of individual behaviour or whether pockets or clusters with certain behaviour or characteristics may form. The spatial connotation is by no means necessary however and alternative structures of links can be considered (see Kirman, Oddou and Weber [1986] and Ioannides [1990)).

Possibly the most interesting challenge in this area is to study not just the behaviour or "states" of individuals who interact in a general or local way but also the evolution of those states and indeed of the communications graph itself and indeed, progress has been made in this direction, (see Kirman and Vriend (2001), Weisbuch et al. (2000), Ioannides (1997), and Jackson and Wolinsky (1996)).

Models which take account of the direct interaction between agents allow us to provide an account of aggregate phenomena which are caused by this interaction at the micro level but are no longer a blown up version of that activity, (see e.g. Glaeser and Scheinkman (2001). Furthermore the sort of behaviour that may occur at the aggregate level is much richer than in standard models. Bubble-like phenomena in financial markets, persistence of inferior technologies, spatial variability of activities or of income levels are among the phenomena that arise naturally in this type of analysis. In addition we do not need to resort to the popular view that such phenomena are just manifestations of irrationality, (see Shiller (2000). However perhaps the most interesting of all is the avenue opened up towards an understanding of how the structure of the links governing the interaction may emerge endogenously.

Furthermore, incorporating the structure of direct interaction into our models allows us to assume that individuals are endowed with rather limited reasoning and calculating capacities. The structure of the interactions emerges as the simple and myopic individuals learn from experience. This seems much more promising than models based on individuals capable of maximizing and in so doing of solving extremely complex problems. Individuals in economies are not blind and without purpose or intent as are ants, but they function is a very limited part of the economic environment and are at best aware of a part of that environment. Worse, they make mistakes and in most standard models this is not allowed for. Yet, together they may coordinate on quite sophisticated solutions to the allocation problem. Put alternatively collective rationality is very different from individual rationality and it is a mistake to talk of market behaviour as that of an individual. Heterogeneity and interaction between heterogeneous individuals can lead to simple aggregate behaviour. Conversely it may be the case that even when we start with identical individuals the evolution of the economy will produce heterogeneity in terms of the states of the individuals, their activity and their welfare. As we have said earlier this point of view is one which will be familiar to physicists and it is now making itself felt in modern economics.

Indeed there has been increasing interest on economics, in recent years, within the community of physicists. The influence of ideas from physics is not new in economics. Classical mechanics has already been taken as a reference paradigm by classic economists.

Interdisciplinary ventures of physicists into domains such as biology and geology are relatively simple because the laws of physics apply to those domains as well. Physics enters from the main door in biophysics, geophysics or astrophysics. But strictly speaking, Newton's as well as other laws of physics have nothing to do with economics, whose fundamental laws have a different origin. The "forces" acting on economic agents are metaphorical ones and are not in such a strict relation as $F = ma$ with the "mass" and the "acceleration" of that agent. Of course one may define F, m and a for economic agents and come-up with a whole dictionary which translates physics into economics. Approaches of this type are typically misleading. They have at best been able to reproduce what was already known in other terms.

Therefore the recent interest of physicists in economics - and the emergence of econophysics - needs some explanation.

While the economy in the prefix of econophysics has been mostly related to financial markets, the physics in the suffix stands for statistical physics. Statistical physics is a branch of physics which studies systems of many interacting degrees of freedom. It originated from statistical mechanics which explains the macroscopic behaviour of matter in thermal equilibrium from the microscopic interactions between atoms. But then statistical physicists started looking at out-of-equilibrium and non-equilibrium phenomena – such as growth of interfaces and fractal structures – and even further to sand-piles, biological evolution, earthquakes etc.

A key passage in all this, has been the study of critical phenomena at second order phase transitions. Physicists have realized that in such particular conditions, the macroscopic behaviour of a system is largely independent of microscopic details. At the same time, in those particular conditions, the system displays scale invariant properties, which take the mathematical form of power law relations.

The independence of collective behaviour on microscopic details means that particular macroscopic statistical laws are shared by a large class of phenomena and of theoretical models. One can then pick up the simplest model in this class to make a theory. This is why physicists tend to prune their models from as many details as possible, retaining only those elementary variables and interactions which are strictly necessary for the emergence of that particular macroscopic behaviour (Per Bak suggested that one should stop cutting details just before "throwing the baby away with the water"). This way of doing may be hard to accept for other disciplines which have worked so much to study those details.

Benoit B. Mandelbrot then observed that scale invariance extends far beyond the limits of critical phenomena in physics. From the galaxies, to lakes and city sizes, scale invariance -fractality as he called it - is the rule rather than the exception. The body of empirical evidence supporting this conclusion has been broadening in several disciplines, including socio-economic sciences. This suggests that the approach which has been so successful in understanding critical phenomena in statistical physics may prove useful to deal with problems which have nothing to do with physics.

This is why physics in econophysics should be meant as "scale invariance statistics" or "statistical physics".

The bulk of the contributions to econophysics have concentrated on financial markets for three main reasons:

1. financial markets are systems of many interacting degrees of freedom (the traders)
2. a wealth of empirical data is available, which thus allows to test theoretical prediction to a great accuracy
3. Fluctuations in financial markets have non-trivial statistical properties.

Research has followed three main lines:

1. The empirical analysis of high frequency data of financial markets aimed at establishing the key statistical regularities - the so-called stylized facts - which a theory of financial markets should explain. The quest for stylized

facts has been also pursued for other issues in economics, such as company size distribution and growth.

2. The application of methods of theoretical physics to financial "engineering", i.e. to pricing theory and to risk management. This includes for example understanding how standard tools used in a world where prices follow simple geometric Brownian motions (e.g. Black and Scholes) change when one moves to the real world of fat tailed returns.
3. Agent based modelling of financial markets (and in general of socio-economic systems) aimed at identifying the main ingredients, which are responsible for the emergence of the observed stylized facts.

The three physicist keynote speakers of WEHIA2002 are the world leaders in these three lines of research. The work of the group of H.E. Stanley is the reference for empirical studies whereas J.-P. Bouchaud has made outstanding contributions on the pricing and risk management aspects as well as to the other two issues. The introduction of the Minority Game by Y.-C. Zhang has been a quite important step in modelling financial markets. His contributions to econophysics are an example of the highly stylized but essential modelling approach to socio-economic systems.

From the economics' side, the challenge of agent based modelling of financial markets has been also taken by T. Lux, though with a more standard modelling approach. T. Lux is also an authority on the first topic (econometrics of financial markets).

On the other hand, J. A. Scheinkmann's contribution has been a remarkable occasion to realize how the subtleties and richness of speculative bubbles in a financial market can be revealed by a traditional though sophisticated approach, based on rational expectations.

The richness of aggregate economic activity is poorly captured by assimilating it to average individual behaviour. The contributions in this volume suggest ways forward to more interesting and realistic explanations of empirical economic phenomena which, after all, is, or should be, the goal of every economist. Physicists have something to offer in this endeavour, but still they can learn a lot from economists.

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Contents

Preface V

Part I Economics

A New Model of Industry Dynamics
M. Aoki 3

Monetary Policy and the Distribution of Wealth in a OLG Economy with Heterogeneous Agents, Money and Bequests
R. Longaretti, D. Delli Gatti 15

Limit Pricing and Entry Dynamics with Heterogeneous Firms
T. Asada, W. Semmler 35

Viability of Innovation Processes, Emergence and Stability of Market Structures
M. Amendola, J-L Gaffard, P. Musso 49

Firms' Bankruptcy and Turnover in a Macroeconomy
M. Bee, G. Espa, R. Tamborini 79

Part II Econophysics: empirical analysis

Statistical Physics and Economic Fluctuations
H. E. Stanley, P. Gopikrishnan, V. Plerou 109

Modeling the Dynamics of a Financial Index after a Crash
F. Lillo, R. Mantegna 143

Price Impact Function of a Single Transaction
F. Lillo, J. D. Farmer,, R. Mantegna 153

Income Distribution and Stochastic Multiplicative Process with Reset Events
M. Nirei, W. Souma 161

Part III Market Models and Stylized Facts

Volatility Clustring in Agent Based Market Model
I. Giardina, J-P Bouchaud 171

A Dynamical Analysis of Speculation across two Markets
C. Chiarella, R. Dieci, L. Gardini 197

A Simple Micro-Model of Market Dynamics
G. Bottazzi 213

Herd Behavior in Artificial Stock Markets
U. Garibaldi, M. Raberto, P. Viarengo 229

Part IV Models of Bounded Rationality

Bargaining with Posterior Opportunities: an Evolutionary Social Simulation
E.H. Gerding, J.A. La Poutr 241

Herding Behavior of Financial Analysts: A Model of Self-organized Criticality
A. Krause 257

Give-and-Take in Minority Games
A. Namatame, N. Tanoura, H. Sato 269

Coordination in Self-organization in Minority Games: Experimental Evidence
G. Bottazzi, G. Devetag 283

Part V Societies of Interacting Agents

Coordination of Decisions in a Spatial Model of Brownian Agents
F. Schweitzer 303

Coalition Formation with Boundedly Rational Agents
D. Fiaschi, P.M. Pacini 319

On the Creation of Networks and Knowledge
R. Cowan, N. Jonard, J-B Zimmerman 337

Impact of Corporate Boards Interlock on the Decision Making Dynamics
S. Battiston, E. Bonabeau, G. Weisbuch 355

Interaction of Heterogeneous Agents in Stochastic Socio-Economic Systems
J. Mimkes, M. Hillebrand, F. Fründ, C.Denk, G.Willis 379

Coordination of Decisions in a Spatial Model of Brownian Agents
W. A. T. Wan Abdullah 395

Part I

Economics

A New Model of Industry Dynamics

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Summary. This paper discusses externalities experienced by firms a sector of an economy, by specifying the types of conditional probabilities for entries and exists of additional production units, or entries of production units of entirely new firms or goods. Such conditional probability specifications may result from assumptions on firms adjusting their output rates in response to expected excess demand or profit changes in environments where such excess demands or profits are functions of own output rates, as well as those of other firms.

After a short comment on how entries and exits are modeled in the standard economic literature, models are set in a framework of continuous-time Markov chain models to determine movements of outputs around an equilibrium distributions of sizes of clusters of firms (agents) of different types. There are one- and two-parameter versions of the conditional probability specifications, due to Ewens and Pitman respectively, set in the context of exchangeable partitions of agents in the models.

Instead of the usual state vector which lists the number of agents by types, we explain the use of partition vector, which list the number of types by cluster sizes, as state vector in models composed of a large number of exchangeable agents of possibly many types. We illustrate the use of this new state vector by describing the equilibrium distribution of agents by types in partition vector form.

1 Introduction

Economists often face problems of modeling collective behavior of a large number of interacting agents, which are possibly of several different types. Models are supposed to be constructed so that economists can draw conclusions on such things as equilibrium size distributions of firms, market shares by different types or kinds of goods, and so on, and finally how some macroeconomic regularities emerge as the number of agents increases toward infinity.

Unfortunately, the current state of (macro)economics is such that economists have not been very successful in modeling dynamic or distributional phenomena in economics when markets involve several types of agents.

M. Gallegati et al. (eds.), *The Complex Dynamics of Economic Interaction*

In this paper, we propose an approach which deals explicitly on the clusters of agents by types, and on the stationary distributions of cluster sizes, and focuses on the order statistics of the cluster sizes in order to draw inferences on market behavior from those of a few large clusters of agents.

We explicitly assume that there are several types of agents in our models, the number of which may or may not be known in advance, and that models are open, that is, agents may enter or exit the models at any time. In addition, agents may change their minds at any time about the decisions or behavioral rules they use. In other words, agents may change their types any time.[1]

In this paper, we interpret the word **types** broadly. The word may refer to the decision rules or behavioral rules adopted by economic agents, to the kinds of goods they choose to produce, or to some risk characteristics, and so forth. We assume that the number of types are at most countable. We cannot assume in advance that we know all of them because new rules or new goods may be invented in the future. This is the so-called problem of unanticipated knowledge in the sense of Zabell, see Zabell (1992).[2]

We discuss some new concepts in this paper that are not currently used in economics. For example, we introduce the notion of partition vector as state vectors, which is different from the empirical distributions, and use the assumption of random exchangeable partitions induced by agents of different types in the models, in the technical sense of exchangeable random variables in the probability literature. We apply the equilibrium distribution discovered by Ewens in the context of population genetics literature into that of cluster sizes of agents by types. We show how the uses of these concepts help us to examine dynamics of interacting agents of several types.

The Ewens sampling formula is specified by a single parameter θ, which controls the rate of entries of new types, and correlations among agents of different types. We also describe its two-parameter extension by Pitman (1992), which is specified by two parameters, α and θ discussed later.

We first illustrate the approach of the traditional economists by sketching how the problem of allocating capital stock between two sectors is formulated by Dixit (1989). After briefly mentioning some problems of this traditional approach, we switch to our modeling procedure in terms of continuous-time Markov chains with a large but finite number of interacting agents.

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[1] There is no lock-step behavior by agents.

[2] Zabell describes the problem faced by statisticians in classifying samples of insects collected in unexplored regions, since they may contain new species of insects, say. The naive Bayesian approach is not applicable. See, however, Antoniak (1974) on non-parametric Bayesian approach who arrived at the Ewens distribution from this point of view. The Ewens sampling formula, Ewens (1972) was discovered by Ewens in connection with the population genetics work.

2 A Traditional Model of Two-sector Economy

We briefly sketch the two-sector economy model by Dixit (1989) as an example of traditional approach.[3] In his model, there are two sectors which produce two goods. The economy is supposed to shift resources from one sector to another as the relative prices of the two goods vary to maximize the economy-wide profit. In our language, there are two types of agents, and some agent change its mind when it shifts from one sector to another. Dixit's model does not, however, involve decentralized decision making by individual agents. When the relative prices cross certain price schedules obtained by solving an optimization problem for the whole economy, agents somehow know that it is its turn to switch types, the detail of the process is not described in the paper.

What are some of the main objections to this analysis? First, there is no explanation about which of the firms decide to move. On the macroeconomic level, it does not matter which firm moves since firms are truly indistinguishable, including managerial abilities. As to firm managers' decision problems, how do they decide that it is their turn to move? Must the central planner of this economy choose one firm, by lining up the firms in some order, or do they draw a lot? There is apparently no uncertainty as to which firm enters next, or exit next. This may only be possible in a planned economy. Problems of imperfect or incomplete information and externalities among firms (agents) are cleverly hidden or abstracted away in his analysis.

3 An Alternative Approach: Basic Setup

Aoki and Yoshikawa (2001), and Aoki (2002, Chapt. 8) are examples of some alternative approaches to that sketched above. Basically, our approach focuses on the random partitions of the set of firms into clusters induced by subsets formed by firms of the same types, and utilizes the conditional probability specifications for new entries and exits to derive equilibrium distributions for cluster sizes. Firms' decisions are made probabilistically, that is firms' behavior is set in terms of transition rates of continuous-time Markov chains. We use the master equation (backward Chapman-Kolomorov equation) as the dynamic equation for the time evolution of probabilities of state vectors.[4]

Given the total number of agents, n, we examine how the n-set, that is, the set $\{1, 2, \ldots, n\}$ is partitioned into K_n clusters, or subsets. This partition is treated as a random exchangeable partition in the sense of Zabell (1992).

[3] Although it has several novel features such as the use of gradient-matching condition, it is nevertheless a traditional model in that the central planner solves an optimization problem for the whole economy.

[4] In the above two-sector model the scalar variable of the number of firms in sector one, say, serves as the state variable. See the binary choice models in Aoki (1996; 2002a). In an open model with K sectors, a K-dimensional vector is used.

See Kingman (1993, 1978a,b) who used the order statistics of the fractions of agents by types, and invented what is known as the paint-box process and the resultant Poisson-Dirichlet distribution to solve this problem. In this exposition we mostly keep n finite, but large.

4 Jump Markov Process Models

Here we follow Aoki (1996, 1998, 2000a,b,c, 2002) and sketch the basic ingredients of our modeling procedure without too much detail. The reader is asked to consult the cited references for detail.

Continuous-time Markov chains, also known as jump Markov processes, are specified by transition rates.

Define a state vector X_t which takes on the value $\mathbf{n} := (n_1, n_2, \ldots, n_K)$, called frequency or occupancy vector, where n_i is the number of agents of type i, $i = 1, 2, \ldots, K_n$, $n = n_1 + n_2 + \cdots + n_{K_n}$.

In our model we need to specify entry rates, exit rates and rates of type changes. Over a small time interval Δ, rates are multiplied by the length of interval to approximate the conditional probabilities up to $O(\Delta)$. Entry rates by an agent of type j is given by

$$w(\mathbf{n}, \mathbf{n} + \mathbf{e_j}) = \phi_j(n_j, \mathbf{n}),$$

where $\mathbf{e}_j$ is a vector with the only nonzero element of one at component j.[5] Exit rates of an agent of type k is specified by

$$w(\mathbf{n}, \mathbf{n} - \mathbf{e_k}) = \psi_k(n_k, \mathbf{n})$$

and transition rates of type i agent changing into type j agent by

$$w(\mathbf{n}, \mathbf{n} - \mathbf{e_i} + \mathbf{e_j}) = \lambda_{i,j} \nu(n_i, n_j, \mathbf{n}).$$

With transition rates between states specified, the dynamics for the probability is given by the following equation, where s, s', and s'' refer to some states

$$dP(\mathrm{s}, t)/dt = \sum_{\mathrm{s}'} w(\mathrm{s}', \mathrm{s}) P(\mathrm{s}', t) - \sum_{\mathrm{s}''} w(\mathrm{s}, \mathrm{s}'') P(\mathrm{s}, t).$$

This is called the master equation in physics, ecology and chemistry, and we follow their usage of the name.

A special example of interest has the transition rates:

[5] For example, $w(\mathbf{n}, \mathbf{n} + \mathbf{e_j})\Delta \approx Pr(X_{t+\Delta} = \mathbf{n} + \mathbf{e_j})$.

$$w(\mathbf{n}, \mathbf{n} + \mathbf{e}_k) = c_k(n_k + h_k),$$

for $n_k \geq 0$,

$$w(\mathbf{n}, \mathbf{n} - \mathbf{e}_j) = d_j n_j,$$

$n_j \geq 1$, and

$$w(\mathbf{n}, \mathbf{n} - \mathbf{e}_j + \mathbf{e}_k) = \lambda_{jk} d_j n_j c_k(n_k + h_k),$$

with $\lambda_{jk} = \lambda_{kj}$, and where $j, k = 1, 2, \dots K$. We assume that $d_j \geq c_j > 0$, and $h_j > 0$, and $\lambda_{jk} = \lambda_{kj}$ for all j, k pairs.

The first transition rate specifies entry rate of type k agents, and the second that of the exit or departure rate by type j agents and the last specifies the transition intensity of changing types by agents from type j to type k. In the entry transition rate specification $c_k n_k$ stands for attractiveness of larger group, such as network externality which makes it easier for others to join the cluster or group, and $c_k h_k$ stands for the innovation effects which is independent of the group size. These transition rates for type changes are in Kelly (1979). We need interactions or correlations among agents. It turns out that parameter θ, to be introduced in connection with (2) below, plays this role. See Aoki (2000a, 2002b). The jump Markov process thus specified has the steady state or stationary distribution in the product form

$$\pi(\mathbf{n}) = \prod_{j=1}^{K} \pi_j(n_j),$$

where

$$\pi_j(n_j) = (1 - g_j)^{-h_j} \binom{-h_j}{n_j} (-g_j)^{n_j}$$

where $g_j = c_j/d_j$.

These expressions are derived straightforwardly by applying the detailed balance conditions to the transition rates. See Kelly (1979, Chapt.1) for example.

To provide simpler explanation, suppose that $g_j = g$ for all j. Then, noting that $\prod_j (1 - g)^{-h_j} = (1 - g)^{-\sum_j h_j}$, the joint probability distribution is expressible as

$$\pi(\mathbf{n}) = \binom{-\sum h_k}{n}^{-1} \prod_{j=1}^{K} \binom{-h_j}{n_j}. \tag{1}$$

In modeling industrial sector with n_i being the number of agents of type i, the word type may refer to the kinds of goods being produced by firm i or n_i may refer to the size of the "production line", that is, a measure of capacity utilization by firm producing type i good. Zabell (1982) proved that under the assumption of exchangeable partitions the functional form of f is specified by

$$f(n_i, n) = \frac{n_i}{n + \theta}, \tag{2}$$

with some positive scalar parameter θ. Therefore, the entry rate of a new type is given by $\theta/(n+\theta)$. More generally, they are of the form

$$w(\mathbf{n}, \mathbf{n}+\mathbf{e}_k) = \frac{\beta + n_k}{K\beta + n},$$

for some positive parameter β, which reduces to (2) in the limit of α going to zero, and K to infinity while their product approaches θ, and

$$w(\mathbf{n}, \mathbf{n}-\mathbf{e}_j) = \frac{n_j}{n}.$$

See Costantini (1979, 2000), and Zabell (1982) for circumstances under which these transition rates arise. See Aoki and Yoshikawa (2001) and Aoki (2002, Sec.8.6) for an application of this type of transition rates in models of economy or sectors of economy.

5 Partition Vectors and Ewens Distribution

Now, we introduce the partition vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, so named by Zabell,[6] where a_k is the number of types or clusters with exactly k agents. Consequently we have an inequality

$$\sum_i a_i = K_n,$$

where K_n is the number of groups or clusters formed by n agents, and

$$\sum_i i a_i = n,$$

which is an accounting identity.

To further simplify our presentation, let us suppose that $h_j = h$ for all j.[7] Then (1) is rewritten as (dropping n from K_n

$$\pi(\mathbf{n}) = \binom{-Kh}{n}^{-1} \prod_{j=1}^{K} \binom{-h}{j}^{a_j}.$$

This is so because there are a_j of the ns which equal j.

To simplify our explanation there is a constant K such that K_n is not greater than K. Now suppose that K becomes very large and h very small, while the product Kh approaches a positive constant θ. We note that the negative binomial expression

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[6] This vector is called by different names by Kingman (1980), and Sachkov (1996).

[7] This assumption is not necessary. All we need is that the product Kh_j has the same positive limit as K goes to infinity and h_j to zero.

$$\binom{-h}{j}^{a_j}$$

approaches $(h/j)^{a_j}(-1)^{ja_j}$ as h becomes smaller. Suppose $K_n = k$, much less than K. Then, there are

$$\frac{K!}{a_1!a_2!\cdots a_n!(K-k)!}$$

many ways of realizing a vector. Hence

$$\pi(\mathbf{a}) = \binom{-\theta}{n}(-1)^n \frac{K!}{a_1!a_2!\cdots a_n!(K-k)!}\prod_j(\frac{h}{j})^{a_j}. \tag{3}$$

Noting that $K!/(K-k)! \times h^k$ approaches θ^k in the limit of K becoming infinite and h approaching 0 while keeping Kh at θ, we arrive, in the limit, at the probability distribution, known as the Ewens distribution, or Ewens sampling formula very well known in the genetics literature, Ewens (1972), and Kingman (1987).

$$\pi_n(\mathbf{a}) = \frac{n!}{\theta^{[n]}}\prod_{j=1}^{n}(\frac{\theta}{j})^{a_j}\frac{1}{a_j!},$$

where $\theta^{[n]} := \theta(\theta+1)\cdots(\theta+n-1)$. This distribution has been investigated in several ways. See Arratia and Tavaré (1992), or Hoppe (1987). Kingman (1980) states that this distribution arise in many applications. There are other ways of deriving this distribution. We next examine some of its properties.

The number of clusters and value of θ

Ewens sampling formula has a single parameter θ. Its value influences the number of clusters formed by the agents. Smaller values of θ tends to produce a few large clusters, while larger values produce a large number of smaller clusters.

To obtain some quick feels for the influences of the value of θ, take $n = 2$ and $a_2 = 1$. All other as are zero. Then

$$\pi_2(a_1 = 0, a_2 = 1) = \frac{1}{1+\theta}.$$

This shows that two randomly chosen agents are of the same type with high probability when θ is small, and with small probability when θ is large. In fact, θ controls correlation between agents' types or classification. Furthermore, the next two extreme situations may convey the relation between the value of θ and the number of clusters. We note that the probability of n agents forming a single cluster is given by

$$\pi_n(a_j = 0, 1, \ldots, (n-1), a_n = 1) = \frac{(n-1)!}{(\theta+1)(\theta+2)\cdots(\theta+n-1)}$$

while the probability that n agents form n singleton is given by

$$\pi_n(a_1 = n, a_j = 0, j \neq 1) = \frac{\theta^{n-1}}{(\theta+1)(\theta+2)\cdots(\theta+n-1)}.$$

With θ much smaller than one, the former probability is approximately equal to 1, while the latter is approximately equal to zero. When θ is much larger than n the opposite is approximately true.

We can show that

$$P_n(K_n = k) = \frac{1}{\theta^{[n]}} c(n,k)\theta^k,$$

where $c(n,k)$ is known as the signless Stirling numbers of the first kind, and is defined by

$$\theta^{[n]} = \sum_1^n c(n,k)\theta^k.$$

See Hoppe (1987) for the derivation. Stirling numbers are discussed in van Lint and Wilson (1992, p.104) for example.

6 Two-parameter Generalization of the Ewens Distribution

Pitman (1992) generalized the Ewens' distribution by using the transition rates

$$w(\mathbf{n}, \mathbf{n} + \mathbf{e_j}) = \frac{n_j - \alpha}{n + \theta},$$

where $\theta + \alpha > 0$, instead of (2), and α between 0 and 1.

With this, the conditional probability that a new type enters in the next Δ time interval is approximately given by $\frac{K_n\alpha+\theta}{n+\theta}\Delta$. Pitman also derived the equilibrium distribution for this two-parameter version.

There are interpretations in terms of what is called size-biased sampling of these, but we will not stop here to explain them. See Kingman (1992), and Pitman (1992, 1995)

7 Clusters in Partition Vector

We cite some examples from Kelly (1979) as being suggestive of other applications of the proposed approach to economic modeling.[8] There are N basic

[8] This subsection is based in part on Kelly (1979, Chap. 8). We use a_i rather than m_i used in Kelly.

units partitioned into distinct clusters or collections, with a_i being the number of groups consisting of i units. Recall that we mean by units some basic building blocks from which objects that cluster are made up.

The transition rate $w(\mathbf{a}, \mathbf{a}+e_i) = \alpha$ represents the process in which a basic unit or singleton (called isolate in Kelly (1979, chapt.8)) joins a group of size i at rate α, $i = 1, 2, \ldots$. The transition rate $w(\mathbf{a}, \mathbf{a}-e_i) = \beta$ refers to the rate at which a singleton (one basic unit) leaves that group to become an isolate. Call a cluster of size i i-cluster.

We assume $\mathbf{a}$ is a Markov process in which the transition rate $w(\mathbf{a}, \mathbf{a}-e_1-e_i+e_{i+1}) = \alpha a_1 a_i$, $i \geq 2$. This refers to the rate at which an isolate joins an i-cluster, hence forming one more $i+1$-cluster. When two isolates form a new group of size 2, the transition rate is $w(\mathbf{a}, \mathbf{a}-2e_1+e_2) = \alpha a_1(a_1-1)$. The rate at which an i-cluster breaks up into an isolate and a cluster of size $i-1$ is represented by $w(\mathbf{a}, \mathbf{a}+e_1+e_{i-1}-e_i) = i\beta a_i$, $i \geq 2$. The transition rate $w(\mathbf{a}, \mathbf{a}+2e_1-e_2) = 2\beta a_2$ refers to one cluster of size 2 divides into 2 isolates. In a more general setting, suppose that the transition rate of one r-cluster and one s-cluster form one u-cluster. It is written as

$$w(\mathbf{a}, \mathbf{a}-e_r-e_s+e_u) = \lambda_{rsu} a_r a_s,$$

when $r \neq s$. With $r = s$, we specify the transition rate by

$$w(\mathbf{a}, \mathbf{a}-2e_r+e_u) = \lambda_{rru} a_r(a_r-1),$$

and

$$w(\mathbf{a}, \mathbf{a}-e_u+e_r+e_s) = \mu_{rsu} a_u$$

is the transition rate of one u-cluster breaking up into one r-cluster and one s-cluster. In the simple example described above, we have

$$\lambda_{1,i,i+1} = \alpha,$$

and

$$\mu_{1,i-1,i} = i\beta,$$

for $i \geq 2$.

Assume that $\lambda_{rsu} = \lambda_{sru}$ and $\mu_{rsu} = \mu_{sru}$. We check the detailed balance conditions and verify that the equilibrium distribution is of the form

$$\pi(\mathbf{a}) = B \prod_r \frac{c_r^{a_r}}{a_r!},$$

provided there are positive numbers $c_1, c_2, \ldots$, such that

$$c_r c_s \lambda_{rsu} = c_u \mu_{rsu}.$$

We can easily verify that the detailed balance conditions are satisfied. In the simple example, we note tht

$$c_r = \frac{\beta}{\alpha r!}$$

satisfies the detailed balance conditions.

For a closed model with N fixed, the equilibrium distribution

$$\pi(\mathbf{a}) = B \prod_{i=1}^{N} \frac{1}{a_i!} (\frac{\beta}{\alpha i!})^{a_i}$$

is an example of assemblies analyzed by Arratia and Tavaré. Here $m_i = (\beta/\alpha)$ serves as the "number" of labelled structures on a set of size i, that is the number of the labelled structure in this example is independent of the size of the block.

In the more general development that follow the example, if we set

$$c_r = \frac{m_r}{r!}$$

then m_r is the number of the labelled structures on a set of size r.

8 Concluding Remarks

This paper proposes a finitary approach to economic modeling, that is to start with a finite number of agents with discrete choice sets, and with explicitly specified transition rates. It discusses several entry and exit transition rates in economic models. In particular, it presented Ewens and related distributions as candidates for distributions of cluster sizes formed by a large number of economic agents who interact in a market. This distribution seems to be very useful in economic modeling. See Arratia, Barbour and Tavaré (1992), and Kingman (1980) strongly suggest that the Ewens' and related distributions are robust and ubiquitous.

Although no application is described in this paper, Aoki (2002a, 2002b) has several simple applications of the approach. With $\theta = .3$, in Aoki (2002b), for example, two largest groups are shown to capture nearly 80 per cent of the market shares and hence dominate the market excess demands for the shares, which in turn determine the stationary distributions of the market clearing price changes. In this way it is also possible to relate the tail distribution of the market clearing prices with distributions of a few large-sized clusters of agents.

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Monetary Policy and the Distribution of Wealth in a OLG Economy with Heterogeneous Agents, Money and Bequests

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Summary. We develop an OLG model in which the distribution of wealth makes monetary policy non-superneutral at the individual level. In other words monetary policy may have distributional consequences. We demonstrate that in a representative agent economy, monetary policy would be superneutral both at the individual and at the aggregate level. On the contrary if agents differ from one another as far as income and wealth are concerned, there exists a mean field effect that makes money non-superneutral at the individual level.

1 Introduction

Macroeconomic theory generally focuses on the aggregate consequences of monetary policy, without considering its distributional effects. We develop a model in which it is income and wealth distribution themselves to make monetary policy non-superneutral at the individual level. More precisely we demonstrate that, if agents were homogeneous, that is in a representative agent economy, monetary policy would be superneutral: output, consumption and wealth (of the representative agent) would be independent of money growth. On the contrary if agents differed from one another as far as income and wealth are concerned, there would exist a mean field effect (see section 4) since individual wealth would depend also on average wealth. The mean field effect captures non-strategic interaction between the individual agent and the rest of the population, proxied by the average agent, and makes money non-superneutral at the individual level. In fact the average levels of output, consumption and wealth are independent of money growth but the individual levels of the same variables are indeed affected by changes in the rate of growth of money. In other words, by increasing the rate of money growth, agents who are relatively poor in income/endowment become wealthier whereas relatively rich agents become less wealthy. However, the relative ranking is not reversed.

M. Gallegati et al. (eds.), *The Complex Dynamics of Economic Interaction*

As a consequence, while the first moments of the distributions of output, consumption and wealth do not depend on money (i.e. the distributions are mean preserving), higher moments are influenced by the rate of money growth. If the variance is thought of as a rough measure of inequality, then inequality is decreasing with money growth. In other words income and wealth distribution is the source of a *financial accelerator* since monetary policy has clear asymmetric effects.

The paper is organized as follows. In Section 2 we present and discuss the main features of an OLG economy with money and bequests. Section 3 focuses on a representative agent economy. In Section 4 we consider the consequences of heterogeneity of income and wealth. Section 5 is devoted to an example of heterogeneous population where agents make an occupational choice. Section 6 concludes.

2 An OLG Economy with Money and Bequests

For the sake of simplicity, we assume that population is constant and consists of N young and N old people (of the previous generation) per period. The i-th individual gets output y_{it} when young, nothing when old. Output is perishable and therefore cannot be stored to be consumed in the future. For simplicity, preferences are uniform across individuals and the young do not receive utility from consumption. Assuming intergenerational altruism, the well behaved utility function is $U = U\left(c_{it+1}, b_{it+1}\right)$ where c_{it+1} is consumption of the agent when old and b_{it+1} is bequest of the old to the young (wealth of the young). In a monetary economy, in order to consume when old, the young at time t sells its output to the old of the previous generation at the price P_t in exchange for money M_{it}:

$$M_{it} = P_t y_{it}$$

or

$$\frac{M_{it}}{P_t} = y_{it} \tag{1}$$

Aggregating across individuals we get:

$$\frac{M_t}{P_t} = Y_t \tag{2}$$

where $M_t \equiv \sum_{i=1}^{N} M_{it}$ is the aggregate demand for money and $Y_t \equiv \sum_{i=1}^{N} y_{it}$ is aggregate output.

Money is a means of payment and a store of value which can be carried on from one period to the next in order to buy goods. When old, the agent spends the money received when young M_{it} plus a money transfer proportional to the *average money holding* $h_t \equiv \frac{H_t}{N}$, where H_t is the aggregate money supply. Assuming that there is equilibrium on the money market in t, i.e. aggregate supply H_t is equal to aggregate demand M_t, we can define the individual money transfer as:

$$T_{it+1} = \mu h_t = \frac{\mu M_t}{N} \tag{3}$$

$0 < \mu < 1$. The transfer is uniform across individuals while money balances are not necessarily the same for each and every agent.

The old spend money to buy consumption goods and leave a bequest to the young:

$$M_{it+1} = M_{it} + T_{it+1} = M_{it} + \mu h_t = P_{t+1}\left(c_{it+1} + b_{it+1}\right) \tag{4}$$

Dividing by P_{t+1} and substituting (1) into (4) we obtain the lifetime budget constraint:

$$RMB_i \equiv \frac{M_{it} + T_{it+1}}{P_{t+1}} = \frac{M_{it}}{P_t}\frac{P_t}{P_{t+1}} + \frac{\mu h_t}{P_{t+1}} = \theta_{t+1}\left(y_{it} + \mu\frac{h_t}{P_t}\right) = c_{it+1} + b_{it+1} \tag{5}$$

where RMB_i stands for real money balances of the old, $\theta_{t+1} \equiv \frac{P_t}{P_{t+1}}$ is the real rate of return of money, $\theta_{t+1} \equiv \frac{1}{1+\pi_{t+1}}$ and π_{t+1} is inflation in t+1.According to (5) real money balances are spent either on consumption goods or bequest.

Equilibrium on the money market is brought about by $M_t = H_t$ so that $\frac{M_t}{P_t} = \frac{H_t}{P_t} = Y_t$. Dividing by N, we get $\frac{h_t}{P_t} = \bar{y}_t$where $\bar{y}_t$ is average output[3]. Therefore, the real money balances of the old can be written as $\theta_{t+1}\left(y_{it} + \mu\bar{y}_t\right)$ and the lifetime budget constraint becomes:

$$\theta_{t+1}\left(y_{it} + \mu\bar{y}_t\right) = c_{it+1} + b_{it+1} \tag{6}$$

Let's assume preferences are represented by a Cobb-Douglas utility function:

$$U = c_{it+1}^{(\gamma)} b_{it+1}^{(1-\gamma)} \tag{7}$$

with $0 < \gamma < 1$. Maximizing (7) subject to (6) yields:

$$c_{it+1} = \gamma\theta_{t+1}\left(y_{it} + \mu\bar{y}_t\right) \tag{8}$$

$$b_{it+1} = (1-\gamma)\,\theta_{t+1}\left(y_{it} + \mu\bar{y}_t\right) \tag{9}$$

Thanks to the Cobb-Douglas utility function, both consumption and bequest are proportional to RMB_i.

Aggregate transfers (to the old) in t+1 is $H_{t+1} - H_t = \sum_{i=1}^{N} T_{it+1} = N\mu\frac{H_t}{N} = \mu H_t$. Hence the supply of money in t+1 is $H_{t+1} = H_t\left(1+\mu\right)$. Thanks to equilibrium on the money market in t $H_{t+1} = M_t\left(1+\mu\right)$.

[3] Thus, the money transfer is proportional to average nominal output: $T_{it+1} = \mu h_t = P_t\bar{y}_t$.

Equilibrium on the money market in t+1 is brought about by $M_{t+1} = H_{t+1}$ or $P_{t+1}Y_{t+1} = M_t(1+\mu)$ .Dividing by P_t , recalling (2) and rearranging we get:

$$\frac{P_{t+1}}{P_t}Y_{t+1} = \frac{M_t}{P_t}(1+\mu) = Y_t(1+\mu)$$

or, dividing by N,

$$\frac{P_{t+1}}{P_t}\bar{y}_{t+1} = \bar{y}_t(1+\mu)$$

and finally

$$\frac{P_{t+1}}{P_t} \equiv 1 + \pi_{t+1} = \frac{1+\mu}{1+g_{t+1}} \tag{10}$$

where g_{t+1} is the rate of growth of aggregate (and average) income: $g_{t+1} \equiv \frac{Y_{t+1}}{Y_t} = \frac{\bar{y}_{t+1}}{\bar{y}_t}$. If (10) holds, equilibrium in the goods market is assured.[4]

Using (10), (9) becomes:

$$b_{it+1} = \frac{1-\gamma}{1+\mu}\frac{\bar{y}_{t+1}}{\bar{y}_t}(y_{it} + \mu\bar{y}_t) \tag{11}$$

As an example, let's assume that output y_{it} is the sum of an exogenous variable ω_i and bequest b_{it} (wealth of the young)

$$y_{it} = \omega_i + b_{it} \tag{12}$$

ω_i is non-inherited wealth. For the moment, it can be thought of as an exogenous endowment. Later on, we will specify it as income earned by workers and entrepreneurs.

The distribution of endowments across agents is the *primary* distribution, while the distribution of income is *secondary* , i.e. derived from the former by adding the bequest. As it will become clear in a moment, also the distribution of bequest is secondary, i.e. derived from the distribution of endowment.

Averaging (12) one gets

$$\bar{y}_t = \bar{\omega} + \bar{b}_t \tag{13}$$

where $\bar{\omega}$ is average endowment and $\bar{b}_t$ average wealth. We can carry on the dynamic analysis in terms of output or wealth. Substituting (12) into (11) we obtain the law of motion of output:

$$y_{it+1} = \omega_i + \frac{1-\gamma}{1+\mu}\frac{\bar{y}_{t+1}}{\bar{y}_t}[y_{it} + \mu\bar{y}_t] \tag{14}$$

[4] Aggregating the budget constraints, it turns out that the sum of aggregate consumption and aggregate bequest must be equal to the aggregate real money balances of the old, i.e. $\theta_{t+1}(Y_t + \mu N h_t) = C_{t+1} + B_{t+1}$ where $C_{t+1} \equiv \sum_i c_{it+1}, B_{t+1} \equiv \sum_i b_{it+1}$. But $h_t = \bar{y}_t$ and $Nh_t = N\bar{y}_t = Y_t$.Therefore: $\theta_{t+1}(1+\mu)Y_t = C_{t+1} + B_{t+1}$.If (10) holds true, then $Y_{t+1} = \theta_{t+1}(1+\mu)Y_t$. Substituting this expression into the previous one yields $Y_{t+1} = C_{t+1} + B_{t+1}$ which is the equilibrium condition on the goods market.

Averaging (14) we obtain

$$\overline{y}_{t+1} = \overline{\omega} + (1-\gamma)\,\bar{y}_{t+1}$$

which simplifies to

$$\overline{y}_{t+1} = \frac{\overline{\omega}}{\gamma}$$

which is constant over time. Therefore we can write

$$\overline{y} = \frac{\overline{\omega}}{\gamma} \tag{15}$$

The law of motion of wealth is:

$$b_{it+1} = \frac{1-\gamma}{1+\mu}\frac{\bar{\omega}+\bar{b}_{t+1}}{\bar{\omega}+\bar{b}_t}\left(\omega_i + b_{it} + \mu\bar{\omega} + \mu\bar{b}_t\right) \tag{16}$$

There is a *mean field effect* at work, here: individual wealth in t+1 depends not only on individual wealth in t but also on average wealth in t and t+1.The mean field effect captures non-strategic interaction between the individual agent and the rest of the population proxied by the average agent (see Aoki 1996 and the references therein).

Averaging (16) we get that average wealth is constant over time

$$\bar{b} = \frac{1-\gamma}{\gamma}\bar{\omega} \tag{17}$$

Finally it is easy to prove that average consumption is

$$\bar{c} = \bar{\omega} \tag{18}$$

3 A Representative Agent Economy

Let's pause now to consider the special case of a representative agent economy. If we adopt the representative agent assumption, (1) simplifies to

$$\frac{m_t}{P_t} = y_t \tag{19}$$

where m_t and y_t are the demand for money and output of the representative agent. The aggregate demand for money, therefore, is: $M_t \equiv Nm_t$.Assuming equilibrium on the money market in t, i.e. $M_t = H_t$,the equation of the individual transfer becomes

$$T_{it+1} = \frac{\mu H_t}{N} = \frac{\mu N m_t}{N} = \mu m_t$$

and (4) boils down to

$$M_{it} + T_{it+1} = m_t (1 + \mu) = P_{t+1} (c_{t+1} + b_{t+1}) \tag{20}$$

where c_{t+1} and b_{t+1} are consumption and bequest of the representative old agent. Dividing by P_{t+1} and substituting (19) into (20) we obtain the lifetime budget constraint of the representative agent:

$$\frac{M_{it}}{P_t} (1 + \mu) \frac{P_t}{P_{t+1}} = \theta_{t+1} (1 + \mu) y_t = c_{t+1} + b_{t+1} \tag{21}$$

Maximizing the Cobb-Douglas utility function (7) subject to (21) yields:

$$c_{t+1} = \gamma \theta_{t+1} (1 + \mu) y_t \tag{22}$$

$$b_{t+1} = (1 - \gamma) \theta_{t+1} (1 + \mu) y_t \tag{23}$$

We recall now that

$$\frac{1}{\theta_{t+1}} \equiv \frac{P_{t+1}}{P_t} \equiv 1 + \pi_{t+1} = \frac{1 + \mu}{1 + g_{t+1}}$$

where $1 + g_{t+1} = \frac{\bar{y}_{t+1}}{\bar{y}_t}$. Substituting this expression into (23) we obtain:

$$b_{it+1} = (1 - \gamma) \frac{\bar{y}_{t+1}}{\bar{y}_t} y_t \tag{24}$$

At this point, we are able to prove the following

Proposition 1. *Output, consumption and wealth (of the representative agent) are independent of money growth, i.e. money is superneutral.*

In the representative agent case, in fact, output and wealth are uniform across individuals and there is no difference between individual and average output or wealth, i.e. $y_t = \bar{y}_t$. As a consequence, (24) boils down to:

$$b_{t+1} = (1 - \gamma) y_{t+1} \tag{25}$$

Equation (25) implies that the ratio of wealth to income must be constant and equal to $1 - \gamma$.This must be true in each period:

$$\frac{b_{t+1}}{y_{t+1}} = \frac{b_t}{y_t} = 1 - \gamma$$

As an example, assume that

$$y_t = \omega + b_t \tag{26}$$

Substituting (26) into (25) we obtain:

$$b = \frac{1 - \gamma}{\gamma} \omega \tag{27}$$

Inflation does not affect the accumulation of wealth: in fact, the real rate of return of money θ_{t+1} – which is the reciprocal of the growth factor of the price level – does not show up in (27). Consumption and output are respectively:

$$c = \omega$$

$$y = \frac{1}{\gamma}\omega$$

Output is equal to a multiple of the endowment. In each period, the young sells to the old his entire output $y = \frac{1}{\gamma}\omega$ – which consists of endowment and wealth – at the price $P_t = \frac{\gamma}{\omega}M_t$. He does not consume and keeps its money balances M_t "idle" until the next period. The old agent buys output $\frac{1}{\gamma}\omega$, consumes the endowment ω and leaves a bequest $\frac{1-\gamma}{\gamma}\omega$ to his son equal to the bequest received from his father. Since output is constant, money is superneutral and the rate of change of money supply affects only inflation:

$$\pi = \mu$$

4 Heterogeneous Agents

In the case of heterogeneous agents, things are not that simple and certainly more interesting. We can summarize our results in

Proposition 2. *Money is superneutral on average but is not superneutral at the individual level. In fact the average levels of output, consumption and wealth are independent of money growth but the individual levels of the same variables are indeed affected by changes in the rate of growth of money. As a consequence, while the first moments of the distributions of output, consumption and wealth do not depend on money (i.e. the distributions are mean preserving), higher moments are influenced by the rate of money growth.*

In order to prove this proposition, let's go back to equations 15, 17 and 18[5].

Notice now that if average output is constant over time, the rate of growth of average output is *zero*, i.e. $\frac{\bar{y}_{t+1}}{\bar{y}_t} = 1$,and the inflation rate is constant and equal to the rate of growth of money: $\pi = \mu$. Therefore $\theta_{t+1} = \frac{1}{1+\mu}$, (14) simplifies to[6]:

$$y_{it+1} = \omega_i + \frac{1-\gamma}{1+\mu}\left(y_{it} + \mu\bar{y}\right) \tag{28}$$

[5] From the equilibrium condition on the goods markete one gets $\bar{c} = \bar{y} - \bar{b}$.These equations happen to coincide with the ones derived for the representative agent case (see the previous section) but in the heterogeneous agents case they refer to the average agent which does not coincide by construction with the representative agent.

[6] Remember that $\bar{y}_t = \bar{y}$, at any t.

and (16) boils down to:

$$b_{it+1} = \frac{1-\gamma}{1+\mu}\left(\omega_i + b_{it} + \mu\bar{\omega} + \mu\bar{b}\right) \tag{29}$$

which is a linear difference equation incorporating a *linear mean field effect.* The mean field effect was already present in (16) but now it is simpler: individual wealth in t+1 depends *linearly* on average wealth. Changes in average wealth play the role of a *positive macroeconomic externality* on individual wealth: the higher average wealth, the higher average output, the higher the money transfer in t+1[7] and the higher individual wealth in t+1, coeteris paribus.

The steady state of (29) is

$$b_i^* = \frac{1-\gamma}{\mu+\gamma}\left(\omega_i + \mu\bar{\omega} + \mu\bar{b}\right) \tag{30}$$

At this level of the analysis, the mean field effect is present also in the steady state: the individual steady state of wealth in fact depends *linearly* on (steady state) average wealth.

Averaging b_i^* from (30) and rearranging, however, one gets[8]

$$\bar{b}^* = \frac{1-\gamma}{\gamma}\bar{\omega}$$

which can be plugged into (29) to obtain

$$b_{it+1} = \frac{1-\gamma}{1+\mu}\left(\omega_i + b_{it} + \frac{\mu}{\gamma}\bar{\omega}\right) \tag{31}$$

This new law of motion makes individual wealth in t+1 *linearly* dependent on average endowment. The phase diagram of (31) is represented in figure 1.

The steady state of (31) is

$$b_i^* = \frac{1-\gamma}{\mu+\gamma}\left(\omega_i + \frac{\mu}{\gamma}\bar{\omega}\right) \tag{32}$$

Applying the same modelling strategy to output and consumption, we obtain the steady state values:

$$y_i^* = \frac{1}{\mu+\gamma}\left[\omega_i\left(1+\mu\right) + \left(1-\gamma\right)\frac{\mu}{\gamma}\bar{\omega}\right] \tag{33}$$

and

[7] In fact, according to (3) $T_{it+1} = \mu h_t$ but $\frac{h_t}{P_t} = \bar{y}_t = \overline{y} = \bar{\omega} + \bar{b}$, hence $\frac{T_{it+1}}{P_{t+1}} = \frac{P_t}{P_{t+1}}\left(\mu\bar{\omega} + \mu\bar{b}\right) = \frac{1}{1+\mu}\left(\mu\bar{\omega} + \mu\bar{b}\right)$. In words, the real value of the money transfer in t+1 for each individual is increasing with average wealth.

[8] Of course this expression is the same as (17).

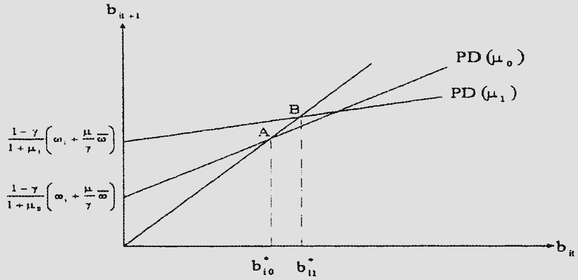

Fig. 1. phase diagram of individual wealth

$$c_i^* = \gamma\frac{1+\mu}{\mu+\gamma}\omega_i + \frac{1-\gamma}{\mu+\gamma}\mu\bar{\omega} \tag{34}$$

Comparing (17)(15)(18) with (32) (33) and (34) it is clear that in the steady state individual wealth, output and consumption are indeed affected by the rate of growth of money (which coincides with inflation) and money is not superneutral while *on average* consumption, output and wealth are independent of money growth, i.e. money is superneutral.

Let's pause now to characterize the relationship between money growth and the steady state of the i-th agent's wealth (output and consumption). In order to do so, notice first of all that an increase in the rate of money growth makes the slope of the phase diagram smaller but it may lead to an increase of the intercept with contrasting effects on steady state wealth.

In order to understand the nature and consequences of these effects, it is necessary to interpret (31) as follows $b_{it+1} = (1-\gamma)\times RMB_i$, where $RMB_i = \frac{M_{it}+T_{it+1}}{P_{t+1}} = \theta_{t+1}\frac{M_{it}+T_{it+1}}{P_t} = \theta_{t+1}\left(y_{it}+\mu\bar{y}\right)$ are the real money balances of the old. Recalling that $\theta_{t+1} = \frac{1}{1+\mu}$, $y_{it} = \omega_i + b_{it}$ and $\bar{y} = \frac{\bar{\omega}}{\gamma}$, RMB_i can be written as follows:

$$RMB_i = \frac{1}{1+\mu}\left(\omega_i + b_{it} + \frac{\mu}{\gamma}\bar{\omega}\right) \tag{35}$$

RMB_i is the product of the real return on money $\theta_{t+1} = \frac{1}{1+\mu}$ times the value of individual money balances in t+1 deflated by P_t (see the expression in brackets) which in turn is the sum of the real money balances carried over from the previous period: $\frac{M_{it}}{P_t} = y_{it} = \omega_i + b_{it}$ and of the money transfer received in t+1 deflated by P_t: $\frac{T_{it+1}}{P_t} = \mu\frac{h_t}{P_t} = \mu\bar{y} = \frac{\mu}{\gamma}\bar{\omega}$.

An increase in money growth has contrasting effects on money holdings in real terms. It

- boosts inflation which reduces the real rate of return on money $\theta_{t+1} = \frac{1}{1+\mu}$ and real money balances, coeteris paribus: this is the *inflation tax effect* of money growth on real money balances;
- implies an increase in the money transfer to the old $\frac{T_{it+1}}{P_t} = \frac{\mu}{\gamma}\bar{\omega}$: this is the *money transfer effect.*

Which effect is prevailing? In order to answer this question, we must compute the derivative of RMB_i with respect to money growth from (35):

$$\frac{\partial RMB_i}{\partial \mu} = \frac{\partial \theta_{t+1}}{\partial \mu} \times \left(\omega_i + b_{it} + \frac{\mu}{\gamma}\bar{\omega}\right) + \theta_{t+1} \frac{\partial \left(\omega_i + b_{it} + \frac{\mu}{\gamma}\bar{\omega}\right)}{\partial \mu} =$$
$$= -\frac{\omega_i + b_{it} + \frac{\mu}{\gamma}\bar{\omega}}{(1+\mu)^2} + \frac{1}{1+\mu}\frac{\bar{\omega}}{\gamma}$$

The first (negative) term captures the inflation tax effect, while the second (positive) term reflects the money transfer effect. After some algebraic manipulation we end up with

$$\frac{\partial RMB_i}{\partial \mu} = -\frac{1}{(1+\mu)^2}\left(\omega_i + b_{it} - \frac{\bar{\omega}}{\gamma}\right) = -\frac{1}{(1+\mu)^2}\left(y_{it} - \bar{y}\right) \qquad (36)$$

According to (36) if the agent has relatively little output $(y_{it} < \bar{y})$, an increase in money growth brings about higher real money balances, i.e. the money transfer effect prevails over the inflation tax effect. If the opposite is true $(y_{it} > \bar{y})$, an increase in money growth yields lower real money balances, i.e. the inflation tax effect prevails over the money transfer effect.

Notice that when $b_{it} = 0$, following an increase in money growth, the money transfer effect prevails if $\omega_i < \frac{\bar{\omega}}{\gamma}$, while the inflation tax effect prevails if $\omega_i > \frac{\bar{\omega}}{\gamma}$.

Let's go back to the phase diagram of 31. As we have already acknowledged, an increase of μ makes the slope $\frac{1-\gamma}{1+\mu}$ smaller, due to the inflation tax effect. As to the intercept $\frac{1-\gamma}{1+\mu}\left(\omega_i + \frac{\mu}{\gamma}\bar{\omega}\right)$, it is easy to realize that an increase of μ makes the intercept greater if $\omega_i < \frac{\bar{\omega}}{\gamma}$ while it leads to a lower intercept if $\omega_i > \frac{\bar{\omega}}{\gamma}$[9]. In other words, the intercept increases if the money transfer effect prevails over the inflation tax effect, it decreases if the opposite is true.

As to the effect of money growth on the steady state (32), it is clear that if the agent can dispose of a relatively big output $\left(\omega_i > \frac{\bar{\omega}}{\gamma}\right)$, an increase of μ yields a smaller steady state, due to the joint effects of a smaller slope and a smaller intercept of the phase diagram. If the agent can dispose of relatively little output $\left(\omega_i < \frac{\bar{\omega}}{\gamma}\right)$, an increase of μ may yield either a greater or a smaller

[9] In fact $\frac{\partial}{\partial \mu}\left(\frac{1-\gamma}{1+\mu}\left(\omega_i + \frac{\mu}{\gamma}\bar{\omega}\right)\right) = -\frac{1-\gamma}{(1+\mu)^2}\left(\omega_i\gamma - \bar{\omega}\right)$

steady state, due to the contrasting effects of a smaller slope and a greater intercept of the phase diagram. In figure 1, we have represented the former case: when the rate of money growth is μ_0, steady state wealth of the i-th agent is b_{i0}^* (point A). An increase of the rate of money growth to μ_1 makes the phase diagram shift upward and rotate so that steady state wealth goes up to b_{i1}^* (point B).

So much for the single agent. Let's now look at the effects of money growth on the distribution of wealth. In figure 2 we have represented equation (32) on the (b_i^*, ω_i) plane.

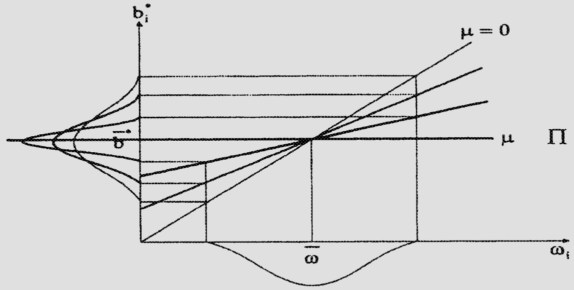

Fig. 2. the effects of μ on the distribution of wealth

Both the intercept $\left(\frac{1-\gamma}{\mu+\gamma}\frac{\mu}{\gamma}\bar{\omega}\right)$ and the slope $\left(\frac{1-\gamma}{\mu+\gamma}\right)$ of the *wealth-endowment line* are positive but the former is increasing while the latter is decreasing with respect to μ. If money were constant ($\mu = 0$), steady state wealth would be represented by the solid line of equation $b_i^* = \frac{1-\gamma}{\gamma}\omega_i$. By increasing μ the line shifts up and rotates around point A, the *average agent point* of coordinates $\left(\bar{\omega}, \bar{b}^*\right)$(see for instance the dotted line). If $\mu \to \infty$ steady state wealth would be represented by the solid bold line of equation $b_i^* = \frac{1-\gamma}{\gamma}\bar{\omega} = \bar{b}^*$.

In other words, by increasing μ, agents who are relatively poor in endowment (i.e. characterized by $\omega_i < \bar{\omega}$) become wealthier (i.e. their b_i^* increases) while relatively rich agents (characterized by $\omega_i > \bar{\omega}$) become less wealthy. However, the relative ranking is not reversed: agents who are relatively poor in endowment $(\omega_i < \bar{\omega})$ remain less wealthy than the average $\left(b_i^* < \bar{b}^*\right)$and tend to the average only asymptotically.

The primary distribution of ω_i can be represented ideally on the x-axis. Let the support of ω_i be $(\min \omega_i, \max \omega_i)$. From the (primary) distribution of the endowment, for each μ the wealth-endowment line generates the (secondary) distribution of wealth, which can be represented ideally on the y-axis. In other words, the wealth-endowment line maps the distribution of the endowment into the distribution of wealth. On the same diagram, we can represent also the secondary distribution of output: the intercept of the straight line of equation $b_i^* = y_i^* - \omega_i$ passing through each point of coordinates (b_i^*, ω_i) belonging to the wealth-endowment line is the steady state individual output. The vertical distance between y_i^* and b_i^* measures consumption c_i^*.

While the primary distribution is independent of money growth (by construction), the secondary distribution of wealth is affected by μ[10]. It is easy to see that increasing μ, the support of the distribution of wealth shrinks and so does the variance. The widest range of $b_i^*$is associated to the constant money scenario $(\min b_i^* \mid \mu = 0, \max b_i^* \mid \mu = 0)$. In the limit, as $\mu \to \infty$ the support of the distribution collapses to a point (corresponding to average wealth) and the variance of the distribution tends to zero[11].

The first moment of the distribution of endowment and wealth (the coordinates of the average agent point) are independent of money growth, while the variance of the secondary distribution is decreasing with money growth. In fact, computing the variance of steady state wealth from (32) one gets:

$$V\left(b_i^*\right) = \left(\frac{1-\gamma}{\mu+\gamma}\right)^2 V\left(\omega_i\right)$$

The variance of wealth falls in the range[12]:

$\min V\left(b_i^*\right) = V\left(b_i^*\right) \mid \mu \to \infty = 0$

$\max V\left(b_i^*\right) = V\left(b_i^*\right) \mid \mu = 0 = \left(\frac{1-\gamma}{\gamma}\right)^2 V\left(\omega_i\right)$

As to output, the variance of output is:

$$V\left(y_i^*\right) = \left(\frac{1+\mu}{\mu+\gamma}\right)^2 V\left(\omega_i\right)$$

and falls in the range:

$\min V\left(y_i^*\right) = V\left(y_i^*\right) \mid \mu \to \infty = V\left(\omega_i\right)$

$\max V\left(y_i^*\right) = V\left(y_i^*\right) \mid \mu = 0 = \frac{1}{\gamma^2} V\left(\omega_i\right)$

Finally the variance of consumption is:

$$V\left(c_i^*\right) = \left(\gamma\frac{1+\mu}{\mu+\gamma}\right)^2 V\left(\omega_i\right)$$

and falls in the range:

[10] The same is true for the distributions of output and consumption, as it is clear from (33) and (34).

[11] It is easy to see that in this case one obtains the the most narrow range also of output and consumption.

[12] Notice that b_i and ω_i are positively correlated and the correlation is perfect (thanks to (32)). Therefore $V(y_i^*) = V(\omega_i) + V(b_i) + 2Cov(\omega_i, b_i)$ and $Cov(\omega_i, b_i) = \sqrt{V(\omega_i)V(b_i)}$ But $V(b_i) = c^2 V(\omega_i)$ where $c = \frac{1-\gamma}{\mu+\gamma}$. Therefore $Cov(\omega_i, b_i) = \sqrt{V(\omega_i)V(b_i)} = cV(\omega_i)$. As a consequence $V(y_i) = V(\omega_i) + c^2 V(\omega_i) + 2cV(\omega_i) = V(\omega_i)\left(1+c\right)^2$.

$\min V(c_i^*) = V(c_i^*) \mid \mu \to \infty = \gamma^2 V(\omega_i)$
$\max V(c_i^*) = V(c_i^*) \mid \mu = 0 = V(\omega_i)$

If the variance is thought of as a rough measure of inequality, then *inequality is decreasing with money growth.*

Another insightful way to look at the effect of money growth on wealth distribution exploits the relation between steady state wealth and money growth. In figure 3 we have represented equation (32) on the (b_i^*, μ) plane.

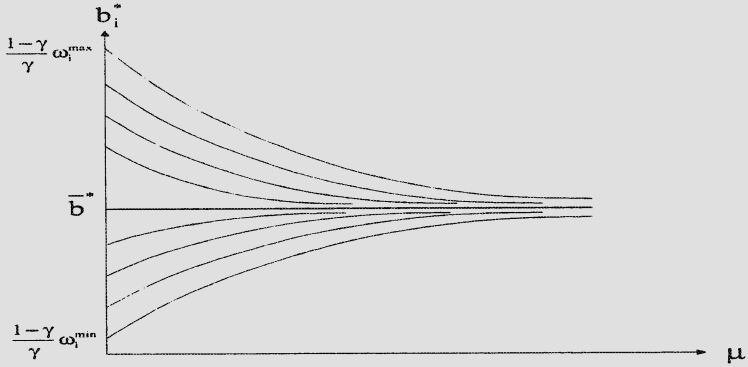

Fig. 3. *wealth money* curve

The shape and location of the *wealth-money curve* depends on individual endowment. In fact, taking the derivative of b_i^* with respect to μ one gets:

$$\frac{\partial b_i^*}{\partial \mu} = -\frac{1-\gamma}{(\mu+\gamma)^2}(\omega_i - \bar{\omega})$$

If the i-th agent is relatively poor $(\omega_i < \bar{\omega})$, the wealth-money curve is an increasing concave function of money growth, asymptotically tending to $b_{i\infty}^* = \bar{b}^*$. Symmetrically, if the agent is relatively rich $(\omega_i > \bar{\omega})$, the wealth-money curve is a decreasing convex function of money growth, asymptotically tending to $b_{i\infty}^* = \bar{b}^*$. In words: a relatively poor (rich) individual becomes less and less poor (rich) – i.e. his wealth keeps increasing (decreasing) if money growth becomes higher and higher,but he remains relatively poor (rich) even if money grows at an infinite rate[13].

In figure 3 we have represented different wealth-money curves associated to different endowments. One can get a picture of the distribution of wealth by sectioning the sheaf of curves at a given growth rate of money. It is clear that increasing μ the first moment of the distribution remains constant but the

[13] By an entirely similar argument, it is easy to conclude that the same is true for output and consumption: if the agent is relatively poor (rich), an increase in money growth increases (decreases) both output and consumption.

second moment goes down. The limiting distribution (associated to $\mu \to \infty$) is characterized by minimum variance (as a matter of fact, the variance tends to zero).

The transition from A to B in figure 1, which leads to a higher steady state wealth as a consequence of an increase in money growth, requires not only that the money transfer effect prevails over the inflation tax effect – which occurs if $\omega_i < \frac{\bar{\omega}}{\gamma}$) – with a corresponding increase of the intercept of the phase diagram, but also that the increase of the intercept more than offset the decrease of the slope of the phase diagram, – which occurs if $\omega_i < \bar{\omega}$).

5 Occupational Choice, Bequests and the Origin of Heterogeneity

In what follows we show the easiest way in order to micro-found income heterogeneity without affecting the results we have derived above. More precisely we consider income heterogeneity as the result of an occupational choice made by heterogeneous agents.

Since their birth, agents are endowed with a level of individual ability/efficiency e_i. When young, they make an occupational choice, which consists in being a worker of the skilled or unskilled type. The young population, therefore, consists of skilled and unskilled workers. Both types of workers supply workhours to the "production sector" which produces output and sells it to the old of the previous generation against money. This generates the income of the young, i.e. the wage, which plays the role of the endowment in previous section. The wage of the unskilled worker is fixed at w. The wage of the skilled worker is proportional to his ability. In order to simplify the argument and without loss of generality we assume that it is equal to his ability. Therefore, unskilled workers have the same wage, skilled workers' wage is differentiated in order to recognize different abilities. The old receive also the money transfer. Finally, as in the previous case, the young receives also a bequest b_{it}. Real output therefore is $y_{it}^u = w + b_{it}$ and $y_{it}^s = e_i + b_{it}$ for the unskilled and the skilled worker respectively.

Preferences are as before: $U = (c_{it+1})^{\gamma} (b_{it+1})^{1-\gamma}$. Therefore the young do not consume. They exchange their output (income and bequest) for money: $P_t y_{it} = M_{it}$. The old receive money transfers ($T_{it+1} = \mu h_t$) and spend their money to consume and leave a bequest. LBC is (see section 2) : $\theta_{t+1}\left(y_{it} + \mu\frac{h_t}{P_t}\right) = c_{it+1} + b_{_{it+1}}$where $\theta_{t+1}\left(y_{it} + \mu\frac{h_t}{P_t}\right)$ are real money balances of the old.

Since, given the preferences, indirect utility is

$$U = \left(\gamma\theta_{t+1}\left(y_{it} + \mu\frac{h_t}{P_t}\right)\right)^{\gamma}\left[(1-\gamma)\,\theta_{t+1}\left(y_{it} + \mu\frac{h_t}{P_t}\right)\right]^{1-\gamma}$$

or

$$U = (\gamma)^{\gamma} (1-\gamma)^{1-\gamma} \theta_{t+1} \left(y_{it} + \mu \frac{h_t}{P_t} \right)$$

and the real rate of return on money and money transfers are uniform across the population, the occupational choice depends on the relative magnitude of income obtained when young as skilled or unskilled worker.

The individual becomes skilled worker if $e_{it} + b_{it} \geq w + b_{it}$ or $e_{it} \geq w \equiv \hat{e}$, i.e. if his ability is high enough to yield a wage as skilled worker higher than the wage of the unskilled worker. $\hat{e}$ is the minimum efficiency a worker must have in order to work as skilled and get a skilled worker wage equal to his efficiency. It turns out, quite simply, that the minimum efficiency of the skilled is equal to wage of the unskilled workers. Let's assume that efficiency is distributed as a uniform random variable with support (0,1). It is clear that $w \equiv \hat{e}$ is also the share of unskilled workers in the population[14]. This share is constant and independent of money growth.

The income of the skilled worker falls in the interval $(\hat{e}, 1)$ where 1 is the maximum efficiency of the skilled (by the assumption above). But $w \equiv \hat{e}$ so that average income of the skilled is

$$\bar{e}^s = \frac{1+\hat{e}}{2} = \frac{1+w}{2} \tag{37}$$

Therefore, average income is

$$\bar{\omega} = w\hat{e} + \bar{e}^s (1-\hat{e}) = \frac{w^2+1}{2} \tag{38}$$

Let's define now the laws of motion. Recalling, as shown above, that $\frac{h_t}{P_t} = \bar{y}_t = \bar{y}$ where $\bar{y}$ is average output of the whole economy, and $\bar{y} = \bar{\omega} + \bar{b}$ where $\bar{\omega}$ is average income and $\bar{b}$ average wealth, each and every unskilled worker has the following law of motion of wealth:

$$b^u_{t+1} = \frac{1-\gamma}{1+\mu} \left(w + b^u_t + \mu\bar{\omega} + \mu\bar{b} \right) \tag{39}$$

The steady state of (39) is

$$b^u = \frac{1-\gamma}{\mu+\gamma} \left(w + \mu\bar{\omega} + \mu\bar{b} \right) \tag{40}$$

The skilled worker of efficiency $e^s_i > w$ has the following law of motion of wealth:

$$b^s_{it+1} = \frac{1-\gamma}{1+\mu} \left(e^s_i + b^s_{it} + \mu\bar{\omega} + \mu\bar{b} \right) \tag{41}$$

whose steady state is

[14] By assumption therefore $w < 1$.

$$b_i^s = \frac{1-\gamma}{\mu+\gamma}\left(e_i^s + \mu\bar{\omega} + \mu\bar{b}\right) \tag{42}$$

Averaging (41) we obtain the law of motion of the average wealth of the skilled

$$\bar{b}_{t+1}^s = \frac{1-\gamma}{1+\mu}\left(\bar{e}^s + \bar{b}_t^s + \mu\bar{\omega} + \mu\bar{b}\right) \tag{43}$$

whose steady state is:

$$\bar{b}^s = \frac{1-\gamma}{\mu+\gamma}\left(\bar{e}^s + \mu\bar{\omega} + \mu\bar{b}\right) \tag{44}$$

Plugging (17) into (39) and (41) we get:

$$b_{t+1}^u = \frac{1-\gamma}{1+\mu}\left(w + b_t^u + \frac{\mu}{\gamma}\bar{\omega}\right) \tag{45}$$

and

$$b_{it+1}^s = \frac{1-\gamma}{1+\mu}\left(e_i^s + b_{it}^s + \frac{\mu}{\gamma}\bar{\omega}\right) \tag{46}$$

and the steady states are

$$b^{u*} = \frac{1-\gamma}{\mu+\gamma}\left(w + \frac{\mu}{\gamma}\bar{\omega}\right) \tag{47}$$

$$b_i^{s*} = \frac{1-\gamma}{\mu+\gamma}\left(e_i^s + \frac{\mu}{\gamma}\bar{\omega}\right) \tag{48}$$

$$\bar{b}^{s*} = \frac{1-\gamma}{\mu+\gamma}\left(\bar{e}^s + \frac{\mu}{\gamma}\bar{\omega}\right) \tag{49}$$

As to the impact of money growth on bequest, from the previous section we know that an increase in the rate of money growth is beneficial for (because it increases the wealth of) the relatively poor ($\omega_i < \bar{\omega}$) and detrimental for the relatively wealthy ($\omega_i > \bar{\omega}$).

The average income of the economy (38) is a quadratic function of w. It is clear that the wage of the unskilled is always lower than average income: therefore an increase in money growth always boosts their wealth. For each w (for instance w_0) the range of the wages of the skilled workers is the distance between w and 1. There are some skilled workers – those with an efficiency which falls in the range $\bar{\omega} > e_i^s > w$ – who gain from an acceleration in monetary expansion, others – those with an efficiency which falls in the range $1 > e_i^s > \bar{\omega}$ – who loose. On average, however, money growth does not affect wealth (output and consumption). If we think of this model in terms of workers *versus* entrepreneurs, the result we have just showed confirms the stylized fact that small entrepreneurs bear the brunt of a deceleration of monetary growth. On average, however, entrepreneurs gain from a deceleration of money growth. This is the simplest illustration of Proposition 4.1 in a context where we explicitly consider the occupational choice: in the steady state, money is superneutral on average but is not superneutral at the individual level. As a consequence, while the first moments of the distributions of output, consumption and wealth do not depend on money, higher moments are influenced by the rate of money growth.

6 Conclusions

In this paper we have developed a model that investigates the distributional effects of monetary policy. This topic has been neglected in macroeconomic theory, that generally focuses on the aggregate consequences of monetary policy. In the model we have developed it is income and wealth distribution themselves to make monetary policy non-superneutral at the individual level. Actually, if agents are homogeneous, that is in a representative agent economy, monetary policy will be superneutral: output, consumption and wealth (of the representative agent) will be independent of money growth. On the contrary if agents are heterogeneous individual wealth will depend also on the average wealth, that is there is a mean field effect at work that captures non-strategic interaction between the agent and the population, proxied by the average agent, and makes money non-superneutral at the individual level. In other words, in the heterogenous agents economy the average levels of output, consumption and wealth are independent of money growth but the individual levels of the same variables are indeed affected by changes in the rate of growth of money. By increasing the rate of money growth, agents who are relatively poor in income/endowment become wealthier whereas relatively rich agents become less wealthy. In other words while the first moments of the distributions of output, consumption and wealth do not depend on money (i.e. the distributions are mean preserving), higher moments are influenced by the rate of money growth. Therefore, if the variance is thought of as a rough measure of inequality, then inequality is decreasing with money growth.

The easiest way to micro-found income heterogeneity is to think of it as the result of an occupational choice made by agents when young. If we think of this model in terms of workers *versus* entrepreneurs, the results we have derived confirm the stylized fact that small entrepreneurs bear the brunt of a deceleration of monetary growth.

A straightforward variant of the environment described above focuses on the choice faced by agents of becoming entrepreneurs or workers. In order to become entrepreneur, an agent must carry on a fixed investment. Capital markets are imperfect. In this context, therefore, an entrepreneur must be not only relatively efficient but also relatively wealthy. A relatively efficient agent who cannot afford incurring the fixed cost of the investment project falls behind in the social ladder and is lumped together with the inefficient agents in the working class. A reasonable conjecture in this context is that when capital markets are imperfect, monetary policy may be non-superneutral both at the individual and at the aggregate level.

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Limit Pricing and Entry Dynamics with Heterogeneous Firms

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Summary. In this paper we study a dynamic model of pricing and investment with heterogeneous firms under imperfect competition. We assume the existence of two types of firms, a dominant firm and fringe firms. We introduce several asymmetries between the two types of firms. The dominant firm is not financially constrained. It has free access to capital markets although it is subject to increasing adjustment cost of investment. On the other hand, the fringe firms are credit constrained and have no access to capital markets so that they are restricted to internal finance of investment. Furthermore, it is assumed that the dominant firm acts as a price setter and it controls both prices and investment while the fringe firms are price takers who can control only their own investment through internal retention. The formal structure of our model is described by a dynamic game. More particularly, it represents an open loop Stackelberg differential game of two firms in which the dominant firm acts as a leader and fringe firms act passively as followers. We investigate both the steady state as well as the dynamic behavior of the model analytically and numerically.

1 Introduction

Recently, a large number of papers have studied the problem of pricing of firms when there is a threat of entry by other firms. In the literature this pricing behavior has been called limit pricing. In this paper, following Gaskins (1971) and Judd and Petersen (1986) we study an industry where there are two types of firms. In the industry there exist dominant firms which we call for short the dominant firm, and fringe firms, henceforth called fringe firm. We study a dynamic model of pricing, investment and firm valuation for those heterogeneous types of firms under imperfect competition.

We introduce several asymmetries between the two types of firms. First, the dominant firm is not financially constrained. It has free access to capital markets although it is subject to increasing adjustment cost of investment. On the other hand, the fringe firm is credit constrained and has no access to capital markets so that it is restricted to internal finance of investment.

M. Gallegati et al. (eds.), *The Complex Dynamics of Economic Interaction*

Second, it is assumed that the dominant firm acts as a price setter and it controls both prices and investment while the fringe firm is a price taker that can control only its own investment through internal retention. As in Judd and Petersen (1986), in our model entry is expressed as the growth of the investment of the fringe firm but unlike Judd and Petersen (1986), in our model the trend growth of demand is determined endogenously. Third, the formal structure of our model is described by a dynamic game. More particularly, it represents an open loop Stackelberg differential game of two firms in which the dominant firm acts as a leader and fringe firms act passively as followers.

The paper is organized as follows. Section 2 sets up the Model, Section 3 solves the model by using Pontryagin's maximum principle. Section 4 studies the implied out-of-steady dynamics and section 5 provides some simulations. Section 6 concludes the paper.

2 The Model

Let us define the present value of the dominant firm as follows

$$\begin{aligned} W &= \int_0^\infty \{(P_t - c)(D_t - x_t) - \varphi(g_t)K_t\}e^{-\rho t}dt \qquad (1) \\ &= \int_0^\infty \{(P_t - c)(E_t - y_t) - \varphi(g_t)\}K_t e^{-\rho t}dt \end{aligned}$$

where P_t, price of the goods, c, average cost of the dominant firm which is assumed to be fixed ($c > 0$), D_t, real market demand of the goods, x_t, real output of the fringe firm, K_t, real capital stock of the dominant firm, g_t, growth rate of capital stock of the dominant firm ($g_t = \dot{K}_t/K_t$), ρ, discount rate of the dominant firm which is assumed to be fixed ($\rho > 0$). Moreover, we presume $\varphi'(0) = 1$, $\varphi''(g_t) > 0$ and denote $E_t = D_t/K_t$, and $y_t = x_t/K_t$,. We assume that the price of the capital good (P_k) is constant, which we normalize so that $P_k = 1$.

We specify the demand function and adjustment cost function as follows.

$$\begin{gathered} D_t = A_t(1 - aP_t); \qquad (2) \\ A_t > 0,\ a > 0,\ 0 \leq P_t \leq 1/a \end{gathered}$$

$$\varphi(g_t) = g_t + \alpha g_t^2;\ \alpha > 0 \qquad (3)$$

Eq. (2) represents a linear downward-sloping demand function, and A_t denotes the 'scale' of the market. We assume that

$$A_t = BK_t; \; B > 0 \tag{4}$$

We presume that only the investment by the dominant firm contribute to the expansion of the market.[5] Substituting eq. (4) into eq. (2), we have

$$\begin{aligned} E_t = D_t/K_t = B(1 - aP_t); \\ B > 0, \; a > 0, \; 0 \leq P_t \leq 1/a. \end{aligned} \tag{5}$$

We suppose that the dominant firm is not financially constrained. It freely can access capital market, but it is subject to increasing adjustment cost, see Uzawa (1969). Eq. (33) is the standard type of adjustment cost function. Substituting eqs. (3) and (5) into eq. (1), we obtain

$$\begin{aligned} W &= \int_0^\infty [(P_t - c)\{B(1 - aP_t) - y_t\} - g_t - \alpha g_t^2]K_t e^{-\rho t} dt \\ &\equiv \int_0^\infty f(P_t, g_t; \; y_t) K_t e^{-\rho t} dt \end{aligned} \tag{6}$$

where

$$f(P_t, g_t; y_t) \equiv (P_t - c)\{B(1 - aP_t) - y_t\} - g_t - \alpha g_t^2. \tag{7}$$

Next, let us consider the behavior of the fringe firm. We assume that the fringe firm acts as a price taker. It produces output up to full capacity. In this case, we have

$$\dot{x}_t = \dot{K}_{ft} m; \; m > 0 \tag{8}$$

where K_{ft} is the capital stock of the fringe firm, and m is the output-capital ratio of the fringe firm in case of full capacity utilization, which is assumed to be constant, see Judd and Petersen (1986). Following Judd and Petersen (1986), we assume that the fringe firm has not free access to capital markets so that the source of finance of the investment of the fringe firm is restricted to its internal finance. Obviously, this is the most strict form of a financial constraint. In this case, we have

$$\dot{K}_{ft} = \overline{s_f} \pi_{ft} = \overline{s_f}(P_t - c_f)x_t, \tag{9}$$

where $\overline{s_f}$ is the rate of internal retention of the fringe firm (it is assumed that it is fixed at the level $0 < \overline{s_f} < 1$), c_f is the average cost of the fringe firm. We presume that c_f is constant and $c_f \geq c$, and $\pi_{ft} = (P_t - c_f)x_t$ is the profit of the fringe firm. Note that it is assumed that $P_k = 1$, where P_k is the price of

[5] As in Asada and Semmler (1995) we assume that this is achieved through sales strategies, such as advertisement expenditure and built up of customer stock.

the capital good. The expression (9) is meaningful only if $P_t > c_f$ which we will assume henceforth. Substituting eq. (9) into eq. (8), we have

$$\dot{x}_t = \overline{s_f}(P_t - c_f)mx_t. \tag{10}$$

Therefore, we obtain

$$\frac{\dot{y}_t}{y_t} = \frac{\dot{x}_t}{x_t} - \frac{\dot{K}_t}{K_t} = \overline{s_f}(P_t - c_f)m - g_t \tag{11}$$

so that it follows

$$\dot{y}_t = \{\overline{s_f}(P_t - c_f)m - g_t\}y_t. \tag{12}$$

Although it is assumed that the fringe firm acts as a price taker, s_f must be the control variable of the fringe firm. However, in the appendix in Asada and Semmler (2002), we showed that the optimal policy of the fringe firm coincides with the policy which maximizes the growth rate $\dot{x}_t/x_t$ so that the optimal retention rate is fixed at the institutionally given upper bound (i.e., the corner solution) under certain assumptions. Because of this reason, we treat $\overline{s_f}$ as a constant through time.

As Judd and Petersen (1986), in our model, "entry" is expressed as the growth of the investment of the fringe firm.

To sum up, the optimal problem of the dominant firm is

$$\max_{P_t, g_t} \int_0^\infty f(P_t.g_t; y_t)K_t e^{-\rho t} dt \tag{13}$$

subject to

$$\dot{K}_t = g_t K_t, \quad K_0 > 0 \tag{14}$$

$$\dot{y}_t = \{\overline{s_f}(P_t - c_f)m - g_t\}y_t, \quad y_0 > 0. \tag{15}$$

At a first glance, this is a dynamic optimization problem of the single agent, the dominant firm. However, we can interpret this model as an open-loop Stackelberg differential game between the dominant and fringe firms. In the appendix in Asada and Semmler (2002), we showed that the optimal policy of the fringe firm which acts as a follower is to keep $s_f = \overline{s_f}$ for all $t \geq 0$ under some conditions. In this case, the solution of the above problem becomes the solution of the open-loop Stackelberg differential game, in which the dominant firm acts as a leader and the fringe firm as a follower.

3 The Solution of the Model

We can solve the above problem (13)-(15) by using Pontryagin's maximum principle[6].

The current value Hamiltonain of the above problem (H_t) reads as follows.

$$\begin{aligned} H_t &\equiv f(P_t, g_t;\ y_t)K_t + \lambda_t g_t K_t \\ &+ \mu_t\{\overline{s_f}(P_t - c_f)m - g_t\}y_t \\ &= [(P_t - c)\{B(1 - aP_t) - y_t\} - g_t - \alpha g_t^2]K_t \\ &+ \lambda_t g_t K_t + \mu_t\{\overline{s_f}(P_t - c_f)m - g_t\}y_t. \end{aligned} \tag{16}$$

where λ_t and μ_t are two costate variables which correspond to two dynamic constraints eq. (14) and eq. (15). A set of the optimal conditions for the dominant firm reads as follows.

$$\max_{P_t, g_t} H_t \quad \text{for all} \quad t \geq 0. \tag{17}$$

$$\dot{\lambda}_t = -\frac{\partial H_t}{\partial K_t} + \rho\lambda_t \quad \text{for all} \quad t \geq 0. \tag{18}$$

$$\dot{\mu}_t = -\frac{\partial H_t}{\partial y_t} + \rho\mu_t \quad \text{for all} \quad t \geq 0. \tag{19}$$

$$\lim_{t\to\infty} \lambda_t e^{-\rho t} = 0 \tag{20}$$

$$\lim_{t\to\infty} \mu_t e^{-\rho t} = 0 \tag{21}$$

The first order condition for the maximization of H_t becomes as follows.[7]

$$[-2aBP_t - y_t + (1 + ac)B]K_t + \mu_t\overline{s_f}my_t = 0 \tag{22}$$

$$[-[1 + 2\alpha g_t) + \lambda_t]K_t - \mu_t y_t = 0 \tag{23}$$

Solving eq.(22) with respect to P_t, we obtain

$$\begin{aligned} P_t &= \frac{(-K_t + \mu_t\overline{s_f}\ m)y_t + (1 + ac)BK_t}{2aBK_t} \\ &= \frac{(-1 + \kappa_t\overline{s_f}\ m)y_t + (1 + ac)B}{2aB} \\ &= \frac{1}{2aB}(-1 + \kappa_t\overline{s_f}\ m)y_t + \frac{1}{2a} + \frac{c}{2} \end{aligned} \tag{24}$$

[6] Chiang (1992)

[7] We can easily see, that the second order conditions are in fact satisfied.

where $\kappa_t \equiv \mu_t / K_t$.
Solving eq. (23) with respect to λ_t, we have

$$\lambda_t = 1 + 2\alpha g_t + \kappa_t y_t. \tag{25}$$

On the other hand, eq. (18) and eq. (19) become

$$\begin{aligned}\dot{\lambda}_t = (P_t - c)\{B(-1 + aP_t) + y_t \\ + g_t + \alpha g_t^2\} + (\rho - g_t)\lambda_t\end{aligned} \tag{26}$$

$$\dot{\mu}_t = (P_t - c)K_t + \{\rho + g_t - \overline{s}_f(P_t - c_f)m\}\mu_t \tag{27}$$

¿From eq. (27) we have

$$\begin{aligned}\frac{\dot{\mu}_t}{\mu_t} &= (P_t - c)\frac{K_t}{\mu_t} + \rho + g_t - \overline{s}_f(P_t - c_f)m \\ &= \frac{P_t - c}{\kappa_t} + \rho + g_t - \overline{s}_f(P_t - c_f)m\end{aligned} \tag{28}$$

so that we obtain

$$\begin{aligned}\frac{\dot{\kappa}_t}{\kappa_t} = \frac{\dot{\mu}_t}{\mu_t} - \frac{\dot{K}_t}{K_t} &= \frac{\dot{\mu}_t}{\mu_t} - g_t \\ &= \frac{P_t - c}{\kappa_t} + \rho - \overline{s}_f(P_t - c_f)m\end{aligned} \tag{29}$$

Eq. (29) implies that

$$\dot{\kappa}_t = P_t - c + \{\rho - \overline{s}_f(P_t - c_f)m\}\kappa_t \tag{30}$$

Next, differentiating eq. (25) with respect to time, we have

$$\dot{\lambda}_t = 2\alpha\dot{g}_t + \kappa_t\dot{y}_t + y_t\dot{\kappa}_t \tag{31}$$

Substituting equations (25) and (31) into eq. (26), we obtain

$$\begin{aligned}g_t = \frac{1}{2\alpha}[(P_t - c)\{B((-1 + aP_t) + y_t + g_t + \alpha g_t^2\} \\ +(\rho - g_t)(1 + 2\alpha g_t + \kappa_t y_t) - \kappa_t\dot{y}_t - y_t\dot{\kappa}_t]\end{aligned} \tag{32}$$

Substituting eq. (24) into equations (15), (30) and (32), we arrive at the following nonlinear three-dimensional dynamic system

(i)

$$\dot{y}_t = [\bar{s}_f \Big\{ \frac{1}{2aB}(-1 + \kappa_t \bar{s}_f\ m) y_t + \frac{1}{2a} + \frac{c}{2} - c_f \Big\} m - g_t] y_t$$
$$\equiv F_1(y_t, \kappa_t, g_t)$$

(ii)

$$\dot{\kappa}_t = \frac{1}{2aB}(-1 + \kappa_t \bar{s}_f\ m) y_t + \frac{1}{2a} - \frac{c}{2} \tag{33}$$
$$+ [\rho - \bar{s}_f \Big\{ \frac{1}{2aB}(-1 + \kappa_t \bar{s}_f\ m) y_t + \frac{1}{2a} + \frac{c}{2} - c_f \Big\} m] \kappa_t$$
$$\equiv F_2(y_t, \kappa_t; \rho)$$

(iii)

$$\dot{g}_t = \frac{1}{2\alpha}[\Big\{ \frac{1}{2aB}(-1 + \kappa_t \bar{s}_f\ m) y_t + \frac{1}{2a} - \frac{c}{2} \Big\} \Big\{ (1 + \frac{\kappa_t \bar{s}_f\ m}{2} y_t)$$
$$+ \frac{1}{2}(-1 - B + caB) + g_t + \alpha g_t^2 \Big\}$$
$$+ ((\rho - g_t)(1 + 2\alpha g_t + \kappa_t y_t)$$
$$- \kappa_t F_1(y_t, g_t . \kappa_t) - y_t F_2(y_t, \kappa_t; \rho)]$$
$$\equiv F_3(y_t, \kappa_t, g_t; \rho)$$

Eq. (33) is a system of the fundamental dynamic equations for our model. Next, let us consider the equilibrium solution (y^*, κ^*, g^*) of the system (33) which satisfies $y^* \neq 0$.
We can see that the economically meaningful equilibrium solution must satisfy the following conditions.
(i)

$$c_f < P^* < 1/a$$

(ii)

$$0 < y^* < B(1 - aP^*) \tag{34}$$

(iii)

$$\rho > g^*$$

where

$$P^* = \frac{1}{2aB}(-1 + \kappa^*\ \bar{s}_f\ m) y^* + \frac{1}{2a} + \frac{c}{2} \tag{35}$$

Inequality (34)(i) implies that the fringe firm can earn positive profit at equilibrium. If this inequality is satisfied, we also have the inequality

$$c < P^* < 1/a \tag{36}$$

because of the assumption $c_f \geq c$. Inequality (36) means that the dominant firm's profit is also positive at equilibrium. Inequality (34) (ii) implies that the dominant firm and the fringe firm coexist at equilibrium. Inequality (34)

(iii) ensures that the net cash flow of the dominant firm (W) becomes finite at equilibrium.

In fact, it follows from eq. (6) that

$$0 < W = \frac{f(P^*, g^*; y^*)}{\rho - g^*} K_0 < +\infty \tag{37}$$

at equilibrium if the inequalities (34) (i) $\sim$ (iii) are satisfied.

Moreover, from eq. (15) we obtain

$$g^* = \overline{s}_f (P^* - c_f) m. \tag{38}$$

Therefore, we get

$$g^* > 0 \tag{39}$$

if the inequality (34)(i) is satisfied. Substituting eq. (38) into eq. (30) and considering $\dot{\kappa}^* = 0$, we obtain

$$\kappa^* = -\frac{P^* - c}{\rho - g^*} \tag{40}$$

Therefore, we have

$$\kappa^* < 0 \tag{41}$$

if the inequalities (34)(i) and (34)(iii) are satisfied.

4 The Dynamics

Next, let us consider the out of steady state dynamics around the equilibrium point by assuming that the economically meaningful equilibrium exists. The Jacobian matrix of the system (33) which is evaluated at the equilibrium point can be written as

$$J = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & 0 \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \tag{42}$$

As for the entries of the Jacobian J, see Asada and Semmler (2002).

The characteristic equation of this system becomes

$$\Delta(z) \equiv | zI - J | = z^3 + b_1 z^2 + b_2 z + b_3 = 0 \tag{43}$$

where

$$b_1 = -\text{trace} J = -F_{11} - F_{22} - F_{33} \tag{44}$$

$$b_2 = \begin{vmatrix} F_{22} & 0 \\ F_{32} & F_{33} \end{vmatrix} + \begin{vmatrix} F_{11} & F_{13} \\ F_{31} & F_{33} \end{vmatrix} + \begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix} \tag{45}$$
$$= F_{22}F_{33} + F_{11}F_{33} - F_{13}F_{31} + F_{11}F_{22} - F_{12}F_{21}$$

$$\begin{aligned} b_3 = -detJ &= -F_{11}F_{22}F_{33} - F_{13}F_{32}F_{21} \\ &+F_{13}F_{22}F_{31} + F_{12}F_{21}F_{33} \end{aligned} \tag{46}$$

$$\begin{aligned} b_1 b_2 - b_3 &= (-F_{11} - F_{22} - F_{33})(F_{22}F_{33} + F_{11}F_{33} - F_{13}F_{31} \\ &+F_{11}F_{22} - F_{12}F_{21}) + F_{11}F_{22}F_{33} + F_{13}F_{32}F_{21} \\ &-F_{13}F_{22}F_{31} - F_{12}F_{21}F_{33} \end{aligned} \tag{47}$$

It is expected that under a wide range of parameter sets, the equilibrium point will become a saddle point. In this case, we must select the convergent path because the divergent path will not satisfy the transversality conditions (equations (20) and (21)).

However, there may be some parameter set under which the Hopf-Bifurcation occurs. We can make use of the following useful criterion

Theorem 1:*Hopf-Bifurcation occurs in the system (33) at $\rho = \rho_0 > g^*$ if and only if a set of conditions*

$$b_1(\rho_0) \neq 0, \quad b_2(\rho_0) > 0, \tag{48}$$
$$b_1(\rho_0)b_2(\rho_0) - b_3(\rho_0) = 0$$

and

$$\frac{\partial\{b_1(\rho)b_2(\rho) - b_3(\rho)\}}{\partial\rho} |_{\rho=\rho_0} \neq 0 \tag{49}$$

are satisfied.

In this case, three characteristic roots of eq. (42) become

$$z = \begin{cases} i\sqrt{b_2(\rho_0)} \\ -i\sqrt{b_2(\rho_0)} \\ -b_1(\rho_0) \end{cases} \tag{50}$$

where $i = \sqrt{-1}$.

(Proof.)

See Asada and Semmler (1995).

If the conditions (48) and (49) are satisfied at $\rho = \rho_0 > g^*$, there exist some non-constant periodic solutions at some parameter values ρ which are sufficiently close to ρ_0. We selected the discount rate ρ of the dominant firm as a bifurcation parameter following the usual procedure of the dynamic optimization theory, yet we could also select another parameter as a bifurcation parameter.

At the closed orbit, y_t, κ_t and g_t are bounded so that λ_t is also bounded from eq. (25). Therefore, at the closed orbit we have $\lim_{t\to\infty} \lambda_t e^{-\rho t} = 0$, which implies that the transversality condition with respect to λ_t (eq. (20) is satisfied at the closed orbit. Furthermore, we have

$$\mu_t = \kappa_t K_t = \kappa_t K_0 e^{\int_0^t g_\tau d_\tau} \tag{51}$$

so that we have

$$\lim_{t\to\infty} \mu_t e^{-\rho t} = \lim_{t\to\infty} (\kappa_t K_0) e^{-\int_0^t (\rho - g_\tau) d_\tau} = 0 \tag{52}$$

if $\rho > g_\tau$ for all $\tau \geq 0$ at the closed orbit. In fact, the closed orbits which satisfy $\rho > g_\tau$ for all $\tau \geq 0$ exist for the parameter values ρ which are sufficiently close to ρ_0 when $\rho > g^*$ at $\rho = \rho_0$.

5 Numerical Simulations

For the convenience of the numerical study let us adopt the transformation

$$\tilde{K}_t \equiv K_t e^{-g^* t} \tag{53}$$

where g^* is the equilibrium growth rate which is endogenously determined by the condition $\dot{y}_t = \dot{\kappa}_t = \dot{g}_t = 0$ in eq. (33). Then the optimization problem which is given by equations (13), (14), and (15) is reduced to the following equivalent expressions

$$\max_{P_t, g_t} \int_0^\infty (P_t, g_t; y_t) \tilde{K}_t e^{-(\rho - g^*)t} dt \tag{54}$$

subject to

$$\dot{\tilde{K}}_t = (g_t - g^*) \tilde{K}_t, \quad \tilde{K}_0 > 0 \tag{55}$$

$$\dot{y}_t = \{\bar{s}_f (P_t - c_f) m - g_t\} y_t, \quad y_0 > 0 \tag{56}$$

Needless to say, at the equilibrium point of the system (15) such that $\dot{\tilde{K}}_t = 0$, we have $g_t = g^*$. We assume that $\rho > g^*$, which is in fact satisfied in our numerical examples. We undertake a numerical study of the above system (54)-(56) by employing a dynamic programming algorithm developed by Grüne (1997).[8]

[8] The dynamic programming algorithm of Grüne (1997) was programmed in "Maple" and is available upon request from the authors.

Periodic Solutions:
With $\overline{s_f}$=0.3, B=7.3, c=1.38, $c_f = 1.40$, α=11.50, m=0.3, a=0.1, ρ=0.12, we get the economically meaningful equilibrium solution

$$\kappa^* = -0.276, y^* = 6.09, g^* = 0.0012,$$

which satisfies system (34). With starting values of y_t and $\widetilde{K}_t$ being 6.0 and 0.2, the simulations are presented in Figure 1a - 1c. From Figures 1 - 3 we can observe that for the above parameters periodic solutions arise.

Convergent Solutions:
With $\overline{s_f}$=0.1, B=0.3, c=0.38, $c_f = 0.40$, α=4.50, m=0.3, a=0.3, ρ=0.12, we get the economically meaningful equilibrium solution

$$\kappa^* = -1.65, y^* = 0.22, g^* = 0.005,$$

which satisfies system (34). With the starting values of y_t and $\widetilde{K}_t$ being 0.8 and 2, the simulations are presented in Figure 4 - 6. We can observe that for this second set of parameters converging solutions arise.

6 Conclusions

In this paper we have studied a model of pricing and investment with heterogenous firms. More specifically we study the interaction of a dominant firm and a fringe firm. The dominant firm that controls the price and its investment is financially unconstrained. The fringe firm that enters the industry and adjusts its output is financially constrained. The trend growth in demand is determined endogenously. The formal structure of the model is described as a Stackelberg game between the dominant firm, the leader and the fringe firm as follower. The model allows to study the dynamics of market shares and the value function of both types of firms. We can show analytically and numerically by employing a dynamic programming algorithm that there are periodic as well as converging solutions feasible under certain parameter constellations. The model thus predicts that considerable fluctuations of market shares as well as the asset prices of firms might arise in industries with heterogenous firms.

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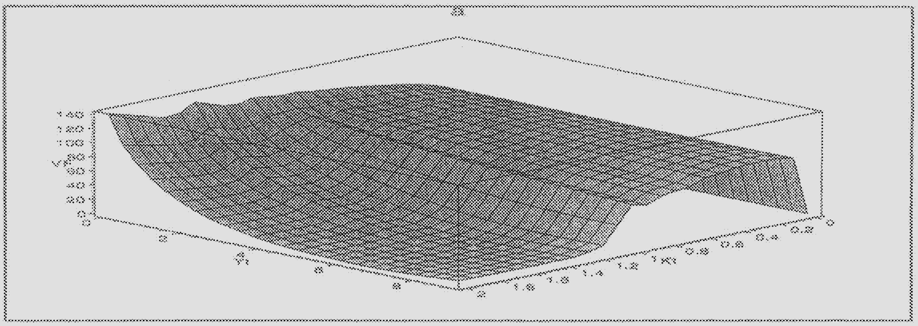

Fig. 1. Value function (Y_t and K_t on the horizontal axes stand for y_t and $\tilde{K}_t$ respectively)

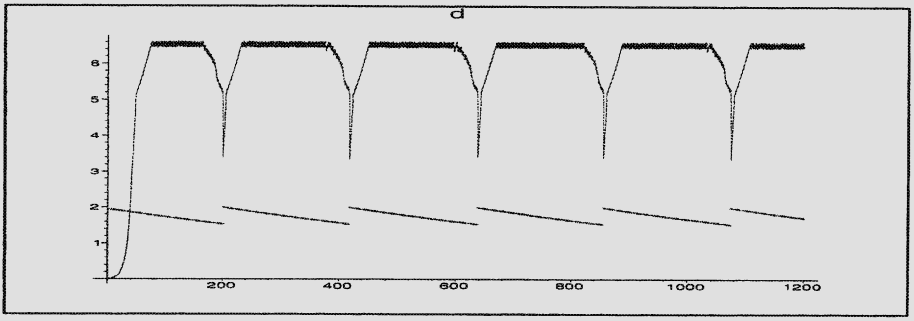

Fig. 2. Paths of y_t (upper part) and $\tilde{K}_t$ (lower part)

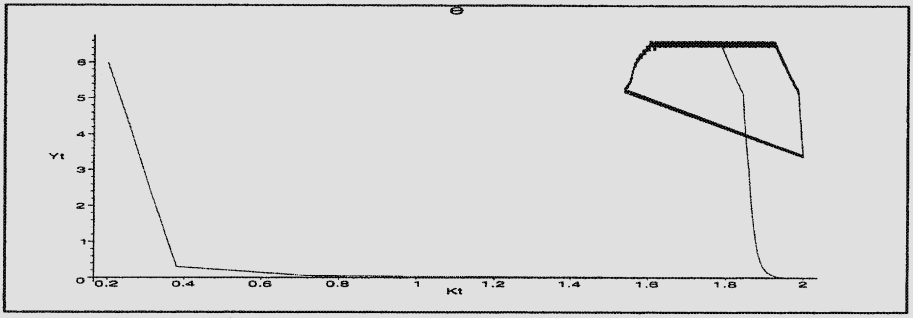

Fig. 3. Phase diagram of y_t (vertical) and $\tilde{K}_t$ (horizontal)

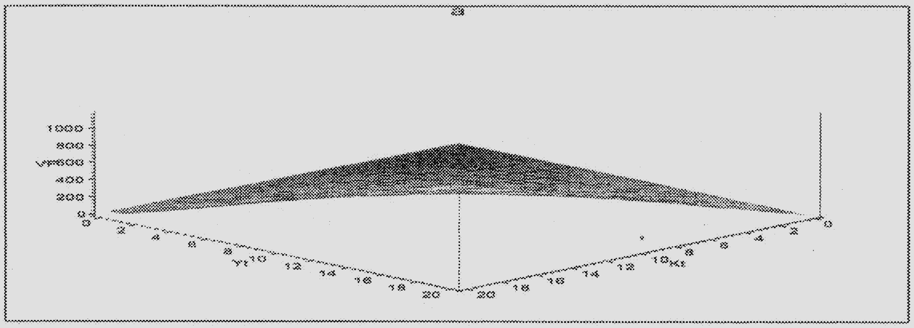

Fig. 4. Value function (Y_t and K_t on the horizontal axes stand for y_t and $\tilde{K}_t$ respectively)

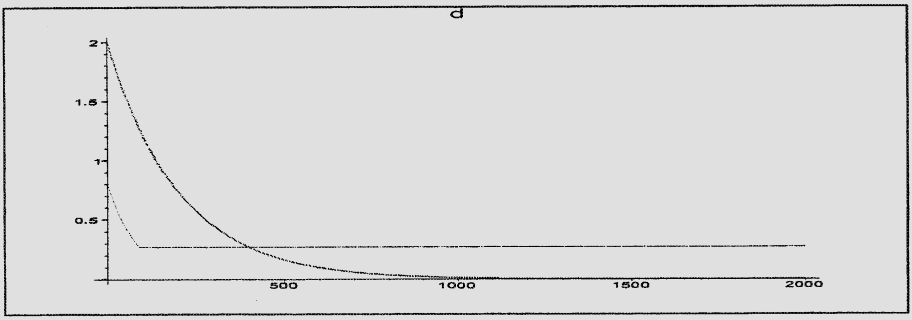

Fig. 5. Paths of y_t (starting value 0.8) and $\tilde{K}_t$ (starting value 2)

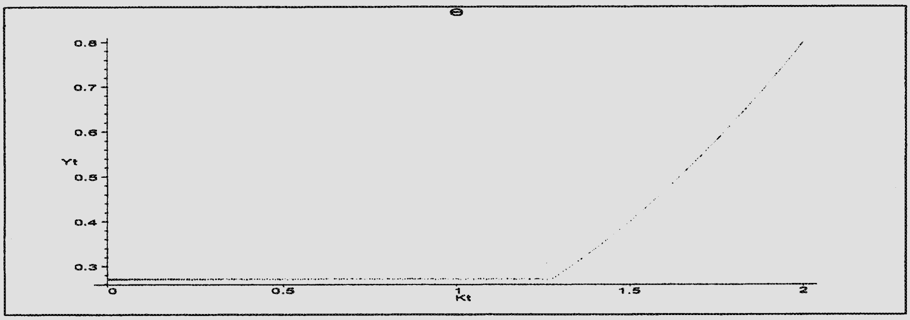

Fig. 6. Phase diagram of y_t (vertical) and $\tilde{K}_t$ (horizontal)

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Viability of Innovation Processes, Emergence and Stability of Market Structures

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Abstract. The paper is devoted at analyzing the co-ordination role that markets and organizations are called to play in order to make viable innovation processes. This analysis reveals that the viability of innovation processes cannot be dissociated from the way market structures emerge and evolve, and hence that there is not a 'new economy' problem referring to the specific character of certain technologies, namely, the information and communication technologies.

JEL code L11 O31.

1. Introduction

Modern economic analysis focuses on first principles and hence on fundamentals (that is, on properties of technologies, preferences and information structures) in order to explain simultaneously firms' performances and market structures.

Within this analytical framework, a superior technology coupled with market rules that determine better incentives to adopt will result in higher performances both at the micro and at the macro level. Co-ordination problems are supposed to be solved *ex ante* by an invisible hand or by the strategic interaction between economic agents. This puts all the weight of the firms' (and of the economy's) performances on the intrinsic character of technology and the prevailing institutions. However, the recent contradictory performance of the so-called `new economy', which was reckoned to allow a great jump forward, has cast doubts on the alleged role of certain technologies and institutional frameworks as a major factor of economic growth. This shifts the focus on the co-ordination mechanisms required to make viable the innovation process on which, rather than on the specific character of particular technologies, the firms' performances and hence the growth of the economy actually depend. As a matter of fact, although there is strong evidence

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that technical and market conditions have deeply changed with the emergence of new technologies, it should also be clear that the new economy has to solve the same basic co-ordination problems than the old one.

Economies of the new age, as those of the older era, are confronted to a biased technological progress, characterized, with reference to a Neo-Austrian analytical framework (Hicks 1973), by an increase in the construction costs of the productive capacity more than compensated by a decrease in its utilization costs[1].

In particular, innovation results in a breaking up of the (time) structure of the existing productive capacity that induces the appearance of co-ordination problems both at the firm level and at the industry level. In a restructuring process costs and proceeds are no longer synchronized, and hence also supply and demand are no longer equal at each moment of time and over time. Reactions to these imbalances result in fluctuations that may be a threat to the viability of the innovation process. The way in which these co-ordination problems are dealt with is what actually determines the performance of the firms involved, The market itself is affected by this process, though. The viability of innovation processes cannot be dissociated from the way market structures emerge and evolve.

Building on the insights of the Marshallian and the Schumpeterian approaches we will focus on the co-ordination role that markets and organizations are called to play in order to make viable the innovation process. We shall interpret competition as the main co-ordination mechanism involved in this process and argue that competition not only may co-exist with increasing returns but helps firms to capture them. We shall further show that a stabilization of market structure, not any particular feature of it, is required for innovations to be successfully carried out (and returns actually obtained). However, competition may, or may not, lead to stabilization. It is really successful when prices and quantity adjustments are carried out, which make it possible for firms to obtain normal profits, that is when these adjustments do not result in a waste of productive resources.

The outline of the paper is as follows. Section 2 will analyse the nature of co-ordination problems involved by innovation processes. Section 3 will lay down a model that allows to deal analytically with these problems. Section 4 will describe, by means of simulations, the behaviour of the model, and will present the main results. Section 5 will conclude with some analytical and policy considerations.

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[1] Clearly, information and communication technologies tend to increase the capacity of the economy to deal with variety. ``This has led to decreases in the cost of switching from one series to another, from one product to another" (E. Zuscovitch 1998, p. 252), that is, to decreases in the utilization costs of a larger gamut of products and services. However "this tendency calls for a massive incorporation of science and technology in specific configurations to match this specified variety", which generally involves higher construction costs.

2. Innovation and Sunk Costs: the Nature of Co-ordination Issues

According to the standard approach, technology is given in that it has an already defined specific character. Innovation is then reduced to a decision of adoption of new technologies to which are associated given results. According to our approach innovation comes down to a process of research and learning that may, or may not, result in the appearance of new productive options and new productive structures. In this sense It is a process of 'creation of technology', which, when (and if) successfully brought about, makes it possible to actually transform the potential increasing returns of technology into growth and monetary gains for the firms and the consumers. This implies, in particular, the breaking up of the existing industrial structure and a modification of the market conditions, followed by a gradual reshaping.

Coordination is required in the first place to re-establish the harmony between construction and utilization, disturbed by the structural modification involved, so as not to have too strong imbalances between costs and proceeds and between supply and demand of final output. However, co-ordination must not be seen only at the single firm's level but at a systemic level. Most innovations do not arise as the result of the independent actions of single innovating firms, but are the result of new forms of co-ordination across several heterogeneous firms. Co-ordination of the production process and co-ordination between firms interact. Co-ordination problems mainly come from the existence of the sunk costs associated with a restructuring of productive capacity. The focus on sunk costs and the two faces of the co-ordination problem involved are common to the New Industrial Organization (NIO) approach, but the role that we attribute to them is quite different.

In the NIO approach, sunk costs are construction (investment) costs, "a fascinating aspect of (which) is their commitment value" (Tirole 1988, p.314). This commitment refers to a multi-period context and represents a credible threat, which is essential to the determination of market structure. However, the sunk costs depend in turn on market structure and are determined simultaneously with the latter. This is so because the sunk costs are determined once the market game that defines the market equilibrium is known. Everything is defined following a backward induction process that implies an analytically instantaneous determination of all the relevant magnitudes. A less extreme version of NIO is that of Sutton (1991,1998) who abandons the aim of identifying a unique equilibrium outcome in a given multi-period context. "Instead we admit some class of candidate models (each of which may have one or more equilibria) and ask whether anything can be said about the set of outcomes that can be supported as an equilibrium of any candidate model" (1998, pp.6-7) This set of outcomes must satisfy two conditions: the viability condition - which means that each firm covers its sunk cost over the multi-period domain - and the stability condition - which allows to preserve a certain structure of the market. Although essentially different our treatment of sunk costs has some relation to Sutton's analysis. This concerns the focus on viability and the market structures compatible with it.

However, sunk costs, in our analysis, are not only the expression of the existence of investment costs, but also of the divorce between costs and proceeds at each step of an evolution process triggered by the breaking of the intertemporal complementary of the production process as the result of the attempt to carry out an innovation. Intertemporal complementary is the main feature of a process of production where the relation between the phase of construction and that of utilization of productive capacity is stressed. It is assured by definition in equilibrium. When it is no longer assured, costs are dissociated in time from proceeds and hence become 'sunk' costs. The characteristic of the sunk costs of the investment in a process which implies a structural change is that they will only be recovered when (and if) the process self is actually established. This means not only to take into account the whole period of construction of the new productive capacity - which is likely to have a considerable length as, before construction in a proper sense, it implies experimenting, pilot plans, and so forth - but to go further beyond that point, until the stream of receipts from the new output has reached a certain size and the change has thus proved viable. The main point to be stressed is that, in a context of gradual reshaping, costs depend not only on the current production but also on the length of construction of the new productive capacity, on the length of utilization of this capacity, and on the total volume of the output produced over the successive periods. These are not data but results of the process self. When, e.g., productive capacities in excess result in lower rates of utilization and/or in scrapping of production processes, then there are changes in production costs, and hence in viability conditions. These changes really express coordination failures that emerge as a consequence of a breaking in the intertemporal complementary of production processes.

Then, firms do not know *ex ante* whether it pays to innovate. "Indeed the answer to this question for any single firm depends on the choices made by other firms, and reality does not contain any provisions for firms to test their policies before adopting them. Thus there is little reason to expect equilibrium policy configurations to arise. Only the course of events over time will determine and reveal what strategies are the better ones" (Nelson and Winter1982 p. 286).

In this perspective competition can no longer be considered as a particular *state of affairs*. It is not only aimed at equalising supply and demand in a given market and technological environment, but "has also to adapt both structure and technology to the fresh opportunities created by expanding markets" (G.B. Richardson 1975 p. 353). It must be viewed as *a process of trials and errors* an essential moment of which is the discovery of the information about the behaviour of competitors and customers. This behaviour has feed-back effects on what happens inside each firm in terms of the relation between costs and proceeds, the relation on which the viability of the innovation process and the emergence of a particular market structure eventually depends.

3. The Model

The course of the events over time resulting from innovative choices will be analysed here by means of a model derived from that built by Amendola and Gaffard (1998), which makes it possible to exhibit the time structure of production processes and to analyse the sequential interaction of competing firms in a process of restructuring of productive capacities.

State and control variables

In the usual way the system is described by state and control variables.

The state variables are for each firm i:

$x^i(t)$, the vector of production processes,

$m(t)$, the money proceeds from sales,

$h^i(t)$, the monetary idle balances,

$o^i(t)$, the stock of final output,

$\omega^i(t)$, the wage fund,

$\psi^i(t)$, the available human resources,

$d^i(t)$, the volume of final demand,

$s^i(t)$, the volume of supply

$\delta^i(t)$, the market share.

The control variables are:

$x_1^i(t)$, the rate of starts of production processes,

$u^i(t)$, the rate of scrapping of production processes,

$\tau^i(t)$, the rate of utilization of productive capacity,

$p^i(t)$, the price of final output,

$w^i(t)$ the wage rate,

$f^i(t)$, the external financial resources which depend on banking policy,

$\eta^i(t)$, the fraction of total real stocks actually put back on the market.

These are either determined exogenously in the simulations (the open-loop control variables) or according to feedback mechanisms (the close-loop control variables).

The structure of productive capacity

In each firm i production is carried out by means of processes of a Neo-Austrian type. An elementary process of production is defined by the input vector:

$$\mathbf{a}^i_j = \left[a^i_{jk}\right];\ \mathrm{k}=1,..,n^c + n^u$$

whose elements represent the quantities of labour required in the successive periods of the phase of construction c (from 1 to n^c) and following it, of the phase of utilization u (from $n^c + 1$ to $n^c + n^u$) of the productive capacity of commodity (technology) j, so that:

$$\mathbf{a}^i_j = \left[\mathbf{a}^{ic}_j, \mathbf{a}^{iu}_j\right]$$

with $a^i_{jk} = a^{ic}_j \ \forall k = 1,\ldots,n^c$ and $a^i_{jk} = a^{iu}_j \ \forall k = n^c + 1,\ldots,n^c + n^u$

and by the output vector:

$$\mathbf{b}^i_j = \left[b^i_{jk}\right]$$

with $b^i_{jk} = 0 \ \forall k = 1,\ldots,n^c$ and $B^i_{jk} = b^i_j,\ \forall k = n^c + 1,\ldots,n^c + n^u$

At each given moment t the productive capacity of a commodity j by a firm i is represented by the intensity vector:

$$\mathbf{x}^i_j(t) = \left[\mathbf{x}^{ic}_j(t), \mathbf{x}^{iu}_j(t)\right]$$

each element of which is a number of elementary production processes of a particular age, still in the construction phase or already in the utilization phase.

At each given moment of time the productive capacity of the firm i is given by the vectors:

$$\mathbf{x}^i(t) = \left[\mathbf{x}^{ic}(t), \mathbf{x}^{iu}(t)\right]$$

Whose elements are the number of processes in construction, $x^{ic}(t)$, and in utilization, $x^{iu}(t)$referring to all the technologies in use - all the commodities ($\mathbf{x} = \sum_j \mathbf{x}_j$).

The productive capacity is subject to ageing and to modifications due to investment and scrapping. Scrapping of production processes u(t) occurs when resource constraints are as stringent as not to allow all the processes inhered from the past to be carried on. An alternative to scrapping is a partial use of utilization processes, which, however, implies a cost, as we shall see when considering the rate of utilization of existing productive capacity. We assume that the firms choose a less than full degree of utilization to allow for an over normal functioning of the existing capacity in order to be able in case to wage a 'capacity competition'(see below).

Resources constraints

In each period the level of activity (both investment and current production) of each firm, which depends on its wage fund $\omega^i(t)$, is constrained by available financial resources or, alternatively, by available human resources.

The available financial resources $F^i(t)$ are:

$$F^i(t) = m^i(t-1) + h^i(t-1) + f^i(t)$$

where the internal financial resources are given by $m^i(t-1)$ the money proceeds from the sales of final output, and $h^i(t-1)$ the idle money balances involuntarily accumulated in the past and ready for use, and the external financial resources by $f^i(t)$.

External financial resources are such that:

$$f^i(t) = \min\left[k^i m^i(t-1), f^i_d(t)\right]$$

where k^i stands for the borrowing power of each firm, and $f^i_d(t)$ is the demand for external financing resulting from the production and investment decisions actually taken.

External financial constraints are formally exogenous in the model. Different financing scenarios, which imply to consider the relation between external finance and the viability of innovation processes have been explored[2].

The available human resources at the industry level depend on a natural growth rate of population and on a wage elasticity:

$$\psi(t) = (1+g)^t L(0) w(t)^{\vartheta}$$

where g is the natural growth rate, w the industry average wage rate, and ϑ the wage elasticity of the labour supply.

A general human constraint may appear due an insufficient growth of the labour force. This constraint can be relaxed by each firm through the wage policy followed.

A competition among the firms exists on the labour market implying that each firm obtains a fraction of the labour supply ψ^i which depends on the relative wage it pays to its workers:

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[2] An interpretative step forward can be made by inferring that when positive results of the simulations are associated with a specific value of k, such a value will express the opinion (and the decisions) of financial markets and/or of bankers.

$$\psi^{i}(t) = \phi^{i}(t)\psi(t)$$

$$\text{with } \phi^{i}(t) = \frac{\left[\phi^{i}(t-1)w^{i}(t-1)\right]^{\varepsilon}}{\sum_{i}\left[\phi^{i}(t-1)w^{i}(t-1)\right]^{\varepsilon}}$$

With ε less than one, the distribution of the labour supply between firms is more or less constant while wages they paid are different. With ε greater than one, on the other hand, the firm which, at one moment, has the greater share of labour supply, benefits a kind of cumulative advantage, which implies that more and more workers prefer to be hired by this firm whatever the wage it pays.

When the human constraint is more stringent than the financial constraint money balances are involuntarily accumulated:

$$h^{i}(t) = \max\left[0, m^{i}(t-1) + h^{i}(t-1) - \omega^{i}(t)\right]$$

Money proceeds, real stocks and current production

Within the sequential setting considered prices are fixed within each given period and can only change at the junction of one period to the next one. As a consequence money proceeds are given by:

$$m^{i}(t) = \min\left[p^{i}(t)d^{i}(t), p^{i}(t)s^{i}(t)\right]$$

Real stock changes $\Delta o^{i}(t)$ are substitutes for the price changes, which cannot take place within the period. Excess supply results in an accumulation of undesired stocks for the firm:

$$\Delta o^{i}(t) = o^{i}(t) - o^{i}(t-1)$$

where $s^{i}(t)$ and $d^{i}(t)$ are current real supply and real demand (for the different and successive commodities or technologies), respectively.

Current final production by firm i is given by:

$$q^{i}(t) = s^{i}(t) - \eta o^{i}(t-1)$$

The rate of utilization τ^{i} of the productive capacity inhered from the past is such that:

$$q^{i}(t) = \tau^{i}(t)\sum_{k=1}^{n^{c}+n^{u}} b_{k}^{i} x_{k}^{i}(t)$$

$$\text{where } b_{k}^{i} = \sum_{j} b_{kj}^{i} \text{ and } x_{k}^{i} = \sum_{j} x_{kj}^{i}$$

Aggregate demand and market shares

The aggregate market demand, D, is determined as follows:

$$D(t) = (1+g)D(t-1)p^{\theta}(t), \ \theta \leq 0$$

where g is a given exogenously determined growth rate, and θ the demand price elasticity.

The average market price is given by:

$$p(t) = \frac{\sum_i p^i(t)s^i(t)}{\sum_i s^i(t)}$$

The market shares are: $d^i(t) = \delta^i(t)D(t)$

with:

$$\delta^i(t) = \frac{\left(\frac{\delta^i(t-1)}{p^i(t-1)}\right)^\beta}{\sum_i \left(\frac{\delta^i(t-1)}{p^i(t-1)}\right)^\beta}$$

that is, a firm's market share depends on the relation of its price to the average market price in the preceding period.

With β less than one, more or less constant market shares obtain when the prices charged by the firms are different. This looks a Chamberlinian competition. With β greater than one, on the other hand, the firm which, at one moment, has the greater market share, has the cumulative advantage which result in the ex of some other firms. This looks like a situation characterized by increasing returns of adoption.

Investment, production and price decisions

The evolution path followed by each firm is actually determined by the behaviour of the decision variables, namely, the rate of starts of new production processes $x_1^i(t)$, the rate of utilization of productive capacity τ^i the price of final output $p^i(t)$ the wage rate $w^i(t)$ the ratio $k^i(t)$ of the external financial resources $f^i(t)$ to the money proceeds from the sales of final output $m^i(t)$ (i.e., the firm's borrowing power), and the rate of scrapping $u_k^i(t)$.

Each firm determines the rate of starts of production processes in such a way that the productive capacity available $n + t^c$ periods later will match the expected demand:

$$x(t) = \max\left[0 \frac{\tilde{d}^i(t+n^c) - \tilde{q}^i(t+n^c)}{\bar{\tau}^i b^i_{n^c+1}(t+n^c)}\right]$$

where $\tilde{d}^{i}(t+n^{c})$, is the real demand expected by the firm i for the period $t+n^{c}$, computed by extrapolating the average growth rate of real demand registered in the previous periods and $\tilde{q}^{i}(t+n^{c})$ is the output that will be obtained from the productive capacity available at the period $t+n^{c}$, the construction of which has begun before t , so that:

$$\tilde{q}^{i}(t+n^{c})=\bar{\tau}^{i}\sum_{k=2}^{n^{u}} b^{i}_{k+n^{c}}(t+n^{c})x^{i}_{k+n^{c}}(t+n^{c})$$

where $\bar{\tau}^{i}$ is the desired rate of utilization of the productive capacity.

Different investment behaviours may be considered by introducing more or less stringent limits to the variations of the *desired* rate of starts from one period to the next: limits which represent more or less aggressive investment behaviours. In fact, firms take investment decisions looking at the expected demand, but they also know that the volatility of investments is a threat to their survival. So the change in the rate of starts of new production processes (whether an increase or a decrease) from one period to the next is bounded, which sets a limit to the 'capacity competition' that would otherwise take place. This kind of competition depends on the fact that in a truly sequential context the firms do not know in advance the result of the market game. Thus, when they make investment decisions looking at the expected demand, they discount the increases in productivity resulting from their own innovations but not those realized by the competitors. This is likely to bring about productive capacities in excess with respect to the existing demand and pushes the competing firms to a `capacity competition' aimed at stealing market shares from each other.

Each firm determines current production by fixing the current rate of utilization of its productive capacity, $\hat{\tau}^{i}$, so as to adjust its current supply to the expected final demand $\hat{d}$:

$$\tau^{i}(t)=\min\left[1,\frac{\hat{d}^{i}(t)-\left[o^{i}(t-1)-o^{i}_{d}(t)\right]}{\sum_{k=1}^{n^{c}+n^{u}} b^{i}_{k}x^{i}_{k}(t)}\right]$$

where $\hat{d}$ is such that:

$$p^{i}(t)\hat{d}^{i}(t)=\frac{m^{i}(t-1)^{2}}{m^{i}(t-2)}$$

that is, the expected final demand is made to depend on the past trend of money proceeds of the firm, and o^{i}_{d} are the stocks that the firms desire to keep.

As the result of the production and investment decisions the actual wage fund is given by:

$$\omega^{i}(t)=w^{i}(t)\Lambda^{i}(t)$$

where Λ^{i} is the labour demand given by:

$$\Lambda^i(t)=\sum_{k=1}^{n^c+n^u} A_k^i(t)x_k^i(t)\rho_k^i(t)$$

where ρ_k^i are the elements of the vector ρ^i, which allows to take into account what are the consequences on the labour demand of a variation in the rate of utilization of the productive capacity:

$\rho^i(t)=\left[\rho_1^i(t),\ldots,\rho_{n^c}^i(t),\ldots,\rho_{n^c+n^u}^i(t)\right]$,

with: $\rho_k^i(t)=1 \;\; \forall 1\leq k\leq n^c$

and: $\rho_k^i(t)=\tau^i(t)+\zeta^i(1-\tau^i(t)) \;\; \forall n^c+1\leq k\leq n^c+n^u$

where ς^i stands for the labour required to maintain a process of production idle.

The price charged by each firm is determined as follows:

$$p^i(t)=\frac{w(t)\sum_{k=1}^{n^c+n^u} a_k^i\rho_k^i(t)}{\bar{\tau}^i\sum_{n^c+1}^{n^c+n^u} b(t)}$$

with $\rho_k^i(t)=1 \quad \forall 1\leq k\leq n^c$ and $\rho_k^i(t)=\tau^i(t)+\zeta^i(1-\tau^i(t)) \; \forall n^c+1\leq k\leq n^c+n^u$

That is, it is determined in such a way as to cover the cost of production when using the productive capacity which is the expression of the technology adopted, at the desired rate of utilization of this productive capacity. This price is determined step-by-step with reference to the new technology adopted each time, at the moment this first reaches the phase of utilization. This is how price competition is implemented by each firm.

This price can be adjusted as mentioned above in order to relax the financial resource constraint:

$$\breve{p}^i(t)=p^i(t)+\sigma^i\frac{\left[\tilde{x}^i(t-1)-\hat{x}_1^i(t-1)a_1^i w^i(t-1)\right]}{\hat{d}^i(t)},\; 0\leq\sigma^i\leq 1$$

where $\tilde{x}_1^i$ is the desired rate of starts, and $\hat{x}_1^i$ is the rate of start constrained by the available financial resources.

It can also (and alternatively) be adjusted in reaction to the market disequilibria perceived in the previous period:

$$\widehat{p}^i(t)=p^i(t)+\chi^i\frac{\left[d^i(t-1)-s^i(t-1)\right]}{s^i(t-1)},\; 0\leq\chi^i\leq 1$$

Moreover changes in price from one period to the next are both upward and downward bounded.

The wage rate is endogenous to the model, being determined by the partially exogenous supply of labour and the endogenous demand for labour. Changes in the wage rate paid by each firm reflect the disequilibria arising on s labour market, that is:

$$w^i(t) = w^i(t-1)\left[1 + \nu^i \frac{\Lambda^i(t-1) - \psi^i(t-1)}{\psi^i(t-1)}\right]$$

where ν^i is a reaction coefficient.

As already mentioned, firms are wage makers on local labour markets. However competition between firms results in different but convergent wage rates charged by each of them.

Innovation, imitation, entry and exit

A firm can introduce a new technology by innovating or by imitating. Innovation means entering a process, which should allow better performances than those of the firms that keep using older technologies. Imitation consists in copying the prevailing best practice. Innovations as well as imitations are generated by probability distributions that are independent from firm to firm, but the same for all firms and over all periods.

The market structure evolves endogenously. On the one hand, as already mentioned, price variations stirred by cost variations result in changes in market shares. Any firm whose market share falls below a given threshold (e.g. 1%), whatever the reason (too high price or lack of resources), exits from the market. On the other hand, only one firm can enter the market in each period of time.

Entry is modelled as a random process, characterized by an independent random variable *new-entry* that takes on the values one or zero according to whether a new firm does or does not enter. Effective entry occurs with the probability:

$$\Pr\{new-entry\} = \pi\,\Gamma(t)$$

with Γ is a rate of industry excess demand calculated on a given number of previous periods.

The size of a new entrant is equal to a targeted market share (e.g. 50% of the existing excess demand at the industry level). This threshold may be considered as figuring the strength of the financial constraint that the new firm has to bear.

Firms' performance and market concentration

The performance of each firm is measured by s unit margins, whereby a unit margin is defined, in each period, as the ratio of the difference between the price (calculated as mentioned above) and the current unit cost of output - obtained by dividing the total cost of production of the amount of output obtained in that period by the same amount - to the price itself:

$$\mu^i(t) = \frac{p^i(t) - c^i(t)}{p^i(t)} \qquad \text{where:} \qquad c^i(t) = \frac{w^i(t)\Lambda^i(t)}{q^i(t)}$$

Unit margins on average equal to zero mean that firms realize normal profits. Unit margins will be instead necessarily negative at the beginning of any innovation process characterized by higher construction costs. This reveals the initial competitive disadvantage suffered by the innovative firm. On the other hand, negative unit margins may also reveal the existence of excess capacities, that is, of a lower degree of utilization of productive capacity with respect to the desired level, and vice-versa.

The market concentration is measured by the Herfindahl index.

4. The Simulation Analysis

In what follows, we shall focus on the conditions required for innovation processes to allow to reap the benefits of technology, and on the evolution of market structures associated with these processes.

With standard models of oligopoly or monopolistic competition usually deal with the degree of competition and the characteristics of industrial structures as determined by given information and cost conditions, our model intends to deal with a dynamic process of rivalry such as determined by *changing* costs and information conditions. This process can result in a waste of productive resources and no real advantage for the customers or, alternatively, may allow firms and/or customers to benefit from increasing returns. It can likewise result in a strongly instable market structure or at the opposite in a fairly stable structure.

Within this framework, the character of the shocks that take place at each period does not really matter. These shocks always come down to a demand for new productive resources which will result in a productive structure functioning in such a way as to allow to reap the benefits of the change undergone only if the coordination problems raised by the shocks themselves are properly dealt with.

We shall consider a market in which two or more firms compete with each other by innovating, whether at the same time or sequentially, one after the other. Technological changes are `forward biased' in a sense similar but not equal to the definition given by Hicks (1973), that is, increasing construction (labour) costs are more than compensated by increasing output rates. At the beginning of the experiment the firms considered have an equal share of the market and face an aggregate final demand which is growing at a given rate (5%). There are no biases in the functioning of the product and the labour markets ($\beta = 1$ and $\varepsilon = 1$). Prices are determined with regards to a structure of productive capacity (embodying the more recent technology) capable of sustaining a steady state: in other words they are fixed at a level which corresponds to the average long unit cost associated with the prevailing technology. Cost changes, not automatically transferred on prices, have therefore an immediate negative effect on unit margins. Finally, there are free entry and ex conditions.

We assume that innovation perturbs an industry that at the beginning of the simulations is in equilibrium. This means in particular that in each firm the investment carried out is such as to keep consistency over time of the phases of construction and utilisation of productive capacity, and that at the same time it is related to the investments of the other firms in such a way as to keep a stable market structure. A technological shock breaks both the internal consistency of the capital structure of the firms involved and the equilibrium relation among firms. What happens within the firms also affects the relation among them, and vice-versa. This means that investments will become either insufficient or excessive with respect to those required to keep both the internal and the external equilibrium structure of productive capacity of the firms. This reflects the existence (or less) of a resource constraint: a financial constraint and/or a human resource constraint in our modelling. Together with the prevailing price and wage change regimes and the specific features of the environment dealt with (in particular, the original number of firms in the market) this will on the other hand determine the viability or less of the adjustment process triggered by the shock considered. A strong resource constraint (whether a financial or a human one) prevents an excessive capacity competition between the incumbents from being too strong, and hence favours the profitable entry of new firms that are supposed to have the required funds, given the targeted market share exogenously determined. With a limed number of firms at the beginning (N=2 in our experiments) and a strong financial constraint (k=0.2) - simulation 1 - the following entry-exit process is characterized by a concentration index that decreases before it is stabilized. Costs are diminishing, although through fluctuations, which means that the productivity gains associated with the new technologies are really obtained. Unit margins, necessarily negative at the beginning of any innovation process characterized by higher construction costs, converge towards a more or less normal level. The robustness of these results is attested by the synthesis carried out from multiple runs corresponding to different values of the variables randomly chosen, which shows an increase followed by a stabilization of the average number of firms existing on the market, and an increase followed by a decrease in the mean dispersion of market shares (simulation 2).

This, when the human resource constraint prevails over the financial one, holds only if the wage reaction coefficient is equal to zero, or enough low (v=0.05), that is, when the scarcity of the labour resource does not bring about strong variations in wages (simulation 3). If the wage reaction coefficient is too high, and this is coupled with a too high wage elasticity of the labour supply, a very unstable market structure prevents innovation processes to be viable (simulation 4).

This also holds whatever the initial number of firms. In the case of an initial atomistic structure (N=50), a shake-out process takes place that results in a stabilized market structure. Then the gains from innovation are obtained (simulation 5). Once again, the robustness of these results is attested by the synthesis carried out from multiple runs (simulation 6). Under specific co-ordination conditions, the industry converges towards a dynamic equilibrium. At the opposite, with few original incumbents (N=2), a weak external financial constraint (k=1) - simulation 7 - by favouring investment on the part of the incumbent themselves, makes it difficult for new firms to enter and simultaneously results in a relatively strong instability in the market shares, which is associated with an increase of the concentration index. Costs and unit margins strongly fluctuate. There are actually no productivity gains from innovations.

With a larger number of initial incumbents the competition conditions dramatically change, and there is a large fluctuation in the number of firms over time, the amplitude of which is all the more important than the initial number of firms is higher. In the case of an initial atomistic structure (N=50), a selection process takes place, which has a cumulative character. As a matter of fact, any exit results in a reduction of the average market price (the exiting firms being of course the less competitive ones, that is, those charging higher prices) which makes marginal firms more fragile and can even push them out of the market. This effect is the stronger the higher the number of initial firms and hence the more contiguous their position. The concentration index increases. Costs and unit margins exhibit strong fluctuations without a falling trend, which means that gains from innovations are not really obtained (simulation 8). The instability of the market structure appears as an obstacle to the viability of the innovation process.

Summing up. Increasing returns - obtained in a context of a sequential competition where each firm introduces at different moments of time new products belonging to the same general market - allow only a transitory competitive advantage. Several heterogeneous firms can coexist on the market, despite the existence of increasing returns, remaining differentiated not so much because they supply differentiated goods, but because they are at different steps of the life cycle of the production process. The latter situation can be really defined as a dynamic equilibrium. This is a situation in which competition causes "the rate of investment in product development to rise or fall towards the level at which this investment yields only a normal return" (Richardson 1998, p. 172). This is a situation in which the prices charged by the firms reflect decreasing average costs so as to allow the benefits from innovation to be also distributed to consumers. This is also a situation in which stability of markets shares obtains: there are neither significant new entries nor significant exits from the market. All these considerations not only qualify a dynamic equilibrium, but also the competitive conditions consistent with increasing returns.

5. Conclusion

What happens to the firms involved in an innovation process - what happens to their cost performances and market shares, and hence what happens to the market structure - has been looked at as a process sketched out step by step by sequentially interacting disequilibria. What essentially matters is the deformation of the structure of productive capacity of the different firms involved, which will be amplified or dampened according to the nature of the co-ordination mechanisms that prevails along the way. The conjecture that we have tested is that the possibility to take really advantage of innovations essentially depends on the ability of each firm to maintain a structure of the productive capacity that sustains a quasi-steady state. And this depends in turn on the working of the market coordination mechanism.

The availability of productive resources, and the constraints that these may impose on production processes, and the equilibrating (or disequilibrating) role performed by price and wage regimes, are the essential elements of the co-ordination mechanism at work.

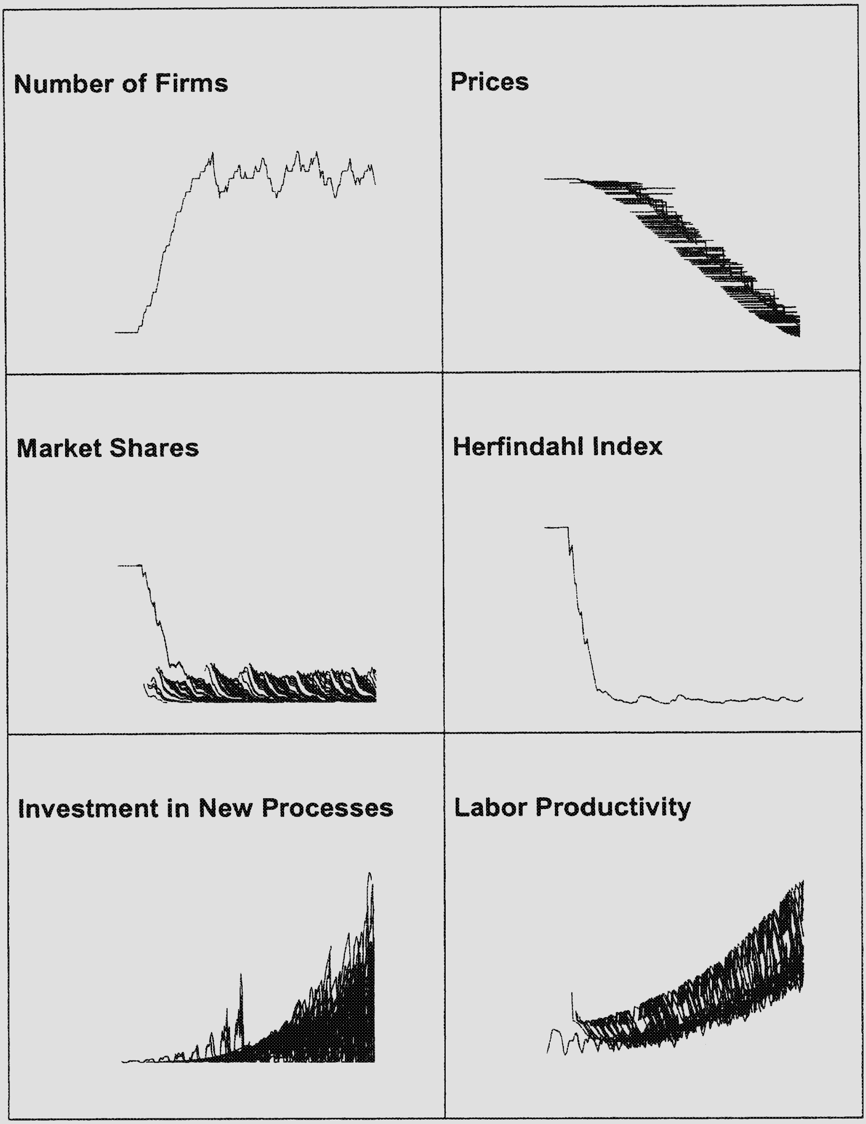
There is not an *ex ante* optimal co-ordination mode. It all depends on how the ingredients just mentioned get combined along the way. And this depends in turn on the specific context within which co-ordination has to be carried out (initial number of incumbent firms, entry and ex conditions, and the like).

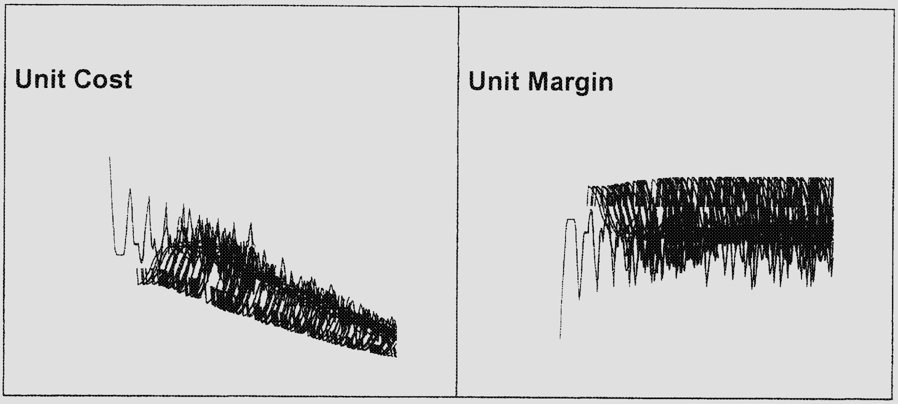
It follows that the success (or less) of the introduction of new technologies and the emergence and evolution of given market structures does not depend on the properties of technology, but on the capacity to coordinate the activity of the different firms participating to the restructuring process involved, which results in a certain degree of stability of the market structures.

Thus technological advances do not determine the dynamics of the number of firms. On the contrary, this is actually identified only once a stabilization of the market structure signals that viability conditions have been fulfilled. Nevertheless, different market structures can emerge from the same kind of innovation process, depending on the effective working of the coordination mechanism.

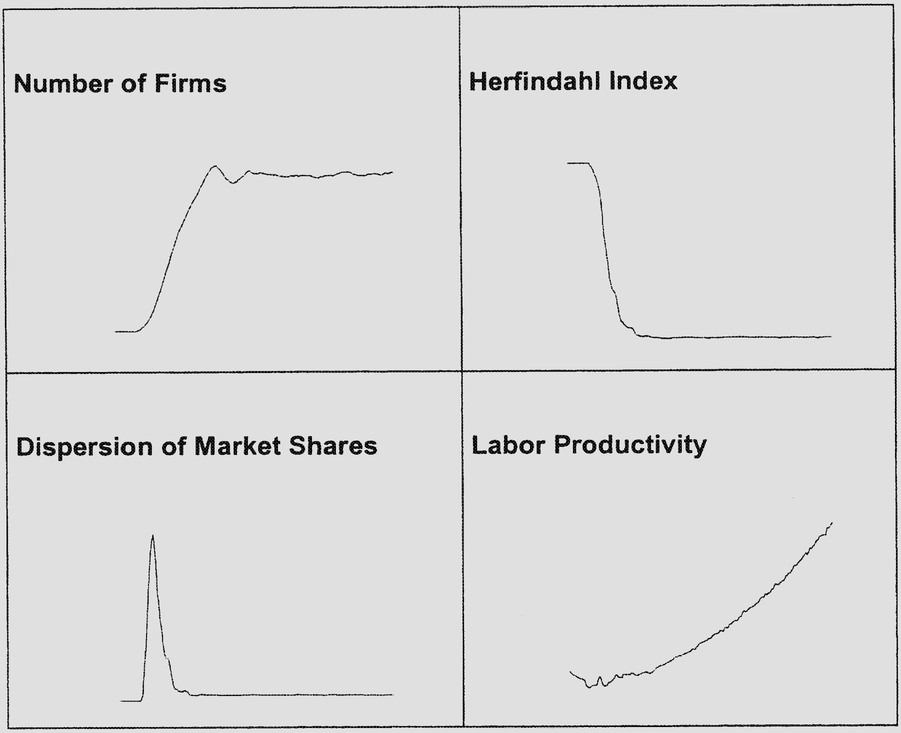
The focus on coordination of innovation seen as an essentially economic process reveals that there is not a 'new economy' problem referring to the specific character of certain technologies, namely, the information and communication technologies. The true reason withy these are reckoned to be a major factor of growth is that their supposed flexibility is believed to remove obstacles to the working of the market and eliminate the possibility of market failures (as the existence of increasing returns or the choice of non optimal scales of production) thus making possible to establish a full competition. We have shown on the contrary that, the more we let the market play in a sense near to the Walrasian equilibrium (by assuming full price flexibility, free entry, and the like) the less probable is to carry out a viable innovation process and to reap the benefits of technology.

- **Simulation 1**

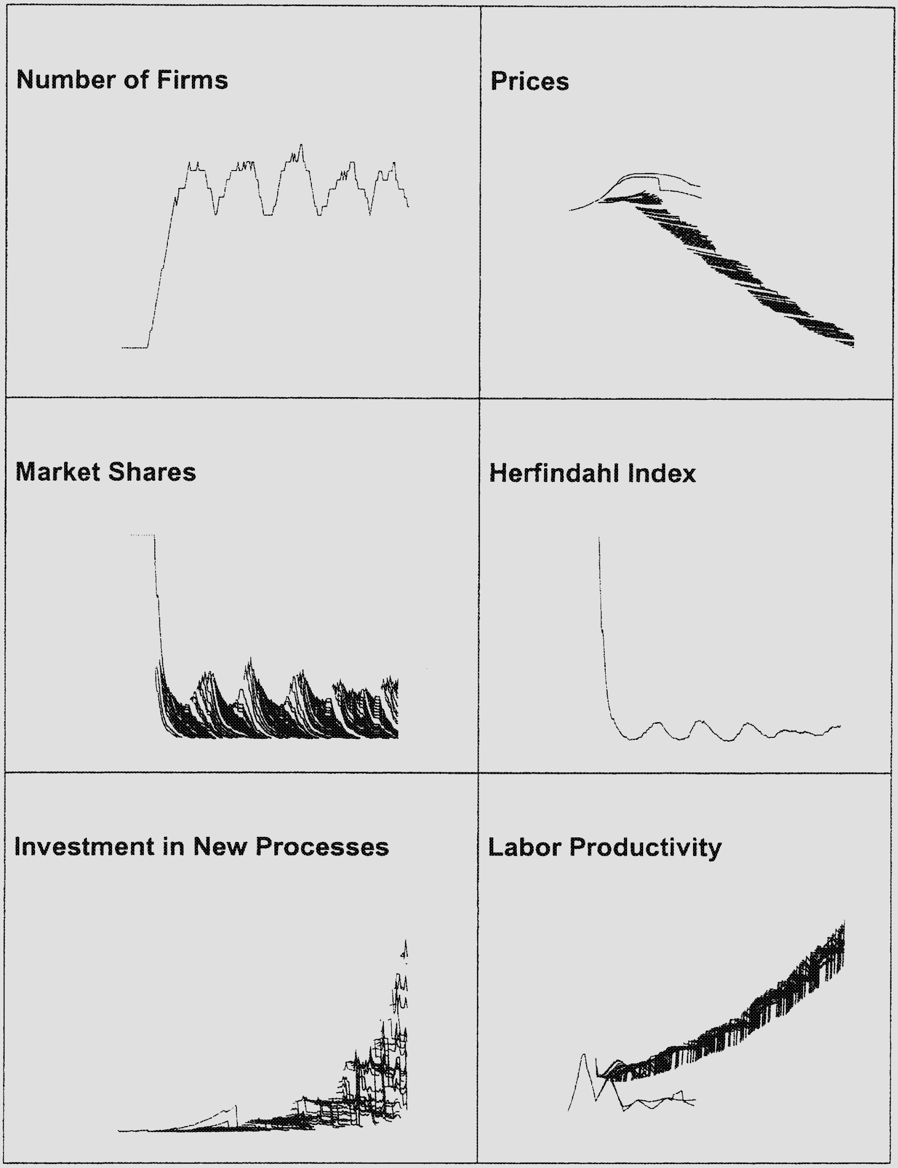


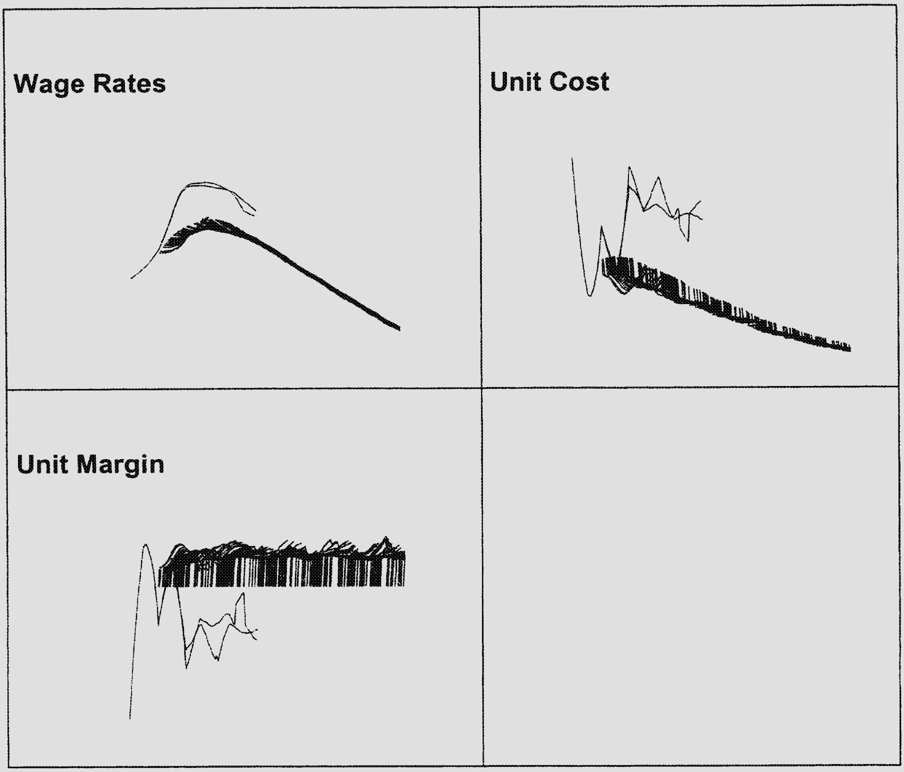

Unit Cost
Unit Margin

- **Simulation 2**

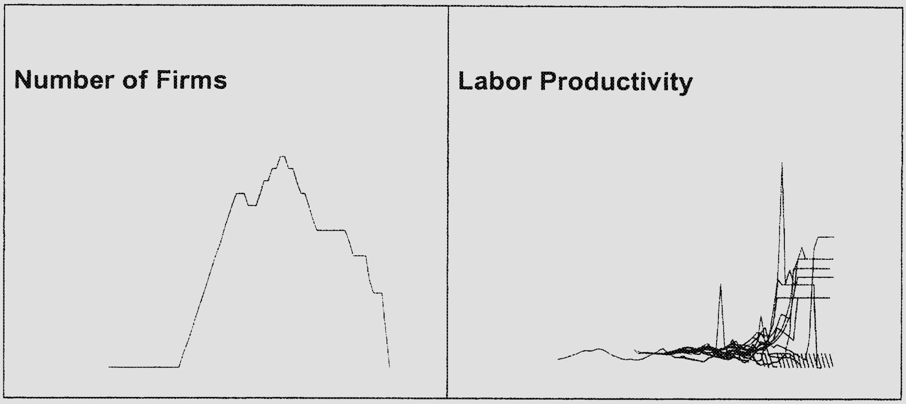


- **Simulation 3**

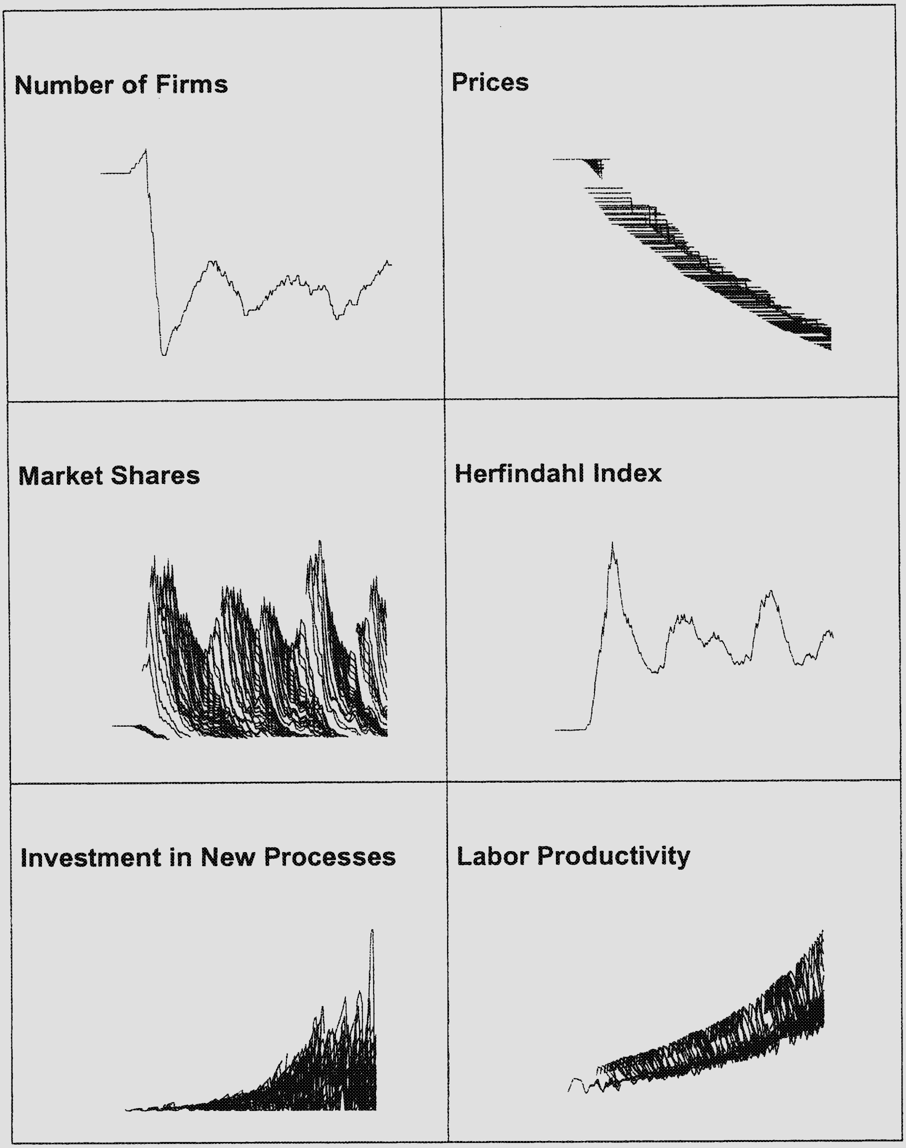


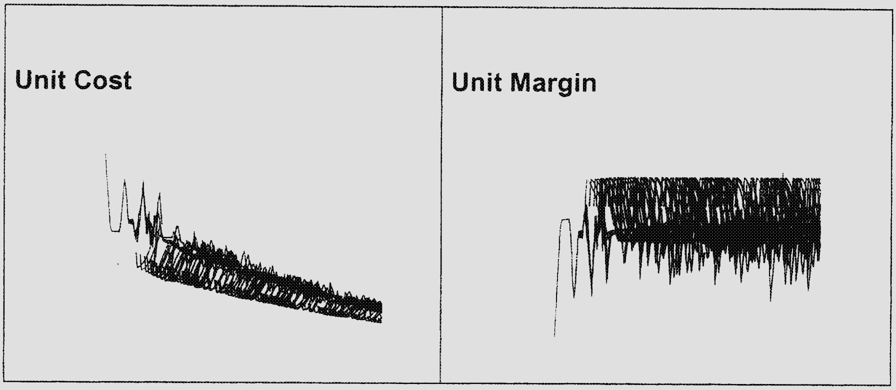

Wage Rates
Unit Cost
Unit Margin

- **Simulation 4**

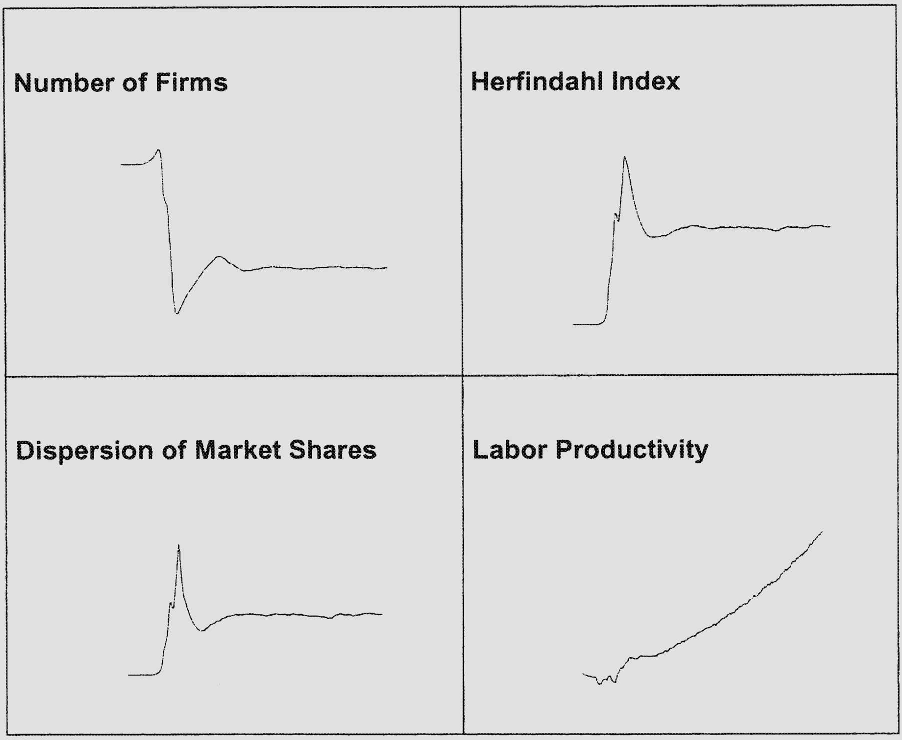


- **Simulation 5**

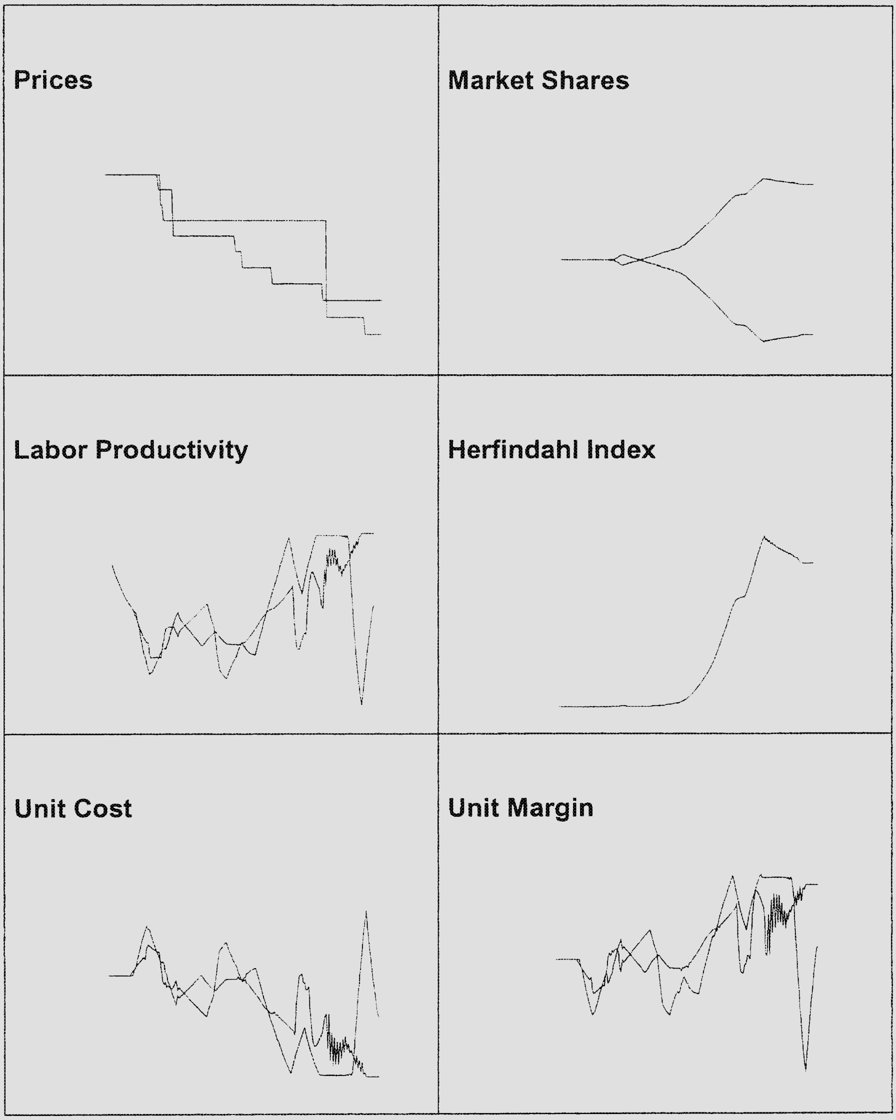



Unit Cost
Unit Margin

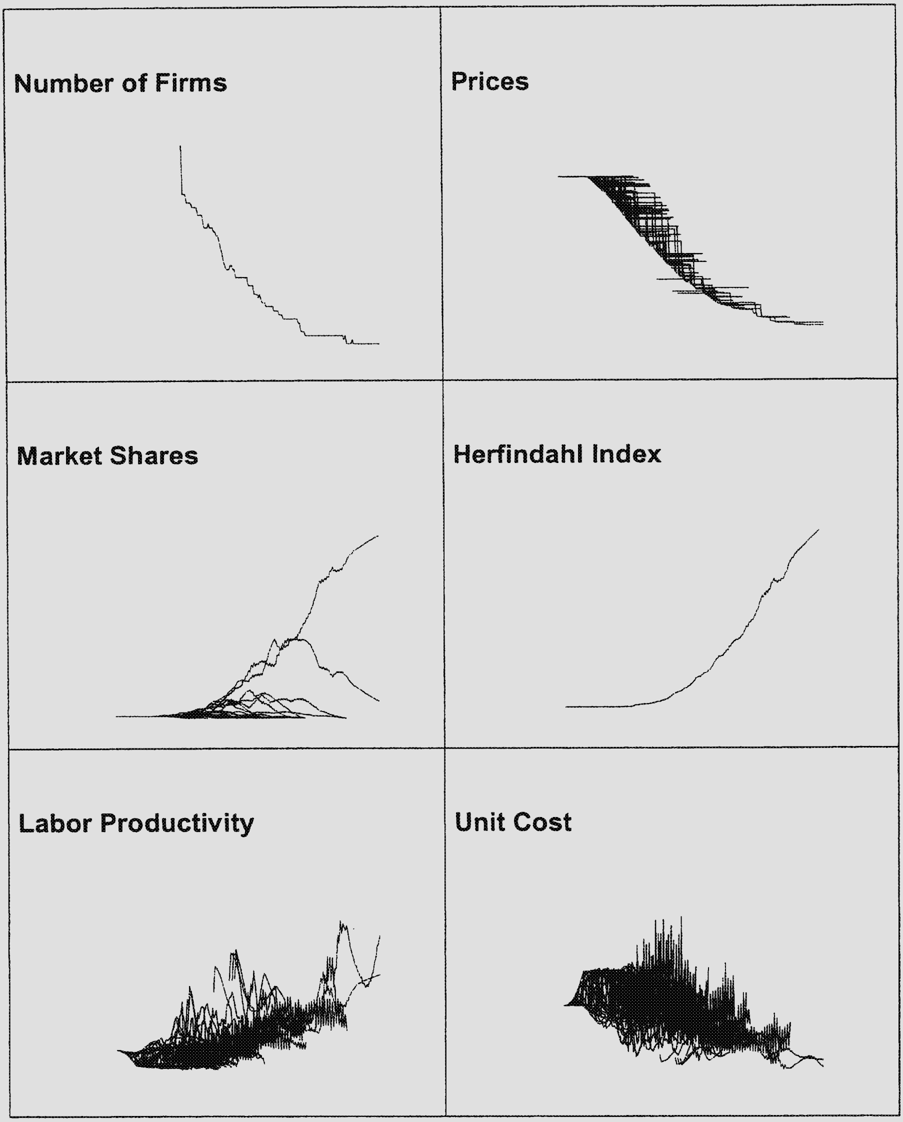
- **Simulation 6**



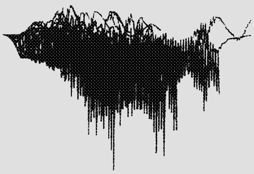
- **Simulation 7**



- Simulation 8



Unit Margin



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Part II

Econophysics: Empirical Analysis

Firms' Bankruptcy and Turnover in a Macroeconomy*

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1. Introduction

The so-called "rational expectations revolution" that has completely reshaped economic theory and general equilibrium theory in the last two decades has, incidentally, brought earlier ideas on the crucial importance of agents' knowledge, information and beliefs to the forefront forcing modern followers of those ideas to reconsider them far more deeply, systematically and rigorously (Arrow (1986), Hahn (1977, 1981)). It soon turned out that when agents act upon beliefs and engage in out-of-equilibrium learning, heterogeneity (of beliefs) and self-referentiality (of market outcomes)[1] may determine large sets of multiple equilibria, and of dynamic paths of the economy, which collapse onto the unique rational-expectations (RE) competitive general equilibrium only under a number of restrictive conditions[2].

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[1]A self-referential system is such that the actual value of a variable is a function of its expected value in the population, which is a function of the distribution of beliefs in the population. See e.g. Pesaran (1987) and Frydman (1983) for introductory treatment.

[2]See the seminal works by Frydman-Phelps (1983), Kirman (1987, 1992), Pesaran (1987), Bray-Kreps (1986), Marcet-Sargent (1989), Arthur (1992)

M. Gallegati et al. (eds.), *The Complex Dynamics of Economic Interaction*

The fact that expectations, let alone out-of-equilibrium beliefs and learning, may give rise to mistakes in decision-making[3], has however been much less considered, and the implications of this fact at the individual as well as systemic level even less investigated. Rational (in a broad sense) expectations are rooted in knowledge, and knowledge is an evolving-adaptive mental representation of the external environment (e.g. Lucas (1987), Arthur (1992), Holland (1975)). Trials and *errors* are an integral part of the evolutionary-adpative process that builds up our knowledge (Holland (1975)). Even in the most formal models of this process, from Bayesian to stochastic recursive ones (e.g. Bray-Kreps (1986), Marcet-Sargent (1989)), errors paly a crucial informative role in steering the process itself. Yet errors not only bring benefits but also costs. In particular, models of expectations formation in economics usually do not include the possibility that decision-makers may fail, that failures are generally costly, that they may happen to be fatal, and that the birth-death turnover of agents may change the environment structure. It is surprising that economic theorists have tended to overlook these facts since "paying for one's mistakes" is a building block of the capitalist way of living, of market ethics and organization, and of Darwinian evolutionary explanations of individual rationality and market efficiency (Alchian (1950), Friedman (1949)). Hence, the possibility of costly errors should have consequences on individual economic behaviour as well as on aggregate outcomes of individual decisions. Here we concentrate in particular on one extreme consequence of economic errors that nonetheless lies at the very core of business life: bankruptcy.

By bankruptcy we simply mean a firm's foreclosure and exit from the market (we speak of bankruptcy because firms in our model are indebted with a bank). Technically, this event may take different forms that are irrelevant for this paper's purposes. What is essential is that a firm may be forced to leave the market as a consequence of wrong decisions, where a firm is identified by its "software" (namely the expectations generation "programme"), not its "hardware". Although we model firms as decision makers that discount the probability of this event, our interest is not for the consequences of bankruptcy at the firm level (on which an enormous literature exists involving law, buisiness finance and business administration) but at the economy level, particularly in a general-equilibrium perspective, where the literature is instead scant (e.g. Hahn (1977), Greenwald-Stiglitz (1988, 1990, 1993), Hahn-Solow (1995)). How does an economic system work when the fact that the firms' population displays a certain birth-death rate is introduced?

We focus on two issues. First, at the micro-level, when forming expectations under uncertainty rational decision-makers (entrepreneurs) should discount the probability of making fatal mistakes that lead to bankruptcy. Second, at the macro-level, in any given time unit the population of agents in the economy consists of "learned" survivors from the past and "ignorant" newcomers that take over failed agents; hence we expect the aggregate outcome of the decisions at that point in

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[3]A fact quite clear even under the strong RE hypothesis, which states that *asymptotically* decisions are not wrong in a systematic manner but does not state that they are systematically right (Lucas (1981)).

time to differ from the one given by a uniform population, and we expect it to evolve over time in a way that differs from the one traced out by the asymptotic "free lunch" error processing *à la* Lucas-Sargent.

We address these problems by means of a model of the Greenwald-Stiglitz type (Greenwald-Stiglitz (1988, 1993)), with the following main features:

- heterogenous population of entrepreneurs (i.e. heteregenous beliefs about the relevant variable, the inflation rate)
- sequential decision making
- self-referentiality
- a positive probability of bankruptcy for firms at each point in time

At the beginning of each period t, entrepreneurs who want to run a firm should receive credit from a bank to pay for the wage bill. We straightforwardly assume a single central bank with a money and a credit department with the latter issuing standard debt contracts with the exclusion of any risk on the part of the bank. This type of debt contract is a crucial element in our model since it introduces a bankruptcy clause in case of insolvency, and hence the selction mechanism in the population of firms. This mechanism is not due to exogenous shocks like in Greenwald-Stiglitz's original models but it is "endogenous" in the sense that it is the direct consequence of entrepreneurs' heterogeneous beliefs on the price-generating machanism after observing a common external signal given by the rate of growth of money supply. These beliefs generate individual expected prices which we assume to be uniformly distributed. Individual expected prices are strictly private information. We identify a given generation of entrepreneurs by three population's parameters: the mean value, the dispersion and the tolerance level of their expected prices. Since individual expected prices are private information, the popolution's parameters are unknown to individual agents. Then we show that, at the market-clearing price in t+1, there will go bankrupt all the t-th generation's entrepreneurs whose expected inflation exceeds a threshold value which is a function of the three population's parameters. In other words, we might say that whether a firm fails or not in this economy *only* depends on *relative expected prices.*

As far as a single generation is concerned, we study:

- the inflation determination mechanism under firms' failure
- the ensuing relationship between the inflation rate and the rate of money supply in comparison with its "fundamental" value that would result under homogeneous rational expectations
- the relationship between the actual inflation rate and the populations' parameters, in particular the mean value of expected inflation
- the relationship between the population's parameters and the bankruptcy probability.

We then move towards the dynamics of this economy as bankrupt firms are driven out of the market and newcomers enter. Reshuffling the population in each period has two consequences: first, it preserves hetereogeneity, second, it changes the economy's structure for incumbent firms. Therefore, the process driving changes in the price level over time, for a given money growth rate, is fully determined by the population's dynamics and, under our assumptions, is totally hidden from individual agents' view. Successful beliefs (as well as the underlying generating "programmes") in one period may no longer be such in the subsequent period.

At the present stage of development of our research, we limit ourselves to the implications of self-referentiality under the population dynamics generated by exits and entries with no specific assumptions on learning. Newcomers that take over bankrupt firms are characterized by beliefs drawn from the same uniform distribution as the previous generation, whereas we substantially rule out incumbents' learning by assuming that all entrepreneurs in solvent firms do not revise the previous generation's successful belief. This only implies that previous generations' successful beliefs have a growing weight, and that the distribution of beliefs is no longer uniform, as the population evolves over time. We then study

- the dynamics and the asymptotic properties of this evolution mechanism
- its consequences on inflation determination and bankruptcy probability over time.
- and in particular whether in the long run bankruptcies tend to disappear or to settle down in a "structural" limit-level.

2. The Model Economy

We consider a sequential monetary economy of production and consumption. Time is introduced in discrete periods indexed by t. In each period t there exists a constant population of agents A which live for two periods, and a government G inclusive of a central bank B that live forever. Agents consume one single good which is distributed as endowment at birth and can also be produced by means of a technology requiring 1 period of production regardless of the scale with one single input (labour) with decreasing return.

2.1. Agents and institutions

- Each agent in the population, $a \in A$, has a lifetime horizon of two periods: one period of activity (t), and one period of retirement (t+1); at birth each agent receives a given quantity of the consumable good as endowment, which is fixed for all t.
- There are two classes in the population that differ in their endowments: the "poor", $a = i, i \in I \in A$, have an endowment just equal to the subsistance level, $x_i = \underline{x}$; the "rich", $a = j, j \in J \in A$, have a quantity of the consumable good that exceeds $x_j > \underline{x}$. For simplicity and without loss of generality we assume that the measure of the two classes of agents, I and J respectively, is equal and constant over time.
- All agents are risk neutral.

• All transactions in the economy should take place against money according to the so-called "Clower rule"[4].

Endowments predetermine the agents' activity choices. The poor, if they wish to consume in both periods of life, can only be workers, i.e. they have to sell all their labour force in the labour market, earn a wage, and transfer it to the second period for consumption[5]. The rich may choose to be entrepreneurs, i.e. to run a firm by transforming the excess of their endowment over subsistance in the first period into a production means in the way that will be explained below. When in the next period the entrepreneur retires, the firm is either taken over by a newborn entrepreneur or is foreclosed in the way that will be explained below. The incentive for the rich to become entrepreneurs is given by the prospect of adding their firm's second-period profit to their total resources. Given risk-neutrality, any positive expected profit is sufficient for the rich to choose to be entrepreneurs. This choice is also Pareto improving since it allows the poor to become workers and to consume in the second period.

Pareto-improving transactions between the two classes in the economy require that the labour market opens at each t when newborn workers wish to exchange labour for wage with newborn entrepreneurs, and the output market opens at $t+1$ when the retired workers wish to exchange their wage earnings for consumption with firms. Given Clower rule, workers want to be paid in money at t and firms want to be paid in money at $t+1$. Consequently, firms at t need to collect the money equivalent of wages. This operation is made possible by opening the credit market at each t[6]. The central bank is the sole banking institution in the economy and performs three functions:

- it issues fiat money according to the Clower rule
- it acts as commercial bank, lending to entrepreneurs and offering deposit services to workers, at the terms that will be specified below
- it finances public expenditure.

2.2. The economy at work

The working of our sequential economy is described in figure 1 and explained below.

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[4]We assume this rule as an insititutional fact which is typical of monetary economies, and which we do not wish to explain here. We are instead interested in investigating the *consequences* of this fact as concerns the working of the economy.

[5]For simplicity we do not model the possible choices of workers concerning work and leisure and the time distribution of consumption of wage earnings. Modelling these choices would only add further parameters which are not important in our context.

[6]This is the same function attributed to banks in Kiyotaki and Moore's (1997) economy.

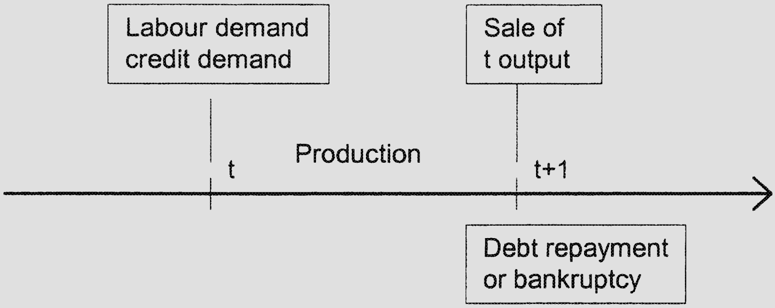

Fig. 1.

The sequence of decisions in the economy results as follows.

In t: a) each firm j plans the output level $y(t)_{jt+1}$
b) it employs the necessary labour input n_{jt}, at the market nominal wage rate w_t, and borrows the resulting wage bill $w_t n_{jt}$ at the nominal gross rate $(1 + r_t) \equiv R_t$;
c) each worker i offers 1 unit of labour in fixed amount at the nominal rate w_t, works and consumes his/her initial endowment $\underline{x}$, and saves income w_t at the rate R_t for consumption in t+1
d) the bank lends to each entrepreneur the wage bill $w_t n_{jt}$ and accepts from each worker the deposit w_t, both at the rate R_t

In t+1: a) each retired worker consumes his/her savings $w_t R_t$
b) each firm sells output at the market price p_{t+1} and may be solvent or insolvent with the bank (see below).

As far as the bank-firm relationship is concerned, we straightforwardly assume a standard debt contract with the exclusion of any risk on the part of the bank[7]. This type of debt contract is a crucial element in our picture since it introduces a bankruptcy clause in case of insolvency, and hence the selction mechanism in the population of firms. Accordingly, each entrepreneur can obtain in t a loan of size $B_{jt} = w_t n_{jt}$ from the bank, for 1 period, at the rate R_t provided that:

- the entrepreneur's excess endowment, $x_j - \underline{x}$, can be given as collateral
- the entrepreneur is commited to the following repayment scheme in t+1:
  a) if $p_{t+1} y(t)_{jt+1} \geq B_{jt} R_t$, the firm is solvent, pays $B_{jt} R_t$ to the bank, and

[7]The standard debt contract (see e.g. Freixas-Rochet (1998)) is now a workhorse in bank-firm models like the present one, though we do not prove that this kind of contract is optimal in our setup. In fact, what we simply need for our purposes is any financial arrangement which shifts the bankruptcy risk onto the firm, since this is the way through which forecast errors *beyond a critical magnitude* produce selection in the population of firms. For the same reason, we also wish to exclude that the bank bears any risk.

the entrepreneur reappropriates his/her collateralized endowment

b) if $p_{t+1}y(t)_{jt+1} < B_{jt}R_t$, the firm is insolvent, the bank seizes the firm's revenue and possibly the collateral up to

$$p_{t+1}(x_{bjt+1} + y(t)_{jt+1}) = B_{jt}R_t + p_{t+1}b$$

where b are fixed bankruptcy costs in real terms, and the firm is declared bankrupt and exits from the market.[8]

Note that the above repayment scheme implies that the entrepreneur faces two possibile lifetime consumption possibilities:

$\pi_{jt+1} + x_j$ in case of solvency

$x_j - x_{bjt+1}$ in case of bankruptcy

where π_{jt+1} is the net real profit given by

$$\pi_{jt+1} = y(t)_{jt+1} - B_{jt}R_t/p_{t+1}$$

If we take the real value of debt as of t, B_{jt}/p_t, substitute the expression of B_{jt}, define $\underline{w}_t \equiv w_t/p_t$ as the real wage rate, and $q_t \equiv p_{t+1}/p_t$ as the growth factor of prices (inflation rate for short), the net real profit can be re-written as follows:

$$\pi_{jt+1} = y(t)_{jt+1} - \underline{w}_t n_{jt} R_t/q_{t+1} \tag{2.1}$$

where $\underline{w}_t n_{jt}$ is the real wage bill and R_t/q_{t+1} the (gross) real interest rate.

3. Statics. The Certainty Case

We first study the competitive general equilibrium solution of the above model economy in the certainty case. Given the sequence structure of decisions, certainty requires perfect foresight. In this case, the solution is trivial, amounting to a simple exemplification of Say's Law with money, and we only give it as a reference point.

Each entrepreneur's objective is to maximize the net real profit[9]

$$\max \pi_{jt+1} = y(t)_{jt+1} - \underline{w}_t n_{jt} R_t/q_{t+1} \tag{3.1}$$

given the production function

$$y(t)_{jt+1} = n^{\alpha}_{jt}$$

with $\alpha \in [0,1)$. The choice variable is n_{jt} and the optimal labour input is given by the function

$$n_{jt} = (\alpha q_{t+1}/\underline{w}_t R_t)^{1/1\text{-}\alpha} \tag{3.2}$$

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[8]Note that, as a consequence, it must be that $x_j - \underline{x} \geq x_{bjt+1}$, which we assume is always satisfied.

[9]Which, in our setup, is equivalent to maximizing lifetime consumption possibilities.

Since each worker offers 1 unit of workforce inelastically, n_{jt} is also the size of employment by firm j, and is equal for all j. Aggregate labour demand is therefore $N_t = Jn_{jt}$. Total employment cannot exceed the measure of the workers' class, I. Since $I = J$, the labour market determines the real wage rate $\underline{w}_t$ at full employment in such a way that

$$(\alpha q_{t+1}/\underline{w}_t R_t)^{1/1\text{-}\alpha} = 1$$

$$\underline{w}^*_t = \alpha q_{t+1}/R_t \tag{3.3}$$

which also implies $n_t = 1$, $y(t)_{t+1} = 1$ for all j.

The bank pegs the nominal interest rate, and hence R_t, lends $I\underline{w}^*_t p_t$ to entrepreneurs and receives the same amount as workers' deposits; hence it always achieves balance-sheet equilibrium.

In t+1, aggregate full-employment output on sale is $Y(t)_{t+1} = J$. Aggregate consumption consists of retired workers' and retired entrepreneurs' consumption.

A worker's consumption is given by his/her real saving from previous period,

$$c(t)_{it+1} = \underline{w}^*_t p_t R_t / p_{t+1}$$
$$= \underline{w}^*_t R_t / q_{t+1}$$

which is therefore equal for all workers. Hence, substituting the value for $\underline{w}^*_t$, and recalling that $I = J$, workers' aggregate consumption is

$$C(t)_{it+1} = \alpha J \tag{3.4}$$

In force of perfect foresight, entrepreneurs realize planned maximum real profits after repaying debt, and recalling that $I = J$, their aggregate (market) consumption is

$$C(t)_{jt+1} = Y(t)_{t+1} - J\underline{w}^*_t R_t / q_{t+1} = (1 - \alpha)J \tag{3.5}$$

It is immediate to notice that, due to the transfer of labour income to t+1 in the form of savings, Say's Law holds in t+1. This, as is well-known, implies that the output market always clears at any price level.

To determine p_{t+1} a monetary equation is introduced on the grounds that, under the Clower Rule, the total (outside) money stock available in each t must exactly meet the demand for money, which in this case amounts to the money value of output with unit velocity, i.e.:

$$M_{t+1} = Y(t)_{t+1} p_{t+1} = J p_{t+1} \tag{3.6}$$

Of course, classical (super)neutrality holds. In fact, let the central bank create money at a gross rate $\omega_{t+1} = M_{t+1}/M_t$. Since in equilibrium $M_t = Jp_t$, dividing both sides of equation (3.6) by M_t, we obtain

$$\omega_{t+1} = p_{t+1}/p_t = q_{t+1} \tag{3.7}$$

that is to say, the price level grows at the growth factor of money.

4. Statics. Uncertainty and Failures

In this section we introduce uncertainty. The only relevant uncertainty in the model of section 3 is entrepreneurs' uncertainty at time t over the inflation rate at time $t+1$, which affects profit maximization as given by problem (3.1).

In this problem, q_{t+1} has to be replaced by an expectation that is uncertain. As to expectation formation, we assume bounded rationality (in Pesaran's sense (1987)), that is,

(A1) Each agent knows the variable's generation process but does not know its exact specification.

A fundamental reason behind this assumption is that the "true" generating process is self-referential (i.e. it may differ from (3.7)) as will become clear in due course.

4.1. Heterogeneous individual beliefs

We translate the previous assumption into our model by associating to each entrepreneur born at t an individual specification of equation (3.7) in the following "conjectural" form

$$q^e_{jt+1} = \omega_{t+1} u_{jt} \tag{4.1}$$

where ω_{t+1} is known with certainty (e.g. is announced by the central bank in advance), and u_{jt} represents the individual belief about the relationship between the growth factor of money and the inflation rate. Note that the "theoretical value" of u_{jt} is 1.

The following additional assumptions complete our characterization of entrepreneurs' individual beliefs.

(A2) Individual beliefs u_{jt} in each period's newborn population of entrepreneurs are a continuous random variable U_t uniformly distributed in the population, with non-negative support $u_{jt} \in [u^L_t, u^H_t]$, and density

$$f(u_{jt}) = \begin{cases} (u_t^H - u_t^L)^{-1} & u \in [u_t^L, u_t^H] \\ 0 & \text{elsewhere} \end{cases} \tag{4.2}$$

We restrict the distribution U_t to a non-negative support because we want individual beliefs to be "theoretically informed" (a specification often present in the bounded-rationality literature, see e.g. Frydman-Phelps (1983)), and the theoretical relationship between money growth and inflation in our economy is non-negative (we instead allow the support to include 0, i.e. the belief that no relationship exists).

From (4.1) individual inflation expectations q^e_{jt+1} also follow a continuous uniform distribution, with support between the lower bound $q^{eL}_{t+1} = \omega_{t+1} u^L_t$ and the upper bound $q^{eH}_{t+1} = \omega_{t+1} u^H_t$. The mathematical expected inflation rate in the economy is therefore

$$\begin{aligned} q^e{}_{t+1} &= E_t(q^e{}_{jt+1}) \\ &= \omega_{t+1} E_t(u_{jt}) \\ &= \omega_{t+1}(u^H{}_t + u^L{}_t)/2 \end{aligned}$$

(A3) Each entrepreneur, in turn, holds his/her inflation expectation with a "tolerance interval" around $q^e{}_{jt+1}$ of equal module $|\delta_t|$, i.e.:

$$\begin{aligned} q^{eL}{}_{jt+1} &= q^e{}_{jt+1} - \delta_t \\ q^{eH}{}_{jt+1} &= q^e{}_{jt+1} + \delta_t \end{aligned}$$

(A4) Individual beliefs u_{jt}, and hence individual inflation expectations $q^e{}_{jt+1}$, are private non-observable information.

Note that, as a consequence, each individual entrepreneur does not know the distribution of inflation expectations in the population, and hence the population's expected inflation rate $q^e{}_{t+1}$ is unknown too.

Now we can completely identify a population of firms in t with its own three parameters:

- δ_t, individual tolerance
- $\mu_t = (u^H{}_t + u^L{}_t)/2$, expected value of beliefs u_{jt}
- $\Delta_t = u^H{}_t - u^L{}_t$, dispersion of beliefs u_{jt}

4.2. Individual decisions

Uncertainty, as defined above, modifies the entrepreneur's decision problem. First, note that so far we have introduced uncertainty in a "subjective" form since each entrepreneur holds his/her inflation expectation within a range of values (assumption (A3)). Rationally, this uncertainty has to be associated with a positive probability of bankruptcy. In fact, assumption (A3) implies that the entrepreneur expects that the actual inflation rate q_{t+1} may turn out to be different from his/her individual expectation $q^e{}_{jt+1}$, and it may happen to be too low for the firm to be solvent with the bank.

We first give a measure of this "subjective" bankruptcy probability implied by assumption (A3). From the debt contract described in section 2, it follows that a firm is insolvent when its net real profit is negative or

$$q_{t+1} < \underline{w}_t n_{jt} R_t / y(t)_{jt+1} \equiv v_{jt} \tag{4.3}$$

i.e. if the actual inflation rate is lower than the firm's real debt-output ratio, that we define v_{jt}.

Now let us define the bankruptcy probability of the firm as

$$\phi_{jt} \equiv \text{Prob}(q_{t+1} \leq v_{jt}) \tag{4.4}$$

Since assumption (A3) describes the distribution of the entrepreneur's inflation expectations, the measure of (4.4) implied by (A3) is:

$$\begin{aligned} F_{jt}(v_{jt}) &= (v_{jt} - q^{eL}{}_{jt+1})(q^{eH}{}_{jt+1} - q^{eL}{}_{jt+1})^{-1} \\ &= 1/2 - (q^e{}_{jt+1} - v_{jt})/2\delta \end{aligned} \tag{4.5}$$

Therefore, the subjective bankruptcy probability of each firm,

- increases with v_{jt} (a high real debt-output ratio makes insolvency more likely)
- decreases with q^e_{jt+1} (a higher expected inflation rate makes insolvency less likely)

We now examine the optimal output and employment decision. Under debt contract, the firm's problem is to choose n_{jt} so as to

$$\max \pi^e_{jt+1} = y(t)_{jt+1} - \underline{w}_t n_{jt} R_t / q^e_{jt+1} - F_{jt}(v_{jt}) b$$

The f.o.c. is the soluton of the following equation

$$\alpha n_{jt}^{\alpha-1} - (1-\alpha)(b\underline{w}_t R_t / 2\delta) n_{jt}^{-\alpha} - \underline{w}_t R_t / q^e_{jt+1} = 0$$

In order to obtain a closed-form solution we impose $\alpha = 0.5$. The optimal labour input results:

$$n^*_{jt} = [(q^e_{jt+1}/2\underline{w}_t R_t) - b/4\delta_t]^2 \tag{4.6}$$

The first addendum is the same as in the case of certainty. The second one is the marginal expected bankruptcy cost $b/4\delta_t$ (the increase in the expected bankruptcy cost due to an increase in planned output, employment and debt). This is obviously increasing in the direct bankrupty cost b while is decreasing in δ_t up to the "certainty equivalent" value of 0 as $\delta_t \to \infty$. We interpret this as a measure of the "degree of tolerance" of forecast errors. For instance, if δ_t is large the entrepreneur operates under less strict forecast precision, and his/her labour demand is more buoyant[10]. Therefore, we conclude that

- *uncertainty reduces each firm's labour demand proportionally to the marginal expected bankruptcy cost* $b/4\delta_t$
- *each firm's labour demand differs by the individual expected inflation rate* q^e_{jt+1}.

4.3. Aggregate results

We have seen above that, owing to different individual expected inflation rates, labour demand is now different across firms. For each firm to employ one unit of labour force so as to ensure full employment as in section 3, each firm should be willing to pay the individual real wage rate

$$\underline{w}^*_{jt} = \beta_t q^e_{jt+1}/2R_t \tag{4.7}$$

so that the mathematical expectation of the real wage rate in the economy is

$$E(\underline{w}^*_{jt}) = \beta_t \mu_t / 2R_t \tag{4.8}$$

where

$$\beta_t = (1 + b/4\delta_t)^{-1} \in]0, 1[$$

---

[10]Think of δ_t as the diameter of the target in a rifle contest. The economic meaning of δ_t is analogous to the degree of risk aversion, though we derive it as an attitude towards errors rather than as a property of the utility function.

Hence, *ceteris paribus*, uncertainty modifies $E_t(\underline{w}^*_{jt})$ relative to the market value $\underline{w}^*_t$ under certainty by the factor β_t. This factor captures the effect of the marginal expected bankruptcy cost. Since $b/4\delta_t > 0$, generally $\beta_t < 1$, and hence uncertainty reduces the average reale wage rate. This effect is larger (smaller) as $b/4\delta_t$ increases (decreases).

In t+1, aggregate output on sale is again $Y(t)_{t+1} = J$. However, as is intuitive, bankruptcies break Say's Law since bankrupt entrepreneurs cannot participate in market consumption. In Keynesian words, bankruptcies operate as an endogenous source of effective demand deficiency; since output is fixed by previous period's decisions, the price level should adjust to clear the output market.

Let us first consider retired workers' consumption. This is given by their real savings. Suppose that the announced increase in money supply is realized in the form of a government per-capita monetary transfer to retired workers m_{it+1} entirely financed by printing money, so that $M_{t+1} = M_t + Im_{it+1} = \omega_{t+1}M_t$, or $Im_{it+1} = M_t(\omega_{t+1} - 1)$. Consequently, the retired workers' real savings in t+1 are equal to the real value of previous period's deposits and the money transfers, and therefore

$$C(t)_{it+1} = [p_t E(\underline{w}^*_{jt})R_t I + M_t(\omega_{t+1} - 1)]/p_{t+1} \tag{4.9}$$

Since $M_t = Y(t\text{-}1)_t p_t$, or $M_t = Ip_t$, and considering the expression of $E(\underline{w}^*_{jt})$ (4.9), we can also write

$$C(t)_{it+1} = I(\mu_t\beta_t/2 + \omega_{t+1} - 1)/q_{t+1}$$

As to retired entrepreneurs, since some firms may be insolvent, and the corresponding share of entrepreneurs has zero income, the retired entrepreneurs' consumption is limited to *aggregate positive real profits*. In order to compute them, let us recall that a firm that operates at the profit-maximizing real debt-output ratio given by

$$\begin{aligned} v^*_{jt} &= \underline{w}^*_{jt}n^*_{jt}R_t/y^*(t)_{jt+1} \\ &= q^e_{jt+1}\beta_t/2 \end{aligned}$$

is solvent if

$$q_{t+1} \geq v^*_{jt}$$

i.e. if

$$q^e_{jt+1} \leq 2q_{t+1}/\beta_t \equiv \hat{q}_{t+1} \tag{4.10}$$

In words, a firm turns out to be solvent in t+1 if its expected inflation rate in t was no greater that the threshold level $q_{t+1}2/\beta_t$, that we define $\hat{q}_{t+1}$. Conversely, in t+1 there fail all firms that had "too optimistic" inflation expectations in t exceeding $\hat{q}_{t+1}$. Note the important points that i) $\hat{q}_{t+1}$ is the same for all firms, and ii) it cannot be known in advance given our assumptions. This is an important preliminary result on which we shall return later. Now we proceed with the computation of aggregate positive real profits.

Aggregate positive real profits Π_{t+1} are the profits of all firms with $q^e_{jt+1} \in [q^{eL}_{t+1}, \hat{q}_{t+1}]$. From (4.1) we can write the following equalities:

$$q^{eL}_{t+1} = \omega_{t+1}u^{L}_{t}$$
$$q^{e}_{jt+1} = \omega_{t+1}u_{jt}$$
$$\hat{q}_{t+1} = \omega_{t+1}\hat{u}_{t+1}$$

Using the definition of solvency, the definition of v^*_{jt}, and these equalities, we can express the aggregate positive real profits in terms of the primitives u_{jt} and therefore

$$C(t)_{jt+1} = J \int_{u_t^L}^{\hat{u}_t} (q_{t+1} - \omega_{t+1}u_{jt}\beta_t/2)(u_t^H - u_t^L)^{-1} du_{jt} \tag{4.11}$$

We can now compute the equilibrium price level at $t+1$, which, given $I = J$, must satisfy,

$$C(t)_{it+1} + C(t)_{jt+1} = J \tag{4.12}$$

The result is a quadratic function in q_{t+1}:

$$\gamma_0 q^2_{t+1} + \gamma_1\omega_{t+1}q_{t+1} + \gamma_2\omega^2_{t+1} = 0 \tag{4.13}$$

In terms of the population's parameters $\{\beta_t, \mu_t, \Delta_t\}$, the coefficients of equation (4.13) result:

$$\gamma_o = -\frac{2\Delta_t + 1}{\beta_t\Delta_t}$$

$$\gamma_1 = -(1 + \mu_t - \Delta_t/2)$$

$$\gamma_2 = \frac{\beta_t(\mu_t + \Delta_t/2)^2}{4\Delta_t} + 1$$

4.4. Determination of the inflation rate

Equation (4.13) has two roots of generic form:

$$q^*_{t+1} = k_t\omega_{t+1} \tag{4.14}$$

where

$$k_t = \frac{1}{2\gamma_o}\left[-\gamma_1 \pm \sqrt{\gamma_1^2 - 4\gamma_o\gamma_2}\right] \tag{4.15}$$

Since $\gamma^2_1 - 4\gamma_0\gamma_2 > 0$, equation (4.13) has two real roots, and since $4\gamma_0\gamma_2 < 0$ and $\gamma_1 < 0$, one root is positive and one is negative. In principle both roots are admissible, the negative one implying that a positive money growth rate $\omega_{t+1} > 0$ generates a certain rate of *deflation* $q^*_{t+1} < 0$. If this result may sound in contrast with general economic principles, it should be reminded that k_t can be thought of as the "structural" monetary multiplier, where the structure is given by the relationships between the population's parameters. Indeed, (4.14) has the same form as the conjectural equation used by entrepreneurs (4.1), and it correponds to the "theoretical" model only when $k_t = 1$. However, the "true" (or better, the structural) generating process is (4.14), not (4.1). Nonetheless, since we have restricted the distribution of beliefs to a non-negative support as explained in section 4, assumption (A2), in order to guarantee at least qualitative consistency between beliefs and the actual value of the structural parameter we will also restrict our analysis to the positive root

$$k_t = -\frac{1}{2\gamma_o}\left[\gamma_1 + \sqrt{\gamma_1^2 - 4\gamma_o\gamma_2}\right] \tag{4.16}$$

Therefore, we can put forward our first proposition

(P1) *The inflation generating process in the economy is self-referential, in that the relevant structural relationship is a function of the parameters characterizing the beliefs of the population about the structural relationship itself.*

To gauge the relationship between the population's parameters and k_t, let us first consider the average belief in the economy μ_t. The relationship between k_t and μ_t is the first indicator of the effect that the beliefs have on the structure of the economy. Three important properties are worth noting:

(i) for any $\mu_t > u^L{}_t$ (i.e. $\Delta_t > 0$), we have $k_t > 0$, which means that any heterogeneous population with individual beliefs as well as average belief of positive sign is consistent with an actual structural parameter of positive sign

(ii) for any $\mu_t \in [u^L{}_t, u^H{}_t]$, $\mu_t > u^L{}_t$, k_t is monotocally increasing in μ_t, which means that beliefs tend to be self-fulfilling[11]

(iii) β_t operates as a shifting parameter of the function k_t in such a way that, cet. par., a weaker uncertainty effect (higher β_t) increases k_t and vice versa

Figure 2 plots k_t as a function of μ_t, taking the lowest bound of the distribution of beliefs $u^L = 0$, and for the two values of $\beta_t = 0.2$ (strong uncertainty effect) and $\beta_t = 0.8$ (weak uncertainty effect). It exemplifies the above properties.

[11] We have verified this property using the *Symbolic Math Toolbox* of MATLAB.

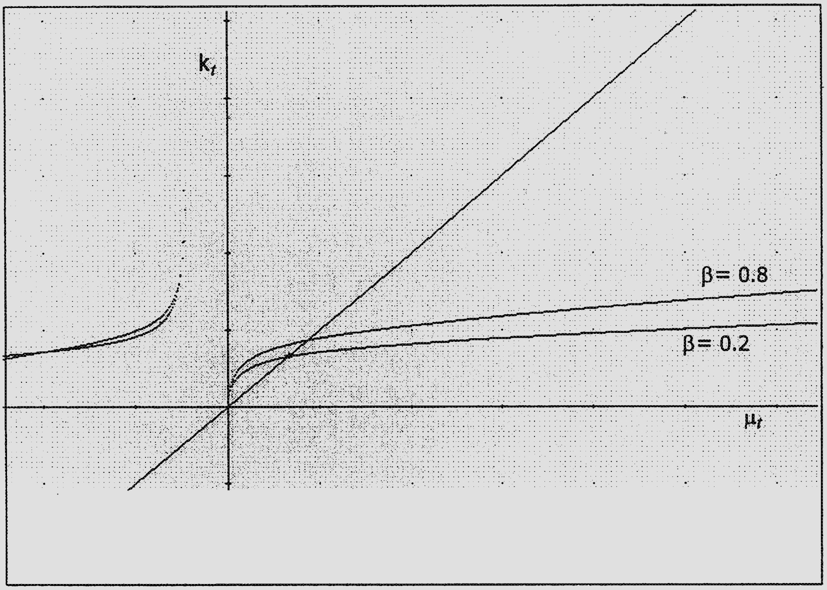

Fig. 2. ($u^{\mathrm{L}} = 0$)

The economic reason of these results lies in the effect that beliefs and the marginal expected bankruptcy cost have on entrepreneurs' decisions. Since beliefs are bounded from below (they cannot fall below $u^{\mathrm{L}}_t = 0$), an increase in μ_t is the result of a greater upper bound u^{H}_t, that is to say a larger tail of high-inflation forecasters. These are prone to demand more labour and pay higher wages which in turn will feed higher demand of retired workers and raise the market-clearing inflation rate in $t+1$. The entrepreneurs' willingness to pay a larger wage bill implies that borrowing increases: this is another way, a way that looks at the increase in "inside" money, to explain the upward pressure on the price level. Likewise, a fall in the marginal expected cost of bankruptcy, i.e. an increase in β_t, affects all entrepreneurs boosting their labour demand and inducing them to pay higher wages with the same previous enhacing effect on the inflation-generating mechanism.

4.5. Fixed points

A critical issue in all self-referential models is the existence of fixed points in the map from the beliefs about a variable to the actual value of that variable. This problem is important for two reasons which relate to the notion of RE. The first is that if such a fixed point exists we may then check wheteher it can act as an attractor of beliefs under some law of motion of beliefs themselves. The second is that, *in a self-referential system*, such a fixed point is a necessary but not sufficient condition for the standard or "*strong*" notion of RE (i.e. the case whereby beliefs coincide with the "theoretical" value of the variable). In other words, we need two conditions for the "strong" RE hypothesis to hold:

(i) $k_t(\mu_t) = \mu_t$

(ii) $k_t = 1$

The function $k_t(\mu_t)$ is determined by two primitive population's parameters ($u^L{}_t$, β_t). Explorations in the space of these parameters reveal that in our economy condition (i) holds in a limited domain of values. In that domain, however, condition (ii) may not hold. An instance is provided by the previous case with $u^L{}_t = 0$: figure 2 shows that a fixed point exists but at a value lower than 1[12]. Numerically, a population born at time t characterized by the average expectation $q^e{}_{t+1} = 0.189\,\omega_{t+1}$, comprised between 0 and $0.378\omega_{t+1}$, and with $\beta_t = 0.2$ will in fact generate an inflation rate $q_{t+1} = 0.189\omega_{t+1}$, which is lower than it would be under "strong" RE, i.e. $q_{t+1} = \omega_{t+1}$.

4.6. Bankruptcies

A key feature of our model is that firms may go bankrupt. This event occurs because an entrepreneur in t may have an individual inflation expectation "too high", that is to say his/her $q^e{}_{jt+1}$ exceeds the threshold value given by $2q_{t+1}/\beta_t$. As already remarked above, this value cannot be known in advance because it depends on what the actual inflation rate q_{t+1} will be. But as we have seen, q_{t+1} is in turn a function of the population's parameters, that is to say the bankruptcy mechanism in the economy is "endogenous" since it arises from the beliefs of entrepreneurs u_{jt} about the inflation generating process, and from their self-referentiality effect. In other words, we might say that whether a firm fails or not in this economy *only* depends on *relative expected prices*.

In order to compute the share of firms that fail in each period, let us recall that the insolvency condition

$$q^e{}_{jt+1} > 2q_{t+1}/\beta$$

can be re-written in terms of the primitives u_{jt},

$$u_{jt}\omega_{t+1} > 2k_t\omega_{t+1}/\beta$$

$$u_{jt} > 2k_t/\beta \equiv \hat{u}_{t+1} \tag{4.17}$$

Consequently, given the distribution of beliefs as defined previoulsy, the share of bankruptcies in the population is:

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[12]Note that $u^L{}_t = \mu_t - \Delta_t/2$ and $\mu_t > u^L{}_t$. Under these constraints, simulations of the function $k_t(\mu_t)$, for $u^L{}_t > 0$, show no fixed points in the range of values of $\beta_t \in [0.2, 0.8]$. that $\mu_t = 1$, requires $u^L{}_t = 1 - \Delta_t/2$, which is less than 1. Values of $u^L{}_t$ between 0 and 1 do not change this result.

$$\phi_{t+1} = 1 - F(\hat{u}_{t+1}) = 1 - (\hat{u}_{t+1} - u^{L}_{t})/\Delta_t \qquad (4.18)$$

i.e. a non-linear function of the population's parameters. We are interested in two information: first, under what conditions failures occur ($\phi_{t+1} > 0$), second the effects of the population parameters on ϕ_{t+1}.

As to the first question, for failures to occur the population should display a critical combination of parameters. Intuitively, failures are more likely to occur as a result of a combination of high average μ_t (and large dispersion Δ_t) of conjuctures, and/or weak uncertainty effect (high β_t). Figure 3 portrays the function $\phi_{t+1}(\mu_t)$ in the usual benchmark case with $u^{L}_{t} = 0$, and for $\beta_t = \{0.2, 0.8\}$.

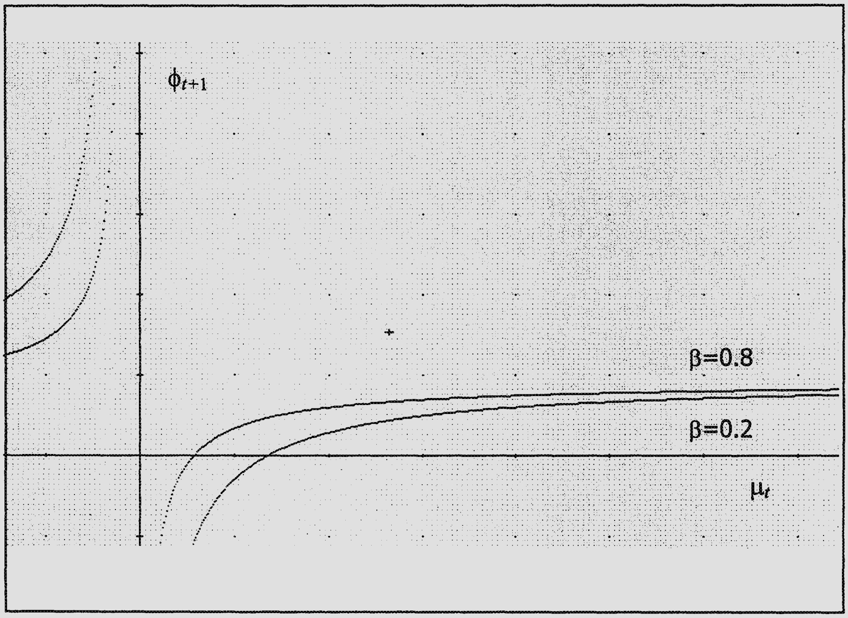

Fig. 3. ($u^{L}_{t} = 0$)

We observe that ($\phi_{t+1} > 0$) if ($\mu_t > 0.58$, $\beta_t = 0.8$), ($\mu_t > 1.36$, $\beta_t = 0.2$).

The factors mentioned above are consistent with the bankruptcy mechanism in our economy: each of them, directly or indirectly, implies a larger share of firms on the high-inflation tail of the population's forecasts, and high-inflation forecasters face a greater probability to trespass the bankruptcy threshold. It is worth stressing that, on this front, two opposite economic forces are at work. On the one side, higher μ_t (and/or high β_t) means greater planned output, more borrowing and hence more bankruptcy risk on each firm. On the other side, as we know, higher expected inflation leads to higher actual inflation, which is beneficial to debtors for it raises $\hat{u}_{t+1}$ and makes it shrink the tail of firms bound to insolvency. The monotonic positive relationship between ϕ_{t+1} and μ_t indicates that as μ_t increases the first force prevails on the second, so that the share of bankruptcies eventually grows. This examplifies a pattern where "inside" inflationary conditions are associated with greater bankruptcy risk and actual bankruptcies.

Finally, it is also worth stressing that the bankruptcy rate *is independent of the (anticipated) rate of outside money growth.* Racalling that the marginal expected bankruptcy cost reduces the real wage bill relative to the certainty case and that actual bankruptcies are a consequence of overproduction by some firms, one might conclude that money transfers to workers should sustain aggregate demand and reduce bankruptcies. This is not the case, however, because the bankruptcy probability eventually depends on the *relative* expected inflation of an entrepreneur: as long as money creation is anticipated, it raises the expected inflation rate of all entrepreneurs uniformly thus leaving their *relative positions* and bankruptcy probability unchanged.

5. Towards Dynamics

The results presented above hold for a single two-period sequence taken in isolation and hence are essentially comparative-static in nature. The next natural step in the presence of failures is to move towards population's dynamics as dictated by the turnover of firms.

5.1. Population's dynamics

At the present stage of development of our research, we limit ourselves to the analysis of the implications of the simplest entry and exit mechanism whereby all insolvent firms are eliminated from the population and replaced with new entrants leaving the measure of the class of entrepreneurs J unchanged. Note that it may now be useful to distinguish between the entrepreneur, who exits from activity after 1 period by assumption, and the firm, which may be thought of as an institution with a "memory" that survives generation after generation. By contrast, bankruptcy means that the institution is destroyed with all its memory. Accordingly, to begin with, as far as entrepreneurs and their beliefs are concerned we make the following operational assumptions:

- newborn entrepreneurs taking control of pre-existing firms also inherit the firm's "memory" of past beliefs (i.e. entrepreneurs in successful firms do not revise previous period's beliefs u_{jt})
- newborn entrepreneurs undertaking new firms are endowed with randomly generated beliefs from the previous generation's distribution (i.e. with density $f(u_{jt+1})$ and $u_{jt+1} \in [u^L{}_t, u^H{}_t]$) and zero memory (i.e. all the the generations of entrepreneurs in new firms are informationally identical).

We also leave the parameters β_t, and Δ_t unchanged over time (the time subscript will be dropped). These assumptions rule out a major issue: learning. This limitation is useful to obtain a first set of neat results, while we postpone learning to future developments of the model. However, though individually beliefs are not revised, as we shall see the population of firms as a whole does display a certain degree of adaptation to the environment via the effect the bankruptcy mechanism exerts on the distribution of beliefs through time.

5.2. The dynamics of the distribution of beliefs

The population dynamics described above brings as a major implication a period by period modification of the distribution of beliefs (and hence inflation forecasts) in the population of firms. We first give a simulative rendition of this process. The analytical treatment is provided in Appendix.

For the sake of comparison with the RE view, let us start with an initial configuration at the beginning of period $t = 0$ such that beliefs are "on average correct" according to theory, i.e. $\mu_0 = 1$, and $\phi_1 > 0$ (see table 1). The latter condition requires $\beta > 0.444$.

	k_0	$\hat{u}_0$	ϕ_1
β= 0.8	0.453	1.13	36.7%

Table 1. Initial configuration: $\mu_0 = 1$ $(u^L = 0.5,\ u^H = 1.5,\ \Delta = 1)$

The new firms' entry mechanism is such that 36.7% of the previous distribution above $\hat{u}_0$ is removed and "spread" over the whole support with the effect that the tail of beliefs below $\hat{u}_0$ is larger, while that above $\hat{u}_0$ is smaller, than before. Therefore, though a mixture of two identical uniform distributions, the resulting distribution is no longer uniform. Figure 4 and table 2 show the evolution of the distribution of beliefs and its parameters over time.

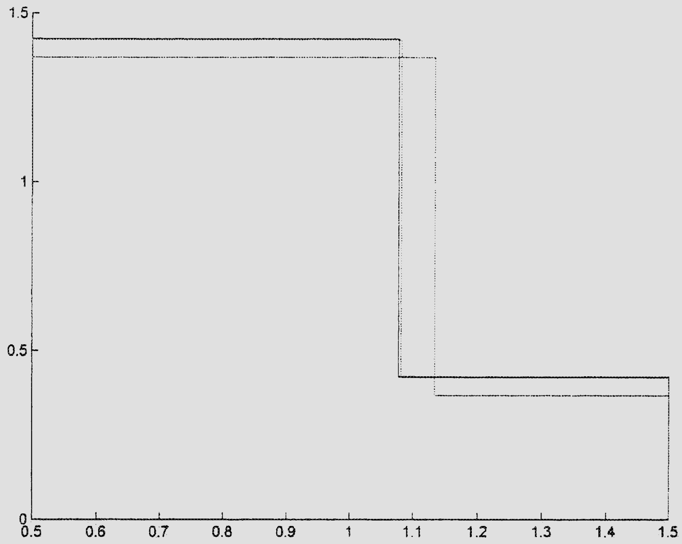

Fig. 4. ($u^L = 0.5$, $u^H = 1.5$, $\beta = 0.8$)

Iteration	$\hat{\mu}_t$	$\hat{u}_t$	k_t
t= 1	0.88384	1.08012	0.4532
2	0.87821	1.07725	0.43205
3	0.87798	1.07715	0.4309
4	0.87798	1.07715	0.43086
5	0.87798	1.07715	0.43086
6	0.87798	1.07715	0.43086
7	0.87798	1.07715	0.43086
8	0.87798	1.07715	0.43086
9	0.87798	1.07715	0.43086
10	0.87798	1.07715	0.43086

Table 2. Dynamics of the populations' and the economy's parameters

Note that owing to the assumption that the previous period's successful beliefs are transmitted to the new entrepreneurs running pre-existing firms while such beliefs are also randomly represented among the new entrepreneurs of new firms, past sucessfull beliefs gain weight in the population. Unsurprisingly, the average belief μ_1 is lower than μ_0 and is decreasing over time. In fact, successful beliefs add weight to the lower tail of the previous period's distribution. Parallely, the bankruptcy threshold $\hat{u}_t$ also rolls back over time. We have argued above that in our economy the firms who are bound to fail are those whose belief lies above this threshold value. Should we expect a fall in the share of bankruptcies, and can we conclude that the population and beliefs dynamics that we have so far examined will, by expelling over-optimistic beliefs period by period, reduce the bankruptcy probability in the economy ending up with zero bankruptcies? Simulation results in table 4 suggest that the evolution of the population parameters seems to converge towards a non-degenerate distribution of beliefs with non-zero asymptotic probability of bankruptcies. In the next section we shall see that these are indeed the long-run properties of our economy.

5.3. Asymptotic properties of the economy

In this section we investigate analytically the behavior of $\hat{u}_0$ in the long run, that is, more formally, we analyze the asymptotic properties of the distribution of beliefs.

First, we look at the evolution of the parameters of the distribution of beliefs over time. At time 0 the expected value is given by $\mu_0 = (u^L + u^H)/2$. From this expected value we get $k_0 = g_1(\mu_0)$(see eq. 0 and $\hat{u}_0 = g_2(k_0)$ (see eq. (4.17)). As we shall see in a moment, it is essential to check whether the functions g_1 and g_2 are non-decreasing. This is obvious for g_2, and also holds for $g_1$as recalled in section 4.

Now let us recall that the entrepreneurs whose belief is larger than $\hat{u}_0 = g_2[g_1(\mu_0)]$ go bankrupt and are excluded, so that the remaining firms are distributed uniformly between $u^L$and $\hat{u}_0$. Then a new sample is drawn from a $U(u^L, u^H)$ distribution. As shown in Apppendix, the sampling scheme at time $t = 1$ is therefore as follows: with probability $\theta_1 = (\hat{u}_0 - u^L)/(u^H - u^L)$ we draw an observation from $U(u^L, \hat{u}_0)$, with probability $(1 - \theta_1)$ we draw an observation from $U(u^L, u^H)$. Formally, this means we sample from a mixture of these two distributions, having density

$$f^{(1)}(u_{j1}) = \theta_1 \frac{1}{\hat{u}_0 - u^L} I_{u^L, \hat{u}_0}(u_{j1}) + (1 - \theta_1) \frac{1}{u^H - u^L} I_{u^L, u^H}(u_{j1}) \tag{5.1}$$

From (5.1) we get new values

$$k_1 = g_1(\mu_1)$$

$$\mu_1 = \theta_1 \frac{u^L + \hat{u}_0}{2} + (1 - \theta_1) \frac{u^L + u^H}{2},$$

$$\hat{u}_1 = g_2(k_1)$$

Notice that

$$\mu_1 = q_1 \frac{u^L + \hat{u}_0}{2} + (1-q_1)\frac{u^L + u^H}{2} \le q_1 \frac{u^L + u^H}{2} + (1-q_1)\frac{u^L + u^H}{2} =$$

$$= \frac{u^L + u^H}{2} = \mu_0 .$$

Now, as the functions g_1 and g_2 are increasing, this implies $\hat{u}_1 \le \hat{u}_0$. If the sequence $\{\hat{u}_t\}_{t\in N}$ is non-increasing, it is not difficult to see that at time t we sample from a mixture

$$f^{(t)}(u_{j1}) = \theta_t \frac{1}{\hat{u}_{t-1} - u^L} I_{u^L,\hat{u}_{t-1}}(u_{jt}) + (1-\theta_t)\frac{1}{u^H - u^L} I_{u^L,u^H}(u_{jt}) \tag{5.2}$$

where $\theta_t = (\hat{u}_{t-1} - u^L)/(u^H - u^L)$

Thus the first point we have to prove is that the sequence $\{\hat{u}_t\}_{t\in N}$ is non-increasing, because in this case the sampling scheme is established for any $t \in N$. After this, we will have to examine whether the sequence converges.

To begin, notice that the functions g_1 and g_2 used for computing $k_t = g_1(\mu_t)$ and $\hat{u}_t = g_2(k_t)$ are known, so that we can put the problem in the functional form $\hat{u}_t = l(\hat{u}_{t-1})$. With this notation we have the following result: the sequence $\{\hat{u}_t\}_{t\in N}$ is non-increasing if and only if $l'(x)$ is non-negative for any $x \in [\hat{u}_t, u^H]$.

This statement can be proved by mathematical induction. We have already shown that $\hat{u}_1 \le \hat{u}_0$. Suppose now that $\hat{u}_t \le \hat{u}_{t-1}$. Then, under the hypothesis that $l'(x) \ge 0 \;\; \forall x \ge \hat{u}_t$, we have $l(\hat{u}_t) - l(\hat{u}_{t-1}) = \hat{u}_{t+1} - \hat{u}_t \le 0$.

To prove that the sequence converges, we use the fixed point theorem[13]. Given the function $g \in C[a,b]$, if $g'(x) \le K < 1 \forall x \in [a,b]$, the sequence $p_t = g(p_{t-1})_{t\in N}$ converges to the unique fixed point in $[a,b]$. When applied to our setup, the theorem says that, for the sequence $\{\hat{u}_t\}_{t\in N}$ to converge, $l'(x)$ must be smaller than one for any $x \in [u^L, u^H]$.

The intersection of these two results implies that we need $l'(x)$ to be non-negative for any $x \in [\hat{u}_t, u^H]$ and smaller than one for any $x \in [u^L, u^H]$. The function $l(x)$ is quite complicated and we used again *the Symbolic Math Toolbox* of MATLAB to differentiate it. The absolute value of the first derivative is smaller than one for all values of interest. In addition, $l'(x)$ is positive for "large enough" values of x: in the examples considered, "large enough" means larger than 1. There

[13]See, for example, Burden and Faires (1993), theorem 2.3

is no a priori guarantee that $\hat{u}_t \geq 1 \forall t$: in general, it will be necessary to check it at each iteration of the algorithm. So the general strategy, given β, u^L and u^H, consists in: (i) checking for which values of x it holds that $0 \leq l'(x) < 1$; (ii) running the process until the difference $\hat{u}_{t-1} - \hat{u}_t$ is smaller than a prespecified tolerance level, checking at each iteration that $\hat{u}_t$ is such that $l'(x) \geq 0$, $\forall x \geq \hat{u}_t$.

Let us consider again the simulation presented in the previous section (tables 1 and 2). In this case the condition $\hat{u}_t \geq 1 \forall t$ is always true. In the table below we report the convergence results, denoted by (*) (in parentheses the number of iterations). The exercise has also been repeated for the same initial configuration assuming a stronger uncertainty effect (β = 0.5), and for a different initial configuration generated as a mean preserving spread of the previous one (table 4)

	k_0	$\hat{u}_0$	ϕ_1	$\hat{u}$ *	ϕ*	μ*	k*
β= 0.8	0.453	1.13	36.7%	1.077 (12)	17.8%	0.878	0.431
β= 0.5	0.354	1.416	8.4%	1.390 (20)	1.2%	0.951	0.347

Table 3. Convergence values from the initial configuration: μ_0 = 1 (u^L = 0.5, u^H = 1.5, Δ = 1).

	k_0	$\hat{u}_0$	ϕ_1	$\hat{u}$*	ϕ*	μ*	k*
β= 0.8	0.482	1.205	35.4%	1.136 (12)	16.2%	0.832	0.454
β= 0.5	0.376	1.505	13.9%	1.464 (18)	2.8%	0.902	0.366

Table 4. Convergence values from the initial configuration: μ_0 = 1 (u^L = 0.3, u^H = 1.7, Δ = 2)

Notice that in all cases the sequence always converges to a value $u^* \in [u^L, u^H]$. This means that the proportion of firms that go bankrupt converges to a constant value in the long run, $\phi^* \in (0, 1)$, and does not collapse to 0 or 1. This can be considered as the long-run or steady-state rate of bankruptcies in the economy to which there corresponds the steady-state structural monetary multiplier k^*. Though, conversely, the economy displays a steady-state rate of successful firms, this fact does not necessitate that either individually, or even on average, firms' beliefs be theoretically correct ($\mu^* = 1$) or structurally correct ($\mu^* = k^*$) or "strongly" rational ($\mu^* = k^* = 1$). One may then ask what is the economic rationale underlying the steady-state beliefs of successful firms: the answer may be that such beliefs (that are concentrated in the lower tail of the distribution) are sufficiently close to k^* so as to pass the acid test of the goodness of fit: survival[14]. As to the comparative statics across steady states it is worth noting that

- a stronger uncertainty effect (lower β), for a given initial configuration, lowers ϕ^* and k^*, and vice versa (the explanation is the same as in section 4)
- a mean preserving spread of beliefs, for a given β, has an ambiguous effect depending on β itself: ϕ^* and k^* raise if the uncertainty effect is stronger (β is lower) and vice versa.

6. Conclusions

We have studied a model of a macro-economy in sequential time where the population of firms at each point in time is characterized by a uniform distribution of individual unobservable beliefs about the mechanism relating an observable market signal (the rate of increase in outside money) and the future inflation rate. As a consequence of heterogeneous beliefs, a certain share of the population of firms can go bankrupt and is driven out the market in each time period. The bankruptcy mechanism is such that the probability for a firm to fail depends on the parameters of the population's beliefs and its own expected inflation relative to average. We have shown that within consistent ranges of parameters, non-zero bankruptcies obtain. The rate of bankruptcies in a given period results to be related to economically palusible effects of the parameters of the population beliefs, but not to the rate of money growth by itself (no money illusion)

Though markets are in equilibrium for incumbent firms' and workers' exchanges, bankruptcies alter equilibrium properties substantially. The system is also self-referential in that the actual inflation rate in each period turns out to be a function of the parameters of the population's beliefs. Given this property, and the share of bankruptcies, the observed relationship between the growth rate of money and the inflation rate is no longer equal to the "fundamental" or "theoretical" one.

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[14]The detachment of *population* properties of "success" or "goodness of fit" from a priori requirements of strong *individual* rationality – though not blatantly in contrast with measures of individual success – is a point of view and a line of reasearch actively pursued by theories of large heterogeneus systems: see e.g. Gode-Sunder (1993), Beltratti et al. (1996). A history of this view is traced back to Alfred Marshall by Leijonhufvud (1993).

In fact, in the same ranges of parameters that yield non-zero bankruptcies, we have explored the existence of fixed points in the map that projects the averege expected inflation onto the actual one, i.e. "cross-sectional" rational expectations, and we have found that where such fixed points exist they do not coincide with the theoretical value of the inflation rate. On the one hand, this result may be added to the class of "self-fulfilling (average) prophecies", on the other, thinking of the self-fulfilled average expected inflation as an "anomaly" with respect to the inflation rate that would prevail under homogenous perfect foresight and no bankruptcies is misleading because heterogeneous beliefs and non-zero bankruptcies *are part of the structure of the economy*. The implication is rather that, normatively, the content of the rational expectations hypotesis should be extended to include the bankruptcy-generating mechanism and the way it modifies the *structural* relationship between the growth rate of money and the inflation rate, which seems however contradictory with the existence itself of bankruptcies.

Finally we have extended our analyis to the turnover of firms along the sequence of periods. We have started form the simplest case: bankrupt firms are "erased" (no information left to posterity) and replaced by "blank" newcomers with beliefs randomly extracetd from the existing distribution. Incumbent successful firms do not change their beliefs nor do they engage in learning. This turnover mechanism generates a dynamics of the distribution of beliefs: changes in the distribution's parameters produce changes in the inflation rate and in the bankruptcy rate period after period (successful firms in one period may no longer be such in the next). Though successful firms are predominant in each period and the bankruptcy rate tends to shrink, we have found by numerical methods that the probability of bankruptcy converges towards a non-degenerate limit value. In other words, in the long run the economy displays a "structural" or "natural" rate of bankruptcy such that all the previous properties described above hold. In this end state, individual as well as average beliefs in subsequent populations may display a tenous, if any, relationship with a priori criteria of "rationality" of beliefs, nonetheless the beliefs of successful firms are sufficiently close to the actual structural parameter so as to guarantee the profitable survival of a share of firms generation after generation.

Appendix

Let us start form the original set of random beliefs U_t uniformly distributed with density function $f(u_{jt})$, $u_{jt} \in [u^L, u^H]$, in period t. The consequence of the exit mechanism is equivalent to truncating the support of U_t at $\hat{u}_t$ so that the beliefs of solvent firms, $U^S{}_t \in U_t$, are realizations from the uniform random variable $U^S{}_t \sim U(u^L{}_t, \hat{u}_t)$. The consequence of the entry mechanism in period t+1 is that the beliefs are realizations from the random variables U_{t+1} and $U^S{}_t$ according to the following law

$$\begin{cases} \text{from } U_t^s \sim U(u^{\mathrm{L}}, \hat{u}_t) & \text{with probabilit y } \theta_t \\ \text{from } U_{t+1} \sim U(u^{\mathrm{L}}, u^{\mathrm{H}}) & \text{with probabilit y } 1-\theta_t \end{cases}$$

where

$$\theta_t = \frac{\hat{u}_t - u^{\mathrm{L}}}{u^{\mathrm{H}} - u^{\mathrm{L}}}$$

Consequently, the density function of beliefs in period $t+1$, $f(u_{jt+1})$, can be written as

$$f(u_{jt+1}) = \theta_t f_1(u_{jt+1}) I_{u^{\mathrm{L}},\hat{u}_t}(u_{jt+1}) + (1-\theta_t) f_2(u_{jt+1}) I_{u^{\mathrm{L}},u^{\mathrm{H}}}(u_{jt+1}) \tag{A.1}$$

where f_1 and f_2 are two density functions defined as follows:

$$f_1(u_{jt+1}) = \begin{cases} (\hat{u}_t - u^{\mathrm{L}})^{-1} & u_{jt+1} \in [u^{\mathrm{L}}, \hat{u}_t] \\ 0 & \text{elsewhere} \end{cases}$$
$$f_2(u_{jt+1}) = \begin{cases} (u^{\mathrm{H}} - u^{\mathrm{L}})^{-1} & u_{jt+1} \in [u^{\mathrm{L}}, u^{\mathrm{H}}] \\ 0 & \text{elsewhere} \end{cases} \tag{A.2}$$

and $I_{u^{\mathrm{L}},\hat{u}_t}(u_{jt+1})$ and $I_{u^{\mathrm{L}},u^{\mathrm{H}}}(u_{jt+1})$ are two indicator functions defined as follows

$$I_{u^{\mathrm{L}},\hat{u}_t}(u_{jt+1}) = \begin{cases} 1 & u_{jt+1} \in [u^{\mathrm{L}}, \hat{u}_t] \\ 0 & u_{jt+1} \notin [u^{\mathrm{L}}, \hat{u}_t] \end{cases}$$
$$I_{u^{\mathrm{L}},u^{\mathrm{H}}}(u_{jt+1}) = \begin{cases} 1 & u_{jt+1} \in [u^{\mathrm{L}}, u^{\mathrm{H}}] \\ 0 & u_{jt+1} \notin [u^{\mathrm{L}}, u^{\mathrm{H}}] \end{cases} \tag{A.3}$$

We are now in a position to compute the expected value of the beliefs in period $t+1$, μ_{t+1}. Since the other two population's parameters, Δ and β, are constant by assumption, μ_{t+1} represents the evolution of the population owing to the exit and entry mechanism. It follows straightforwardly from (A.1) that

$$\mu_{t+1} = \theta_t \mu_1 + (1-\theta_t)\mu_2 \tag{A.4}$$

where

$$\mu_1 = (u^{\mathrm{L}} + \hat{u}_t)/2$$
$$\mu_2 = (u^{\mathrm{L}} + u^{\mathrm{H}})/2$$

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Statistical Physics and Economic Fluctuations

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Summary. We present an overview of recent research—much of it carried out in collaborations between economists and physicists—which is focussed on applying ideas of statistical physics to try to better understand puzzles regarding economic fluctuations. One of these puzzles is how to describe outliers, phenomena that lie outside of patterns of statistical regularity. We review evidence consistent with the possibility that such outliers may not exist. This possibility is supported by extensive numerical analysis of a huge database, containing every trade, which results in power law descriptions of a number of quantities whose fluctuations are of interest. It is also supported by recent analysis by Plerou et al. of a database containing the bid, ask, and sale price of each trade of every stock. Further, the Plerou et al. analysis is consistent with a possible theoretical framework for understanding economic fluctuations in which a financial market alternates between being in an "equilibrium phase" where market behavior is split roughly equally between buying and selling, and an "out-of-equilibrium phase" where the market is mainly either buying or selling.

1 Introduction

Collaborative work joining economists and physicists has begun to lead to modest progress in answering questions of interest to both economists and physicists. In particular, these collaborations have the potential to change the paradigm for understanding economic fluctuations. Until relatively recently, theories of economic fluctuations invoked the label of "outlier" (bubbles and crashes) to describe fluctuations that do not agree with existing theory. These outliers are of interest, as they correspond to extremely large and unpredictable changes of magnitude sufficient to wreak havoc.

The paradigm of "statistical regularity plus outliers" does not exist in the physical sciences. Indeed, if events occur that do not conform to predictions of the appropriate theory, then that theory is immediately relegated to the dust bin and new theories are sought. An example are the "outliers" that led to the demise of classical mechanics, eventually replaced by the theory of relativity.

M. Gallegati et al. (eds.), *The Complex Dynamics of Economic Interaction*

Traditional economic theory is not always sufficient to predict all the outliers. Recent analysis of extremely gargantuan databases suggests that classic theories fail not only for a few outliers, but that there occur similar outliers of every possible size. In fact, if one analyzes only a small data set (say, 10^4 data points), then outliers appear to occur as "rare events." However, when orders of magnitude more data (10^8 data points) are analyzed, one finds orders of magnitude more outliers—thus ignoring them is not a responsible option, and studying their properties becomes a realistic goal. One finds that the statistical properties of these "outliers" are identical to the statistical properties of everyday fluctuations. For example, a histogram giving the number of fluctuations of a given magnitude x for fluctuations ranging in magnitude from everyday fluctuations to extremely rare fluctuations ("financial earthquakes") that occur with a probability of only 10^{-8} is a perfect straight line in a double-log plot.

An analogy with earthquake research is perhaps not entirely inappropriate. If one studies limited data sets, which correspond to every day experience, an entirely natural paradigm arises in which there are everyday (unnoticeable except by sensitive seismometer) "tremors," punctuated from time to time by rare events ("earthquakes"). Thanks to the empirical work, we now know that the partition of shocks into "tremors" and "earthquakes" is not valid. Rather, if one examines enough data, one sees that the shocks occur for all possible magnitudes. The empirical ",law" named after Gutenberg and Richter, refers to a statistical formula that gives all the data from the smallest tremors all the way up to the "big one." This empirical law states that the histogram giving the number of shocks of a given size is a straight line in a log-log plot [1, 2]. Since this straight line fits all the data, there are no outliers in earthquake research. The utility of this empirical law is at least twofold: (i) it motivates understanding "the big ones" (that matter!) by extensive analysis of small ones (that do not matter), and (ii) it enables *quantitative* estimates of the "risk" of large earthquakes, thereby making possible safer design of buildings by guiding engineers to have just enough vibration protection appropriate to reducing the risk to an acceptable level.

Our experience with thinking about earthquakes teaches us that an inappropriate paradigm can arise when a limited quantity of data are considered, since if one has only a handful of rare events, it is natural to partition data into everyday events (often describable by one statistical law) and rare events (which, since they are not described by the law, are termed outliers). Has an inappropriate paradigm arisen in economic research? In economic research, there are fluctuations in stock prices, in number of shares trading hands, and in total number of fluctuations, and so forth. Recent empirical studies calculating histograms for all recordable quantities are linear on log-log plots (albeit with different slopes). In mathematical language, the occurrence probability of such a quantity's fluctuations appear to be described by a power law.

In economics, neither the existence of power laws nor the exact exponents measured (the slopes of the above-mentioned log-log plots) has an accepted theoretical basis. Professionally, empirical laws such as power laws are called "stylized facts," a term that would in physics be regarded as being dismissive since physics is a discipline grounded on empirical facts, and facts that are observed and confirmed by independent observation are called "laws". Accordingly, some theoretical understanding is urgently needed or else these empirical laws will continue to be regarded as largely uninteresting if not irrelevant. Of course facts, even facts without any interpretation, may have practical value. If the Gutenberg-Richter law enables one to accurately calculate the risk of a shock (tremor or earthquake) of a given magnitude, and hence informs the building codes of Los Angeles and Tokyo, could the empirical laws governing economic fluctuations perhaps enable one to accurately calculate the risk of an economic shock of a given magnitude.

The lack of a coherent theory is unfortunate—especially in economics, where to be faced with facts bereft of any theoretical foundation is considered deplorable. Accordingly, my collaborators and I have been seeking to develop a theoretical framework within which to interpret these new empirical facts, and recently some progress is beginning to occur [3, 4]. This work is potentially significant since it provides a theoretical framework within which to interpret the new empirical laws. Specifically, the model fulfills these requirements for such a basic "microscopic" model of the stock market. It is founded on realistic features of the stock market, and reflects the view that market participants have of the functioning of the market, as well as the main determinants of their trading behavior.

2 First Discovery of Scaling and Universality

That at least *some* economic phenomena are described by power law tails has been recognized for over 100 years, ever since Pareto investigated the statistical character of the wealth of individuals by modeling them using the *scale-invariant* distribution

$$f(x) \sim x^{-\alpha}, \tag{1}$$

where $f(x)$ denotes the number of people having income x or greater than x, and α is an exponent that Pareto estimated to be 1.5 [5, 6]. Pareto noticed that his result was *universal* in the sense that it applied to nations *"as different as those of England, of Ireland, of Germany, of the Italian cities, and even of Peru"*. A physicist would say that the universality class of the scaling law (1) includes all the aforementioned countries as well as Italian cities since, by definition, two systems belong to the same universality class if they are characterized by the same exponents.

In the century following Pareto's discovery, the twin concepts of scaling and universality have proved to be important in a number of scientific fields [7, 8, 9, 10, 11, 12, 13]. A striking example was the elucidation of the puzzling behavior of systems near their critical points. Over the past few decades it has come to be appreciated that the scale-free nature of fluctuations near critical points also characterizes a huge number of diverse systems also characterized by strong fluctuations. This set of systems includes examples that at first sight are as far removed from physics as is economics. For example, consider the percolation problem, which in its simplest form consists of placing pixels on a fraction p of randomly-chosen plaquettes of a computer screen. A remarkable fact is that the largest connected component of pixels magically spans the screen at a threshold value p_c. This purely geometrical problem has nothing to do, at first sight, with critical point phenomena. Nonetheless, the fluctuations that occur near $p = p_c$ are scale-free and functions describing various aspects of the incipient spanning cluster that appears at $p = p_c$ are described by power laws characterized by exponent values that are universal in the sense that they are independent of the details of the computer screen's lattice (square, triangle, honeycomb). Nowadays, the concepts of scaling and universality provide the conceptual framework for understanding the geometric problem of percolation.

It is becoming clear that almost any system comprised of a large number of interacting units has the potential of displaying power law behavior. Since economic systems are comprised of a large number of interacting units, it is perhaps not unreasonable to examine economic phenomena within the conceptual framework of scaling and universality [7, 8, 9, 10, 11, 12, 13]. We will discuss this topic in detail below.

3 Scaling and Universality: Two Concepts of Modern Statistical Physics

Statistical physics deals with systems comprising a very large number of interacting subunits, for which predicting the exact behavior of the individual subunit would be impossible. Hence, one is limited to making statistical predictions regarding the collective behavior of the subunits. Recently, it has come to be appreciated that many such systems consisting of a large number of interacting subunits obey universal laws that are independent of the microscopic details. The finding, in physical systems, of universal properties that do not depend on the specific form of the interactions gives rise to the intriguing hypothesis that universal laws or results may also be present in economic and social systems [7, 12].[1]

[1] An often-expressed concern regarding the application of physics methods to the social sciences is that physical laws are said to apply to systems with a very large number of subunits (of order of $\approx 10^{20}$), while social systems comprise a much smaller number of elements. However, the "thermodynamic limit" is reached in practice for rather small systems. For example, in early computer simulations of gases or liquids, reasonable results are already obtained for systems with 20–30 atoms.

3.1 Background

Suppose we have a small bar magnet made up of 10^{12} strongly-interacting subunits called "spins." We know it is a magnet because it is capable of picking up thumbtacks, the number of which is called the order parameter M. As we heat this system, M decreases and eventually, at a certain critical temperature T_c, it reaches zero. Since M approaches zero at T_c with infinite slope, the transition is remarkably sharp, hence M is not an analytic function. Such singular behavior is an example of a "critical phenomenon." Recently, the field of critical phenomena has been characterized by several important conceptual advances, two of which are scaling and universality.

Scaling

The scaling hypothesis has two categories of predictions, both of which have been remarkably well verified by a wealth of experimental data on diverse systems. The first category is a set of relations, called *scaling laws*, that serve to relate the various critical-point exponents characterizing the singular behavior of functions such as M.

The second category is a sort of *data collapse*, which is perhaps best explained in terms of our simple example of a uniaxial magnet. We may write the equation of state as a functional relationship of the form $M = M(H, \tau)$, where M is the order parameter, H is the magnetic field, and $\tau \equiv (T - T_c)/T_c$ is a dimensionless measure of the deviation of the temperature T from the critical temperature T_c. Since $M(H, \tau)$ is a function of two variables, it can be represented graphically and M vs. τ for a sequence of different values of H. The scaling hypothesis predicts that all the curves of this family can be "collapsed" onto a single curve provided one plots not M vs. τ but rather a *scaled* M (M divided by H to some power) vs. a *scaled* τ (τ divided by H to some different power).

The predictions of the scaling hypothesis are supported by a wide range of experimental work, and also by numerous calculations on model systems. Moreover, the general principles of scale invariance used here have proved useful in interpreting a number of other phenomena, ranging from elementary particle physics and galaxy structure to finance [7, 14, 15].

Universality

The second theme goes by the name "universality." It was found empirically that one could form an analog of the Mendeleev table if one partitions all critical systems into "universality classes." Consider, e.g., experimental MHT data on five diverse magnetic materials near their respective critical points. The fact that data for each material collapse onto a scaling function supports the scaling hypotheses, while the fact that the scaling function is the *same* (apart from two material-dependent scale factors) for all five diverse materials is truly remarkable. This apparent universality of critical behavior motivates the following question: *"Which features of this microscopic interparticle force are important for determining critical-point exponents and scaling functions, and which are unimportant?"*

Two systems with the same values of critical point exponents and scaling functions are said to belong to the same universality class. Thus the fact that the exponents and scaling functions are the same for all five materials implies they all belong to the same universality class. Hence we can pick a tractable system to study and the results we obtain will hold for all other systems in the same universality class.

3.2 Scaling and Universality in Systems outside of Physics

At one time, many imagined that the "scale-free" phenomena are relevant to only a fairly narrow slice of physical phenomena [8, 9, 16]. However, the range of systems that apparently display power law and hence scale-invariant correlations has increased dramatically in recent years, ranging from base pair correlations in noncoding DNA [17, 18], lung inflation [19] and interbeat intervals of the human heart to complex systems involving large numbers of interacting subunits that display "free will," such as city growth, and even populations of birds [20].

4 Towards a "Theory of the Firm"

Having embarked on a path guided by these two theoretical concepts, what does one do? Initially, critical phenomena research—guided by the Pareto principles of scaling and universality—was focused on finding which systems display scaling phenomena, and on discovering the actual values of the relevant exponents. This initial empirical phase of critical phenomena research proved vital, for only by carefully obtaining empirical values of exponents such as α could scientists learn which systems have the same exponents (and thus belong to the same *universality class*). The fashion in which physical systems partition into disjointed universality classes proved essential to such later theoretical developments as the renormalization group [9]—which offered some insight into why scaling and universality seem to hold; ultimately, it led to a better understanding of the critical point.

Similarly, our group's initial research in economics—guided by the Pareto principles—has largely been concerned with establishing which systems display scaling phenomena, and with measuring the numerical values of the exponents with sufficient accuracy that one can begin to identify universality classes if they exist. Economic systems differ from often-studied physical systems in that the number of interacting subunits is much smaller. The macroscopic samples in physical systems contain a huge number of interacting subunits (as many as Avogadro's number 6×10^{23}). In contrast, in an economic system, one initial work was limited to analyzing time series comprising of order of magnitude 10^3 terms, and nowadays, with high frequency data being the standard, one may have 10^8 terms. Scaling laws of the form of (1) are found that hold over a range of a factor of $\approx 10^6$ on the x-axis [21, 22, 23, 24, 25]. Moreover, these scaling laws appear to be universal in that they, like the Pareto scaling law, hold for different countries [26], for countries themselves, [27, 28], for other social organizations [29], and even for bird populations [20, 30].

One aspect of this work has come to the attention of a particular group of social scientists: the research community that studies S & T (Science and Technology) systems. Earlier findings for the growth of economic organizations [10, 5, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40] prompted investigations to ask if similar laws may hold for the time evolution of S&T systems. To this end, Plerou et al. [41, 42, 43] analyzed the fluctuations in the growth rates of university research activities, using five different measures of research activity. Plerou et al studied the production of research both from the point of view of inputs (R&D funding) and outputs (publications and patents).

Recent attempts to make models that reproduce the empirical scaling relationships suggest that significant progress on understanding firm growth may be well underway [45, 44, 46, 47, 48, 49, 50], leading to the hope of ultimately developing a clear and coherent "theory of the firm." One utility of the recent empirical work is that now any acceptable theory must respect the fact that power laws hold over typically six orders of magnitude; as Axtell put the matter rather graphically: *"the power law distribution is an unambiguous target that any empirically accurate theory of the firm must hit"* [21].

5 Inverse Cubic Law of Stock and Commodity Price Fluctuations

With this background on power laws and scale invariance in geometry and in economics, we now turn to the well-studied problem of finance fluctuations, where a consistent set of empirical facts is beginning to emerge. One fact that has been confirmed by numerous, mostly independent, studies is that stock price fluctuations are characterized by a scale-invariant cumulative distribution function of the power law form (1) with $\alpha \approx 3$ [51, 52, 53]. This result is also universal, in the sense that this inverse cubic law exponent is within the error bars of results for different segments of the economy, different time periods, and different countries—and is the same for stock averages as different as the S&P and the Hang Seng [54].

This "inverse cubic law" disagrees with the classic work of Ref. [7] on price fluctuations of cotton, which appear to have display scale free behavior ("no outliers") but with much fatter tails characterized by $\alpha \approx 1.7$; this work is of interest because if $\alpha < 2$, then the distribution is of the Lévy form. To understand this discrepancy, Matia and collaborators have suggested that perhaps cotton has fatter tails because it is a commodity; commodities exist in limited supply, and one must sometimes pay exhorbitant prices (e.g., electricity in California). Accordingly, they analyzed a large number of commodities, but found that these commodities have tails described not by $\alpha < 2$ but rather by $\alpha \approx 3$ [55, 56]. Another possible reason is that Mandelbrot analyzed three data sets, each containing only about 2000 points, while the results on stocks typically contain about 40,000 points per stock (and 1000 stocks, or 40,000,000 total data points). This possibility was tested by randomly choosing 2000 points to analyze, but again $\alpha < 2$ could not be obtained. A third possible explanation of this discrepancy is that the cotton market was "out of equilibrium," and that such out-of-equilibrium markets have fatter tails—a possibility consistent with a recent analysis of stock price fluctuations [57, 58]. A fourth possible explanation is that, at the time Mandalbrot collected his data, commodities were intrinsically different from what commodities are today. The modern commodities market, the source of the Matia data, is much more similar to the stock market than it was in 1963. Still another possibility is that the cotton distribution has $\alpha < 2$ in the central region analyzed in 1963, but ultimately crosses over to a power law in the distant tails (which were not analyzed in 1963). This disagreement led to the development of a class of mathematical processes called *truncated* Lévy distributions [59, 14, 60, 61, 62, 63, 64]—which has attracted the attention of a number of mathematicians and is actually taught in Columbia University's Graduate School of Finance. In any case, one of the challenges of econophysics is to resolve current results with the classic 1963 analysis of Mandelbrot [7].

6 Databases Analyzed

Our results are based on the analysis of different databases covering securities traded in the three major US stock exchanges, namely, (i) the New York Stock Exchange (NYSE), (ii) the American Stock Exchange (AMEX), and (iii) the National Association of Securities Dealers Automated Quotation (Nasdaq).

For studying short time scale dynamics, they are analyze the Trades and Quotes (TAQ) database, from which we select the 4-year period January 1994 to December 1997. Nasdaq and AMEX have merged on October 1998, after the end of the period studied in this work. The TAQ database, which is published by NYSE since 1993, covers *all* trades at the three major US stock markets. This huge database is available in the form of CD-ROMs. The rate of publication was one CD-ROM per month for the period studied, but has recently increased to two to four CD-ROMs per month. The total number of transactions for the largest 1000 stocks is of the order of 10^9 in the 4-year period studied. They analyze the largest 1000 stocks, by capitalization on January 3, 1994, which survived through December 31, 1995. From the set of these 1000 stocks, they select a subset consisting of 880 stocks which survive through the further two years 1996–97.

The data are adjusted for stock splits and dividends. The data are also filtered to remove spurious events that occur due to inevitable recording errors. The most common error is missing digits which appears as a large spike in the time series of returns. These are much larger than usual fluctuations and can be removed by choosing an appropriate threshold. We tested a range of thresholds and found no effect on the results.

To study the dynamics at longer time horizons, they also analyze the Center for Research and Security Prices (CRSP) database. The CRSP Stock Files cover common stocks listed on NYSE beginning in 1925, the AMEX beginning in 1962, and the Nasdaq Stock Market beginning in 1972. The files provide complete historical descriptive information and market data including comprehensive distribution information, high, low, and closing prices, trading volumes, shares outstanding, and total returns. In addition to adjusting for stock splits and dividends, they have also detrended the data for inflation.

The CRSP Stock Files provide monthly data for NYSE beginning in December 1925 and daily data beginning July 1962. For the AMEX, both monthly and daily data begin in in July 1962. For the Nasdaq Stock Market, both monthly and daily data begin in July 1972.

They also analyze the S&P 500 index, which comprises 500 stocks chosen for market size, liquidity, and industry group representation in the US. In our study, we first analyze high-frequency data that covers the 13-year period 1984–1996, with a recording frequency of less than 1 min. The total number of records in this database exceeds 4.5×10^6. To investigate longer time scales, we also study daily records of the S&P 500 index for the 35-year period 1962–1996, and monthly records for the 71-year period 1926–1996.

7 The Distribution of Stock Price Fluctuations

The nature of the distribution of price fluctuations in financial time series has been a topic of interest for over 100 years [65]. A reasonable *a priori* assumption, motivated by the central limit theorem, is that the returns are independent, identically Gaussian distributed (*i.i.d.*) random variables, which results in a Gaussian random walk in the logarithm of price.

Empirical studies [7, 14, 51, 52, 53, 66, 67, 68, 69, 70, 71] show that the distribution of returns has pronounced tails, in striking contrast to that of a Gaussian. In addition to being non-Gaussian, the process of returns shows another interesting property: "time scaling"—that is, the distributions of returns for various choices of Δt, ranging from one day up to one month have similar functional forms [7]. These results together would suggest that the distribution of returns is consistent with a Lévy stable distribution [5, 68, 7, 72], the rationale for which arises from the generalization of the central limit theorem to random variables which do not have a finite second moment. Empirical studies suggest, however, that the tails of the return distribution are inconsistent with the stable Paretian hypothesis [14, 51, 52, 53, 54, 67, 73, 74, 75, 76, 77].

In particular, alternative hypotheses for modeling the return distribution were proposed, which include a log-normal mixture of Gaussians [77], Student t-distributions [73, 74, 75], and exponentially-truncated Lévy distributions [14, 59, 60].

7.1 "Universality" of the Distribution of Returns

Conclusive results on the distribution of returns are difficult to obtain and require a large amount of data to study the rare events that give rise to the tails. We analyze approximately 40 million records of stock prices sampled at 5 min intervals for the 1,000 leading US stocks for the 2-year period 1994–1995 and 30 million records of daily returns for 6,000 US stocks for the 35-year period 1962–1996.

The basic quantity studied for individual companies is the price $S_i(t)$. The time t runs over the working hours of the stock exchange—removing nights, weekends and holidays. For each company, we calculate the return

$$G_i \equiv G_i(t, \Delta t) \equiv \ln S_i(t + \Delta t) - \ln S_i(t)\,. \tag{2}$$

For small changes in $S_i(t)$, the return $G_i(t, \Delta t)$ is approximately the forward relative change, $G_i(t, \Delta t) \approx [S_i(t + \Delta t) - S_i(t)]/S_i(t)$. For time scales shorter than one day, we analyze the data from the TAQ database.

We then calculate the cumulative distributions—the probability of a return larger than or equal to a threshold—of returns G_i for $\Delta t = 5$ min. For each stock $i = 1, \ldots, 1000$, the asymptotic behavior of the functional form of the cumulative distribution is consistent with a power-law,

$$P\{G_i > x\} \sim \frac{1}{x^{\alpha_i}}\,, \tag{3}$$

where α_i is the exponent characterizing the power-law decay. In order to compare the returns of different stocks with different volatilities, we define the normalized return $g_i \equiv (G_i - \langle G_i \rangle_T)/v_i$, where $\langle \ldots \rangle_T$ denotes a time average over the 40,000 data points of each time series for the 2-year period studied, and the time-averaged volatility v_i of company i is the standard deviation of the returns over the 2-year period $v_i{}^2 \equiv \langle G_i{}^2 \rangle_T - \langle G_i \rangle_T{}^2$. Values of the exponent α_i can be estimated by a power-law regression on each of these distributions $P\{g > x\} \sim x^{-\alpha}$, whereby we obtain the average value for the 1000 stocks,

$$\alpha = \begin{cases} 3.10 \pm 0.03 & \text{(positive tail)} \\ 2.84 \pm 0.12 & \text{(negative tail)} \end{cases}, \tag{4}$$

where the fits are performed in the region $2 \leq g \leq 80$. These estimates of the exponent α are well outside the stable Lévy range, which requires $0 < \alpha < 2$, and is therefore consistent with a finite variance for returns. However, moments larger than three, in particular the kurtosis, seem to be divergent [66, 76]. Our results are consistent with the results of the analysis of the daily returns of 30 German stocks comprising the DAX index [51], daily CRSP returns [66], and foreign exchange rates [67].

In order to obtain an alternative estimate for α, we use the methods of Hill [66, 51, 52, 53]. We calculate the inverse local slope of the cumulative distribution function $P(g)$, $\gamma \equiv -\left(d\log P(g)/d\log g\right)^{-1}$ for the negative and the positive tail. We obtain an estimator for γ by sorting the normalized increments by their size, $g^{(1)} > g^{(2)} > ... > g^{(N)}$. The cumulative distribution can then be written as $P(g^{(k)}) = k/N$, and we obtain for the local slope

$$\gamma = \left[(N-1)\sum_{i=1}^{N-1} \log g^{(i)}\right] - \log g^{(N)}\,, \tag{5}$$

where N is the number of tail events used. We use the criterion that N does not exceed 10% of the sample size, simultaneously ensuring that the sample is restricted to the tail events [66]. We thereby obtain the average estimates for 1000 stocks,

$$\alpha = \begin{cases} 2.84 \pm 0.12 & \text{(positive tail)} \\ 2.73 \pm 0.13 & \text{(negative tail)} \end{cases}. \tag{6}$$

Removing overnight events yields the average values of $\alpha = 3.11 \pm 0.15$ for the positive tail and $\alpha = 3.03 \pm 0.21$ for the negative tail. Currently, we are also investigating the dependence of the exponent α on the time of day by splitting a trading day into three equal parts of 130 min each. A parallel analysis on the S&P 500 index shows consistent asymptotic behavior [54], although the central part of the distribution seems to display Lévy behavior for short time scales ($<$ 30 min) [14]. One reason for a different behavior at the central part of the distribution of S&P 500 returns is the discreteness of the prices of individual stocks (which causes a cut-off for low values of returns) that comprise the S&P 500 index.

7.2 Scaling of the Distributions of Returns and Correlations in the Volatility

Since the values of α we find are inconsistent with a statistically stable law, we expect the distribution of returns $P(G)$ on larger time scales to converge to Gaussian. In contrast, our analysis of daily returns from the CRSP database suggests that the distributions of returns retain the same functional form for a wide range of time scales Δt, varying over three orders of magnitude, 5 min$\leq \Delta t \leq$ 6240 min = 16 days. The *onset* of convergence to a Gaussian starts to occur only for $\Delta t >$ 16 days [53, 54]. In contrast, n-partial sums of computer-simulated time series of the same length and probability distribution display Gaussian behavior for $n \geq 256$ [13, 54]. Thus, the rate of convergence of $P(G)$ to a Gaussian is remarkably slow, indicative of time dependencies that violate the conditions necessary for the central limit theorem to apply.

To test for time dependencies, we analyzed the autocorrelation function of returns, which we denote as $\langle G(t)G(t+\tau)\rangle$, using 5 min returns of 1000 stocks. Our results show pronounced short-time (< 30 min) anti-correlations, consistent with the bid-ask bounce [78]. For larger time scales, the correlation function is at the level of noise (for some portfolios of common stocks Lo [79] has reported long memory), consistent with the efficient market hypothesis [69, 80, 81]. Lack of linear correlation does not imply independent returns, since higher-order correlations may exist. Our recent studies [82] show that the amplitude of the returns measured by the absolute value or the square has long-range correlations with persistence [83, 84] up to several months,

$$\langle |G(t)|\,|G(t+\tau)|\rangle \sim \tau^{-a}\,, \tag{7}$$

where a has the average value $a = 0.34 \pm 0.09$ for the 1000 stocks studied. In order to detect genuine long-range correlations, the effects of the U-shaped intra-day pattern [85] for $|G|$ has been removed [82]. This result is consistent with earlier studies [86, 87, 88] which also noted long-range correlations. In addition to analyzing the correlation function directly, we are applying power spectrum analysis and the recently-developed detrended fluctuation analysis [82, 89]. Both of these methods yield consistent estimates of the exponent a. We are also applying estimators such as those developed in Ref. [90] to obtain accurate estimates of the exponent a.

7.3 Statistics of Trading Activity

In order to understand the reasons for slow decaying tails in the return distribution and long-range correlations in volatility, we follow an approach in the spirit of models of time deformation proposed by Clark [77], Tauchen and Pitts [91], Stock [92], Lamoureux and Lastrapes [93], Ghysels and Jasiak [94], and Engle and Russell [95].

Returns G over a time interval Δt can be expressed as the sum of several changes δp_i due to the $i = 1, \ldots, N_{\Delta t}$ trades in the interval $[t, t + \Delta t]$,

$$G_{\Delta t} = \sum_{i=1}^{N_{\Delta t}} \delta p_i\,. \tag{8}$$

If Δt is such that $N_{\Delta t} \gg 1$, and δp_i have finite variance, then one can apply the classic version of the central limit theorem, whereby one would obtain the result that the unconditional distribution $P(G)$ is Gaussian [77]. It is implicitly assumed in this description that $N_{\Delta t}$ has only *narrow* Gaussian fluctuations i.e., has a standard deviation much smaller than the mean $\langle N_{\Delta t}\rangle$.

Our investigation of $N_{\Delta t}$ suggests stark contrast with a Gaussian time series with the same mean and variance—there are several events of the magnitude of tens of standard deviations which are inconsistent with Gaussian statistics [77, 91, 92, 95, 96, 97, 98, 99, 100]. For each stock analyzed, we chose sampling time intervals Δt such that it contains sufficient $N_{\Delta t}$; for actively-traded stocks $\Delta t = 15$ min, and for stocks with the least frequency of trading, $\Delta t = 390$ min (one day) [99]. We find that the distribution of $N_{\Delta t}$ appears to display an asymptotic power-law decay

$$P\{N_{\Delta t} > x\} \sim x^{-\beta} \quad (x \gg 1)\,. \tag{9}$$

For the 1000 stocks that we analyze, we estimate β using Hill's method and obtain a mean value $\beta = 3.40 \pm 0.05$. Note that $\beta > 2$ is outside the Lévy stable domain $0 < \beta < 2$ and is inconsistent with a stable distribution for $N_{\Delta t}$, and with the log-normal hypothesis of Clark [77].

7.4 Price Fluctuations and Trading Activity

Since we find that $P\{G_{\Delta t} > x\} \sim x^{-\alpha}$, we can ask whether the value of β we find for $P\{N_{\Delta t} > x\}$ is sufficient to account for the fat tails of returns. To test this possibility, we implement, for each stock, the ordinary least squares regression

$$\ln |G_{\Delta t}(t)| = a + b \ln N_{\Delta t}(t) + \psi(t)\,, \tag{10}$$

where $\psi(t)$ has mean zero and the equal time covariance $\langle N_{\Delta t}\psi(t)\rangle = 0$. Our results on 30 actively-traded stocks yield the average value of $b = 0.57 \pm 0.09$.

Values of $b \approx 0.5$ are consistent with what we would expect from Eq. (8), if δp_i are *i.i.d.* with finite variance. In other words, suppose δp_i are chosen *only from the interval* $[t,\ t+\Delta t]$, and let us hypothesize that *these* δp_i are mutually independent, with a common distribution $P(\delta p_i | t \in [t, t+\Delta t])$ having a finite variance $W^2_{\Delta t}$. Under this hypothesis, the central limit theorem, applied to the sum of δp_i in Eq. (8), implies that the ratio

$$\epsilon \equiv \frac{G_{\Delta t}}{W_{\Delta t}\sqrt{N_{\Delta t}}} \tag{11}$$

must be a Gaussian-distributed random variable with zero mean and unit variance. We can test this hypothesis by analyzing the distribution $P(\epsilon)$ and the correlations in ϵ.

Our results on 30 actively-traded stocks seem to indicate that the distribution $P(\epsilon)$ is consistent with a Gaussian, with mean values of excess kurtosis ≈ 0.1. This is noteworthy since, for the unconditional distribution $P(G_{\Delta t})$, the kurtosis is divergent (empirical estimates yield mean values ≈ 80 for 1000 stocks).

If our hypothesis that $P(\epsilon)$ is consistent with a Gaussian is borne out by the data, this would imply that the fat tails of $P\{G_{\Delta t} > x\} \sim x^{-\alpha}$ cannot be caused solely by $P\{N_{\Delta t} > x\} \sim x^{-\beta}$ because, by conservation of probabilities, $P\{\sqrt{N_{\Delta t}} > x\} \sim x^{-2\beta}$ with $2\beta \approx 6.8$. Equation (11) then implies that $N_{\Delta t}$ alone cannot explain the value $\alpha \approx 3$.

Since $N_{\Delta t}$ is not sufficient to account for the fat tails in $G_{\Delta t}$, one other possibility is that it arises from $W_{\Delta t}$. By definition, $W_{\Delta t}$ is the variance of all

δp_i in Δt, which is difficult to estimate when one does not have sufficient $N_{\Delta t}$. We can investigate the statistics of $W_{\Delta t}$ and examine if the distribution of $W_{\Delta t}$ is sufficient to explain the value of α found for $P\{G_{\Delta t} > x\}$. Our results on 30 actively-traded stocks suggest that

$$P\{W_{\Delta t} > x\} \sim x^{-\gamma}, \tag{12}$$

where we obtain rough estimates $\gamma = 2.85 \pm 0.20$, consistent with the estimates of α for the same 30 stocks. Estimates of γ are obtained by choosing $\Delta t =$ 15 min for these stocks, at the same time ensuring that $N_{\Delta t} > 20$.

7.5 Volatility Correlations and Trading Activity

Thus far we have discussed Eq. (11) from the point of view of distributions. Next, we analyze time correlations in $N_{\Delta t}$ and relate them to the time correlations of $|G_{\Delta t}|$. Our studies on the same 30 actively-traded stocks indicate that the autocorrelation function $\langle N_{\Delta t}(t) N_{\Delta t}(t+\tau)\rangle \sim \tau^{-\nu}$, with a mean value of the estimates of $\nu = 0.32 \pm 0.09$ using the detrended fluctuation analysis method [89]. To detect genuine long-range correlations, the marked U-shaped intra-daily pattern [85] in $N_{\Delta t}$ is removed [82]. We substantiate this analysis using semi-parametric estimators such as those due to Robinson [90]. We can then test the dependence of the exponent ν on (i) the type of industry sector and (ii) the market capitalization.

Our long-term goal is to relate the exponent ν of the autocorrelation function of $N_{\Delta t}$ to that of $|G_{\Delta t}|$. To this end, we also estimate, in parallel, the time correlations in $W_{\Delta t}$ and $|\epsilon|$. Since our investigations on the 30 stocks seem to indicate the absence of long-range correlations in $W_{\Delta t}$, the above investigation of correlations could yield the interesting statement that the long-range correlations in volatility are due to those of $N_{\Delta t}$. Together with the above discussion on distribution functions, these results suggest an interesting result—that the fat tails of returns $G_{\Delta t}$ arise from $W_{\Delta t}$ and the long-range volatility correlations arise from trading activity $N_{\Delta t}$.

7.6 Statistics of Share Volume Traded

Understanding the equal-time correlations between volume and volatility and, more importantly, understanding how the number of shares traded impacts the price has long been a topic of great interest [77, 91, 101, 98, 99, 100, 102, 103]. The number of shares traded in Δt is the sum

$$Q_{\Delta t} \equiv \sum_{i=1}^{N_{\Delta t}} q_i\,, \tag{13}$$

where q_i traded for all $i = 1, \ldots, N_{\Delta t}$ transactions in Δt. So it is clear that $Q_{\Delta t}$ must be positively correlated with $N_{\Delta t}$.

The results of Plerou et al. on 30 actively-traded stocks suggest that the probability distributions $P\{Q_{\Delta t} > x\}$ are consistent with a power law asymptotic behavior

$$P\{Q_{\Delta t} > x\} \sim x^{-\lambda} . \tag{14}$$

Using Hill's estimator, they obtain an average value $\lambda = 1.7 \pm 0.2$, which is within the Lévy stable domain $0 < \lambda < 2$. This result suggests that $Q_{\Delta t}$ can be effectively described using a one-sided (fully asymmetric) stable distribution. A parallel analysis for $P\{q_i > x\}$ (from Eq. (13)) yields consistent values of exponents within the Lévy stable domain, suggesting a divergent second moment.

As a further test for Lévy stability of $Q_{\Delta t}$, we can investigate the scaling behavior of the sum

$$Q_n \equiv \sum_{i=1}^{n} q_i, \tag{15}$$

where n is a fixed number of trades. We first analyze the asymptotic behavior of $P(Q_n)$ for increasing n. For a Lévy stable distribution, $n^{1/\lambda}\, P([Q_n - \langle Q_n \rangle]/n^{1/\lambda})$ should have the same functional form as $P(q)$, where $\langle Q_n \rangle = n\, \langle q \rangle$ and $\langle \ldots \rangle$ denotes average values. We can also perform an independent test and estimate λ by analyzing the scaling behavior of the moments

$$\mu_r(n) \equiv \langle |Q_n - \langle Q_n \rangle|^r \rangle, \tag{16}$$

where $r < \lambda$. For a Lévy stable distribution

$$[\mu_r(n)]^{1/r} \sim n^{1/\lambda}. \tag{17}$$

Hence, by regressing $[\mu_r(n)]^{1/r}$ as a function of n, we obtain an inverse slope which would yield an estimate of λ.

7.7 Share Volume Traded and Number of Trades

If our hypothesis is true that $Q_{\Delta t}$ and q_i are consistent with a one-sided Lévy stable process, then $N_{\Delta t}^{1/\lambda}\, P([Q_{\Delta t} - \langle q \rangle\, N_{\Delta t}]/N_{\Delta t}^{1/\lambda})$ from Eq. (13) should have the same distribution as any of the q_i. Thus, we hypothesize that the dependence of $Q_{\Delta t}$ on $N_{\Delta t}$ can be separated by defining

$$\chi \equiv \frac{Q_{\Delta t} - \langle q \rangle N_{\Delta t}}{N_{\Delta t}^{1/\lambda}}, \tag{18}$$

where χ is a one-sided Lévy-distributed variable with zero mean and exponent λ. To test this hypothesis, we first analyze $P(\chi)$ for consistent asymptotic behavior to $P(Q_{\Delta t})$.

7.8 Time Correlations in Share Volume Traded

We also study extensively the time correlations in $Q_{\Delta t}(t)$. A difficulty arises due to the divergent second moment of the distribution $P(Q_{\Delta t})$. To circumvent this problem, we consider the family of correlation functions $\langle [Q_{\Delta t}(t)]^a [Q_{\Delta t}(t+\tau)]^a \rangle$, where the parameter a ($< \lambda/2$) is required to ensure that the correlation function is well defined. Instead of analyzing the correlation function directly, we apply detrended fluctuation analysis [89], which has been successfully used to study long-range correlations in a wide range of complex systems. Our results suggest that $Q_{\Delta t}(t)$ has strong long-range correlations, while the number of shares traded in each transaction q_i (Eq. 13) displays only short-range correlations, suggesting that long-range correlations in $Q_{\Delta t}$ can in turn be related to those of $N_{\Delta t}$, if Eq. (18) is found to be valid.

7.9 Returns and Share Volume Traded

An interesting implication is an explanation for the previously-observed [101, 102, 103] equal-time correlations between $Q_{\Delta t}$ and volatility $V_{\Delta t}$, which is the local standard deviation of price changes $G_{\Delta t}$. Now, $V_{\Delta t} = W_{\Delta t}\sqrt{N_{\Delta t}}$ from Eq. (11). Consider the equal-time correlation $\langle Q_{\Delta t}\, V_{\Delta t}\rangle$, where the means are subtracted from $Q_{\Delta t}$ and $V_{\Delta t}$. Since $Q_{\Delta t}$ depends on $N_{\Delta t}$ through $Q_{\Delta t} = \langle q \rangle N_{\Delta t} + N_{\Delta t}^{1/\zeta}\,\chi$, and if the equal-time correlations $\langle N_{\Delta t}\, W_{\Delta t}\rangle$, $\langle N_{\Delta t}\,\chi\rangle$, and $\langle W_{\Delta t}\,\chi\rangle$ are small (orrelation coefficients ≈ 0.1), it follows that the equal-time correlation $\langle Q_{\Delta t}\, V_{\Delta t}\rangle \propto \langle N_{\Delta t}^{3/2}\rangle - \langle N_{\Delta t}\rangle\langle N_{\Delta t}^{1/2}\rangle$, which is positive due to the Cauchy-Schwartz inequality.

8 Are Power Laws Really There?

Newcomers to the field of scale invariance often ask why a power law does not extend "forever" as it would for a mathematical power law of the form $f(x) = x^{-\alpha}$. This legitimate concern is put to rest by reflecting on the fact that power laws for natural phenomena are not equalities, but rather asymptotic relations of the form $f(x) \sim x^{-\alpha}$. Here the tilde denotes *asymptotic equality*. Thus $f(x)$ is not "approximately equal to" a power law, making the notation $f(x) \approx x^{-\alpha}$ inappropriate. Similarly, $f(x)$ is not proportional to a power law, so the notation $f(x) \propto x^{-\alpha}$ is also inappropriate. Rather, asymptotic equality means that $f(x)$ becomes increasingly like a power law as $x \to \infty$. Moreover, crossovers abound in financial data, such as the crossover from power law behavior to simple Gaussian behavior as the time horizon Δt over which fluctuations are calculated increases beyond about a year (i.e., the power law behavior holds for time horizons up to a month or even a year, but for horizons exceeding a year there is a distinct crossover to Guassian behavior). Such crossovers are also characteristic of other scale-free phenomena in the physical sciences [8, 9], where the Yule distribution often proves quite useful.

For reasons of this sort, standard statistical fits to data are inappropriate, and often give distinctly erroneous values of the exponent α. Rather, one reliable way of estimating the exponent α is to form successive slopes of pairs of points on a log-log plot, since these successive slopes will be monotonic and converge to the true asymptotic exponent α. One finds that successive slopes for the empirical data converge rapidly to a value $\alpha \approx 3$, while successive slopes for the model diverge. While it is clear that a simple three-factor model [104] cannot generate power law behavior, it is less clear why the empirical data analyzed appear at first glance to be well approximated by the model. The first fact is that the region of linearity of the data is not as large as in typical modern studies because the total quantity of data analyzed is not that large, since only a low-frequency time series comprising daily data is used. Only 28,094 records are analyzed [104] (not 4×10^7 as in recent studies [53, 54]) and the model simulations are presented for a limited sample size. The second fact is that when one superposes a curved line (the model) on a straight line (the data), the untrained eye is easily tempted to find agreement where none exists—and closer inspection of Figs. 2–5 of Ref. [104] reveals actually a rather poor agreement between model and data due to the pronounced downward curvature of the model's predictions [105].

9 Other Scale-Invariant Quantities Describing Economic Fluctuations

Other quantities characterizing stock movements (such as the volatility, share volume traded, and number of trades) also display a range of power law behavior (typically $\approx 10^2$) [82, 99, 100, 106]. The exponents characterizing the power law decays are different for different quantities; it is tempting to conjecture that in finance there may exist a set of relations among the power law exponents found, just as relations exist among the exponents characterizing different quantities near the critical point. Finally, it is well known that while the autocorrelation function of price returns decays rapidly, the autocorrelation function of the absolute values of price returns is power-law correlated in time (see [82] and extensive earlier work cited therein).

Consider, for example, the volatility. There are several possible definitions of this quantity, all of which seem to give the same scale invariant properties. But why care about volatility at all? On the cover of the 15 May 2000 issue of Forbes magazine is a large photograph of Henk Paulson, CEO of Goldman Sachs, and the headline quotation *"Volatility is Our Friend."* Why is this the case? Because it is known that volatility *clusters*, i.e., there are time correlations in this quantity. Our group has attempted to quantify these correlations, and has found evidence of power law behavior [82, 107, 108, 109]. If we plot an economic earthquake such as Black Monday (19 October 1987), a date on which most worldwide stock indices dropped 25–50 percent, and then plot and compare the volatility (the absolute value of the fluctuations), we see a big peak in the volatility curve on Black Monday. But even prior to Black Monday the value of the volatility on our graph seems to be particularly unstable; there is some precursor to Black Monday evident in its behavior. One can imagine a computer program that would monitor volatility, not necessarily for the entire market but certainly for an individual stock, and the volatility calculation would need to be updated in real time.

There are correlations in the stock price change, but those correlations have a very short range—on the order of a few minutes—and they decay exponentially in time. Our group calculated the autocorrelation function of stock-price changes and plotted the logarithm of the function linearly in time; since the logarithm of e^{-x} is $-x$, we get a straight line. In contrast, for the volatility we find that the autocorrelation function is linear on log-log paper, meaning that the correlations in the volatility are power-law in nature. That, in turn, translates to mean they are much, much longer-range in time.

In order to quantify long-range power law volatility correlations, we developed a method of analyzing a non-stationary time series. The volatility of a financial market is non-stationary: there are days when the volatility is quiet and days when it is active. The statistical properties of a volatility time series are changing in time. The standard deviation of that time series is fluctuating wildly on every scale, which is the reason conventional methods are not effective. The method our group has been developing—detrended fluctuation analysis (DFA)—gets rid of trends in the raw data [89, 110, 111]. We take a graph of the volatility expressed in absolute values (i.e., it is always positive) in which we see the peaks that indicate it is a very "noisy" or non-stationary time series, we integrate this time series, and we subtract the mean. This produces an up-and-down "landscape." We then look for correlations in this landscape. We do this by partitioning the landscape into "windowboxes" of a fixed size (e.g., 200) and asking whether the regression fits the fluctuations in that windowbox. We then calculate for each box the RMS fluctuation around the regression line. Finally, we average the RMS fluctuation for all 40,000 windowboxes of the entire series. With that many windowboxes, we get a very accurate measurement. We call the quantity f. We repeat the entire calculation for windowboxes one-half as big (size 100). Obviously, the smaller the windowbox, the less the fluctuation. This gives us the circle for size 100. We repeat this a number of times. When that fluctuation is plotted as a function of windowbox size we find, contrary to what we might expect—that in almost all correlated signals the fluctuations increase as the square root of the windowbox size—the fluctuations instead increase more rapidly than that. That means there is some positive correlation in the signal. This analysis method produces results with very little noise. The data fall very close to the straight line, and the exponent can be obtained with a high degree of accuracy. All this allows us to analyze quantitatively the behavior of the volatility as a function of time and elucidate its correlations. This could be very useful information for people actually working in financial markets.

The distribution of volatility fluctuations has also been the object of extensive study. It was at one time believed by many that the volatility follows a log-normal distribution—i.e., the number of times the volatility has a certain value follows not a Gaussian but a log-normal distribution (one has $e^{-(\log x)^2}$ not e^{-x^2}). But until our group's work, no one had studied *all* the data, i.e., *every* trade [82]. Our doing it meant we could study relatively rare events, those occurring much less frequently than everyday events. What we find is that the log-normal part of the curve—the middle—though true for the middle, does not describe the tails. The huge volatilities in the tails are described by a different exponent μ. We also see that volatility clusters—i.e., that volatility is correlated in time.

10 Cross-correlations Among Fluctuations of Different Stocks

Another capability of an appropriately-designed software package could be the ability to determine how the fluctuations of one stock price correlate with those of another. This question of cross-correlation is one we have been studying [112, 113, 114, 115]. To quantify cross-correlations, we draw a circle corresponding to the stock price x and draw a second circle corresponding to the stock price x, e.g., five minutes later. If we make the difference in the radii proportional to G, the stock price change, then we can think of the market as thousands of circles, each growing and shrinking—a kind of pulsation that is a function of time. The key is that these correlations change in time. Car sales by Ford and GM may be anti-correlated during some time periods and positively correlated during others.

The standard approach to this problem is to calculate, by brute force, a huge square matrix that has as many rows as there are companies in the database. Each element of the matrix is the correlation between the price change of company i and the price change of company j, but to find a genuine correlation we have to be able to distinguish between correlations from coincidences. In order to do that we draw on something developed by Wigner in his work in nuclear physics—random matrix theory. Random matrix theory compares the matrix calculated by brute force from stock market data with a random matrix that also has 1000 rows and 1000 columns—but with every number generated randomly. Somewhere hidden in the huge matrix calculated by brute force from stock market data are the true correlations. To uncover them, we first diagonalize the matrix in order to determine its eigenvalues, and then make a histogram that gives the number of times each given eigenvalue is found. The histogram curve of a random matrix, unlike this one from real data, can be predicted exactly. For a random matrix there is never an eigenvalue larger than 2.0. The histogram of the empirical stock price data, on the other hand, contains a significant number of eigenvalues larger than

2.0. Some are as big as 5.0. These eigenvalues of necessity must correspond to genuine correlations.

The eigenvalue of a matrix has a corresponding eigenvector—a column matrix of 1000 elements—each element of which is a different weight from each of the 1000 stocks. Thus we can look at the column vectors that correspond to these deviating, genuinely-correlated eigenvalues and ask: what kind of stocks entered into each of these eigenvectors? What we found, fortunately, has implications for portfolios. If we restart the graph at 2.0—removing the distortions of the random values—and look at the 20 eigenvalues larger than 2.0, we see that the stocks that make up most of the weights in the corresponding eigenvectors are almost entirely transportation stocks in the first case, almost entirely paper in the second, almost entirely pharmaceuticals in the third, and so on. In other words, the market *automatically* partitions itself into separate business sectors [114, 115, 116]. Thus a physicist who knows nothing about the stock market can mathematically partition the economy into separate business sectors!

The sectors and the quantitative degree to which each constituent firm conforms to the sector can be monitored and updated as a function of time, e.g., every 15 minutes. Firms that belong to the same business sector can be monitored in a kind of rainbow spectrum. The "good" firms sticking to the business sector are assigned to the "violet" end of the spectrum, and the "bad" firms deviating from the sector are assigned to the "red." When a firm first starts to move to the red end of the spectrum, i.e., starts to deviate, the trader is alerted to consider action.

11 Equilbrium vs. Out-of-Equilibrium Market Phases

Before concluding, we ask what sort of understanding could eventually develop if one takes seriously the power laws that appear to characterize finance fluctuations. It is tempting to imagine that there might be analogies between finance and known physical processes displaying similar scale-invariant fluctuations. One initially promising analogy was with turbulence: In turbulence, one adds energy at a large scale and this energy is dissipated at smaller and smaller scales in a scale-invariant fashion. Similarly, if external news is added at a large scale, then this news is dissipated by traders at smaller and smaller scales in a scale-invariant fashion. Despite some initial claims, these similarities are not borne out by quantitative analysis—although one finds non-Gaussian statistics, and intermittency, for both turbulence fluctuations and stock price fluctuations, the time evolution of the second moment and the shape of the probability density functions are different for turbulence and for stock market dynamics [70, 71].

More recent work pursues a rather different analogy: phase transitions in spin systems. One might be tempted to say that the set of all firm fluctuations is like a set of subunit fluctuations in a physics system such as a spin glass (the simplest spin glass is a set of Ising spins, each interacting with other spins by means of an interaction whose sign and range are chosen from a probability distribution). Similarly, in economic fluctuations, the sign of the fluctuation can be up or down, and can be of any magnitude; fluctuations interact with one another via interactions that are certainly long-range and of both signs. Further, the interactions change with time. A given subunit fluctuation is influenced (a) by other fluctuations (so the exchange interactions among spins is somewhat like the "herd effect"), and (b) by forces external to the system (so the external field is somewhat like "news" which plays a role in determining the sign and magnitude of fluctuations). Economists all appreciate these (and all other!) facts—but for physicists they are truly reminiscent of a cooperative system for which emergent behavior can result from collective behavior of subunits, each obeying relatively simple and "local" rules.

If this crude analogy were to hold even approximately, then a first step should perhaps be to seek to identify the analogs for the price fluctuation problem of field and temperature in the magnetic problem. Stock prices respond to demand, just as the magnetization of an interacting spin system responds to the magnetic field. Periods with large number of market participants buying the stock imply mainly positive changes in price, analogous to a magnetic field causing spins in a magnet to align. Recent work [117] quantifies the relations between price change and demand fluctuations, and finds results reminiscent of phase transitions in spin systems (Fig. 1), where the divergent behavior of the response function at the critical point (zero magnetic field) leads to large fluctuations [8]. More precisely, buying and selling behavior in complex financial markets are driven by demand, which can be quantified by the imbalance in the number of shares transacted by buyers and sellers over a time interval Δt.

If demand is the analog of the magnetic field, what is the analog of temperature? To answer this question, Plerou et al. [57, 58] analyze the probability distribution of demand, conditioned on its local noise intensity Σ, and find the surprising existence of a critical threshold Σ_c separating two market phases (Fig. 2). Their findings for the financial market problem are identical to what is known to occur in all phase transition phenomena, wherein the behavior of a system undergoes a qualitative change at a critical threshold K_c of some control parameter K. Plerou et al interpret these two market phases as corresponding to two distinct conditions of the financial market: (a) The "$\Sigma < \Sigma_c$ market phase," where the distribution of demand is single-peaked with the most probable value being zero, they interpret to be the market *equilibrium phase*, since the price of the stock is such that the probability of a transaction being buyer initiated is equal to the probability of a transaction being seller initiated, and (b) the "$\Sigma > \Sigma_c$ market phase", where the distribution of demand is bimodal, they interpret to be the *out-of-equilibrium phase*, since the price of the stock is such that there is an excess of either buyers or of sellers and there is a non-zero net demand for the stock.

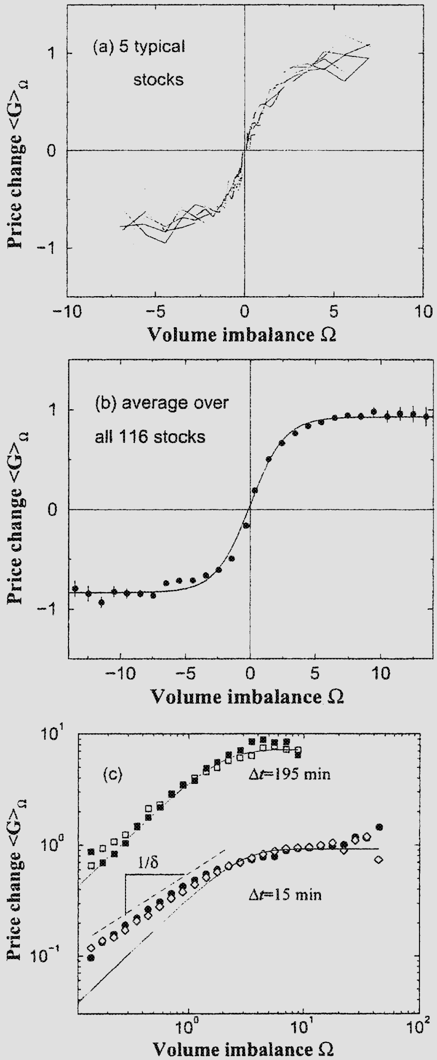

Fig. 1. (a) Conditional expectation $\langle G\rangle_\Omega$ for 5 typical stocks over a time interval $\Delta t = 15$ min, where Ω is defined as the difference in number of shares traded in buyer and seller initiated trades. We normalize G to have zero mean and unit variance. Since Ω has a tail exponent $\zeta = 3/2$ which implies divergent variance, we normalize Ω by the first moment $\langle|\Omega - \langle\Omega\rangle|\rangle$. (b) Conditional expectation $\langle G\rangle_\Omega$ averaged over all 116 stocks studied. We calculate G and Ω for $\Delta t = 15$ min. The solid line shows a fit to the function $B_0 \tanh(B_1\Omega)$. (c) $\langle G\rangle_\Omega$ on a log-log plot for different Δt. For small Ω, $\langle G\rangle_\Omega \simeq \Omega^{1/\delta}$. For $\Delta t = 15$ min find a mean value $1/\delta = 0.66{\pm}0.02$ by fitting $\langle G\rangle_\Omega$ for all 116 stocks individually. The same procedure yields $1/\delta = 0.34 \pm 0.03$ at $\Delta t = 5$ min (interestingly close to the value of the analogous critical exponent in mean field theory). The solid curve shows a fit to the function $B_0 \tanh(B_1\,\Omega)$. For small Ω, $B_0 \tanh(B_1\,\Omega) \sim \Omega$, and therefore disagrees with $\langle G\rangle_\Omega$, whereas for large Ω the fit shows good agreement. For $\Delta t = 195$ min ($\frac{1}{2}$ day) (squares), the hyperbolic tangent function shows good agreement.

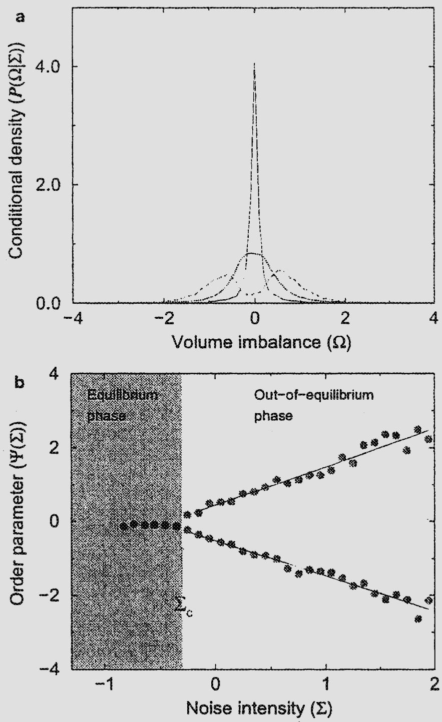

Fig. 2. Empirical evidence supporting the existence of two distinct phases in a complex financial market. (a) Conditional density $P(\Omega|\Sigma)$ for varying local noise intensity Σ computed using data for all stocks. For each stock, Ω and Σ are normalized to zero mean and unit first centered moment. The distribution displays a single peak for $\Sigma < \Sigma_c$ (solid line). For $\Sigma \approx \Sigma_c$ (dotted line), the distribution flattens near the origin, and for $\Sigma > \Sigma_c$, $P(\Omega|\Sigma)$ displays two peaks (dashed line). (b) Order parameter, Ψ (positions of the maxima of the distribution $P(\Omega|\Sigma)$), as a function of Σ. For small Σ, $P(\Omega|\Sigma)$ displays a single maximum whereas for large Σ two maxima appear. To locate the extrema as accurately as possible, we compute all probability densities using the density estimator of Ref. [133]. Also shown, via shading, is a schematic phase diagram representing the two distinct market phases. Here $\Delta t = 15$ min; we have tested that our results hold for Δt ranging from 15 min up to approximately 1/2 day, beyond which we have insufficient insufficient.

It should be possible to design a software package that could be on every trader's desk allowing instant access to data on any firm in which time is partitioned into two different phases: equilibrium and out-of-equilibrium. Qualitatively and informally many people use those terms in reference to the stock market, but in this case we would be actually *quantifying* the extent to which the market is in or out of equilibrium. If we graph the price-change of a particular stock as a function of time for a sequence of 15-minute intervals and use two different symbols for data points when the market is in equilibrium and for those for when it is out of equilibrium, we notice that, in general, a stock price is not changing when the market is in equilibrium and is changing when the market is out of equilibrium. This could be useful in that it could be an indicator of the relative stability of an individual stock. When the market is out of equilibrium, the probability that a stock price is going to change is higher than when the market is in equilibrium.

12 Discussion

Since the evidence for an analogy between stock price fluctuations and magnetization fluctuations near a critical point is backed up by quantitative analysis of finance data, it is legitimate to demand a theoretical reason for this analogy. To this end, we discuss briefly one possible theoretical understanding for the origin of scaling and universality in economic systems. As mentioned above, economic systems consist of interacting units just as critical point systems consist of interacting units. Two units are correlated in what might seem a hopelessly complex fashion—consider, e.g., two spins on a lattice, which are correlated regardless of how far apart they are. The correlation between two given spins on a finite lattice can be partitioned into the set of all possible topologically linear paths connecting these two spins—indeed this is the starting point of one of the solutions of the two-dimensional Ising model (see Appendix B of [8]). Since correlations decay exponentially along a one-dimensional path, the correlation between two spins would at first glance seem to decay exponentially. Now it is a mathematical fact that the total number of such paths grows exponentially with the distance between the two spins—to be very precise, the number of paths is given by a function which is a product of an exponential and a power law. The constant of the exponential *decay* depends on temperature while the constant for the exponential *growth* depends only on the geometric properties of the system [8]. Hence by tuning the temperature it is possible to achieve a threshold temperature where these two "warring exponentials" balance each other, allowing a previously-negligible power law factor that enters into the expression for the number of paths to dominate. Thus power law scale invariance emerges as a result of canceling exponentials, and universality emerges from the fact that the interaction paths depend not on the interactions but rather on the connectivity. Similarly, in economics, two units are correlated through a myriad of different correlation paths; "everything depends on everything else" is the adage expressing the intuitive fact that when one firm changes, it influences other firms. A more careful discussion of this argument is presented, not for the economy but for the critical phenomena problem, in Ref. [9].

Consider the following thought experiment. Measure the temperature in an extremely tiny region of a glass of water—say in a cube measuring only 10 Åon a side. Set the macroscopic water temperature to be exactly at water's critical temperature (about 647 K). The measured "local" temperature will fluctuate in time from being above or below the actual crtical temperature. Hence the tiny cube will fluctuate from being in the liquid or the gas phase, as a function of time. Similarly, it appears that the market fluctuates from being in one phase to another as a function of time [57, 58].

13 Concluding Remarks

In summary, physicists are finding this emerging field fascinating. For a long time, physicists and economists had noticed the potential for useful collaborations but, notwithstanding some stellar insights [118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130], not too many new results were discovered by physicists. A major reason for this is that, until recently, the amount of data routinely recorded concerning financial transactions was insufficient to be useful to physicists. That fact is no longer true. Now every trade is recorded, along with bid-ask quotes for every trade, and these data are available.

Part of the reason for the invention [131] of the neologism "econophysics" (in the tradition of the neologisms "biophysics," "astrophysics," "geophysics"...) was to enable our physics students to persuade the departmental administrators that their dissertation research topics actually belonged in the physics department. The neologism seems to have caught on, and there are now several conferences each year with the word "econophysics".

Finally, a word of humility with respect to our esteemed economics colleagues is perhaps not inappropriate. Physicists may care passionately if there are analogies between physics systems they understand (like critical point phenomena) and economics systems they do not understand. But why should anyone else care? One reason is that the scientific understanding of earthquakes moved ahead after it was recognized [1, 2] that extremely rare events—previously regarded as statistical outliers requiring for their interpretation a theory quite distinct from the theories that explain everyday shocks—in fact possess the identical statistical properties as everyday events; i.e., all earthquakes fall on the same straight line on an appropriate log-log plot. Since economic phenomena possess the analogous property, the challenge is to develop a coherent understanding of financial fluctuations that incorporates not only everyday fluctuations but also those extremely rare "financial earthquakes."

According to the economist Neil A. Chriss—who is affiliated with ICor Brokerage Incorporated in New York:

> "The aim of modern financial theory (or at least that part of modern finance having to do with financial markets) might be described as an attempt to produce theoretical models describing the behavior of financial markets, with an eye toward causal mechanisms, statistical laws, and even predictive power. Starting with assumptions about the behavior of rational economic agent, one makes restrictions on the set of possible laws describing financial markets. Adding simplifying assumptions such as frictionless markets, an absence of transaction costs, and unlimited short selling, the analysis is brought into the realm of the tractable. By observing the behavior of actual financial markets, through the collection and analysis of time series of financial data, one ultimately eliminates many models that are *a priori* possible but contrary to observed behavior" [132].

Thus one prevalent paradigm in economics is to marry finance with mathematics, with the fruit of this marriage the development of clever models, some of which are used in everyday trading. In physics, we also develop and make use of models (or "artificial worlds"). However a large number of physicists are fundamentally empirical in our approach to science—indeed, some physicists never make reference to models at all (other than in classroom teaching situations). This empirical approach has led to advances when theory has grown out of experiment; one such example is the understanding of phase transitions and critical phenomena. Such a basic and deep grounding in empirical facts could have an influence on the way physicists approach economics. Our approach has been to follow the paradigm of experimental physics, i.e., to first examine the empirical facts as thoroughly as possible before we begin to construct models.

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Modeling the Dynamics of a Financial Index after a Crash

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1 Introduction

Supply and demand are perhaps the most fundamental concepts in economics. In a financial market they reflects the orders of the agents to buy or sell a given asset. In turn the fluctuations of supply and demand influence the dynamics of the price of an asset, as, for example, a stock or a financial index. Therefore the dynamics of the price of an asset is affected by the actions and of the beliefs of the agents. It is known that the dynamics of the price of an asset is far from simple. Several stylized facts has been empirically discovered such as, for example, the fat tails in the return distribution and the clustered volatility. These stylized facts has been detected by considering long time series of returns and by computing statistical quantities, such as probability density function and correlation coefficient, over these long period of time. As a matter of fact this implies that one considers the time series as stationary in these statistical analyses. It is well known that basic financial indicators, such as for example the volatility, display non-stationary patterns [1]. Statistical tests on the volatility fluctuation strongly reject the hypothesis of constant volatility [2]. It is therefore worth investigating the presence of stylized facts in financial markets which specifically refer to non-stationary periods. In this paper we report a recently discovered statistical regularity found in the empirical analysis of the dynamics of the time series of a financial index just after a major financial crash [3]. The time period after a financial crash is characterized by a highly non-stationary dynamics of return time series. Moreover large negative returns act as a leverage increasing the volatility after the crash [4, 5]. In this paper we show how it is possible to statistically characterize the dynamics of the volatility after a major crash. To this end we investigate the number of times the absolute value of index return exceeds a given threshold value. We find that the frequency of exceedances decays as a power law in time after a market crash. The discovered statistical regularity is analogous to a statistical law discovered in geophysics more than a century ago and known

M. Gallegati et al. (eds.), *The Complex Dynamics of Economic Interaction*

today as the Omori law [6]. The Omori law states that the frequency of aftershock earthquakes measured at time t after the main earthquake decays with t as a power-law function. The observation of this specific functional form in the stochastic relaxation of the market statistical indicators to their typical values implies that the relaxation dynamics of a market index just after a financial crash is not characterized by a typical scale. Moreover this empirical finding suggests that the volatility of the index decays as a power law in time after the crash.

The outline of the paper is the following. In Section 2 we introduce the new empirical stylized fact describing the behavior of a financial index in the aftercrash period. We also show that simple autoregressive model are unable to capture such a behavior. In Section 3 we introduce a simple model of the aftercrash dynamics of the index, and we validate the model with numerical simulations. Section 4 is devoted to the discussion of the results and to conclusions.

2 Empirical Study of Aftercrash Time Series

In this section, we characterize the time series of ultra high frequency returns after a major market crash. The variable investigated is the one-minute logarithm changes of a financial index $r(t)$. A direct characterization of the time evolution of this random process is extremely difficult in the time period after a market crash. This is due to the fact that the aftercrash period is highly non-stationary because the financial market needs some time to be back to a "normal" period. In order to characterize the aftercrash return time series we make use of a statistically robust method. Specifically, we quantitatively characterize the time series of index returns by investigating the number of times $|r(t)|$ is exceeding a given threshold value in the non-stationary time period [3].

A similar approach is used in the investigation of the number $n(t)$ per unit time of aftershock earthquakes above a given threshold magnitude measured at time t after the main earthquake. This quantity is well described in geophysics by the Omori law [6]. The Omori law $n(t) \propto t^{-p}$ says that the number of aftershock earthquakes per unit time measured at time t after the main earthquake decays as a power law. In order to avoid divergence at $t = 0$, Omori law is often written as

$$n(t) = K(t + \tau)^{-p}, \tag{1}$$

where K and τ are two positive constants. An equivalent formulation of the Omori law, which is more suitable for comparison with real data, can be obtained by integrating Equation (1) between 0 and t. In this way the cumulative number of aftershocks after the main earthquake observed until time t is

$$N(t) = K[(t + \tau)^{1-p} - \tau^{1-p}]/(1 - p), \tag{2}$$

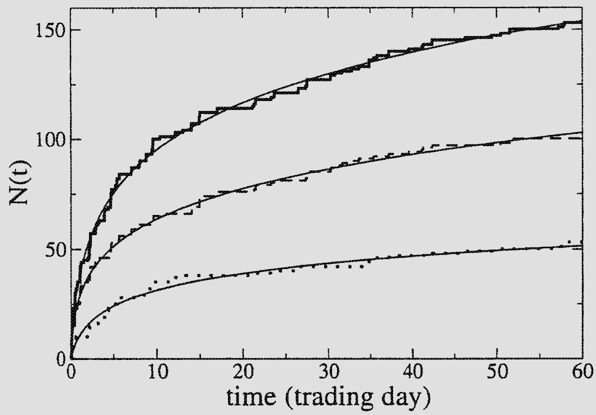

Fig. 1. The thick line is $N(t)$, the cumulative number of times $|r(t)|$ is exceeding the threshold $\ell = 5\sigma$ during the 60 trading days immediately after the Black Monday financial crash. The dashed (dotted) line is $N_+(t)$ ($N_-(t)$), the cumulative number of times that $r(t) > \ell$ ($r(t) < -\ell$) in the same time period. The thin lines are the best fit of Equation (2). The value of the exponent p is 0.90 for $N(t)$, 0.85 for $N_+(t)$ and 0.99 for $N_-(t)$.

when $p \neq 1$ and $N(t) = K \ln(t/\tau + 1)$ for $p = 1$. The value of the exponent p for earthquakes ranges between 0.9 and 1.5.

When the random process is stationary the frequency of aftershock $n(t)$ is on average constant in time and therefore the cumulative number $N(t)$ increases linearly in time. We have tested that $N(t)$ increases approximately linearly in a market period of roughly constant volatility such as, for example, the 1984 year. For independent identically distributed random time series it is possible to characterize $n(t)$ in terms of an homogeneous Poisson process [7].

As an example of the behavior of $N(t)$ empirically observed after a market crash, we discuss the results obtained by investigating the index returns during the time period just after the Black Monday crash (19 October 1987) occurred at New York Stock Exchange (NYSE) [3]. This crash was one of the worst crashes occurred in the entire history of NYSE. The Standard and Poor's 500 Index (S&P500) went down 20.4% that day. In our investigation, we select a 60 trading day aftercrash time period ranging from 20 October 1987 to 14 January 1988. For the selected time period, we investigate the one-minute return time series of the S&P500 Index. The unconditional one-minute volatility is equal to $\sigma = 4.91 \cdot 10^{-4}$. In Figure 1 we show the cumulative number of events $N(t)$ detected by considering all the occurrences observed when the the absolute value of index return exceeds a threshold value $\ell = 5\sigma$. We observe a nonlinear behavior in the entire period. We observe a similar behavior when we set the threshold value ℓ equal to 4σ, 6σ and 7σ. The best fit of the exponent p slightly increases when ℓ increases and eventually converges to a constant value. Figure 1 also shows our best nonlinear fit performed with the functional form of Equation (2). The agreement between empirical data and the functional form of the Omori law is pretty good.

We also consider the contribution to $N(t)$ coming from positive and negative values of $r(t)$ in a separate way. In Figure 1 we show $N_+(t)$ ($N_-(t)$), i.e. the cumulative number of times $r(t)$ is larger than $\ell = 5\sigma$ (smaller than $\ell = -5\sigma$). Obviously it is $N(t) = N_+(t) + N_-(t)$. For both $N_+(t)$ and $N_-(t)$ a behavior consistent with the Omori law of Equation (2) is still observed. The value of p obtained with a best fit with Equation (2) is $p = 0.84$ for $N_+(t)$ and $p = 0.99$ for $N_-(t)$. These values are to be compared with the value $p = 0.90$ obtained by a best fit of $N(t)$. It is also worth noting that $N_+(t) > N_-(t)$. This means that the distribution of return is positively skewed in the aftercrash period. This result is not specific of the aftercrash periods. In fact empirical analysis performed on high frequency data shows that the skewness of the density function of index return is positive. This is in contrast with the negative value of the skewness found in the analysis of daily data (see, for example, [8]).

This paradigmatic behavior is not specific of the Black Monday crash of the S&P 500 index. In fact, we observe similar results also for a stock price index weighted by market capitalization for the time periods occurring after the 27 October 1997 and the 31 August 1998 stock market crashes [3]. This index has been computed selecting the 30 most capitalized stocks traded in the NYSE and by using the high-frequency data of the *Trade and Quote* (TAQ) database issued by the NYSE.

The empirical finding that the time series of index returns follows the Omori law after a major market crash cannot be explained by a simple autoregressive model [9]. We consider here the GARCH(1,1) model [10] described by the equation

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \tag{3}$$

where r_t is the return and σ_t is the volatility. To this end we estimate the best value of the parameters of the GARCH(1,1) model. The estimation has been performed with the G@RCH 2.3 package The package and its documentation can be downloaded at the web site http://www.egss.ulg.ac.be/garch/. The best estimation of the parameters on the time series of one minute logarithmic price change during 60 trading days after Black Monday crash are $\alpha_0 = 2.87 \cdot 10^{-8}$, $\alpha_1 = 0.38$ and $\beta_1 = 0.54$. With these parameters we generate 10^4 surrogate time series according to Equation (3) and we compute the average behavior of $N(t)$. To mimic the dynamics after the crash, we set a large value of return as an initial condition in each realization. Figure 2 shows $N(t)$ versus time for real data and GARCH(1,1) model simulations (dotted line) with a

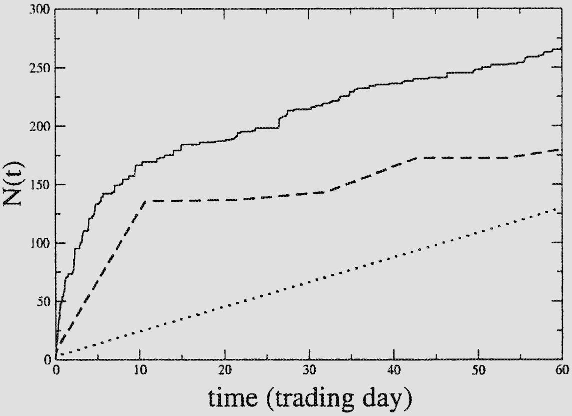

Fig. 2. The dotted line is the mean value of the cumulative number $N(t)$ of the number of times $|r(t)|$ is exceeding the threshold $\ell = 4\sigma$ over 10^4 simulations obtained with a GARCH(1,1) model with parameters estimated in a period of 60 trading days immediately after the Black Monday financial crash. The dashed line is the mean of $N(t)$ over 10^4 simulations obtained with a GARCH(1,1) model with parameters estimated in 6 nonoverlapping windows of 10 days each. The solid line is the empirical $N(t)$ for the same threshold value.

threshold $\ell = 4\sigma$. The GARCH(1,1) time series converges to its stationary phase very quickly and it is unable to show a significant nonlinear behavior.

In order to take into account the non-stationary behavior of the return time series after a crash we have performed a different investigation. Specifically we divide the 60 trading day period in 6 non-overlapping time intervals of 10 trading days and we estimate the GARCH(1,1) parameters specific for each interval. We then generate 10^4 GARCH(1,1) surrogate time series using the estimated parameters for each subinterval. The mean value of $N(t)$ as a function of time obtained with this procedure is also shown in Figure 2 as a dashed line. In each subinterval $N(t)$ increases linearly for almost all the period. This is due to the fact that the non-stationary part is extremely small compared with the time scale of the figure.

In fact numerical simulations of GARCH(1,1) model show that the dynamics of $n(t)$ after an extreme event is essentially an exponential decay to the unconditional value. These results show that a simple autoregressive model is unable to capture the dynamics of the transition of the index time series from an aftercrash to a normal period. There is need for a different modelization. In [3] we introduced such a model. In the next section we describe the properties of this model and we validate the model through numerical simulations.

3 Stochastic model of aftercrash time series

3.1 Description of the model

We assume that the stochastic variable $r(t)$ describing the index return can be written as

$$r(t) = \gamma(t)\; r_s(t) \tag{4}$$

where $\gamma(t)$ is a deterministic time dependent scale parameter and $r_s(t)$ is a stationary stochastic process. Under these assumptions, the frequency of events of $|r(t)|$ larger than ℓ observed at time t is proportional to

$$n(t) \propto 1 - F_s\left(\frac{\ell}{\gamma(t)}\right), \tag{5}$$

where $F_s(r_s)$ is the cumulative distribution function of the random variable $r_s(t)$.

By assuming that the stationary return probability density function behaves asymptotically as a power-law

$$f_s(r_s) \sim \frac{1}{r_s^{\alpha+1}}, \tag{6}$$

and that the scale of the stochastic process decays as a power-law $\gamma(t) \sim t^{-\beta}$, the number of events above threshold is power-law decaying as $n(t) \sim (\gamma(t)/\ell)^{\alpha} \sim 1/t^{p}$. The exponent p is given by

$$p = \alpha\; \beta. \tag{7}$$

The previous relation links the exponent p governing the number of events exceeding a given threshold to the α exponent of the stationary power-law return cumulative distribution and to the β exponent of the power-law decaying scale. It is direct to show that if we relax one of the two conditions of (i) Equation (6) and (ii) of power-law decaying $\gamma(t)$, the Omori law is not consistent with the model. For example, if the scale $\gamma(t)$ decays exponentially (as in autoregressive models), then $n(t)$ decays exponentially too.

3.2 Validation of the model

The hypotheses of our model are consistent with recent empirical results. In fact, a return probability density function characterized by a power-law asymptotic behavior has been observed in the price dynamics of several stocks [8, 11] and a power-law or power-law log-periodic decay of implied volatility has been observed in the S&P500 after the 1987 financial crash [12]. In Ref. [3] we have tested the hypotheses of our model by measuring independently the exponents α and β from real data and we have shown that the relation between exponents described by Equation (7) is satisfied in all the three investigated crashes. In the cited reference, by using the ordinary least square method, we fitted the absolute value of return with the functional form $f(t) = c_1 t^{-\beta} + c_2$ in the 60 day time period after each considered market crash. We checked that the relation $c_1 t^{-\beta} >> c_2$ is verified in the investigated period. By using the relation of Equation (4), we found that the β exponent obtained with this procedure is also the exponent controlling the scale $\gamma(t)$. The fitted value of the exponent β is in the range $0.2 \div 0.3$ in the three investigated crashes. In order to estimate the α exponent governing the stationary part of the return evolution we defined a new variable $r_p(t)$ obtained dividing $r(t)$ by the moving average of its absolute value. In Ref. [3] the averaging window was set to 500 trading minutes and it was shown that the quantity $r_p(t)$ is a good proxy for the stationary return $r_s(t)$. We investigated the asymptotic properties for large absolute values of the stochastic process $r_p(t)$ by computing the Hill's estimator [13] of the process. To assess the reliability of the α estimate obtained with this method we also computed its 95% confidence interval. With our procedure, we obtained a value of the exponent α which is 3.18 ± 0.34 (1987 crash), 3.67 ± 0.40(1997 crash) and 3.49 ± 0.37 (1998 crash). These values are consistent with the values found in unconditional financial time series. By using the estimated value of the three exponents α, β and p, we proved that the matching condition between exponents of Equation (7) is verified in all the three investigated crashes. The agreement proves that the empirical data can be described by our model.

In order to explicitly confirm that our model of Equation (4) is able to describe the dynamics of a financial index after a crash, here we perform numerical simulations of the model. We consider the Black Monday crash and we generate artificial time series obtained by multiplying a deterministic time dependent scale of the form $\gamma(t) = c_1 t^{-\beta} + c_2$ times a time series obtained by bootstrapping the proxy $r_p(t)$. Specifically, for each time step of our numerical simulations we extract randomly one of the values of the empirically determined $r_p(t)$. In such a way we are simulating a stationary time series $r_s(t)$ with the same unconditional properties of the proxy $r_p(t)$ but we remove all the correlations by randomizing the sequence of records. The scale $\gamma(t)$ is generated by using the values $c_1 = 6.3 \cdot 10^{-4}$, $c_2 = 2.8 \cdot 10^{-6}$ and $\beta = 0.32$ obtained from a best fit of real data as explained above.

Figure 3 shows the cumulative number of aftershocks $N(t)$ corresponding to a threshold value $\ell = 5\ \sigma$ for the Black Monday crash (same data as in Figure 1). The dashed line in Figure 3 is the average over 100 realizations of $N(t)$ for the artificial time series and the dotted lines are the one standard deviation error bars. The agreement between real data and model is quite good. This implies that our modeling of the index return time series as the product of a stationary power-law distribution times a deterministic power law decaying volatility is rather accurate in an aftercrash period.

The numerical simulations are also able to take into account (at least qualitatively) the asymmetry between $N_+(t)$ and $N_-(t)$ shown in Figure 1. Since the proxy $r_p(t)$ is positively skewed, also in the numerical simulations we

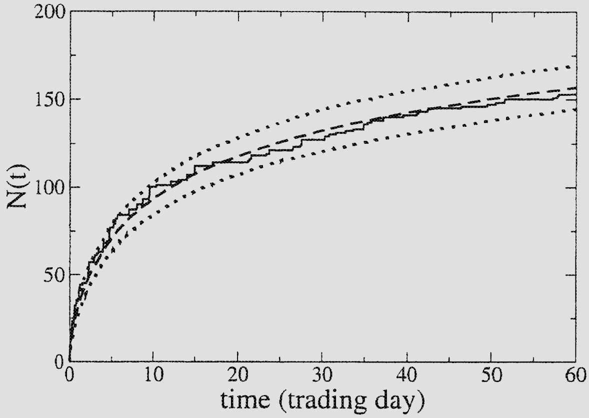

Fig. 3. The dashed line is the mean over 100 numerical realizations of $N(t)$ obtained by using the model of Equation (4) with a threshold $\ell = 5\sigma$. The dotted lines are the error bars corresponding to one standard deviation. As a solid line we also show the cumulative number of times $|r(t)|$ is exceeding the threshold $\ell = 5\sigma$ during the 60 trading days immediately after the Black Monday financial crash (same data as in Figure 1).

observe $N_+(t) > N_-(t)$. However the effect is less pronounced in simulations than in real data. In other words the numerical simulations underestimate $N_+(t)$ and overestimate $N_-(t)$, whereas $N(t)$ is correctly estimated (see Figure (3)).

4 Conclusions

Our empirical observations show that the statistical properties of index return time series after a major financial crash are essentially different from the ones observed far from the crash. Other examples of statistical properties of market which are specific of the aftercrash period have been observed in the investigation of cross-sectional quantities computed for a set of stocks before, at and after financial crashes [14, 15]. Simple autoregressive model, such as GARCH(1,1), are not able to reproduce the Omori law in financial markets. On the other hand a simple model which takes into account (i) a return probability density function characterized by a power law asymptotic behavior and (ii) a deterministic power law decay of volatility is able to reproduce quite well the empirical properties of a financial index after a crash.

It is worth noting that our results are relevant for risk management. We have shown that it is rather accurate to model the dynamics of the volatility as a deterministic process after a big crash. This suggests the possibility of devising a risk management strategy able to quantify the time dependent risk in the turbulent period after a major crash. Specifically in [9] we have shown that the Value at Risk of a financial portfolio measured just after a financial crash evolves as a power-law time evolution which is lacking any typical scale.

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Price Impact Function of a Single Transaction

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1 Introduction

Although supply and demand are perhaps the most fundamental concepts in economics, finding any general form for their behavior has proved to be elusive. Here we discuss our recent findings [1] on the price impact function empirically detected in the New York Stock Exchange (NYSE). Our study builds on earlier studies of how trading affects prices [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. In particular, we look at the short term response to a *single* trade. This is done by using huge amounts of data and by measuring the market activity in units of transactions rather than seconds, so that we can more naturally aggregate data for many different stocks. This allows us to find regularities in the response of prices to new orders that have previously been hidden by the extremely high noise levels that dominate financial prices. We study the 1000 largest stocks of the NYSE, from (1995-1998), and we find that, by appropriate averaging and rescaling, it is possible to collapse the price shift caused by a transaction onto a single curve [1]. In our study, we showed that the price shift grows slowly with the transaction size, with a power that is the order of 1/2 in the small volume limit and much more slowly in the large volume limit. These findings are observed across many stocks and for four different years.

2 Empirical Analysis of the Price Impact Function

The response of prices to orders is a key property of a market. If an attempt to buy or sell results in a small change in price, then the traded asset is considered *liquid*; otherwise it is considered *illiquid*. One expects liquidity to depend on properties of the asset, such as trading volume, or for stocks, the market capitalization (the total worth of the company, i.e. the total number of shares times their price).

M. Gallegati et al. (eds.), *The Complex Dynamics of Economic Interaction*

The study is based on the Trades And Quotes (TAQ) database, which contains the prices for all transactions as well as price quotations (the best offers to buy and sell at a given price at any given time) for the US equity markets. For illustrative purposes we present here results obtained for the 1996 year. The complete study presenting the investigation of the period 1995-1998 can be found in Ref. [1].

Our goal is to understand how much the price changes on average in response to an order to buy or sell of a given size. Of course, in each trade there is both a buyer and a seller. Nonetheless, one often loosely refers to a trade as a "buy" or a "sell" depending on whether the initiator of the trade was buying or selling. By *initiator* we mean the agent who placed the more recent order. Buy orders tend to drive the price up, and sell orders tend to drive it down. It is this *price impact* that we are interested in.

Based on only transactions and quotes it is not possible to know with certainty whether trades are initiated by buyers or sellers. However, we can infer this indirectly by using an algorithm developed by Lee and Ready [12]. This algorithm identifies the correct sign of trades by comparing the prices of transactions with recent quotes. The Lee and Ready algorithm is able to classify the sign of approximately 85% of the trades of our database. An order by a single party may trigger transactions with multiple counterparties; from the TAQ database we can only see transactions. To cope with this, we lump together all transactions with the same timestamp and treat them as a single trade.

We study the shift in the midquote price caused by the most recent transaction. For each transaction of volume ω occurring at time t we observe two cases: (i) When the next event is a quote revision, we compare the next quote to the prevailing (previous) quote, and compute the difference in the logarithm of the midquote price. Letting the logarithm of the midquote price be $p(t)$, we compute the price shift $\Delta p(t_{i+1}) = p(t_{i+1}) - p(t_i)$, where t_i is the time of the prevailing quote and t_{i+1} is the time of the next quote following the transaction; (ii) When the next event is a new transaction we set the price shift $\Delta p(t_{i+1})$ to zero [13]. We then investigate the average price shift as a function of the transaction size ω measured in dollars, doing this separately for buys and sells.

For individual transactions this behavior is quite noisy. Plotting price shift vs. order size for individual transactions results in something that is hard to distinguish from a random scatter plot. In addition, prices are strongly quantized. So that in most cases the best bid or ask either remains the same or changes by only one tick. The impact of a trade on prices is felt both in the size of the price change, but also in the probability of there being any price change at all. To investigate the average behavior we bin the data based on transaction size and compute the average price shift for the data in each bin. To put all stocks on roughly the same footing, we normalize the transaction size by dividing by its average value for each stock in each year. The results of doing this for two representative stocks are shown in Figure 1.

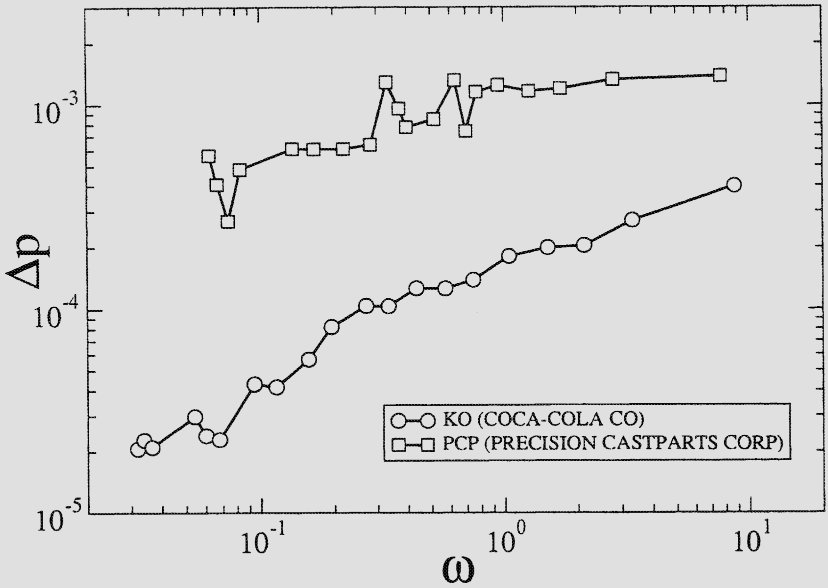

Fig. 1. Price shift vs. normalized transaction size for buyer initiated order for two representative stocks, Coca Cola Co. (circles) and Precision Castparts Corp. (squares) in 1996.

For one of the highest capitalization stocks (Coca Cola Co.) the average price impact increases roughly as $\omega^{0.6}$ throughout almost the entire volume range. In contrast, for a mid-capitalization stock such as Precision Castparts Corp. (PCP) it increases roughly as $\omega^{0.3}$. The behavior is roughly the same for both buy and sell trades.

3 The Role of Market Capitalization

To understand more systematically how this behavior varies with market capitalization, we group the 1000 stocks of our sample into 20 groups. The groups are ordered by market capitalization, and the number of stocks in each group is chosen to keep roughly the same number of transactions in each group. The groups are labeled with letters from A to T. The group size varies from the highest market capitalization group (T) with 4 stocks, to the least capitalized group (A) with 315 stocks [14]. We then bin each transaction based on size, choosing bin widths to maintain roughly the same number of trans-

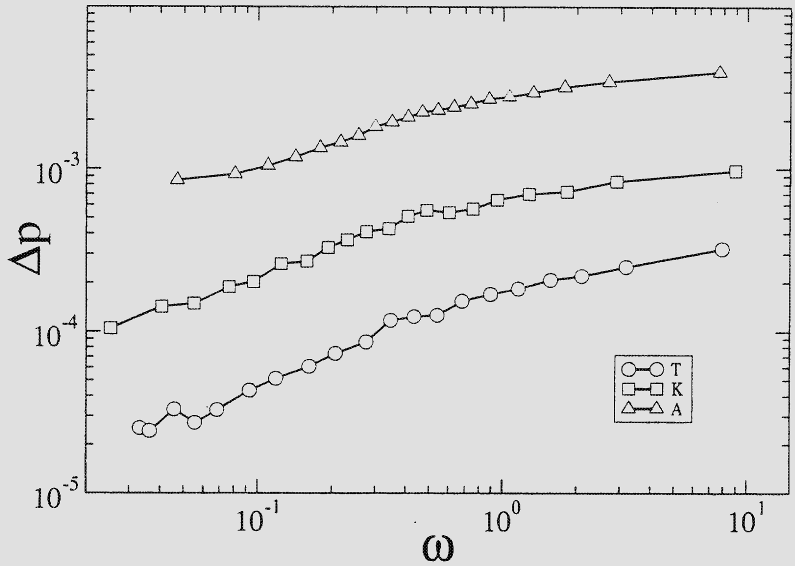

Fig. 2. Price shift vs. normalized transaction size for buyer initiated trades for 3 representative groups of stocks sorted by market capitalization. Set T is the set of most capitalized stocks (circles), set K includes intermediate capitalized stocks (squares) and set A is the set of least capitalized stocks (triangles). The investigated year is 1996.

actions in each bin (22,000 in 1996), and plot the average price impact vs. transaction size for each group. The results obtained for 1996 for three representative sets of most (T, circles), intermediate (K, squares) and least (A, triangles) capitalized stocks are shown in Figure 2. The price impact functions naturally separate themselves from top to bottom in increasing order of market capitalization. The slope of each curve varies from roughly 0.5 (for small transactions in higher capitalization stocks) to ≈ 0.2 (for larger transactions in lower capitalization stocks).

It is clear from these results that higher market capitalization stocks tend to have smaller price responses for the same normalized transaction size. Naively, one might have expected liquidity to be proportional to volume; the fact that the price impact for higher capitalization stocks is lower, even though we are normalizing the x-axis by average transaction size, says that larger capitalization stocks are even more liquid than one would expect. To investigate this aspect, in Ref. [1] we obtain liquidity parameter λ of each group of stocks by performing a best fit of the impact curves for small values of the normalized transaction size with the functional form $\Delta p = sign(\omega)|\omega|^{\beta}/\lambda$. Our empirical investigation shows that the liquidity parameter λ is a function of the mean market capitalization of the group which increases roughly as $C^{0.4}$, where C is the average market capitalization of each group.

4 Single Curve Collapse of the Impact Function

The power law dependence of liquidity parameter λ with the mean market capitalization of the group suggested us that one can collapse the data of Figure 2 onto a single curve by using an appropriate capitalization scaling. In Ref. [1] we perform a scaling of the x and y axes of each group of stocks according to the transformations

$$x \to x/C^{\delta} \qquad y \to y\ C^{\gamma} \tag{1}$$

We hypothesize that $\Delta p(\omega, C) = C^{-\gamma} f(\omega\ C^{\delta})$ and we rescale the ω and Δp axes of each group according to the transformations $\omega \to \omega/C^{\delta}$, $\Delta p \to \Delta p\ C^{\gamma}$. We then determine the values of δ and γ that do the best scaling. To do this we find values that minimize the mean of the two dimensional variance $\epsilon = (\sigma_y/\mu_y)^2 + (\sigma_x/\mu_x)^2$, where σ denotes the standard deviation and μ denotes the mean, and y is the renormalized return and x is the renormalized transaction size. In all investigated years there is a clear minimum for $\delta \approx \gamma \approx 0.3$ (to be precise $\gamma = 0.3 \pm 0.05$ for all years and $\delta = 0.3 \pm 0.05$ for 1995, 1997 and 1998 whereas $\delta = 0.4 \pm 0.05$ for 1996). The resulting rescaled price impact curves for buys in 1996 are shown in Figure 3.

The collapse is pretty clear in 1996 and it is quite good also for the other investigated years of 1995, 1997 and 1998. The resulting master function spans three decades. It increases slower than a power law, and decreases more slowly in 1998 than in 1995.

This slow rate of increase of the price impact function shown here is surprising. Naive arguments [15, 16] predict that the price impact function should increase at least linearly for positive ω. In contrast, many previous empirical studies of price impact suggest concave behavior [2, 5, 6, 7, 10, 11]. However, this result has not been observed universally [17], and none of these studies have given a clear indication as to functional form. We have investigated the problem in the NYSE [1] by focusing on the most elementary response, which is the price impact following a *single* trade. This was done by analyzing a huge amount of data, aggregating across different stocks and by scaling the data based on market capitalization. A less than square root increase of the price impact function has also been observed by Potters and Bouchaud in a recent study of the Paris stock exchange [18]. In their study, they propose to model the price impact function with a logarithmic function. In our analysis we note that a logarithmic price impact function works well only through part of the

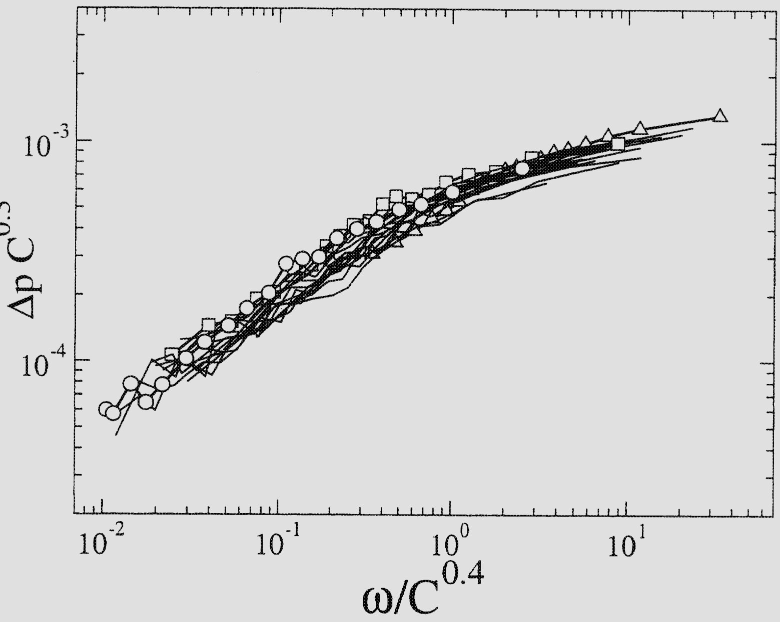

Fig. 3. The scaled price shift vs. scaled transaction size, for buy orders in 1996. The scaling procedure is described in the text. Circles refer to the T set of stocks (most capitalized ones), squares to the K set of stocks (intermediate capitalization) and triangles to the A set of stocks (least capitalized stocks). The other lines represents the remaining 17 set of stocks (which have not been shown in Figure 1 for the sake of clarity). The scaling exponents are $\gamma = 0.3$ and $\delta = 0.4$. Results for sell orders are very similar.

investigated range whereas the price impact does not appear to be described by any simple standard functional form in the entire investigated range.

5 Discussion and Conclusions

The traditional approach in economics to deriving demand curves is to assume that agents maximize their utility under assumptions about cognitive ability and access to information. The standard interpretation of our results would be that the size dependence of price impact is due to differences in the information content of trades. In other words, some trades are based on more information than others, and this is known by market participants and factored into the price setting process. This hypothesis suffers from the problem that the information content of trades is difficult to assess *a priori*, making the hypothesis unfalsifiable. In contrast, an alternative approach is to study the mechanism for making transactions in detail, under the hypothesis that order placement and cancellation are largely random. This results in predictions of price impact that are qualitatively consistent with those seen here [15, 16]. If these predictions are born out quantitatively it will be significant in demonstrating that it is important to model financial institutions in detail, and that for some purposes it is may be more useful to model human behavior as random rather than rational.

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Income Distribution and Stochastic Multiplicative Process with Reset Event

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Summary. This paper examines a stochastic multiplicative process with reset event to explain the power law in the tail of personal income distribution. The tail part of the income distributions in post-war Japan persistently exhibits a power-law distribution with an exponent around 2. We find that a multiplicative process with reset events can explain this pattern. By using a default rate of corporate fundings as a hazard rate of the reset event, we obtain the correct exponent for the power-law in Japanese income.

1 Introduction

It has long been noticed since a work of V. Pareto that the tail of the income distribution obeys a power-law distribution: $\Pr(X > x) \propto x^{-\alpha}$. Recent research revealed that the tail exponent fluctuates in a certain interval over years. Using tax returns data, Souma [7] and Feenberg and Poterba [3] reported that the Pareto exponents α of Japan and the U.S. income distributions hover within an interval $(1.5, 2.5)$ with mean around 1. This paper gives an account for the emergence of the power law distribution and a possible cause of the Pareto exponent.

Many researchers have attempted to explain the power law in income distribution by utilizing a multiplicative stochastic process. A multiplicative process yields a log-normal distribution if the process is unconstrained. Levy and Solomon [5] showed that a power-law distribution emerges when there is a reflective lower barrier that shifts with the mean of the entire distribution. In their model, the exponent of the power-law is determined by the relative size of the lower barrier to the mean income. However, we could not find a good proxy of the lower bound that would explain the exponent in Japanese data.

An alternative way to yield a power-law distribution is a stochastic multiplicative process with reset events. Manrubia and Zanette [6] presented the power-law stationary distributions of this class of models with analysis and simulations. Gabaix [4] solved the stationary distribution of a normalized version of this process, and proposed that the variety of the exponents in income distribution may be accounted by

M. Gallegati et al. (eds.), *The Complex Dynamics of Economic Interaction*

the dependence of the exponents on parameters of the process. We take this avenue to explain the power law in Japanese income distribution.

We view income process as follows. There are two kinds of income source: risky ones and risk free ones. Risky income source includes a part of capital and human capital. Risk free income source is typically labor and risk free assets. Risky income bears two kinds of risks: randomness in growth rates and the possibility that the income source vanishes suddenly. Hence it obeys a stochastic multiplicative process with reset events. Risk free income source serves as a lower bound of the individual income process, since most people are endowed with labor. The risk free income may vary across jobs, but the distribution has a thin tail, and the entire distribution is shifting along with mean income. If the distribution of risky income has a power-law tail, as we will show shortly, then the tail of the entire distribution will be dominated by the power-law distribution.

2 Model

In this section we formulate the process of the personal risky income. Suppose that the income $w(t)$ follows a geometric Brownian motion with "killing". Hence we are imposing a normality of the shocks to the process. The killing rate is μ. A reset point is denoted by $w_0(t)$. The reset point is also a reflective lower barrier.

$$dw_t/w_t = \begin{cases} \gamma dt + \sigma dZ_t & \text{with probability } (1-\mu)dt \\ w_t^0 & \text{with probability } \mu dt \end{cases} \tag{1}$$

$$w_t \geq w_t^0 \tag{2}$$

where Z_t is a Wiener process. Inequality (2) means that w_t^0 is the reflective lower bound. Namely, whenever $w_t < w_t^0$ happens in equation (1), w_t is reset to w_t^0. Suppose further that the lower bound w_t^0 grows as fast as the drift of risky income.

$$dw_t^0/w_t^0 = \gamma dt \tag{3}$$

The economic intuition behind this specification is as follows. Risky capital income naturally obeys a multiplicative process, as we often observe that the capital grows exponentially. By the term capital income we mean the income derived from accumulative resource broadly. Thus it includes an income from human capital (education etc.), and also the wage of managers that are often associated with the growth of the capital the managers are in charge of.

The lower bound of capital income w^0 is the income level below which the person's consumption cannot be sustained. The bound is reflective, since the consumers can utilize their labor resource to maintain the minimum income level. This minimum income level may be random in reality, but Manrubia and Zanette [6] showed by simulation that the randomness does not alter the tail exponent of the power-law.

The growing lower bound is consistent with the empirical regularity that the ratio of labor income to capital income is roughly stable. Also, it is natural in terms of risk sharing that the capital income takes all the risk of income process and the labor income takes none when the asset holders are risk neutral and workers are risk averse.

The income growth process is "killed" by the rate μ. The killing represents the evaporation of some income source. We view bankruptcy of firms or default of corporate bonds as such evaporation. Risky assets or well-paying managerial positions disappear from economy with such events. Default is an unnegligible risk of an economy. About 1% of corporate funding defaults in a typical year in Japan. We will see later that this level of default is large enough to generate a power-law distribution of income with exponent 2. The interpretation of μ as a default rate requires the default rate to be independent of the size of the firms. This is consistent with the fact that the size of firms' income and the size of defaulted firms both follow a Zipf's law (Aoyama et.al [1] and Axtell [2]). Also, our interpretation of μ implicitly assumes that the recipients of the risky income do not hedge the risk. It is natural, as we imagine the top tier income group as managers, entrepreneurs, and founding family of corporations rather than rentiers.

We normalize the income process w_t by the lower bound w_t^0.

$$v_t = w_t/w_t^0 \tag{4}$$

$$v^0 = 1 \tag{5}$$

Then, $dv/v = dw/w - \gamma dt$ by Ito formula.

$$dv_t/v_t = \begin{cases} \sigma dZ_t & \text{with probability } (1-\mu)dt \\ v^0 & \text{with probability } \mu dt \end{cases} \tag{6}$$

$$v_t \geq 1 \tag{7}$$

The process v_t has a stationary distribution when $\mu > 0$. We draw on Gabaix [4] in solving the stationary distribution. First we write the Kolmogorov forward equation as:

$$\frac{\partial}{\partial t}\phi(v,t) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial v^2}v^2\phi(v,t) - \mu\phi(v,t) \tag{8}$$

Suppose that a Pareto distribution $\phi = Cv^{-1-\alpha}$ solves the equation for the stationary distribution. Then $\alpha > 0$ solves the characteristic equation

$$\alpha^2 - \alpha - 2\mu/\sigma^2 = 0. \tag{9}$$

The positive root is

$$\alpha = \frac{1+\sqrt{1+8\mu/\sigma^2}}{2}. \tag{10}$$

This is the Pareto exponent of the stationary distribution of v. The exponent carries over to the distribution of income w.

Numerical simulations verify this formula. Figure 1 shows the evolution of distribution of 100000 points for 2000–10000 periods where each point follows a discretized version of the v_t process (6,7). It is clear that the distribution converges to a power-law distribution with exponents determined by (10)

We show that the formula (10) matches the Japanese data. We suppose that the mean of the yearly Pareto exponent is determined by a stationary exponent α when μ and σ is given. We use as μ the default rate data which is quarterly announced by Bank of Japan. The default rate is the ratio of defaulted bonds over total funds of corporate sectors. The ratio fluctuates from 0.2% to 1.7% around mean 0.9705% during the period from 1970 to 1996. As for σ, we use a constant $0.1/1.022 = 0.0978$ for entire periods. The value is estimated from the sales growth rates of Nikkei

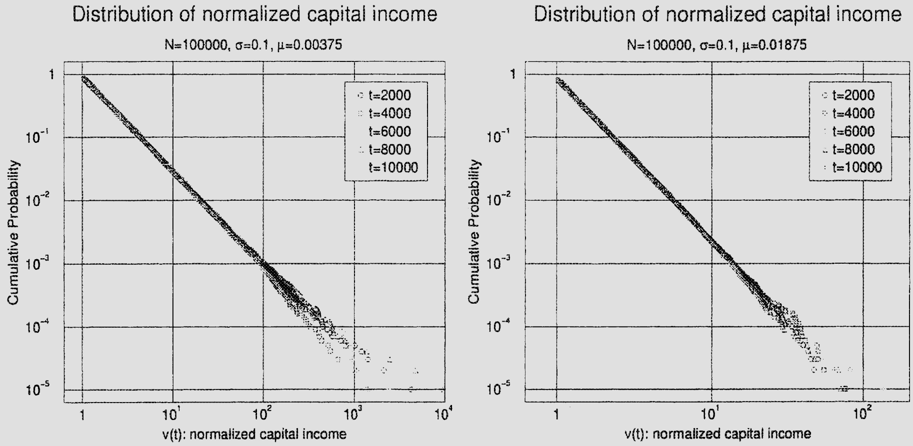

Fig. 1. Stationary distributions from simulation of the process for various values of μ

225 companies during 1996–2000. The growth rate has mean 1.022 and standard deviation 0.1. Then the normalized income in the model has an annual standard deviation 0.1/1.022.

Applying $\mu = 0.009705$ and $\sigma = 0.0978$ to the formula (10), we obtain a theoretical prediction $\alpha = 2.0097$. The historical mean of Pareto exponent is 2.003 for the period 1970–1996. Hence our model offers a good quantitative match with the Japanese income distribution.

The empirical time series of the default rate also exhibit an interesting correlation with the yearly Pareto exponent. Suppose that a yearly Pareto exponent is determined by (10) given the default rate of the year as μ and a constant $\sigma = 0.0978$ for the period. Figure 2 plots the predicted and actual exponents from 1970 to 1996. This theoretical prediction exhibits quite a good fit with empirical observation, as is seen in the scatter plot of the same series (Figure 3).

Figure 4 shows the same relationship as Figure 2 for a longer time series. The data for the default rate was constructed from the total value of defaulted bonds (Tokyo Shoko Research) and the total funding of corporate sectors (Ministry of Finance). The computed default rate does not exactly match with the default rate reported by Bank of Japan.

The correlation shown in Figures 2, 3, 4 cannot be taken as an evidence for that the formula (10) directly governs the year-to-year fluctuations of α, since with our parameter values of μ and σ, it takes longer periods than our data period for the income to converge to the stationary distribution.[3] Nonetheless, it is possible that the default affects other parameters of the model such as the growth rate of risky income or the number of consumers in the power-law part so that it accelerates the convergence of the income distribution.

One may think that the Pareto exponent is driven by business cycles and thus the fit obtained above merely reflects a spurious correlation. Figure 5 plots the time

[3] We owe J.P. Bouchaud on this point.

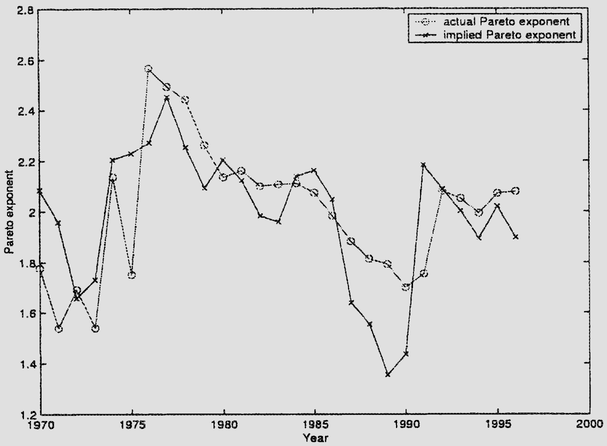

Fig. 2. Pareto exponents of income distribution in Japan. The circle shows the actual exponent and the cross shows the predicted exponent.

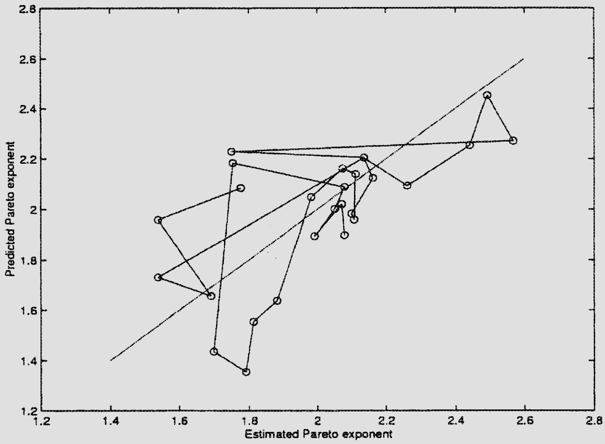

Fig. 3. Scatter plot of the actual and implied Pareto exponents

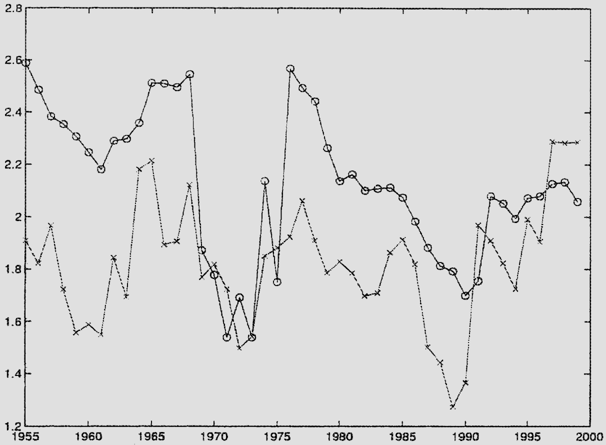

Fig. 4. Pareto exponents for 1955–1999

series of the default rate as well as the Pareto exponent and the periods of recession. It is seen that the default rate comoves with Pareto exponent, whereas recession does not correlate with either of them. This shows that the default is a non-trivial factor in determining income distributions.

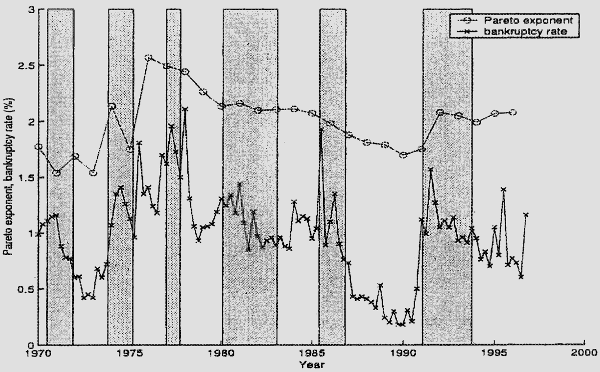

Fig. 5. Bankruptcy rates. Shaded area designates the period of recession.

The model may also explain why the exponent of the personal income distribution is volatile and large (1.5–2.5) whereas the exponent of the firm income distribution is stable and low (around 1). The important difference between a personal income process and a firm income process is that the firm income does not have a lower bound growing as fast as the entire distribution, whereas the personal income has one because labor is endowed to most persons. If the lower bound does not grow, we obtain a modified formula for the exponent as:

$$\alpha = \left(1 - 2\gamma/\sigma^2 + \sqrt{(2\gamma/\sigma^2 - 1)^2 + 8\mu/\sigma^2}\right)/2 \tag{11}$$

When the income drift is large enough so that $\gamma > \sigma^2/2$, the right hand side is approximated by the first order Taylor expansion as $\mu/(\gamma - \sigma^2/2)$. Thus the exponent 1 of the firms' income distribution is obtained when $\mu = \gamma - \sigma^2/2$. Let us notice that μ here can incorporate the growth in the number of firms in addition to the default rate. This is because an entrant firm at the lower barrier reduces the density of other income levels proportionally to the density. It is natural to think that the number of firms grows as fast as the total income of firms. Also, the magnitudes of default rate and σ^2 are much smaller than the growth rate of firms' income. Hence we obtain a power-law distribution of exponent near 1 which is irresponsive to the fluctuation of the default rates.

3 Conclusion

By using a stochastic multiplicative process with reset events and lower reflective barrier, we obtain an explicit formula for the power-law exponent of income distribution. The exponent is dependent on the default rate (default rate of corporate funding) and the diffusivity of the income process. The model fits well to the historical exponent in Japanese income distribution. It is also pointed out that the default rate plays an important role to determine the year-to-year fluctuation of the exponent.

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Part III

Market Models and Stylized Facts

Volatility Clustring in Agent Based Market Model

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Summary. We define and study a rather complex market model, inspired from the Santa Fe artificial market and the Minority Game. Agents have different strategies among which they can choose, according to their relative profitability, with the possibility of not participating to the market. The price is updated according to the excess demand, and the wealth of the agents is properly accounted for. Only two parameters play a significant role: one describes the impact of trading on the price, and the other describes the propensity of agents to be trend following or contrarian. We observe three different regimes, depending on the value of these two parameters: an oscillating phase with bubbles and crashes, an intermittent phase and a stable 'rational' market phase. The statistics of price changes in the intermittent phase resembles that of real price changes, with small linear correlations, fat tails and long range volatility clustering. We discuss how the time dependence of these two parameters spontaneously drives the system in the intermittent region. We analyze quantitatively the temporal correlation of activity in the intermittent phase, and show that the 'random time strategy shift' mechanism that we proposed earlier allows one to understand the observed long ranged correlations. Other mechanisms leading to long ranged correlations are also reviewed.

1 Introduction

It is now well known that the statistics of price changes in financial markets exhibit interesting 'stylized facts', which are to some extent universal, i.e. independent of the type of market (stocks, currencies, interest rates, etc.) and of the epoch [1, 2, 3, 4]. Price changes are in a good approximation uncorrelated beyond a time scale of the order of tens of minutes (on liquid markets). Their distribution is strongly non Gaussian: they can be characterized by Pareto (power-law) tails with an exponent in the range $3-5$. Another striking feature is the 'intermittent' nature of the fluctuations: localized outbursts

M. Gallegati et al. (eds.), *The Complex Dynamics of Economic Interaction*

of the volatility, i.e. the amplitude of the price fluctuations (averaged over a given time interval), can be identified. This fact, known as *volatility clustering* [5, 6, 2, 3], can be analyzed more quantitatively: the temporal correlation function of the daily volatility σ_t can be fitted by an inverse power of the lag τ, with a rather small exponent in the range $0.1 - 0.3$ [6, 7, 8, 9, 10]. This suggests that there is no characteristic time scale for volatility fluctuations: outbursts of market activity can persist for short times (a few hours), but also for much longer times, months or even years. The slow decay of the volatility correlation function leads to a multifractal-like behaviour of price changes [11, 12, 13, 10, 14], and has important consequences for option pricing. Other stylized facts have been reported, such as the leverage effect that leads to skewed distribution of price changes [15], or the apparent increase of inter-stock correlations in volatile periods.

It is now very clear to many that these features are very difficult to explain within the traditional framework of 'rational expectations', where all agents share the same information, have an infinite computation power and act in a perfectly rational way (see e.g the clear discussion in the introduction of refs. [16, 17] and in [18]). Another route, much less formalized and still very much in an exploratory stage, is followed by an increasing number of academics. The aim is to assume as little as possible about agents preferences and abilities, and to explore *generic* classes of models, with the hope of finding some *plausible* mechanisms that reproduce at least part of the stylized facts recalled above. In this endeavor, one should not be constrained by preexisting prejudices or established frameworks. The 'grand unification' of different mechanisms which would lead in fine to a logically consistent and simultaneous understanding of all the empirical facts is deferred to later times. Similarly, it is premature to ask for rigorous proofs, but leave space for hand waving arguments and numerical simulations.

In this paper, we report [19] the results of an artificial market that bears some similarities with many previous attempts [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38]. Although the detailed behaviour of our artificial market depends on the value of the different parameters entering the model, only a few market typologies are identified, and some qualitative features (such as the long range volatility correlations) are robust to parameter changes, at least in some regions of parameter space. We explain in particular that the general mechanism proposed in [39] is indeed responsible for volatility clustering in our model. We also discuss, in this context, other agent based models proposed so far in which this phenomenon has been observed. We identify several other interesting features (appearance of bubbles and crashes, influence of the price fixing procedure and of wealth constraints, of transaction costs, etc.) that may be of some relevance to real markets.

In our model, we have not included herding, or imitation effects. Each agent acts independently of other agents. The correlations between their actions is entirely mediated by the price history itself. Direct herding might also be important to account for the phenomenology of real markets (see

[40, 41, 29, 42] and references therein). We leave to a future study the extension of the present model to account for these herding effects.

The aftermath of our study (and of all other similar studies) is the following paradox: in order to get a 'good looking' price chart, one has to tune quite a bit the important parameters of the model. What are the mechanisms tuning the parameters in real markets to make them look all alike ? There must be some generic self-organization mechanisms responsible for this selection. We discuss in section 6, in view of our results, what could be the 'evolutionary' driving forces relevant for this fundamental issue.

2 Set up of the Model

2.1 Basic ingredients

In line with the original idea of the Santa Fe artificial market [21, 22], which was later simplified and popularized as the Minority Game (MG) [24, 43, 30, 44, 45], in our model agents do not follow a rational expectation paradigm but rather act inductively, adapting their behaviour to their past experience. As in the MG each agent has a certain fixed number of strategies, each of which converts some *information* into a *decision*. We will assume a world where there are only two tradable assets: a stock, with fluctuating price, and a bond, yielding a certain (known) risk free rate ρ. The information on which the agents decide their action will be the past history of the price itself. The decision is to buy stocks (converting bonds in cash), to sell stocks, or to be inactive (i.e. to hold bonds). Each strategy is given a score, which is updated according to its performance. The strategy played at time t by a given agent is the one, among those available to him, which would have best performed in a recent past. We take proper account of the wealth balance of each agent, and proper market clearing (i.e matching supply and demand) is enforced.

2.2 Notations and definitions

Each agent i, $i \in \{1, ..., N\}$ has $S-1$ active strategies plus an inactive one. He owns, at time t, a number $\phi_i(t)$ of stocks and $B_i(t)$ of bonds. The price of the stock is $X(t)$, and therefore the total wealth of agent i is $B_i(t) + \phi_i(t)X(t)$. The dynamics of the model, between t and $t+1$, is defined by the following set of rules:

- Information:

  We assume that all agents rely on the *same* information $\mathcal{I}_t$, given by the m last steps of the past history of the return time series (m is for 'memory'). We choose the information to be qualitative and to only depend on the *sign* of the previous price changes:

$$\mathcal{I}_t = \{\chi(t-m), \cdots, \chi(t-1)\} \qquad \chi(t) = \text{sign}\left[\log(\frac{X(t)}{X(t-1)}) - \rho\right]. \quad (1)$$

In this sense, our traders are 'chartists' on short time scales, and take their decision based on the past pattern of price changes. (We will add below the possibility for the agents not to follow their systematic strategies, and act as 'fundamental' traders if the price is too far from a reference value, or even act at random).
A comment on the value of the time scale '1' is in order here. Clearly, different agents observe the price time series on different time scales, from several minutes for intra day traders to months for long term pension funds. Here, we assume for simplicity that all agents coarse-grain the price time series using the same clock, and consider as meaningful price variations on – say – a day or a week. All the parameters below were chosen such as one time step roughly corresponds to a week of trading. One interesting outcome of our model is that even if all agents have the same intrinsic clock, a broad range of time scales is spontaneously generated.

- Strategies:

  Each agent i is endowed with a certain number S of fixed strategies, that convert information $\mathcal{I}_t$ into decision $\epsilon_i(\mathcal{I}_t) = \pm 1, 0$ (buy, sell, inactive). For example, a 'trend following' strategy could be to choose $\epsilon_i = 1$ as soon as there is a majority of $+$'s in the signal $\mathcal{I}_t$. Another example is the inactive strategy, for which $\epsilon_i \equiv 0$, $\forall \mathcal{I}$.
  Each agent is given the possibility to remain inactive, i.e. has an inactive strategy. The $S-1$ other strategies are chosen at random in the space of all strategies (there are 2^{2^m} of them), in order to model heterogeneity in the agents capabilities. One can however give a bias to this random choice, and favor 'trend following' or 'contrarian' strategies. This is done by defining the 'magnetization' $M \in [-1, 1]$ of the string $\chi(1), \cdots, \chi(m)$, defined as:

$$M = \frac{1}{m} \sum_{j=1}^{m} \chi(j), \qquad (2)$$

  and choose the corresponding decision to be $\epsilon = +1$ with probability $(1 + PM)/2$ and $\epsilon = -1$ with probability $(1 - PM)/2$. The parameter $P \in [-1, 1]$ can be called the 'polarization' of the strategies. If $P = 1$, trend following strategies are favored, whereas $P = -1$ corresponds to contrarian strategies. The choice $P = 0$ is means no bias in the strategy space.

- Decision and buy orders:

  Knowing the strategy used by agent i and the information $\mathcal{I}_t$ allows one to compute the decision $\epsilon_i(t)$ of each agent. Depending on the value of $\epsilon_i(t)$, the agent buys/sells a quantity $q_i(t)$ proportional to his current belongings. More precisely, we set:

$$\begin{aligned} q_i(t) &= g\frac{B_i(t)}{X(t)} && \text{for } \epsilon_i(t) = +1 \\ q_i(t) &= -g\phi_i(t) && \text{for } \epsilon_i(t) = -1 \\ q_i(t) &= 0 && \text{for } \epsilon_i(t) = 0. \end{aligned} \tag{3}$$

This means that we consider 'prudent' investors who change their positions progressively: only a fraction g of the cash is invested in stock between t and $t+1$ if the signal is to buy, and the same fraction g of stock is sold if the signal is to sell. Typical values used below are $g \sim 1\%$.
The normalized total order imbalance, which will be used to determine the change of price, is denoted $Q(t)$:

$$Q(t) = \frac{1}{\Phi}\sum_{i=1}^{N} q_i(t) = Q^+(t) - Q^-(t), \tag{4}$$

where Φ is the total number of outstanding shares (that we assume to be constant), Q^+ is the fractional volume of buy orders and Q^- the fractional volume of sell orders.
Agents sometimes choose to abandon their 'chartist' strategies when the price reaches levels that they feel unreasonable: when the price is too high, they are likely to sell, and vice versa. More precisely, we construct a long term average of past returns as:

$$\overline{r}(t) = \frac{1}{1-\alpha}\sum_{t'<t}\alpha^{t-t'-1}\,r(t') \qquad r(t') = \log\left(\frac{X(t'+1)}{X(t')}\right), \tag{5}$$

where $\alpha < 1$ defines the time scale $T_0 = 1/\log(1/\alpha)$ over which the averaging is done, and $r(t)$ is the instantaneous stock return. When $\overline{r}$ is larger than a certain reference return ρ_0, related to economy fundamentals, the stock can be deemed as overvalued and 'fundamentalists' will sell ($\epsilon_i = -1$). Conversely, if $\overline{r} < \rho_0$, the stock is possibly undervalued, and $\epsilon_i = +1$. We model the occurrence of fundamental trading as stochastic, by assigning a certain probability p_f for every agent to follow a fundamental strategy rather than a technical (chartist) strategy. We want p_f to increase with $|\overline{r} - \rho_0|$, and have chosen the following simple relation:

$$p_f = \min\left(1, f\frac{|\overline{r} - \rho_0|}{\rho_0}\right), \tag{6}$$

where f is a certain parameter describing the confidence of agents in fundamental information. Since p_f increases when the price goes up too fast, fundamentalists have a stabilizing role and give to the price a mean reverting component. In the following, we will assume that on the long run, the

overall economy growth ρ_0 and the interest rate ρ are equal, and impose $\rho_0 = \rho$, although in practice the two fluctuate with respect to each other. In the above rule, we have again assumed that all agents use the same time scale to determine the past average trend. This is probably very far from reality, where one expects that this time scale could be very different for different agents.
Finally, it can be useful to consider the influence of 'irrational' traders, who take their decisions on the basis of random coin tossing only. We define p_i as the probability for an agent to take a random decision. In this case, the probability to be a fundamentalist is $(1 - p_i)p_f$.

- Price formation and market clearing mechanism:

  Once the aggregate order imbalance $Q(t)$ is known, we update the price following a simple linear rule [20, 21, 26, 41, 27]:

$$r(t) = \log\left(\frac{X(t+1)}{X(t)}\right) \simeq \frac{X(t+1)}{X(t)} - 1 = \frac{Q(t)}{\lambda}, \tag{7}$$

  where λ is a measure of the 'stiffness' of the market. There has been recent empirical studies of this relation, which was shown to hold for individual stocks for small enough Q, on a sufficiently large time interval [47]. For larger order imbalance, the price response appears to bend downward, a possible consequence of the structure of the order books. We have included this effect, with no noticeable effect on the qualitative results presented below.
  From a more microscopic point of view, i.e. on time scales smaller than the time unit that we have chosen, agents place orders of different types in the market: market orders and limit orders. Market orders allows the order to be executed with certainty, but at the current market price. Limit orders ensures a maximum price for buy orders (and a minimum price for sell orders) but can be unexecuted, or only partially executed, depending on the history of the price. Therefore, in general, the order put down by an agent will be only partially filled. We assume that the fraction of unfulfilled orders is the same for all agents. Market clearing is then ensured by the following rule: the global amount of sell orders is $Q^-(t)$, and the total number of shares that can be bought at price $X(t+1)$ is:

$$\tilde{Q}^+(t) = Q^+(t)\frac{X(t)}{X(t+1)}. \tag{8}$$

  The fraction of filled buy orders φ_+ (resp. filled sell orders φ_-) is therefore:

$$\varphi_+ = \min\left(1, \frac{Q^-}{\tilde{Q}^+}\right) \qquad \varphi_- = \min\left(1, \frac{\tilde{Q}^+}{Q^-}\right). \tag{9}$$

  From these quantities, one determines the actual number of shares $\delta\phi_i$ bought or sold by agent i:

$$\delta\phi_i(t) = g\varphi_+ \frac{B_i(t)}{X(t+1)} \qquad \text{for } \epsilon_i(t) > 0$$
$$\delta\phi_i(t) = -g\varphi_- \phi_i(t) \qquad \text{for } \epsilon_i(t) < 0 \tag{10}$$

- Wealth dynamics:

  We now have all the ingredients to update the number of stocks and bonds of agent i, i.e.:

$$\phi_i(t+1) = \phi_i(t) + \delta\phi_i(t)$$
$$B_i(t+1) = B_i(t)(1+\rho) - \delta\phi_i(t)\, X(t+1), \tag{11}$$

  where the last line adds the interest gained on the bonds between t and $t+1$ to the cash need to finance new stocks, or gained through stock selling. Note that there is an *injection* of wealth due to the positive interest rate ρ. We will discuss this further in the following.

- Update of the scores:

  Each agent assigns scores to his strategies to measure their performance and uses at time t the best strategy, i.e. the one with highest scores. We need now to specify how the scores of the different strategies are updated. The score of the αth strategy of agent i at time t is denoted $\mathcal{S}_i^\alpha(t)$, whereas the decision associated to this strategy when the available information at time t is $\mathcal{I}_t$ is $\epsilon_i^\alpha(\mathcal{I}_t)$. When the agent i decides at time t to trade, the price at which the trade takes place is $X(t+1)$. Therefore, the *virtual* profit he makes due to this trade is only known at time $t+2$ and is $\epsilon_i^\alpha(\mathcal{I}_t)[X(t+2) - X(t+1)]$. We choose to update the score of the active strategies proportionally to the relative profit, corrected by the interest rate [5]:

$$\mathcal{S}_i^\alpha(t+1) = (1-\beta)\mathcal{S}_i^\alpha(t) + \beta\epsilon_i^\alpha(\mathcal{I}_{t-1})[r(t) - \rho], \quad \alpha = 1, ..., S-1, \tag{12}$$

  whereas the score of the inactive strategy is identically zero. The parameter $\beta \leq 1$ defines a memory time: the performance of the strategies is only computed using the recent part of the history. Note that the update of the scores is not weighted by the actual transaction volume: good decisions are valued independently of the current wealth of the agent. Therefore, the score of the strategy is not proportional to the actual profit and loss curve.
  One should keep in mind that only the best strategy $\alpha^*(t)$ is played by the agent at time t. Nevertheless, he updates the scores of all strategies *as if* they had been played. In other words, market impact is neglected here, since the very fact of using a given strategy influences the price itself.

[5] A similar rule for the update of scores was recently considered in [38]

The history of the price would have been different if a different strategy had been played. We do not take into account the market impact for two reasons: first, the update of the score is delayed as compared to the action itself (see Eq. (12)) – therefore, the main source of systematic bias discussed in the context of the Minority Game in, e.g. [46], is removed. Second, market impact is in practice very hard to estimate for traders themselves (although some recent studies start addressing this issue [47, 48, 49]), and strategies are often backtested under the assumption that the market impact is small.
A related point is that of the virtual profit computed above. Taking profit means closing one's position, at a price that is not known in advance. Again there will be some market impact and the actual price of the transaction is on average less than the current price. This effect is well known to active market participants and, as mentioned above, has been recently the subject of some studies. A way to model this is to add a transaction cost to the above update of the score, independent of whether one buys or sells. (This cost should also be taken into account in the above wealth balance).

2.3 Summary of the parameters and main results

The model contains a rather large number of parameters: the interest rate ρ, the memory length used for technical trading m, the 'polarization' of strategies P, the fraction of invested wealth g, the time scale used to trigger fundamental trading α, the propensity of fundamental trading f and the fraction of irrational agents p_i, the stiffness of the market λ, and the memory time of agents when the score of the strategies are updated β. However, the only truly important parameters are g/λ and the polarization P, which determine the qualitative behaviour of price changes. The other parameters influence the quantitative results, but not the qualitative features, which is the appearance of three qualitatively different regimes (see Fig. 1):

- An *Oscillatory Regime*, corresponding to 'weak coupling': $g/\lambda \lesssim 0.4$, and $P \geq 0$, where speculative bubbles are formed, and finally collapse in sudden crashes induced by the fundamentalist behaviour. In this regime, markets are not efficient, and a large fraction of the orders is (on average) unfulfilled.
- A *Turbulent regime* ($g/\lambda \gtrsim 0.4$, $P \geq -|P_0|$) where the 'stylized' facts of liquid markets are well reproduced: the market is efficient (although some persistent or antipersistent correlations survive), the returns follow a power law distribution, and volatility clustering is present.
- A *Stable regime*, which arises if the polarization P is sufficiently negative (predominance of contrarian strategies). In this case, the fluctuations of the price are mild and mean reverting (see Fig. 1 c), as one would expect in a 'rational market' where the trading price is always close to the fundamental price.

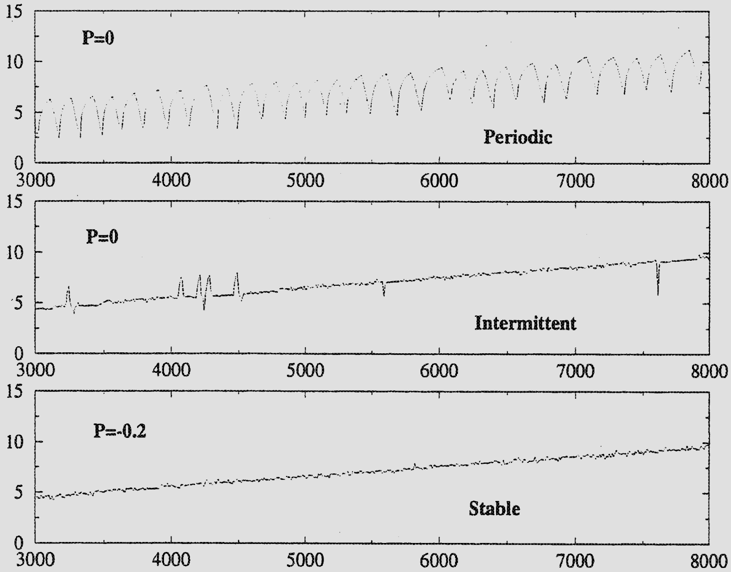

Fig. 1. Typical price charts in the three regimes: periodic, intermittent, stable (efficient). The parameters are: $S = 3$, $m = 5$, $g = 0.005$, $f = 0.05$, $N = 1001$, $1 - \beta = 10^{-2}$ and $1 - \alpha = 10^{-4}$. The top graph corresponds to $g/\lambda = 0.1$, the two bottom graphs to $g/\lambda = 0.6$.

A (rather schematic) phase diagram of the model in the plane $(g/\lambda, P)$ for a fixed value of all the other parameters is shown in Fig. 2. This qualitative phase diagram is the central result of our study.

3 Kinematics of the Model: fully Random Strategies

Before embarking to analyze the influence of strategies, it is important to calibrate the bare version of our model where agents take purely random, uncorrelated decisions at each instant of time. In this case, the price fluctuations will only reflect the wealth constraints. We show in Fig. 3 the price chart for some values of the parameters. The log-price performs a mean reverting random walk around the fundamental price $X_f(t) = \exp(\rho t)$. In the inset we show the log-price variogram, defined as:

$$\mathcal{V}(\tau) = \left\langle \left(\log \frac{X(t+\tau)}{X(t)} - \rho\tau \right)^2 \right\rangle, \tag{13}$$

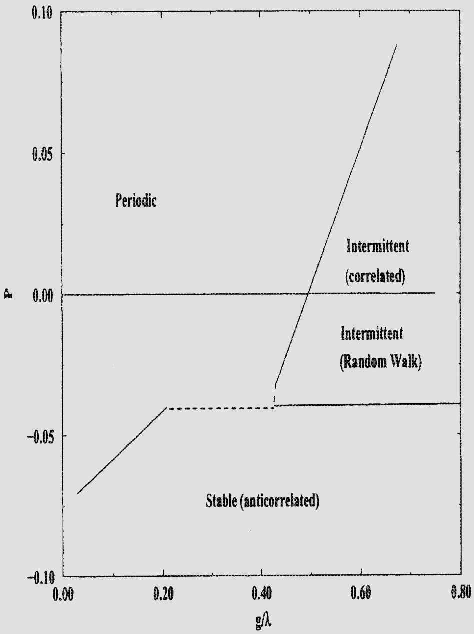

Fig. 2. Phase diagram of the model. The region $g/\lambda > 0.4$, $P \ll 1$ corresponds to an intermittent regime, with small linear correlations but strong volatility fluctuations. The dashed lines correspond to crossover regions, where a mixed behaviour is observed.

together with an Ornstein-Uhlenbeck fit:

$$\mathcal{V}(\tau) = \mathcal{V}_\infty \left(1 - \exp(-\tau/\tau_0)\right), \tag{14}$$

that describes a mean reverting random walk with a reverting time τ_0, and mean square excursion from the mean equal to $\mathcal{V}_\infty$. For small τ, the behaviour of $\mathcal{V}(\tau)$ is linear in τ, as for a free random walk, indicating that random trading leads, as expected, to unpredictable price changes. However, on larger time scales, the limitation of wealth and of stocks (agents cannot borrow nor short sell stocks) prevents the price from wandering infinitely far from the fundamental price, and leads to a mean-reverting behaviour. This mechanism will also operate for more complicated trading rules and will be discussed again in section 5.

It is simple to understand how these quantities depend on the parameters g, λ. From the price fixing mechanism, one can write a Langevin equation for the price that reads:

$$r(t) = \frac{d \log X}{dt} = \frac{g}{2\lambda\phi} \left(\frac{B(t)}{X} - \phi + \xi(t) \right), \tag{15}$$

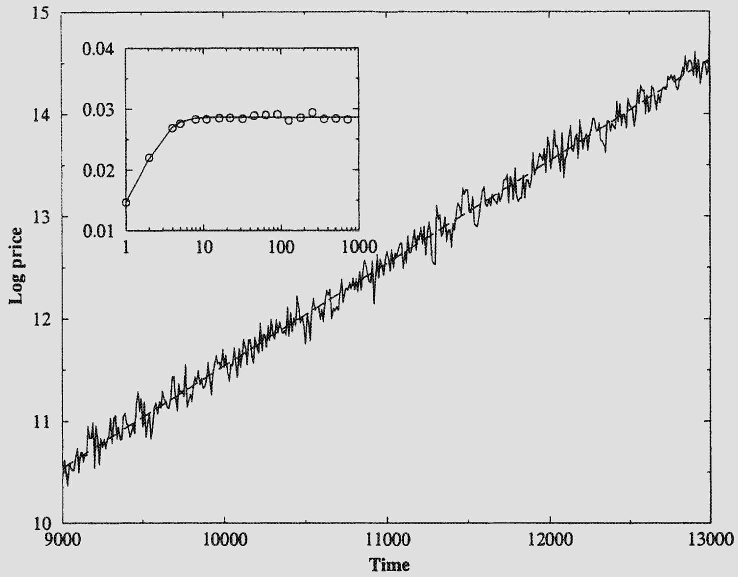

Fig. 3. Behaviour of the price as a function of time for purely random strategies. Inset: Variogram of the price fluctuations, and Ornstein-Uhlenbeck fit.

where $B(t)$ is the average of an agent wealth in bonds, ϕ the average number of stocks per agent, and ξ is a random binomial variable measuring the small offset from a perfect 50-50 division between buyers and sellers. The variance of $\xi(t)$ is therefore of order N^{-1}. Equation (15) indeed describes a mean reverting process around $B(t)$: when the price is too high, the demand goes down due to budget constraints and the price goes down, and vice-versa. The long term increase of price is only due, in our model, to the continuous injection of cash through the interest rate.

Eq. (15) defines a mean reverting process, that allows one to obtain in particular:

$$\tau_0 \propto \frac{\lambda\phi}{g} \qquad \mathcal{V}_\infty \propto \frac{g}{N\lambda\phi}, \tag{16}$$

in agreement with numerical results. The short time volatility of the market, for $\tau \ll \tau_0$, is given by $\sigma^2 = \mathcal{V}_\infty/\tau_0 \propto g^2/N\lambda^2\phi^2$, and is small for 'stiffer' markets, or if the fraction of invested wealth is smaller, as expected. Also, the volatility decreases when the number of agents increases. For a market with 10^4 participants such that $g = 10\%$, this formula gives a reasonable volatility of 1% per week, when the market stiffness is $\lambda \sim 0.1$. However, in this case, the return time is also of the order of a week and the total variability of prices is 1%, both being far too small compared to reality.

Let us finally note that from the simulations, the market liquidity, measured as the fraction c of fulfilled orders, improves when the market stiffness λ decreases. For example, we find $c = 0.9$ for $g = 1\%$, $\lambda = 1$, and $c = 0.98$ for $g = 1\%$, $\lambda = 0.1$.

4 The Oscillatory Regime

This regime is characterized by the presence of regular bubbles followed by rapid 'crashes'. The period of the bubbles is a function of the model parameters. In Fig. 4 we show the dependence of the period on g/λ, for some fixed values of the other parameters. The period decreases as g/λ increases, and vanishes when the market enters the turbulent regime. On the same plot, we have also shown the fraction c of fulfilled orders, which is very low in the periodic phase and increases with g/λ.

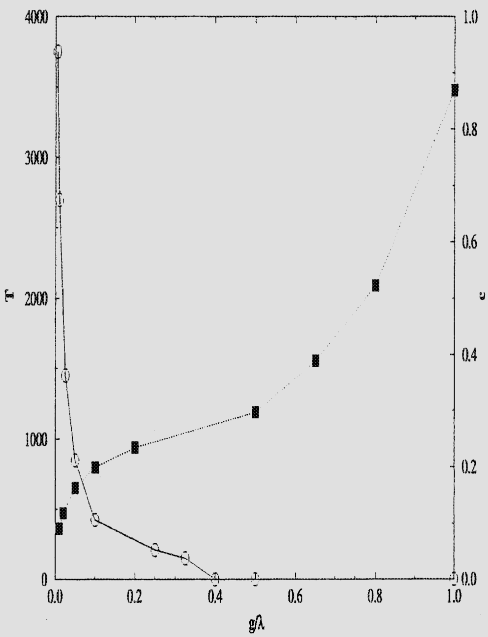

Fig. 4. Period T of the oscillations (open circles, left scale), and fraction c of fulfilled orders (black squares, right scale), as a function of g/λ. All other parameters are as in Fig. 1.

We want to understand the mechanism underlying the creation and persistence of bubbles, and their final collapse.

To answer this question, one has to look back at Eq. (7) that describes the price dynamics. Since the strategies are randomly distributed among agents, we can expect that at the beginning of the game one half of the agents is willing to buy and the other half to sell. The price increment is therefore $r = \delta X/X \sim g\lambda^{-1}N/(2\Phi)[B/X - \Phi]$. Since in general $B/X \neq \Phi$, the price rises or decreases. Suppose that the initial conditions are such that it increases. If g/λ is small, both the wealth and number of shares of the agents change quite slowly, and therefore $B/X - \Phi$ will keep a constant sign for a while. This will generate a history of price of the form $..,1,1,1,1,1....$ (See eq. (1)). This is the initial stage of the bubble. Now, the population of agents fall into two categories: those who have at least one strategy such that $..,1,1,1,1,1$ sends a buy signal, and those who do not have such a strategy available. The buy-strategies keep being rewarded, while the sell ones keep loosing points: agents who can buy will continue buying, while agents who cannot will soon become inactive since all active strategies have a negative score. In this way the bubble is self-sustained and the price keeps increasing. This 'majority' mechanism was discussed in the context of the Minority Game in [36, 38]. However, the buying power of the buyers decreases, because the price increases and the available cash decreases. Hence, the relative return of the stock over the risk-free rate diminishes, and the score of the buying strategies only become marginally positive (because the initial large gains are forgotten, only the recent past is included in the calculation of the scores). Since the return exceeds the reference rate ρ_0, a certain fraction of agents become fundamentalist and act contrarily to the main trend, i.e. sell. As soon as their action is such that the price drops, the history will change to $..,1,1,1,1,-1$. At this point, one half of the active agents still receive a buy signal, but the other half receive a sell signal (since strategies are random, and for now unbiased – $P = 0$). Now, these selling agents have a lot of stocks, since they have been buying for a long time, whereas the still buying agents have a poor buying power. This obviously results in a series of -1, and an 'anti-bubble' is created. This anti-bubble has a large initial negative slope because the system starts in a highly unbalanced state. There is no symmetry between a bubble and an antibubble because of the presence of a non zero interest rate, which is a source of cash and favors, on average, positive trends. Once again, it is the presence of fundamentalist agents which at some point triggers the end of the descending trend, and re-establish a bubble. The precise time at which the bubble collapses is random, and the way the collapse is triggered is very similar to a 'nucleation' process.

One interesting quantity is the fraction of fulfilled orders. In the oscillating regime, one understands from the above arguments that the unbalance between buy orders and sell orders is in general very large, leading to great amount of unfulfilled orders. The fraction c of fulfilled orders increases with g/λ: see Fig. 4. As will be discussed below, this is a driving force to escape from this oscillating regime, which obviously does not look at all like real markets.

The effect of the polarization P, which can be appreciated in the phase diagram of Fig. 2, is quite easy to understand in the light of this microscopic analysis: a positive value of P increases the average number of trend followers and therefore is almost irrelevant for the bubble dynamics which is based on an endogenous trend following behaviour. On the other hand $P < 0$ acts contrarily to the bubble creation forcing a percentage of agents to act against the trend. Interestingly enough, as it can be seen from the phase diagram, even a small $P < 0$ is able to prevent the appearance of a bubble. As a function of $|P|$, the transition between an oscillating behaviour and a stable behaviour is first-order like, in the sense that the period of the oscillation is still finite when the transition occurs.

The above mechanism also allows us to understand the role of the parameters f or the fraction of irrational agents p_i. The numerical simulations indicate that increasing the value of these parameters may at first *stabilize* the oscillating phase, while one would intuitively expect an opposite behaviour since fundamentalists/irrational players reduce the relative number of trend followers. This effect can be explained in the following way. We have seen that, as the price increases during the bubble, the available cash of active players and therefore their buying power decreases bringing the bubble to saturation. However, each active agent has a certain probability proportional to f to become a fundamentalist and sell, increasing his wealth in cash. The buying pressure can then stay higher for a longer period of time, determining more stable bubbles. Of course, as f increases substantially, its other effect of nucleating opposite trends becomes the most important one, and the system behaves much as if the polarization parameter P was strongly negative, and thus prevents the appearance of bubbles. Finally, the presence of irrational agents who buy or sell randomly, and therefore statistically drop out of the order imbalance, can be seen as reducing the effective value of g, and thus – somewhat surprisingly – stabilizing the oscillating regime.

Note finally that the oscillating regime tends to die out for very long times. This is due to the fact that agents that have a buying strategy during the bubble tend to underperform on the long run. Their wealth is, at long times, insufficient to sustain the bubble. In order to determine the oscillation period more precisely, we have artificially given to each agent both its initial strategy and its perfect mirror image. This prevents the appearance of two groups with systematically different wealths, without changing the basic mechanism leading to bubbles and crashes.

5 The Intermittent Regime

5.1 Results from the simulation

Upon increasing the parameter g/λ, the oscillating regime disappears and gives rise to an interesting market behaviour, where different market 'states' coexist: bubbles and crashes, periods of very small activity, and periods of very large activity, intermixed with each another. A typical chart of the returns and of the volume as a function of time is plotted in Figure 5. From visual inspection, it is quite clear that the return time series exhibits volatility clustering. We now turn to a more quantitative analysis of the price series statistics. The data we analyze below corresponds to $N = 10000$ agents, with 'unpolarized' strategies $P = 0$, and for $g/\lambda = 0.75$, deep in the intermittent regime (see Fig. 2). Other parameters are identical to those in Fig 1. We note that the qualitative effects we report below have not been seen to depend on N, at least up to $N = 10000$ which is the largest size we have investigated. Strong size effects, reported in the Lux-Marchesi model [29] for instance, seem to be absent in our case. However, we expect that when N becomes comparable to the total number of strategies (i.e. 2^{2^m}), the phenomenology will change since many agents will share exactly the smae strategies.

First, we look at the price variogram, defined by Eq.(13). This is shown in Fig. 6, together with an Ornstein-Uhlenbeck fit. The saturation time τ_0 is of the order of 100; taking the unit time in our model to be the week, this corresponds to one or two years, with a volatility of roughly 20% per year, which is quite reasonable. As explained in section 3 the saturation is a direct consequence of the bounded wealth of agents. Is there a similar mean-reverting trend in real markets ? The analysis of the variogram of the Dow-Jones index, for example, in the period 1950-2000, shows no convincing sign of saturation on the scale of the year, although a slight bend down wards for longer time lag is visible, but the data become noisy. There has been reports in the literature of a systematic mean-reverting effects on the scale of 5-10 years, perhaps related to the mechanism discussed here. Remember that the fundamental price in our model has zero volatility. The short time volatility is the result of pure trading, which leads to a random walk like behaviour of the price; the saturation occurs because of insufficient resources to sustain a large difference between the fundamental price and the speculative price.

The two other quantities that we have systematically studied are the volume variogram and the absolute return variogram, defined as:

$$\mathcal{V}_{v,\sigma}(\tau) = \left\langle \left(O(t+\tau) - O(t)\right)^2 \right\rangle, \tag{17}$$

where $O(t)$ denotes, respectively, the fraction of active agents and the absolute return $|r(t)|$. We show these two quantities in Fig. 7 as a function of $\sqrt{\tau}$, together with a fit inspired from the theory explained in the next subsection:

$$\mathcal{V}(\tau)|_{SQRT} = \mathcal{V}_\infty \left(1 - \exp(-\sqrt{\frac{\tau}{\tau_0}})\right). \tag{18}$$

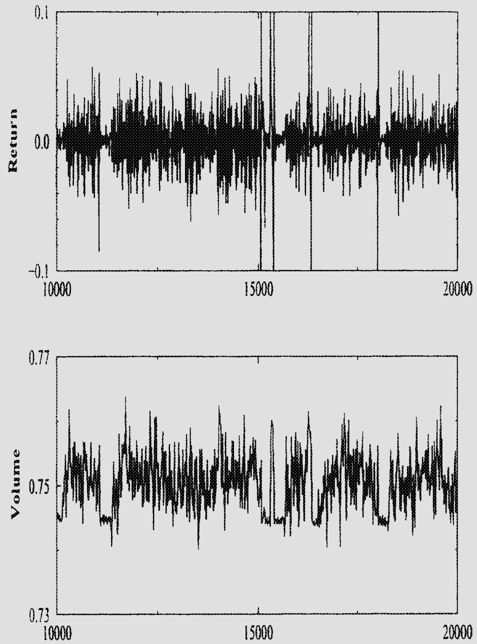

Fig. 5. Time series of the returns, truncated to $\pm 10\%$ (top panel), and of the fraction of active agents (bottom panel), for $g/\lambda = 0.75$.

Note that the short time behaviour of the above function is $\sqrt{\tau}$, in contrast with the regular (linear) behaviour of the Ornstein-Uhlenbeck form. We will explain the origin of this singularity below, and explain why Eq. (18) fits very well the volume variogram. For the variogram of the absolute returns, we have added to Eq. (18) a non zero constant, that takes into account the fact that $|r(t)|$ is a noisy estimate of the volatility; this extra noise is uncorrelated for different days and adds a contribution proportional to $1 - \delta_{\tau,0}$. We have also shown a power-law fit

$$\mathcal{V}(\tau)|_{PWR} = \mathcal{V}_\infty \left(1 - \left(\frac{\tau}{\tau_0}\right)^{-\alpha}\right), \tag{19}$$

which has been advocated in many empirical studies, with $\alpha \sim 0.1 - 0.3$. As can be seen from Fig. 7, the two fits are of comparable quality. Note that the value of α found for $\mathcal{V}_v(\tau)$ is significantly smaller that that for $\mathcal{V}_\sigma(\tau)$, as also found for real market data. The reason for this will be explained in the next subsection.

We have also studied the distribution of returns. Not surprisingly, this distribution is found to be highly kurtic, which is expected since the volatility

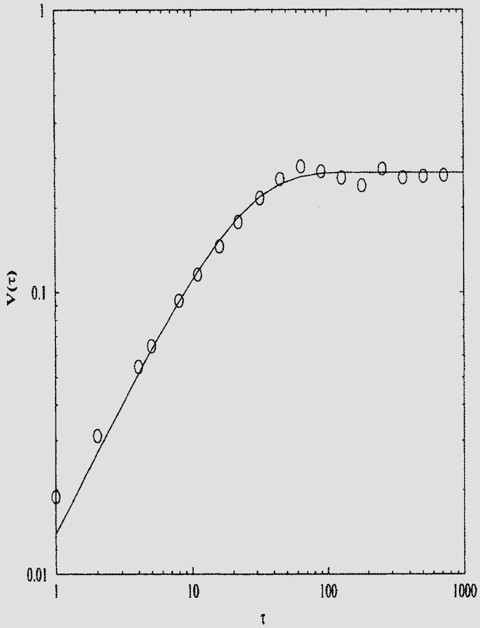

Fig. 6. Variogram of the price as a function of the time lag, together with an Ornstein-Uhlenbeck (mean-reverting) fit.

is fluctuating. The tail of the distribution can be fitted by a power-law, with (for the particular value $g/\lambda = 0.5$) an exponent $\mu \approx 3.5$, similar to the value reported in [50]: see Fig 8. The point here is of course not to claim quantitative agreement, but to show that the ingredients of the model are sufficient to give rise to these tails. Note also that the negative tail is slightly fatter than the positive tail.

The role of a non zero polarization of strategies P is, for small enough P, to induce some correlations (or anticorrelations) in the returns. For $P < 0$, as was the case for the oscillating regime, there is a first-order (discontinuous) transition toward a stable market, with small, strongly anticorrelated fluctuations that track the fundamental price. This transition occurs for rather small values of $|P| \approx 0.03$. For $P > 0$, on the other hand, the intermittent phase survives but the variogram of price fluctuations shows significant positive correlations. For sufficiently large P, the oscillating phase reappears in a continuous way.

The conclusion of this subsection is that volatility clustering appears for large values of g/λ. Qualitatively similar effects are seen for different choices of m (memory time for the strategies), β (memory time for the scores), and S (number of strategies per agent), provided g/λ is large enough to be in

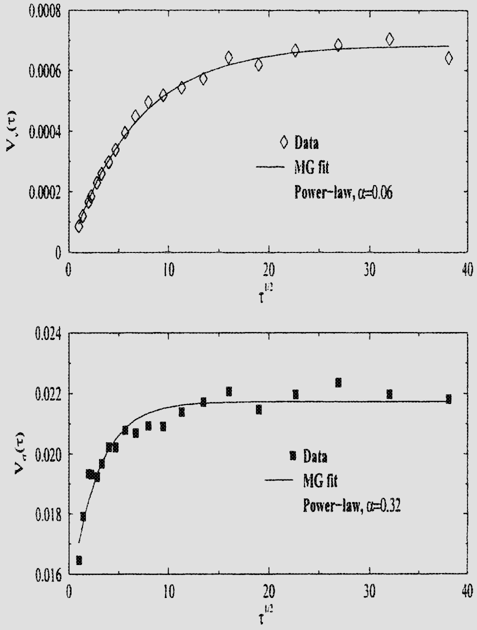

Fig. 7. Variogram of the volume (top panel) and of the absolute return (Bottom panel) as a function of the time lag, together with the fits given by Eq. (18) and Eq. (19).

the intermittent phase. A crucial ingredient, however, is the existence of an inactive strategy, i.e., the fact that the volume of activity is allowed to fluctuate. We have not been able to obtain long term volatility correlations of the type reported in Fig. 7 when all strategies are active. This observation has motivated us to propose a simple mechanism for non trivial volume (and volatility) fluctuations [39, 19], that we discuss now in the present context.

5.2 A simple mechanism for long-ranged volume correlations

Random time strategy shifts

In the above model, as in the Minority Game, scores are attributed by agents to their possible strategies, as a function of their past performance. In particular, the inactive strategy is adopted when the score of all active strategies are negative.

In the turbulent regime we have seen that the market is 'quasi-efficient': the autocorrelation of the price increments is close to zero. To a first approximation no strategy can on the long run be profitable. This implies that the

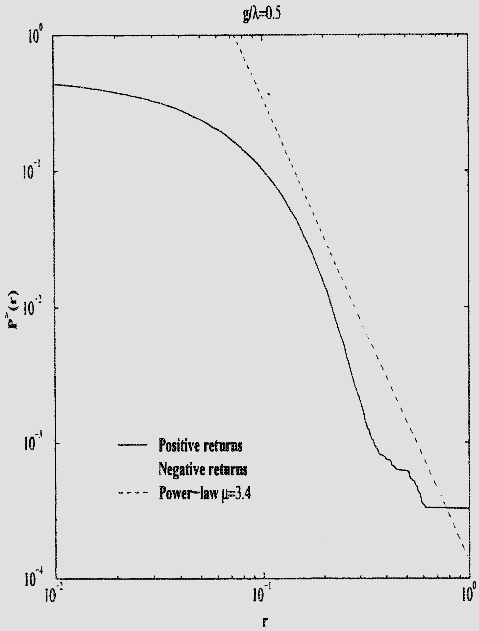

Fig. 8. Distribution of returns for $g/\lambda = 0.5$. A power-law tail with $\mu = 3.4$ is shown for comparison.

strategy scores locally behave, as a function of time, as random walks. This very fact enables us to explain the fluctuations of volume. Let us consider for simplicity the case $S = 2$ (one active strategy and one inactive strategy per agent). Since the switch between two strategies occurs when their scores cross, the activity of an agent is determined by the survival time of the active strategy over the inactive one, that is by the return time of a random walk (the score of the active strategy) to zero. The interesting point is that these return times are well known to be power-law distributed. This leads immediately to the non trivial behaviour of the activity variogram shown in Fig.7. The same argument was used to explain the volume fluctuations in the Minority Game with an inactive strategy, in the efficient phase [39].

More formally, let us define the quantity $\theta_i(t)$ that is equal to 1 if agent i is active at time t, and 0 if inactive. The total activity is given by $v(t) = \sum_i \theta_i(t)$, and the activity variogram is given by

$$V_v(t,t') = \langle [v(t) - v(t')]^2 \rangle = C_v(t,t) + C_v(t',t') - 2C_v(t,t'). \qquad (20)$$

where $C_v(t',t)$ denotes the volume correlation function.

One can consider two extreme cases which lead to the same result, up to a multiplicative constant: (a) agents follow completely different strategies and have independent activity patterns, i.e. $\langle\theta_i\theta_j\rangle \propto \delta_{i,j}$ or (b) agents follow very similar strategies, in which case $\theta_i = \theta_j$. In both cases, $C_v(t,t')$ is proportional to $\langle\theta_i(t)\theta_i(t')\rangle$ and can be exactly expressed in terms of the distribution $P(s)$ of the survival times s of the active strategies [51, 39]. For an unconfined random walk, the return time distribution $P(s)$ decays as $s^{-3/2}$ for large s. Note that this $s^{-3/2}$ behaviour is super-universal and only requires short range correlation in the score increments, not even a finite second moment [52, 53]. However, in our model, the finite memory with which the scores are updated (i.e. the value of $\beta < 1$ in Eq. (12)) leads to a truncation of the $s^{-3/2}$ beyond a time $\tau_0 \simeq 1/\log(1/\beta)$. Without this truncation, the volume $v(t)$ would never become a stationary process, i.e. $C_v(t,t')$ would still depend on both t and t' at long times, a phenomenon called 'aging' in the physics literature [54]. For finite τ_0, one can show analytically that [39, 19]:

$$V_v(t+\tau,t) = V_v(\tau) \propto \sqrt{\frac{\tau}{\tau_0}} \qquad \tau \ll \tau_0. \tag{21}$$

The formula given by Eq. (18), was found to reproduce very well the crossover from this exact short time singular behaviour to the saturation regime.

The interesting conclusion is the following: the very fact that agents compare the performance of two strategies on a random signal leads to a multi-time scale behaviour of the volume fluctuations. This argument accounts very accurately for our numerical data both for the Minority Game and the present market model (see Fig. 7) [39, 19], and also reproduces quite well the empirical variogram of activity in real markets [39].

Volume and volatility

Let us now discuss the relation between volume and volatility. In real markets, the two are known to be correlated; more precisely, it has recently been shown in [55] that the long run correlations in volatility come from the long range correlation in the volume of activity (see also [56]). In our artificial market, a scatter plot of the logarithm of the absolute return, $\log|r(t)|$ versus the volume of activity $v(t)$ shows nearly no correlations when the volume has small fluctuations around its average value $\overline{v}$, but is strongly correlated with $v(t)$ when $v(t)$ has large excursions above $\overline{v}$. This means that periods of high activity are also periods of large volatilities. One therefore expects that the structure of temporal correlations of the volume discussed above is also reflected in the volatility. If the relation between $|r(t)|$ and $v(t)$ was linear, or weakly non linear, one would in fact expect exactly the same shape for the variogram. The fact that this relation is highly non linear, i.e., nearly no correlations for $v(t) \approx \overline{v}$, and a roughly exponential relation for $v(t) > \overline{v}$, adds an instantaneous noise contribution proportional to $1 - \delta_{\tau,0}$ to the variogram of absolute returns, and leads to a strong distortion of the shape of the relaxation. The fact that the effective power-law α, defined in Eq. (19), is larger for $|r(t)|$ than for $v(t)$ can be understood in details in the context of the multifractal random walk model of Bacry et al. [10].

Other mechanisms for long-ranged correlations

Recently, many agent based models have been proposed to account for the stylized facts of financial markets, in particular volatility clustering and long range dependence [57, 31, 32, 58, 37, 59, 60]. From the analysis of these models, one can distinguish three main mechanisms for this long-range dependence:

- Subordination of the strategies to performance. This is the mechanism explained above: as soon as each agent has different strategies with different levels of activities, and that the choice between these strategies is subordinated to their performance, one expects to see long range dependence of the type described above, whenever these strategies lead to identical long term performance. A similar mechanism is found in the models of [57, 32, 58], where agents switch between different trading styles (e.g. fundamentalists/chartists) as a function of their perceived performance and of herding effects. The basic prediction of this scenario is the short time square root singularity of the volume variogram. This prediction is very well obeyed in the Lux-Marchesi model, as shown in Fig. 9.
- Subordination of the volatility to the price. In many models, the level of activity depends on the difference between the current price and a fundamental price. For example, in the model considered by Bornholdt [59], the volatility is a growing function of the absolute difference of these prices. In the model recently studied by Alfarano and Lux [60], on the contrary, the volatility is a decreasing function of this difference. In these models, the price is mean reverting toward the fundamental price. Calling $y(t) = \log(X(t)/X_f(t))$, where X_f is the fundamental price, a schematic equation for $y(t)$ is:
$$\frac{dy}{dt} = -\kappa y + \sigma(y)\xi(t), \tag{22}$$
where ξ is a white noise and $\sigma(y)$ a certain function. The corresponding temporal correlation function of the volatility can, for some specific forms of $\sigma(y)$, be exactly computed, and generically leads to a non exponential decay that can mimic long term dependence. Empirically, one can indeed detect, on the Dow Jones index over a century, some correlation between the volatility and the difference between the current level and the average level of the index. However, empirical studies show that volatility clustering exists even when the market is close to its average level.
- Heterogeneity of the agents time scales. Another mechanism, close in spirit to the 'HARCH' model [61] or the cascade models proposed recently [11, 12, 13, 9, 10], comes from the different time horizons used by the agents to set up their strategies. For example, in the model of Raberto et al. [37], agents place orders at a distance from the current price proportional

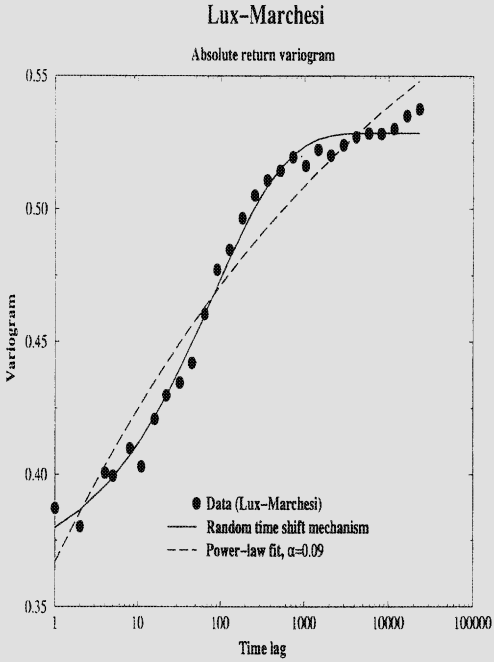

Fig. 9. Variogram of the absolute returns as a function of the time lag in the Lux-Marchesi model with parameters as in [32]. We compare with the fit suggested by the random time strategy shift mechanism (Eq. (18)) and a power-law fit Eq. (19), as suggested by multifractal models [10].

to a sliding average of the past volatility. The time scale used by the different agents is uniformly distributed between 10 day and 100 days. Correspondingly, this induces a non exponential decay of the volatility correlation function when $\tau \leq 100$.

All these three mechanisms are expected to play a role in real markets. More precise statistical tests will hopefully allow one to distinguish between them and estimate their relative contribution to the long-range dependence effects.

6 Dynamical Evolution of the Parameters and Self-organization

As discussed above, our model *a priori* involves a large number of parameters. However, the precise value of most of them is not crucial. The only two important parameters are the ratio g/λ, that measures the impact of trading on prices, and P, that measures the tendency of the agents for trend following ($P > 0$) versus contrarian strategies ($P < 0$). For small values of g/λ, the volatility of price changes is very small, and, as we have discussed in section 4, bubbles form in such a way that very visible trends appear. Furthermore, for small g/λ, these bubbles have a very long lifetime. This means that the agents will naturally increase the fraction of the wealth they invest in this market, since the perceived risk is small. Hence g will spontaneously increase. Simultaneously, small g/λ's lead to rather small execution rates: see Fig. 4. Therefore, the microstructure of the market will evolve such as to make λ smaller, so that the market becomes more liquid. Both effects lead to an increase of the ratio g/λ, and the corresponding destruction of clear trends. This scenario therefore suggests that g/λ enters the intermittent region, where the market becomes 'quasi-efficient' (i.e. returns are uncorrelated on short time scales), but where interesting statistical anomalies appear. The ratio g/λ then stops growing, since the market becomes very volatile, without clear trends. The agents therefore limit their investment. The above scenario is somewhat related to that proposed in the context of the Minority Game in [44]: increasing the number of players leads to a more efficient game, but reduces the incentive for new players to enter, such as a 'marginally efficient' state is reached.

The role of the parameter P is quite interesting. If agents were on average only slightly contrarian, the behaviour of the market would be completely different, with boring small mean reverting fluctuations around the fundamental price, that we have here modeled as a deterministic growth $X_f(t) = \exp(\rho t)$. If this fundamental price was itself randomly fluctuating, our model market would be very close to the standard 'efficient market' picture, where rational agents systematically correct past excess returns such as to lock the market price to the fundamental price. The fact that human psychology seems to favor mimetic, trend following strategies seems to keep the market in the intermittent region of the phase diagram shown in Fig. 2.

An interesting question in the context of financial markets is to understand how the introduction of transaction costs (as the Tobin tax for example) might stabilize the behaviour of these markets. We have studied a generalization of our model in which a non zero proportional fee is paid at each transaction. This drives the system toward a much more stable regime. Transaction costs play a similar role to a negative polarization (more contrarian) strategies.

7 Summary and Conclusion

In this paper, we have presented a detailed study of a rather complex market model, inspired from the Santa Fe artificial market and the Minority Game. Agents have different strategies among which they can choose, according to their relative profitability, with the possibility of *not* participating to the market. The price is updated according to the excess demand, and the wealth of the agents is properly accounted for. The set up of the model involves quite a large number of parameters. Fortunately, only two of them play a significant role: one describes the impact of trading on the price, and the other describes the propensity of agents to be trend following or contrarian.

The main result of our study is the appearance of three different regimes, depending on the value of these two parameters: an oscillating phase with bubbles and crashes, an intermittent phase and a stable 'rational' market phase. The statistics of price changes in the intermittent phase resembles that of real price changes, with small linear correlations, fat tails and long range volatility clustering. We have discussed how the time dependence of these parameters could spontaneously lead the system in the intermittent region. We have analyzed quantitatively the temporal correlation of activity in the intermittent phase, and have shown that the 'random time strategy shift' mechanism proposed in an earlier paper [39] allows to understand these long ranged correlations. Other mechanisms leading to volatility clustering have been reviewed.

Many extensions of the model could be thought of, for example, the diversity of time horizons used by the different agents, the introduction of time dependent fundamental factors, or a larger number of tradable assets. A more interesting path, in our mind, would be to simplify the model sufficiently as to be able to analytically predict the phase diagram of Fig. 2, and reach a similar level of understanding as in the Minority Game (for some work in this direction, see [36].)

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A Dynamic Analysis of Speculation across two Markets

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Summary. A discrete time model of a financial market is proposed, where the time evolution of asset prices and wealth arises from the interaction of two groups of agents, fundamentalists and chartists. Each group allocates its wealth between a risky asset (stock) and an alternative asset (bond), and the two groups have heterogeneous expectations about returns. We assume that chartists compute expected returns by extrapolating past price changes, while fundamentalists form their expectations on the basis of their superior knowledge of fundamentals. Under the assumption that agents have CRRA utility, investors' optimal demand for each asset depends on their wealth, and this results in growing price and wealth processes. The time evolution of the prices is modeled by assuming the existence of a market maker, who sets excess demand of each asset to zero at the end of each trading period by taking an off-setting long or short position. The market maker is assumed to adjust the price, in each period, partly on the basis of the excess demand and partly according to particular stabilizing policies. The model results in a high dimensional nonlinear discrete-time dynamical system with growing prices and wealth. Although the model is nonstationary, suitable changes of variables lead to a stationary model where the dynamic variables are actual and expected returns, fundamental/price ratios, and wealth proportions of chartists and fundamentalists. The steady states and other invariant sets of the model are determined, and important global dynamic phenomena are studied through numerical investigations. Stochastic simulations are also performed, that show the ability of the model to generate some of the characteristic features of financial time series.

1 Introduction

In recent years several models of asset price dynamics based on the interaction of *heterogeneous agents* have been proposed (Day and Huang 1990, Brock and Hommes 1998, Lux 1998, Chen and Yeh 1997, Gaunersdorfer 2000, Chiarella and He 2001, 2002a,b, Fernandez-Rodriguez et al. 2002). Most of these models, some of which allow the size of the different groups of agents to vary

M. Gallegati et al. (eds.), *The Complex Dynamics of Economic Interaction*

according to the evolution of the financial market, are of necessity not very mathematically tractable. In a previous paper, Chiarella et al. 2002a, whose antecedents are Chiarella 1992, Beja and Goldman 1980, and Zeeman 1974, we developed a two-dimensional discrete time model of asset price dynamics containing the essential elements of the heterogeneous agents paradigm whilst still remaining mathematically tractable. In that model we assumed a financial market with a risky asset and an alternative riskless asset, consisting of two types of traders, *fundamentalists* and *chartists*. However, the model studied in Chiarella et al. 2002a is a partial one since it leaves in the background the dynamics of the market for the alternative asset. This results in the equilibrium not being at the fundamental value of the risky asset. Moreover, in that model agents' optimal demands for the risky asset are independent of their wealth, as a result of the underlying CARA utility functions, and therefore the time evolution of agents' wealth has no effect on price dynamics.

In the present paper, again assuming a financial market with fundamentalists and chartists, we develop a more complete model where the dynamics of the price of the alternative asset, and its dependence on agents' investment decisions, is also taken into account. This avoids the unintuitive steady state of the earlier model. Moreover, we consider a more realistic framework where investors' optimal decisions depend on their wealth (as a result of underlying CRRA utility functions) and both price and wealth processes are thus growing. Each group forms expectations about asset returns and allocates its wealth between the risky asset and the alternative asset. The time evolution of the prices is modelled by assuming the existence of a *market maker*, who sets excess demand to zero at the end of each trading period by taking an off-setting long or short position. The market maker also decides the next period prices partly on the basis of the excess demand (because of inventory reasons) and partly in order to steer prices back to their fundamental values (to ensure an "orderly" adjustment of prices).

The model results in a high-dimensional nonlinear discrete-time dynamical system that describes the time evolution of actual and expected returns, fundamental/price ratios and wealth proportions of the two groups. Despite the high dimension of the model, analytical results can be obtained in some particular lower-dimensional cases, and these help to understand the global behaviour of the dynamical system.

The structure of the paper is as follows. Section 2 derives the asset demand functions for each asset by each investor type. Section 3 describes how demands are aggregated by the market maker via a price adjustment rule in the market for each asset. Section 4 describes the resulting dynamical system for the time evolution of actual and expected returns, fundamental/price ratios and wealth proportions. Section 5 focuses on the equilibria of the model as well as on other important *invariant* subsets of the phase-space. Section 6 performs some global analysis of the out-of-equilibrium dynamics of the model, focusing on phenomena of regular or chaotic oscillatory dynamics, intermittent behaviour, and coexistence of attracting sets in the phase space.

Section 7 performs some stochastic simulations in order to show how the interaction of the nonlinear deterministic phenomena of the model with simple noise processes can give rise to some of the typical distributional features of asset returns. Some conclusions are contained in Section 8.

2 Asset Demand and Expectation Formation

We label with 1 the risky asset (stock) and with 2 the alternative asset (bond). For $i = 1, 2$ we denote by $P_{i,t}$ the price of the i-th asset at time t, by g_i the dividend (or coupon) yield, assumed constant, produced by the i-th asset from t to $t+1$; we use the subscript $j \in \{f, c\}$ to denote fundamentalists or chartists. In each time period each group of agents is assumed to invest some of its wealth in the risky asset and some in the risk-free asset. Denote by $\Omega_t^{(j)}$ and $Z_{i,t}^{(j)}$ the wealth of agent j and the fraction of that wealth that agent j decides to invest in the i-th asset at time t, respectively, with $Z_{1,t}^{(j)} + Z_{2,t}^{(j)} = 1$. The time evolution of the wealth of agent j satisfies:

$$\begin{aligned} \Omega_{t+1}^{(j)} &= \Omega_t^{(j)} + \Omega_t^{(j)} Z_{1,t}^{(j)} \left(\frac{P_{1,t+1} - P_{1,t}}{P_{1,t}} + g_1\right) + \\ &\quad + \Omega_t^{(j)} (1 - Z_{1,t}^{(j)}) \left(\frac{P_{2,t+1} - P_{2,t}}{P_{2,t}} + g_2\right) , \end{aligned} \tag{1}$$

where $(P_{i,t+1} - P_{i,t})/P_{i,t}$ is the capital gain on the i-th asset over $(t, t+1)$.

We denote by $E_t^{(j)}$, $Var_t^{(j)}$ the "beliefs" of investor type j, at time t, about conditional expectation and variance, respectively. It is assumed that all the investors have the same attitude to risk with the same CRRA utility of wealth function $u(\Omega) = \log(\Omega)$. Agent j seeks the fractions $Z_{i,t}^{(j)}$, $i = 1, 2$, so as to maximize $E_t^{(j)}[\log(\Omega_{t+1}^{(j)})]$, the expected utility of wealth at time $t+1$.

For $j \in \{f, c\}$, $i = 1, 2$, denote by $m_{i,t}^{(j)} = E_t^{(j)}[(P_{i,t+1} - P_{i,t})/P_{i,t}]$ and $V_{i,t}^{(j)} = Var_t^{(j)}[(P_{i,t+1} - P_{i,t})/P_{i,t}]$ the agent j's conditional expectation and variance of the capital gain on the i^{th} asset, respectively. As it can be shown[4], under some not very restrictive simplifying assumptions the maximization problem of agent j, viz. $\max_{Z_{1,t}^{(j)}} E_t^{(j)}[\log(\Omega_{t+1}^{(j)})]$, results in the following optimal investment fractions ($j \in \{f, c\}$):

$$Z_{1,t}^{(j)} = \frac{(m_{1,t}^{(j)} - m_{2,t}^{(j)}) + (g_1 - g_2)}{V_{1,t}^{(j)}} , \qquad Z_{2,t}^{(j)} = 1 - Z_{1,t}^{(j)} . \tag{2}$$

The two groups of agents differ in the way they update their expectations over successive time intervals. We assume that chartists follow the trend, so

[4] See Chiarella et al. 2002b and Chiarella and He 2001.

that their conditional expectations $m_{i,t}^{(c)}$, $i = 1, 2$, evolve over time according to the adaptive rule $m_{i,t+1}^{(c)} = (1 - c_i)m_{i,t}^{(c)} + c_i[(P_{i,t+1} - P_{i,t})/P_{i,t}]$, where $0 \leq c_i \leq 1$. On the other hand we assume that the fundamentalists, with their superior knowledge of the economy, are able to estimate the *fundamental* price of each asset at time t ($Y_{i,t}$, $i = 1, 2$) as the discounted present value of expected future payments. We also assume that they believe in a return to the fundamental price, so that their expectation of the capital gain on asset i over $(t, t+1)$ includes a time-varying short-run component $\eta_i(Y_{i,t} - P_{i,t})/P_{i,t}$, $0 \leq \eta_i \leq 1$, which is proportional to the (relative) deviation of the actual price $P_{i,t}$ from the fundamental $Y_{i,t}$, and a constant component ψ_i, that reflects their expectation about the growth rate of the fundamental in the long-run. Therefore the fundamentalist expected capital gain on the i-th asset may be written as $m_{i,t}^{(f)} = \eta_i(Y_{i,t}/P_{i,t} - 1) + \psi_i$, so that: $m_{1,t}^{(f)} - m_{2,t}^{(f)} = \eta_1(Y_{1,t}/P_{1,t} - 1) - \eta_2(Y_{2,t}/P_{2,t} - 1) + \delta$, where $\delta = \psi_1 - \psi_2$ represents the difference between the expected growth rates of the fundamentals.

3 Price Setting Rules

We assume the existence of a market maker, who is sufficiently "rational" to know the fundamental prices at time t as well as their growth rates over the next period (denote them by $\phi_{i,t+1}$, $i = 1, 2$). We assume that he/she also knows the wealth fractions that "in equilibrium" each type of agents will invest in each asset (denote them by $\widetilde{Z}_1^{(f)}, \widetilde{Z}_1^{(c)}, \widetilde{Z}_2^{(f)}, \widetilde{Z}_2^{(c)}$, with $\widetilde{Z}_2^{(j)} = 1 - \widetilde{Z}_1^{(j)}$, $j \in \{f, c\}$)[5]. The market maker sets the excess demand of each asset to zero at the end of each trading period, by taking an off-setting long or short position, and adjusts the prices partly according to the out-of-equilibrium demand, and partly in order to steer the prices back to their fundamental values. The above assumptions about the market maker's behavior are expressed by the following price setting rule in the i-th market, $i = 1, 2$:

$$\begin{aligned} P_{i,t+1} - P_{i,t} = \alpha_i(Y_{i,t} - P_{i,t}) + \phi_{i,t+1}Y_{i,t} + \\ + P_{i,t}\beta_i \left[\frac{\Omega_t^{(f)}(Z_{i,t}^{(f)} - \widetilde{Z}_i^{(f)}) + \Omega_t^{(c)}(Z_{i,t}^{(c)} - \widetilde{Z}_i^{(c)})}{\Omega_t^{(f)} + \Omega_t^{(c)}} \right] \end{aligned} \tag{3}$$

where $\beta_i > 0$, $0 < \alpha_i < 1$. The price setting rule (3) can be interpreted in the sense that the market maker varies the price from period to period so as to adjust his/her inventory of the asset, by raising the price when excess demand reduces the inventory and by lowering the price when excess supply determines inventory accumulation; at the same time, the market maker corrects his/her

[5] We could for instance think of the market maker using his/her knowledge of the time series of the order flow from the two groups of market participants to estimate these fractions as a long-run average.

response so as to bring about "orderly" price movements in the market, this being one of the other assumed roles of the market maker[6]. In equation (3), the first term on the right hand side describes the market maker's policy that seeks to steer the price of the i-th asset back to fundamental price $Y_{i,t}$ (with adjustment coefficient α_i). The second term on the right hand side of (3) specifies the price change due to the underlying trend in the fundamental, assumed to be known to the market maker. The third term on the right hand side is the portion of the price change depending on agents' demand at time t: consistently with our assumptions about the market maker's behavior, this latter term is proportional to the out-of equilibrium demand for the i^{th} asset[7]. The coefficient β_i represents the market maker's speed of adjustment of the i^{th} price.

4 The Dynamical System and the Map

The dynamics of the model derived in the previous sections can be summarized as:

$$\begin{aligned} P_{i,t+1} - P_{i,t} &= P_{i,t}\beta_i \left[\frac{\Omega_t^{(f)}(Z_{i,t}^{(f)} - \widetilde{Z}_i^{(f)}) + \Omega_t^{(c)}(Z_{i,t}^{(c)} - \widetilde{Z}_i^{(c)})}{\Omega_t^{(f)} + \Omega_t^{(c)}} \right] + \\ &\quad + \alpha_i(Y_{i,t} - P_{i,t}) + \phi_{i,t+1}Y_{i,t} \;, \qquad i = 1,2 \\ m_{i,t+1}^{(c)} &= (1 - c_i)m_{i,t}^{(c)} + c_i[(P_{i,t+1} - P_{i,t})/P_{i,t}] \;, \qquad i = 1,2 \qquad (4) \\ \Omega_{t+1}^{(j)} &= \Omega_t^{(j)} \left[1 + \sum_{i=1}^{2} Z_{i,t}^{(j)} \left(\frac{P_{i,t+1} - P_{i,t}}{P_{i,t}} + g_i \right) \right], \; j \in \{f, c\} \quad (5) \end{aligned}$$

where $Z_{1,t}^{(f)} = [\eta_1(Y_{1,t}/P_{1,t} - 1) + \delta - \eta_2(Y_{2,t}/P_{2,t} - 1) + (g_1 - g_2)]/V_{1,t}^{(f)}$, $Z_{1,t}^{(c)} = [(m_{1,t}^{(c)} - m_{2,t}^{(c)}) + (g_1 - g_2)]/V_{1,t}^{(c)}$, and $Z_{2,t}^{(j)} = 1 - Z_{1,t}^{(j)}$, $\widetilde{Z}_2^{(j)} = 1 - \widetilde{Z}_1^{(j)}$, $j \in \{f, c\}$. For the moment we do not specify the way agents update their beliefs about the conditional variance. In order to avoid adding more dynamic equations to the model, we assume[8] that agent type j calculates $V_{1,t}^{(j)}$ as a function of the expected returns $m_{1,t}^{(j)}$ and $m_{2,t}^{(j)}$.

[6] In order to focus on the dynamics resulting from agents' interaction, the role played by the market maker in this model is highly stylised. The literature on the market maker behaviour suggests that he/she may play a more complex role in price formation, being not only a dealer who adjusts the quoted prices, but also an active investor (see, for instance, Madhavan 2000).

[7] Since in this model agents' wealth, and therefore average demand for each asset, are growing over time, in eq. (3) we normalize the out-of-equilibrium demand, dividing it by total asset demand, i.e. by total agents' wealth $\Omega_t = \Omega_t^{(f)} + \Omega_t^{(c)}$.

[8] The same assumption is made in Chiarella et al. 2002a.

Eqs. (3), (4), and (5) define a dynamical model where both prices and wealth processes are growing, due to the underlying growth of the fundamentals in each market and due to the fact that the optimal demand for each asset depends on agents' wealth. A stationary system can be obtained by formulating the model in terms of *returns* and *wealth shares*. To do this, we denote by $\Omega_t = \Omega_t^{(f)} + \Omega_t^{(c)}$ the total wealth at time t, and we consider the new variables[9]: $w_t^{(j)} = \Omega_t^{(j)}/\Omega_t$, $j \in \{f, c\}$, $\rho_{i,t+1} = (P_{i,t+1} - P_{i,t})/P_{i,t}$, $y_{i,t} = Y_{i,t}/P_{i,t}$, $i = 1, 2$, where $w_t^{(f)}$ and $w_t^{(c)} = 1 - w_t^{(f)}$ are the wealth shares of the two groups at time t, $y_{i,t}$ is the ratio between the fundamental value and the price of the i-th asset, $\rho_{i,t+1}$ is the capital gain on the i-th asset over $(t, t+1)$, and therefore $(\rho_{i,t+1} + g_i)$ represents the return on the i-th asset, $i = 1, 2$. We also denote by $\omega_{t+1}^{(j)}$, $j \in \{f, c\}$, the growth rate of the wealth of agent type j over $(t, t+1)$: $\omega_{t+1}^{(j)} = (\Omega_{t+1}^{(j)} - \Omega_t^{(j)})/\Omega_t^{(j)} = (1 - Z_{1,t}^{(j)})(\rho_{2,t+1} + g_2) + Z_{1,t}^{(j)}(\rho_{1,t+1} + g_1)$.

Using the foregoing changes of variables, equations (5) may be rewritten in terms of wealth shares and rates of growth of the agents' wealth (see Chiarella et al. 2002b for details), giving:

$$w_{t+1}^{(f)} = \frac{w_t^{(f)}(1 + \omega_{t+1}^{(f)})}{w_t^{(f)}(1 + \omega_{t+1}^{(f)}) + (1 - w_t^{(f)})(1 + \omega_{t+1}^{(c)})} \; ; \quad w_{t+1}^{(c)} = (1 - w_{t+1}^{(f)}) \; . \qquad (6)$$

As far as the time evolution of the fundamental prices $Y_{i,t}$ is concerned, we assume that their growth rates are constant over time, $\phi_{i,t+1} = \phi_i$, $i = 1, 2$. It follows that the fundamental/price ratios evolve according to

$$y_{i,t+1} = y_{i,t} \frac{(1 + \phi_i)}{(1 + \rho_{i,t+1})}, \qquad i = 1, 2. \qquad (7)$$

Finally, under the assumption that agent j calculates $V_{1,t}^{(j)}$ as a function of the expected returns $m_{1,t}^{(j)}$ and $m_{2,t}^{(j)}$, the agents' conditional variance (and optimal investment fractions) can be represented as functions of the state variables y_1, y_2 and $m_1^{(c)}$, $m_2^{(c)}$, respectively:

$$V_{1,t}^{(f)} = v^{(f)}(y_{1,t}, y_{2,t}) \; ; \quad V_{1,t}^{(c)} = v^{(c)}(m_{1,t}^{(c)}, m_{2,t}^{(c)}) \; ;$$

$$Z_{1,t}^{(f)} = \zeta^{(f)}(y_{1,t}, y_{2,t}) = \frac{\eta_1(y_{1,t} - 1) + \delta - \eta_2(y_{2,t} - 1) + (g_1 - g_2)}{v^{(f)}(y_{1,t}, y_{2,t})} \; ;$$

$$Z_{1,t}^{(c)} = \zeta^{(c)}(m_{1,t}^{(c)}, m_{2,t}^{(c)}) = \frac{(m_{1,t}^{(c)} - m_{2,t}^{(c)}) + (g_1 - g_2)}{v^{(c)}(m_{1,t}^{(c)}, m_{2,t}^{(c)})} \; .$$

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[9] Similar changes of variables for prices and wealth are used, in order to get a stationary model of asset price dynamics, in Chiarella and He 2001. An evolutionary model of a financial market formulated in terms of *wealth shares* of the market participants is also developed, in a different framework, by Blume and Easley 1992.

The foregoing assumptions and changes of variables allow to reduce the the dynamic model defined by (3), (4), and (5) to a stationary system where the dynamic variables are ρ_i, y_i, $m_i^{(c)}$, $i = 1, 2$, and $w^{(f)}$. Denoting with the symbol $'$ the unit time advancement operator, and recalling also that $Z_{2,t}^{(j)} = 1 - Z_{1,t}^{(j)}$ and $\widetilde{Z}_2^{(j)} = 1 - \widetilde{Z}_1^{(j)}$, $j \in \{f, c\}$, the time evolution of the stationary system is given by the iteration of the following 7-dimensional nonlinear map $T : (\rho_1, \rho_2, y_1, y_2, m_1^{(c)}, m_2^{(c)}, w^{(f)}) \longmapsto (\rho'_1, \rho'_2, y'_1, y'_2, m_1^{(c)\prime}, m_2^{(c)\prime}, w^{(f)\prime})$:

$$T : \begin{cases} \rho'_1 = \beta_1[w^{(f)}(Z_1^{(f)} - \widetilde{Z}_1^{(f)}) + (1 - w^{(f)})(Z_1^{(c)} - \widetilde{Z}_1^{(c)})] + (\alpha_1 + \phi_1)y_1 - \alpha_1 \\ \rho'_2 = \beta_2[w^{(f)}(\widetilde{Z}_1^{(f)} - Z_1^{(f)}) + (1 - w^{(f)})(\widetilde{Z}_1^{(c)} - Z_1^{(c)})] + (\alpha_2 + \phi_2)y_2 - \alpha_2 \\ y'_1 = \frac{(1+\phi_1)}{(1+\rho'_1)}y_1; \quad y'_2 = \frac{(1+\phi_2)}{(1+\rho'_2)}y_2 \\ m_1^{(c)\prime} = (1 - c_1)m_1^{(c)} + c_1\rho'_1; \quad m_2^{(c)\prime} = (1 - c_2)m_2^{(c)} + c_2\rho'_2 \\ w^{(f)\prime} = \frac{w^{(f)}[1+(1-Z_1^{(f)})(\rho'_2+g_2)+Z_1^{(f)}(\rho'_1+g_1)]}{1+(1-Z_1^{(c)})(\rho'_2+g_2)+Z_1^{(c)}(\rho'_1+g_1)+w^{(f)}(Z_1^{(f)}-Z_1^{(c)})(\rho'_1+g_1-\rho'_2-g_2)} \end{cases} \tag{8}$$

where:

$$Z_1^{(f)} \equiv \zeta^{(f)}(y_1, y_2) = \frac{\eta_1(y_1 - 1) + \delta - \eta_2(y_2 - 1) + (g_1 - g_2)}{v^{(f)}(y_1, y_2)} \; ;$$

$$Z_1^{(c)} \equiv \zeta^{(c)}(m_1^{(c)}, m_2^{(c)}) = \frac{m_1^{(c)} - m_2^{(c)} + (g_1 - g_2)}{v^{(c)}(m_1^{(c)}, m_2^{(c)})} \; ,$$

We notice that when prices and agents' investment fractions are at their "equilibrium" levels ($y_i = 1$, $i = 1, 2$, $Z_1^{(j)} = \widetilde{Z}_1^{(j)}$, $j \in \{f, c\}$), then actual returns and chartists' expected returns are equal to the rates of growth of the fundamentals: $\rho_i = m_i^{(c)} = \phi_i$, $i = 1, 2$. It follows that the "equilibrium" investment fractions are given by:

$$\widetilde{Z}_1^{(f)} = \frac{\delta + (g_1 - g_2)}{v^{(f)}(1, 1)} \; , \quad \widetilde{Z}_1^{(c)} = \frac{(\phi_1 - \phi_2) + (g_1 - g_2)}{v^{(c)}(\phi_1, \phi_2)} \tag{9}$$

and: $\widetilde{Z}_2^{(j)} = 1 - \widetilde{Z}_1^{(j)}$, $j \in \{f, c\}$. Eqs. (9) state that the equilibrium investment fraction in the risky asset is given by the expected equilibrium risk-adjusted excess return.

5 Invariant Subsets of the Phase Space and Fundamental Steady States

In this section we focus on some general properties of the map (8), that help to understand the dynamic behaviour of the system. An important feature of the map is the existence of a one-dimensional invariant subset of the phase space, associated with the "fundamental" levels of the state variables

ρ_i, $m_i^{(c)}$, y_i, $i = 1, 2$. In fact, assume that such variables are at their equilibrium levels, i.e. $\rho_1 = m_1^{(c)} = \phi_1$, $\rho_2 = m_2^{(c)} = \phi_2$, $y_1 = y_2 = 1$. Then it is easy to check that such variables do not vary under iteration of T, i.e. $T(\phi_1, \phi_2, 1, 1, \phi_1, \phi_2, w^{(f)}) = (\phi_1, \phi_2, 1, 1, \phi_1, \phi_2, w^{(f)\prime})$, which means that the dynamics of the system are constrained in a one-dimensional *invariant* subset of the phase space (let us denote it by E). The time evolution of the system along the invariant manifold E is obtained by iteration of a one-dimensional nonlinear map governing the dynamics of the wealth fractions, say $T^{(w)}$: $w^{(f)} \longmapsto w^{(f)\prime}$, given by $w^{(f)\prime} = w^{(f)}(1 + \widetilde{\omega}^{(f)})/[1 + \widetilde{\omega}^{(c)} + w^{(f)}(\widetilde{\omega}^{(f)} - \widetilde{\omega}^{(c)})]$, where $\widetilde{\omega}^{(j)} = (1 - \widetilde{Z}_1^{(j)})(\phi_2 + g_2) + \widetilde{Z}_1^{(j)}(\phi_1 + g_1)$, $j \in \{f, c\}$, are the rates of growth of the wealth of fundamentalists and chartists along the invariant manifold. Notice that, apart from the particular case where $\widetilde{Z}_1^{(f)} = \widetilde{Z}_1^{(c)}$, and thus $\widetilde{\omega}^{(f)} = \widetilde{\omega}^{(c)}$, the one-dimensional map $T^{(w)}$ admits two fixed points, $w^{(f)} = 1$ and $w^{(f)} = 0$. We can conclude that the 7-dimensional system has at least two steady states (let us denote them by *fundamental* steady states, F and C), both lying in the invariant subset E, that are characterized by equilibrium values of the state variables $\rho_i = m_i^{(c)} = \phi_i$, $y_i = 1$, $i = 1, 2$, and by equilibrium wealth shares $w^{(f)} = 1$ or $w^{(f)} = 0$, where only fundamentalists or, respectively, chartists survive in the market[10].

Other invariant subsets of the phase space are those defined by $w^{(f)} = 1$ and $w^{(f)} = 0$, where only fundamentalists, respectively chartists, participate to the market. In particular, in this latter case the dynamics of the system is obtained by iteration of a 6-dimensional map. As we shall see in the next section, the case $w^{(f)} = 0$ is a good starting point to understand the dynamic behavior of the full system.

6 Out-of-equilibrium Dynamics and Coexistence of Attracting Sets

Throughout the present section, we analyse numerically some global dynamic phenomena of the system: these include regular or chaotic oscillatory dynamics, intermittency, and coexistence of attracting sets. In order to perform the numerical simulations, we specify the analytical form of the fundamentalist and chartist investment fractions $Z_1^{(f)} = \zeta^{(f)}(y_1, y_2)$ and $Z_1^{(c)} = \zeta^{(c)}(m_1^{(c)}, m_2^{(c)})$. Similarly to the basic model developed in Chiarella et al. 2002a, we assume that the fundamentalist estimate of the conditional variance of the risky return is constant ($V_1^{(f)} = \overline{v}_1^{(f)}$), so that the fundamentalist optimal investment proportion in asset 1 is $\zeta^{(f)}(y_1, y_2) = a_1(y_1 - 1) - a_2(y_2 - 1) + b$,

[10] Of course such steady states cannot be considered as "true" equilibrium situations: the possibility that the system converges to such equilibria is mainly due to the fact that our dynamic model does not allow the size of the two groups to vary over time according, e.g., to the realized profits.

where $a_i = \eta_i/\overline{v}_1^{(f)}$, $i = 1, 2$, and $b = (\delta + g_1 - g_2)/\overline{v}_1^{(f)}$. Unlike the fundamentalists, the chartists are assumed to change their estimate $V_1^{(c)}$ of the conditional variance according to the magnitude of the expected excess return $\left|m_1^{(c)} - m_2^{(c)}\right|$. As this quantity becomes larger they expect greater volatility and increase their estimate $V_1^{(c)}$, so that the optimal investment proportion in asset 1 results in a nonlinear S-shaped increasing function of the expected excess return. For the numerical experiments we assume $\zeta^{(c)}(m_1^{(c)}, m_2^{(c)}) = (\gamma/\theta)\arctan[\theta(m_1^{(c)} - m_2^{(c)} + g_1 - g_2)]$, $(\gamma, \theta > 0)$[11].

6.1 Steady state bifurcations and attracting limit cycles

For particular parameter values, the long-run evolution of the system may be characterized by stable oscillations along a limit cycle, where prices and returns fluctuate around their fundamental levels. Let us first illustrate such phenomenon in the subcase $w^{(f)} = 0$, where only chartists operate in the market and therefore, as stressed in the previous section, the time evolution of the system is obtained by iteration of a 6-dimensional map. *Fig. 1a* shows the projection, in the plane of the state variables ρ_1, y_1, of a trajectory converging with damped oscillations to the "fundamental equilibrium", which is an attracting focus. By increasing the values of the adjustment parameters c_1, c_2, β_1 and β_2, with respect to the ones used in *Fig. 1a*, the fundamental equilibrium becomes a repelling focus and trajectories converge to an attracting limit cycle around the equilibrium. We have numerical evidence that the creation of such a limit cycle occurs through a supercritical Neimark-Hopf bifurcation. Figs. *1b* and *1c* show the attracting invariant closed curve existing for increased values of the parameters c_1 and, respectively, β_1 starting from the stable case represented in Fig.*1a*.

The lower dimensional system obtained in the case $w^{(f)} = 0$ is a good starting point to understand the dynamic behavior of the full system: in fact, the limit cycles represented in *Fig. 1* may also attract trajectories starting with positive initial fundamentalists' wealth share $w_0^{(f)}$. As an example, with initial condition $y_{1,0} = y_{2,0} = 0.9$, $\rho_{1,0} = m_{1,0}^{(c)} = 0.025$, $\rho_{2,0} = m_{2,0}^{(c)} = 0.02$, and $w_0^{(f)} = 0.08$, the system again converges to the limit cycle of Fig. *1b*, while starting with higher values of the $w_0^{(f)}$ the system converges to the "fundamental" invariant set E. This is a phenomenon of *coexistence* of attracting sets, where the numerical study of the basins of attraction becomes crucial, with particular attention to the initial wealth shares of the two groups and to the initial deviation of the prices from the fundamentals. Further *global* dynamic phenomena will be analysed in the next section, by use of numerical simulations.

[11] The parameter γ (strength of chartist demand) represents the slope of the sigmoid demand function $h(x) = \frac{\gamma}{\theta}\arctan(\theta x)$ when the expected excess return $x = m_1^{(c)} - m_2^{(c)} + g_1 - g_2$ is zero.

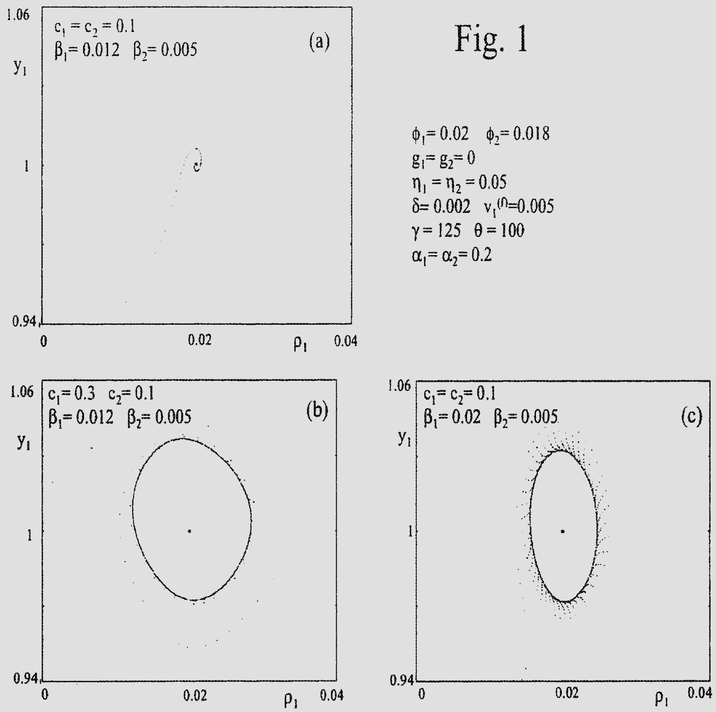

Fig. 1. market dominated by chartists; (a) low values of the adjustment parameters: convergence to the "fundamental" equilibrium; (b), (c) higher values of the adjustment parameters: convergence to a limit cycle.

6.2 Global dynamics and bifurcations

In this section we focus on some important *global* phenomena, such as the creation of new attractors, characterized by regular or chaotic oscillatory dynamics or by phenomena of intermittency.

The first phenomenon we consider concerns the existence of a new attracting limit cycle (different from the one that may exist in the invariant subset $w^{(f)} = 0$), characterized by *long-run fluctuations of the agents' wealth shares*, as well as of the other dynamic variables. The creation of such an attractor is in general due to a *global* bifurcation, i.e. it is not related to the eigenvalues of the Jacobian matrix of the map T evaluated at the equilibria F and C. An example of an attracting limit cycle of this kind is represented in *Fig. 2a*. Along the limit cycle both groups of agents survive in the long-run, with wealth proportions fluctuating around average values. *Figs. 2b* and *2c* represent the effect on the limit cycle of increasing the *strength* of chartist demand γ. Looking at the projection of the attractor in the (ρ_1, y_1) plane, we observe a transition from regular to chaotic fluctuations and an increasing average wealth proportion of fundamentalists as long as the fluctuations become chaotic. This phenomenon suggests that the adaptive rule used by chartists to forecast asset returns is more succesful in the case of regular, rather than erratic, fluctuations. A similar effect on the nature of the cyclic attractor and on the average wealth proportions in the long-run is obtained if we increase the adjustment parameters c_1 and c_2.

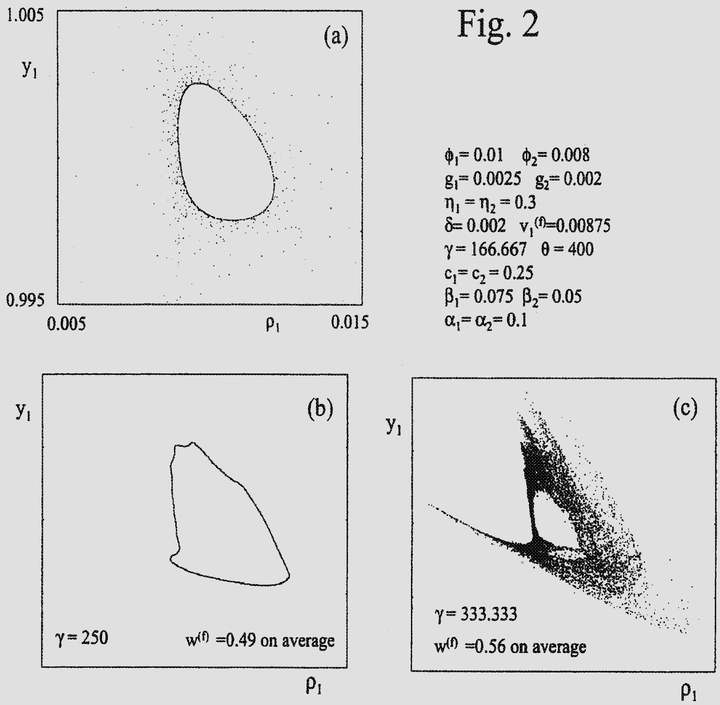

Fig. 2. (a) prices, returns and wealth shares fluctuate on a limit cycle; (b), (c) effect of increasing the strength of chartists demand.

Numerical simulations show that for particular parameter values such an attractor may coexist with other asymptotic behaviors: in particular, according to the initial condition, the system may converge to the cyclic attractor of *Fig. 2* (with long-run fluctuations of all the dynamic variables), or to the invariant set E (with the dynamic variables settling down on their fundamental levels).

A second dynamic phenomenon that is worth considering is the existence of asymptotic behaviors where the system may switch between different regimes characterized by fluctuations of different amplitude. Such phenomenon may consist in a regular switching between phases of periodic fluctuations of different size, as in the case represented in *Fig. 3a*, or in chaotic behavior characterized by sudden "bursts" of erratic fluctuations (*intermittency*), as in the case represented in *Fig. 3b*. It may also happen that such bursts characterize only the transient part of the trajectory, before the system settles down on an attractor with fluctuations of almost constant amplitude (as in the case of *Fig. 3c*, obtained with small changes in the parameters of *Fig. 3b*). As shown in the next section, the interaction between such deterministic nonlinear dynamic phenomena and simple exogenous stochastic factors may generate the characteristic features of financial time series, such as *kurtosis*, *skewness* and *fat tails* of return distributions.

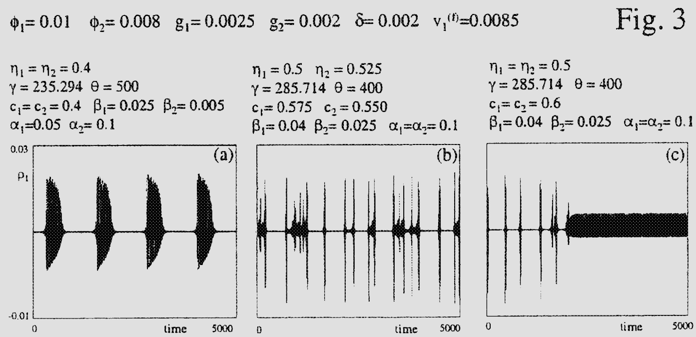

Fig. 3. cases of intermittent behavior: (a) periodic switching to phases of large fluctuations; (b) chaotic "bursts"; (c) intermittency in the transient part of a trajectory.

7 Stochastic Simulations

It is of interest to see how the dynamic features of the nonlinear heterogeneous agents' model established in this paper are affected by simple noise processes. The aim of this section is to show that such an interaction may generate some of the basic characteristics of time series of returns in financial markets such as *fat tails*, *skewness*, *peaked distributions* and *volatility clustering*. In our examples, normal i.i.d random disturbances are added in the dynamic equation for the chartists' expected return $m_1^{(c)}$ in the map (8), so that the equation becomes: $m_1^{(c)\prime} = (1-c_1)m_1^{(c)} + c_1\rho_1' + \xi$, $\xi \sim N(0, \sigma^2)$. This captures the notion that chartists adjust their estimate of the risky return according to randomly arriving good or bad news in the market.

Figs. 4a,b display the time series of returns on asset 1 and the return distribution (compared with the corresponding normal distribution) resulting

Fig. 4

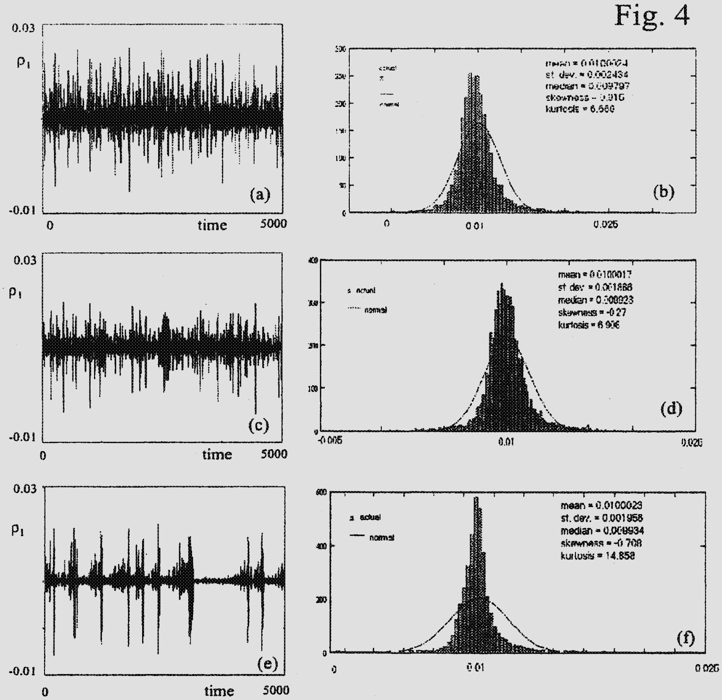

Fig. 4. effect of a normally distributed shock: (a), (b) stochastic time series and distribution of ρ_1 in the case of convergence to a chaotic attractor; (c), (d) and (e), (f) stochastic time series and distribution of ρ_1 in the case of irregular chaotic bursts.

from a simulation of (8) with a normally distributed noise with $\sigma = 0.0006124$. The parameters are the one relevant for the chaotic regime illustrated in *Fig. 2b.* The stochastic version of the model is able to generate fat tails, skewness and kurtosis.

Fig. 4c,d display the time series of returns and the return distribution obtained adding a normally distributed noise ($\sigma = 0.0006532$) to the deterministic situation of intermittency shown in *Fig. 3b.* Again the stochastic model clearly shows some distributional characteristics of real financial data. Similar features are present in *4e,f* ($\sigma = 0.0002041$), associated to the deterministic trajectories represented in *Fig. 3c.* In particular, the phenomenon of volatility clustering observed in the latter cases is related to the underlying deterministic switching between phases with fluctuations of different amplitude.

It is worth noting how such distributional characteristics do not appear if we allow normal random disturbances when the deterministic dynamics

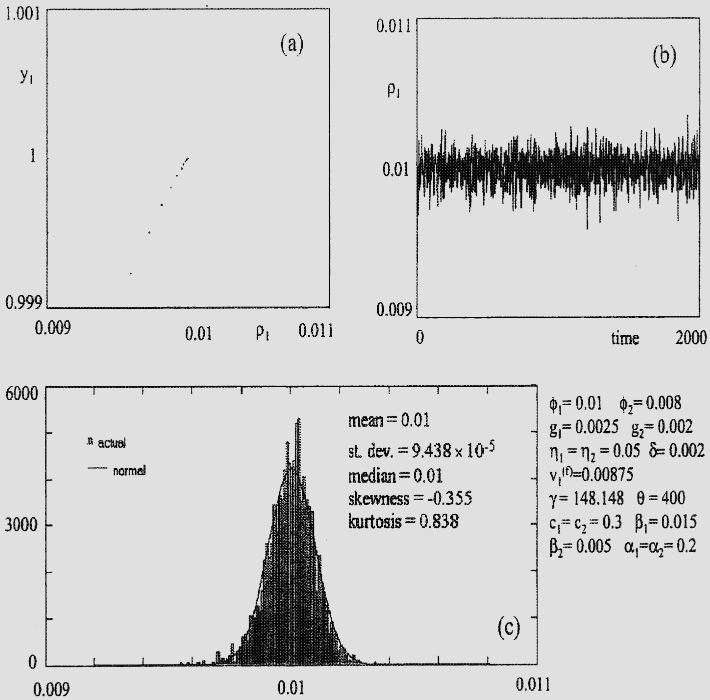

Fig. 5. effect of a normally distributed shock in the case where the system converges monotonically to the fundamental equilibrium: (a) deterministic trajectory, (b), (c) stochastic time series and distribution of ρ_1.

exhibit a monotonic convergence to steady state, as in the case of *Fig. 5* ($\sigma = 0.001306$). In this case the distribution of the capital gain ρ_1 is approximately normal.

Of course such simulations are not expected to mimic the real data, but they have been chosen to illustrate qualitatively how non normally distributed returns similar to the ones observed in financial markets may be the result of the interaction of nonlinear deterministic dynamic phenomena, such as chaotic fluctuations and intermittency, shocked by normally distributed noise.

8 Conclusions

In the present paper we have set up a model of heterogenous agents (fundamentalists and chartists) investing in a portfolio of a risky asset and a risk-free asset. Each group forms expectations about asset returns and allocates its wealth between the two assets according to one-period expected utility maximization. The investors differ with respect to their "beliefs" about the conditional expected returns of the two assets and the variance of the risky return. Market clearing is effected by a market maker whose price adjustment rules take account of agents' excess demand but also seek to maintain asset prices close to their exogenously determined fundamental values in the long-run. Due to the assumed CRRA utility functions, investors' optimal decisions depend on their wealth and this results in growing prices and wealth.

We set up the high-dimensional dynamical system arising from the interaction and dynamic updating of beliefs of the various agents across the markets for the two assets, and we reduce it to a stationary system where the dynamic variables are actual and expected returns, ratios between fundamental values and prices, and wealth proportions of the two groups of agents. Using both analytical and numerical techniques, we are able to characterize the steady states of the model, as well as other invariant sets on which the dynamics are described by lower dimensional maps.

We then focus on the out-of-equilibrium dynamics of the model: here analytical results seem difficult, so that we mainly use numerical simulations to study the out-of-equilibrium behavior. The main characteristics are phenomena of coexistence of attracting sets, phenomena of intermittent behaviour as well as other phenomena of chaotic dynamics: these latter phenomena seem to emerge mainly when the chartists' demand function is sufficiently sloped or when they update sufficiently fast their expectations. Stochastic simulations, performed with parameter regimes where the underlying deterministic system is chaotic, show the ability of the model to generate some of the characteristic features of financial time series.

It still remains to undertake a more thorough numerical study of the effect of changes of key parameters of the model, such as the ones characterizing the agents' expectations formation and the market maker's price adjustment rule, as well as the impact of exogenous stochastic factors. This kind of analysis will require an interplay among theoretical and numerical methods, which is typical for the study of the global dynamic properties of nonlinear dynamical systems of dimension greater than one, as stressed in Mira et al. 1996.

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A Simple Micro-Model of Market Dynamics

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Summary. We present a simple agent based model aimed at the qualitative description of trading activity in a "stylized" financial market. A two assets economy is considered, with a bond providing a riskless constant return and a risky stock, paying constant dividends, whose price is fixed via Walrasian auction. The market participants are speculators described as myopic utility maximizers provided with limited forecasting ability. If one varies the parameters describing the market and the agents behavior, the model presents many distinct "phases". In particular, the no-arbitrage "fundamental" price can emerge as a stable fixed point, while for different parameterizations the market shows chaotic dynamics with speculative bubbles and crashes.

1 Introduction

This paper presents an agent-based model of a two assets exchange economy: one riskless bond and one risky equity whose price is determined via Walrasian auction. The market participants are described as myopic utility maximizers provided with limited forecasting abilities. They act as short-horizon speculators and choose their portfolio composition starting from forecasted price dynamics.

The present analysis extends previous contributions [6, 7, 15, 13] by considering traders who explicitly take in consideration the risk implicit in their market positions. One finds that, with this extension, the set of possible market dynamics is strongly enriched. In particular, we will show that even in the limit of single representative agent the market displays non-trivial (constant or explosive) dynamics.

As to this problem, notice that a large part of the agent-based models of financial markets present in literature assume the coexistence of different "families" of traders operating on the market. Both the models inspired by the "complex system" paradigm [3], and the more recent contributions [20, 22, 11, 18], based on the introduction of some form of "bounded rationality" (or better "inductive rationality" as in [2]) to account for the agents decisions

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(see [20] for a critical review), share the same idea that only an heterogeneous population of traders, characterized by a switching dynamics between different "trading strategies" (or, more generally, "visions of the world") induced by an (apparent) difference in their relative "rewardingness" [6, 8] or by imitative behavior [17] can actually lead to interesting market dynamics.

On the contrary, we will show that the hypothesis of "heterogeneous traders" and "evolving believes" are not necessary to generate volatile bubble-crash dynamics, very similar to the ones observed in real markets. At the same time, the analysis of the "homogenous agents" limit suggests that under very natural assumptions, an infinitesimal deviation from perfect rationality can generate finite-size effect in the market dynamics and an enormous reduction in its "efficiency".

The outline of the paper is as follows: in Sec. 2 the model is introduced and the various assumptions discussed. In Sec. 3 the analytical and numerical study of the system is performed while in Sec. 4 some final remarks and some future possible developments are discussed.

2 Model Structure and Homogeneous Agents Dynamics

We suppose that each agent shapes his trading activity only on the basis of his possible wealth one step in the future. At the beginning of the trading round at time t the agent forms his personal demand function, deciding the amount of risky asset he wants to buy or sell for any possible "ex-ante" value of the transaction price.

Suppose that at the end of period t, after its participation to the market, the agent possesses $B(t)$ riskless assets and $A(t)$ risky assets. The agent wealth then reads

$$W(t) = B(t) + A(t)p_h \tag{1}$$

where p_h is the stock price (for now "hypothetical") fixed by the market at time t.

Let x be the fraction of agent wealth invested in the risky stock. The future wealth of the agent portfolio (i.e. its wealth at the beginning of the next time step) depends on the hypothetical return on the stock price $h(t) = p(t+1)/p_h - 1$ and reads

$$W(t+1; h(t)) = x\,W(t)\,(h(t) - r + d/p_h) + W(t)\,(1+r) \tag{2}$$

where the dividends d are payed after the payment of the riskless interest at the end of time t.

The future value of the portfolio depends on the future price of the stock. Supposing that the agent possesses some forecasting ability concerning the future price return h he would be able to formulate expectations on his own future wealth. The problem of the agent becomes to maximize its utility U consistently with his expectations.

To this purpose, we suppose that the agent utility depends on his forecast of expected price return and variance. This choice is consistent with the "mean-variance portfolio theory" (see, for instance, [10]), that can be considered a "standard" procedure to compare different investment possibilities. With respect to a more general "expected utility theory" (EUT) approach [12] this choice allows to model an agent who takes in consideration a "finite" (possibly small) amount of information in his decision processes and it guarantees the possibility of performing some analytical study of the resulting dynamics.

It is interesting, however, to observe that, as some empirical investigations have shown [19], in real application the use of a mean-variance approach with respect to a more demanding utility maximization leads to a reduction of efficiency in the portfolio of less than 5%.

Reassured by this "practical equivalence" between the two approaches, we choose as the expression of the agent utility the simplest function of the expected return and variance [6, 15, 17]

$$U(t) = E_{t-1}[W(t+1)] - \frac{\beta}{2} V_{t-1}[W(t+1)] \tag{3}$$

where $E_{t-1}[.]$ and $V_{t-1}[.]$ stand respectively for the expected return and variance computed at the beginning of time t, i.e. with the information available at time $t-1$, and where β is the "risk-aversion" parameter.

Using the expression for W in (1) one obtains

$$E_{t-1}[W(t+1)] = x\, W(t)\, (E_{t-1}[h(t)] - r + d/p_h) + W(t)\,(1+r) \tag{4}$$

and

$$V_{t-1}[W(t+1)] = x^2\, W(t)^2\, V_{t-1}[h(t)] \quad . \tag{5}$$

The portfolio position chosen by the agent is the one that maximizes its utility. Using (4) and (5) and remembering the definition of x from the first order condition one obtains

$$\Delta A(t) = -A(t-1) + \frac{E_{t-1}[h(t)] - r + d/p_h}{\beta\, V_{t-1}[h(t)]\, p_h} \tag{6}$$

where the quantity of stock $\Delta A(t)$ the agent is willing to trade (i.e. to buy if it is positive or to sell if it is negative) at time t is related to the "ex-ante" price p_h.

The demand curve discussed above represents the model of market participation for a single "prototypical" agent. Even keeping constant the form of the utility function, one can in principle vary the parameters from agent to agent to obtain an heterogeneous population. In the present analysis, however , we will focus on the case in which the traders are "almost" identical and "almost" completely described by a representative agent. The word "almost" is used because it is obvious that in the case of perfect homogeneity, even assuming that agents have different initial endowments, the actual trading can stand only for one step, then the market position of each agent is identical and no more trading can take place.

In fact the situation under study is different. We consider a population of agents whose demand curves are a stochastic perturbation around the demand curve defined in (6). In general, the dynamics of the model will depend on the realizations of the stochastic noise involved. These noisy terms keep the market constantly "out of equilibrium" so that the trading can indefinitely go on. The homogeneous approximation we want to consider is defined as taking the no-noise limit of the dynamics "after" the dynamics did evolve. This can be considered as the *deterministic skeleton* of a noisy dynamical system [1].

To describe this limit, we consider a market composed of N agents, each agent possessing a personal demand curve $\Delta A_i(p_h)$ that follows (6).

The model assumes that the stock price is determined via a Walrasian auction. The personal demand curves $\Delta A_i(p)$ are "aggregated" in a global demand curve and the asset present price $p(t)$ is computed through the market clearing condition $\sum_i \Delta A_i(p(t)) = 0$. From (6) one obtains

$$\gamma V_{t-1}[h(t)]p_h^2 + (r - E_{t-1}[h(t)])p_h - d = 0 \tag{7}$$

where $\gamma = \beta A_{TOT}/N$ with A_{TOT} total number of stock shares. The positive root of the previous equation is the price at time t fixed by the market. It depends on the agent expectations about the average return and its variance. To this respect, we model the agents as "naive econometricians" who obtains forecasted variables using EWMA (exponentially weighted moving averages) predictors. The expression for the expected returns and variance then becomes:

$$\begin{aligned} E_t[h(t+1)] &= (1-\lambda)\sum_{\tau=0} \lambda^\tau h(t-\tau) \\ V_t[h(t+1)] &= (1-\lambda)\sum_{\tau=0} \lambda^\tau h(t-\tau)^2 - E_{t-1}[h]^2 \end{aligned} \tag{8}$$

where $\lambda \in [0,1]$ is a weighting coefficient setting the "time scale" on which the averaging procedure is performed. Notice that the expression for $V_{t-1}[h]$ is analogous to the one proposed by RiskMetrics group (see the RiskMetrix Technical Manual), and widely applied by the real operators in their forecasting activity[2].

Using the positive root of (7) and (8) we can finally write the dynamical equations governing the evolution of the market

[1] This must be understood as an instrumental simplification, introduced for the purpose of obtaining analytical tractability. Its validity resides in the ability of providing a reliable description of the market dynamics that is preserved, at least qualitatively, when differences among agents are introduced. The analysis of this issue constitutes the topic of a forthcoming work [5] but we can anticipate that this is actually the case.

[2] The RiskMetrics group actually proposes an EWMA estimator of the volatility, defined as the second moment of the returns distribution. The expression above represents its natural extension to central moment

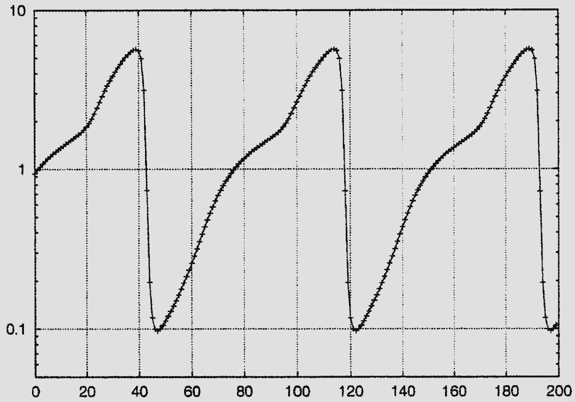

Fig. 1. The price history computed with $\beta = 2.5$, $d = 0.01$, $\lambda = .95$, $r = 0.01$ and with initial condition $p(0) = 100.$, $E_{t-1}[h]_0 = .01$ and $V_{t-1}[h^2]_0 = .0001$ after a transient of 1000 time steps.

$$
\begin{aligned}
p(t) \qquad\quad &= \left(E_{t-1}[h] - r + \sqrt{(E_{t-1}[h] - r)^2 + 4\gamma V_{t-1}[h]d}\right) / \left(2\gamma V_{t-1}[h]\right) \\
E_t[h(t+1)] &= \lambda E_{t-1}[h(t)] + (1-\lambda)h(t-1) \\
V_t[h(t+1)] &= \lambda V_{t-1}[h(t)] + \lambda(1-\lambda)(h(t-1) - E_{t-1}[h(t)])^2
\end{aligned}
\tag{9}
$$

where $h(t) = p(t+1)/p(t) - 1$ stands for the realized price return at time t.

As expected, the system described in (9) possesses a fixed point in $(\bar{p}, 0, 0)$ where $\bar{p} = d/r$ corresponds to the no-arbitrage "equilibrium" price, i.e. the present value of the future stream of dividends. Quite interestingly, however, this fixed point is not always stable. The dependence of the stability of the "no-arbitrage" fixed point and, more in general, of the system trajectories on the values of the various parameters will be analyzed in the next Section. For now, however, it is useful to mention a couple of qualitative features that generally shape the model behavior.

First of all, notice that the dynamics described in (9) is bounded. Indeed if the forecasted return tends toward a constant value, the variance is progressively reduced and the price increases. This behavior rules out the possibility of an indefinite steady increase of the price. On the other hand, if the forecasted return $E_{t-1}[h]$, after a period of explosively increasing prices, scales of a factor a the forecasted variance $V_{t-1}[h]$ scales of a factor a^2 but then the price scales down of a factor $1/a$.

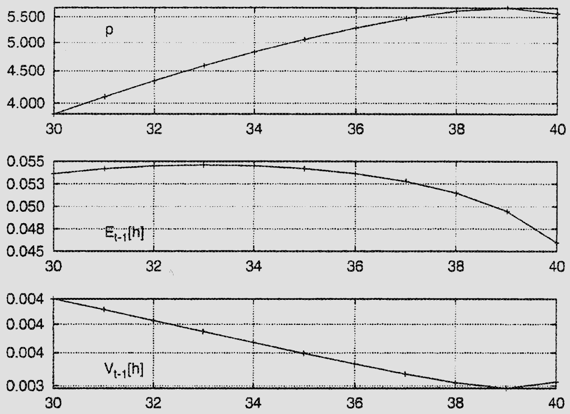

Fig. 2. The same simulations as in Fig. 1. Here price (top), forecasted return (middle) and forecasted variance (bottom) are shown for few time steps preceding a sudden crash.

A typical[3] price history is shown in Fig. 1: with the chosen parameters (see caption) the dynamics is stuck in a periodic cycle. The boundedness of the dynamics manifests itself as a relative slow (but "explosive" i.e. more then exponential) rise in price followed by a sudden fall, that reminds the "crashes after speculative bubbles" dynamics found in financial markets.

To see how these "crashes" are generated, we can inspect the few steps that precede one of them. Figure 1 reports the price, forecasted return and forecasted variance of the same simulation as in Fig. 1 for the time interval $30-40$ that precedes one price crash at around $41-42$. As can be seen from the first steps, the constant increase in price comes both from an increase in forecasted returns and a decrease in the forecasted variance. Indeed in the computation of the forecasted variance the high contribution from the last price crash is progressively discounted. Nevertheless, the contribution from the progressively increasing returns keeps $V_t[h(t+1)]$ bounded away from zero so that, at a given point, the progressive decrease in the forecasted variance starts to slow down. This slowing down, in turn, decreases the price growth rate and consequently the value of $E_t[h(t+1)]$ generating a feedback effect on the same forecasted variance and strengthening its slowing trend. After few steps, the reversed trend in returns is so high that the same variance starts to increase. At this point, the price starts to go down. This generates a big jump in the forecasted return value and, consequently, on the forecasted variance

[3] In the next Section we will see that in fact the model possesses many phases and, depending on the parameters values, shows quite different trajectories. In this respect, here "typical" has to be intended as both "not strange" and "not trivial".

and, thanks to the feedback mechanism, a sudden price change is generated in a very short time.

Incidentally, notice that in previous works [6, 15] the same form (3) for the agents utility function has been used but the authors did not introduce an agent forecasting rule for the price variance, simply assuming that all the agents equated it to a given constant value. From the discussion above, it is clear that this approximation, apart from being inconsistent with the same time series generated by these models, which typically show strong volatility dynamics, would essentially change the nature of the model[4].

3 The Deterministic Dynamical System

In order to simplify the analysis of the market dynamics it is convenient to rewrite the system in (9) as

$$\begin{aligned} x(t+1) &= f(y(t), z(t)) = \tfrac{y(t)-r+\sqrt{(y(t)-r)^2+4sz(t)}}{2z(t)} \\ y(t+1) &= \lambda y(t) + (1-\lambda)\left(\tfrac{f(y(t),z(t))}{x(t)} - 1\right) \\ z(t+1) &= \lambda z(t) + \lambda(1-\lambda)\left(\tfrac{f(y(t),z(t))}{x(t)} - 1 - y(t)\right)^2 \end{aligned} \tag{10}$$

where $x(t) = \gamma p(t)$, $y(t) = E_{t-1}[h(t)]$, $z(t) = V_{t-1}[h(t)]$ and $s = d\gamma$. This system depends on three parameters r, s and λ. Notice that the "risk-aversion" parameter γ has been absorbed in a rescaling of both the prices and the paid dividends.

With these new variables, the "fundamental" price becomes $\bar{x} = s/r$ and $(\bar{x}, 0, 0)$ is a fixed point for the dynamics[5]. It is easy to check that the system doesn't posses any other fixed point. But what about the stability of this point? Does there exist a region in parameters space where the system evolves constantly towards this equilibrium price? This would imply that it is possible to recover a market equilibrium around the "fundamental" price with an ecology made exclusively of "technical" traders.

To this purpose, notice that one can check immediately that the partial derivatives

$$\begin{aligned} f_y(y,z) &= f(y,z)/\sqrt{(y-r)^2+4sz} \\ f_z(y,z) &= (s/\sqrt{(y-r)^2+4sz} - f(y,z))/z \quad . \end{aligned} \tag{11}$$

are continuous for the domain $\mathcal{D} = \{y \geq 0, z > 0\} \bigcup \{y < r, z = 0\}$. In particular $\partial_y f(0,0) = s/r^2$ and $f_z(0,0) = -s^2/r^3$. Since the dynamics described in (10) is bounded in $\{x > 0, z > 0\}$ one can conclude that there is a neighborhood of the fixed point $(\bar{x}, 0, 0)$ such that in its intersection with the

[4] For a discussion of this approximation and a comparison with (9) see [4]

[5] Even if (10) is only defined for $z > 0$ and $y \geq 0$ it can be extended continously to $z = 0$ when $y < r$

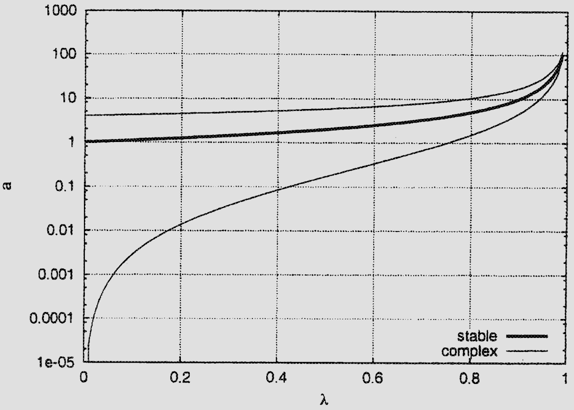

Fig. 3. (λ, a) parameter space. The fixed point stable region is the bottom right region delimited by the thickest (denoted with "stable" in the legend) line. The region inside the two other lines (denoted with "complex" in the legend) is where the eigenvalues are complex. Notice that both these regions are unbounded from above.

largest invariant set of the dynamics, the system is differentiable with continuous derivatives. This is enough to use the following theorem, which can be applied to slightly more general cases the than one at hand.

Theorem 1. *Suppose a system dynamics is described by a set of equations analogous to (10) with a generic function f continuous in $(0,0)$ with continuous first derivatives. Then if $a = \partial_y \ln(f(0,0))$ the point $(f(0,0),0,0)$ is stable when*

$$a < \frac{1}{1-\lambda} \tag{12}$$

Moreover, the stability of the fixed point is lost by an Hopf bifurcation [16] (i.e. by two complex conjugate eigenvalues that cross the unit circle) when $a = 1/(1-\lambda)$.

Proof. The eigenvalues of the Jacobian of the system computed in the fixed point $J(\bar{x},0,0)$ reads

$$\begin{aligned}\mu_0 &= \lambda \\ \mu_+ &= (\lambda + (1-\lambda)a + \sqrt{(\lambda + (1-\lambda)a)^2 - 4(1-\lambda)a})/2 \\ \mu_- &= (\lambda + (1-\lambda)a - \sqrt{(\lambda + (1-\lambda)a)^2 - 4(1-\lambda)a})/2\end{aligned} \tag{13}$$

from the stability condition [14] $||\mu_i|| < 1$ for $i \in 0,+,-$ after little algebra the statement follows. □

The curve defined in (12) is plotted in Fig. 3 together with the region in which the Jacobian of the system computed in the fixed point possesses two conjugated complex eigenvalues.

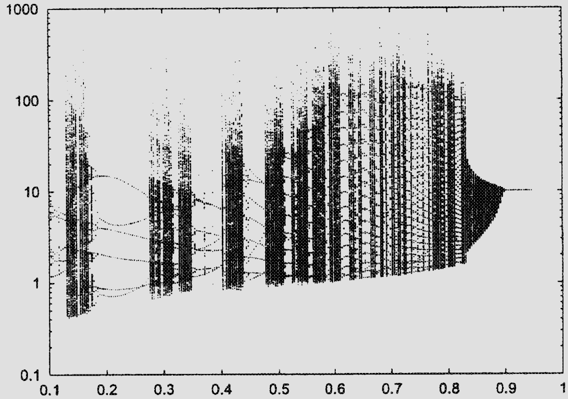

Fig. 4. Bifurcation diagram. The x support of a 500 steps orbit (after a 1000 steps transient) is shown for 800 distinct values of λ in $[0,1]$ ($r = .1$ and $s = 1$). The initial condition is $(1, .01, .0001)$.

The foregoing result suggests some considerations:

- The validity of the theorem for a "generic" function f unties the obtained result from our choice for the utility function in (3). The existence of a stability region for the fixed point, when the agent evaluates future prices starting from forecasted returns and variances, is thus guaranteed whatever expression one chooses for the utility as long as $\partial_y f(0,0) > 0$. This is a rather general assumption in a speculative trading framework.
- A market can be perfectly stable, i.e. at "equilibrium", even when only "trend followers" traders are present. This suggests that the general idea of technical trading as a destabilizing force of the market is not always true or , at least, is not enough to generate highly volatile dynamics, when risk evaluation is taken in account.
- Relatively long memory agents, i.e. agents smoothing their forecasts on time scales that are large if compared to a, behave like fundamentalist, even if they base their choices only on the forecasted price movement.
- With the expression of f as in (10) and following (11), $a = 1/r$ such that (12) becomes $\lambda > 1-r$. Thus the market tends toward the equilibrium fixed point when the riskless return is relatively high and the agents forecasting behavior sufficiently "smooth".
- Quite surprisingly, for the expression of f defined in (10) , the s parameter does not play any role in the stability of the fixed point. This means that the existence of a stable fixed point does not depend either on the dividend d or on the value of the "aggregate" risk aversion γ.

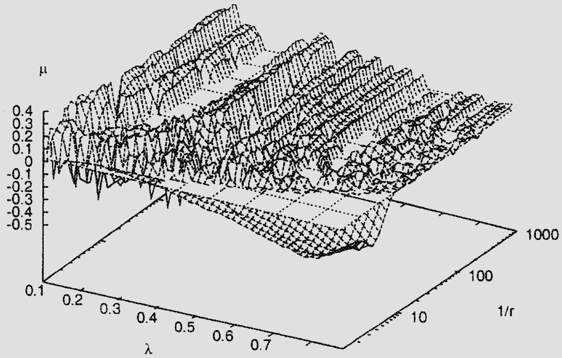

Fig. 5. The system largest Lyapunov exponent as a function of λ and $a = 1/r$ with $s = 1$. The value are obtained with a simulation length of 3000 steps, after discarding the first 1000 as transient.

The next two natural questions are what is the basin of attraction of the fixed point and what happens when the fixed point is no more stable. We are not able to discuss these points in general terms and, in the following parts of the present paper, we will refer to the expression of f defined in (10). Moreover, since from now on we will trace a global analysis of this system, we will mainly rely on numerical methods.

Let us postpone the discussion of the fixed point attraction domain and proceed with a straightforward inspection of the system behavior when one leaves the stability region in the parameters space. Keeping fixed $r = .1$ and $s = 1$ we plot the support for the x values (after a suitable "transient" period) when the λ parameter is varied, to obtain a bifurcation plot. The result is reported in Fig. 4. As can be seen, for $\lambda > .9$ the system is stationary in the stable fixed point. This is in fact our expectation, following the previous analysis and the chosen value for r.

As the nature of the bifurcation suggests, when the level of λ crosses the .9 boundary, the system moves toward quasi-periodic, multi frequency orbits (this cannot be detected from Fig. 4 due to its coarse grain, but can be directly checked). Moreover, when λ keeps moving towards lower values, we see the subsequent appearance of regions in which the system shows clear periodic behavior intermixed with other regions where the density of the support suggests the presence of strange attractors (i.e. attractor whose dimension is not integer) and chaos.

This can be confirmed studying the values of the system Lyapunov exponents for different parameterizations. In Fig. 5 the largest Lyapunov exponent is shown as a function of both λ and r. This plot confirms again the presence

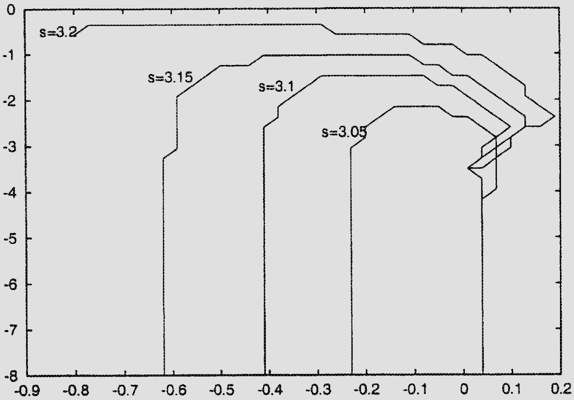

Fig. 6. The boundaries of the fixed point domain of attraction in the $y - z$ space. Each point represents an initial condition for the y and z values. The initial condition for x is chosen equal to $\bar{x}$. The system is then iterated for 210^5 steps and the initial condition is assumed to belong to the domain of attraction if $|x - \bar{x}| + |y| + |z| < 10^{-5}$. The chosen values for the parameters are $r = .01$ and $\lambda = .91$. Please note that the choice of the threshold value and the form of the distance function are asymptotical irrelevant but introduce noticeable effect at finite time lengths. Thus, the lines in the present plot must be read as a qualitative guess of the real boundaries. No attempt has been made to obtain any estimate of the error.

of "periodic" regions (with below 0 largest exponent) and "chaotic" regions, heavily intermixed. Even if the Jacobian is a smooth function of λ, the Lyapunov exponents show, at least as a first inspection, non-smooth behavior with respect to this parameter (this fact is reported as typical in [9]).

Another fact to be noted is that the "mountains landscape" of Fig. 5 seems to show rather stable valleys or hills along the r direction. This would suggest that the central role in the determination of the attractor structure is played by λ much more than by r.

As an extensive numerical investigation shows, this is actually the case. The two parameters mostly shaping the global structure of the system are s and λ. It turns out that even if the parameter s does not play any role in the stability of the fixed point, its role is of major relevance in the characterization of its domain of attraction.

Let us start with the discussion of the effect of s in the region where the fixed point is stable, i.e. for $l > 1 - r$. In general, if one takes not too large values for s, the stable fixed points is a global attractor. When the parameter s increases, however, a new attractor constituted by a periodic orbit appears and the domain of attraction of the fixed point shrinks rapidly to a small neighborhood of $\bar{x}$. This can be directly checked considering simulations with different initial conditions $(\bar{x}, y(0), z(0))$ and plotting the trajectory average

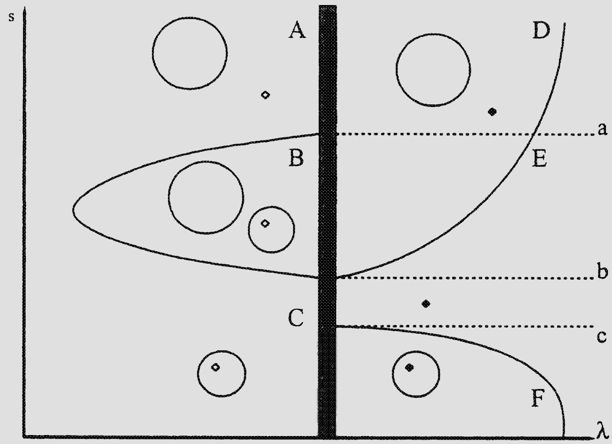

Fig. 7. A summary description of the system behavior. See the text for comment. The various regions are indicated with capitals letter and separated by continuous lines. This picture has been obtained with the use of numerical simulations. In particular we have chosen $r = .1$ and we have studied the system near the bifurcation point $\lambda = .9$ along the lines $\lambda = .8991$ and $\lambda = .9001$ (being nearer to the bifurcation generally implies waiting for longer transients). With these choices the values for the boundaries of the various regions are $a = 2.39$, $b = 1.88$ for higher λ, $b = 1.758$ for lower λ and $c = .33$. The region B disappears for $\lambda \sim .89$ and region F for $\lambda \sim 9.05$.

distance from $(\bar{x}, 0, 0)$ after a sufficiently high number of steps. The results of this analysis for $r = .1$ and $\lambda = .91$ have been reported in Fig. 6. The boundaries reported there delimit the fixed point attracting region for different values of s. As can be seen, when s increases above a given threshold, the attraction domain rapidly shrinks. For $s > 3.3$ it becomes a small neighborhood of the fixed point while for $s < 3.03$ the fixed point is a global attractor. This threshold value is an increasing function of λ and diverges for $\lambda \to 1$ (where the system dynamics is definitely frozen). Two attractors coexist also for low values of s, if λ is slightly higher then r. We can conclude that a quite complex picture emerges. We will try a qualitative description in what follows.

We will refer generically as "orbits" the various structures appearing in the analysis since the actual topological nature of these objects, i.e. periodic orbits, quasi periodic orbits or strange sets, depends generally in a non smooth way on the parameters values as suggested by Fig. 5.

The qualitative behavior of the system for $\lambda \sim 1 - r$ is depicted in Fig. 7. For $\lambda > 1 - r$ and moderate values of s, only the global attractor constituted by the stable fixed point exists (region E in the plot). When s is relatively high two attractors coexist: the fixed point and an orbit (region D). The fixed point, in the $x - y$ plane, is external to the orbit (characterized by prices constantly lower than the equilibrium one). When s is low and λ near to the

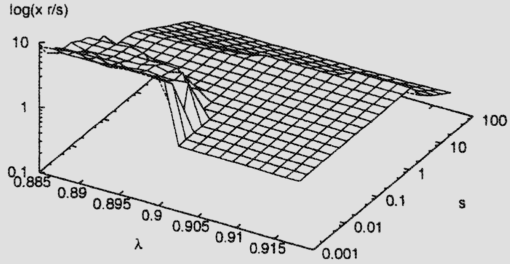

Fig. 8. Rescaled average price $x/\bar{x}$ as a function of λ and s in the fixed point stability region with $r = .1$ and initial condition $(\bar{x}, -.1, 0)$. Averages are computed for 1.000 steps after a transient of 50000.

threshold value, a new orbit appears containing the fixed point in its interior, so that along this orbit the price oscillates around the equilibrium price (region F). For large enough values of λ this region disappears.

When λ cross the $1-r$ boundary, the fixed point looses its stability and for large (region A) or small (region C) value of s the orbits keep the same characteristics. Interestingly, for moderate values of s and for λ near the boundary (region B) two stable orbits coexist.

In order to understand the nature of the different phases it is interesting to look at the average price generated by the dynamics. In Fig. 8 and Fig. 9 we report the average price computed after a suitable transient as a function of λ and s for $r = .1$ and for values of λ respectively above and below the fixed point stability threshold. In both these plots the prices are rescaled by the equilibrium value. In Fig. 8, both regions D and F of Fig. 7 clearly show up and are associated respectively to lower and higher average (rescaled) prices. In the second case, even if the price moves "around" the equilibrium price as mentioned above, its value is on average much higher. Another interesting feature is the appearance, for quite low values of λ and moderate values of s of a region in which the price dynamics becomes "extreme": the big mountains in the average prices signals the presence of very large cycles in the $x - y$ plane. The typical trajectory is analogous to the one in Fig. 1 but with prices varying over many orders of magnitude.

For different values of r the stability region boundary moves, so that all the regions in Fig. 7 shift but they remain qualitatively unchanged.

Finally, and mainly for esthetic purposes, let us plot in Fig. 10 a "typical" strange attractor. It is from the C region of Fig. 7 and, if plotted on the x-

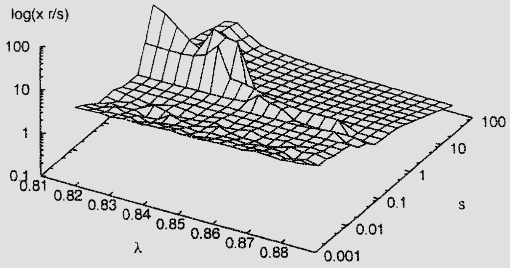

Fig. 9. Rescaled average price $x/\bar{x}$ as a function of λ and s outside of fixed point stability region with $r = .1$ and initial condition $(\bar{x}, -.1, 0)$. Averages are computed for 1.000 steps after a transient of 50000.

y planes, the fixed point $(.1, 0, 0)$ clearly appears in its "interior", while the average price is almost 10 times larger.

4 Conclusions and Outlook

The interesting part of the foregoing analysis is constituted by the richness of the dynamic scenarios one has been able to generate starting from very simple assumptions about the agents behaviors and the structure of the market. The apparently "harmless" hypothesis of describing traders as utility-maximizing agents updating their expectations on the past market history leads to huge movements in price and to an high degree of "inefficiency".

We can draw two lessons from this discovery:

- first, that the notion of equilibrium expressed by the Efficient Market Hypothesis is in fact extremely weak and can be easily made unstable with very mild assumption about the agents behavior (in some sense, this conclusion is analogous to [1] where the more general idea of economic equilibrium is analyzed)
- second, that in order to destroy EMH stability is not necessary to suppose the existence of a complex ecology of strategies together with an high-frequency switching dynamics of agents behaviors.

The present paper constitutes, just a first step in the study of the market model presented. In the author view, the more interesting part of this kind of studies resides in the analysis of the effect of heterogeneity on the dynamics of

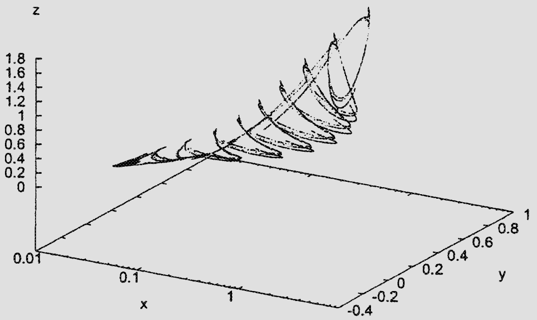

Fig. 10. The shape of the strange attractor reconstructed with 20.000 points for $\lambda = .8, r = .1$ and $s = .01$. The associated Lyapunov exponents are $2.1e-02$ $-6.3e-02$ $-2.8e-01$.

the system[6]. Nevertheless the investigation of the "homogeneous" limit helps in disentangling the contributions to the market dynamics that are generated by the chosen strategies and market structure and the ones that are introduced by the presence of heterogeneity in the agents population.

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[6] We will try to pursue this analysis in the forthcoming [5]

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Herd Behavior in Artificial Stock Markets

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Abstract. Herd behavior in Economics can be fruitfully represented by a generalization of the well-known Ehrenfest urn model to correlated clustering. The strategies of an agent in a stock market (planning to buy, to sell or to be inactive) are represented by three urns, and the accommodation of each agent in one of them is ruled by a random mechanism that may depend strongly on the behavior of the other agents. This mechanism is introduced in the "Genoa Artificial Stock Market" [14]. At each step, each agent chooses its strategy following the Ehrenfest-Brillouin model [10]. Given the old price, the demands of bulls and the supply of bears intersect, and generate the new price. Fat tails of price returns are obtained directly as a function of the herding parameter, without introducing any individual distinction among agents (like initial wealth, or risk-propensity). Time correlation is driven by a simple mechanism, and volatility clusters can be introduced as temporal variations of the herding parameter. The relationship between the active strategy excess (the difference "bulls minus bears") and the price returns is studied.

1 Introduction

Herd behavior in Economics can be fruitfully represented by a generalization of the well-known Ehrenfest urn model [6] to correlated clustering. This has been shown by some of us in previous papers [8],[9],[10], where some typical cases like Kirman ants [11], or bulls and bears in financial markets [4], or clustering of agents in open markets [1],[2],[3], are treated as offsprings of the Ehrenfest old fleas that jump from dog to dog. While some of these examples are essentially pedagogic, the case of bulls and bears in financial markets is far more concrete. In fact the natural tendency to imitation (or to herding) in this case is undoubtedly counterbalanced by a lot of constraints, like money availability, price values, matching between demand and supply, so that the economic impact of the crowd effect is not trivial. In some previous works we assumed the basic hypothesis of R.Cont and J.P. Bouchaud [4] about the linearity of price increments (or returns) and excess demand. To challenge this hypothesis here we introduce our treatise into "The Genoa Artificial Stock Market" [12], that is a realistic market, where agents place buy and sell orders, within the limits of their available cash and asset portfolios. Orders are

M. Gallegati et al. (eds.), *The Complex Dynamics of Economic Interaction*

processed by a module which builds the supply and demand curves whose intersection point determines the new asset price. Orders, whose limit prices are compatible with the value of the new price are satisfied, others are discarded. Each trader's cash and portfolio are updated and a new simulation step begins. The main idea of the paper is that of use the Ehrenfest-Brillouin mechanism in order to choose to be bull or bear (or neutral), leaving all the rest invariant. Here we have two main perspectives: a) from the point of view of Cont and Bouchaud, is it a realistic hypothesis to maintain the linearity of price increments (or returns) and excess demand? b) from the point of view of Artificial Markets in general: is the Ehrenfest-Brillouin herding mechanism a sufficient explanation of fat tails in the equilibrium distribution of returns? As we will see, both questions seem to have a positive answer.

2 Ehrenfest-Brillouin Strategy Changes

Consider a dynamical system composed of n entities and g cells (urns). The state of the system is described by the occupation number vector $\boldsymbol{n} = (n_1, ..., n_i, ..., n_g)$, $n_i \geq 0$, $\sum_{i=1}^{g} n_i = n$. The most elementary dynamical event we take into account is: "an entity changes its state from the cell i to the cell k". Hence the state of the system is submitted to the transition $\boldsymbol{X}_t = \boldsymbol{n} \rightarrow \boldsymbol{X}_{t+1} = \boldsymbol{n}_i^k$, where $\boldsymbol{n} = (n_1, ..., n_i, ..., n_k, ..., n_g)$ denotes the initial state and $\boldsymbol{n}_i^k := (n_1, ..., n_i - 1, ..., n_k + 1, ..., n_g)$ the final one in terms of the coordinates of the starting vector. This transition can be split into two distinct operations. The first (the "Ehrenfest term") is the destruction of an entity belonging to the ith-cell, with probability

$$P(\boldsymbol{n}_i|\boldsymbol{n}) = \frac{n_i}{n} \tag{1}$$

where $\boldsymbol{n}_i = (n_1, n_i - 1, .., n_k, ..., n_g)$. The second term (that we call the "Brillouin term" for historical reasons, see [5]) is the creation of a particle in the k-th cell given the vector $\boldsymbol{n}_i$, whose probability is:

$$P(\boldsymbol{n}_i^k|\boldsymbol{n}) = \begin{cases} \frac{\alpha_k + n_k}{\alpha + n - 1} & \text{for } k \neq i \\ \frac{\alpha_k + n_k - 1}{\alpha + n - 1} & \text{for } k = i \end{cases} \tag{2}$$

where $\alpha = \sum_i \alpha_i$, and $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_g)$ is a vector of parameters.

The meaning of α_i is tied to the probability of an accommodation on the cell i if void. Here we limit ourselves to the case where all $\alpha_i > 0$. The resulting transition probability is:

$$P(\boldsymbol{n}_i^k|\boldsymbol{n}) = P(\boldsymbol{n}_i|\boldsymbol{n})P(\boldsymbol{n}_i^k|\boldsymbol{n}_i) = \frac{n_i}{n} \frac{\alpha_k + n_k}{\alpha + n - 1} \tag{3}$$

with the above said correction for repeated indices. It is easy to see that (3) represents the transition probability $w(\boldsymbol{n}, \boldsymbol{n}') = P(\boldsymbol{n}'|\boldsymbol{n})$ of a Markov chain, that is irreducible and aperiodic. It means that all possible states (whose number is $\binom{n+g-1}{n}$) are reachable from each other without any cyclical constraint, and they are all persistent. Elementary considerations [13] impose that it exists the (unique) invariant measure of the chain is also the equilibrium distribution on the ergodic set. The main difficulty in this case is finding out the invariant measure, that turns out to be the solution $\pi(\boldsymbol{n})$ of the detailed balance conditions, that is for any couple of first neighbors $\boldsymbol{n}'$ and $\boldsymbol{n}$

$$w(\boldsymbol{n},\boldsymbol{n}')\pi(\boldsymbol{X}=\boldsymbol{n}) = w(\boldsymbol{n}',\boldsymbol{n})\pi(\boldsymbol{X}=\boldsymbol{n}') \tag{4}$$

Now $\pi(\boldsymbol{n})$ is invariant and it is also the equilibrium distribution, that is $\lim_{t\to\infty} P(\boldsymbol{X}_t = \boldsymbol{n}) = \pi(\boldsymbol{n})$.

Posing

$$\frac{\pi(\boldsymbol{n}_i^k)}{\pi(\boldsymbol{n})} = \frac{n_i}{n_k+1}\frac{\alpha_k+n_k}{\alpha_j+n_j-1}$$

that is satisfied by the generalized g-dimensional *Polya* distribution.

$$\pi(\boldsymbol{n};\boldsymbol{\alpha}) = \frac{n!}{\alpha^{[n]}}\prod_{i=1}^{g}\frac{\alpha_i^{[n_i]}}{n_i!}, \quad \boldsymbol{n}: \sum_i n_i = n \tag{5}$$

In the domain $\{\alpha_i > 0\}$ creations are positively correlated with the occupation numbers $\boldsymbol{n}$. The equilibrium probability (5) is the usual g-dim-*Polya*. The correlation is large for small α, while it tends to zero for $\alpha \to \infty$, where creation probability is $p_i = \alpha_i/\alpha$, independent of $\boldsymbol{n}$. The marginal distribution of a category is $Polya(n_i, n\text{-}n_i; \alpha_i, \alpha\text{-}\alpha_i)$, whose first two moments are the following

$$\mu_i = n\frac{\alpha_i}{\alpha} \tag{6a}$$

$$\sigma_i^2 = n\frac{\alpha_i}{\alpha}(1-\frac{\alpha_i}{\alpha})\frac{\alpha+n}{\alpha+1} \tag{6b}$$

The mechanism can be complicated, as nothing prevents $m \leq n$ statistical units changing category at any step, following appropriate generalizations of (1) and (2). If m units are extracted without replacement from $\boldsymbol{n}$ and then replaced following a Polya scheme, the transition $\boldsymbol{X}_t = \boldsymbol{n} \to \boldsymbol{X}_{t+1} = \boldsymbol{n}_{ij..}^{kl..}$ has a known probability. What is relevant is that the equilibrium probability is always (5), the only difference being the rate of approach, rapidly increasing with m.

In order to tackle this problem, we introduce the occupation random variable K_t of a particular cell at time t: it is easy to show [10] that the regression of $K_t - \mu$ is linear, that is $E(K_{t+1}|k_t) - \mu = (1-r)(k_t - \mu)$, and the rate is

$$r = \frac{m\alpha}{n(\alpha+n-m).} \tag{7}$$

Then $Cov(K_t, K_{t+s}) = (1-r)^s \sigma^2$, where μ and σ^2 are given by (6a) and (6b). Note that the rate r does not depend on the particular cell.

All we said is exact, but (5) is not so easy to deal with. But a very simple limit (the Gibbs limit) of (5) is achieved when $\alpha = \sum_i \alpha_i >> 1, n >> 1, \chi = n/\alpha = const$. The marginal of (5) with respect to the i-th category becomes

$$P(K_i = a) = \frac{\alpha_i^{[a]}}{a!} \left(\frac{1}{1+\chi}\right)^{\alpha_i} \left(\frac{\chi}{1+\chi}\right)^a \sim NegBin(\alpha_i, \chi),$$

where $\chi = n/\alpha$. The joint distribution of two categories factorize in the Gibbs limit, so that they become independent. The moments of the equilibrium distribution of the occupation numbers are very simple functions of α_i and χ, and also the Kurtosis Excess is simple; in fact

$$\begin{aligned} \mu_i &= \alpha_i \chi \\ \sigma_i^2 &= \alpha_i \chi (1+\chi) \\ Kurt_i &= \frac{1}{\alpha_i}\left(6 + \frac{1}{\chi(\chi+1)}\right) \end{aligned} \tag{8}$$

These mathematical properties are sufficient to describe herding of agents in stock markets.

3 Herding of Agents in Stock Markets

The strategies (or market positions) of an agent in a stock market (planning to buy, to sell or to be inactive) are represented by three urns, and the accommodation of each agent in one of them is ruled by a random mechanism that may depend strongly on the behavior of the other agents. Let the state of the system at some initial time to be $\mathbf{n}_{t=0} = (n_1, n_2, n_3)$, where n_1 is the number of "bulls", n_2 is the number of "bears" and n_3 is the number of "neutrals", $\sum n_i = n$. At each step a number $m \le n$ of agents choose their strategy following the Ehrenfest-Brillouin model. What is the meaning of a strategy change ruled by (3)? Equation (3) describes the strategy change of a single agent, but the meaning is the same for any number of changes. The destruction term is easy to understand, as it means a random choice of an agent. The creation term is more interesting: the probability of entering the strategy k is $(\alpha_k + n_k)/(\alpha + n - 1)$, that you can re-write as

$$\frac{\alpha\frac{\alpha_k}{\alpha} + (n-1)\frac{n_k}{n-1}}{\alpha + n - 1}.$$

Thus the choice probability among g strategies can be considered a weighted mean of the two distributions, an "initial" (theoretical) distribution $(p_1, ..., p_g)$, $p_i = \alpha_i/\alpha$, with weight α, and the empirical $\left(\frac{n_1}{n-1}, ..., \frac{n_g}{n-1}\right)$ with weight $n-1$.

The theoretical and empirical distributions are the extreme points of a simplex, which is able to represent both independence ($\alpha >> n$) and extreme correlation or herding ($\alpha << n$). But the mixture can be interpreted as a randomization of two strategies, following Kirman's suggestion [11]. Suppose that the choice is performed in two stages: first the choice of the basic attitude (independence or herding), followed by the choice of the strategy from the distribution associated to the basic attitude. That is

$$P(k) = P(k|Theor)P(Theor) + P(k|Emp)P(Emp), \tag{9}$$

where $P(k|Theor) = p_k = \frac{\alpha_k}{\alpha}$ and $P(k|Emp) = \frac{n_k}{n-1}$, while $P(Theor) = \frac{\alpha}{\alpha+n-1}$ and $P(Emp) = \frac{n-1}{\alpha+n-1}$. If the agent chooses to use the theoretical distribution (he behaves as a "fundamentalist"), he is not influenced by his colleagues. In this case we have self-conversion. If the agent chooses the empirical distribution (he behaves as a "chartist"), he selects a colleague at random and he converts to its strategy. This is exactly the ant foraging behavior model introduced by Kirman in the very special case of two symmetric sources of food. Our generalization admits g sources of food, and a quite general initial probability $(p_1, .., p_g)$, that may reflect objective differences ("fundamentals") among the sources. The weight of the "herd behavior" is just the size of the herd at the moment of the choice. It must be compared with α, the weight of the "self-conversion".

The model can be applied to price increments or to returns directly [4],[8], [9],[10], supposing that the price returns at the step t are proportional to the aggregate excess strategy $S_t = (n_1\text{-}n_2)_t$, that is

$$R_t = \frac{S_t}{k} = \frac{(n_1 - n_2)_t}{k} \tag{10}$$

This allows us to treat the number of active agents $(n_1 + n_2)_t$ as a random variable. The parameters α_1, α_2, α_3, $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, associated with the three strategies, determine the transition probability of the chain. If we pose α_3 (the weight of the neutral strategy) $>> \alpha_1, \alpha_2$, and the number of agents is large, we are in the Gibbs limit, and we know everything about the equilibrium distributions of n_1 and n_2, given by (8), and due to independence about the excess strategy S. If we are given empirical date about mean, standard deviation and kurtosis of (suppose) daily returns of some stock index, we can estimate the parameters α_1, α_2 and χ through (8), the sole indeterminacy being introduced by the estimate of k, the excess strategy needed to move R_t by one unit, that it was assumed to be 1 in [10]. Further we can estimate the rate (7) from the daily autocorrelation of returns, and from r we can derive m, the number of daily changes. In this simple way we can fit the equilibrium distribution and the temporal evolution of our model to any set of data. What is interesting is that the fit chooses values of $\alpha << n$, that indicate a very strong herding among agents.

4 Herding in GASM

This mechanism is now introduced in the Genoa Artificial Stock Market [12] (GASM). In the GASM each agent has a total wealth made with an available cash and an amount of the stock. These individuals quantities act as trivial constraints on the amount of his actions, but his strategy (to buy, to sell or to be neutral) are independent of his wealth. The initial conditions of the random trader i are: an amount $c_0^{(i)}$ of cash, an amount $a_0^{(i)}$ of the stock, a strategy $s_0^{(i)}$, that are individual quantities, and the stock price P_0. In general you can consider all that as the final state of the previous step (day). The trading mechanism begins with the reallocation of agents on the three strategies. From this point of view, we suppose that the probabilistic dynamics from $\boldsymbol{n}_{t=0} = (n_1, n_2, n_3)$ to $\boldsymbol{n}_{t=1} = (n_1', n_2', n_3')$ is ruled by the 3-dim transition probability (3) generalized to m changes. That is to say m agents are extracted at random from the population of initial composition $\boldsymbol{n}_0$, and then they are inserted back into the three urn system, to give the final composition $\boldsymbol{n}_1$ with respect to strategies. Then each agent has the new strategy $s_1^{(i)}$, and he can place buy and sell orders (or nothing at all) $\phi_1^{(i)}$, taking into account the starting price P_0 and within the limits of his available cash and asset portfolios. Orders are processed by a module which builds the supply and demand curves whose intersection point determines the new asset price P_1. Orders, whose limit prices are compatible with the value of the new price are satisfied, others are discarded. Each trader's cash and portfolio are updated and a new simulation step begins. With respect to simpler models like [9] and [4] now we have two macro-variables, $S_t = \sum_{i=1}^{n} s_1^{(i)}$ and $\Phi_t = \sum_{i=1}^{n} \phi_t^{(i)}$, whose correlation with respect to the price increment (or return) is worth of study. In the following we limit our attention to S_t. To set up the model we need three weights for the strategies, α_1 for buyers, α_2 for sellers, α_3 for neutrals. In order to have a mean fraction of the active strategies equal to 2% we need the following orders of magnitude $\alpha_1{:}\alpha_2{:}\alpha_3 = 1 : 1 : 98$, that is $p_1 = p_2 = 1/100$, $p_3 = 98/100$. Note that $\alpha_i = \alpha p_i$, therefore these mean values are independent on the total theoretical weight $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, that is the crucial point of our model. The strategy process $\boldsymbol{n}_0, \boldsymbol{n}_1, \ldots, \boldsymbol{n}_t, \ldots$ is independent of all the rest of the market, and then also the excess strategy S_t, whose equilibrium distribution in the Gibbs limit has very simple moments deducible by (8), that is

$$\begin{aligned}
\mu_S &= (\alpha_1 - \alpha_2)\chi = n(p_1 - p_2) \\
\sigma_S^2 &= (\alpha_1 + \alpha_2)\chi(1+\chi) = n(p_1 + p_2)\left(1 + \frac{n}{\alpha}\right) \\
Kurt_S &= \frac{1}{\alpha_1 + \alpha_2}\left(6 + \frac{1}{\chi(\chi+1)}\right) = \frac{1}{\alpha(p_1 + p_2)}\left(6 + \frac{\alpha^2}{n(\alpha + n)}\right)
\end{aligned} \tag{11}$$

Further we know the rate of approach to equilibrium is given by (7). The condition of stationarity of μ_S implies $p_1 = p_2$. All these theoretical relationships suggest to try the following simulation.

Example 1. The number of agents is $n = 2000$, the weight of the bullish strategy is $\alpha_1 = 1$, the weight of the bearish strategy is $\alpha_2 = 1$, the weight of the neutral strategy is $\alpha_3 = 98$. It is equivalent to put $\alpha = 100$, $p_1 = p_2 = .01$, $p_3 = .98$; the number of agents that choose a possible new strategy every day is $m = 1984$, the number of days is 5000. The value $m = 1984$, associated to $n = 2000$, gives raise to a daily autocorrelation of the excess strategy equal to .15, that is assumed to be realistic [7]. Note that $m = n$ implies independence. In this model the "seed" and the amount of the daily autocorrelation is obtained imposing that 16 (random) agents conserve the same strategy of the previous day. The remaining unspecified feature of the model are all uniform, that is all agents are initially similar (same cash, same amount of the stock). No "volatility cluster" is introduced in the generation of sell or buy orders, whose standard deviation around the mean sell or buy order is fixed for ever. The results of this simulation is given in Fig.1, where fat tails are evident.

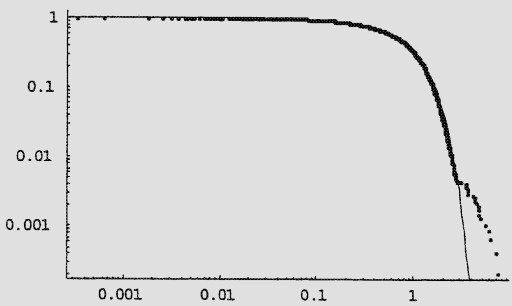

Fig. 1. First 1000 steps of log-prices increments; log-prices standardized distribution (dotted) compared with the normal

The equilibrium distribution and the daily autocorrelation of the excess strategy have the following statistics, compared with theoretical values:

Table 1. Theoretical and empirical values of the excess strategy

	μ_s	σ_s	$Kurt_s$	q_s
Th.	0.0	28.98	3.00	0.15
Emp.	-0.325	28.27	2.33	0.14

The equilibrium distribution and the daily autocorrelation of the log-price increments have the following statistics, compared with reasonable expected values, is they exist:

Table 2. Theoretical and empirical values of the log-price increments

	μ_r	σ_r	$Kurt_r$	q_r
Th.	0.0	unknown	unknown	unknown
Emp.	0.0	0.10	2.34	0.03

Note that if (10) were exact we should have $\sigma_r = \sigma_s/k$, and we could estimate $k = \sigma_s/\sigma_r$ through the empirical $\frac{28.9828}{0.103872} = 272.168$. But now the question can be posed in a fair realistic way. In facts the relationship between the excess strategy and the log-price increment is probabilistic in character, as the trading mechanism is probabilistic. We can interpret (10) as a linear regression of returns on the excess strategy, that is

$$E(R_t|s_t) = \frac{s_t}{k} = \frac{(n_1 - n_2)_t}{k},$$

that implies $E(R_t) = \frac{1}{k}E(S_t)$, and $E(R_t S_t) = E((R_t|s_t)s_t) = \frac{1}{k}\sigma_s^2$. Hence $1/k$ is the angular coefficient of the best linear estimate of $E(R_t|s_t)$, and its theoretical value is

$$\frac{1}{k} = \frac{\sigma_r}{\sigma_s}\rho_{rs}$$

We calculate the empirical cross-correlation between the excess strategy and the price increments $\rho_{rs} = Corr(R_t, S_t)$, that results .69.

Then the best linear estimate is $k = \dfrac{\sigma_s}{\sigma_r \rho_{rs}} = 391.76$.

A second observation deserves the theoretical value of q_r, the daily autocorrelation of price returns. Suppose that prices are self-correlated only due to the cross-correlation with the excess strategy, that is autocorrelated in turn. In this case we can prove that

$$Corr(R_t, R_{t+1}) = Corr(S_t, S_{t+1}) \left(Corr(R_t, S_t)\right)^2,$$

The cross-correlation $\rho = Corr(R_t, S_t)$ happens to be very stable in all simulations, its order of magnitude being .7, so that we expect that q_r is roughly one half of q_s.

5 Conclusions

Turning to our two perspectives: a) from the point of view of Cont and Bouchaud, the hypothesis of the linearity of price increments (or returns) and excess strategy, once it is introduced in the GASM, seems to be realistic, given that ρ^2 is roughly one half. It means that half of the price variance is accounted by the excess strategy. Of course this hypothesis must be translated in probabilistic jargon, and becomes a hypothesis on the linear regression of prices on excess strategy; b) from the point of view of Artificial Markets in general: the Ehrenfest-Brillouin herding mechanism a sufficient explanation of fat tails in the equilibrium distribution of returns, being all remaining unspecified feature of the model uniform for all agents. Herding depends only on the total weight α: it is an increasing function of $1/\alpha$ (Fig. 2a) and it vanishes in the limit $\alpha >> n$ (Fig. 2b).

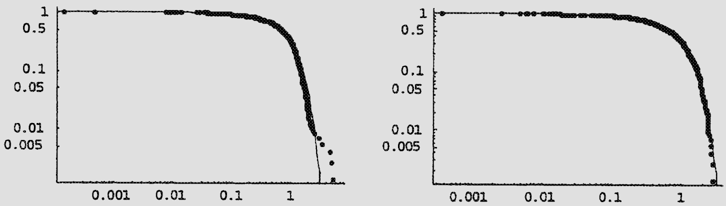

Fig. 2. Log-prices standardized distribution (dotted) compared with the normal, $n = 2000$, for (**a**) $\alpha = 50$ and (**b**) $\alpha = 10^8$

Fig. 1 is an intermediate case. Observe that agents are exchangeable, in the sense that they can behave differently, but this difference is possible to anyone. Further we are able to modulate temporal evolution of prices (and excess strategy) easily, although no volatility cluster is introduced. A simple way to introduce them in the treatise explicitly is that of admitting some temporal fluctuation of the total weight α of the fundamentalist strategy. A decrease of α indicates periods of uncertainty, and it leads to an increase of σ_r and $Kurt_r$. This is very simple to simulate, but we are unwilling to leave our exact model, almost for excess strategy, until we have exploited it.

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Part IV

Models of Bounded Rationality

Bargaining with Posterior Opportunities: An Evolutionary Social Simulation*

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Summary. Negotiations have been extensively studied theoretically throughout the years. A well-known bilateral approach is the ultimatum game, where two agents negotiate on how to split a pie or a "dollar": the proposer makes an offer and responder can choose to accept or reject. In this paper a natural extension of the ultimatum game is presented, in which both agents can negotiate with other opponents in case of a disagreement. This way the basics of a competitive market are modelled where for instance a buyer can try several sellers before making a purchase decision. The game is investigated using an evolutionary simulation. The outcomes appear to depend largely on the information available to the agents. We find that if the agents' number of future bargaining opportunities is commonly known, the proposer has the advantage. If this information is held private, however, the responder can obtain a larger share of the pie. For the first case we also provide a game-theoretic analysis and compare the outcome with evolutionary results. Furthermore, the effects of search costs and allowing multiple issues to be negotiated simultaneously are investigated.

1 Introduction

In the advent of ubiquitous application of agent technology, bargaining agents are expected to play an essential role in electronic market places. Automated negotiations are therefore becoming an important field of research [4,2,10,11]. The agents in a competitive market are self-interested and can be equipped with the ability to autonomously search for products and services and negotiate the terms of an agreement. In this paper we focus on strategic aspects of bilateral bargaining within a market-like setting.

Bilateral bargaining has been extensively researched, for instance in game theory [6,13,14]. Negotiations are often stylised using the ultimatum game, a two-stage game in which an offer is proposed by player one (the proposer) in the first stage, and the second player (the responder) can only choose to accept or reject the offer. The ultimatum game has been extensively researched, both theoretically and by experiments using human subjects [14,15].

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The ultimatum game models a negotiation between an isolated pair of players. In a market setting, however, an agent's behaviour can change if future opportunities are taken into account. This paper introduces a natural extension of the basic ultimatum game in which fallback opportunities are explicitly modelled. Both the proposing and the responding agents have several bargaining opportunities with different opponents before their payoff is determined. In this way a market place is modelled where several sellers and buyers are available.

The game is further extended to allow several issues to be negotiated simultaneously; not only the price, but also other important attributes such as delivery time, package deals, warranty, and other product-related aspects can be taken into account. This can reduce the competitive nature of the game since trade-offs can be made to obtain win-win solutions. Furthermore, the paper considers the effect of search costs if an offer is refused and a new opponent needs to be found.

An important aspect within this setting is the information available to the agents about their opponents. We distinguish between the complete information case, where an agent's current number of future bargaining opportunities is common knowledge, and the incomplete information case, where this information is known to the protagonist but hidden from the opponent.

The complete information case can be easily approached theoretically using game theoretic sub-game perfect equilibrium given reasonable assumptions. The incomplete information case, on the other hand, seems much more difficult to analyse. We therefore apply an evolutionary simulation to investigate this setting. We also compare the evolutionary and the theoretical approach in the complete information case.

In the field of computational economics, evolutionary simulations are increasingly applied to study the dynamic process of locally interacting, adaptive agents, particularly in the area of agent-based computational economics (ACE) [5,16]. In contrast to for instance game theory, no explicit rationality assumptions are made; the agents are naive optimisers acting on limited information. In these simulations, the agents are also myopic: they do not have any forward-looking ability or memory. Nonetheless, surprisingly rational behaviour often emerges from such "low-rational" agents.

In the paper we present the evolutionary results for the settings described above and a game-theoretic analysis for the extended game with complete information. Sub-game perfect results predict an extreme split of the surplus similar to the ultimatum game: the proposer claims the entire surplus, and the responder accepts this deal. The evolutionary outcomes show a good match with these game-theoretic results. Moreover, the simulation shows that results differ significantly if information about the opponent's future bargaining opportunities is not available: if the number of bargaining opportunities is sufficiently high, the responder now obtains the largest share.

The outcomes in the incomplete information case, however, also depend on the existence of positive search costs. Search costs stimulate agents to reach agreements early and discourage both players to exploit the additional opportunities. In the evolutionary simulation, the agreements are then similar to the one-shot ultimatum game.

The remainder of the paper is organised as follows. In Section 2 the bargaining game with multiple bargaining opportunities is described. Section 3 provides a game-theoretic analysis of the game in case of complete information. Section 4 outlines the evolutionary simulation and Section 5 discusses the obtained results from the simulation. Lastly, Section 6 concludes.

2 Description of the Bargaining Game

The modelled market consists of buyers and sellers who exchange a single good through bilateral negotiations. Each bargaining opportunity, an ultimatum-like game is played, where a seller proposes an offer and a buyer can reject or accept the seller's offer[1]. In our model an offer consists of one or more issues. If an agreement is reached, both agents obtain a payoff equal to their utility of the offer. In case of multiple issues, the utility is calculated as the weighted sum of the share obtained for each issue. The weights determine the *competitiveness* and are explained further in Section 5.3. The utilities of the agents are normalised between 0 and 1.

Each agent initially has up to m bargaining opportunities to reach an agreement. In case of a disagreement the agents are newly matched with randomly selected opponents, until no more bargaining opportunities remain. The number of remaining bargaining opportunities we call an agent's *bargaining state*, denoted by $\gamma_s \in \{0, 1, \ldots, m\}$ for a seller and $\gamma_b \in \{0, 1, \ldots, m\}$ for a buyer. If an agent's bargaining state reaches zero, the agent obtains a disagreement payoff which is set to zero.

An example is shown in Fig. 1 from a buyer's perspective. The buyer, whose initial bargaining state is $\gamma_b = 2$, first encounters a seller, seller 1, with bargaining state $\gamma_s = 1$. The seller proposes $o = (0.5, 0.5)$ and the buyer refuses this offer. Because the seller has no more bargaining opportunities his bargaining game ends and he obtains the disagreement payoff. The buyer, on the other hand, can continue bargaining when matched with another opponent, seller 2. In the example this opponent with $\gamma_s = 2$ offers $(0.6, 0.6)$. The buyer now accepts and the bargaining game ends for both agents.

Note that although the agents initially have equal bargaining opportunities, the matched agents can have different bargaining states. Having agents

[1] Alternatively, a more complex bargaining game such as the alternating-offers game [14] involving multiple rounds can be used. Outcomes are equivalent to the ultimatum game, however, if no time pressure exists; agreements are delayed until in the final round a take-it-or-leave-it offer is made. This deadline effect was studied in [6] using an EA simulation.

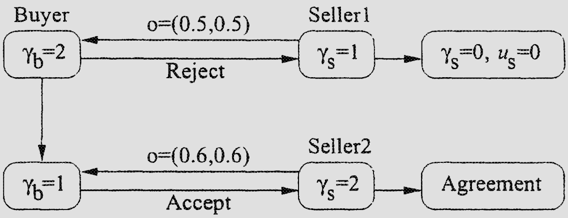

Fig. 1. A two-issue negotiation example in a market where each agent has two initial bargaining opportunities ($m = 2$).

with different states is an important aspect of the market game, particularly when agents are unaware of their opponent's remaining opportunities. Furthermore, once an offer is rejected, agents cannot go back on a previous offer[2]. Finally, for simplicity there are an equal number of buyers and sellers in the market. This in contrast to the work in e.g. [13], where markets are studied with unequal number of buyers and sellers.

3 Game-Theoretical Approach

This section considers the game-theoretic sub-game perfect equilibrium (SPE) of the above game where the agents' bargaining states are common knowledge. A game-theoretical analysis seems to be very difficult if the agents have incomplete information of their opponent's bargaining state. We will, however, drop the complete information assumption in the evolutionary approach (Section 4). In the following analysis we assume all agents of a specific type (i.e., buyer or seller) apply the same negotiation strategy. This assumption is reasonable since the preferences are homogeneous for a given type.

In case of a single opportunity, the bargaining game is reduced to the ultimatum game. The ultimatum game has a unique SPE where the seller (here the proposer) claims the total share for each issue, and the buyer (the responder) accepts this take-it-or-leave-it deal [14]. This result can be obtained by applying backward induction. Intuitively, a rational buyer will accept any positive amount, which is always better than obtaining the zero payoff in case of a disagreement. The SPE is precisely the point where the buyer is indifferent between accepting and refusing.

We argue that the SPE for multiple bargaining opportunities and complete information has the same outcome as the ultimatum game: the seller obtains the whole share, and the buyer receives the disagreement payoff, which is set to zero. Consider a buyer with $\gamma_b = 1$, i.e. with a final bargaining opportunity remaining. The buyer will then accept any positive amount offered by the seller. An anticipating seller will then claim the entire share,

[2] Agents are said to have no recall [18].

as in the ultimatum game, independent of γ_s. In SPE, the buyer's payoff for $\gamma_b = 1$ therefore equals zero.

If $\gamma_b = 2$, the buyer has two bargaining opportunities. Using the above, we can replace the payoff for refusing the seller's offer when $\gamma_b = 2$ by the disagreement payoff. The situation for $\gamma_b = 2$ is now equal to $\gamma_b = 1$: the buyer is indifferent between accepting and refusing a value of zero and in SPE the buyer accepts this deal, independent of γ_s. By backward induction the same holds for $\gamma_b = m$.

4 Evolutionary Approach

Evolutionary algorithms (EAs) are powerful search algorithms based on Darwin's theory of natural selection. In recent years, more and more the evolutionary approach has been applied within the field of computational economics as a model for both social and individual decision making. A number of related papers have demonstrated that, using an EA, artificial agents can learn effective negotiation strategies in similar negotiation games [6,17]. An important advantage of EAs is that they do not make any explicit assumptions or use of rationality. Basically, the fitness (i.e., quality) of the individual agents is used to determine whether a strategy will be used in future situations.

4.1 Evolutionary Algorithm

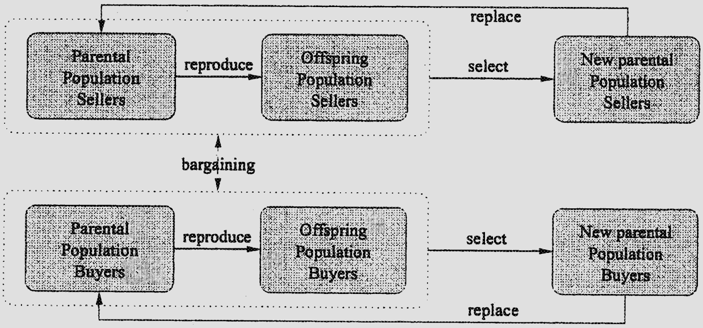

Fig. 2. Iteration loop of the evolutionary algorithm.

The evolutionary simulation works as follows. Sellers and buyers are grouped into separate populations. This way the two types of agents co-evolve. The system starts with randomly initialised "parental" populations of bargaining agents, having random bidding strategies. The EA is subsequently executed for a number of iterations or "generations". An iteration, depicted in Fig. 2, consists of three consecutive stages: reproduction, fitness evaluation, and selection.

In the reproduction stage, offspring agents are generated by first (randomly, with replacement) selecting an agent in the parental population, and then mutate its strategy to create a new offspring. The mutation operator is explained in more detail below.

In the second stage, the outcomes of a series of bargaining games assess the quality or "fitness" of the agents. The parental and offspring populations are combined to form the group of seller and the group of buyer agents. Each bargaining opportunity, two agents are randomly selected (with replacement) and play the one-shot game. An agent obtains a payoff in case an agreement is reached or if no more opportunities are available. Because an outcome depends on many random factors, each strategy is evaluated a number of times and the fitness is the average of r payoff values. The parameter r is called the evaluation frequency. This way the fitness becomes a more accurate measure of the expected payoff.

Because both buyers and sellers start with the same bargaining state, in the first periods the opponent's bargaining states do not represent an ongoing bargaining society. To prevent so-called initiatory effects and to model an on-going bargaining society, a strategy's fitness is only measured after the first payoff is determined. A strategy is thus evaluated at least $r + 1$ times. Furthermore, we model a market situation where the number of agents remains constant over time, also called a steady-state market in [13]. Therefore, once the fitness of a strategy has been established, the strategy can still be selected to play again but its fitness is no longer affected by the outcome. The bargaining games are continued until the fitness for each strategy has been established.

In the third and final stage (see Fig. 2), the fittest strategies (i.e., with the highest average payoff) from each group are selected as the new parents for the next iteration. This selection scheme is also known as $(\mu+\lambda)$-selection for evolutionary strategies (ES) [1], where μ is the number of parents and λ is the number of generated offspring. In our simulation, we take $\mu = \lambda$.

4.2 Strategy Encoding

An agent's strategy is encoded on a so-called chromosome. The implementation of the EA is based on "evolution strategies"(ES) [1], using real-encoding of the chromosome[3]. The chromosome specifies either an offer or a threshold for each bargaining state, depending on the type of the agent (i.e., seller or buyer). The threshold determines whether an offer of the opponent is accepted or rejected: if the utility falls below the threshold the offer is refused; otherwise an agreement is reached. A similar approach was used in [6,17].

[3] The widely-used genetic algorithms (GAs) are more tailored toward binary-coded search spaces [9,12,7].

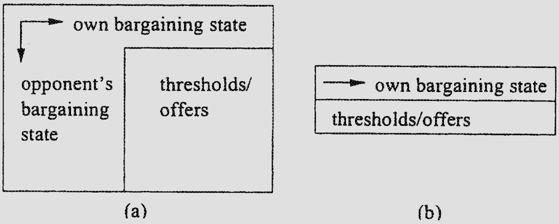

Fig. 3. Strategy representation for the complete information case (a) and incomplete information case (b).

We distinguish between the complete information setting and the incomplete information setting (see Section 2). The strategy representation for each setting is schematically depicted in Fig. 3. In the incomplete information case (Fig. 3b), different offers or thresholds are specified for different bargaining states of the agent. In case of complete information (Fig. 3a), the offers and thresholds also depend on the opponent's bargaining state.

4.3 Mutation Operator

The mutation operator produces random changes in a chromosome in the following way. Each real value x_i on the chromosome position i is mutated by adding a zero-mean Gaussian variable with a standard deviation σ [1]. Formally, $x_i' := x_i + \sigma N_i(0,1)$. All resulting values larger than unity (or smaller than zero) are set to unity (respectively zero). The standard deviation is initially set to 0.1 and gradually decays such that every t generations their value is reduced to half the size. We call t the half-life parameter. Although other mutation models such as self-adaptive control of the standard deviations [1,6] were tried as well, this model showed a closest match with game-theoretic results. Due to lack of space, we exclude the results of other models in this paper.

Parental population size (μ)	30
Offspring population size (λ)	30
Initial standard deviations (σ)	0.1
Standard deviation half-life (t)	400
Number of generations	4000
Number of runs per experiment	30
Strategy evaluation frequency (r)	20

Table 1. Default settings of the evolutionary simulation.

5 Evolutionary Simulation Results

The results are organised as follows. First, the game with complete information is studied in Subsection 5.1 and the results are compared to the game-theoretic (SPE) predictions. Subsection 5.2 studies the incomplete information case. Subsection 5.3 introduces a measure of competitiveness for multi-issue negotiations and compares results for different levels of integrative negotiations. Finally, in Subsection 5.4 considers the effects of fixed search costs in the market game.

5.1 Game-Theoretic Validation

This section considers a competitive (i.e., single-issue) scenario with complete information of the agents' bargaining opportunities and compares the evolutionary algorithm (EA) outcomes to SPE predictions. Default parameter settings for the EA are shown in Table 1. Note that because of random fluctuations, the EA results are averaged over 30 runs using the same settings.

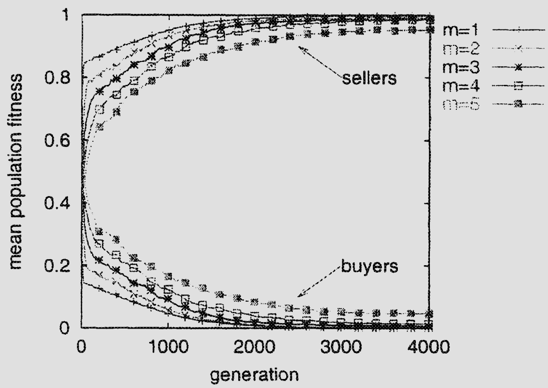

Fig. 4. Development of the mean fitness (averaged over 30 runs) for complete information setting with varying initial number of bargaining opportunities (m).

In SPE the share of the buyers is zero and the sellers obtain the whole surplus in case the initial number of bargaining opportunities of the players is equal and finite, and the bargaining state of the opponent is common knowledge (see also Section 3). Figure 4 shows the EA outcomes for different values of m (initial bargaining opportunities). The results indicate an almost perfect match between evolutionary outcomes after 4000 generations and game-theoretic outcomes, particularly when m is small.

For larger values of m we find that, using the same EA parameter settings, the evolutionary outcomes become somewhat less extreme. This is because as m becomes larger, the complexity of the problem increases due to a larger search space, making learning by an EA more difficult. However, a better match for larger values of m also appears by adjusting EA parameters, such as the evaluation frequency and the population size, to handle the increased complexity. We will not give the details here, due to space limitations. Henceforth, we present only experiments using uniform settings in this paper. We refer the interested reader to previous research [17,6], in which different EA settings are systematically studied for an alternating-offers bargaining game.

5.2 Incomplete Information

We now examine the results when the agents do not know their opponent's bargaining states; the agents only know their own bargaining states. Although no explicit information is available, the agents implicitly learn the distribution of the bargaining states in the opponent's population. This distribution is endogenously determined by the strategies of the agents. The strategies, in turn, adapt to the distribution of the bargaining states. This complex interaction is one reason why theoretical analysis is difficult. An EA, on the other hand, is well suited to find outcomes that emerge from such local interactions.

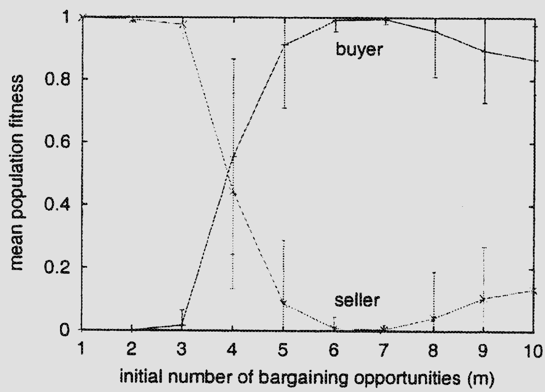

Fig. 5. Results after 4000 generations (averaged over 30 runs) for incomplete information settings with various m.

Results produced after 4000 generations of the EA for the incomplete information case are shown in Fig. 5, for different values of m (the initial number of bargaining opportunities). These results are averaged over 30 runs. The error bars indicate the standard deviation. Whereas in the complete information case the seller obtains almost the entire surplus, the responder (i.e., buyer) has the best bargaining position in the incomplete information case (see Fig. 5). This holds as long as the initial number of bargaining opportunities are sufficiently large (i.e., ≥ 5). Note that these results are obtained even though the buyers' and sellers' initial settings are equal.

The results can be explained as follows. If the buyer is in her final state, she will accept any deal (as in the ultimatum game). In other states, however, the buyer can try to find a better deal elsewhere. Consider a seller in his last bargaining state. Because he does not know the buyer's bargaining state, he can no longer anticipate the buyer's behaviour. In order to prevent a disagreement, the sellers will then concede in the last bargaining state. The expected payoff in case of a disagreement and the offers in earlier bargaining states will then also decrease. After many generations, the simulation converges to an outcome where the seller concedes almost his entire surplus in each bargaining state. We also observe that the seller concedes slightly less if he has more bargaining opportunities remaining, resulting in less extreme deals if m becomes large, as shown in Fig. 5.

In the incomplete information setting the first-mover (here the seller), has no information about his opponent. The responder, on the other hand, can make a relatively more informative decision based on the seller's offer. Whereas in the ultimatum game the proposer seems to dominate the outcome, a more competitive setting allows the responder to obtain a considerable advantage. This result, however, holds only if the number of bargaining opportunities is finite. Furthermore, the players incur no costs for refusing a deal. As we will show in Section 5.4, even slight costs completely change these results.

In the above experiments, each strategy is evaluated $r = 20$ times. We find that if r is much lower the fitness measure becomes exceedingly stochastic and as a result outcomes are less extreme. If a high value of r is used, the average utility becomes a more accurate measure of the expected payoff (see Section 4.1). In the experiments we find that more extreme outcomes are found by increasing r, particularly when m is large and the final outcomes depend on a large number of interactions. Note that, in game theory, the payoff is based on the expected outcome, i.e., the average outcome of a strategy evaluated infinitely often.

When the number of initial bargaining opportunities is set higher than three, a sudden transition in the long-term outcomes can be observed in Fig. 5: up to $m = 3$, the seller obtains almost all, whereas the buyer obtains the largest share if $m > 3$. By increasing m, the number of possible states increases, making the buyer's behaviour less predictable for the seller. The value for which the transition occurs depends on game parameters such as r and the competitiveness of the negotiation. The latter will be discussed further in the next section.

5.3 Integrative Negotiations

An advantage of bilateral negotiation is the ability to negotiate complex contracts with several issues. When mutually beneficial solutions are available, negotiations are called *integrative [8]*. We consider integrative two-issue negotiations in this section and introduce the notion of competitiveness. We show that the information in the integrative case has a very similar impact as in the competitive case. Due to increased complexity, however, the evolutionary results are less extreme when the number of bargaining opportunities is large.

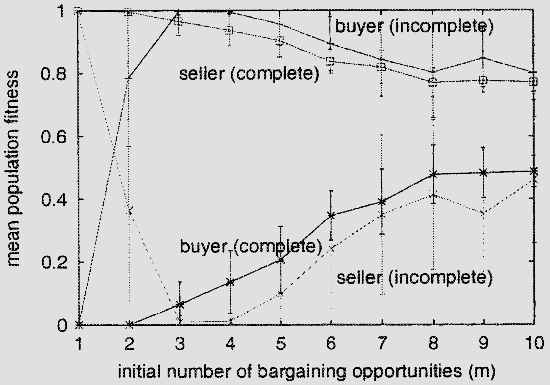

Fig. 6. Mean long-term outcomes for two-issue negotiations and $\alpha = 0.2$.

The utility of an offer is an additive, weighted function of the share obtained for each issue (see also Section 2). The weights for sellers and buyers for the two issues are $w_s = (0.5 - \alpha, 0.5 + \alpha)$ and $w_b = (0.5 + \alpha, 0.5 - \alpha)^T$ respectively, where $\alpha \in [0.0, 0.5]$ is the so-called degree of *competitiveness*. When the parameter α is set equal to 0, negotiations are purely competitive; if $\alpha = 0.5$ there is no competition at all. Note that the maximum social welfare, i.e. the maximum total utility that a seller and a buyer can achieve together equals $2 \cdot (0.5 + \alpha)$, where each agent obtains $(0.5 + \alpha)$.

Results for $\alpha = 0.2$ are visualised in Fig. 6. The results show that, as in the competitive case, a transition occurs to a buyer-dominated outcome for sufficiently large m and incomplete information. We find, however, that this transition already occurs when $m = 2$ (see Fig. 6). Only two bargaining opportunities are needed to obtain an advantage for the responder, as supposed to four in the single-issue game (Fig. 5).

Figure 6 also shows a less extreme split compared to competitive negotiations, particularly for large m. This occurs firstly since the strategy search space is increased (a value for each issue needs to be learned), making learning more difficult. Moreover, the win-win possibilities are fully exploited: if one of the agents slightly concedes, the other agent can obtain a relatively large gain by negotiating a Pareto-efficient deal. This effect becomes stronger as α increases. In the extreme case, where $\alpha = 0.5$, both agents can obtain the full surplus without any concession.

Note that the EA parameters are fixed for the various game settings. As we mentioned in Section 5.1 we can adjust the parameters to handle more complex bargaining settings as a result of a larger m and an increased number of issues. By increasing the population size and adjusting other parameters of the EA, we obtain results which are closer to game-theoretic predictions.

5.4 Search Costs

We further extend the bargaining game in this section and introduce search or negotiation costs each time an offer is refused and agents engage in a new negotiation. These costs can represent the amount of money, time, or effort that an agent may incur for finding a new opponent. It is shown theoretically that if buyers have search costs, the sellers charge monopolistic prices in equilibrium [3, Ch.7]. We consider the impact of search costs on the bargaining game where both buyers and sellers have equal search costs β. The final utility is reduced by fixed search costs β for each new bargaining opportunity. Only the first bargaining opportunity has no costs.

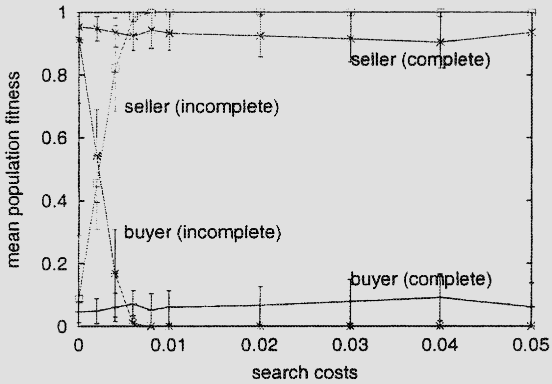

Fig. 7. Mean long-term results as a function of the search costs (β) for $m = 5$.

Evolutionary outcomes for the complete and incomplete information settings with different search costs are depicted in Fig 7. Negotiations are competitive and buyers and sellers each have 5 initial bargaining opportunities. Search costs seem to have little impact on the fitness in the complete information case; variations are not statistically relevant. Although the fitness does not change, the actual behaviour of the agents does: most agreements are now reached immediately. Without search costs, agreements reached are distributed over the various bargaining states.

In the incomplete information case, on the other hand, even small search costs have a drastic impact on the fitness of the agents, see Fig. 7. The sellers claim almost the entire share even if search costs are very small (e.g. 0.01) and equal for both agents. Results are robust for different settings of the EA. These outcomes are consistent with economic theory, which states that prices become monopolistic even if search costs are infinitely small.

As in the complete information case, both buyers and sellers are stimulated to reach agreements early in case of search costs. The final opportunity of the seller is therefore almost never reached, removing the advantage for the buyer. The game changes from a game with incomplete information, to a game where almost all players complete a deal in their first bargaining opportunity. Now the seller can again claim the entire surplus as in the one-shot game.

6 Conclusion

We study the evolutionary dynamics of a market-like game in this paper, where a seller sells a single good and has several opportunities to do so. At the same time, a buyer wishes to buy an item by trying several sellers. The terms of an agreement are negotiated using an ultimatum-like game, where the seller proposes an offer and the buyer can choose to accept or reject the offer. The game is extended to allow for multiple opportunities for both the seller and the buyer if the deal is rejected. This way a competitive market is modelled. We furthermore investigate multi-issue integrative negotiations and the effects of search costs if a disagreement occurs.

The game-theoretic outcome using sub-game perfect equilibrium (SPE) for the one-shot ultimatum game predicts an extreme split of the surplus: the seller obtains the whole surplus whereas the buyer obtains her disagreement payoff. We extend the analysis for multiple bargaining opportunities with complete information of the opponent's bargaining state and find an equivalent outcome. A theoretical analysis seems to be very difficult, however, if the bargaining states of the agents are not common knowledge. An evolutionary simulation, on the other hand, is very well suited to investigate such games with incomplete information.

We first compare the evolutionary results with the game-theoretical outcomes for the game with complete information to validate the EA approach. If the number of bargaining opportunities is small, a very good match is found. In larger games or when the negotiations become less competitive, the EA shows somewhat deviating outcomes due to larger search space and the limited computational capacity of the EA. We note that we mainly report experiments using uniform EA settings in this paper. However, adjusting EA settings appear to improve results even further for more complex games.

The evolutionary simulation shows a large impact of the additional bargaining opportunities if the agents have no information on their opponent's number of future opportunities. Whereas in the complete information game the seller dominates the market, the buyer is better off in the incomplete information setting, as long as the number of bargaining opportunities is sufficiently high. By increasing the initial number of bargaining opportunities a sudden transition is observed where the buyer obtains the largest share instead of the seller. This occurs because the seller can then no longer anticipate the buyer's response and gives in to avoid a disagreement.

A similar large impact is found for two-issue integrative negotiations. At the same time, integrative negotiations produce less extreme evolutionary outcomes, both in the game with complete and incomplete information, particularly if the number of initial bargaining opportunities is large. This mainly occurs since the space of possible deals increases. Moreover, the agents find win-win situations which benefit one agent without affecting the payoff obtained by the opponent.

An integrative setting also already affects small games with incomplete information: we find that for certain settings, a transition from a seller to a buyer dominated payoff occurs even in case both agents merely have two initial bargaining opportunities, whereas in the competitive case more bargaining opportunities are needed to achieve the same result.

We also study the effect of search or negotiation costs in case a negotiation fails and the agent needs to find a new opponent. Search costs induce players to reach an agreement in the very first bargaining opportunity. This changes an incomplete information game into an ultimatum-like game with only a single bargaining opportunity. Even very small search costs result in an extreme split where the seller obtains almost the entire share, similar to the ultimatum game outcome. This is consistent with economic theory which states that even infinitely small search costs produce monopolistic prices.

In this paper we have shown that evolutionary simulations are extremely useful to investigate negotiations with incomplete information, which are unwieldy to analyse theoretically. Using evolutionary algorithms, we can simulate complex interactions involving a large number of agents, as is the case in bargaining with posterior opportunities. It is interesting to further refine the model to specific real-world settings, where for instance agents have incomplete information about their own future number of bargaining opportunities. Another interesting extension is allowing agents to return to previously encountered opponents.

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Herding Behavior of Financial Analysts: A Model of Self-Organized Criticality

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Abstract: We investigate a simple model of investment recommendations issued by financial analysts in different industries. Financial analysts are influenced by the recommendations of some of their colleagues in the same industry as well as their own recommendations in certain other industries. Describing the relations with other financial analysts and among industries as a graph, we derive the complex dynamics of investment recommendations as relations change over time according to a simple rule, resulting in self-organized criticality. We show that financial analysts are more likely to be optimistic about industries that are unrelated to other industries than about industries that are related to others.

It is a widely accepted empirical fact that recommendations issued by financial analysts are overly optimistic. One piece of evidence in support of this view is that the number of buy recommendations exceeds sell recommendations by far, although the fundamental value does not necessarily support this view. The most prominent explanation for this result is the conflict of interest between financial analysts and other areas of investment banking, like brokerage, Initial Public Offerings or Mergers & Acquisitions advisory [16, 19]. This became especially apparent during the "internet bubble" 1998-2000, where the behavior of financial analysts has been investigated by the U.S. House of Representatives as well as the Securities and Exchange Commission (SEC). Measures to reduce the conflict of interest have been proposed and financial institutions have taken steps to change their internal rules to address the public criticism of their behavior.

A complementary result of empirical investigations is that financial analysts follow similar trends, e.g. by being overly optimistic on the prospects of certain industries or countries [5-7, 25-26]. An identical result has been found for macroeconomic forecasts [1-2, 14, 23]. This observation is in the literature known as *herding behavior*.

Although several models have been suggested to explain herding behavior [4, 21], the mechanism causing herding and the sudden emergence and disappearance of these trends has thus far not been explained satisfactory.

The basis for our model will be the motivations of financial analysts in issuing their recommendations rather than the fundamental value of the securities, which

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we neglect for simplicity here by assuming that securities are always priced at their fundamental value.[1]

Financial analysts are much concerned about their future careers. The performance of a financial analyst is evaluated relative to the performance of their colleagues, where more senior financial analysts and colleagues covering the same or similar securities serve as reference points. The performance, usually measured by the precision of forecasts, forms the basis for promotions as well as their remuneration, besides the attraction of other business in investment banking which we do not consider here in detail.

Issuing recommendations that deviate substantially from those of their colleagues expose them to the risk of showing a low relative performance and thus reduce the chances of a promotion as well as a large bonus. When issuing recommendations similar to those of their colleagues, they are likely to show an average performance. For risk averse financial analysts this situation is usually preferred. Hence the financial analyst has strong incentives not only to evaluate the fundamental value of the securities, but also take the recommendations issued by his colleagues into account.[2] This tendency to conform with the majority view is further reinforced from investors following these recommendations making them easily self-fulfilling.

The framework used in this paper will be the Bak-Sneppen model [3], whose properties have been extensively analyzed in recent years [8-13, 15, 17-18, 20, 22, 24].

We proceed by introducing in detail the model in the next section. The second section then describes the simulations conducted and section three evaluates the properties of this model. Section four concludes our findings.

1 The Model

We consider a market with s financial analysts issuing recommendations on n industries, where $n < s$. Assume that all financial analysts cover every industry and issue appropriate recommendations. Each financial analyst observes the recommendations of all other financial analysts in every industry.

As has been mentioned above, financial analysts will take into account the recommendations made by other financial analysts. Naturally there will be financial analysts whose recommendations are more important than those of others. If the importance is large, we note this as the existence of an influence from this financial analyst, otherwise there does not exist an influence. Influences do not have to be symmetric as the recommendation issued by financial analyst j may be impor-

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[1] We could easily include the fundamental value into our model without changing the general properties of our results by redefining a recommendation as optimistic if it exceeds the fundamental value.

[2] Note that with the neglect of the fundamental value these recommendations are not providing additional information for the decision making process.

tant for financial analyst i, but the recommendation issued by financial analyst i may be of no concern for financial analyst j. This can for example be the result of their different positions as senior and junior financial analysts in a company. Let us finally assume that influences from other financial analysts are confined to a single industry, hence there are no (direct) influences from the recommendations issued by other financial analysts in different industries.

It is therefore possible to interpret financial analysts as nodes and any interactions among them as vertices of a directed graph. Define $A^k = (a_{ij,k})$ as the adjacency matrix of this graph in industry k, where $a_{ij,k} = 1$ for $i \neq j$ if there exists an influence from financial analyst j towards financial analyst i, and zero otherwise. We exclude self-enforcing influences by setting $a_{ii,k} = 0$ for all $i = 1,...,s$.

The individual industries are also more or less closely related to each other, e.g. through the existence of common factors. Hence the recommendations in two related industries issued by the same financial analyst should influence each other. These relations among industries can again be interpreted as a graph, which obviously has to be symmetric. We define the adjacency matrix of this industry graph by $D = (d_{ij})$ with $d_{ij} = 1$ for $i \neq j$ if industries i and j are related and zero otherwise. Furthermore we set $d_{ii} = 0$ for all $i = 1,...,s$ to exclude a self-enforcing mechanism and $d_{ij} = d_{ji}$.

The Bak-Sneppen model assigns a measure of fitness to each node of the graph. In our model this fitness can be interpreted as the strengths of the recommendations issued by financial analysts for a specific industry, the higher this value the more optimistic the financial analyst is for an industry. The strength of the recommendation issued by financial analyst i for industry k is denoted by $y_{i,k}$. Following the Bak-Sneppen model we assume the dynamics of these beliefs to evolve as follows:

$$\dot{y}_{i,k} = \sum_{j=1}^{s} a_{ij,k} y_{j,k} + \gamma \sum_{l=1}^{n} d_{kl} y_{i,l}\,, \tag{1}$$

where $\gamma \geq 0$ denotes the relative importance of the influence generated by the recommendations of other financial analysts in the same industry (first term) and the recommendations by the same financial analyst in other industries (second term).

For $\gamma \longrightarrow 0$ the recommendations in each industry are issued independently and the properties of our model would be identical to the Bak-Sneppen model, while for $\gamma \longrightarrow \infty$ the financial analysts would issue recommendations independently of each other, only taking into account their recommendations in other industries. As we had excluded the fundamental value from our considerations and assumed securities to be priced fairly, we are here concerned about the influences of both, other financial analysts as well as other industries, and thus will choose an intermediate value for γ.

We can define $A = diag\{A^k\}$ such that

$$C = A + \gamma D \otimes I_s \tag{2}$$

is the adjacency matrix of the supergraph including influences arising from other financial analysts as well as other industries. Here ⊗ denotes the operator of the Kronecker product and I_s is the s-dimensional identity matrix. The nodes of this supergraph would be the financial analysts in a specific industry, while the vertices still represent the influences between the nodes.

With $y_k = (y_{1,k}, \ldots, y_{s,k})'$ and $y = (y_1, \ldots, y_n)'$ equation (1) can be rewritten as

$$\dot{y} = Cy. \tag{3}$$

As in the Bak-Sneppen model it is assumed that the equilibrium in the recommendations is reached immediately, such that only equilibrium values for y are considered in the analysis. The equilibrium values of y will be normalized by the following transformation:

$$y_{i,k} \rightarrow \frac{y_{i,k}}{\sum_{j=1}^{s} y_{j,k}}. \tag{4}$$

This normalization ensures that the sum of the recommendation strengths in each industry is unity. We will say that a financial analyst i is *optimistic* for industry k if $y_{i,k}$ is above the average, i.e. $y_{i,k} > 1/s$, otherwise he is *neutral*. Given the pressure imposed on financial analysts to issue buy recommendations, i.e. to be optimistic, we do not include pessimistic financial analysts.[3]

While the industry graph does not change over time, the graph capturing the interactions of financial analysts within a single industry does evolve over time according to a fixed rule.

Due to the pressure on financial analysts to issue buy recommendations, we suggest here that in each industry the least optimistic financial analyst, i.e. the financial analysts that has the lowest value for $y_{i,k}$, is replaced by a new financial analyst using different reference points or, equivalently, reviews his reference points by making random changes. All vertices emerging or ending at the node of financial analyst i in industry k, which is the least optimistic in that industry, are eliminated and replaced by new vertices according to the following rule: for all $i \neq j$ $a_{ij,k} = 1$ with probability p and 0 with probability $1-p$. We assume that the new vertices in each sector are chosen independently from each other. As before it is $a_{ii,k} = 0$. Based on these new adjacency matrices A^k the new equilibrium is determined in the coming time period.

Given the way financial analysts operate, we could identify one time period in our model with approximately one week. The developments in markets are followed continuously and any updates on recommendations regularly issued, especially when significant events trigger a change, making such a time setting the most realistic.

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[3] Usually a neutral recommendation such as "hold" or "underweight" is regarded as pessimistic and interpreted as "sell".

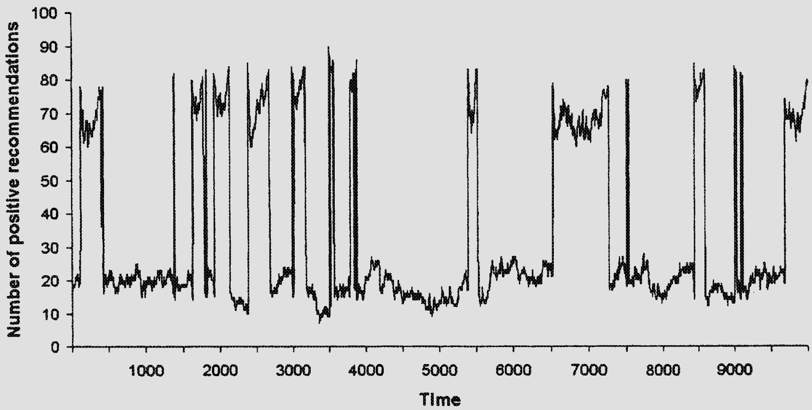

Fig. 1. Example of the number of positive recommendations

2 Numerical Simulation

The model is investigated with a simulation study based on specific parameter constellations using 500 simulations.

For each simulation an industry graph is determined randomly, where $d_{ij} = 1$ with probability q, $d_{ij} = 0$ with probability $1-q$ for $i \neq j$ and $d_{ii} = 0$.

The initial adjacency matrices A^k of the interactions of financial analysts in an industry k are chosen randomly and independently of each other such that for every i and j $a_{ij,k} = 1$ with probability p, $a_{ij,k} = 0$ with probability $1-p$ for $i \neq j$ and $a_{ii,k} = 0$. The probability p is the same as that chosen for updating the graph in subsequent periods. The initial values for $y_{i,k}$ are taken from independent uniform distributions and renormalized accordingly.

In every simulation we used 15,000 time periods, of which the first 5,000 time periods were not further analyzed to eliminate any influences from the choice of the initial matrices. The parameter constellation investigated has been as follows: $s = 100$, $n = 10$, $p = 0.0025$, $q = 0.1$, and $\gamma = 2$.

Were the issue of recommendations purely random, the model would suggest that the number of optimistic financial analysts should fluctuate around 50 with no further structure to be found. Fig. 1 shows a representative example of the evolution of positive recommendations over time. We clearly see the sharp transitions between periods of overwhelmingly neutral and positive recommendations, which obviously is not a random fluctuation around 50 positive recommendations. This result is very much in accordance with the Bak-Sneppen model which differs by lacking the interaction between different industries.

3 Results

We define an industry to be in a *group* if all other members of the group are reachable from this industry. Thus all industries of a group are directly or indirectly related with each other. The number of industries in a group is called the *group size.* An industry belonging to a group of size two or more is called *integrated*, and *isolated* otherwise. We furthermore call a situation with more than 50 financial analysts being optimistic a *bias*. The number of financial analysts being optimistic in a bias, we define as the *size of the bias.*

We can use the group size to which an industry belongs as the explanatory variable for the average time length of a bias in each industry, the number of such events in the 10,000 time periods considered, the fraction of time spent with a bias and the average size of the bias. These relations are shown in fig. 2 together with their 95% confidence intervalls.[4] Using the number of direct links from an industry or the average distance to other industries as the explanatory variable does not change the results significantly.

For integrated industries, the group size has no statistically significant influence on these variables, while for isolated industries the differences are statistically significant.

The time length of individual biases is longer for isolated industries and biases are more frequently observed. We also see that the average size of the bias is smaller for isolated industries. Table 1 shows the descriptive statistics for isolated and integrated industries. All differences are statistically significant at any reasonable level.

The intuition for this result is straightforward. As we know from the Bak-Sneppen model, it happens from time to time that "keystone" financial analysts issue the least optimistic recommendation and hence are replaced by another financial analyst in the same industry. This can cause the entire network structure established among financial analysts in that industry to break down, causing the bias to fall suddenly as financial analysts become less optimistic (self-organized criticality). In our model, this mechanism works for all industries, hence also isolated industries see these sudden changes in the bias. If industries are integrated, however, any such development in one industry can be further propagated into other industries via the industry graph and makes it more likely for the bias to disappear, although the network structure in those industries is not affected. Thus the average length of the bias is smaller in integrated than in isolated industries. This causes the time it takes for the bias to vanish to be reduced. The structure of the networks is however not that vulnerable to these disturbances in other industries that the group size has a significant impact.

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[4] We excluded biases with a length of less than 10 time periods. The reason for this is that in cases where the beliefs of financial analysts are very homogeneous, i.e. $y_{i,k}$ □ $1/s$, small variations in $y_{i,k}$ can cause large fluctuations in the number of financial analysts issuing positive recommendations. These fluctuations are erratic and have no further meaning. We also conducted investigations by excluding all biases with a length below 100 and did not exclude any biases. In both cases the results were qualitatively identical.

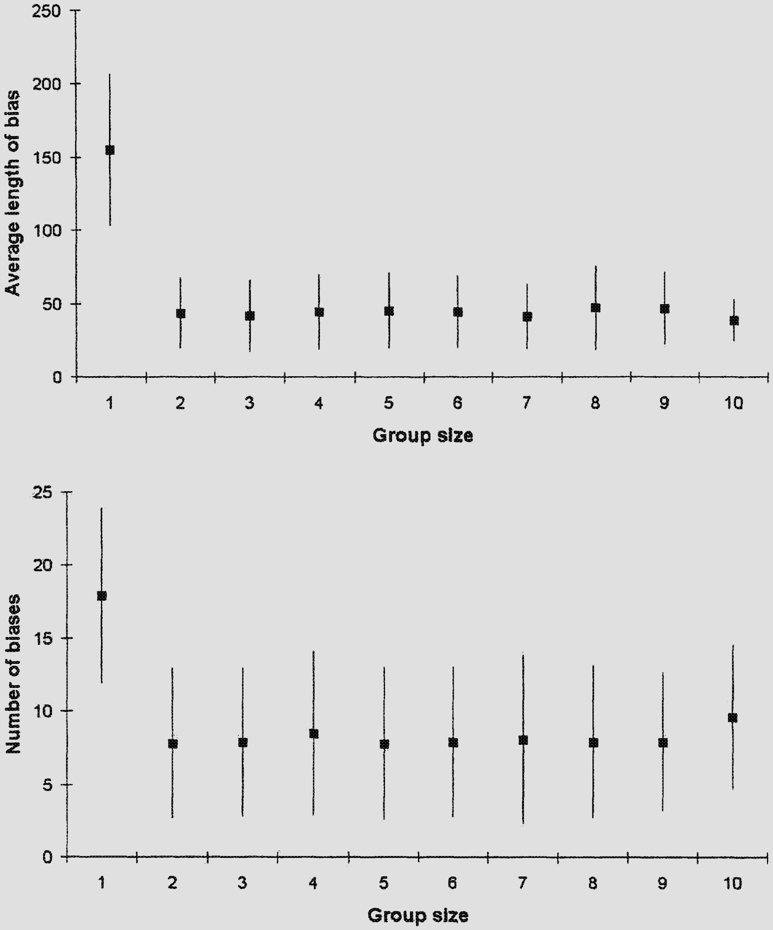
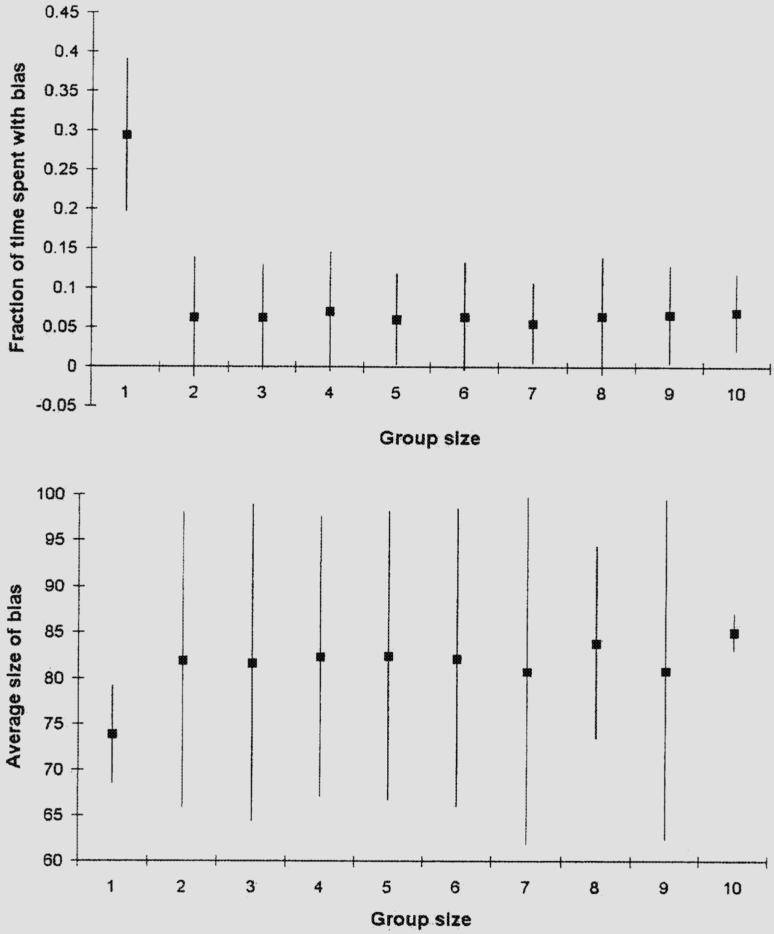


Fig. 2. Characteristics of the bias



Fraction of time spent with bias
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
1
2
3
4
5
6
7
8
9
10
Group size
Average size of bias
100
95
90
85
80
75
70
65
60
1
2
3
4
5
6
7
8
9
10
Group size

Fig. 2. (cont.)

Table 1. Descriptive statistics of biases

	Isolated industries	Integrated industries
Average time length of a bias	154.78	43.60
Number of biases	17.89	7.94
Fraction of time spent with bias	0.294	0.063
Average size of bias	73.64	85.14

In order to establish a positive bias, isolated industries only have to develop an appropriate structure within their industry through the random changes in the network structure. Integrated industries, however, have to overcome the feedback arising from other industries, too. This makes the emergence of positive biases much less likely and reduces the number of biases observed.

Taking these two effects together, it is apparent that the time spent with a bias is longer for isolated than for integrated industries. When observing the recommendations of financial analysts at a certain point of time, we are therefore more likely to observe a bias in isolated than in integrated industries.

On the other hand, once these obstacles are overcome and a bias has been established, the integrated industries enforce the bias through their feedback mechanism such that it is larger than for isolated industries.

4 Conclusions

The main conclusion to be derived from our model is that it is more likely that a substantial proportion of financial analysts are optimistic about isolated than integrated industries. We can attribute this result to the herding behavior of financial analysts that emerges from their interactions with each other. Additionally we saw that biases in the recommendations of financial analysts vanish as quickly as they appear.

If we assume that investors tend to follow the recommendations of analysts, we should observe bubbles, i.e. excessive valuations, more frequently in isolated industries, while bubbles in integrated industries should be much rarer as should be a bubble in the entire market.

When looking at historic bubbles, this picture is confirmed. There is a large number of bubbles occurring in isolated industries: Tulipmania 1634-1637, Mississippi-Bubble 1716-1720, South-Sea Bubble 1717-1720, English Canals 1792, Railways 1847, Automobiles 1922, Internet 1998-2000, to name only the best known examples. Similarly there are a large number of examples of markets in not well integrated countries, usually developing countries, to experience a sudden influx of speculative capital that causes the stock market to increase significantly or the currency to be overvalued. Bubbles in integrated industries as well as countries are much rarer found and usually less pronounced. Most market wide bubbles, e.g. before the stock market crashes 1929 and 1987, or the stock market bubble 1998-

2000, were fuelled mainly by an isolated industry, like the internet stocks in the most recent period. Hence we have a first indication that our model produces some realistic features.

It is worth stressing that our model represents the dynamics of opinions (investment recommendations issued by financial analysts) rather than trading decisions or stock price dynamics. In order to model actual trading decisions, much more complex considerations have to be included, like the risk associated with the decisions for investors or the relative valuation of securities in different industries.

Besides an empirical verification of the model, future research could extend the model in various ways, e.g. by enabling financial analysts to concentrate on certain industries rather than covering all industries, by allowing the graphs of financial analysts to be correlated across industries or generalizing the industry and analyst graph by including different degrees of relations, i.e. d_{ij}, $a_{ij,k} \in [0;1]$ instead of d_{ij} , $a_{ij,k} \in \{0;1\}$ as in our model. Since similar variations in the Bak-Sneppen model did not change the results significantly, we cannot expect fundamentally new insights from any of these changes.

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Give-and-Take in Minority Games

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Abstract. There is no presumption that collective behavior of interacting agents leads to collectively satisfactory results. In this paper, we attempt to probe deeper understanding at this issue by considering the asymmetric coordination problems formulated as minority games. We introduce a new adaptive model based on the concept of give-and-take, in which agents yield to others if they gain and randomize their actions if they lose or do not gain. We show that both efficiency and equality of collective behavior are significantly improved if agents adapt with the give-and-take strategy. We also investigate how agents co-evolve their give-and-take strategies from the bottom-up.

Keywords: asymmetric coordination, give-and-take, social efficiency, co-evolution, meta-rule

1 Introduction

There are many situations where interacting agents can benefit from coordinating their actions. Social interactions pose many coordination problems to individuals. For example, individuals face problems of sharing and distributing limited resources in an efficient way. Coordination implies that increased effort by some agents leads the remaining agents to follow suit, which gives rise to multiplier effects. We classify this type of coordination as symmetric coordination [3]. Coordination is also necessary to ensure that their individual actions are carried out with little conflicts. We classify this type of coordination as asymmetric coordination [7]. Consider the following situation: A collection of agents has to travel using either the route A or route B. Each agent gains the payoff if he chooses the route that is also determined by what the majority does. This type of coordination is classified as symmetric coordination. On the other hand, each agent gains a payoff if he chooses the opposite route to what the majority does. This type of coordination is classified as asymmetric coordination.

Coordination problems are characterized with many equilibria, and they often face the problem of coordination failure resulting from their independent inductive processes [1][4]. An interesting problem is then under what circumstances will a collection of agents realizes some stable situations, and whether they satisfy the conditions of social efficiency. In recent years, this issue has been addressed by

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formulating the minority games (MG) [2][10]. However, the growing literature on the MG treats agents as automata, merely responding to changing environments without deliberating about individuals' decisions [13]. There is no presumption that the self-interested behavior of agents should usually lead to collectively satisfactory results [8][9]. How well each agent does in adapting to its social environment is not the same as how satisfactory a social environment they collectively create for themselves. An interesting problem is then under what circumstances will a society of rational agents realize social efficiency? Solutions to these problems invoke the intervention of an authority that finds the social optimum and imposes the optimal behavior to agents. While such an optimal solution may be easy to find, the implementation may be difficult to enforce in practical situations. Self-enforcing solutions, where agents achieve optimal allocation of resources while pursing their self-interests without any explicit agreement with others are of great practical importance.

We are interested in the bottom-up approach for leading to more efficient coordination with the power of more effective learning at the individual levels [11]. Within the scope of our model, we treat models in which agents make deliberate decisions by applying rational learning procedures. We explore the mechanism in which interacting agents are stuck at an inefficient equilibrium. While agents understand that the outcome is inefficient, each agent acting independently is power less to manage the collective activity about what to do and also how to decide. The design of efficient collective action is crucial in many fields. In collective activity, two types of activities may be necessary: Each agent behaves as a member of society, while at the same time, it behaves independently by adjusting its view and action. At the individual level, it learns to improve its action based on its own observation and experiences. At the same level, they put forward their learnt knowledge for consideration by others. An important aspect of coordination is the learning rule adapted by individuals.

2 Formalisms of the EL Farol Problem and Minority Games

The EL Farol bar problem and its variants provide a clean and simple example of asymmetric coordination problems [1][4]. Brian Arthur used a very simple yet interesting problem to illustrate effective uses of inductive reasoning of heterogeneous agents. There is a bar called El Farol in downtown Santa Fe. Many agents are interested in going to the bar each night. All agents have identical preferences. Each of agents will enjoy the night at El Farol very much if there is no more than the threshold number of agents in the bar. However, each of them will suffer miserably if there is more than the threshold number of agents. In Arthur's example, the total number of agents is $N = 100$, and the threshold number is set to 60. The only information available to agents is the number of visitors to the bar in previous nights.

What makes this problem particularly interesting is that it is impossible for each agent to be perfectly rational, in the sense of correctly predicting the attendance on any given night. This is because if most agents predict that the attendance to be low (and therefore decide to attend), the attendance will actually be high, while if they predict the attendance will be high (and therefore decide not to attend) the attendance will be low. Arthur investigated the number of agents attending the bar over time by using a diverse population of simple rules adapted by agents. One interesting result obtained is that over time, the average attendance of the bar is about 60. Agents make their choices by predicting ahead of time whether the attendance on the current night will exceed the capability and then take the appropriate course of action. Arthur examined that the dynamic driving force behind this equilibrium.

The Arthur's "El Farol" model has been extended in the form as Minority Games (MG), which show for the first time how equilibrium can be reached using inductive learning [2]. The MG is played by a collection of rational agents $G = \{A_i : 1 \le i \le N\}$. Without losing the generality, we can assume N is an odd number. On each period of the stage game, each agent must choose privately and independently between two strategies $S = \{S_1, S_2\}$. We represent the action of agent A_i at the time period t by $a_i(t)=1$ if he chooses S_1, and $a_i(t)=0$ if he chooses S_2. Given the actions of all agents, the payoff of agent A_i is given by

[Payoff Scheme 1]

(i) $u_i(t) = 1$ if $a_i(t) = 1$ and $p(t) = \sum_{1\le i\le N} a_i(t)/N \le \theta$

(ii) $u_i(t) = 1$ if $a_i(t) = 1$ and $p(t) > \theta$ (2.1)

where θ is the capacity rate, and $\theta = 0.6$ with the El Farol problem, and with $\theta = 0.5$ with the MG. Each agent first receives aggregate information $p(t)$ of all agents' actions, and then decides whether to choose S_1 or S_2. Each agent is rewarded with a unitary payoff whenever the side chosen happens to be chosen by the minority of the agents, while agents on the majority side receive nothing.

Since $A(t) \equiv \sum_{1\le i\le N} a_i(t)$ represents the total number agents to choose S_1 (the total attendance) the time period t, the payoff scheme in (2.1) can be summarized as follows:

$$u_i(t) = -a_i(t)\,\mathrm{sgn}(A(t)) \tag{2.2}$$

We have another payoff scheme specified as follows:

[Payoff scheme 2]

$$u_i(t) = -a_i(t)A(t)/N\,,\quad A(t) = \sum_{1\le i\le N} a_i(t) \tag{2.3}$$

The payoff function in (2.3), is linear with respect to the proportion of the attendances $p(t) = \sum_{1\le i\le N} a_i(t)/N$. Each agent also gets aggregate information $p(t)$ that aggregate all agents' actions, and then he decides whether to chooses S_1 or

S_2. Each agent is rewarded with a payoff which is linearly decreasing function of the proportion of the attendance, $p(t)$. With the payoff in scheme 1, whenever the side an agent chooses happens to be chosen by the minority of agents, they receive a unitary award, while agents on the majority side receive nothing. With the payoff in scheme 2, whenever the side an agent chooses happens to be chosen by the minority of the agents, they receive an award that is proportional to the level of the crowdedness.

The game presents a unique symmetric mixed strategy Nash equilibrium in which each agent selects the two sides with an equal probability. We analyze the structure of the MG to see what to expect. The social efficiency can be measured from the average payoff of one agent over a long-time period. Consider the extreme case where only one agent take one side, and all the others take the other side at each time period. The lucky agent gets a reward, nothing for the others, and the average payoff per agent is $1/N$. Equally extreme situation is that when $(N-1)/2$ agents on one side, $(N+1)/2$ agents on the other side where the average payoff is about 0.5. From the society point of view, the latter situation is preferable. Several methods have been suggested to lead an efficient outcome when agents learn from each other [2][15]. All agents have access to public information of $p(\tau)$, $\tau \le t$. The past history available at the time period t is represented by $\mu(t)$. How do agents choose actions under the common information $\mu(t)$? Agents may behave differently because of their personal beliefs on the outcome of the next time period $p(t+1)$, which only depends on what agents do at the next time period $t+1$, and the past history $\mu(t)$ has no direct impact on it.

3 Decomposition of Minority Games into 2x2 Asymmetric Games

The matching methodology also plays an important role in the outcome of the game. Agents interact with all other agents, which are known as the uniform matching, or they interact with a randomly chosen agent. Agents are not assumed to be knowledgeable enough to correctly anticipate all other agents' choices, however they can only access on the information about the aggregate behavior of the society with the random matching.

Agents are rewarded a unitary payoff whenever the side chosen happens to be chosen by the minority of the population. The El Farol problem and Minority Games have a common feature that an agent's utilities depend on the number of total participants. We now show the MG can be represented as 2x2 games in which an agent play with the aggregate of the society of the population N with payoff matrix in Table 1. Let suppose each agent plays with all other agents individually with the payoff matrix in Table 1. The payoffs of agent A_i from the play with all other agents with S_1 and S_2 are given:

$$\overline{U_i}(S_1) = -n + N - n - 1 = -A(t) - 1\,,\ \overline{U_i}(S_2) = n - (N - n - 1) = A(t) + 1 \tag{3.1}$$

where n represents the number of agents to choose S_1. Dividing the above payoffs by N, we obtain the average payoff of each interaction with one agent as:

$$U_i(S_1) = \overline{U_i}(S_1)/N \cong -A(t)/N\,,\ U_i(S_2) = \overline{U_i}(S_2)/N \cong A(t)/N \tag{3.2}$$

We denote the proportion of agents to choose S_1 at the time period t by $p(t) = \sum_{1\le i\le N} a_i(t)/N$. The payoffs of agent A_i from the play with one randomly chosen agent (random matching) with S_1 and S_2 are given:

$$U_i(S_1) = 1 - \sum_{1\le i\le N} a_i(t)/N = 1 - p(t)\,,\ U_i(S_1) = \sum_{1\le i\le N} a_i(t)/N = p(t) \tag{3.3}$$

Table 1. The payoff matrix of the minority games

Other's Strategy / Own Strategy	S_1 (Go)	S_2 (Stay)
S_1 (Go)	1 \ 0	1 \ 1
S_2 (Stay)	1 \ 1	0 \ 0

Table 2. The payoff matrix of the general minority games

Other's Strategy / Own Strategy	S_1 (Go)	S_2 (Stay)
S_1 (Go)	0 \ 0	θ \ $1-\theta$
S_2 (Stay)	$1-\theta$ \ θ	0 \ 0

The El Farol model is about the equilibrium, the MG is about fluctuations, these two models can be treated with the following generic formulation: Let suppose each agent play the two-person game using the payoff matrix in Table 2 with all other agents or the aggregate of the society. The payoff when agent A_i chooses S_1 and the aggregate chooses S_2 is given by θ ($0 < \theta < 1$), and the payoff when agent A_i chooses S_2 and the aggregate chooses S_1 is given by $1-\theta$. The El Farol problem can be modeled with θ = 0.6, which is the ratio of the bar, and the MG is formulated with θ = 0.5.

Social efficiency of the MG also depends on the payoff scheme. First of all, we obtain with the payoff scheme 1. Let suppose there exists some central authority, and it leads that a little bit larger number of agents than $N\theta$ choose S_1 if $\theta \ge 0.5$, and a little bit fewer agent than $N\theta$ choose S_1 if $\theta < 0.5$. In this case the average payoff per agent is obtained as $Max(\theta, 1-\theta)$. Similarly, if the central authority leads that a little bit fewer number of agents than $N\theta$ choose S_1 if $\theta \ge 0.5$, and a little bit larger than $N\theta$ choose S_1 if $\theta < 0.5$. In this case the average payoff per

an agent is obtained as $Min(\theta,1-\theta)$. Then we have the following average payoff as the best case and the worst case:
[The payoff the best case]

$$\underset{0\le\theta\le1}{Max}(0,1-\theta) \tag{3.4}$$

[The payoff the worst case]

$$\underset{0\le\theta\le1}{Min}(\theta,1-\theta) \tag{3.5}$$

Therefore the average payoff of the best case and the worst case become the same at $\theta = 0.5$.

We now consider with the payoff scheme 2 in Eq. (2.3). The expected payoff of an agent who chooses S_1 is given $1-\theta$, and that of an agent who chooses S_2 is θ, where θ denotes the proportion of agents who choose S_1. Therefore the average payoff of an agent is given by $2\theta(1-\theta)$, which takes the minimum value 0.5 at $\theta = 0.5$.

4 Models of Individual Learning

Game theory is typically based upon the assumption of a rational choice. In our view, the reason for the dominance of the rational-choice approach is not that scholars think it to be realistic. Nor is game theory used solely because it offers good advice to a decision maker, because its unrealistic assumptions undermine much of its value as a basis for advice. The real advantage of the rational-choice assumption is that it often allows deduction. The main alternative to the assumption of rational choice is some form of adaptive behavior. The adaptation may be at the individual level through learning, or it may be at the population level through differential survival and reproduction of the more successful individuals. Either way, the consequences of adaptive processes are often very hard to deduce when there are many interacting agents following rules that have nonlinear effects.

We specify how agents adapt their behavior in response to others' behavior in strategic environments. Among the adaptive mechanisms that have been discussed in the learning literature are the following [5][6][12][14]. An important issue in strategic environment is the learning strategy adapted by each individual.

1. **Reinforcement learning:** Agents tend to adopt actions that yielded a higher payoff in the past, and to avoid actions that yielded a low payoff. Payoff describes choice behavior, but it is one's own past payoffs that matter, not the payoffs of the others. The basic premise is that the probability of taking an action in the present increases with the payoff that resulted from taking that action in the past [6].

2. **Best response learning:** Agents adopt actions that optimize their expected payoff given what they expect others to do. In this learning model, agents choose the best replies to the empirical frequencies distribution of the previous actions of the others.

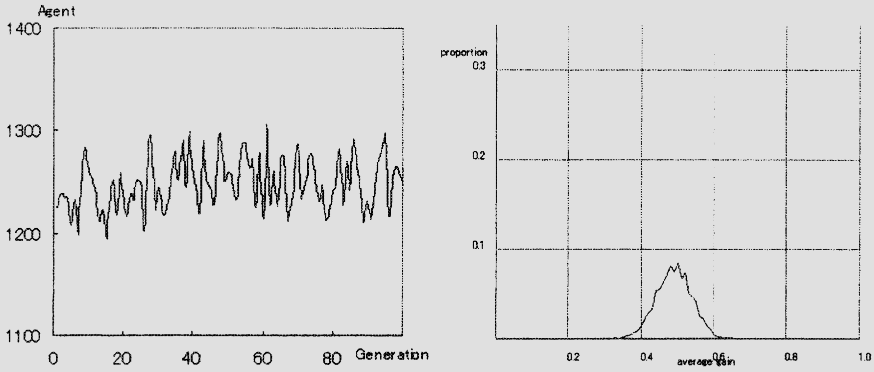
3. **Evolutionary learning:** Agents who use high-off payoff strategies are at a productive advantage compared to agents who use low-payoff strategies, hence the latter decrease in frequency in the population over time (natural selection). In the standard model of this situation agents are viewed as being genetically coded with a strategy and selection pressure favors agents that are fitter, i.e., whose strategy yields a higher payoff against the population.

4. **Social learning:** Agents learn from each other with social learning. For instance, agents may copy the behavior of others, especially behavior that is popular to yield high payoffs (imitation). In contrast to natural selection, the payoffs describe how agents make choices, and agents' payoff must be observable by others for the model to make sense. The crossover strategy is also another type of social learning.

5 Collective Behavior with Mixed Strategies

In this section, we provide simulation results when each agent adapts the same strategy *RND*(x), which represents the mixed strategy $x = (x, 1-x)$ of choosing S_1 with the probability x and S_2 with 1 - x. In the section 2, we showed that the MG can be analyzed by the 2x2 games. The rational behavior of an agent in the MG becomes to be the same as the one when each agent interacts to all other agents with the payoff matrix in Table 1. The payoff matrix in Table 1 has the unique symmetric mixed strategy Nash equilibrium in which each agent selects the two sides with the equal probability. If all agents adapt the mixed Nash equilibrium strategy, *RND*(0.5), each agent can expect the payoff 0.5 of each time period, and the society payoff follows a binomial distribution with the mean equal to $N/2$ and the variance $N/4$. The variance is also a measure of the degree of social efficiency. The higher the variance, the higher magnitude of the fluctuations around $N/2$ and the corresponding aggregate welfare loss.

We consider a population of agents with N = 2,500 with the capacity rate θ = 0.5. In Figure 6, we showed the simulation result when all agents adapt the same mixed Nash equilibrium strategy *RND*(0.5). The Figure 5(a) shows the number of agents having chosen S_1 and S_2 over time, and it is shown that the average number of agents who choose S_1 (Go) converges to the capacity of the bar, indicating that collective behavior satisfies the constraint. In Figure 5(b), we showed the proportion of agents with the same average payoff. The average payoff per agent ranges from 0.3 to 0.7, and the difference of payoff for lucky agents and that of unlucky agents becomes large. This indicates that the social inequality spreads throughout the society.



(a) (b)

Fig. 1. The simulation result under Nash equilibrium strategies: (a) The number of agents to choose S_1 and S_2, (b) The proportion of agents with the same average payoff under the mixed Nash strategy

6 Collective Behavior with Give-and-Take Strategies

Several learning rules have been found to lead an efficient outcome when agents learn from each other [2][15]. In this section, we propose the give-and-take strategy that departs from the conventional assumption such that agents update their behaviors in order to improve their measure functions such as payoffs. It is commonly assumed that agents tend to adopt actions that yield a higher payoff in the past, and to avoid actions that yield a low payoff. With the give and take learning, on the contrary, agents are assumed that they yield to others if they receive the payoff by taking the opposite strategy at the next time period, and they choose randomly if they do not gain the payoff. We formalize the payoff scheme with give-and-take strategy as follows:

We denote the status of the collective choice by all agents with the following state variable $\omega(t)$ as follows:

$$\omega(t) = 0 \text{ if } A(t) < N\theta, \quad \omega(t) = 1 \text{ if } A(t) \geq N\theta. \qquad (6.1)$$

(Minority) (Majority)

Each agent receives common information on $\omega(t)$ which aggregate all agents' actions of the last time period, and then decides whether to choose S_1 or S_1 at the time period $t+1$ by considering whether he is rewarded at the previous time t: The action $a_i(t+1)$ of agent A_i at the next time period $t+1$ is determined by the following rules:

(i) $(\omega(t)=0)\wedge(a_i(t)=1)\Rightarrow a_i(t+1)=0$

(ii) $(\omega(t)=1)\wedge(a_i(t)=0)\Rightarrow a_i(t+1)=1$

(iii) $(\omega(t)=1)\wedge(a_i(t)=1)\Rightarrow a_i(t+1)=RND(x)$

(iv) $(\omega(t)=0)\wedge(a_i(t)=0)\Rightarrow a_i(t+1)=RND(y)$ (6.2)

where $RND(x)$ represents the mixed strategy $x=(x,1-x)$ of choosing S_1 with the probability x and S_2 with $1-x$.

We should consider how agents choose the random strategy $RND(x)$ when they are in the majority by choosing S_1, the random strategy $RND(y)$ when they are in the majority by choosing S_2, so that they eventually converge to the capacity. The expected number of agents to choose S_1 at the next time $t+1$ if they use the rules in (6.3) is given as

$$A(t+1)=xA(t)+N-A(t) \tag{6.3}$$

Therefore, if they choose $RND(x)$ so that

$$A(t+1)=N\theta. \tag{6.4}$$

Then we can obtain x as

$$x=\{N\theta-(N-A(t))\}/A(t). \tag{6.5}$$

Here we assume the condition $N-A(t)\le N\theta$ is satisfied. In the case when the attendance is more that the capacity (over crowded), the number of agents to stay at home is smaller than the capacity. This assumption is easily satisfied with θ = 0.5. Similarly we obtain the mixed strategy $RND(y)$ when they are in the majority by choosing S_2. The expected number of agents to choose S_1 at the next time $t+1$ if they use the rules in (5.3) is given as

$$A(t+1)=y(N-A(t)). \tag{6.6}$$

Then, if we choose $RND(y)$ so that the following condition is satisfied

$$A(t+1)=N\theta. \tag{6.7}$$

Then we obtain as y as

$$y=N\theta/(N-A(t)) \tag{6.8}$$

Here we assume the condition of $N - A(t) \geq N\theta$ is satisfied. In the case when the attendance is below than the capacity, the number of agents to stay at home is greater than the capacity. This assumption is also easily satisfied with $\theta = 0.5$.

We also consider a population of agents with $N = 2{,}500$ with the capacity rate $\theta = 0.5$. Figure 1 shows the simulation result when all agents adapt the give-and take learning rules in (6.2). Figure 1(a) shows the number of agents having chosen S_1 and S_2 over time, and it is shown that the average number of agents who choose S_1 (Go) converges to the capacity, indicating that collective behavior satisfies the constraint. Figure 1(b) shows the proportion of agents with the same average payoff. The majority of agents receive the average payoff 0.5. This result indicates that not only social efficiency, but also social equality is achieved with give-and-take strategy.

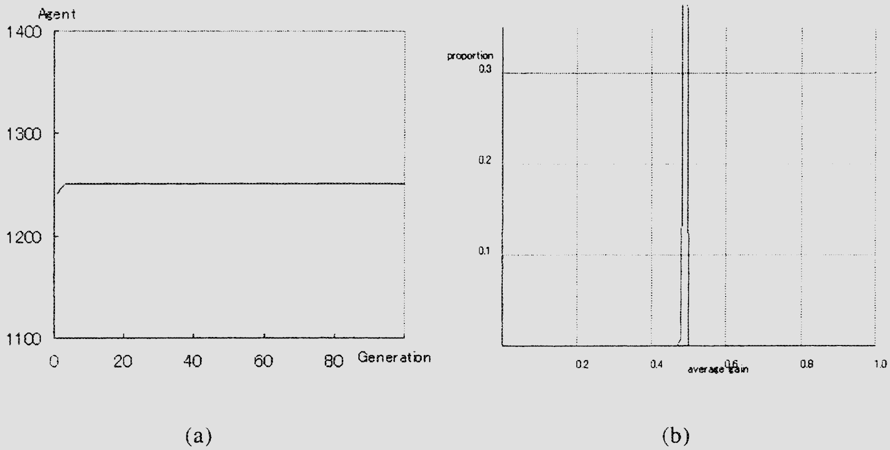
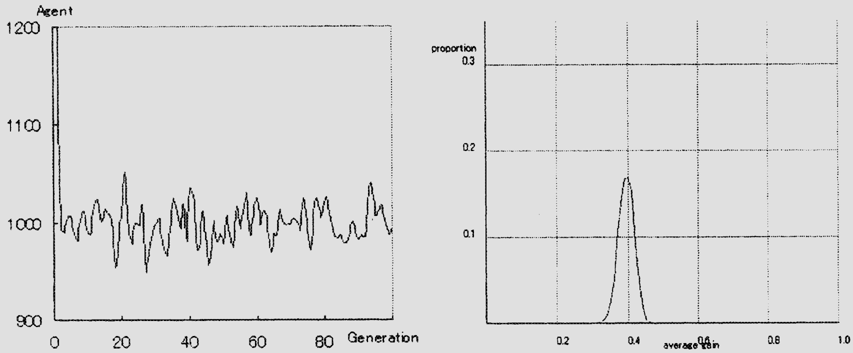


Fig. 2. Simulation result of give-and-take strategy with deliberation ($\theta = 0.5$):
(a) The dynamic changing of numbers of agents of having chosen S_1 and S_2,
(b) The proportion of agents with the same payoff

As shown in Section 3, the average payoff per agent is given by $2\theta(1-\theta)$, which takes the minimum value at the capacity rate $\theta = 0.5$. We evaluate the performance of give-and take learning with the capacity rate $\theta = 0.6$, the capacity rate of the El Farol problem. In Figure 2, we showed the simulation result. The Figure 2(a) shows the number of agents of having chosen S_1 and S_2 over time. It is shown that the average number of agents who choose S_1 (Go) converges to the capacity given by $N\theta$. In Figure 2(b), we showed the proportion of agents with the same average payoff. The majority of agents received the payoff less than 0.5, which is less than the average payoff at the efficient collective behavior.

This result indicates that the problems of inefficiency and inequity become to be crucial if the capacity rate θ deviates from 0.5, and increasing the asymmetry of the minority and majority sides. This implies that we may need the central authority in order to achieve both social efficiency and equity in asymmetric situations.



(a) (b)

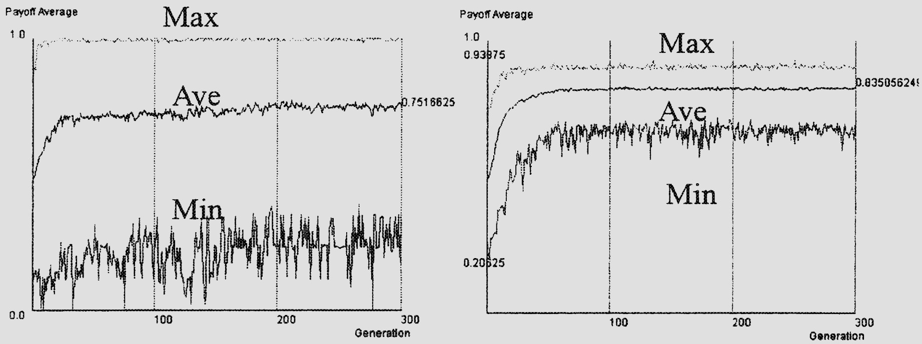
Fig. 3. Simulation result of give-and-take strategy with deliberation (θ = 0.4):
(a) The dynamics change of numbers of agents of having chosen S_1 and S_2,
(b) The proportion of agents with the same payoff

8 Evolutionary Learning with Local Matching

In this section, we investigate evolutionary learning where agents learn from the most successful neighbors, and they co-evolve their strategies over time. Each agent adapts the most successful strategy as guides for their own decision (individual learning). Hence their success depends in large part on how well they learn from their neighbors. If the neighbor is doing well, its strategy can be imitated by all others (collective learning) [11][15].

A part of the list is replaced with that of the most successful neighbor. An agent's decision rule is represented by the N binary string. At each generation *gen*, $gen \in [1,...,lastgen]$, agents repeatedly play the game for T iterations. An agent A_i, $i \in [1,...,N]$, uses a binary string i to make a decision about his action at each iteration t, $t \in [1,...,T]$. A binary string consists of 22 positions (genes). Each position p_j, $j \in [1,...,22]$ is represented as follows. The first and second position, p_1 and p_2, encodes the action that the agent takes at iteration t = 1 and t = 2. A position p_j, $j \in [3,...,6]$, encodes the history of mutual hands (cooperate or defect) that agent i took at iteration $t - 1$ and $t - 2$ with his neighbor (opponent). A position p_j, $j \in [7,...,22]$, encodes the action that agent i takes at iteration $t > 2$, corresponding to the position p_j, $j \in [3,...,6]$.

We also consider the error at the choice of the strategy. Agents choose their strategy that is specified by the meta-rule. However, we assume there exists small probability of choosing the wrong strategy. We showed the simulation results without any error and with the error rate 5% in Figure 4. Consequently, we can conclude that evolution learning leads to a more efficient situation in the strategic environments. Significant differences were observed when agents have small chances of making mistakes. As shown Figure 4(a), the highest payoff and the lowest payoff become to be close, which imply that each agent acquires the almost the same.



(a)Without any mistake

(b)Error rate : 5%

Fig. 4. The average payoff under evolutionary learning

At beginning, each agent has a different meta-rule that is specified by the 16 bits information. In Figure 5, we showed the meta-rules acquired by 400 agents, which are aggregated into 15 types. The numbers of the blanket represent the numbers of agents who acquired the same types of the meta-rules. Those 15 meta-rules have also the commonality as shown in Figure 5. If agents choose $S_1(0)$ and their opponent chooses $S_2(1)$ at the previous time period, then they choose $S_2(1)$. If agents choose $S_2(1)$ and their opponent chooses $S_1(0)$ at the previous time period, then they choose $S_1(0)$. These rules represent the following action: if they gain then they change their strategy, and this is the principle of give-and-take strategy as discussed at the section 7. From this result we can conclude that the evolutionary learning of meta-rules with some mistakes help agents to acquire the give-and-take strategy in the long run, which lead a society of agents to be both efficient and equitable.

Type01: 0 1 0 1 1 1 0 0 0 1 0 1 1 1 0 0	(10)	Type09: 1 1 0 0 1 0 0 0 0 1 0 1 0 1 0 0	(43)
Type02: 0 1 0 1 1 1 0 0 0 1 0 1 0 1 0 0	(31)	Type10: 0 1 0 1 1 0 0 0 0 1 0 1 0 1 0 0	(31)
Type03: 1 1 0 0 1 1 0 0 0 1 0 1 0 1 0 0	(72)	Type11: 1 1 0 1 1 0 0 0 0 1 0 1 0 1 0 0	(35)
Type04: 0 1 0 0 1 1 0 0 0 1 0 1 0 1 0 0	(61)	Type12: 0 1 0 0 1 0 0 0 0 1 0 1 1 1 0 0	(2)
Type05: 0 1 0 0 1 1 0 0 0 1 0 1 1 1 0 0	(10)	Type13: 1 1 0 1 1 1 0 0 0 1 0 1 1 1 0 0	(12)
Type06: 1 1 0 0 1 1 0 0 0 1 0 1 1 1 0 0	(13)	Type14: 0 1 0 1 1 0 0 0 0 1 0 1 1 1 0 0	(1)
Type07: 0 1 0 0 1 0 0 0 0 1 0 1 0 1 0 0	(40)	Type15: 1 1 0 0 1 1 0 0 0 1 1 1 0 1 0 0	(2)
Type08: 1 1 0 1 1 1 0 0 0 1 0 1 0 1 0 0	(37)		

Fig. 5. The types of the meta-rules acquired by 400 agents: Then number of the blanket represent the number of agents who acquired the same types of the meta-rules

	Memory				
	t – 2		t - 1		
Position	own	opponent	own	opponent	action
#2	0	0	0	1	1
#3	0	0	1	0	0
#5	0	1	0	0	1
#7	0	1	1	0	0
#8	0	1	1	1	0
#9	1	0	0	0	0
#10	1	0	0	1	1
#11	1	0	1	0	0
#12	1	0	1	1	1
#14	1	1	0	1	1
#15	1	0	1	0	0
#16	1	1	1	1	0
#6	0	1	0	1	0/1

Fig. 6. The commonalties among the meta-rules acquired by co-evolution

9 Conclusion

In this paper we addressed questions such as: 1) how a society of selfish agents self-organizes, without a central authority, their collective behavior to satisfy the constraints? 2) How does learning at individual levels generate more efficient collective behavior? 3) How does co-evolution in a society put in its indivisible hand to promote self-organization of emerging collective behaviors? We considered specific strategic environments in which a large number of agents have to choose one of two sides independently and those on the minority side win, which is known as minority game. A rational approach is helpless in our minority game by generating a large-scale social inefficiency. We introduced a new learning model at the individual level, give-and-take strategy. It was shown that emergent collective behavior is more efficient than that generated from the mixed Nash equilibrium strategies. We also proposed collaborative learning based on Darwinism. It is shown that in strategic environments where every agent has to keep improving their meta-rule in order to survive, if agents learn from each other, then the social efficiency is realized without central of authority.

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Part V

Societies of Interacting Agents

Coordination and Self-organization in Minority Games: Experimental Evidence

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Summary. This work presents experimental results on a coordination game in which agents must repeatedly choose between two sides, and a positive fixed payoff is assigned only to agents who pick the minoritarian side. We conduct laboratory experiments in which stationary groups of five players play the game for 100 periods, and manipulate two treatment variables: the amount of 'memory' M that players have regarding the game history (i.e., the length of the string of past outcomes that players can see on the screen while choosing), and the amount of information about other players' past choices. Our results show that, at the aggregate level a quite remarkable degree of coordination is achieved. Moreover providing players with full information about other players' choice distribution does not appear to improve efficiency significantly. At the individual level, a substantial portion of subjects exhibit 'inertial' behavior.

1 Introduction

Many social interaction phenomena imply some advantage that arises from acting in opposition to the modal behavior in a population. For example, if deciding when to leave for vacation with our car, we will be better off avoiding rush hours and travelling when only few other people are on the road. If we want to spend some hours surfing on the web, we will prefer to do that when few other people are surfing so that the connection is faster. If a long-waited new movie hit is being shown at our local theater, we will prefer to go on those nights in which spending two hours in line for tickets is *least* likely, and so on.

Thomas Schelling in *Micromotives and Macrobehavior* (1978) described and modelled a vast array of social situations that fall under this broad category, from micro-phenomena like the ones just mentioned to far more relevant ones such as racial segregation and house migration. Clearly, what matters in this type of interactive settings is the interplay between the individuals' expectations and iterated expectations (i.e., expectations of expectations), and aggregate behavior in terms of goodness of coordination. The key to the whole

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problem in fact consists not simply in trying to guess what the majority will do, and do the opposite, but rather to guess what the majority will think that the majority will do, and so on ad infinitum. Intuitively, the more heterogeneous individual expectations are, the 'smoother' - and hence efficient - the aggregate outcome of interaction is likely to be (e.g., no traffic jams on the highways, no band-width congestion, etc.), because different beliefs will induce differentiated enough behaviors so as to produce a kind of self-organization at the population level. Too similar expectations, on the contrary, may be likely to result in large overall inefficiency. This point had been first made long ago by B. Arthur (1994) in his 'El Farol Bar' model of interaction. Similar coordination problems have been extensively investigated within the experimental economics literature, usually going under the heading of 'market entry games' (e.g., Meyer et al. 1992; Ochs 1990, 1995 and references therein).

In this work we present experiments on a very simple repeated game in which the payoff to each player is based on a minority rule. In the Minority Game, first introduced by Challet and Zhang (1997, 1998) a fixed group of N players (where N is odd) must privately and independently choose each round between two actions available to them. The players choosing the action that is chosen by the minority of the players earn a fixed, positive payoff, while players who end up on the majority side earn nothing. Hence, the two alternatives are perfectly symmetric and individual payoff is solely based on how players distribute between them[3]. Formally, the model is an N-person non-cooperative game with multiple Nash equilibria in pure strategies; the pure strategy Nash equilibria are Pareto-equivalent and asymmetric with respect to players, in the sense that players differentiate from one another on the basis of the side chosen; further, the game has a unique symmetric equilibrium in mixed strategies, requiring that all agents pick the two sides with equal probability.

This work is aimed at answering the following research questions: what is the degree of efficiency of the system when groups of players interact repeatedly in a Minority Game compared to the benchmark value offered by the mixed strategy equilibrium solution? What is the impact - in terms of efficiency - of varying the amount of information that players have regarding the game history and the past actions of the other players? Do players ramdomize between rounds, as prescribed by the Nash equilibrium solution? Or are they able to detect and exploit predictable patterns in others' behavior, when they are given more information?

In order to answer these questions, we conducted experiments in which stationary groups of players played a minority game repeatedly and under different information conditions (see section 3 for a description of the experimental design).

[3] For a large collection of papers concerning both the analytical and numerical explorations of the original minority game and various extensions see also `http://www.unifr.ch/econophysics/`.

Our preliminary results can be roughly summarized as follows: first, efficiency is higher on average than the benchmark value corresponding to the mixed strategy Nash equilibrium in all treatments, suggesting that a quite remarkable degree of coordination is achieved; second, providing players with more information does not appear to improve efficiency significantly. At the individual level, a substantial portion of subjects exhibit 'inertial' behavior, i.e., the tendency to replicate their previous round choice with a higher frequency than the one prescribed by randomizing behavior.

The paper is organized as follows: Section 2 describes the game-theoretic framework and discusses previous related work. Section 3 describes the experimental design, and section 4 analyzes the results. Finally, section 5 offers some concluding remarks and directions for future research.

2 The Minority Game

The minority game is played repeatedly by a stationary set of N players, where N is an odd number. On each period of the stage game, each player must choose privately and independently between two actions or sides which will label 0 and 1. The payoff function, which is the same for all players is given by

$$\pi_i = \begin{cases} 1 \text{ if } k_i \leq (N-1)/2 \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

where $i \in \{0, 1\}$ and k_i is the number of players choosing i. In descriptive terms, players are rewarded with one point each whenever the side they choose is chosen by the minority, while players who end up on the majority side obtain a null payoff.

It is straightforward to see that the game has $N!/(((N-1)/2)!)^2$ Nash equilibria in pure strategies, in which exactly $(N-1)/2$ players choose either one of the two sides. The pure strategy Nash equilibria are Pareto equivalent and they are not strict, as players on the majority side are just indifferent between deviating and sticking to equilibrium play. Besides, they imply a payoff-asymmetry between players belonging to the different sides, which can rule out the possibility that groups may develop a simple form of tacit coordination based on historical precedent, as it may occur in analogous games (e.g., Meyer et al. 1992). The game also has a unique symmetric mixed-strategy Nash equilibrium, in which each players selects the two actions with equal probability. Finally, there are infinite asymmetric mixed strategy equilibria.

From a game-theoretic standpoint, the game is analogous to the market entry games studied experimentally by Ochs (1990), Meyer et al. (1992), Sundali et al. (1995) and Rapoport et al. (1998). In most of these games, a groups of N players must decide whether or not to enter a market that has a certain fixed capacity k, where k is typically $\leq N^4$. Some of the parameters of the payoff function may vary so as to render the 'staying out' option more or less attractive to players (in some cases players receive a fixed payoff from not entering). The payoff to entrants usually declines linearly in their number. In all such games there is usually a unique symmetric mixed strategy equilibrium and several pure strategy equilibria. The minority game, although belonging to the same class of coordination games, differs from the previously studied market entry games in some crucial aspects: its equilibrium structure is extremely unstable relative to other games, in that all pure strategy equilibria, whereby players 'lock in' in the choice of the same side, besides being non strict, and hence subject to being disrupted by minor deviations, imply a payoff-asymmetry between the two resulting parties which strongly increase their instability; for this reason, MG may be the ideal setup to study the emergence of collective efficiency resting on some forms of dynamic coordination between players. Moreover, while in market entry games the value of the market capacity usually changes from period to period thus changing the probability distribution implied by the mixed strategy equilibrium, in the MG this remains fixed throughout the game and it is very simple, as it assigns equal probability to the two available options; this, combined with the large numbers of rounds, the small group size and the strongly competitive nature of the game, should provide a strong incentive to random behavior and should allow for learning, thus making the unique mixed strategy equilibrium a natural 'benchmark' against which to confront actual behavior. Deviations from the mixed strategy hypothesis hence may be interpreted as the result of decision rules - other than randomization - which may depend on the game structure and to the history of interaction rather than as deriving from subjects' 'imperfect' ability to generate random series (e.g., Budescu 1987).

3 The Experiment: Design and Implementation

The information treatments

We conducted experiments in which stationary groups of five players played a minority game for 100 rounds, and we applied a $2X3$ factorial design in which the two treatment variables were the amount of 'memory' M that players could have regarding the game history (i.e., the number of past rounds for which information was made visible on the subjects' computer screen), and the amount of information that players had on the past choices of the other players.

A total of 120 subjects participated in the experiment. Four groups of five players participated in the single treatments. A group size of five seems large

[4] In other cases, as in Meyer et al. (1992), there are actually two markets between which players must allocate their entry decisions.

enough as to prevent subjects from developing repeated game strategies aimed at achieving tacit coordination. Indeed, such a form of dynamic coordination is not trivial to achieve in the minority game even with a small group, because it would require a complex rotation in the chosen side between players[5].

Table 1 summarizes the experimental design.

As previously stated, in this work we are mainly interested in studying the impact of information on aggregate efficiency and individual behavior.

Several previous simulation studies on the MG (e.g., Challet and Zhang 1997, 1998) have focussed on the role of the information available to players in determining aggregate efficiency. In most of these studies the amount of information given to artificial agents was varied by changing the value of the 'memory' parameter M, defined as the length of the string of past outcomes that agents were able to retain and process when making their choices. Indeed, these works have shown that the amount of agents' memory (and hence the amount of information on the 'history' of interaction) has a significant impact on the system's degree of self-organization when artificial agents are designed to behave according to experience-based strategies.

In line with previous simulation studies, we also chose to focus on the role of information; moreover, in this study we decided to jointly investigate the impact of both information about the series of past outcomes (i.e., the players' available memory capacity) and information (or lack thereof) about the behavior of single players in the population.

As far as the first parameter is concerned, in the case of human agents memory itself cannot be directly controlled for; hence, a rough 'proxy' for it was devised by varying the number of past outcomes for which information was disclosed and remained visible on the subjects' computer screens while playing. We chose three levels of memory: short ($M = 1$), medium ($M = 4$), and long ($M = 16$). In the last case, we assume that the amount of information made available is considerably more than what individuals are able to process, due to well known human working memory constraints (e.g., Miller 1956).

The second treatment variable was the type of information provided to subjects at each round of play. In the *aggregate* information treatment, subjects were only allowed to know which side (0 or 1) was the winning side at each round (i.e., they only knew the aggregate outcome of interaction at every time step), while in the *full* information treatment, subjects could also see the entire distribution of individual choices in their group. Besides, the choices of players always appeared in the same order, and this fact was explicitly stated in the instructions, so subjects were able to keep track of every individual history of choices in their group.

Previous experimental studies have shown that providing information about other players' choices usually has a significant impact on individual behavior in repeated games, although such impact tends to be game-specific and not necessarily efficiency-enhancing. Duffy and Hopkins (2001) showed

[5] This form of tacit cooperation was in fact never observed in our experiments.

	Memory (M)		
	$M=1$	$M=4$	$M=16$
Information			
Aggregate (A)	4	4	4
Full (F)	4	4	4

Table 1. The $3X2$ factorial design of the experiments. Numbers in the cells indicate the number of groups that were assigned to each treatment

	$M=1$		$M=4$		$M=16$	
steps	A	F	A	F	A	F
25	1.31	1.05	.99	1.45	1.25	.91
50	1.25	1.29	.83	1.13	1.29	.71
75	1.19	1.31	.93	.85	.89	.97
100	.99	1.03	.91	.77	.89	.61

Table 2. Values of σ averaged over successive 25 time steps and across groups, reported separately for values of M and for the aggregate (A) and full (F) information treatments

that when players had full information about others' choices, they converged much more quickly to equilibrium play in a repeated market entry game with strict pure strategy equilibria. On the contrary, providing subjects with full information about others' choices lowered efficiency in a large-group coordination game with multiple Pareto-ranked equilibria (Van Huyck, Battalio and Beil 1990). Hence, information about others' behavior, though used by players to modify their choices, not necessarily leads to better collective outcomes compared to the case in which only minimal or no information is available. In the case of a minority game, it would be reasonable to expect that when players have full information (and such condition is common knowledge), they might try to render their behavior less predictable, knowing that other players could exploit patterns in their choices, and might at the same time try to exploit patterns in others' behavior, provided they can detect any of them. Thus, it is plausible to expect that a full information treatment would increase randomizing behaviour.

Implementation

Subjects on each round had to choose between two actions that were labelled A and B in the experiment. The information that each player received at the end of every round was the visualization of the action chosen in that round (A or B), the payoff gained (1 or 0), the additional information prescribed by the experimental treatment, and his or her cumulative payoff, expressed as a percentage of successful rounds over the total at that point[6].

The experiment was ran at the Computable and Experimental Economics Laboratory of the University of Trento. Subjects were recruited from different departments of the University of Trento and had never participated in experiments of this type before. The experiment was conducted in six sessions of 20 players each, and it was computerized. Subjects were randomly assigned a

[6] Subjects were explicitly told that in case all five members chose the same action, they would all earn a payoff of zero, and the computer would display the action not chosen by anyone as the winning action.

seat at the lab computer room upon arrival, and were given written instructions[7]. They could see each other, but no form of communication was allowed during the entire session.

Subjects were told that they would be randomly divided in four groups of five players at the beginning of the experiment and that such division would be known only to the computer program. Then they would participate in a repeated decision-making experiment with the same group of people for a total of 100 periods. Each subject received a fixed show up fee plus anything he or she could earn in the experiment. Payoffs in the game were expressed in 'experimental points'. At each round a subject could earn either 1 point or zero, and the monetary exchange rate was fixed at E.18 for each point, for a theoretical maximum of E25.82 per subject (show up fee included). When all members of a group entered their choice, the computer calculated each player's payoff and disclosed information about the round. Sessions lasted fortyfive minutes on average, including instruction time.

4 Results

4.1 Aggregate behavior

We first define as a measure of *allocative efficiency* the squared deviation Σ of the number of agents choosing the 0 side from the half population value:

$$\Sigma = \sum_t (N_0(t) - \frac{N}{2})^2 \tag{2}$$

where $N_0(t)$ is the number of players choosing party 0 at time t and N is the total number of players participating in the game. The value of Σ can be comprised between 0.25 and 6.25. Clearly, the lower its value, the higher the efficiency.

Tab. 2 reports the values of σ over time separately by the value of M and averaged across groups in both the aggregate and full information treatments.

The 'benchmark' value for σ in the case of random symmetric players is equal to 1.25. Note that the degree of allocative efficiency in all treatments is significantly higher than the benchmark value. Furthermore, a general increasing trend in efficiency is clearly detectable (see also Tab. 3 and 4).

Hence, all groups in the experiment are able to achieve a quite remarkable degree of self-organization, with a higher efficiency than the one attainable through purely random play. Recall that the highest degree of efficiency would be obtained in correspondence of any of the pure strategy Nash equilibria, whereby players repeatedly choose the same side. However, as such equilibria are not strict and payoff-asymmetric, they were in fact never observed.

[7] The complete set of instructions is available from the authors upon request.

time steps	A	F
25	1,18	1,14
50	1,12	1,04
75	1,00	1,04
100	0,93	0,80

Table 3. Values of σ over time averaged across values of M for the aggregate and full information treatments

time	$M = 1$	$M = 4$	$M = 16$
25	1,18	1,22	1,08
50	1,27	0,98	1
75	1,25	0,89	0,93
100	1,01	0,84	0,75

Table 4. Values of σ over time averaged across information treatments for different values of M

Given the high degree of allocative efficiency obtained by all groups, and hence the high level of dynamic coordination between players, one might ask whether such coordination produces 'predictable' patterns in the series of aggregate outcomes. In fact, such a good level of coordination could be produced by the players' use of repeated choice patterns that could reflect themselves into regularities at the aggregate level.

Indeed, if this was the case, the observed level of allocative efficiency would not be 'robust' because it could be in principle decreased by some hypothetical arbitraging behaviors aimed at exploiting the predictability in the time series of outcomes.

In order to investigate the issue, we introduce a measure of *informational efficiency*, which describes the amount of 'informational content' of the history of interaction. In fact, if players behave randomly or if there is a significant degree of heterogeneity and change in their behavior, the aggregate information represented by the history of interaction - i.e., the string of successive winning sides at each round - should be perfectly described by a series of i.i.d. draws and hence render the outcome in the next period of play perfectly unpredictable. The resulting 'market' would be, in a sense, perfectly efficient. Otherwise, the presence of 'structure' in the time series would indicate the presence of some 'slack' in the players' coordination patterns that could in principle be exploited.

Note that the degree of informational efficiency of the resulting 'market' is an aggregate measure of closeness to the mixed strategy equilibrium solution. In fact, mixed strategy equilibrium behavior implies that the series of winning sides be a sequence of unrelated draws.

We introduce a measure of *informational efficiency* A defined as

$$A(l) = \sum_{h \in I_l} max\{F(0|h), F(1|h)\} \tag{3}$$

where I_l is the set of all the binary strings of length l and $N(i|h)$ with $i \in \{0, 1\}$ is the number of times i has been the winning side when h was the preceding history of length l (i.e. the string of the previous l winning sides). From its definition, $A(l)$ represents the average payoff of the (ex-post defined) best strategy of a hypothetical player who has access to the last l rounds of the history of play. It follows that $A(l)$ is a non decreasing function of l.

	Aggregate Info.			Full Info.		
	Mean	St. dev.	K.S. test	Mean	St. dev.	K.S. test
All	32.6	6.67	0.039	33.15	6.41	0.008
$m = 1$	31.7	5.46	0.56	31.9	5.22	0.56
$m = 4$	33.7	9.13	0.016	32.75	4.89	0.11
$m = 16$	32.6	4.77	0.16	34.8	8.48	0.03
Theoretical	31.25	4.63				

Table 5. Payoff distribution analysis

The results for the informational efficiency are shown in Fig. 1 and Fig. 2 for the aggregate and full information treatments respectively. We calculated the values of Λ for different values of the history length l. The dotted lines represent the 1 st.dev. band relative to the usual 'benchmark' case. Interestingly, as the figures show, irrespectively of the information and memory amounts, the observed informational efficiency departs from the 'benchmark' values for less than one standard deviation, suggesting a substantial lack of structure in the outcome time series.

4.2 Analysis of payoffs distribution

The last aggregate analysis concerns the total payoff distribution over the players population in all treatments. The natural benchmark to which compare the observed distribution is again the "random players" case where all the players involved choose randomly with equal probabilities between the two sides.

If N is the number of players the winning side can be of size $0, \ldots, (N-1)/2$ and the probability for any given player of belonging to a winning side of size k is k/n. Then the probability to be a winner can be computed as

$$p_w = \sum_{h=0}^{(N-1)/2} 2 \binom{N}{h} \frac{h}{N} \tag{4}$$

where the factor 2 accounts for the symmetry of the game and the probability of winning k matches out of M is given by a binomial distribution:

$$p(k; M, p_w) = \binom{M}{k} p_w^k (1 - p_w)^{M-k} \tag{5}$$

In our case $N = 5$ and since we are concerned with the total payoff in each treatment, $M = 100$. Tab. 5 shows the mean and average payoff for all treatments and for the subpopulation obtained considering only treatments with a given m. The distributions so obtained are then compared with the theoretical distribution defined in Eq. 5 by using the Kolmogorov Smirnov statistics (von Mises 1964). As can be seen, the null hypothesis of identity between the observed distributions and the theoretical prediction based on purely random players can never be disproved with high significance, even if the variance of the observed payoffs seems larger especially in the $m = 4$ case.

Hence, concluding the section on aggregate behavior, we can summarize the main results as follows: our groups of players were able to achieve a high degree of coordination - and hence were fully able of exploiting the game's gain opportunities - without producing structure in the times series of aggregate outcomes, i.e., producing a correspondingly high level of informational efficiency. Finally, the analysis of payoffs reveals that their distribution upon the population does not depart significantly from the theoretical benchmark of perfect randomizing players. In the next section we will analyze how players did implement individually the observed collective self-organization.

4.3 Individual Behavior

In this section we analyze individual behavior in the game. First, we check whether choices are biased toward one of the two possible options A and B due to their labels. One may in fact hypothesize that players react to choice labels in various "irrational" ways or in the effort of using them as a coordination device. A binomial test performed on choices pooled across treatments failed to reject the null hypothesis of no association between action label and choice[8]. Hence, the results of the subsequent analysis is collapsed across the label variable.

We next compare individual data with the theoretical benchmark represented by the symmetric mixed strategy equilibrium solution. There is considerable evidence that individuals have difficulties in generating random sequences, even when explicitly asked to do so. Many of them, in fact, exhibit a negative recency bias, generating series with too many alternations compared to those of a purely random series (e.g., Budescu 1987). This phenomenon seems related to the representative heuristic (Kahneman, Slovic and Tversky 1982) according to which a binary series with many alternations is generally judged as very representative of a random series, although it is not random. However, in competitive game-theoretic settings in which subjects repeatedly play games with a unique, mixed strategy equilibrium, the performance is generally better, as subjects have an incentive to render their future actions unpredictable (Rapoport and Budescu 1992), although the negative recency bias does not disappear completely (Rapoport and Boebel 1992).

In the minority game, the amount of information available to players might influence the extent to which they behave randomly; in particular, in the full information treatment, in which everybody's choice is disclosed to all group members, players may be more induced to render their choice unpredictable

[8] Choices of A are 50.2% of all choices (binomial test, $p = .05$). From now on, all statements on statistical significance based on nonparametric tests will always be based on the conventional significance level of .05.

and may try to exploit predictable patterns in others' choices, and this is more likely to occur the higher the value of M (the longer the string of the game history).

However, a longer 'memory' of one's own past performance associated to each choice may induce players to take more into account their personal history of gains and losses when choosing, which might go against randomizing behavior (which intuitively would require a 'memoryless' selection procedure).

In the aggregate information treatment, a binomial test performed on each individual sequence of choices allows to reject the null hypothesis of a binomial distribution in the 35% of cases (5 of 20 players in both the $m = 1$ and $m = 16$ treatments, and 6 of the 20 players in the $m = 4$ treatment). Few of these players followed almost pure strategies, i.e., they almost always chose the same side. The vast majority of the players did not follow pure strategies. However, only few of the mixed strategies can be accounted for by the mixed-strategy equilibrium.

In the full information treatment, the percentage of players' strings for which the binomial distribution can be rejected rise up to 52% (9 of 20 players in the M=1 treatment, 5 of 20 players in the M=4 treatment, and 17 of 20 players in the M=16 treatment), a datum that seem to go against the intuition that increased information would increase randomizing behavior.

Let us then introduce a finer-grained investigation of individual choice patterns. We will next detect the presence of 'structure' in the time series of players' choices, where by 'structure' we mean any deviation from the null hypothesis of mixed strategy equilibrium.

In order to provide a synthetical description of possible general relations between a player's choice and the information he possesses about the game (i.e., his previous choices and the information about past aggregate outcomes and, in the full information treatment, the individual choices of others) we use the measure of *entropy*, which defines the informational content of a probability distribution.

In order to clarify the analysis that follows, let us first review a few formal definitions:

Given a random variable X taking discrete values with probabilities $\{p_1, \ldots, p_N\}$ its entropy is defined as

$$H(X) = -\sum_{i=1}^{N} p_i \log(p_i) \tag{6}$$

and characterizes its information content (see Shannon 1948). In the same way, considering two random variables X and Y, one can compute the entropy of the the joint distribution $H(X, Y)$ and the entropy of one variable with respect to the other, for instance of the variable Y with respect to the variable X which is defined as $H(Y|X) = H(X,Y) - H(X)$. This quantity can be thought to represent the increase in information obtained observing the realizations of the variable Y when the variable X is already known. If $H(Y|X) = H(Y)$ the

l	m=1	m=4	m=16	bench.
1	-4.39 (-6.44,-2.95)	-6.41 (-7.39,-4.53)	-4.74 (-5.24,-3.23)	-6.06 (-7.22,-4.7)
2	-3.22 (-4.19,-2.53)	-4.00 (-4.99,-3.18)	-3.42 (-4.02, -2.38)	-4.14 (-4.68,-3.44)
3	-2.49 (-3.04,-1.89)	-2.84 (-3.33,-2.41)	-2.52 (-3.16,-1.95)	-3.05 (-3.4,-2.64)
4	-1.77 (-2.14,-1.50)	-1.97 (-2.34,-1.69)	-1.74 (-2.08,-1.52)	-2.09 (-2.33,-1.83)
1	-4.94 (-5.45,-3.20)	-4.61 (-5.53,-2.65)	-4.84 (-6.01,-3.60)	-6.06 (-7.22,-4.7)
2	-3.56 (-4.40,-2.99)	-3.41 (-4.11,-2.44)	-3.51 (-4.42,-2.83)	-4.14 (-4.68,-3.44)
3	-2.71 (-3.16,-1.97)	-2.43 (-3.02,-1.94)	-2.34 (-2.78,-1.88)	-3.05 (-3.4,-2.64)
4	-1.94 (-2.31,-1.56)	-1.78 (-2.19,-1.48)	-1.65 (-1.92,-1.50)	-2.09 (-2.33,-1.83)

Table 6. Mean and st. dev. for the informational content U of the string of choices with respect to the string of past outcomes for the aggregate info. (top) and the full info. (bottom) cases. The values for four different depths $l = 1, 2, 3, 4$ are reported.

increment in information is exactly the information content of the variable itself, in other words, the previous knowledge of X doesn't reduce the uncertainty on Y and hence the two variables can be considered to be unrelated. Indeed, it is easy to show that the relation $H(X,Y) = H(X) + H(Y)$ holds if and only if X and Y are independently distributed random variables.

Following from the previous properties the "uncertainty coefficient" of a variable Y with respect to a variable X can be defined as:

$$U(Y|X) = \frac{H(Y) - H(Y|X)}{H(Y)} = \frac{H(X) + H(Y) - H(X,Y)}{H(Y)} \tag{7}$$

This "coefficient" represents the fractional contribution to the knowledge of Y given by the previous knowledge of X, i.e. which fraction of the possible realizations of the variable Y can be predicted knowing the actual realizations of X. If $U(Y|X) = 0$ the variables are independent, while if $U(Y|X) = 1$ they are identically distributed; therefore U can be considered a measure of probabilistic dependence between X and Y. Notice that in this method no ad-hoc assumption is made on the nature of the probability distributions and on the form of the relation between them.

For what concerns our analysis, we fix a "memory depth" l and consider the joint probability distribution $p(c,h)$ where $c \in \{0,1\}$ is a player's choice at a given round and $h \in \times^l\{0,1\}$ is the past history of winning sides of length l at that time. This distribution can be computed for each player in each treatment from the observed frequencies and analogously can be computed the probability distribution of choices $p(c)$ and of past histories $p(h)$.

Using Eq. (7) we can define an "uncertainty coefficient" $U_{\text{hist}}(l) = U(c|h)$ of choices with respect to previous outcomes for each player in each treatment and obtain, for a given memory m, a distribution of $5 \times 4 = 20$ points. In Tab.6 we compare their means with a "benchmark" distribution obtained by a Monte Carlo simulation using group of 5 perfectly random player. The same procedure is repeated using, instead of the history of past aggregate

l	m=1	m=4	m=16	bench.
1	-3.14 (-3.49,-2.27)	-3.96 (-4.80,-3.21)	-4.36 (-5.12,-2.73)	-6.06 (-7.22,-4.7)
2	-2.44 (-2.99,-2.06)	-2.81 (-3.55,-2.18)	-3.00 (-3.84,-2.40)	-4.14 (-4.68,-3.44)
3	-2.11 (-2.57,-1.80)	-2.13 (-2.76,-1.70)	-2.36 (-2.69,-1.93)	-3.05 (-3.4,-2.64)
4	-1.54 (-1.71,-1.48)	-1.55 (-1.81,-1.42)	-1.75 (-2.02,-1.40)	-2.09 (-2.33,-1.83)
1	-4.94 (-5.45,-3.20)	-4.61 (-5.53,-2.65)	-4.84 (-6.01,-3.60)	-6.06 (-7.22,-4.7)
2	-3.56 (-4.40,-2.99)	-3.41 (-4.11,-2.44)	-3.51 (-4.42,-2.83)	-4.14 (-4.68,-3.44)
3	-2.71 (-3.16,-1.97)	-2.43 (-3.02,-1.94)	-2.34 (-2.78,-1.88)	-3.05 (-3.4,-2.64)
4	-1.94 (-2.31 -1.56)	-1.78 (-2.19,-1.48)	-1.65 (-1.92,-1.50)	-2.09 (-2.33,-1.83)

Table 7. Mean and st. dev. for the informational content U of the string of choices with respect to the string of past choices for the aggregate info. (top) and the full info. (bottom) cases. The values for four different depths $l = 1, 2, 3, 4$ are reported.

outcomes, the string of the l past choices of the player himself, obtaining a distribution for $U_{\text{self}}(l)$ compared with the Monte Carlo distribution in Tab.7. Notice that, due to the relatively short length of the time series, the length parameter$l$ cannot take too large values. Therefore we limit our analysis to $l \leq 4$.

The first thing to notice in Tab. 6 and Tab. 7 is that, irrespectively of l and m, the relationships between a player's present choice and his own past choices results stronger that the one between present choices and previous outcomes. It might appear that players tend to repeat, quite irrationally, their previous choices rather than trying to spot and exploit possible patterns in the series of group outcomes, a behavior that would be suggested by any "sophisticated" approach to the game. Moreover, no substantial difference can be noticed between treatments, with one exception: the $m = 4$ aggregate information treatment, which is the one characterized by the highest allocative efficiency, is characterized by a substantial lack of "short range" ($l = 1, 2$) relationship between players' present and past outcomes (see Tab. 6).

A qualitative analysis of choices over time confirms these observations. No statistical differences were found at this level of analysis between treatments[9]. Players in the game choose an action equal to their previous period action, on average, 65% of times, and such percentage is significantly different from the mixed strategy equilibrium hypothesis (binomial test). Hence, players exhibit a certain degree of inertia in their behavior. Such persistence is reinforced when the previous period action has been successful (i.e., when players chose the "winning side"). In fact, in this case players repeat their last period action 73% of the times on average. The percentage after a "losing" round decreases to 61%, but it is still significantly different from the mixed strategy equilibrium hypothesis.

[9] Kruskal-Wallis and Kolmogorov-Smirnov tests. Each group value was treated as a single independent observation.

Moreover, the 'inertia' and 'reinforcement' effects seem to appear even strongly in the full information treatments, as Tab. 9 shows, and they reach their maximal values in correspondence of the F(M = 16) treatment, i.e., in the treatment in which the highest amount of information was given to subjects, as shown in Tab.8. Hence, increasing information seem to increase the tendency toward 'inertial' behavior.

	M=1		M=4		M=16	
	A	F	A	F	A	F
c(t+1)=c(t)	.63 (.11)	.68 (.12)	.62 (.14)	.65 (.14)	.61 (.12)	.74 (.13)
c(t+1)=c(t) if $\pi(t) = 1$	.70 (.18)	.77 (.19)	.70 (.18)	.70 (.20)	.70 (.21)	.83 (.14)
c(t+1)=c(t) if $\pi(t) = 0$	.59 (.14)	.65 (.13)	.58 (.13)	.61 (.15)	.55 (.11)	.70 (.15)
c(t+1)=winner(t)	.49 (.12)	.49 (.08)	.51 (.06)	.49 (.12)	.52 (.09)	.51 (.07)

Table 8. Frequencies of choices of actions that were equal to the previous period action and to the previous period winning action separately for each treatment (standard deviations are reported in parenthesis)

P$\{c(t+1) = c(t)\}$	if $\pi(t) = 1$		if $\pi(t) = 0$	
	A	F	A	F
mean	.70	.76	.57	.65
st.dev.	.13	.18	.07	.15

Table 9. Mean and standard deviation for the probability of choosing an action equal to the previous period action on the pooled population in the aggregate and full information treatments

On the other hand, players do not seem to react to the previous period "aggregate information" , i.e., to what action was actually the winning one. In fact, players choosing an action equal to the last round winning action is overall equal to 50.3%, which is perfectly consistent with the hypothesis of randomizing behavior.

However, the presence of heterogeneity in players' behavior is likewise revealed, both by the variances reported in Tab. 8 and 9 and, finally, by the analysis reported in Fig. 3, where $p_i(a|b,c)$, the probability that player i chooses $a \in \{0,1\}$ given his choice was $b \in \{0,1\}$ in the previous round and the winning side was $c \in \{0,1\}$, computed starting from the choices string of each player, is plotted. The two tendencies of 'inertia' and 'reinforcement' clearly appear from the plots; however, despite these central tendencies, heterogeneity is relatively high, especially in the (b) and (c) cases.

5 Conclusion and Outlooks

The main research question of this work was to investigate, first, the extent to which groups of human players are able to coordinate efficiently in a minority game, and, second, if and to what extent the amount of information available to them makes a difference in terms of both their individual choice patterns and the resulting aggregate efficiency. Our main results can be summarized as follows: at the aggregate level, the degree of both allocative and informational efficiency achieved by our groups are remarkably high in all treatments when compared to the theoretical benchmark represented by the case of symmetric players who follow the unique mixed strategy equilibrium. The lack of a significant difference between treatments suggests that players only need minimal information to coordinate efficiently, and disclosing more information does not necessarily lead to improved aggregate performance.

Second, analysis of individual behavior in the game reveals that such efficiency is not achieved by means of 'rational' (i.e., randomizing) behaviors, but rather through a co-evolution in players' - 'suboptimal' - choice rules. In fact, analysis of individual strings of choices reveals the presence of structure; in particular, players on average exhibit 'inertial' behavior, i.e., the tendency to replicate their previous round choice more often than what would be implied by pure randomization, and this tendency increases when the previous round choice was successful, suggesting the presence of a 'reinforcement' effect (e.g., Erev and Roth 1998).

On the contrary, there is no significant relation between players' choices and the previous rounds' 'winning' sides, suggesting that players focus relatively more on their personal series of choices than on the behavior of others. Evidence on reinforcement effects confirms previous experimental studies on market entry games showing that subjects seemed to be affected in their current choices by their personal histories of gains and losses (Zwick and Rapoport 1999).

An additional treatment whereby groups' compositions vary from round to round (random assignment) would allow to measure the extent to which the high level of coordination observed rests upon some forms of "self-organization", generated by players' ability to learn over time to adapt to one another.

In addition, further research might involve a sound confrontation with experimental data from similar coordination games (like the aforementioned market entry games), tests of learning models and further simulations studies in order to pin down the behavioral and institutional conditions that favor the emergence of spontaneous, decentralized coordination in this large class of interactive settings.

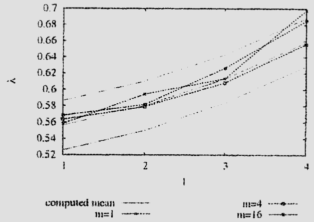

Fig. 1. Average informational efficiency for $m = 1, 4, 16$ computed with different values of l in the aggregate information case.

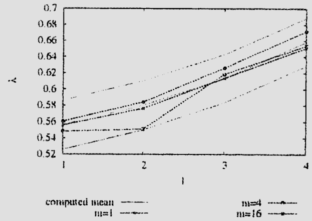

Fig. 2. Average informational efficiency for $m = 1, 4, 16$ computed with different values of l in the full information case.

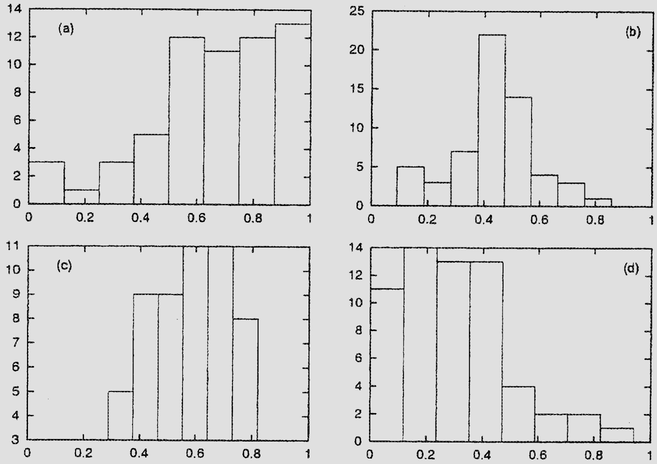

Fig. 3. Binned frequency for $p(0|a; b)$. **(a)** When the last choice was 0 and the winning side 0. **(b)** When the last choice was 1 and the winning side 0. **(c)** When the last choice was 0 and the winning side 1. **(d)** When the last choice was 1 and the winning side 1. The persistence is made stronger by a "reinforcement" effect

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Coordination of Decisions in a Spatial Model of Brownian Agents

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Abstract. Brownian agents denote a particular class of heterogeneous agents that combines features of reactive and reflexive agent concepts. As one major advance, the Brownian agent concept allows the derivation of macroscopic equations from the agent dynamics, which can be used to analyze and predict the behavior of the MAS. As an application of the concept, we discuss a binary choice problem where individual decisions are based on different local information generated by the agents. The spatial coordination of decisions in a multi-agent community is investigated both analytically and by means of stochastic computer simulations. We find that dependent on two essential parameters describing the local impact and the spatial dissemination of information either a definite stable minority/majority relation (single-attractor regime) or a broad range of possible values (multi-attractor regime) occurs. In the latter case, the outcome of the decision process becomes rather diverse and hard to predict, both with respect to the fraction of the majority and their spatial distribution. We also show that a more "efficient" information dissemination of a subpopulation provides a suitable way to stabilize their majority status and to reduce "diversity" and uncertainty in the decision process.
Keywords: spatial structures, collective phenomena, communication, decision processes, multi-agent system, phase separation, *PACS:* 05.40.+j, 82.40.-g

1 Introduction

Discrete, individual-based or *agent-based* modeling has become a very promising and powerful methodology to describe the occurence of complex behavior in economic and social systems [1, 11, 24]. While the patterns emerging are observable only on the "macroscopic" system level, the modelling effort aims to understand their emergence from the "microscopic" level of interacting individuals [22]. The advantage of such an individual-based approach is given by the fact that it is applicable also in cases where only a small number of agents govern the further evolution. Here deterministic approaches or mean-field approximations are not sufficient to describe the behavior of the complex system. Instead, the influence of history, i.e. irreversibility, path dependence, the occurence of random events play a considerable role.

As one example, in this paper we investigate the spatial coordiation of decisions in a multi-agent system. Decision making is one of the fundamental processes in economy, but also in social systems. Among the various factors

M. Gallegati et al. (eds.), *The Complex Dynamics of Economic Interaction*

that may influence the decision of agents we mention the *information* available on a particular subject – such as the price or the quality of a particular product, in an economic context, or the benefits and harms that might result from the decision, in a social context, but also information about the decisions of others. Considering the *bounded rationality* of agents, decisions are not taken upon complete a priori information, but on incomplete, limited information that involves uncertainties and is disseminated with finite velocity. This however would require to model the information flow between the agents explicitely. A possibile approach to this problem is given by the *spatio-temporal communication field* [25, 26], that is also used in this paper (cf. Sect. 2).

Based on incomplete information, how does an agent make her decision on a particular subject? The "rational" agent usually calculates her private utility and tries to maximize it. But in a world of uncertainty it turns out that the maximization of private utilities can be only achieved by some supplemented strategies. In order to reduce the risk of making the wrong decision, it often seems to be appropriate just to copy the decisions of others. Such an *imitation* strategy is widely found in biology, but also in cultural evolution and in economics, where late entrants quite often size markets from pioneers [21]. In the case where agents can observe the *payoffs* generated by other agents *information contagion* [2] has been presented as an explanation for particular patterns of macrobehavior in economic systems, for example path-dependence and lock-in-effects [13, 28]. Information contagion however involves the transmission of two different information, the *decision* made by an agent, and the *payoff* received. The situation becomes different, when agents only observe the choices of other agents and tend to imitate them, without complete information about the possible consequences of their choices. This is commonly denoted as *herding behavior* which plays a considerable role in economic systems [3, 10], in particular in financial markets [16], but also in human and biological systems where *panic* can be observed [5].

In social systems, herding behavior may result from the many (internal or external) interdepencencies of an agent community that push or pull the individual decision into a certain direction, such as peer pressure or external influences. The *social impact theory* [7, 18] that intends to describes the transition from "private attitude to public opinion" has covered some these collective effects in a way that can be also formalized within a physical approach [12, 15]. One of the modelling impacts of the social impact theory was the "rediscovering of physical space" in sociology, i.e. distance matters for social influence [6, 14, 17]. Hence, instead of mean-field approaches where all actions of agents are coupled via a mean field, spatial models become of increasing interest. In addition to the question of how individual decisions of agents may affect the macrobehavior of the system, now the question becomes important how these desisions may organize themselves *in space*, i.e. what kind of *spatial patterns* may be observed on the global level.

2 Agent Model of Decision Making

2.1 Concept of Brownian Agents

Recently, different computer architectures in distributed artificial intelligence have been developed to simulate the collective behavior of interacting individuals or agents (cf. for instance the SWARM project at `http://www.swarm.org/`). However, due to their rather complex simulation facilities many of the currently available simulation tools lack the possibility to investigate systematically and in depth the influence of specific interactions and parameters. Instead of incorporating only as much detail as is *necessary* to produce a certain emergent behavior, they put in as much detail *as possible*, and thus reduce the chance to understand *how* emergent behavior occurs and *what* it depends on.

Therefore, it would be feasible to have multi-agent systems (MAS) that can be also investigated by means of *analytical methods* (from statistical physics or mathematics) – in addition to their computational suitability. The concept of *Brownian agents* [23] is one of the possible approaches to serve for this purpose. It denotes a particular class of agents that combines features of reactive and reflexive agent concepts.

A Brownian agent is characterized by a set of state variables that include also internal degrees of freedom. The change of these state variables is in general described by a stochastic dynamics that further considers direct and indirect interactions with other agents and external influences. To be specific, let us consider a 2-dimensional spatial system with the total area A, where a community of N agents exists. In general, N can be changed by birth and death processes but A is assumed fixed. Each agent i shall be treated as a rather autonomous entity which is assigned two individual variables: its position in space, r_i, which should be a continuous variable, and its current "opinion", θ_i (with respect to a definite aspect or problem). The latter one is a discrete valued variable representing an *internal degree of freedom* (which is a rather general view of "opinion").

For $N = \text{const.}$, the community of agents may be described by the time-dependent canonical N-particle distribution function

$$P(\underline{\theta}, \underline{r}, t) = P(\theta_1, \boldsymbol{r}_1, ..., \theta_N, \boldsymbol{r}_N, t), \tag{1}$$

which gives the probability to find the N agents with the opinions $\theta_1, ..., \theta_N$ in the vicinity of $\boldsymbol{r}_1, ..., \boldsymbol{r}_N$ on the surface A at time t. Different from [25] we assume in this paper that the agents do *not* migrate, i.e. their positions $\boldsymbol{r}_i$ do not change stochastically. The time depentent change of $P(\underline{\theta}, \underline{r}, t)$ is then given by the following master equation [29]:

$$\frac{\partial}{\partial t} P(\underline{\theta}, \underline{r}, t) = \sum_{\underline{\theta}' \neq \underline{\theta}} \Big[w(\underline{\theta} | \underline{\theta}') P(\underline{\theta}', \underline{r}, t) - w(\underline{\theta}' | \underline{\theta}) P(\underline{\theta}, \underline{r}, t) \Big] \tag{2}$$

Eq. (2) describes the "gain" and "loss" of agents with the coordinates $\boldsymbol{r}_1, ..., \boldsymbol{r}_N$ due to opinion changes, where $w(\underline{\theta}|\underline{\theta}')$ means any possible transition within the opinion distribution $\underline{\theta}'$ which leads to the assumed distribution $\underline{\theta}$. After specifying the possible opinion changes in the next section, eq. (2) can be solved by means of stochastic computer simulations [23]. We note however the possibility to derive from eq. (2) *macroscopic* equations for e.g. the spatial distribution or the global fraction of agents sharing a particular opinion [25, 29].

2.2 Modeling Communication

As one example, let us imagine the separate disposal of recycling material. Each agent in the system needs to decide whether she will cooperate in the recycling campaign or defect. Then, there are only two (opposite) opinions, i.e. $\theta_i \in \{+1, -1\}$, or $\{+, -\}$ to be short. $\{+\}$ shall indicate the cooperating agent, and $\{-\}$ the defecting agent.

From the classical economic perspective, the agents' decision about her opinion may depend on an estimate of her *utility*, i.e. what she may gain compared to her own effort, if she decides to cooperate or not. Here, we neglect any question of utility and may simply assume that the agent will more likely do what others do with respect to the specific problem, i.e. she will decide to cooperate in the recycling campaign if most of her neighbors will do so, and defect if most of their neighbors have the same opinion in this case. This type of herding behavior in decision processes – a special kind of the imitation strategy – is well known from different fields, as discussed in Sect. 1.

This example raises the question about the interaction between agents at different locations, i.e. how is agent i at position r_i affected by the decisions of other agents at closer or far distant locations? In a checkerboard world, commonly denoted as cellular automaton, a common assumption is to consider only the influence of agents, which are at the (four or eight) nearest neigbour sites or also at the second-nearest neighbor sites, etc. Contrary, in a *mean-field approximation*, all agents are considered as influencial via a mean field, which affects each agent at the same time in the same manner.

Our approach will be different from these ones in that we will consider a continuous space and a gradual, time delayed interaction between all agents. We assume that agent i at position r_i is not directly affected by the decisions of other agents, but only receives information about their decisions via a *communication field* generated by the agents with the different opinions. This field is assumed a scalar *multi-component spatio-temporal field* $h_\theta(\boldsymbol{r}, t)$, which obeys the following equation:

$$\frac{\partial}{\partial t} h_\theta(\boldsymbol{r}, t) = \sum_{i=1}^{N} s_i \, \delta_{\theta,\theta_i} \, \delta(\boldsymbol{r} - \boldsymbol{r}_i) \; - \; k_\theta h_\theta(\boldsymbol{r}, t) \; + \; D_\theta \Delta h_\theta(\boldsymbol{r}, t). \quad (3)$$

Every agent contributes permanently to this field with her personal "strength" or influence, s_i. δ_{θ,θ_i} is the Kronecker Delta indicating that the agents contribute only to the field component which matches their opinion θ_i. $\delta(\boldsymbol{r}-\boldsymbol{r}_i)$ means Dirac's Delta function used for continuous variables, which indicates that the agents contribute to the field only at their current position, $\boldsymbol{r}_i$. The *information* generated this way has a certain life time $1/k_\theta$, further it can spread throughout the system by a diffusion-like process, where D_θ represents the diffusion constant for information dissemination. We have to take into account that there are two different opinions in the system, hence the communication field should also consist of two components, $\theta \in \{+1, -1\}$, each representing one opinion. Note, that the parameters describing the communication field, s_i, k_θ, D_θ do not necessarily have to be the same for the two opinions.

The *spatio-temporal communication field* $h_\theta(\boldsymbol{r},t)$ is used to reflect some important features of communication in social systems:

(i) the existence of a *memory*, which reflects the past experience. In our model, this memory exist as an external memory, the lifetime of which is determined by the decay rate of the field, k_θ.
(ii) the *dissemination of information* in the community with a *finite* velocity. It means that the information will eventually reach each agent in the whole system, but of course at different times.
(iii) the influence of *spatial distances* between agents. Thus, the information generated by a specific agent at position r_i will affect agents at a closer spatial distance earlier and thus with larger weight, compared to far distant agents.

The communication field $h_\theta(\boldsymbol{r},t)$ influences the agent's decisions as follows: At a certain location $\boldsymbol{r}_i$ agent i with e.g. opinion $\theta_i = +1$ is affected by two kinds of information: the information $h_{\theta=+1}(\boldsymbol{r}_i,t)$ resulting from agents who share her opinion, and the information $h_{\theta=-1}(\boldsymbol{r}_i,t)$ resulting from the opponents. The diffusion constants D_θ determine how fast she will receive any information, and the decay rate k_θ determines, how long a generated information will exist. Dependent on the information received locally, the agent has two opportunities to act: she can *change her opinion* or she can keep it. A possible ansatz for the transition rate to change the opinion reads [25]:

$$w(-\theta_i|\theta_i) = \eta \exp\left\{-\frac{h_\theta(\boldsymbol{r}_i,t) - h_{-\theta}(\boldsymbol{r}_i,t)}{T}\right\} \tag{4}$$

The probability to change opinion θ_i is rather small, if the local field $h_\theta(\boldsymbol{r}_i,t)$, which is related to the support of opinion θ_i, overcomes the local influence of the opposite opinion. Here, η defines the time scale of the transitions. T is a parameter which represents the *erratic circumstances* of the opinion change, based on an incomplete or incorrect transmission of information. Note, that T is measured in units of the communication field. In the limit

$T \to 0$ the opinion change rests only on the difference $\Delta h(\boldsymbol{r}_i, t) = h_\theta(\boldsymbol{r}_i, t) - h_{\theta'}(\boldsymbol{r}_i, t)$, leading to "rational" decisions (cf also [4]), i.e. decisions that are totally determined by the external information. In the limit $T \to \infty$, on the other hand, the influence of the information received is attenuated, leading to "random" decisions. We note that T can be also interpreted in terms of a "social temperature" [8, 25], i.e. it is a measure for the randomness in social interaction.

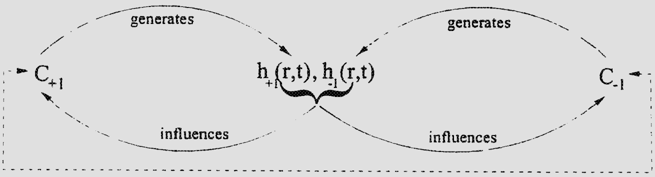

Fig. 1. Circular causation between the agents, C_{-1}, C_{+1}, and the two-component communication field, $h_\theta(\boldsymbol{r}, t)$.

In order to summarize our model, we note the non-linear feedback between the agents and the communication field as shown in Fig. 1. The agents generate the field, which in turn influences their further decisions. In terms of synergetics, the field plays the role of an order parameter, which couples the individual actions, and this way initiates coherent behavior within the agent community.

3 Fast Information Dissemination

In this section, we will neglect any spatial effects of the agents distribution and the communication field [25]. This case may have some practical relevance for communities existing in small systems with short distances between different agents. In particular, in such small communities a very fast dissemination of information may hold, i.e. spatial inhomogenities in the communication field are equalized immediately. Thus, in this section, the discussion can be restricted to subpopulations with a certain opinion rather than to agents at particular locations. Let us define the fraction x_θ of a subpopulation θ and the respective mean density $\bar{n}_\theta$ in a system of size A consisting of N agents:

$$x_\theta(t) = \frac{N_\theta(t)}{N} \; ; \quad \bar{n}_\theta(t) = \frac{N_\theta(t)}{A} \tag{5}$$

where the total number of agents sharing opinion θ at time t fulfils the condition $N_+(t) + N_-(t) = N = \text{const.}$, $x_+(t) = 1 - x_-(t)$. In the mean-field approach, the communication field $h_\theta(\boldsymbol{r}, t)$ can be approximated by a mean value $\bar{h}_\theta(t)$ which obeys the following dynamic equation:

$$\frac{\partial \bar{h}_\theta(t)}{\partial t} = -k_\theta \bar{h}_\theta(t) + s_\theta \bar{n}_\theta \tag{6}$$

Here, we have assumed that agents with the same opinion θ will have the same influence $s_i \longrightarrow s_\theta$. The dynamic equation for the fraction of subpopulation θ can be derived from eq. (2) in the mean–field approximation as follows [25]:

$$\dot{x}_\theta = (1 - x_\theta)\, \eta \, \exp(a) - x_\theta \, \eta \, \exp(-a) \; ; \quad a = \left[\bar{h}_\theta(t) - \bar{h}_{-\theta}(t)\right] / T \quad (7)$$

Via $\Delta\bar{h}(t) = \bar{h}_\theta(t) - \bar{h}_{-\theta}(t)$, this equation is coupled to eq. (6). Let us for the moment assume that the parameters decribing the communication field are the same for both components, i.e.

$$s_+ = s_- \equiv s \; ; \quad k_+ = k_- \equiv k \; ; \quad D_+ = D_- \equiv D \quad (8)$$

The stationary solutions for the fraction of each subpopulation can be obtained from $\dot{x}_\theta = 0$, $\dot{h}_\theta = 0$. It is shown in Fig. 2 for x_+^{stat} dependent on a parameter κ that results from the value a, eq. (7). In the stationary limit a can be expressed as:

$$a = \kappa \left(x_+ - \frac{1}{2} \right) \quad \text{with} \quad \kappa = \frac{2 s \, \bar{n}}{k \, T} \quad (9)$$

The parameter κ plays the role of a bifurcation parameter that includes the specific *internal conditions* within the community, such as the population density, the individual strength of the opinions, the life time of the information generated or the randomness T.

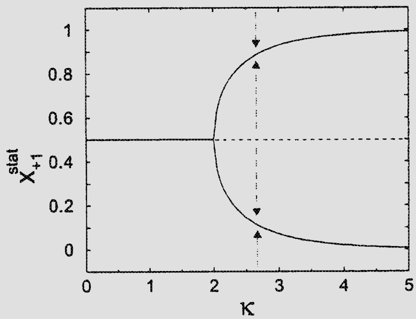

Fig. 2. Stationary solutions for x_+ (eq. 7) for different values of κ. The bifurcation at the critical value $\kappa^c = 2$ is clearly visible. For $\kappa = 2.66$ used for some of the computer simulations we find in the mean field limit the stationary values $x_+ = 0.885$ and $x_+ = 0.115$ for the majority and the minority status, respectively. [27]

In [25] we found that depending on κ different stationary values for the fraction of the subpopulations exist. For $\kappa < 2$, $x_+ = 0.5$ is the only stationary solution, which means a stable community where both opposite opinions have the same influence. However, for $\kappa > 2$, the equal distribution of opinions becomes unstable, and a separation process towards a preferred opinion is obtained, where $x_{\pm} = 0.5$ plays the role of a separatrix. Then, two stable solutions are found where both opinions coexist with different fractions in the community, as shown in Fig. 2. Hence, each subpopulation can exist either as a *majority* or as a *minority* within the community. Which of these two possible situations is realized, depends in a deterministic approach on the initial fraction of the subpopulation.

From the critical condition $\kappa^c = 2$ we can derive by means of eq. (9) a *critical population size*,

$$N^c = k\,A\,T/s, \tag{10}$$

where for larger populations an equal fraction of opposite opinions is certainly unstable.[1] I.e., after a certain population growth, the community tends towards one of these opinions, thus necessarily separating into a majority and a minority.

4 Spatial Influences on Decisions

4.1 Results of Computer Simulations

The previons section has shown within a mean-field approach the emergence of a minority/majority relation in the agents community. With respect to the example of the recycling campaign adressed in the beginning, it means that *either* most of the agents decide to cooperate *or* most of them defect. The question remains how the cooperators and the defectors organize themselve in space. In order to consider the spatial dimension of the system explicitely, let us consider N agents randomly distributed in a system of size A with random initial opinions. They get information about the opinions of other agents by means of the two-component communication field $h_\theta(\boldsymbol{r}, t)$, eq. (3), which now explicitely considers space and therefore "diffusion" of information. The two-dimensional system is here treated as a torus, i.e. we assume periodic boundary conditions. Further, for the parameters we use again eq. (8).

Fig. 3 shows a snapshot of the spatial distribution of the cooperators and the defectors after a sufficient simulation time. We note that besides some stochastic fluctuations the observed coordination pattern remains stable also in the long run. Evidently, we find again the emergence of a minority and a majority, but interestingly their fractions could be not very different, as in the case of Fig. 3. The two different groups organize themselve in space in such a way that they are separated. Thus, besides the existence of a global majority, we find regions in the system which are dominated by the minority. From this we can conclude a *spatial coordination of decisions*, i.e. agents which share the same opinion are spatially concentrated in particular regions. With

[1] We note that this critical value has been derived based on a mean field analysis and therefore does not consider finite size or discrete effects.

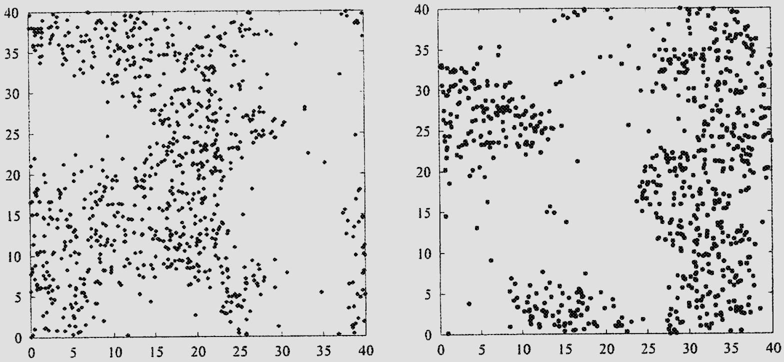

Fig. 3. Snapshot of the spatial distribution of cooperators (◇ left) and defectors (○ right) at $t = 5 \cdot 10^4$. System size $A = 1600$, total number of agents $N = 1600$, $s = 0.1$, $k = 0.1$, $T = 0.75$, i.e. $\kappa = 2.66$, $D = 0.06$. In this particular realization, the frequency of collaborators is $x_+ = 0.543$ and the frequency of defectors is $x_- = 0.456$, respectively, which is very different from the mean field limit, Fig. 2. [27]

respect to the example of the recycling campaign this means that those agents who cooperate (or defect in the opposite case), are mostly found in a spatial domain of a like-minded neighborhood.[2]

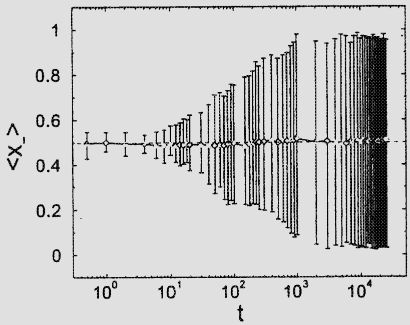

Fig. 4. Relative subpopulation size $\langle x_- \rangle$ averaged over 20 runs together with minimum/maximum values. $N = 400$, $A = 400$, other parameters see Fig. 3. [27]

Running the simulations of the agent system different times with the same set of parameters but just different initial random seeds reveals an

[2] This result might remind on the famous simulations of segregation in social systems [19, 20] - however, we would like to note that in our case the agents do *not migrate* toward supportive places; they rather *adapt* to the opinion of their neighborhood based on the information received.

interesting effect that can be observed in Fig. 4. It shows the global fraction of agents of subpopulation $\{-1\}$ over time averaged over 20 runs. The mean value gives an estimate of the chance that subpopulation $\{-1\}$ becomes the minority or majority in the system, while the error bars give an estimate about the possible values. We find that the chance to become majority is about 50 percent, i.e. only random events decide about its status. Regarding the possible values, we see that instead of a single fixed majority/minority relation for the spatially extended system a *large range* of such relations exist, which even in the presence of fluctuations are stable over a very long time. We call this the *multi-attractor regime* and conclude from these simulations that the spatially extended system – under certain conditions – possesses multiple attractors for the collective dynamics that makes the outcome of the decision process hard to predict. This holds not only for the global minority/majority ratio, but also for the possible spatial patterns that correspond to the different attractors. This shall be discussed in more detail in the following section.

4.2 Results of Analytical Investigations

In [27] we have investigated analytically the attractor structure of the spatially extended system. While the mean-field case is characterized by just one bifurcation parameter, $\kappa_1 = 2$, we found that in the spatial case a new bifurcation parameter $\kappa_2(D/k)$ appears. It depends on the scaled diffusion constant D/k, a measure of the spatial coupling, as shown in Fig. 5.

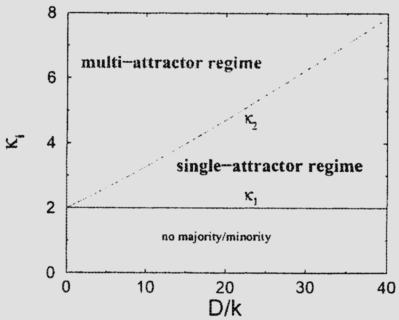

Fig. 5. Change of the critical bifurcation values κ_i' (i=1,2) dependent on D/k.

κ, eq. (9) includes the specific internal conditions within the agent community, namely the population density $\bar{n}$, the production rate of information per agent s, the lifetime of information k and the randomness T that can be envisioned as a measure of the incompleteness or incorrect transformation of information. Defining $\nu = s\bar{n}/k$ as the net information density, $\kappa = \nu/T$ describes the relation between the mean information available at any location and its impact $\sim 1/T$ – in other words, the *efficiency* of the information produced. We recall that the limit $T \to 0$ means a large impact of the available information leading to "rational" decisions, whereas in the limit $T \to \infty$ the influence of the information is attenuated, leading to "random" decisions.

In order to gain at least some impact of the available information, a supercritical value of $\kappa > \kappa_1 = 2$ is needed. For $\kappa < \kappa_1$ the equal distribution of both opinions is the stable state also for the spatially heterogeneous system and only random decisions occur. For $\kappa > \kappa_1$ a majority and a minority within the agent community emerges, which organizes itself in space in the way shown e.g. in Fig. 3. Whether this minority/majority relation is characterized by a fixed value (*single-attractor regime*) or by a multitude of possible values (*multi-attractor regime*) further depends on the spatial coupling, D/k.

For a given value of κ (i.e. fixed internal conditions), we find the multi-attractor regime only for $\kappa > \kappa_2(D/k)$, i.e. in the case of *small* values of D. This means that the spatial couplings, expressed in terms of D, are not large enough to *globally* organize the system. Since κ characterizes the average *local* situation in a spatially extended system in terms of a net information production, this can be also interpreted in a way that the impact resulting from the information *dissemination* does not overcome the impact resulting from the *local* information production. Then a variety of possible spatial decision patterns can be found, and the outcome of the decision process becomes certainly unpredictable, *both* with respect to the fraction of the majority *and* to the spatial distribution.

For values of κ below $\kappa_2^c(D)$, however, these local effects become smaller, and the spatial couplings are able to organize the whole system. Thus only one minority/majority relation occurs on the global level, which relates to randomly different, but very similar spatial patterns. If we put these results in the context of a social system, we could conclude that strong local influences, expressed in a high information efficiency, can prevent the global system from being equalized and "globalized" by some ruling information. While such a *diversity* might be among the wanted effects, we note again that this on the other hand makes the system difficult to predict.

5 Breaking the Symmetry of Decisions

5.1 Influence of External Support

From the investigations in the previous section, we found the emergence of different majority/minority relations in a spatially extended agent system. So far, however, fluctuations during the initial period may decide whether the cooperators or the defectors will appear as the majority. If we start from an unbiased initial distribution, i.e. an equal distribution between both opinions, there are different ways to break the symmetry towards e.g. cooperation.

In [25] we have considered two similar cases: (i) the existence of a *strong leader* in the community, who possesses a strength s_l which is much larger than the usual strength s of the other individuals [8, 9, 25], (ii) the existence of an external field, which may result from government policy, mass media, etc. which support a certain opinion with a strength s_m. In the case of fast information dissemination discussed in Sect. 3 the additional influence $s^\star := \{s_l/A, s_m/A\}$ mainly effects the mean communication field, eq. (6), due to an extra contribution, normalized by the system size A. We found within the mean-field approach that at a critical value of $s^\star$ the possibility of a minority status completely vanishes. This is shown in Fig. 6 that shall be compared to the bifurcation diagram in Fig. 2.

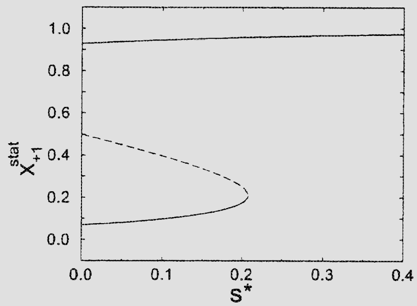

Fig. 6. Stationary solutions for x_+ for different values of $s^\star$ and a fixed supercritical value $\kappa^c < \kappa = 3$. The dashed line represents the separation line for the initial conditions, which lead either to a minority or to a majority status of the supported subpopulation. [25]

Hence, for a certain supercritical external support, the supported subpopulation will grow towards a majority, regardless of its initial population size, with no chance for the opposite opinion to be established. This situation is quite often realized in communities with one strong political or religious leader ("fundamentalistic dictatorships"), or in communities driven by external forces, such as financial or military power ("banana republics").

5.2 Increase the Efficiency of Communication

Another possibility to break the symmetry of decisions exploits the different properties of the information dissemination in the system, as expressed in terms of the parameters s_θ, k_θ, D_θ of the communication field. For instance, we may assume that the information generated by of one of the subpopulations is distributed *faster* in the system than the information generated by the other one. Alternatively, we may also consider different life times of the different components of the communication field. To be consistent, we have to choose [27] that both the ratios

$$\frac{k_\theta}{s_\theta} = \beta \,; \quad \frac{D_\theta}{s_\theta} = \gamma \tag{11}$$

need to be constant for both components $\theta = \{+1, -1\}$. In this case, eq. (3) for the dynamics of the multi-component communication field can be rewritten as:

$$\frac{\partial}{\partial \tau} h_\theta(\boldsymbol{r}, \tau) = \sum_{i=1}^{N} \delta_{\theta,\theta_i}\, \delta(\boldsymbol{r} - \boldsymbol{r}_i) \; - \; \beta\, h_\theta(\boldsymbol{r}, \tau) \; + \; \gamma\, \Delta h_\theta(\boldsymbol{r}, \tau). \tag{12}$$

where the time scale τ is now defined as $\tau = t\,(D_\theta/\gamma)$. If both parameters β and γ are kept constant, the dynamics of the respective component of the communication field occurs on a different time scale τ, dependent on the value of D_θ. An increase in the diffusion constant D_θ then models indeed the information dissemination on a faster time scale. This effect can be understood by means of computer simulations where the ratio

$$d = \frac{D_+}{D_-} \tag{13}$$

is varied.

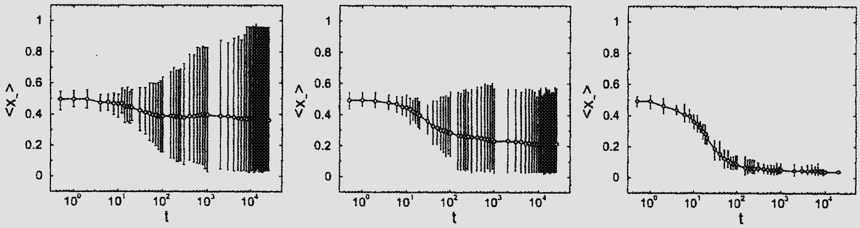

Fig. 7. Relative subpopulation size $\langle x_- \rangle$ averaged over 20 runs together with minimum/maximum values: (left) $d = 1.1$, (middle) $d = 1.2$, (right) $d = 1.5$. Other parameters see Fig. 4 ($d = 1.0$). [27]

Fig. 7 shows the total fraction of agents of subpopulation $\{-1\}$ over time averaged over 20 runs, for different values of d and shall be compared to Fig. 4 for $d = 1$. Again, the mean value gives an estimate of the chance that subpopulation $\{-1\}$ becomes the minority or majority in the system, while the error bars give an estimate about the possible values. For $d = 1$ the chance for subpopulation $\{-1\}$ to become the majority in the system is about 50 percent (see Fig. 4), but with increasing d (i.e. with an increasing information diffusion of the *other* subpopulation) there is a clear trend towards the minority status for subpopulation $\{-1\}$. Inspite of this trend, we find that for $d \in \{1.1; 1.2\}$ there are still possibilities that the subpopulation $\{-1\}$ ends up as the majority in the system – even with a slower communication. Only for $d > 1.4$, theses possibitities vanish, i.e. the deviations become small enough to allow only one stable size of the minority subpopulation.

This on the other hand means that an increase/decrease of the ratio $d = D_+/D_-$ forces a crossover between the *multi-attractor* regime, where different values for a stable minority/majority ratio are possible, and the *single-attractor* regime, where only one stable minority/majority ratio exists. More efficient communication (in terms of d) enables the supported subpopulation to largely reduce the chance to become the minority and also largely reduce the uncertainty about their total fraction in the system. Another feature to be noticed by comparing Fig. 4 with Fig. 7 is the decrease of the initial time lag when the decision about which subpopulation becomes the majority is yet pending. I.e. with increasing d there is a considerably reduced period of time for early fluctuations to break the symmetry toward one of the subpopulations.

6 Conclusions

In this paper, we have investigated the coordination of (binary) decisions in a spatially distributed agent community. We were mainly interested in how the majority and the minority of agents making a particular decision emerge in a spatially heterogeneous system and how they organize themselves in space. We observed that the formation of minority/majority subpopulations goes along with a *spatial separation process*, i.e. besides the existence of a global majority, there are regions that are dominated by the minority. Hence, a *spatial coordination* of decisions among the agents occurs. Further, we found that – different from the mean-field case – a large range of possible global minority/majority relations can be observed that refer to different spatial coordination patters.

We have investigated analytically and by means of computer simulations, under which conditions these multiple steady states occur and stable exist: (i) there should be a supercritical population density, cf. eq. (10), i.e. $\kappa > \kappa_1 = 2$, (ii) the spatial coupling in terms of information dissemination should be weak enough to prevent the system from being "globalized", i.e. $\kappa > \kappa_2(D/k)$, (iii) the dissemination of information generated by the different subpopulations should occur on comparable time scales, i.e. $d \approx 1$, in order to prevent the system from beeing "enslaved" by a dominating opinion. Tot put it differently, "efficient" information dissemination provides a suitable way to stabilize the majority status of a particular subpopulation – or to avoid "diversity" and uncertainty in the decision process.

Finally, we want to add that the toy model of communicating agents investigated in this paper may be easily modified or extended to describe other processes. Without giving up the whole framework, we may consider e.g. other types of information distribution in the system, i.e. eq. (3) for the communication field may be replaced for example by a more network-type communication among the agents. Another possible modification is regarding the decision process described in this paper by means of eq. (4). Here, we may envision various dependences on the information received from likeminded or opponent agents.

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Coalition Formation with Boundedly Rational Agents*

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Abstract

We analyse the process of coalition formation in which agents' expectations evolve through repeated interactions in a large population setting. The selection of equilibrium outcomes strongly depends on agents' initial beliefs and individual learning speed; the efficient coalition structure is reached starting from a very limited set of initial beliefs.

Introduction

Cooperative behaviour often emerges at a group, rather than social level; in many instances we observe the formation of independent groups, teams, clubs, cooperatives (coalitions for short) each of them persecuting the same goal (in turn provision of commodities, maximization of profits, raising of public funds, standards of behaviour etc.). This behaviour has been mostly analyzed within the theoretical apparatus of cooperative games; however this approach has recently undertaken a substantial revision in order to explain the emergence of coalitional structures (**CS**) as the outcome of a bargaining process in which agents cannot be committed to binding pre-play agreements; therefore the **CS** formation game has been recast within the framework of non-cooperative games (pioneering works are Farrell-Scotchmer (1988) and Greenberg (1990))[1] Adopting this non-cooperative framework we analyze the problem of how a population of individuals structures in

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[1] This type of games well adapts to represent situations concerning the decentralized supply of public goods (e.g. see Guesnerie-Oddou (1988)), the formation of cooperative firms (see Farrell-Scotchmer (1988)) or multilateral bargaining; more generally they can be applied to all those situations in which there is a problem of coordination and imperfect information (as to this point see also Fiaschi-Pacini (1998)).

M. Gallegati et al. (eds.), *The Complex Dynamics of Economic Interaction*

coalitions when agents and groups are subject to two competing forces: on one hand an increasing returns to scale technology that incentivates aggregation, on the other the non-monitorability of actions in formed coalitions that incentivates free-riding behaviours, thus reducing the incentive to form and act in groups[2]. As it will be shown in Section 2, this situation is compatible with the presence of a multiplicity of Nash equilibria, in the sense that there are many, qualitatively different, partitions of society in cooperation groups where no agent has an incentive to deviate from his actual course of action. Traditionally this indeterminacy has been reduced appealing to a sound refinement proposed by Bernheim et al. (1987), i.e. *coalition proofness*. However coalition-proofness requires the possibility of a pre-play communication stage, in which agents can agree at no cost to correlate their strategies (for example a simultaneous exit from a coalition to form a new one), a stage that is difficult to justify when the population is large, as in the case we are going to study. To overcome this difficulty, we take another point of view and substitute the implicit pre-play communication phase with an explicit dynamic process in which anonymous agents repeatedly interact without the possibility of making jointly agreed deviations. At each stage every agent plays the strategy that is best for him, *given his beliefs* about the behaviour of the others. The outcomes of individual strategies provide the information on the basis of which beliefs are revised, thus moving to a further stage of the game. An equilibrium is found when a resting state is encountered in which beliefs and actions no further change. In this way we try to provide an answer to the following questions: when the population is so large that it is not reasonable to assume that pre-play agreements can be reached, does a system in which agents act rationally find an equilibrium position? And, provided that it does, what are the characteristics of the equilibrium positions? (see also Mailath (1998))

Our main finding is that agents play the strategies sustaining a coalition proof Nash equilibrium (**CPNE** henceforth) only if population start out with *a priori* beliefs that are close to those sustaining coalition-proof outcomes; in other words, in this dynamic setting coalition-proof positions can be reached only if a large fraction of the population initially share the conviction that all other people will act and cooperate in large coalitions; the role of learning will be just to size down the dimension of the equilibrium coalitions to that consistent with individual incentives to cooperate. On the contrary, if initial beliefs are dispersed and there is not such "optimistic" initial state of expectations, the interaction among people will drive the system toward equilibrium positions that are still Nash situations, but society is divided into a larger number of groups than that corresponding to **CPNE**.

The paper is organized as follows: Section 2 describes the basic characteristics of the model, the strategies available to agents and the kind of beliefs they initially have about the state of the world. Section 3 describes the dynamic process of

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[2] For example, take the case of the formation of work cooperatives (research groups as well). The larger the group the greater the potential output because of the integration of different competencies and the cutting of administrative costs; but the larger the group the more a single has an incentive to shirk, benefiting of collective output and saving on the individual effort directed to the coalitional goal.

coalition formation and how, in this process, expectations are revised and decisions are taken. Section 4 describes the calibration of the artificial experiments, the results and their main characteristics. Conclusions close the paper.

1 The Model

The basic characteristics of the model are the following[3]: there is a population $\Im$ of I agents indexed by i. Agents are identical in all physical characteristics and are endowed with 1 unit of time that they can use either working ($l_i = 1$) or as leisure time ($l_i = 0$). They receive utility from the consumption of a commodity y and leisure time, according to the following utility function

$$U_i(y_i, l_i) = y_i + (1 - l_i) \cdot \omega$$

where ω measures the pleasure of not working.

Agents can form coalitions, but an agent can participate into one and only one coalition. A coalition is a group of agents that agree to share the output they produce by means of the total labour input that they provide. Labour is the only productive input and the technology for the production of the coalitional output Y is given by the production function $Y = L^{\alpha}$, where L is the number of labour units and α is assumed to be greater than 1. The production within a coalition has no external effect on the production of other coalitions; commodity Y deteriorates in a single period.

We assume that the agreement within a group entails an *equal sharing* distribution of the coalitional output, i.e. in a coalition of N agents anyone receives an amount $y_i = \dfrac{Y}{N}$ of the produced output Y. While participation into a group is publicly observable, the contribution of the individual working time to the coalitional production process cannot be monitored, so that defection ($l_i = 0$) cannot be punished.

Within this setting an agent has to make two decisions: (i) which coalition to participate into and (ii) which action to perform in a coalition, once formed. The first action determines the formation of a **CS** σ, i.e. a partition of the population $\Im$ in coalitions S_k, while the second determines the individual payoffs within a coalition S_k.

1.1 Strategies

In this framework a strategy for an agent i must concern (i) the formation of a coalition and (ii) the action to perform in a coalition once formed. Let Θ be the set of possible strategies. Every strategy θ is made up of two components, i.e.

[3] For the complete description of the model and proofs of statements see Fiaschi-Pacini (1998).

$\theta = \{\theta', \theta''\}$: θ' is a *signal* indicating the *maximum cardinality* of the coalition that an agent is ready to form, i.e. $\theta' \in \{1, \ldots, I\}$ [4]. θ'' is a complete *contingent plan* indicating the action l_i that agent i is willing to take conditional on the cardinality of the coalition he may happen to belong to, i.e. $\theta'' = [\theta''_1, \ldots, \theta''_{\theta'}]$ and $\theta''_j \in \{0,1\}$. As an example a strategy θ can take the form

$$\theta = \{\theta', \theta''\} = \{N, [1,1,1,0,\ldots]\}$$

meaning that agent i is ready to participate into any coalition of cardinality at most N and cooperate ($l_i = 1$) in all coalitions with cardinality less or equal to 3, while he will not cooperate ($l_i = 0$) in coalitions of greater cardinality. Finally we assume that agents cannot play mixed strategies.

1.2 Equilibria

A configuration of strategies $\Theta^* = \{\theta^*(i)\}_{i \in \Im}$ is a Nash equilibrium of the game if it gives rise to a **CS** σ^* and a corresponding profile of actions l^* such that no agent has an incentive to deviate from his strategy. Indeed this game may have many Nash equilibria and the following example can give an intuition of this fact:

Example 1 *Suppose that $I = 4$, $\alpha = 0.635$ and $\omega = 1.428$. The individual payoffs from cooperation and defection are reported in Table 1, where rows represent the cardinality of a possible coalition and columns the number of other cooperators in the corresponding coalition*

Table 1. Payoffs to cooperation and defection

	Cooperation				Defection			
	0	1	2	3	0	1	2	3
1	1	.	.	.	0.6350	.	.	.
2	0.5000	1.3454	.	.	0.6350	1.1350	.	. .
3	0.3333	0.8969	1.6003	.	0.6350	0.9683	1.5319	.
4	0.2500	0.6727	1.2002	1.8100	0.6350	0.8850	1.3077	1.8382

Clearly the grand coalition with full cooperation can never form as a Nash equilibrium, because everyone has an incentive to defect (1.8382 vs 1.8100). However each of the following strategy profiles is a Nash equilibrium:

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[4] This is motivated by the consideration that agents are identical in all physical respects and differ only as to unobservable characteristics, so that their identity does not matter for the formation of coalitions. The fact that θ' is the maximal acceptable cardinality implies that an agent, signalling θ', refuses to participate in all coalitions larger than θ' but is ready to belong to coalitions of smaller cardinality.

$$\theta^{*}(i)=\{1,[1,0,0,0]\}\ \forall i$$

$$\theta^{**}(i)=\{2,[1,1,0,0]\}\ \forall i$$

$$\theta^{***}(i)=\{3,[1,1,1,0]\}\ \forall i$$

They give rise to the following partitions of society in groups characterized by full cooperation

$$\theta^{*}\rightarrow\sigma^{*}=\{S_k\}_{k=1}^{4},\ |S_k|=1$$

$$\theta^{**}\rightarrow\sigma^{**}=\{S_k\}_{k=1}^{2},\ |S_k|=2$$

$$\theta^{***}\rightarrow\sigma^{***}=\{S_1,S_2\},\ |S_1|=3,|S_2|=1$$

Indeed any partition of society in groups $\{S_1,\ldots S_k\}$ such that $1\leq|S_k|\leq\bar{N}(\alpha)$, where $\bar{N}(\alpha)$ is the cardinality of the maximal coalition in which cooperation is incentive compatible for all its members[5], is a Nash equilibrium for a suitable configuration of strategies; for example the **CS** $\sigma'=\{S'_1,\ldots,S'_K\}$, $1\leq|S'_k|\leq\bar{N}(\alpha)$, is certainly a Nash equilibrium if i's strategy is $\theta=\{\theta',\theta''\}=\{|S'_k|,[1,\ldots 1,0,\ldots]\}$, $S'_k\ni i$, where the last 1 in the conditional action part of θ is in the $|S'_k|^{th}$ position.

Among all Nash equilibria, the **CS** in which all but possibly one coalitions have cardinality $\bar{N}(\alpha)$, is particularly interesting because it implies the maximum aggregate output and furthermore it is a *coalition-proof Nash equilibrium* (see Fiaschi-Pacini (1998)). However the implementation of this refinement presumes a lot of communication and coordination capacities on the side of agents.

In this paper we analyze this equilibrium selection problem by adopting an evolutionary approach. We suppose that agents play repeatedly the game; at each stage they choose their strategies on the basis of their beliefs on the other agents' behaviour, having as time-horizon only one period (i.e. they play a series of one-shot games). Then, at the end of every period, agents revise their beliefs on the basis of experience. While the fact that agents will finally play Nash equilibria is an

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[5] Provided $\frac{1}{2}<\omega<1$, $\bar{N}(\alpha)$ is the integer part of the value of N solving the equation $N^{\alpha}-(N-1)^{\alpha}-\omega\cdot N=0$. $\bar{N}(\alpha)$ is monotonically increasing in α, so that all agents merge and cooperate in the grand coalition provided $\alpha\geq\alpha^{*}>1$, where α^{*} solves $I^{\alpha^{*}}-(I-1)^{\alpha^{*}}-\omega\cdot I=0$. Furthermore, since $\bar{N}(\alpha)<2$ for $\alpha\leq 1$, it follows that increasing returns are a necessary condition to observe cooperation in groups of at least 2 agents. Finally, for $\alpha\in(1,\alpha^{*})$ cooperation can emerge just in coalitions that are proper subsets of $\Im$; this is the case we will deal with in the rest of the paper.

expected outcome, our focus will be on the question on which Nash equilibrium agents will actually play.

2 The Design of the Simulation

The evolution of the game is represented in a sequence of periods. At the beginning of each period every agent plays his strategy. The choice of a strategy is the outcome of a maximization process *conditional on the current period beliefs* about the actions of the others (see Section 3.3). Given the strategies $\theta = \{\theta(1), \ldots, \theta(I)\}$ played by the agents, a **CS** σ is the result of a random matching process. Agents' actions $l = \{l(1), \ldots, l(I)\}$ of the current period are then determined (see Section 3.4). Finally, given σ and l, every player receives his payoff and uses this new information to upgrade his beliefs (see Section 3.5). This concludes the period. In the following we describe these steps in more details[6].

2.1 Beliefs and agents' type

In this game agents play their "best" strategy *given* their expectations on others' behaviour; according to the two parts of a strategy, agents have two forms of expectations:

1. the probability $p_i^t(N|S)$ that an agent i assigns to the event in which a coalition of size N forms in period t if he communicates his willingness to participate in coalitions of size at most S.
2. the probability $Q_i^t(n|N)$ with which an agent i, in period t, expects to find n other cooperators ($0 \leq n \leq N - 1$) in a coalition of N persons ($1 \leq N \leq I$); to keep things as simple as possible, we assume that
   (a) in any period t, an agent i can assess the probability $q_i^t(N)$ with which another randomly chosen agent will be a cooperator in a coalition of cardinality N, and that
   (b) in any period t, the probability $Q_i^t(n|N)$ is the value at n of the binomial distribution with probability $q_i^t(N)$ and N - 1 trials[7].

[6] The software used for the simulation is available at the authors' websites (http: //www-dse.ec.unipi.it/fiaschi and http: //www-dse.ec.unipi.it/pacini).

[7] This assumption is an intuitive way of modelling the probability distribution of cooperators within a coalition, when agents are identical; in short we assume that the probability distribution of cooperators is always binomial. Experience could show that this distribution is not binomial and a possible extension is to consider more complex distributions

It is straightforward to show that the more optimist an agent is (i.e. the higher the value of q_i) the greater is the cardinality of the maximal coalition in which i is ready to cooperate (clearly always within the limit of $\bar{N}(\alpha)$). If we take the values of the parameters as in Example 1, it is easy to verify that an agent will cooperate in any coalition up to cardinality 3 when $1 \geq q_i'(N) \geq 0.77$, while he will sustain cooperation in coalitions of cardinality up to 2 when $0.77 > q_i'(N) \geq 0.39$; finally for lower values of $q_i'(N)$ i will cooperate only in the singleton coalition. We will use this correspondence between beliefs about others' cooperative attitude and individual behaviour to classify agents in three different types, i.e.:

- *optimistic* agents: agents of this type assign high confidence to the fact that other people are cooperators in a formed coalition; in particular we assume that an agent has optimistic beliefs when $1 \geq q_i'(N) \geq 0.77$. As we have seen, agents of this type are ready to support cooperation in larger coalitions, so that we can say that optimistic beliefs sustain a *strongly associative* (**SA**) behaviour.
- *mildly optimistic* agents: agents of this type assign a lower confidence to the cooperative attitude of the other agents and, in particular, beliefs are mildly optimistic when $0.77 > q_i'(N) \geq 0.39$. As we have seen, agents of this type sustain cooperation only in lower coalitions beliefs, i.e. they will give rise to what will be termed a *weakly associative* (**WA**) behaviour.
- *pessimistic* agents: agents of this type assign a very low probability to the fact that anyone else will be a cooperator in a formed coalition and we will assume that their beliefs are such that $0.39 > q_i'(N) \geq 0$. Again we know that for these values of the probabilities of cooperation anyone will cooperate only in the singleton coalition, i.e. he will sustain only a *non associative* (**NA**) behaviour.

In the sequel the terms (i) optimistic and strongly associative, (ii) mildly optimistic and weakly associative, (iii) pessimistic and non associative will be used as synonyms when referred to agents.

2.2 Characteristics of individual expectations

As we have seen in the previous paragraph, expectations of an agent i about the possible behaviour of other agents are summarized by two probabilities referring respectively to the two components of his strategy. The structure of these two expectations will be characterized by means of the following assumptions:

(e.g. gamma); for the sake of simplicity we restrict ourselves to the simpler case of the binomial.

H 1. $p_i^t\left(N|S\right)$ is the probability with which agent i believes that a coalition of cardinality N will form *conditional* on the fact that in period t he played S as the first component of his strategy and, for any t, it satisfies the following conditions

a. $\sum_{N=1}^{S} p_i^t\left(N|S\right)=1$;

b. $p_i^t\left(N|S\right)=p_i^t\left(N|S+z\right)$ for all S, N and z such that $S>N$ and $z>0$.

H 2. $q_i^t\left(N\right)$ is the probability with which agent i expects any other agent j ($j\neq i$) belonging his coalition of cardinality N to have an 1 in the N^{th} position of his own strategy, i.e. $\theta_N''\left(j\right)=1$, and we assume that:

$$q_i^t\left(N+z\right)\leq q_i^t\left(N\right) \text{ for all } t\geq 0,\ I\geq N\geq 1,\ z\geq 1.$$

H 3. The probabilities $p_i^t\left(N|S\right)$ and $q_i^t\left(N\right)$ are *independent.*

Assumption H1.a is nothing but the assertion that $p_i^t\left(N|S\right)$ is a probability distribution over the interval [1, S]. Assumption H1.b is a sort of independence of irrelevant alternatives: the probability of forming a coalition strictly smaller than i's willingness to accept a given cardinality must not change if i increases this willingness; in other words the formation of a small coalition is due to someone else behaviour and this is not affected when i shows ready to accept greater coalitions.

Assumption H2 on individual beliefs about cooperation aims at guaranteeing that expectations are consistent: indeed it grants that the incentives to deviate, given the number of cooperators, are always not decreasing in the coalitional size.

Finally Assumption H3 states that the expected size of a coalition and the expected number of cooperators in that coalition are independent. The presence of increasing returns to scale could question this assumption.

2.3 Choice of strategy

Given individual expectations $q_i^t\left(N\right)$ (and hence $Q_i^t\left(n|N\right)$) and $p_i^t\left(N|S\right)$, agent i can compute his expected payoff of any feasible strategy. Let $EU_i(\theta)$ be the expected payoff of a strategy θ[8] and denote by θ^* the best strategy for i, i.e.

$$\theta^*=\arg\max_{\theta\in\Theta} EU_i\left(\theta\right) \tag{1.1}$$

The expected payoff $EU_i(\theta)$ to agent i of playing a strategy $\theta=\left\{\theta',\theta''\right\}$ is given by

$$EU_i\left(\theta\right)=\sum_{N=1}^{\theta'}\left[p_i\left(N|\theta'\right)\cdot\sum_{n=0}^{N-1}Q_i\left(n|N\right)\cdot\left(\frac{\left(n+\theta_N''\right)^{\alpha}}{N}+\left(1-\theta_N''\right)\cdot\omega\right)\right] \tag{1.2}$$

[8] Here, when strategies are concerned, the reference to agent i is suppressed for brevity.

To clarify expression (1.2), observe that the term $\frac{\left(n+\theta''_N\right)^{\alpha}}{N}+\left(1-\theta''_N\right)\cdot\omega$ is the utility that i would get in a coalition of N persons if n other agents will cooperate and he behaves according to the prescription of his strategy, as indicated in the N^{th} component of θ''. The probability of finding n other cooperators in i's coalition is $Q_i^t\left(n|N\right)$, so that the term

$$EU_i^N\left(\theta\right)=\sum_{n=0}^{N-1}Q_i\left(n|N\right)\cdot\left(\frac{\left(n+\theta''_N\right)^{\alpha}}{N}+\left(1-\theta''_N\right)\cdot\omega\right) \quad (1.3)$$

is the utility that agent i expects to receive in a coalition of size N, given the beliefs $q_i^t\left(N\right)$ on others' cooperative attitude. Finally, this coalition forms with a probability $p_i^t\left(N|S\right)$ that is the leftmost term in the expression in square bracket in (1.2).

The optimal strategy θ^* is obtained as follows. Let $l^*=\left[l_1^*,l_2^*,l_3^*,\ldots\right]$ be agent i's *optimal contingent plan*, i.e. that part of a strategy in which cooperation/defection is chosen so as to maximize i's expected payoff independently of the coalition size, i.e. any component of l^* is such that

$$l_j^*=\arg\max_{l\in\{0,1\}}\sum_{n=0}^{j-1}Q_i\left(n|j\right)\cdot\left(\frac{\left(n+l\right)^{\alpha}}{j}+\left(1-l\right)\cdot\omega\right) \quad (1.4)$$

The optimal contingent plan $\left[l_1^*,l_2^*,l_3^*,\ldots\right]$ is unique and independent of p_i (the latter does not appear in (1.4)). Therefore the probabilities p_i enters only in the choice of the first component of a strategy, i.e. N, and the optimal acceptable size N^* will be given by

$$N^*=\arg\max_N EU_i\left(\left\{N,l^*\right\}\right)$$

Combining previous results we get the optimal strategy for an agent i

$$\theta^*=\left\{N^*,\left[l_1^*,l_2^*,l_3^*,\ldots\right]\right\}.$$

To rephrase this maximization problem in words we can say that the optimal individual strategy comes out of the following process. Since the decision whether to cooperate or not (i.e. l_j^*) does not depend on the probabilities p_i, the maximization process (1.1) can be divided into two steps: (i) agent i chooses the best action as to cooperation, independently of any coalition size (1.2) and then (ii) agent i chooses the coalition size which maximizes his expected payoff (1.3), given the actions that he has already chosen to perform and the expectations on others' actions.

In order to introduce some degree of randomness in individual choices, we assume that agents will play the optimal strategy θ^* with a probability π, the second best strategy in terms of expected payoff with probability $(1 - \pi)\cdot\pi$, the third best with probability $(1 - \pi)\cdot\pi^2$, etc., and finally the worst one with probability $1-\pi-(1-\pi)\sum_{i=1}^{l-1}\pi^i$.[9]

2.4 Coalition formation

Once strategies are announced, agents randomly match. If two agents i and j match and $\min\{\theta'(i),\theta'(j)\} \geq 2$ then the coalition $S = \{i, j\}$ forms; this coalition is ready to accept another (randomly chosen) agent h provided $\min\{\theta'(i),\theta'(j),\theta'(h)\} \geq 3$, otherwise h will be the first member of a new coalition S' and so on. A **CS** σ is obtained when $\Im$ is partitioned in groups S in such a way that $\min\{\theta'(i)\}_{i\in S} \geq |S|$ ($\forall S \in \sigma$) and there is no couple of groups S and S' in σ such that $\min\left\{\min\{\theta'(i)\}_{i\in S}, \min\{\theta'(i)\}_{i\in S'}\right\} \geq |S|+|S'|$, i.e. a **CS** σ is formed whenever no agent is compelled to participate into a coalition of greater cardinality than the one he is willing to accept and no two groups are compelled to remain separated when they could join without the objection of any participant.

2.5 Update of beliefs

Individual expectations are revised through time. Starting from prior beliefs, posterior distributions are formed taking into account observations; the latter are relative to the local experience of an agent, i.e. he can know only the cardinality of the coalition he happens to be into and the profile of actions of the other members. We assume that the mechanism governing the process of revision of expectations is of a very simple type and takes the characteristics of an *adaptive learning*.

To formalize this process of beliefs revision let us start considering the expectations $q_i^t(N)$ about the cooperative attitude of the other agents and suppose that, at at the end of period t a **CS** σ^t is formed; let denote by $H_i^t = \dfrac{n_i^t}{N_i^t}$ be the proportion of cooperators in the coalition which agent i belongs to (n_i^t is the number of cooperators and N_i^t the size of the coalition of agent i). Then we assume that

[9] Indeed this randomness has a twofold purpose; on the one side it tends to represent possible mistakes in performing the complicated calculations implicit in (1.1); on the other hand it allows agents to experiment with their strategies, verifying the performance of actions that pure rationality would have excluded; this will play a role in the individual learning process discussed later on in the text.

$$q_i^{t+1}(N) = \begin{cases} \left(1-\delta^{COOP}\right)\cdot q_i^t(N) + \delta^{COOP}\cdot H_i^t & \text{if } N = N_i^t \\ q_i^t(N) & \text{otherwise} \end{cases} \tag{1.5}$$

where δ^{COOP} is a parameter measuring the importance of experience in updating the beliefs about cooperation. Finally, according to Assumption H2 we check that the probability of cooperation is not increasing in the size of coalitions, i.e. $q_i^{t+1}(N+j) \le q_i^t(N)$ for $j > 0$; if it happens $q_i^{t+1}(N+j) > q_i^t(N)$, then we set $q_i^{t+1}(N+j) = q_i^t(N)$. Therefore experimenting a coalition of size N can affect not only $q_i^{t+1}(N+j)$, but also $q_i^{t+1}(N+j)$ with $j > 0$.

In the same manner every agent revises $p_i^t(N|S)$ in light of experience. In particular we assume that

$$p_i^{t+1}(N|S) = \begin{cases} p_i^t(N|S) + \delta^{CARD} & \text{if } \theta''(i) = S \\ p_i^t(N|S) & \text{otherwise} \end{cases} \tag{1.6}$$

where δ^{CARD} is a parameter measuring the importance of experience in the formation of beliefs about the possible cardinality of a coalition. Then we normalize the new beliefs as required by Assumption H1.a. If the new beliefs violate Assumption H1.b we correct them accordingly. Therefore playing a strategy $\theta' = S$ and experimenting a coalition of size N affect not only $p_i^{t+1}(j|S)$ with $j = 1,\ldots,S$, but also $p_i^{t+1}(j|z)$ with $2 \le z \le I$ and $1 \le j < z$.

Equations (1.5) and (1.6) define a dynamic process of beliefs revision; what is needed is the specification of the initial conditions, i.e. the prior beliefs $p_i^0(N|S)$ and $q_i^0(N)$ with which agents start interacting. To this purpose we assume

H 4. At period 0

a. $p_i^0(N|S) = \frac{1}{S}$, $\forall S \in [1,\ldots,I]$ and

b. $q_i^0(N) = \begin{cases} 0.9 & \text{for optimistic agents} \\ 0.6 & \text{for mildly optimistic agents} \\ 0.3 & \text{for pessimistic agents} \end{cases}$.

By means of H4.a we assume that agents, prior to presenting themselves to the interaction process, are absolutely uncertain about the possible realizations out of their own choice of an acceptable coalition. Assumption H4.b determines conveniently the initial agents' types, choosing the initial belief about cooperation of any type as the mid value of the relevant intervals of q (see Sections 2.1).

3 Computational Experiments and Results

Our computational experiments can be divided into two parts; in the first we analyse how a population of agents with heterogeneous initial beliefs evolves once the parameters π, δ^{COOP} and δ^{CARD} are kept fixed (see Section 3.2). In the second we analyse how our findings are robust to changes in the degree of learning speed (i.e. δ^{COOP} and δ^{CARD}). In any case we run several simulations of 1000 periods (i.e. $t = 1,...,1000$) changing, for a given distribution of initial beliefs, the seed of random numbers. This is necessary to eliminate possible random disturbances, deriving from the fact that agents' strategies are chosen with a certain randomness.

3.1 Parameterization

For simplicity we limit our attention to an economy with just 16 agents; even if the economy is so simple, the set of possible strategies is very large being made up by 2^{20} elements. As to the parameters of the model we set $\alpha = 1.428$ and $\omega = 0.635$; these values respect all the constraints of Section 1, i.e. $1 < \alpha < \alpha^*$ (in the present case the condition $I^{\alpha^*} - (I-1)^{\alpha^*} - \omega \cdot I = 0$ gives $\alpha^* = 1.6608$) and $\frac{1}{2} < \omega < 1$. Given these parameters, the cardinality of the greatest coalition capable of sustaining cooperation is $\bar{N}(\alpha) = 3$, so that every partition of 16 agents in coalitions with 3 or less agents with full cooperation are Nash equilibria. The **CPNE** is given by 5 coalitions of 3 agents and 1 of one agent; it is easy to check that this configuration implies maximum aggregate output, subject to the individual incentive constraints. Finally we set $\pi = 0.98$.

3.2 Equilibrium selection

In this section we study the profile of equilibrium strategies as a function of agents' initial beliefs. To this purpose we set $\delta^{COOP} = \delta^{CARD} = 0.1$ and consider the 153 different compositions of the initial population defined on the basis of agents' initial beliefs (any possible permutation of 16 agents for the possible three types of initial beliefs). For every possible composition of the initial population we ran 50 simulations, each time modifying the seed of the random numbers, so that we have $153 \times 50 = 7650$ observations. The results are reported in the following figures.

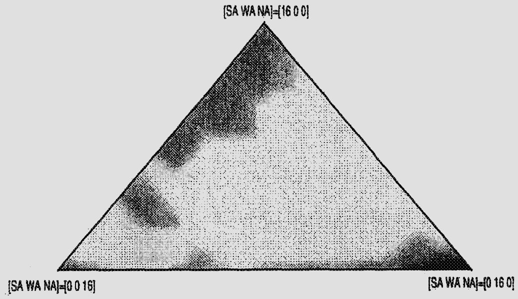

Fig. 1. Rest points of the simulations in terms of the composition of the final population

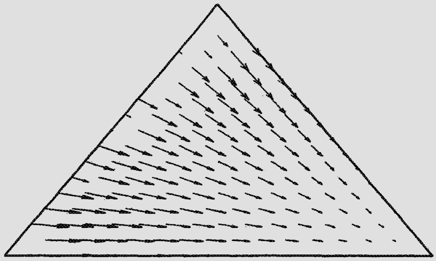

Fig. 2. Direction and magnitude of changes from every initial population

Figure 1 reports in a three dimensional simplex the frequency of the rest points of the simulations. Every vertex corresponds to a population composed by only one type of agents; in particular the vertex on the top corresponds to a population with only **SA** agents, the bottom-right vertex corresponds to a population only made up only by **WA** agents and, finally, the bottom-left vertex corresponds to a population with only **NA** agents. Every point in the simplex corresponds to a mixture of these three types of agents and the darker the colour of that point the higher the frequency with which the corresponding combination of types occurs in the simulation. Figure 1 shows that most of the rest points are characterized by a population of **WA** agents and only few simulations converge to a population with a significant amount of **SA** agents. In particular the shadow zone on the top of the triangle collects less than 3.2% of the total rest points, the one on the bottom-left 2.6%, while the shadow zone on the bottom-right more than 88.8%. We can notice that populations composed only by **SA** and **NA** agents (all the rest points near to the left edge of the simplex) and those with only **WA** and **NA** agents (all the rest points near to the right edge of the simplex) are the most frequent, while population composed by **SA** and **WA** agents show difficult to coexist (all the rest points near to the bottom edge). However Figure 1 does not show how the rest points are related to the initial composition of the population.

This is accomplished in Figure 2 where the direction and the magnitude of change from each initial population (the longer the vector the stronger the change) are portrayed. As we can observe, the equilibrium with a majority of **WA** is the most likely result from any initial population with the exception of some population composed only by **SA** and/or **NA** agents.

To quantify this phenomenon we calculate the number of times a simulation converged to a given point in the grid (the possible rest points) and consider only those for which this number is greater than 50 out of the total number of simulations. We find that most simulations converge to three main sets, one on the bottom-left, denoted as **A**, one on the bottom-right, denoted as **B**, and the last one on the top, denoted as **C**.[10] Figure 3 reports these results.

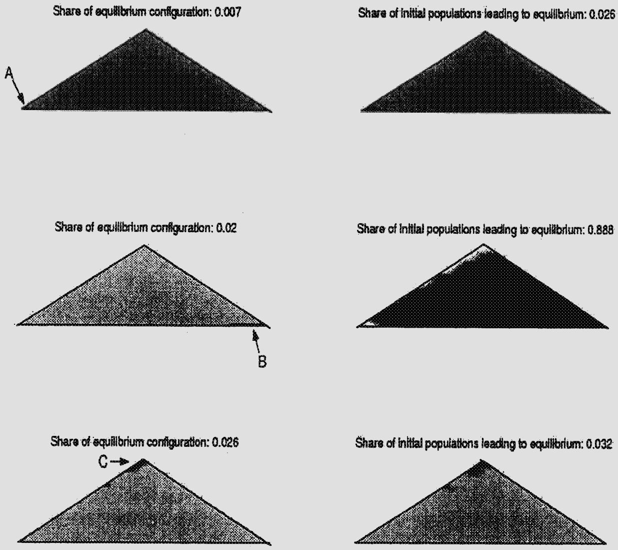

Fig. 3. Attractors and basins of attraction

The left column in Figure 3 represents the three attractor sets (**A**, **B** and **C**) while the right columns reports the respective basins of attraction; the depth of the grey in any point is proportional to the probability with which the corresponding initial population converged to the corresponding attractor set on the left; we can notice that the basin of attraction of the set **B** is absolutely greater than those corresponding to the sets **A** and **C** (precisely 88% of all possible initial conditions showed convergence toward the set **B**).

From Figures 2 and 3 we see that the presence of few **WA** agents can be sufficient to lead to a rest point where **WA** agents are the majority. In particular mixed populations with **SA** and **WA** agents converge to a population of only **WA**, while mixed population of **NA** and **WA** agents converge to a population of a majority of **WA** agents with a limited number of **NA** agents (in Figure 3 this is represented by set **B**). This result is due to the free-riding behaviour of **WA** agents in coalitions of size 3 (indeed this is their best action, given their beliefs), which causes the **SA**

[10] Notice that these three attractor sets count for more than the 93% of the total number of simulations.

agents to revise their beliefs downturn. On the other hand, **NA** agents quickly learn that cooperation is the best action in a coalition of size 2, when they happen, by a random error, to experiment this situation.

Attractor set **C** shows that mixed population of **NA** and **SA** agents are possible if initial population is composed by a majority of **SA** agents. In this case the high number of **SA** agents allows the formation of large coalition among themselves, thus limiting the possible free-riding behaviour of **NA** agents. Finally, attractor set **A** is made up by a population with all **NA** agents; here their free-riding behaviour discourages any form of cooperation. We note that its basin of attraction is very limited (see Figure 3).

To summarize **WA** behaviour seems to be an *evolutionarily stable strategy*, while **SA** behaviour tends to be stable if initial populations include at most a **WA** agents and a very limited number of **NA** agents. We further investigate this point in the next Section.

3.3 Agents' learning speed

There are important contributions in the literature showing the importance of agents' learning speed (see e.g. Gale et. al (1995)). In our context the speed of learning is measured by δ^{COOP} and δ^{CARD}. In particular we take into account 8 different initial populations (the most interesting to our scope) and for each of them we consider 30 different values of $\delta = \delta^{COOP} = \delta^{CARD}$ in the range [0.005, 0.15]. For each value of δ we ran 20 simulations. Figure 4 shows the results in terms of average payoffs for each initial population.

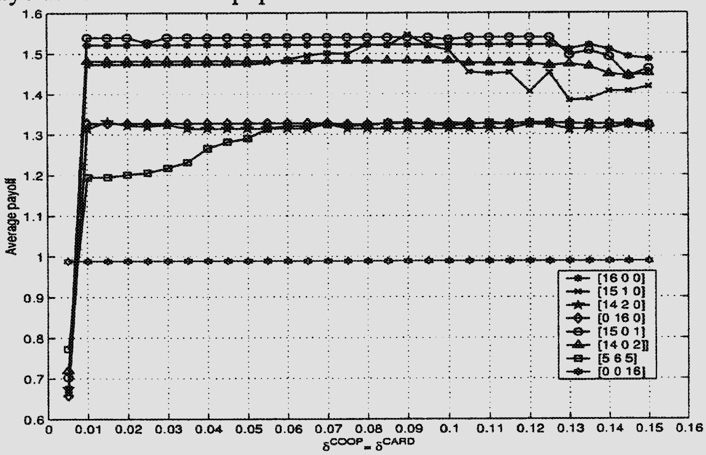

Fig. 4. Average payoffs for different initial population and for alternative values of the learning speed.

The initial population with only **NA** agents (in Figure 4 denoted as [0 0 16]) converges to a population made up only by **NA** agents independent of the learning speed (in fact 1 is the payoff of staying alone). Differently, the rest points corresponding to an initial population of 5 **SA** agents, 6 **WA** agents and 5 **NA** agents (in Figure 4 denoted as [5 6 5]) are characterized by different associative behaviour according to the level of the learning speed. An increase in the latter from 0.03 to 0.07 leads to an increase in the average payoffs because every agent easily learns a **WA** behaviour. On the contrary, when the learning speed is very low (i.e. $\delta^{COOP} = \delta^{CARD} < 0.02$), **NA** agents are unable to learn more favourable strategies, though they may experience them by chance.

The analysis of the rest points from initial populations with a majority of **SA** agents and a small number of **WA** or **NA** agents (in Figure 4 denoted as [15 1 0], [15 0 1], [14 2 0] and [14 0 2]) provides the intuition that **SA** strategy is stable if in the population there are no more than 2 **NA** agents or 1 **WA** agent independently of the learning speed. In terms of evolutionary game theory these values can be interpreted as the threshold levels of mutation allowed to maintain **SA** behaviour as an evolutionary stable strategy. Indeed if initial population were composed by 14 **SA** agents and 2 **WA** agents (in Figure 4 [14 2 0]), all rest points would be characterized by the presence of an entire population of **WA** agents. On the contrary, if the initial population is composed by 15 **SA** agents and 1 **WA** agents (in Figure 4 denoted as [15 1 0]), all rest points are characterized by a population in which most agents are **SA** with the exception of possibly some **WA** agents. Here learning speed plays a crucial role; indeed an increase in the latter causes initially an increase and then a decrease in the share of **SA** agents in the population. This pattern can be explained considering that an increase in the learning speed makes **WA** agents more able to readily learn the advantages of cooperation in a coalition of size 3 (when they happen to match **SA** agents), but a too high level of this learning speed has a contrary effect on the cooperative attitude of the **SA** agents since it induces a sharp reduction of the expected payoff of cooperation when a **SA** agent matches a **WA** agent, so that it makes easier the transformation of **SA** agent in a **WA** agent. Finally the intuition behind the fact that **SA** behaviour is more robust with respect to **NA** than to **WA** behaviour lies in that there is a very low probability that a coalition between a **SA** agent and a **NA** agent forms (the former try to form coalition of size 3 the latter of size 1). This implies that a downturn revision of the expectations of cooperation on the side of the **SA** agents is very unlikely and every agent tends to maintain or foster its initial beliefs.

Conclusions

In this paper we have shown that, if agents adjust their behaviour on the basis of their experience, then they play the **CPNE** strategies only if the population starts out with a priori beliefs that are close to those sustaining coalition-proof outcomes; in this case the role of the learning process will be just to size down the dimension of the equilibrium coalitions to the one consistent with individual incentives to cooperate. The composition of the initial population strongly affects the selection of equilibrium. The computational experiments highlight how there are three main attractor sets, the first one characterized by a population composed by a majority of strongly associative agents and a minority of not associative agents, the second one made up of only not associate agents, while the third one is characterized by a population composed by a majority of weakly associative agents and a minority of not associative agents. This suggests that equilibria with a population of strongly associative and weakly associative agents are not "sustainable" because of the free-riding behaviour in large coalitions of the weakly associative agents. Moreover we observe that the possibility that weakly associative agents become strongly associative is limited to the case in which weakly associative agents are a strict minority of the population with a large majority of associative agents (in our analysis 1 out of 16). On the contrary, if the number of weakly associative agents is sufficiently high (more than 2 out of 16), we observe convergence toward an equilibrium in which there are no strongly associative agents. Not associative behaviour tends to disappear if strong associative and/or weakly associative agents form a large majority in the population. Overall weakly associative behaviour seems the most likely result, while the system-wide efficiency corresponding to the emergence and persistence of **CPNE** strategies seems to be relegated to few and very particular initial conditions in terms of the distribution of beliefs.

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On the Creation of Networks and Knowledge

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Summary. This paper examines the evolution of networks when innovation takes place as a result of agents bringing together their knowledge endowments. Agents freely form pairs creating a globally stable matching. paired agents combine their existing knowledge to create new knowledge. We study the properties of the dynamic network formed by these interactions, and the resultant knowledge dynamics. Each agent carries an amount of knowledge of a certain type, and the innovative output of a pair is a function of the partners' endowments and types. We find evidence that the pattern of substitution between quantity and type of knowledge in the innovation function is vital in determining the growth of knowledge, the emergence of expertise and the stability of a number of network structures. Network structure itself exhibits a phase change when the relative importance of diversity compared to quantity increases beyond a threshold value.
Keywords: Knowledge, Networks, Innovation, Stable Matching, Small Worlds.

1 Introduction

This paper is concerned with three issues: knowledge creation; knowledge diffusion, and the creation and evolution of networks.

In recent years many empirical studies of the creation of knowledge argue that innovation is to a very great extent the recombination of existing knowledge.[4] At the same time, economists have observed that the knowledge base underlying a firms productive and innovative activities has broadened. That is, to operate competitively, or to innovate effectively a firm has to have access to more and more types of knowledge. A strategy that firms now commonly employ to address these to problems is for form alliances with other firms for the purpose of producing knowledge.[5]

[4] See for example Kodama (1986,1992); Gibbons et al. (1994); Sutton and Hargadon (1996) or Hargadon and Sutton (1997).

[5] See for example Zimmermann (1995); Cowan and van de Paal (2000). Smith (2000); Hagedoorn (2001) or Antonelli (1999).

M. Gallegati et al. (eds.), *The Complex Dynamics of Economic Interaction*

To model diffusion economists first turned to epidemiology, and adapted the epidemic model of diffusion. Populations mix, any agent being equally likely to contact any other, and innovations spread by "infected agents" contacting "uninfected agents".[6] More recently, however, economists have observed that this sort of global mixing does not describe well the way knowledge or information is passed among agents. Economists now recognize that much knowledge transmission takes place in bilateral or small multi-lateral interactions.[7] If this is generally the case, then the nature and structure of these interactions will be central in determining the speed and extent of diffusion in an economy. Thus, to study diffusion will involve studying the networks over which the diffusion takes place. There are growing numbers of network models in economics now, but many of them take the network structure as fixed, and even if agents have many interactions, they are all with the same small subset of the population. This is reasonable in many instances, but a more general approach, wherein the network is allowed to evolve over time would be more general, since an agent will typically change the set of agents with whom he interacts as he gains experience about where the highest rents lie.

The literature contains models in which networks evolve.[8] This literature tends to have an equilibrium approach, with rational agents whose actions produce a static equilibrium, in which the network structures that emerge tend to be driven by the costs of form ing links. There is a small contrasting literature that is more evolutionary in spirit.[9] In this literature agents learn about the value of interacting with other agents over time, and the network evolves, typically through some form of reinforcement learning. The present paper includes features of both approaches. In this paper we treat networks as evolving as agents interact with each other, changing their own and each other's characteristics and consequently changing partners as time passes. In the model we develop here, firms form alliances in order to create knowledge, and in so doing, transmit knowledge between them. Firms can change partners, and this is the mechanism that provides the global diffusion of knowledge. In modelling partner choice we use the stable Marriage Matching algorithm of Gale and Shapley (1962), but because agents characteristics change as a result of these partnerships, the network can continually evolve. Thus this model lies between the pure search for Nash equilibria of evolutionary game theory and the melioration models of social network dynamics.

In this paper we develop a model in which each period every firm in the population seeks a partner for knowledge creation. We restrict attention to bilateral partnerships. Firms come together and pool their existing knowledge

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[6] See for example Griliches (1957) or Mansfield (1961).

[7] See for example David and Foray (1992); Valente (1995); Steyer and Zimmermann (1998); or Cowan and Jonard (1999 and 2000).

[8] See for example Debreu (1969); Haller (1994); Qin (1996); Bala and Goyal (1998, 2000) (1997). For a recent survey of the game theoretic work, see Jackson and Dutta (2001).

[9] see for example Plouraboué et al. (1998); or Weisbuch et al.

to create new knowledge which they then both absorb. Partnerships last for one period exactly, at the end of which each firm seeks a new partnership, possibly with the previous partner. That firms search for (possibly) new partners each period implies that over time a network of knowledge flows can form, as firms create links with several different firms. The central question in this paper has to do with the structural properties of these networks and whether the properties change under different knowledge production technologies.

When innovation is the outcome of agents' pairwise interactions, rational agents will seek partners to maximize the expected outcome of the partnership. Put another way, at each period of time, an agent will rank each of his potential partners on the basis of the expected benefits from that interaction. The issue then is to divide the population into $n/2$ pairs such that each agent's partner is the best he can have, given everyone else's preferences. This is known as the room-mate problem. In general stable matches are not guaranteed in the room mate problem, but as we show below in our model they are. This implies that in every period the network of paris that forms is stable in a Nash sense.

2 The Model

Innovation and pair formation take place in parallel. Regarding innovation, individuals' knowledge is combined to produce new knowledge. A production function encompassing both agents' amounts and types of knowledge determines how much new knowledge is generated. Agents rank each other depending on the expected output of a partnership. Hence rankings change over time because innovation changes individual knowledge profiles. Two broad categories of issues are examined: the process of network formation; and the properties of knowledge growth.

2.1 Knowledge and matching

Consider a finite set $S = \{1, \ldots, n\}$ of individuals engaged in repeated interactions. Each individual $i \in S$ is characterized by a real-valued knowledge endowment, v_i, in the form of a quantity of knowledge $\rho_i > 0$ and a composite index representing a type of knowledge $\theta \in (0, \pi/2)$: $v_i = (\rho_i, \theta_i$. This allows a very simple polar representation of individuals, as well as the existence of a continuum of knowledge types. Individual knowledge level and type are randomly drawn from a uniform distribution over $(0, 1) \times (0, \pi/2)$ at the beginning of the process. At each period, agents seek partners with whom to innovate, after which knowledge is absorbed and agents start again a round of matching, based on their new knowledge profiles.

Innovation then, is a process that combines the knowledge of two agents and produces new knowledge. In this process, all else equal, more knowledge will be better. But diversity of knowledge inputs may also matter. Two agents whose expertise lies in exactly the same field are likely not to find many synergies. Thus in general diversity is beneficial to the innovative process, and enters positively into the production function. To fix ideas we model this as a constant returns Cobb-Douglas function, the arguments being the lengths of the knowledge vectors of the two agents (how knowledgeable they are), and the extent to which their expertise differs, that is, the difference between their knowledge "angles". The extent to which diversity matters is a parameter we vary as the exponent on the "difference in angles" argument in the Cobb-Douglas function.

2.2 Knowledge and innovation

There are many ways to characterize knowledge, none of them without its pitfalls. For our purposes two aspects of knowledge are important—quantity and type. Recent views of innovation as recombination make it imperative that any formal characterization of knowledge permit that equally knowledgeable agents may know different things. That is, the knowledge of an agent comprises many different types of knowledge. One approach then would be to formalize knowledge as a long vector, each element representing a different type of knowledge. Agents differ then along many dimensions. This proves cumbersome in implementation however, and is more detailed than necessary for the issues we are exploring here. A simplification of this approach is to represent an agent's knowledge as a pair: one element signifying a quantity of knowledge; the other signifying a type. This representation has the drawback that it takes what is clearly a complex thing, an agent's knowledge stock, and represents it as a pair. But it has the great benefit of simplicity; it is easily generalized; and does capture some important elements of the nature of knowledge.

A representation of the innovation process must satisfy several minimal requirements. Consider two individuals i and j with knowledge stocks (ρ_i, θ_i) and (ρ_j, θ_j) respectively. Suppose they are paired in a stable matching. As innovation is jointly conducted, after innovation has taken place, the following should be true:

- the knowledge amounts held by i and j have increased;
- the knowledge types of i and j have changed;
- the distance between the knowledge types of i and j has fallen.

In a geometric sense, both vectors are longer after the innovative episode, and the angle between them has decreased. A fourth assumption is necessary for the description of innovation to be complete:

- most often, innovation produces knowledge of a type which is "between" the types of the two contributors; however, innovation sometimes produces very different knowledge.

Operationally, each pair of agents creates an amount of new knowledge determined by the production function, and this amount is simply added to their existing knowledge endowments. The type of knowledge determined is a random variable $\widetilde{\theta}$ distributed according to a truncated normal, with mean, $\overline{\theta} = (\theta_i + \theta_j)/2$ and standard deviation s. The knowledge type of an individual is modified by innovation and becomes a weighted mean of his previous knowledge type and the type of new knowledge generated by innovation. This accounts for the relative importance of old and new knowledge in an individual's knowledge stock.

The standard deviation of $\widetilde{\theta}$ is a parameter we vary. If a technological system is deeply entrenched in a trajectory, we would expect the standard deviation to be small. On the other hand, in a case in which there is wide scope for technological discovery in many areas, the standard deviation of this distribution would be large. The smaller the standard deviation the more likely the new knowledge will lie "between" the knowledge of the partners. Varying this parameter permits us to explore the dynamics of these different situations.

Formally, the innovation function is $r : S^2 \to \mathbb{R}_+$, with

$$r(i,j) = |\theta_i - \theta_j|^\beta (\rho_i \rho_j)^{1-\beta}, \beta \in (0,1).$$

Parameter β measures the importance of diversity in knowledge types in the innovative process. Individual i's new knowledge endowment is simply $\rho_i(t+1) = \rho_i(t) + r(i,j)$, and given the innovation is in type $\widetilde{\theta}$-knowledge, i's new knowledge type is

$$\theta_i(t+1) = \frac{\rho_i(t)}{\rho_i(t+1)}\theta_i(t) + \frac{r(i,j)}{\rho_i(t+1)}\widetilde{\theta},$$

the weighted mean of his previous knowledge type and the new knowledge type, where the weights are the relative amounts of knowledge involved.

2.3 The knowledge dynamics

Each period pairs are formed, innovation takes place and endowments change. At the end of the period the partnership is dissolved. This process is repeated, and we investigate the long run behaviour of the system. In principle knowledge vectors grow in length indefinitely, but types of knowledge are bounded between 0 and $\pi/2$. Eventually, in the creation of knowledge, only amounts matter as that argument will dominate in the production function. This is clearly unreasonable. To solve this problem we re-normalize every period so that the population average knowledge vector has length one. What this implies is that data on aggregate knowledge levels have no meaning. However, we can still reasonably measure growth rates, and variance in knowledge levels across agents. In examining knowledge, two issues are of concern: its growth and its distribution. and allocation of knowledge in this economy. The economy's knowledge growth rate is simply the growth rate of the average knowledge level per period. It will be evaluated along the simulation horizon. Since the mean level of knowledge is by definition 1, the variance of knowledge levels $\sigma^2 = \frac{1}{n}\sum_i \rho_i^2 - 1$ provides a good measure of the equality of the distribution.

2.4 Network dynamics

A relational network exists in this economy, but in a dynamic rather than a static sense. To study its properties, we focus on the frequency of contacts (in practice, the frequency of meetings over 2,000 rounds). $G(S, V_t)$ is the graph associated with the stable matching achieved at time $t = 0, 1, 2, \ldots$, that is $V_t(i,j) = 1$ if $(i,j) \in \Gamma_t$ and $V_t(i,j) = 0$ otherwise. The weighted graph recording past interactions is denoted $G(S, W)$, where $W(i,j) = W(j,i)$ is the frequency of activation of the connection between i and j over the final ℓ periods. For this graph several quantities are of interest. We study the frequency distribution of collaborations and, following Watts and Strogatz (1998), two structural parameters: the average path length and the average cliquishness.[10]

The value associated to a path between i and j is the product of the interaction frequencies along that path. The length of the shortest path (or distance) between any pair of individuals is the number of interactions in the path having largest value. Formally, defining $d(i,j)$ as the distance or length of the shortest path between i and j, the average path length is

$$L = \frac{1}{n}\sum_i \sum_{j \neq i} \frac{d(i,j)}{n-1} \tag{1}$$

and simply measures how distant vertices are on average, which is a global property of the graph. Average cliquishness C is a measure of local connectivity capturing the share of active links between any given vertex's neighbours. It is written

$$C = \frac{1}{n}\sum_i \sum_{j,l \in W(i)} \frac{W(j,l)}{n_i(n_i - 1)/2}, \tag{2}$$

where $n_i = \#\{j \mid d(i,j) = 1\}$ is the size of i's neighbourhood. These statistics give a reasonably complete description of the structural properties of the underlying network. We add one simple measure, namely the average degree of the graph $D = \frac{1}{n}\sum_i n_i$, as a measure of the density of the interaction structure.

[10] If one thinks of social networks representing friendship, both have intuitive interpretations. The path length is the number of friendships in the shortest chain connecting two agents. Cliquishness reflects the extent to which the friends of one agent are also friends of each other.

3 Existence: Stable Matching

Before turning to the emergence of network structure and the associated knowledge dynamics, we discuss the matching mechanism present in this model. Because any pair of agents assigns the same cardinal value to their match, a unique stable match always exists. We can prove this proposition by construction.

Proposition 1. *For any innovation function $r : S^2 \to \mathbb{R}_+$ generating a strict preference ordering $\succ$, the matching problem $(S, \succ)$ has a unique stable matching μ.*

Proof The algorithm to construct the stable matching is as follows. Let $S_0 = S$ and $\mu_0 = \{\varnothing\}$. Consider the roommate matching problem $(S_0, \succ)$, where the profile of preference orderings $\succ$ is defined by the innovation function r, as stated in equation (??). As preferences are strict, there is a single pair (a_1, b_1) such that $r(a_1, b_1) = \max_{(i,j)\in S_0^2, i\neq j} r(i,j)$. No matching which does not involve this pair could be stable, as a_1 and b_1 both prefer each other to any other person they might be matched with. Hence the pair (a_1, b_1) is necessarily part of a stable matching. Let $\mu_1 = \mu_0 + (a_1, b_1)$ and $S_1 = S_0 - \{a_1, b_1\}$. Consider then the new matching problem $(S_1, \succ)$. It only involves $n-2$ individuals with exactly the same preferences (modulo those concerning a_1 and b_1) as before this operation. Again there is a single pair (a_2, b_2) such that $r(a_2, b_2) = \max_{(i,j)\in S_1^2, i\neq j} r(i,j)$. Let then $\mu_2 = \mu_1 + (a_2, b_2)$ and $S_2 = S_1 - \{a_2, b_2\}$. Repeat until everyone belongs to a pair: the set $\mu \equiv \mu_{n/2}$ is a stable matching as no pair can blockade it, and it is unique as the sequence $(a_1, b_1), (a_2, b_2), \ldots, (a_{n/2}, b_{n/2})$ is uniquely defined. □

In case of a tie (that is to say when individual i can achieve the same innovative output with two or more different partners), existence is still guaranteed but uniqueness is lost. The elimination algorithm to find a stable matching is unchanged, except that when two (or more than two) pairs achieve the same output, one of them is picked up randomly and the procedure just iterates.

4 Numerical Analysis

There are in the model two parameters that could affect the evolution of the network structure: s, the standard deviation of the distribution from which the angle of innovation is drawn; and β, the value of diversity in knowledge production. In the experiments that follow we simply examine two representative values for s, running each experiment twice, once for high and once for low s. The effects of β we examine in more detail. Random values of β are drawn from a uniform distribution over $(0,1)$. For each β value $n = 100$ individuals repeatedly interact over a horizon of length 3,000 periods. We build a data sample of 100 different β-values and two s-values and the associated statistics. We extract the underlying structure from these data using non-parametric estimation techniques.

4.1 Network dynamics

The emergence of network structure can be described using three indicators: local order or cliquishness; path lengths; and density or degree of the graph. We examine the effects of two parameters, β, a measure of the importance of diversity in innovation, and s, which measures the extent to which new knowledge lies between the expertise of the innovators. In the discussion that follows, we restrict ourselves to the 0/1 graph. Analyzing the weighted graph, using the measures discussed above creates a problem of interpretation, since the measures are not bounded. Graphically, however, the patterns are identical to those produced by analysing the 0/1 graph.

Density

Figure 1 plots the relationship between density, as measured by the average degree, and β for two values of s, $s = 0.1$ (focussed trajectory) and $s = 3$ (diversified trajectory).

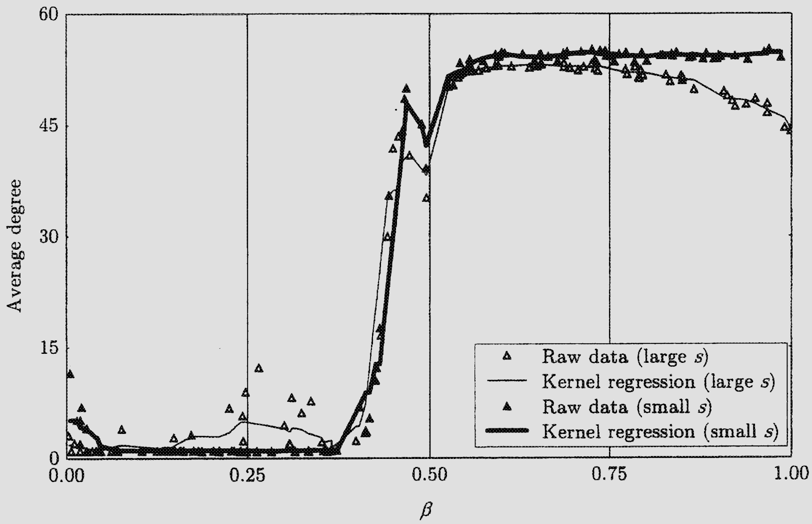

Fig. 1. Average number of connections per individual

What we observe is that there is a phase change in the structure of the graph for $\beta^* \approx 0.4$. When $\beta \leq \beta^*$ the degree is very low. This implies that agents in general have had very few partners over history, and have tended to interact repeatedly with the same partners. For $\beta > \beta^*$, the network is very different. Agents interact with many different partners over time, no longer sticking to the same ones. It is worth mentioning that for small values of β (for both values of s) and for β close to 0.25 in the case $s = 0.1$, the degree can be significantly larger than 1 although not systematically. This suggests a small region (roughly speaking for $0.2 < \beta < 0.4$) where two types of regimes can exist: a world of pairs (degree equal to one) and a more complex world of degree above 1 and below 15, depending on the path taken by the process. After β^* by contrast there is only one regime with degree around 50. Also note that when β is further increased after 0.75, the degree tends to decline when innovation is very localized (small s), while it remains constant when innovation is less focussed (large s).

In Figure 2 we depict the number of individuals involved only in pairs that are stable over time, i.e., of individuals having a unique life-long innovation partner.

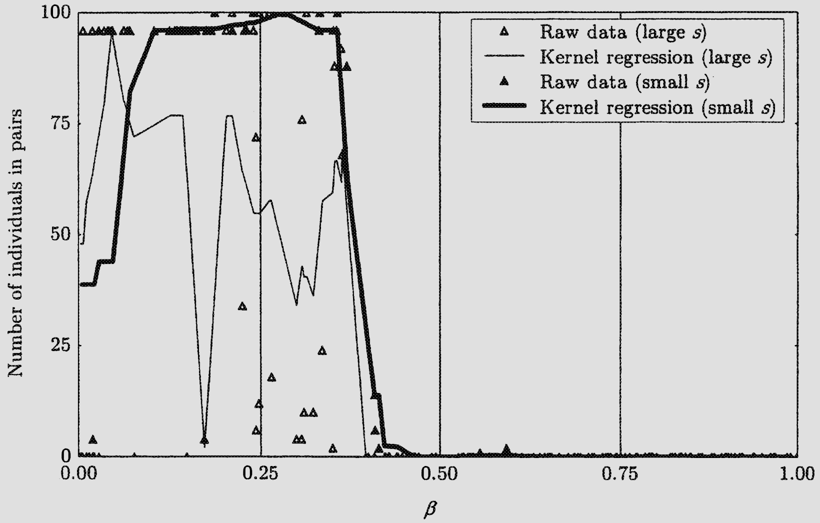

Fig. 2. Number of individuals involved in pairs

The phase change that exists for the degree is also present when the number of paired individuals is considered, with a sharp decline from a large number (between 96 and 100) below β^* to 0 for $\beta > \beta^*$. There is also some noise in the region $[0.2, 0.4]$, with a number of paired individuals that doesn't follow a clear pattern, a phenomenon similar to what we observed for the degree on Figure 1. Above β^* persistent pairs do not form. Interestingly, for very small β and a diversified innovative trajectory (large s), no pairs form and, as we will see when the path length is considered, we even get a connected graph.

Path Length

We see in Figure 3 the same phase change, at the same value of β^* in average path length as in the density graph.

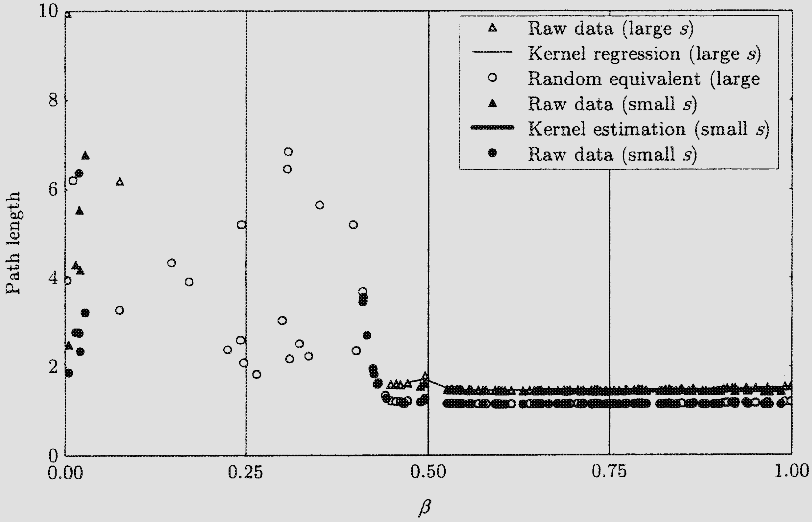

Fig. 3. Average path length between any pair of individuals

For $\beta < \beta^*$, the graph is disconnected. That is, there are isolated groups of agents who interact within groups but not across groups. In principle, this makes the average path length infinite.[11] To keep the figure informative, we only consider a small range over the y-interval so between 0 and β^* most of the points are invisible. When $\beta > \beta^*$, path length is very small. The lowest

[11] Recall that average path length refers to the average path length between pairs of nodes in the graph. If there is no path between two agents, this creates an infinitely long path between them, making the average path infinite. In our calculation of path length, two disconnected agents have a path length of 10,000 rather than infinity. This makes the calculation possible.

possible path length is 1, and exists only in a complete graph. The networks we obtain are thus not complete (their degree is below 55, and a complete graph must have degree 99 in our case) but do have relatively short paths. What this implies is that knowledge flows relatively rapidly around the graph, and indirect exploitation of distant knowledge, through successive rounds of innovation, can be an active feature of the economy. It is known that for a fixed density, random graphs have very short average path lengths. The formula (which is valid for infinite size graphs) for the average path length over a random graph with n vertices and average degree d is $\ln(n)/\ln(d)$. In Figure 3 we represent the path length of the equivalent random graph (that is the random graph with 100 vertices and a degree a given in Figure 1). The path length in our case is consistently above the path length of the equivalent random graph. The networks that emerge from our dynamic process are not random, but do have path lengths approaching those of a random graph of the same degree. Interestingly, and related to the previous paragraph, it is worth mentioning that in a diversified trajectory (and to a smaller extent in a focussed one) there are situations with low β ($\beta \approx 0$) in which path length is finite (between 2 and 7 in the figure).

Local Order and Network Structures

To describe local order we use the standard measure of cliquishness: the share of my partners who are also partners of each other.

Like the previous structural measures, cliquishness shows a phase change at the same value of β^*. Cliquishness is not defined for a graph of disjointed pairs, so our convention is to assign a value of zero for cliquishness of a pair. At the other extreme, a complete graph has a value of 1, and so does a collection of complete disconnected components. In figure 4 we see for small values of β many networks with C close to 0, but most often strictly positive. Since we know that the graph is disconnected here (and thus not complete) this means that we must be observing the emergence of a "caveman graph", that is, a graph of disconnected, complete subgraphs. From figure 1 it is clear that the caves are very small, since the average number of connections per agent is between 1 and 3. When the value of the average cliquishness gets very small, is that there are many isolated pairs in the network. In this part of the parameter space "only length matters" so the two agents with the most knowledge will partner, and create the largest innovation, thus maintaining their positions as the most knowledgeable agents. Their partnerships is stable, as is the partnership of the next most knowledgeable pair, and so on. There is a small range, roughly $0.3 < \beta < 0.45$ in which the graph consists of several, but a falling number of, subgraphs which are themselves highly cliquish. We no longer have a caveman graph, but rather several connected, but not complete subgraphs.

The cliquishness of the equivalent random graph almost perfectly tracks the observed cliquishness, whatever the value of s.

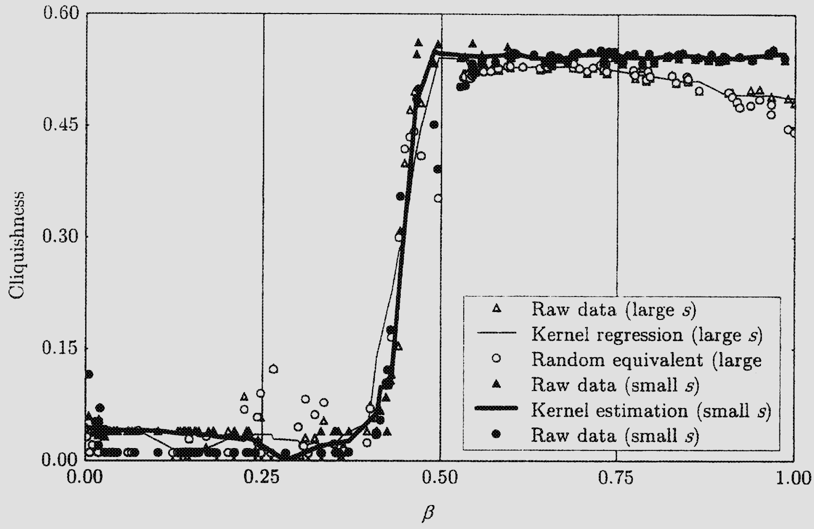

Fig. 4. Average cliquishness

4.2 Knowledge dynamics

Figure 5 depicts the relationship between the long-run economy-wide knowledge growth rate and β.

For both values of s there seem to be, again, two regimes of knowledge growth — one associated with a disconnected graph, and one with a connected graph. In both regions knowledge growth rates as diversity becomes more important. The contrast between large and small s is explained by the fact that with a small value of s, there is a natural convergence over time in the knowledge type of the population. Since innovations tend to occur "between" the knowledge of the members of the innovating pair, knowledge types converge over time. When β is large, diversity is important, and this convergence effect reduces growth. When s is large, however, diversity is preserved over time, since there is a large variance in the type of knowledge created, so the natural convergence is mitigated by the randomness here.

Again, though, there is a small ambiguous region, for $0.25 < \beta < 0.4$, where there is some intermediate behaviour between that of the connected and disconnected regimes. As discussed above, in this region we see the emergence of something resembling a caveman graph — small connected (sometimes complete) sub-graphs within the disconnected total graph. Within each cave, we have the behaviour of a connected graph, which, holding β constant, is good for knowledge production.

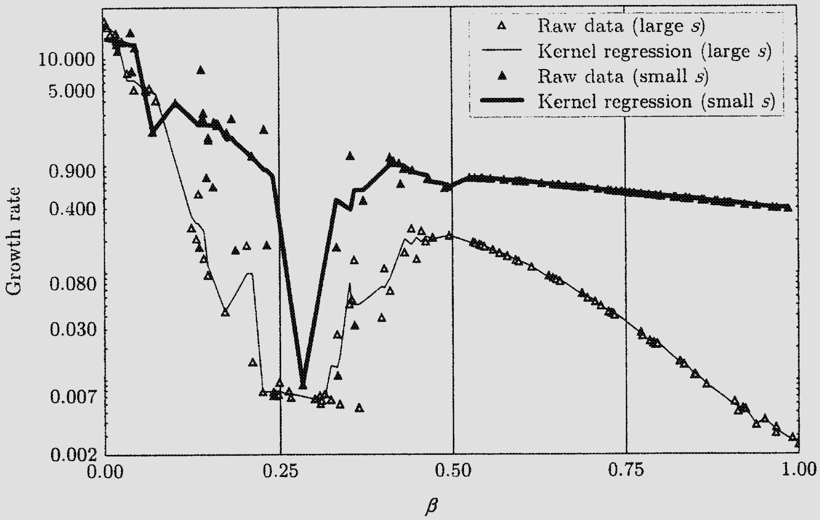

Fig. 5. Growth rate of the average knowledge level

The issue of equity is addressed by considering $\gamma = \sigma/\overline{p}$, the coefficient of variation of individual knowledge levels. Coefficient γ measures how dispersed around the average individual the population is, relative to this average individual. As we re-normalize at each time step, the coefficient of variation reduces to the standard deviation. Figure 6 depicts the way γ varies with β.

For $\beta < \beta^*$, γ is approximately equal to 7, which corresponds to a situation in which knowledge is concentrated in the hands of exactly 2 individuals.[12] There is a first fall in dispersion a bit before the critical β, and then a very marked one down to a point where under both small and large s things stabilize. For small s the coefficient of variation stays around 1 whereas it stabilizes in the vicinity of 0.5 for large s. Hence dispersion decreases with β in a non-smooth way, and the decrease is more marked when the trajectory is more dispersed, undergoing a phase change which again is very comparable to what could be observed concerning network structure. In the beginning of the parameter space "only length matters" so the two agents with the most knowledge will partner, and create the largest innovation, thus maintaining their positions as the most knowledgeable agents. Their partnerships is stable, as is the partnership of the next most knowledgeable pair, and so on. As renormalization carries on, we end up with all the "renormalized" knowledge

[12] Given that we renormalize knowledge levels each period, the standard deviation is equal to 7 if and only if 98 individuals have 0 (or almost 0) and 2 have 50, for an average 1.

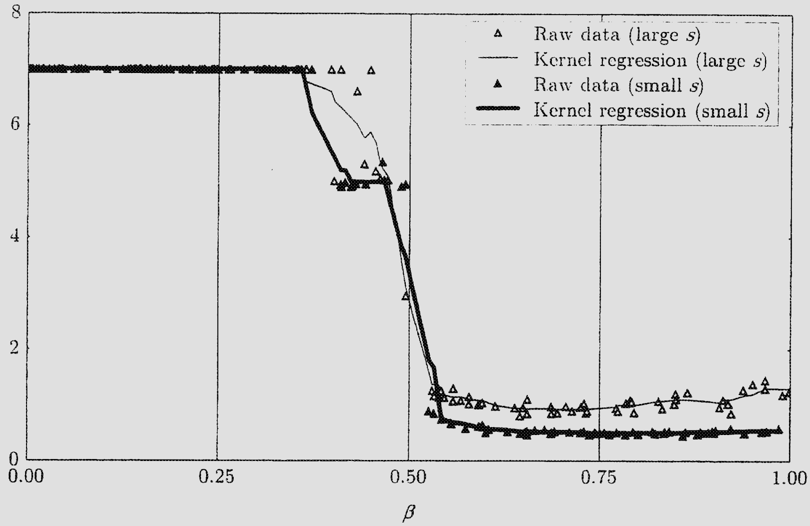

Fig. 6. Coefficient of variation of individual knowledge levels

being held by the two most knowledgeable individuals. It takes a significant weight on variety to combat this tendency and to allow for pairs to fluctuate, entailing that homogeneity in terms of levels decreases. It is worth mentioning that at the other end of the parameter space, "only angle difference matters", which means that the two most distant (in angle) individuals will be matched, so will the next more different individuals, and so on. However the size of their innovation does not matter, only angles do. So with time, because innovation tends to take place "between" the knowledges of the innovating pair, there is a trend of decreasing diversity in the population, as all agents lie between the two most extreme agents of the previous period. This effect is mitigated, of course, as s increases.

5 Conclusion

In this paper we have focussed on issues of knowledge dynamics and emerging network structures when agents create knowledge through partnership agreements. We have emphasized that knowledge creation through cooperation can be a complex process that involves pooling the competencies of the partners in different ways, depending on the nature of the innovation process. Similarly, given a pooling structure, different types of knowledge can be either substitutes or complements for each other in the innovation process itself. The former consideration is represented by the θ parameter in our model, the latter by the elasticity of substitution β in a constant elasticity of substitution production function. Both parameters affect the results, both in terms of the rate of knowledge production, and in terms of the network structure. Knowledge creation is fastest when the knowledge creation production function exhibits complementarity in its inputs. The nature of knowledge pooling has no noticeable effect however. By contrast, heterogeneity among agents regarding their knowledge levels is, generally, decreasing in θ. As it becomes more possible for one partner to dominate in the pooling process (or perform more of the innovation activity) heterogeneity across agents decreases.

The nature of knowledge pooling turns out to be crucial in determining the emergent structure of the network. When θ is very low, stable matchings create pairs of agents whose expertise lies in the same fields fields — agents can be seen as substituting for each other. Innovation tends to occur in that single field of expertise, and the two agents remain good partners for each other in the next round. This implies the emergence of stable pairs of agents; the network becomes a set of $n/2$ disconnected pairs. By contrast, when θ approaches 1, stable matchings create pairs of agents whose respective expertise lies in different fields. Innovation in this case does not reinforce existing expertise of the two agents and thus make them more similar, but rather reducing the differences between them. Thus a pair matched in this round, because they are more similar to each other after innovation are less likely to be a good match in the next round. What this implies is that pairs of agents will form and disintegrate rapidly, agents will constantly search for new partners, and the emergent network will approach a random network. With θ values between these extremes, the network structure too is between the extremes. Here knowledge pooling creates a mixture of the two agents' knowledge with each knowledge type. As θ grows from its minimum value the diameter of the stable groups or cliques increases from two. The emergent cliques tend to be relatively complete, but globally the graph remains disconnected. Cliquishness increases, and is maximal at $\theta = 0.5$. For intermediate values, $0.5 < \theta < 1$, cliquishness falls, as the islands of communicating agents make external connections, opening themselves to each other by establishment of "shortcut" links. Global path length falls in this region and it may be that small worlds, as defined by Watts and Strogatz (1998), emerge.

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Impact of Corporate Boards Interlock on the Decision Making Dynamics

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Summary. Members of boards of directors of large corporations are connected in networks through their multiple appointments on different boards. We focus on those members of a board who also serve together on an outside board, forming the so called *interlock graph* of the board. We investigate the extent to which a minority of well connected directors can drive the decision of the whole board. We present a first model of decision making dynamics inspired to herd behavior and a second one more realistic in the context of boards. We study how the size and the topology of the interlock graph affect the probability that the board approves a strategy proposed by the Chief Executive Officer. We propose a measure of the impact of the interlock on the decision making, which could help identifying boards susceptible to be influenced by a minority. We apply our models and methodology to the data of the boards of the largest US corporations in 1999.

Keywords: social networks, opinion dynamics, directorate interlock, phase transition, Ising model.

1 Introduction

The boards of a set of companies together with their directors form a bipartite network. The director network is the network obtained taking the directors as nodes, and a membership in the same board as a link. It is well known that the director network of the largest companies in the US and in other countries has a high degree of interlock, meaning the fact that some directors serve on several boards at the same time so that many boards are connected by shared directors. Interlock convey information and power. For example a bank that lends money to an industry can use interlocked directors in industries of the same domain to get additional information about the real risk of the loan.
As a consequence of economic power concentration over the last decades, "a special social type emerges spontaneously, a cohesive group of multiple directors tied together by shared background, friendship networks, and economic

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interest, who sit on bank boards as representative of capital in general" [1] Now, while part of the public opinion has been since long ago concerned about the fact that the corporate elite would represent a sort of "financial oligarchy controlling the business of the country" [2], stockholders are more concerned about the effectiveness of boards in overseeing management. Board's directors should in fact monitor managers's strategies and decisions to the interest of stockholders. So, after the recent highly visible cases of bankruptcy in the US, the role of boards in the decision making process is now largely debated and more sophisticated forms of corporate control are advocated.

In this regard there are some works in sociology, investigating whether boards have adequate knowledge and information to make meaningful contributions to strategic decision making. Authors try to assess how multiple boards appointments affect directors' ability to contribute to strategy [3] [4]. This kind of study is usually done by means of surveys and no modelling of the dynamics is involved.

Some authors have studied the topological properties of the corporate elite network: Davis and collaborators have shown [5] that the directors network and the boards network of the largest US corporations has Small World properties [6]. German firms too, turn out to form a Small World [7]. Vedres [8] has analyzed the social network composed by directors of the the largest Hungarian companies, banks and government leaders relating the power of social actors to their properties as nodes of the network.

Finally, some authors have studied the diffusion of governance practices such as the so called 'poison pill' and 'golden parachute' [9], throughout the board network, with an epidemiological approach.

In our work we combine the study of the topological properties of the interlock with the modelling of the dynamics of decision making. Directors have to vote in order for the board to take a decision. It is clear that the social ties linking a director to the other directors in the board influence the formation of his/her opinion. Two directors in the board who also serve in another outside board are likely to take each other's opinion into account more seriously than the opinion of another director (see below). If there are several directors that, within a board, share additional ties among them, they form a sort of "lobby". The question we address here is whether the lobby can influence significantly the decision making process of the whole board. The problem concerns of course not only the boards of large corporations, but many governance structure in social institutions and it is of general interest in social science modelling. We study the boards of the largest corporations because it is a relatively well defined framework and there are data available about the social connections among agents.

Now, one can think of two kinds of decisions a board is faced to: there are decisions regarding a topics specific to a board, such as the appointment of a candidate member. For such decisions, we might suppose that different boards don't influence each other. There are also decisions about topics related to general trends in the economy such as whether to fire part of the employees, depending on the forecast of economical recession or whether to adopt some governance practice [9]. In those cases, decisions previously taken in some boards might influence other boards. The present work only considers decision of the first kind when a single board decides on some issue independently of other boards decisions.

In general, models of social choice assume that agents form their opinion according to the information available to them about the state of the world and to the opinions of other agents [10] [11] [12]. As we said, the interlock comes in the decision making process because we will assume that two directors serving at the same time on several boards have stronger influence on each other. One of the rationale for this assumption is the fact the recruiting mechanism itself relies on personal familiarity: a candidate member is proposed and supported by members who already know him/her because they serve or have served together in another board [4] [5]. As a result, interlocked members are likely to be more influential on each other's opinion.

There exist a large literature about committees and collective decision making, but little numerical or analytical modelling. We start from the standard assumptions of herd behavior [10], which describe individual decisions as based on the successive surveys of other agent opinions; we call this first model to be later fully described as a survey model. A second model is based on the succession of interventions of speakers during the board session, each speaker influencing other directors during his (or her) intervention; this model is called a broadcast model.

We then investigate, for the two models, the effect of the size and topology of the graph of interlocked directors of a board, on the final board decision.

This paper is organized as follow: we first present some statistics based on empirical results concerning the US Fortune 1000 companies. We then describe the survey model, the relevant quantities to be monitored in simulations and check the simulation results with standard mean field results in the absence of interlock. The next section is devoted to a search for a good predictor of the dynamics in the presence of interlock, and to simulation results obtained with test interlock graphs and with empirical board interlock graphs. Similar tests are done for the broadcast models. In the last section we compare and discuss the results.

2 Interlock Graphs

In the literature, the topological properties of interlocking directorate are studied for the director network as a whole [5]. We here focus instead on the interlock inside each single board. We call ***interlock graph of a Board*** the graph obtained by representing directors of a board as nodes and drawing an edge between two directors if they serve together on an outside board (**Figure**

1).
Before investigating how the structure of the interlock graph affects the decision making process, we want to know how a typical interlock graph looks like in real boards. We have analyzed data that have been kindly provided by G.Davis [5], about the boards of the US Fortune 1000 companies (year 1999). We found that 321 boards out of 821 have a non empty interlock graph. 20 per cent of all boards have a 1-link interlock graph, another 20 per cent have a more complex interlock graph. An example of a board with a complex interlock graph, the board of directors of the Bank of America Corp. is shown in Figure 1. Within the 321 interlock graphs there are chains, cliques (subgraphs in which each node is connected to all the others) and various combinations of these components. In particular, we looked at the largest clique in the interlock graph and we found 25 boards with a clique of three nodes, 9 with a clique of 4 nodes. We also looked at the largest connected component (LCC) in the graph and we found 65 boards with a LCC of 3 nodes, 31 boards with a LCC of 4 nodes, 9 boards with a LCC of 5 nodes, 4 boards with a LCC of 6 nodes and 2 boards with a LCC of 8 nodes.
We present in **Figure 2** the histograms of board size (number of directors in the board), lobby size (number of directors involved in the interlock graph) and number of links of the interlock graph. Only the 321 boards with non empty interlock graph are considered in the histograms. The average board size is 12.4 ± 3.6, the distribution is unimodal, skewed to the right. The smallest and largest board have size 5 and 35 respectively. Lobby size ranges up to 12 nodes. Another interesting quantity is the ratio between size of the lobby and size of the board (bottom left) which has a mean of .19. The distribution is obviously non gaussian with a long tail.
From the above analysis we see that the fraction of boards of the 1000 Fortune companies, that exhibit a complex interlock graph, is far from being negligible. It is therefore of great interest to try to model its effect on the decision making dynamics.

3 The Survey Model

We want to model the process of decision making on a single board. We first consider the most standard model in economics used to model herd behavior, which we here call the survey model. The model is basically an iterated voting process.

At each time step, one director randomly chosen polls opinions of other agents and makes his opinion accordingly, most often taking the opinion of the majority.

More precisely, at the board meeting, the CEO proposes a strategy for the company. The board directors discuss the strategy and at the end take a decision by voting. We stylize the situation saying that there are only 2 opinions: opinion +1 corresponds to approving CEO's strategy, and -1 to

refusing it. The CEO always sticks to opinion +1. The other directors can have opinion +1 or -1. Directors discuss between each other and get to know the opinions of all their colleagues, which they take into account to formulate a new opinion. Other colleagues' opinions define a field: the field is a weighted sum of colleagues' opinions, where the weights depend on the number of boards on which two directors sit together. The new opinion depends stochastically on the intensity of the field.
In formulas the model reads as follows. The opinion of director i is a binary variable $s_i = \pm 1$. The field acting on director i is:

$$h_i = \sum_{j=1}^{m} J_{ij} s_j \tag{1}$$

m being the size of the board, $J_{i,j}$ being the number of boards on which directors i and j sit together. Obviously, directors take into account their own opinion, hence $J(i,i)$ in equ. 1 must be non zero. Setting $J(i,i)$ to 1 is not very realistic, since it implies that a director with some interlock ties assigns a larger weight to his colleagues' opinion than to his/her own opinion. We chose to set $J(i,i)$ as the number of boards where director i serve with at least one other director of the same board.

The probability that director i takes some opinion ± 1 at time $t+1$ is given by:

$$P\{s_i(t+1) = \pm 1\} = \frac{\exp(\pm\beta h_i(t))}{\exp(\beta h_i(t)) + \exp(-\beta h_i(t))} \tag{2}$$

Parameter β in the opinion update acts as the inverse of a temperature. It measures the degree of independence of a director's opinion from the field. At T=0 the opinion dynamics becomes deterministic, at infinite T the dynamics becomes random. The Boltzmann formalism, often referred to by economists as the logit function, can be justified by several considerations such as errors in opinion propagation and random fluctuations of some external conditions. What is meaningful for us is that a small amount of fluctuation is sufficient to remove the system from spurious attractors.
In the next we will refer to dynamics with CEO and without CEO, meaning respectively, that there is a director with a constant opinion +1, or not.
Formally the model is analogous to an Ising magnetic system.

3.1 Variables Characterizing the State of the System

We here define some macroscopic variables describing the state of the system. The value of the opinion averaged over directors of the board is called M:

$$M \equiv \frac{1}{m} \sum_{j=1}^{m} s_j \tag{3}$$

M, is the analog of the magnetization in the Ising model and is a function of time. We denote the magnetization at time 0 and at large time T as respectively: $M^0 \equiv M(t=0)$ and $M^* \equiv M(t=T)$. In order to evaluate the impact of the interlock on the decision making process we consider the probability that the board votes +1 at large time T:

$$P_+ = P\{M^* > 0\} \tag{4}$$

If the board is neutral at the beginning i.e. $M^0 = 0$, then in the absence of interlock and CEO there are equal chances of outcome $M^* > 0$ or $M^* < 0$. One way to measure the impact of the interlock is to consider the probability that the board votes +1, conditional to the initial average opinion M^0 in the board being zero:

$$P_+^0 = \{M^* > 0 \ | \ M^0 = 0\} \tag{5}$$

The fact that the CEO sticks to opinion +1 can be regarded as a constant external field: $h_{CEO} = J_{CEO,j}s_j = J_{CEO,j}$

3.2 Dynamics in the Case of no Interlock

In the absence of interlock ($J_{ij} = 1 \ \forall i = 1 : N$), and in the absence of the CEO, the dynamics is equivalent to ferromagnetism in the mean field approximation.

The absolute value of M^*, as a function of beta, shows a clear phase transition around $\beta = 1$, as predicted by the mean field theory, even for a small number of directors $N_d = 10$.

Another way to visualize the phase transition is with a bifurcation diagram: **Figure 3** shows the probability distribution of values of M^* obtained in 1000 runs as a function of beta. When $\beta = 0$ one observes a small magnetization due to the CEO: $M^* \frac{1}{N_d} > 0$.

Let us note that for $\beta > 2$, in practice the only possible values of M^* are ± 1, so that the probability that M^* is positive coincides with the probability that $M^* = 1$.

This means that, at high beta, the board ends up with deciding at unanimity whether to approve or reject CEO's proposal (unanimity minus one in case of rejection). For $\beta = 4$ the attractor is reached within 25-30 steps. This is a realistic scenario because typical discussions end up with very large majority in less than about 3 intervention per director. All the results shown in the following are obtained with $\beta = 4$. In order to compute M^*, we run the dynamics for 50 steps and we average the magnetization over time steps 25-50.

3.3 Measuring the Impact of Interlock Graphs

We want to investigate the effect on the dynamics due to the presence of a group of directors which serve together on one or more outside boards. The value of J_{ij} is the number of boards on which directors i and j serve together. Obviously J_{ij} worths at least 1 for all the directors in the board. The subset of directors of the board for which $J_{ij} > 1$ form a graph, the interlock graph of the board as we called it in the previous sections. We will refer to it as the 'lobby'. This graph consists of one or more *Connected Components*(CC). As a result of the connection structure, a director belonging to a CC feels a stronger influence from a colleague within the CC than from a colleague outside the CC. This fact will lead in the following section to a definition of force of the lobby.

We first tested our methodology with the set of all graphs with 10 directors connected with at most 3 links. **Figure 4** shows all the possible graphs obtained with 1,2,3 links, with a maximum of 2 links per node. They are referred to as interlock graph 1,...,15. In each box the nodes represent the 10 directors of the board. The edges represent the ties between directors which serve together on an outside board. The black node is the CEO. Note that graphs 6 and 8 consist of a fully connected subgraph of 3 directors. In the latter case the CEO belongs to the graph.
For a given initial condition the outcome of the dynamics can be either 0 or 1, because of the stochastic nature of the opinion update. Then, in order to estimate the probability P_+ that a given board approves the CEO's proposal, we repeat the dynamics for a large number of runs. We therefore obtain values of P_+ for each type of interlock graph, as a function of the initial magnetization inside and outside the lobby.

Now, in order to compare results for lobbies with different sizes and topologies, we seek for a scalar quantity that predicts the impact of an interlock graph. It is clear that the impact of a lobby must depend on the number of nodes in it and on the number of links. The number of links alone does not predict the probability of approving CEO's proposal. The number of nodes alone doesn't do better. The fact is that with a same number of links one can build a clique or a chain, so the topological structure must play an important role, too. Moreover the initial opinions of the directors in the interlock graph count a great deal, so we need a quantity which can take them into account. The best predictor we found is the quantity:

$$F = \frac{1}{m^2} \sum_{ij \in G} J_{ij} s_j(t=0) \tag{6}$$

which we call the force. This scalar quantity is in fact the intensity of the field exerted at time $t = 0$, by all the directors in the interlock graph on themselves. The field is normalized with respect to the size m of the whole board, because we want to estimate the impact of the interlock graph with respect to the whole board. The same interlock graph will affect more strongly a small board than a large board.

To have an intuition of the notion we want to capture, suppose that at time $t = 0$ all the directors of the interlock graph have opinion $+1$, but the board as a whole has magnetization $M = 0$. The stronger the field the interlock members exert on themselves as compared to the field exerted by the directors outside the interlock graph, the more chances that the directors of the interlock graph stick to their initial opinion at $t \geq 0$. They would then act as an external field driving the board towards positive values of magnetization (although in principle the directors of the interlock graph can change opinion at any time; only the CEO has a fixed opinion).
The force can take several different values according to the different initial opinions $+1$ and -1 in the interlock graph. Hence, each graph has a set of possible value of force. For each value of the force the dynamics has a certain probability to reach the attractor $M^* = 1$ (as we said, for $\beta > 2$ whenever M^* is positive, it is equal to $+1$).

Figure 5 displays P_+^0 (the probability of approving CEO's proposal when the board is neutral at time 0) as a function of the force of all the interlock graphs. The fact the P_+^0 is an increasing function of the force was quite expected, by construction. What was not clear a priori was that lobbies with different number of links and different topology but with similar value of the force do have similar value of P_+^0 which means that the force is a good predictor of the influence of the lobby on the result of the board decision making. Moreover, it is a linear predictor.

3.4 Voting Dynamics Simulations in Real Boards

We ran simulations of the voting dynamics using the interlock graphs that we found in the real boards. As for the elementary interlock graphs, here for each board and for each initial condition, we repeat the dynamics a large number of times in order to estimate the probability P_+ that the board will approve CEO's proposal, as a function of the initial conditions in the lobby. In **Figure 6** we show P_+^0 (P_+ conditional to having $M = 0$ at time $t = 0$) vs the force, for the real boards of the US 1000 Fortune companies.

To simplify the graph, we considered only points relative to two initial conditions: apart from the CEO, the directors in the interlock graph either have all opinions $+1$ or all -1 at time 0. Hence each board is represented by 2 points: one with a positive value of the force and one with a negative one. **Figure 6** displays a strong linear correlation between P_+^0 and the force.

We left out boards with too large interlock graph i.e. when the number c of nodes in the interlock graph is larger than half the number m of directors. In fact in this case, when all the directors in the lobby start with a same opinion, then, no matter what is the opinion of the directors outside the lobby, there is no configuration with M_0 equal to 0. In this case, we set all directors that are not in the lobby as against the lobby, but since $M_0 > 0$, the corresponding data points are not comparable with the ones of the other boards. Boards with $c > m/2$ (not shown) have P_+^0 values close to 1.

A complementary set of data is obtained by taking the histogram of the fraction of boards which would agree with the CEO with a given probability. The set of boards is reduced to boards with an interlock graph, and the initial condition are $M_0 = 0$ with all lobby members voting initially as the CEO. **Figure 9** displays these histograms for four sets of simulations concerning the two models. The top histograms correspond to the survey model with and without a lobby, for the sake of comparison. One reads the histograms in the following way: with the survey model for example (top right frame), 25 per cent of the boards have 75 per cent chances to approve CEO's proposal, if the directors in the lobby are initially in favor of it. Moreover one can say that 40 per cent of the boards have at least 75 per cent chances of approving the CEO.

4 The Broadcast Model

We consider now a different model for the voting dynamics, based on the idea that at each time step one director takes his turn to speak while the other directors listen to him/her and are influenced by his/her opinion. We will refer in the following to this model as the broadcast model.

At the board meeting, the CEO proposes a strategy for the company. Again this is stylized saying that there are only two opinions: opinion +1 corresponds to approving CEO's strategy, and −1 to refusing it. The CEO always sticks to opinion +1. The other directors can have opinion +1 or -1.

- One director j at a time is chosen to speak. His own opinion is evaluated, as usual, based on the field he experiences according to the logit equation (equ. 2).
- Only the individual field evaluation is changed. When director j speaks, all directors i update their individual field according to:

$$h_i^{new} = (1-\gamma)h_i + \gamma J_{ij} s_j \qquad \forall i \tag{7}$$

  γ is a parameter which determines the memory length of the agent. At the beginning, the field of the agent i is initialized as equal to $J_{ii} s_i$.

As a result, the field experienced by an agent, only takes into account the discounted opinion of the other agents, at the time when they spoke (which might be different from their actual opinion now). This scheme is closer to a class of models based on the Pòlya urn, also used by economists [13]. We might then expect some sensitivity to the ordering of agents' interventions during the board meeting.

In fact, the broadcast model requires to choose at each time step who is going to speak. As modelers we are tempted to use a random order, but in real boards the order is probably far from being random: more convinced directors will likely try to speak first, and moreover the CEO or the chairman plays a role in deciding the order of the speakers. In order to understand the impact of the way in which directors are chosen to speak, two extreme strategies are investigated here:

1. Strategy 1. The speaker is chosen randomly.
2. Strategy 2. For $t \leq c$ (c being the size of the interlock graph), the speaker belongs to the interlock graph. For $t > c$ the speaker is chosen randomly.

We have run simulations of the broadcast dynamics on the elementary interlock graphs and on the real boards of the US 1000 Fortune companies. Similar results were observed for different γ values ($\gamma = 0.1$, 0.3). We performed the same analysis as for the survey model: we estimated the probability P_+^0 that the board will approve CEO's proposal, conditional to having $M = 0$ at time $t = 0$, as a function of the initial conditions in the lobby. As before only the two extreme cases for which the directors in the lobby are all in favor of the CEO, or all against him/her are taken into account. P_+^0 versus the force is shown in **Figure 7,8** for the two strategies of choosing the speakers. The histograms of the fraction of boards which would agree with the CEO with a given probability are shown in **Figure 9**.

5 Discussion

We have investigated the impact of different structures of corporate directors interlock on the outcome of the decision making process of boards of directors. We have considered two models of decision making process, and we have studied the probability of board approval of the CEO's strategy as a function of the topology and the size of the interlock structure. We have applied the models on a set of test interlock graphs or lobbies, in order to find a good predictor of the interlock impact, and then we have applied the models to the boards of the largest US corporation.

Figure 10 summarizes our results: the existence of a lobby does influence the vote as compared to the absence of a lobby. The probability P_+^0 that the board approves CEO's proposal when the board is initially neutral, is plotted against the force of the lobby for the different models (for the purpose of comparison, values of the force are grouped in bins of width 0.05 and the corresponding values of P_+^0 for different boards inside the bin are averaged together). What surprised us is that this influence is of comparable magnitude for the survey and random broadcast models, at least for small values of γ, the time discount factor. In the broadcast model when directors of the lobby speak first, the influence of the lobby is enhanced, and even more so in the neighborhood of zero force. This means that a strategic sequence of interventions may enhance the power of the lobby on the decision making process. The discontinuity at $F = 0$ increases with γ.

We have focused our attention on the case in which initially the whole board is neutral about the decision, that is $M_0 = 0$, while the directors in the lobby have the same opinion, either all +1 or all -1. In this case the probability of approval is related to the power of the lobby, where "power" is used in accordance to Weber's definition: "power of an actor in a social network is the probability that this actor will carry on his/her will despite resistance of the other actors" [14].

The interest of this investigation for the social sciences consists in offering a framework in which it is possible to make quantitative predictions about the power of a lobby within a board: given the topology of the social ties, we can compute a quantity, the force, which is a good predictor of the power of the lobby. In principle the board should take decisions on the interest of all investors, based on the available information. From our results, a well connected lobby of a minority of directors can drive the decision of the board, and the chances that the board will finally agree with the lobby can be predicted measuring the force of the lobby.

Having a powerful lobby inside the board simply means that the opinion of some directors has counted more than the opinion of others, which is not necessarily bad if, for example, the directors in the lobby were the most competent about the matters in discussion.

But suppose now the lobby rather represents the interest of some minority. This minority could consist of officers of the company itself, reluctant to a change of management or officers of another company that owns a minority of stocks and want to attack the company. This could be seen as a dangerous situation for the company and the majority of investors. In this perspective, norms could be introduced to limit the force of the lobby e.g. when a new director is proposed for an appointment in the board.

Of course, the prediction of the outcome of the decision making process assumes some simple hypotheses about the influence of board directors on each other's opinion. The main hypothesis of our models is that the influence J_{ij} of a director i on another director j is a linear function of the number n_{ij} of boards on which the two directors serve together. Some different functional relationships between J_{ij} and n_{ij} could be assumed, provided that the influence is a monotonic increasing function. There is no a priori justification for our linear choice, other than the fact that it is a simple approximation to start with.

For our models, we do not have an estimate of the real value of β. We made simulations for different values of β, then we focussed on the case $\beta = 4$, in order to avoid meta-stable states. But for any value $\beta >> 1$ the dynamics converges very rapidly to unanimity, and for $\beta = 4$ in less than 30 steps, i.e. after about 3 interventions on average of each director. This is a quite realistic scenario: in fact this is the typical number of interventions for a board discussion. Moreover, a typical discussion ends up with a consensus or a large majority.

The two models that we have investigated differ in the mechanism of opinion update. The opinion update mechanism we adopted for the survey model is analogous to what is known as "herd behavior" in the literature of opinion dynamics, but it is also analogous to what is well-known in statistical physics as magnetic system dynamics at finite temperature. In the broadcast model we propose a more realistic mechanism of opinion update, which takes into account the fact that in a real discussion one is not informed of everybody else's opinion at each step in time. Instead, participants speak once at a time, so that each agent only knows the opinion that another agent had at a certain time, which may differ from the opinion he has at the current time.

The present study focuses on boards of directors, because of the availability of empirical data. Our conclusions can be also applied to the decision dynamics of any political committee or academic board.

One possible extension of this investigation is the study, now in progress, of the dynamics of the decision making process of boards when the decision taken at one board influences the decision process of other boards. In fact, in the case of discussions about adoption of governance practices [9] or decisions that require prior forecasting of economic trends, directors of a board are likely to take into account decisions made in interlocked boards.

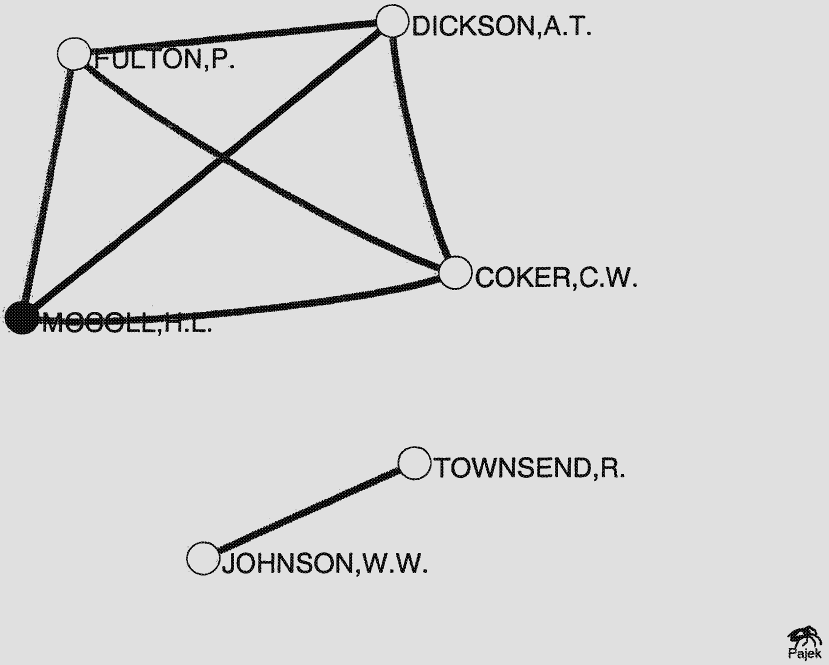

Fig. 1. Example of an interlock graph: The board of directors of the Bank of America Corporation. White nodes represent directors that are not in the management, black nodes represent directors that are also executive of the company. Two directors are connected by a gray edge when they serve on one same outside board. The edge is black when they serve together on more than one outside board.

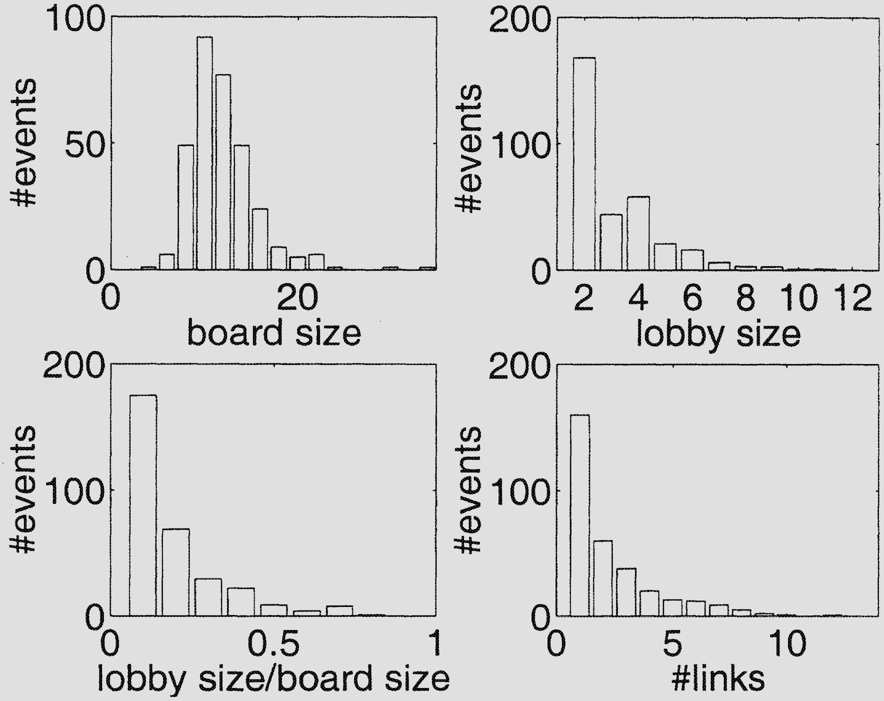

Fig. 2. Histograms of board and interlock characteristics. Top left: board size (number of directors in the board). Top right: lobby size(number of directors involved in an interlock tie with some other directors of the same board). Bottom left: ratio between lobby size and board size. Bottom right: number of links in the interlock graph.

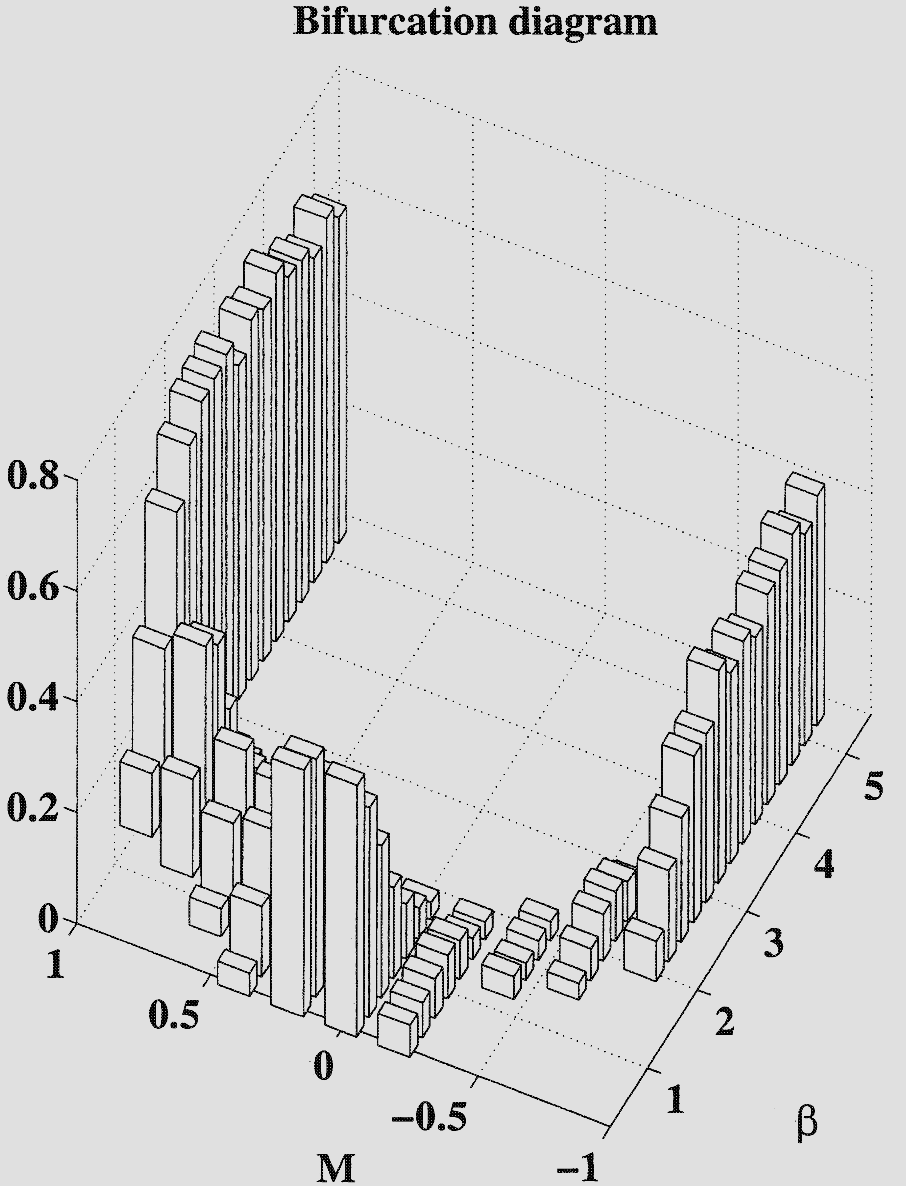

Fig. 3. Bifurcation diagram of the magnetization. Frequency distribution of the final value of the magnetization $M^* = M(t = T)$, obtained in 1000 runs, as a function of β, the temperature parameter. The board has 10 directors. The asymmetry of the diagram is due to the presence of the CEO who always vote +1. Note that for $\beta > 2.5$ the probability that $M^* = 1$ coincides with the probability that $M^* > 0$.

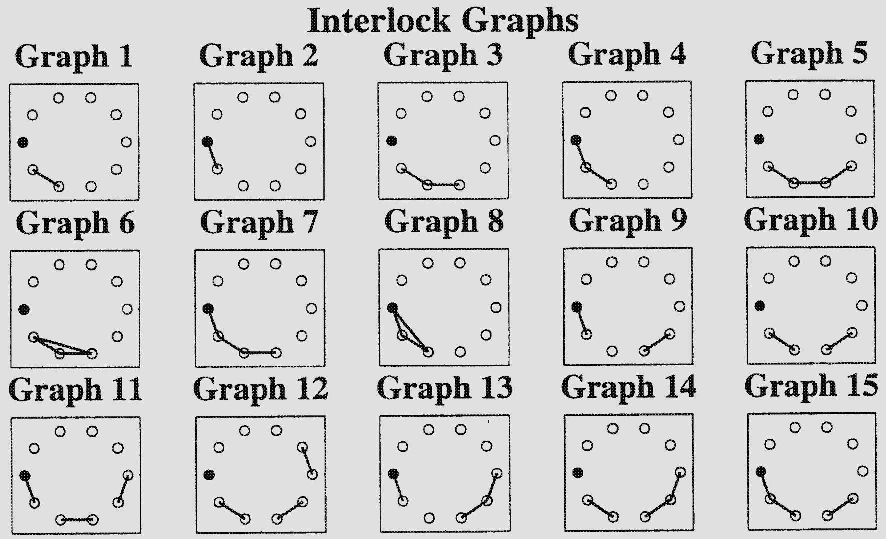

Fig. 4. The simplest interlock graphs for boards with 10 directors. There are 15 different graphs that can be drawn with a maximum of 3 links and with up to 2 links per node. The black node is the CEO.

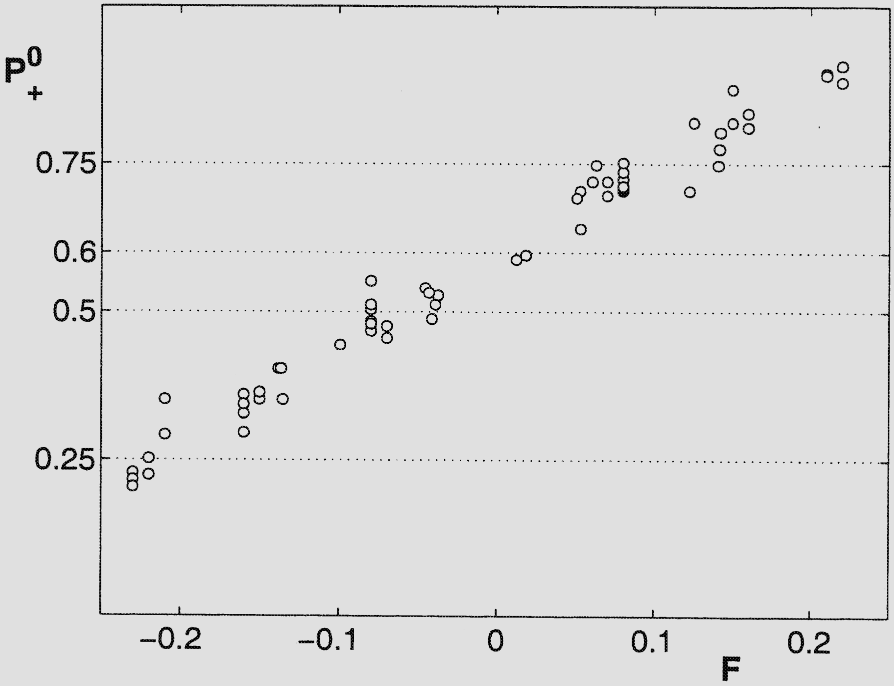

Fig. 5. Survey model simulation of the 15 elementary interlock graphs. Ordinate: probability P_+^0 that the board approves CEO's proposal, conditional to the board being initially neutral ($M^0 = 0$). Abscissa: force of the interlock graph. Data points are the average of 500 runs. Each data point corresponds to a given initial magnetization value of a single interlock graph.

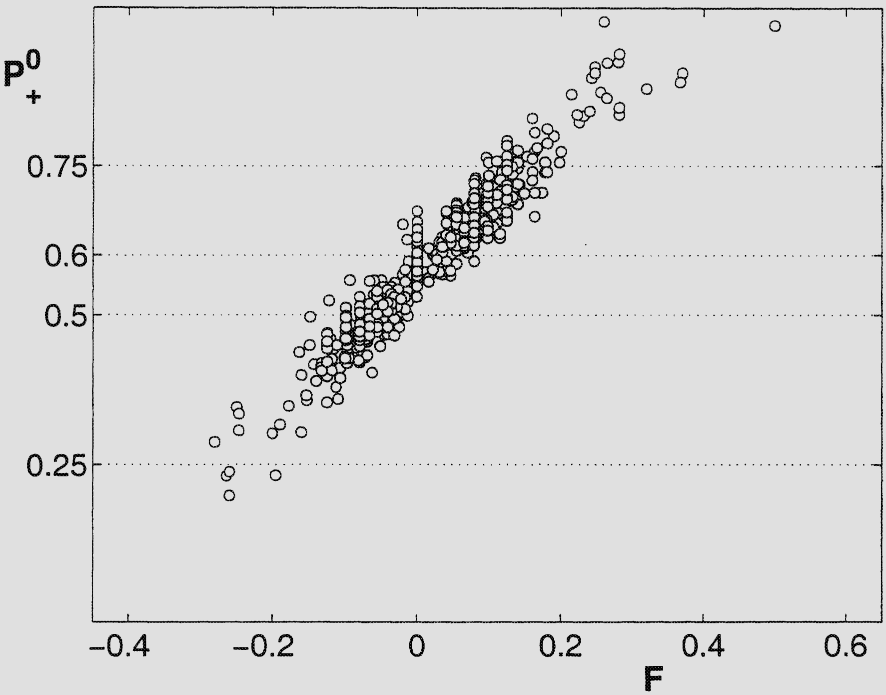

Fig. 6. Survey model simulation on the boards of the Fortune 1000 companies. Ordinate and abscissa as in figure 5, each point is an average over 500 runs. Only 2 initial magnetization values of the lobby are considered (all +1, all -1).

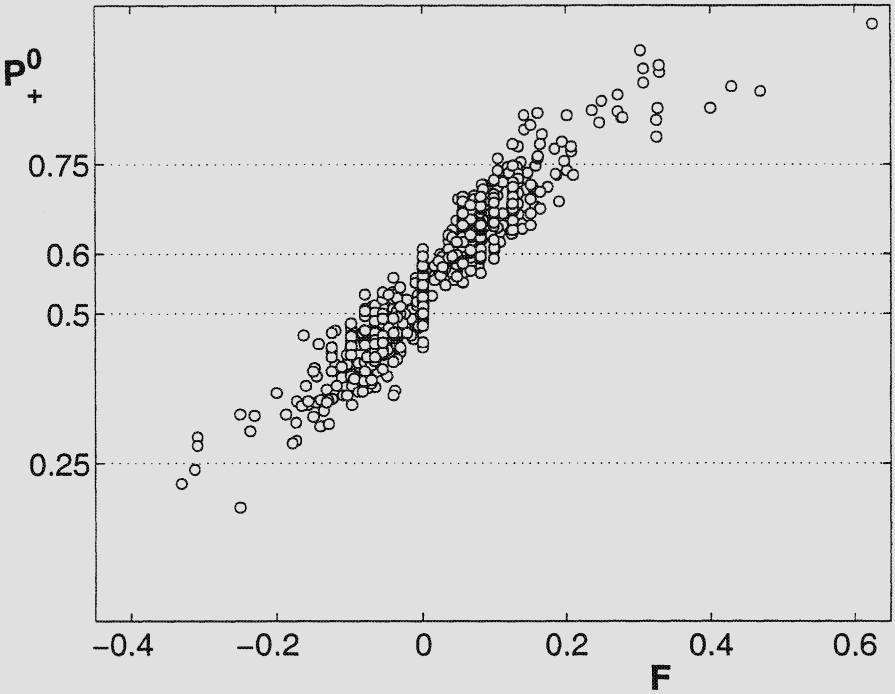

Fig. 7. Broadcast model simulation on the boards of the Fortune 1000 companies. The order of the speakers is random. Ordinate and abscissa as in figure 5, each point is an average over 500 runs. Only 2 initial magnetization values of the lobby are considered (all +1, all -1).

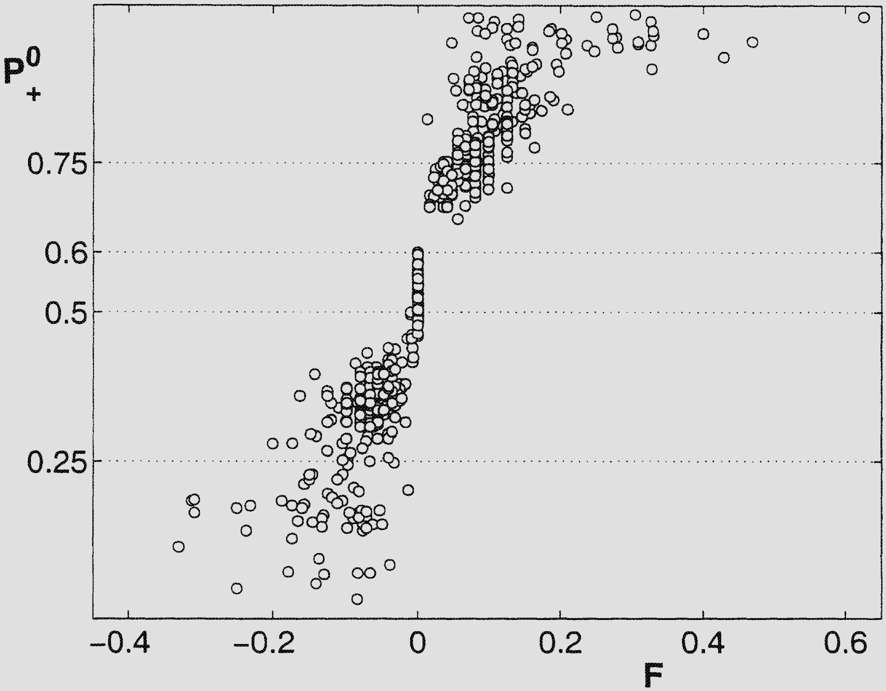

Fig. 8. Broadcast model simulation on the boards of the Fortune 1000 companies. The order of the speakers is as follows: directors of the lobby speak first, then the speaker is chosen randomly. Ordinate and abscissa as in figure 5, each point is an average over 500 runs. Only 2 initial magnetization values of the lobby are considered (all +1, all -1).

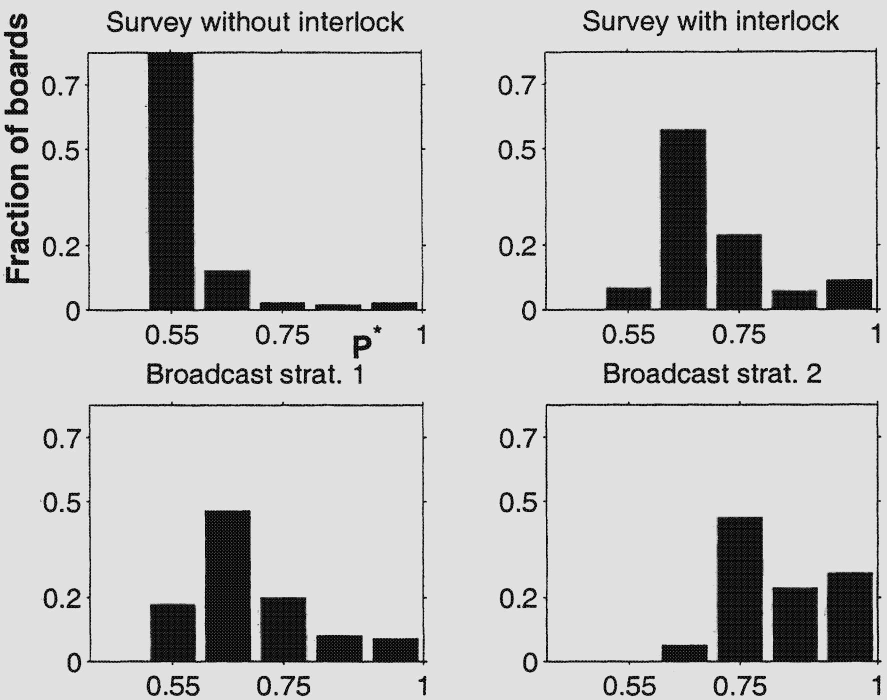

Fig. 9. Histograms of the fractions of boards which would approve the CEO with a probability given in abscissa.
Top left: values for the survey model without interlock.
Top right: survey model.
Bottom left: broadcast model, strategy 1 (random order of speakers).
Bottom right: broadcast model, strategy 2 (directors in the lobby speak first).

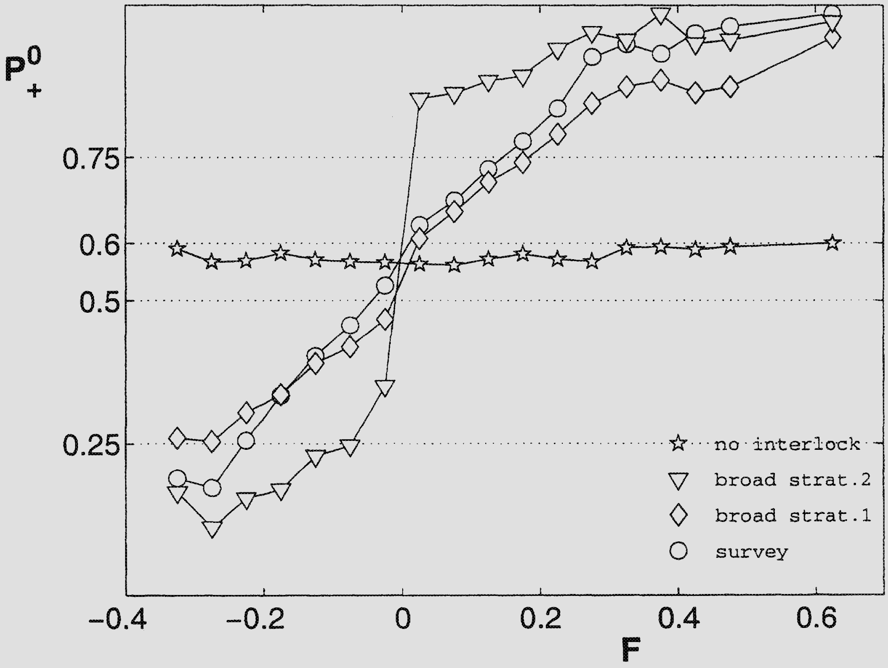

Fig. 10. Comparison of the results of the different models. Ordinate and abscissa as in figure 5. Data points correspond to the 321 boards with non-empty interlock graph. Pentagons: survey model with no interlock for control purposes. Circles: survey model. Diamonds: broadcast model with $\gamma = 0.1$, strategy 1 (random order of speakers). Triangles: broadcast model with $\gamma = 0.1$, strategy 2 (directors in the lobby speak first).

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Interactions of Heterogeneous Agents in Stochastic Socio-economic Systems

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Abstract

Interactions of heterogeneous agents have been calculated from probability theory with constraints (Lagrange principle):

$$L = E + T \ln P$$

L is the Lagrange function, the constraints E are the interactions of agents, P is the probability of a distribution pattern of the agents. The laws of interaction stand for order, the entropy ln P for disorder of the system, T is the ordering parameter.

In natural systems the Lagrange function L corresponds to the negative free energy. The function E represents the chemical interactions, ln P the entropy (S) and T the temperature. At low temperatures the system (solid) is well ordered. At higher temperatures the system (liquid, gas) is disordered. In binary systems the model of chemical interactions leads to solubility or insolubility of materials.

In social systems the Lagrange function L corresponds to the common benefit. The function E represents the social interactions of the system. The entropy ln P reflects the disorder of the system, depending on the Lagrange parameter or quasi-temperature T. The parameter T may be interpreted as tolerance of disorder and is proportional to the standard of living (GNP per capita). At low standard of living a society will be in hierarchic order, aggressive and segregated. At high standard of living a society will be democratic, chaotic and integrated. Order-disorder transformations from hierarchy to democracy are called revolutions. They start at a critical standard of living T_C and require energy of transformation.

In economic systems the Lagrange function L corresponds to the common profit of an economic system. The function E represents the capital of the system. The temperature T may again be interpreted by the standard of living (GNP per capita). At low standard of living the economic structure will be hierarchic, only at a higher standard of living the economy will become capitalistic. These relationships may be observed in all agent systems, in states, companies or families.

M. Gallegati et al. (eds.), *The Complex Dynamics of Economic Interaction*

Introduction

Interactions of heterogeneous agents have become a major field in the last years. Annual international conferences (SocioPhysics 2002; WEHIA 2002; SimSocV 2001) as well as numerous authors (Foley 1994; Schweitzer et. al. 2000; Solomon et. al. 2000) and others indicate the broad interest of a large scientific community in simulations of socio-economic systems.

The present work started with the investigation of alloys. The map of Bosnia of 1991 in fig.1 had been compared to a routine etching of a brass surface, fig. 2, (Mimkes 1995, 1997, 2000).

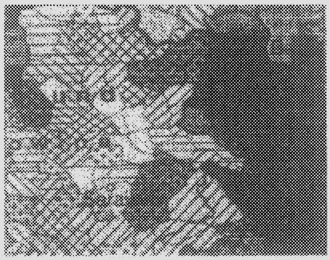

Fig. 1. *Map of Bosnia 1991. The Bosnian society is segregated into regions with predominantly Moslem Bosnians, areas with predominantly Orthodox Serbs and other regions again with predominantly Roman Catholic Croats. (With kind permission, Dierke Weltatlas, Westermann Schulbuch Verlag, Braunschweig.)*

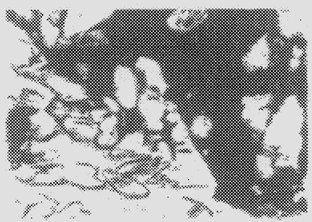

Fig. 2. *Surface of a brass probe after etching. The brass alloy is segregated into areas with predominantly zinc (bright regions) and areas with predominantly copper (dark regions).*

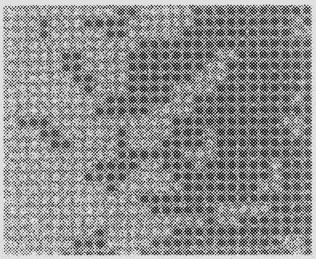

Fig. 3. *Simulation of a binary system according to the present model of stochastic heterogeneous agent interactions. A period of integration followed by a period of segregation leads to a pattern similar to figs. 1 and 2.*

Bosnia in fig. 1 is a mixture of Moslems, Orthodox and Catholic people. Brass in fig. 2 is a mixture of copper and zinc atoms. The structural similarities in figs.1

and 2 started the discussion, whether aggression in countries at war like Bosnia could be explained by models of physical chemistry. Fig. 3 shows a simulation according to the model of atomic interactions according to Bragg and Williams, (Becker, 1966). Atoms and people like and attract each other, dislike and repel each other or are indifferent. This model will be the basis of social interactions of heterogeneous agents. The idea actually goes back to Empedokles of Acragas (495–435 B. C.), who related the mixing of social groups to the solubility of liquids: *people, who love each other mix like water and wine; people, who hate each other segregate like water and oil.* J. W. Goethe (1809) discussed this idea in his novel "*Die Wahlverwandtschaften*". More recently, W. Weidlich (1971) has introduced sociodynamics by theoretical means of modern thermodynamics.

Probability with Constraints

The distribution of N elements in K categories is determined by the statistics of combinations. For two categories, e.g. the left and right side of a street we find N_L people on the left side and N_R people on the right side with $N_L + N_L = N$,

$$P(N_L, N_R) = \frac{N!}{N_R!N_L!} \bullet \frac{1}{2^N} \tag{1}$$

For large numbers N the probability function $P(N_L,N_R)$ leads to a normal distribution. This result is valid for all calculations of probability in stochastic multi – agent systems without constraints.

Joseph de Lagrange (1736 – 1813) has formulated a principle that applies to stochastic systems with interactions (constraints),

$$L = E + T \ln P \quad \rightarrow \quad \text{maximum!} \tag{2}$$

The Lagrange function L is the sum of the functions E and T ln P. The function E stands for the constraints of the system, e. g. the interactions of the agents. The function P is the probability in Eq.(1). ln P is the natural logarithm of probability and is called entropy. The parameter T is called Lagrange factor and may change from zero to infinity. T and L will have the same dimension as the constraint E, the entropy function ln P is without dimension.

The arrow in Eq. (2) indicates the dynamics of the system. L is not necessarily at maximum or at the most probable state, but will always try to get there. At equilibrium the arrow turns into "equal", and we obtain $L = E + T \ln P = \max!$ *Stochastic system will be stable, if the Lagrange function is at maximum!*

The meanings of the functions L, E and T will depend on the type of systems investigated: In atomic systems E will be determined by the energy of atomic interactions, L will be the free energy and T the mean energy (temperature). In social systems E will be given by the emotions of social interactions, L will be the common happiness and T the tolerance of disorder. In economic systems E will be

the property value or price of goods, L the common profit and T the average wealth or standard of living (Mimkes 2000):
Atomic systems will be stable, if the negative free energy L is at maximum!
Social systems will be stable, if the common happiness L is at maximum!
Economic systems will be stable, if the common profit L is at maximum!

Order – Disorder

Without knowledge of the ordering laws (E) of the systems in figs. 4 to 7 we can immediately tell, which system is at order and which one is not:

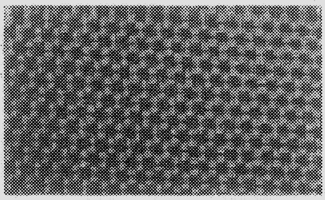

Fig. 4. *At low temperature T atomic systems are ordered, single crystals (model).*

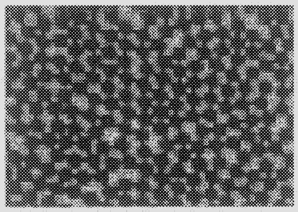

Fig. 5. *At high temperature T atomic systems generally turn into a disordered liquid (model).*

In atomic systems we have at low temperatures (T) crystalline order and at high temperatures disordered liquids or gases. The same is true in social systems:

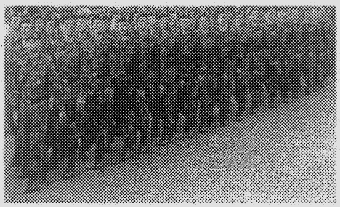

Fig. 6. *At a parade (low tolerance of disorder) soldiers march in perfect order.*

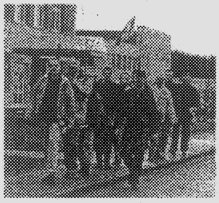

Fig. 7. *After work (high tolerance of disorder) the soldiers relax in disorder. (With kind permission of the Bundeswehr)*

$$\begin{aligned} L &= E + T \ln P \rightarrow \text{maximum!} \\ L &= \text{order} + T \text{ disorder} \rightarrow \text{maximum!} \end{aligned} \qquad (2a)$$

Eq. (2 a) is valid in all stochastic systems with constraints. At low values of T order will be at maximum, at high values of T disorder will have a maximum.

The Interaction Model of Heterogeneous Agents

Atomic systems interact by chemical bonds, atoms may attract or repel each other, or they are indifferent. Social agents interact by emotional bonds like love or hate, they may like and attract each other, dislike and repel each other or they are indifferent. This is in agreement with the ideas of Empedocles. The Lagrange function will depend on the K different agent groups N_i. In this system L, E and T have the dimension of an emotion and they may be regarded as the common happiness (L), love or hate (E) and tolerance (T). The function E may be developed into a Taylor series of the relative group size $x_i = N_i / N$:

$$E(x_i) = N\{ e_0 + \Sigma E_i x_i + \Sigma \Sigma \varepsilon_{ik} x_i x_k + . \} \tag{3}$$

$E_0 = N e_0$ is an emotion that keeps all groups together to build the system or society. In America this may presently be the memory of September 11, that will unite all different groups. E_i may be an emotion specific to the different groups, Afro Americans are especially attached to their songs and heritage, American Indians to their traditions. The parameter ε_{ik} is the emotional interaction between different agent groups. This may the Black – White relationship in the US or the emotions of Catholics – Protestants in Northern Ireland.

The probability function $P(N_i)$ is determined by the laws of combinations, Eq.(1). For K different groups with the relative size x_i and $i = 1, 2, .. K$ we find

$$\ln P(N_i) = \ln(N!) - \Sigma \ln(N_i!) = - N\{\Sigma x_i \ln(x_i) + \ln K\} \tag{4}$$

We can now write down the Lagrange function of a system of K different groups of interacting agents:

$$\begin{aligned} L(N_i) = N\{ & e_0 + \Sigma E_i x_i + \Sigma \Sigma \varepsilon_{ik} x_i x_k + \dots \\ & - T[\Sigma x_i \ln(x_i) + \ln K]\} \quad \rightarrow \text{maximum!} \end{aligned} \tag{5}$$

$L(N_i)$ is the model function for interactions of N heterogeneous agents, x_i is the relative number of agents in each group. The functions e_0, E_i, ε_{ik} represent the emotional interactions of the agents in the system. The agents may be in any probable or improbable state, the system will be at equilibrium only, if the Lagrange function (common happiness of the agents) is at maximum.

The time dependence of the function L may be simulated by optimizing the happiness for each agent. This is the basis of the Monte Carlo computer simulations, fig. 3. Each agent will be randomly asked, if trading positions with a given neighbor will increase his happiness. If it does, he may change positions. Fig. 3 has been obtained in this way to show the distribution of Serbs and Bosnians in Bosnia. The present model of interacting heterogeneous agents, Eq. (5) includes the Ising model of magnetic interactions (Becker 1966) as well as Schelling's model of social interactions (Schelling 1971).

Application: Systems of Homogeneous Agents

In homogeneous atomic systems we have two different simple structures,
$E_{AA} \neq 0$: *chemical bonds and order (solid)*
$E_{AA} = 0$: *no chemical bonds and disorder (gas or liquid at pressure $p > 0$).*

Solid, liquid and gas state depend only on the order parameter T. At low temperatures we have the ordered state of a solid, at high temperatures we obtain the disordered liquid or gas state. The critical temperature T_C of the phase transition depends on the values of E_{AA} of the system.

In homogeneous socio-economic systems we again have two structures,
$E_{AA} \neq 0$: *socio-economic bonds and order* (*hierarchy* like in all non democratic states, in many organizations and most companies)
$E_{AA} = 0$: *no socio-economic bonds, disorder (global system or democracy).*
Hierarchy, democracy and global state of a socio-economic system depend only on the order parameter T. At low standard of living we have the "ordered" state of a hierarchy (most European states 100 years ago, most African states today), at high standard of living we find democratic states (USA, Europe, Japan), fig. 8.

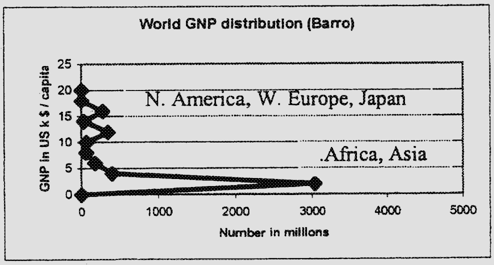

Fig. 8. *The distribution of the world Gross National Product (GNP) in US \$ per person (1995) is quite uneven. N. America, W. Europe. Japan are at the top and democratic, C. Africa, S. E Asia are in a very poor, hierarchic state (after Barro 1995).*

This figure indicates a transition point T_C from hierarchy to democracy at a living standard between 5 k\$ and 10 k\$ per person. Below 5 k\$ / person countries are not or not fully democratic, above 10 k \$ per head all countries are stable democracies. A more detailed discussion (Fründ 2002) shows that the critical living standard T_C of the phase transition (revolution, crisis) depends on the magnitude of cultural and religious interactions E_{AA} of the system. Fundamental religious systems like Saudi Arabia are not democratic even at very high income levels.

A similar distribution of hierarchy and democracy is observed in economic systems. Most people like blue or white collar workers are employed under hierarchic conditions in companies or state organizations at lower wages. They follow the orders of the employer and have little freedom of decision. Only few people like managers, lawyers, medical doctors at a top income are free and work at their own responsibility.

Hierarchy, democracy and free (global) state are part of our own growing up. Children are bound to their parents, they cannot care for themselves. After puberty children have more duties and freedom, but they still live with the family. Only with the first fulltime job after graduation young people are free to move where ever they want to.

Crisis, revolution, phase transitions

In atomic systems a change of phases is called melting or evaporation. It is a structural change from order to disorder, which requires energy (heat of melting). In socio – economic systems the change from hierarchy to democracy is called revolution or crisis. It is a structural change from strict order to freedom and requires human energy and high costs. In economic systems a start of self employment requires a lot of energy and money. In families puberty and graduation change the structure of the family and cost emotional energies and money. The costs (ΔE) of the transition from order, $E \neq 0$ to disorder, $E = 0$ at $L_{order} = L_{disorder}$ may be calculated from Eq. (2),

$$\Delta E = T_c \ln P_{disorder} / P_{order} = T_c \Delta S. \tag{6}$$

In atomic systems T_c is the critical transition point, e.g. melting temperature, in political systems T_c is the critical standard of living that is needed to change from hierarchy to democracy, from feudalism to capitalism.

Application: Systems of Heterogeneous Agents

If we restrict the model to binary agent systems, e. g. countries with two different groups like catholic – protestant, black – white, citizen – foreigner etc. we have four emotional interactions ε_{ik} between the two agent groups A and B: E_{AA}, E_{BB}, E_{AB} and E_{BA}.

The happiness L of a system of two agent groups, Eq.(5), is now simplified,

$$L(x) = N\{E_{AA} + (E_{BB} - E_{AA})\,x + \varepsilon\, x\,(1 - x) \tag{7}$$

$$- T\,[x \ln x + (1 - x) \ln (1 - x) + \ln 2]\,\}$$

$$\varepsilon = (E_{AB} + E_{BA}) - (E_{AA} + E_{BB}) \tag{8}$$

The variable x represents the relative size of the minority group and $(1 - x)$ the majority group. The parameter ε_{ik} has turned into ε in Eq.(8) and is determined by the difference in emotions towards the group of the others $(E_{AB} + E_{BA})$ and towards the own group, $(E_{AA} + E_{BB})$.

With $\varepsilon = (E_{AB} + E_{BA}) - (E_{AA} + E_{BB})$ we find at least six different emotions and the corresponding social structures in heterogeneous agent systems:

1. $\varepsilon > 0$ and $E_{AB} \neq E_{BA} > 0$ humility and hierarchy
2. $\varepsilon > 0$ and $E_{AB} = E_{BA} > 0$ love and cooperation
3. $\varepsilon < 0$ and $E_{AB}, E_{BA} > 0$ self-conceit and segregation
4. $\varepsilon < 0$ and $E_{AB}, E_{BA} < 0$ hate and aggression
5. $\varepsilon = 0$ and $E_{AB}, E_{BA} > 0$ mutual respect and democracy
6. $\varepsilon < 0$ and $E_{AB}, E_{BA} = 0$ selfishness and free (global) state

Humility and Hierarchy

Hierarchy is found at positive values of ε and numbers of coordination $N_A/N_B < 1$. The profit of both parties is positive, but not be equal, one profits more than the other, $E_{AB} \neq E_{BA} > 0$. This usually leads to humility. We may think of a company director and workers, a leader and his group, a monarch and his country, a teacher and his students. Even though atomic interactions are always symmetric, hierarchy is also found in crystal growth, the symmetry of the nucleus dominates the entire system.

Love and Cooperation

Cooperation is determined by positive ε, if the sympathy to the partner group is stronger than to the own group, $(E_{AB} + E_{BA}) > (E_{AA} + E_{BB})$, like in marriage between men and women or trade between buyer and seller. The coordination number usually is $N_A/N_B = 1$. Both partners profit equally from cooperation, $E_{AB} = E_{BA}$. However, we have to realize that two functions $E_{AB} = E_{BA}$ are rarely ever equal up to the last digit. True partnership like marriage or trade partners are ideal states. In atomic systems partnership is found in chemical compounds like NaCl or GaAs.

Self-conceit and Segregation

Self-conceit, a stronger attraction to the own group than to other groups, or $0 < (E_{AB} + E_{BA}) < (E_{AA} + E_{BB})$ leads to segregation and competition. Even though there are positive feelings E_{AB} between group A and B, people of both groups will prefer to be with their own group most of the time. This behavior is very common, it leads to segregation into the different countries of the EU, or predominantly black and non black areas in the US. Segregation leads to multicultural societies and to competition in markets. In atomic systems segregation corresponds to limited solubility of alloys or chemical solutions.

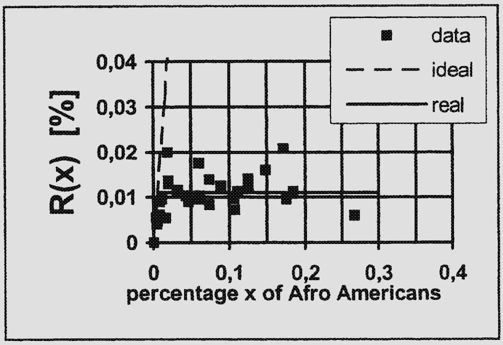

Fig. 9. *Rate (R) of intermarriage between African Americans and Whites in 33 US states (US 1988). Intermarriage is ideal below the "solubility limit" x = 0,55 % African Americans in a state. Above the "solubility limit" we find a constant rate of intermarriage, $T_C = R = 1,1$ %, indicating the "social equilibrium temperature" of the different US states.*

Fig. 9 shows the intermarriage of Afro-Americans and Whites in 33 US states. Intermarriage corresponds to solubility. At equilibrium rate of intermarriage (R = 1.1 %) about x = 0,5 % Afro Americans are integrated (dissolved), completely. The equilibrium rate of intermarriage R = 1.1 % is the same all over USA, it is independent of the number of Afro Americans in a state and corresponds to the mean US standard of living, $R \equiv T_C$. Fig. 9 gives the first experimental evidence for the existence of a "social equilibrium temperature".

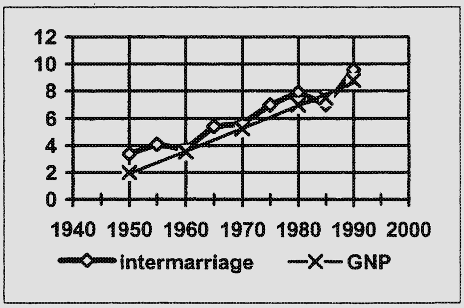

Fig. 10. *The development of Gross National Product (GNP) and rate of intermarriage of citizen and non citizen in Germany between 1950 and 1990.*

The equilibrium temperature may be interpreted as the standard of living of a country. This is demonstrated in fig. 10. The equilibrium rate of intermarriage between citizen and non citizen in Germany has been plotted with time. The data show the close relation between integration rate R and standard of living T (GNP per person).

Fig. 3 is another example for the interactions of heterogeneous agents at $\varepsilon < 0$. The simulation corresponds to the situation in Bosnia before the war in 1991.

Integration may be compared to solubility of sugar in tea. In cold tea the solubility limit will be low, at high temperatures more sugar may be dissolved. However, the solubility will be independent of the amount of sugar. If we add too much sugar, the system will segregate into sweet tea in the cup and sugar at the bottom.

Hate and Aggression

If the emotions towards other groups become negative, like for hate, envy, distrust or aggression, we have $\varepsilon < 0$ and $(E_{AB} + E_{BA}) < 0 < (E_{AA} + E_{BB})$. Negative emotions to others will lead to total separation of the groups, and vice versa, a total separation of groups will generally lead to aggression. This is presently observed in many places with binary populations like Bosnia, Northern Ireland or Israel.

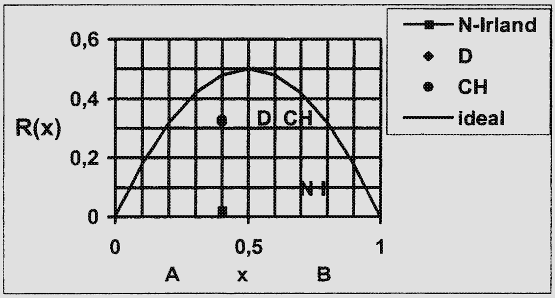

Fig. 11. *Rate of intermarriage between Catholics and Protestants in Northern Ireland R = 2,3 %, compared to R = 32 % in Germany(D) and Switzerland (CH).*

Fig.11 shows the intermarriage diagram of 40 % Catholics and 60 % Protestants in Germany, Switzerland and Northern Ireland. The intermarriage probability R (or social temperature T_C) is 34 % in Germany and Switzerland and 2.3 % in Northern Ireland! This low "social temperature" indicates the low mutual tolerance and the high potential of aggression in Northern Ireland.

The close relation between separation and negative emotions is also observed in the popular prisoner dilemma of game theory (Axelrod 1984). The separation of prisoners generally leads to distrust, $\varepsilon < 0$ and defection.

Mutual Respect and Democracy

The slogan of the French revolution, *liberté, égalité, fraternité* means mutual respect, equal, positive interactions of people, which may be expressed by the condition $\varepsilon = 0$ and $(E_{AB} + E_{BA}) = (E_{AA} + E_{BB}) > 0$.

Christian commandment „*love thy neighbor as thyself*" ($E_{AB} = E_{AA}$) is again equivalent to the same equation and asks for mutual respect of an integrated community of believers. The condition $\varepsilon = 0$ and $(E_{AB} + E_{BA}) = (E_{AA} + E_{BB}) > 0$ is an ideal equation and cannot be obtained in nature. We cannot find four different interactions to be exactly equal.

Christianity, democracy, fully integrated societies, markets of equal opportunities are ideal states, which never can be reached, completely. However, equality can be defined by law: every person or company is equal before the law! The enforcement of this law is necessary to obtain true democracy or equal opportunities. In atomic systems ideal solutions can only be approximated.

Selfishness and Global State

Indifference $\varepsilon = 0$ and $E_{AB} = E_{BA} = E_{AA} = E_{BB} = 0$ is, in contrast to the condition of democracy, very common. People at the beach, at shopping or at a restaurant do not interfere with each other. Rich people and big companies try to be independent of taxes and other obligations to any state. The global state corresponds to the gas – it is a selfish, capitalistic society, ideally integrated, free, independent and chaotic. We will now apply the model of selfish interacting heterogeneous agents to three examples of free capitalistic markets: property distribution, automobiles and jobs in Germany and the USA, respectively.

Boltzmann Distribution of Property in Germany 1993

Property data in Germany (1993) have been published by the German Institute of Economics (DIW, 1993). The data in table 1 show the number N of households and the amount of capital K in each property class (E_k).

Striking rich, selling a car, finding a job is not only a question of economic reasoning, but also very much a matter of luck and chances. This leads to the idea, that the distribution of income may also be a question of Lagrange probability Eq.(5). In free markets we have $\varepsilon = 0$. Instead of social or economic bonds the economic constraints are now given by the price E_k of item k or the property E_k of household number k,

$$L(N_i) = N\{ \Sigma E_k x_k - T[\Sigma x_k \ln(x_k) + \ln K]\} \rightarrow \text{maximum!} \qquad (9)$$

At equilibrium the derivative of Eq. (9) with respect to N_k is zero and leads to the Boltzmann distribution

$$N_k = (N_0/T) \exp(-E_k/T) \qquad (10)$$

The number N_k of persons in a property class E_k decreases exponentially with growing property. N_0 is the total number of households, the order parameter T is given by the mean property per household.

Table 1. Property distribution in Germany 1993 (DIW, estimated)

Total property or capital	9920	Bill. DM
Number of households	35,6	Mill.
Mean property	278	kDM / Hh
Property class in kDM	Number N(k) of Hh in %	Property K(k) of Hh in %
0 : = 0	1,5	0
1 : <100	44,5	9,5
2 : 100-250	24,7	17,6
3 : 250-500	20,3	28,1
4 : 500-1000	6,3	16,8
5 : >1000	2,7	28

Distribution of property in five different property classes for households in Germany 1993, (DIW 1993).

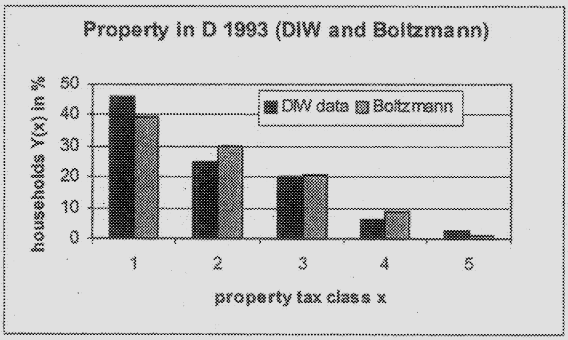

Fig. 12. *Property distribution in Germany 1993. Data estimated by the German Institute of Economy, (DIW 1993), calculation according to Boltzmann.*

Fig. 12 shows the number of households in the five property classes of table 1. The data agree rather well with the calculations according to Boltzmann, Eq. (10). The property market, apparently, is a system of free, selfish agents.

In economic literature property distributions are generally given by a Lorenz curve, showing the relative amount of capital Y as a function of the relative number of households X, fig. 13.

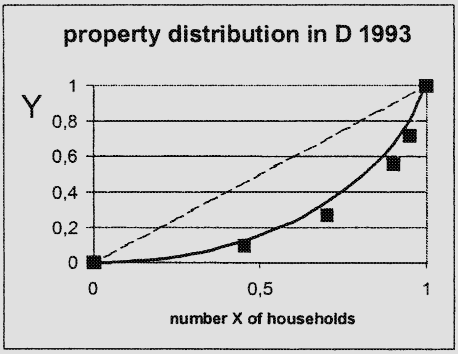

Fig. 13. *Lorenz curve Y(X) of relative property Y as a function of the relative number of households X. Data and calculations according table 1 and Eq.(10).*

The Lorenz function may be calculated by

$$X = \int N(E) / N_0)\, d E = \int (1/T) \exp(-E/T)\, d E = 1 - \exp(-E/T) \quad (11)$$

$$Y = \int E * N(E, T) / T N_0)\, d E = \int (E/T^2) \exp(-E/T)\, d E$$

$$= 1 - \exp(-E/T) - (E/T) \exp(-E/T)$$

$$Y = X + (1-X) \ln (1-X) \quad (12)$$

The calculation in Eq. (12) shows the Lorenz curve of relative property Y(X) as a function of the relative number of households X. The function has been plotted in fig. 13 shows a satisfactory agreement with the data in table 1.

Automobile Market in Germany 1998

Table 2 shows the distribution of new cars in Germany, 1998. The classes give the combustion volume in cm^3, price in DEM and the number of units in millions.

Table 2. Distribution of new cars (1998) in price classes

Class k	cm^3	price class in DM	units in Mill.
1	< 1500	19.000	0,54
2	> 1500	26.000	3,25
3	> 2000	50.000	0,72
4	> 2500	68.000	0,54

Price classes (k) of automobiles produced in Germany 1998, (Statist. Bundesamt 2000).

Fig. 14 shows the number N(E) of new cars as a function of unit price (E). The data of new cars can be fitted to the Boltzmann distribution except for the lowest value (k = 1, 20.000 DM), where a number of 7 Mill. units would be required.

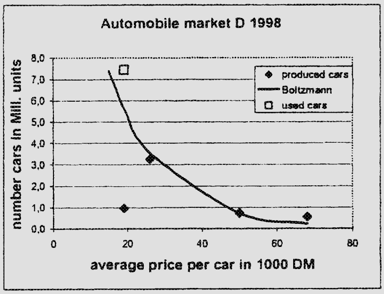

Fig. 14. *Number N(E) of new and used cars in four price classes (E) in Germany 1998. Data according Stat. Bundesamt (1998), calculation according to Boltzmann, Eq.(10).*

Apparently the new car market is not a complete market, and we have to add the used car market, as many buyers prefer cheaper used cars. Fig. 14 shows the complete market. The number of used cars sold in 1998 is nearly 7 Mill. units, as predicted by the Boltzmann distribution.

Job Market in USA

The distribution of people seeking jobs is a matter of chance and should be calculated according to the Boltzmann Eq. (10). The number of job offers N will decrease with the wage expectations (E). However, jobs below a wage minimum (E_0) will not be accepted. Fig. 15 shows the wage distribution for manufacturing in the US in 1995. A similar function has been obtained for services.

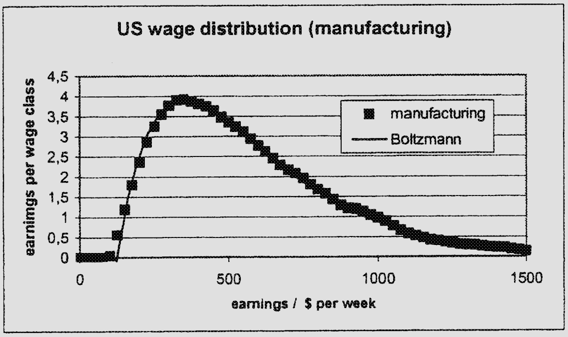

Fig. 15. *Number of people in wage class (E) of manufacturing in the US (Cleveland 1996). The data have been fitted by the Boltzmann type distribution, Eq.(13).*

The data have been fitted by the function

$$N(E, T) = (E - E_0) \exp(-E/T) \tag{13}$$

Similar results have been obtained for jobs in other countries (Fründ 2002).

Conclusion

The interaction model of heterogeneous agents, Eq. (5) has been applied to six different socio-economic systems. The discussion includes political, social and economic problems. Freedom, integration, segregation, aggression, wealth, buying a car or finding a job are a matter of probability with constraints. The general validity of the Lagrange equation in socio-economic systems corresponds to the wide range of applications of the Lagrange principle in natural systems, in thermodynamics, physics, chemistry, metallurgy and engineering. Apparently, the laws of probability are one of the common roots of natural and social sciences.

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Learning Strategies for Global Games with Delayed Payoffs

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Abstract. Economic ensembles can be modeled as networks of interacting agents whose behaviors are described in terms of game theory. The evolutionary paradigm has been applied to two-person games to discover strategies in this context. Subsequently, many-player games, and specifically global games (where payoffs depend collectively on all the rest of the players) as with the minimal game, have been studied. The minority game is attractive because it has intuitive similarities to e.g. securing niche businesses. We enhance this intuitive similarity by extending the game through the introduction of delayed payoffs. Payoffs depend on the values of future moves; we reward choices which later on become popular.

We study agents' moves in such global game with delayed payoff. Instead of an evolutionary approach we allow learning in the agents. We study strategies that may emerge through learning in agents in such games.

1 Introduction

From a physics viewpoint, dynamical systems can be understood to behave asymptotically in three ways described by point attractors, cyclic attractors and strange attractors. Economic ensembles, having large numbers of degrees of freedom, and probably nonlinear interactions, are dynamical systems. Classical economics can be associated with point attractors, while phenomena like business and economic cycles can perhaps be seen as manifestations of cyclic attractors. Of current interest is the regime of strange attractors, where complex dynamics can either produce virtually random trajectories or complex trajectories with fractal characteristics.

The study of the complex dynamics of economic ensembles consisting of interacting agents whose behaviors are described in terms of games, has been initiated through the application of the evolutionary paradigm to the two-person prisoners' dilemma game [1]. Economic ensembles can be modeled as networks of many interacting agents playing games, suggesting the use of many-player games. There have been studies [5] of many-player prisoners' dilemma, but prisoners' dilemma being inherently two-player, each player here interacts only with his immediate neighbors. From the physics point of view, elements in the network interact only

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first-order and/or locally. One interesting many-player game studied is the minority game [2] where payoffs depend on the actions of all other players, thus yielding a network with higher-order or global interactions. The minority game is attractive because it has intuitive similarities to e.g. securing niche businesses.

In this paper, we introduce a new class of games, in which in addition to having global interactions, payoffs are delayed in the sense that they depend on the values of the players' future moves. In particular, subscribing to some economic intuition, we design payoffs in such a way that we reward (near-) minority choices which later on become popular. We study agents' or players' long-time moves in such a game. We use learning agents instead of evolutionary ones and we vary the amount of information available from which the agents can learn. We also explore the effect of culture, understood as direct transmission of knowledge between agents.

2 The Model

N players or agents k each chooses one of n options i. If $i_k(t)$ is k's choice at timestep t, then payoffs p_i for moves i at t is taken to be

$$p_i(t) = \sum_k \delta\big(i, i_k(t+1)\big) - \sum_k \delta\big(i, i_k(t)\big) \tag{1}$$

where the Kronecker delta $\delta(x,y) = 1$ when $x = y$ and 0 otherwise.

Agents learn using a neural network without hidden units, by employing a variant of the back-propagation algorithm [6], or equivalently, Hebbian learning with nonlinearity. For an input u contributing to the output $i_k(t)$, the synaptic strength T^k_{ui} between them changes by

$$\Delta T^k_{ui} = \eta(p_i - h^k_i) f'(h^k_i) + \chi\iota \tag{2}$$

where η is the learning rate, h^k_i is the field for the neuron representing i_k,

$$h^k_i = \sum_u \; T^k_{ui}\, u, \tag{3}$$

$f'(x)$ is the differential of the neuron transfer function (the nonlinearity), taken to be

$$f'(x) \; = 1/(x^2+1), \tag{4}$$

and χ is a 'creativity' factor incorporating 'noise' or some 'non-zero temperature' effects via the random number ι picked from a uniform random distribution between 0 and 1. Initial synaptic values are small random numbers equally likely to be positive or negative. Without learning ($\eta = 0$), option choices are random.

Agents decide on options through a competitive network [4] involving the final layer neurons; in particular, the agent k chooses the option I if $h^k_I = \max(\{h^k_i\}_i)$. We further defined the memory M of agents as the number of timesteps back that they remember of payoffs and moves. Cases where $M = 0$ then are equivalent to random choices.

Inputs u to the field for decision and to the learning may consist only of each respective agent's own previous moves ('self learning'). A more plausible scenario is where u also include previous resulting payoffs ('learning with payoff'), and with information, there is also the case where in addition to these, the previous moves of other agents contribute to the decision-making and the learning ('learning with information'). A model of culture is also explored where there is a direct contribution to the 'knowledge' stored in an agent's synapses from those of the others:

$$T^{k}_{ui} := \varepsilon\,(\textstyle\sum_{k' \neq k} T^{k'}_{ui}) + (1-\varepsilon)\, T^{k}_{ui} \qquad (5)$$

With ε being the 'culture factor' and this inclusion of culture is carried out after learning (with information) is done ('learning with culture).

3 Simulation and Results

We carried out studies on the model using computer simulations. Specifically, 10 agents choose between 5 options. The following parameters were used: initial randomness (maximum magnitude of initial random value for synapses) 0.1, creativity 0.1, culture factor 0.2 and learning rate 0.1. We studied (100 random trials) long-time behavior (after 1000 timesteps) of cases of no learning, self learning, learning with payoff, learning with information and learning with culture, each with zero memory, and memories of 1, 3 and 7. We investigated the mean (between agents) average (over timesteps) payoffs obtained by agents, which measures the average performance of the players, and the standard deviation (between agents) of the average payoffs, which gives a measure of variety of the performances.

For the cases of no learning, we obtain a well-defined cluster of means of average payoff at around -0.8 and standard deviations between about 0.02 to 0.08. This is shown in the figure below. Learning with zero memory also yields a similar output, as expected.

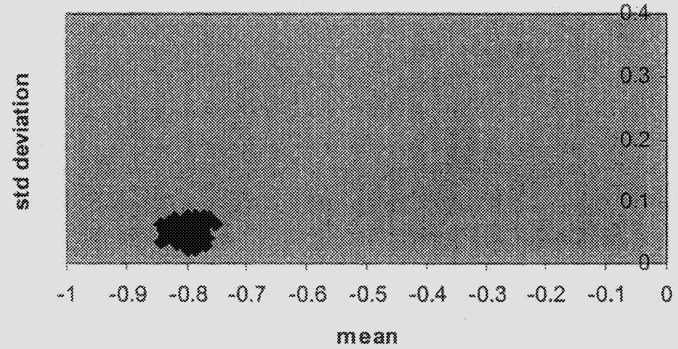

Fig. 1. Standard deviation vs mean of average cumulative payoffs for the case of no learning and memory 0

Self learning generally betters the average performances without significantly increasing variances. This is portrayed in the following graphs. An interesting thing to note is the better average performance when only immediately recent states are remembered.

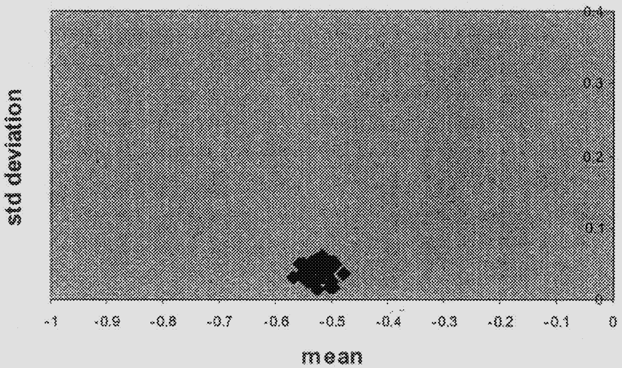

Fig. 2. Standard deviation vs mean of average cumulative payoffs for the case of self learning and memory 1

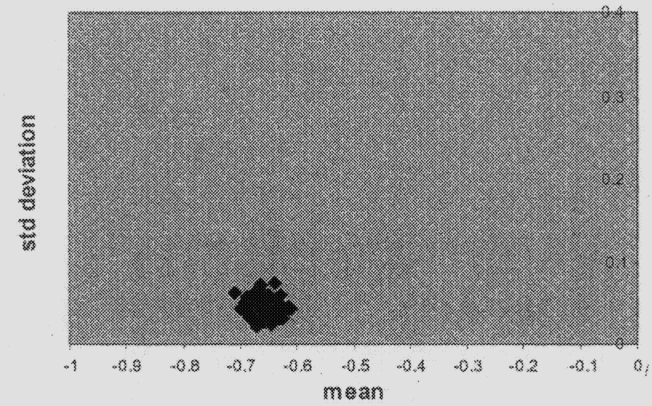

Fig. 3. Standard deviation vs mean of average cumulative payoffs for the case of self learning and memory 3

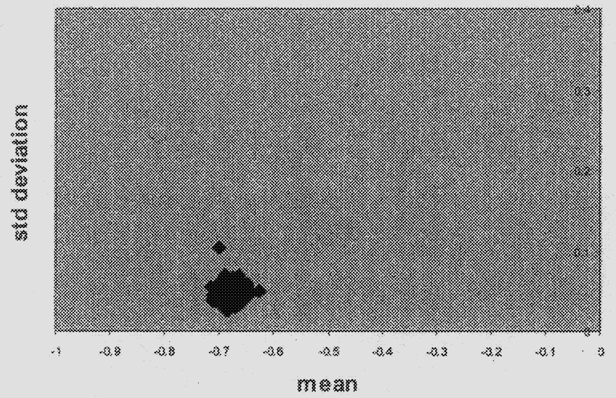

Fig. 4. Standard deviation vs mean of average cumulative payoffs for the case of self learning and memory 7

Given knowledge of resulting payoffs, agents should learn to perform better. This is depicted by higher means, as is evident in the following graphs. Increased standard deviations, reflecting emergence of variety, can also be seen, although with more memories, this seems to decrease.

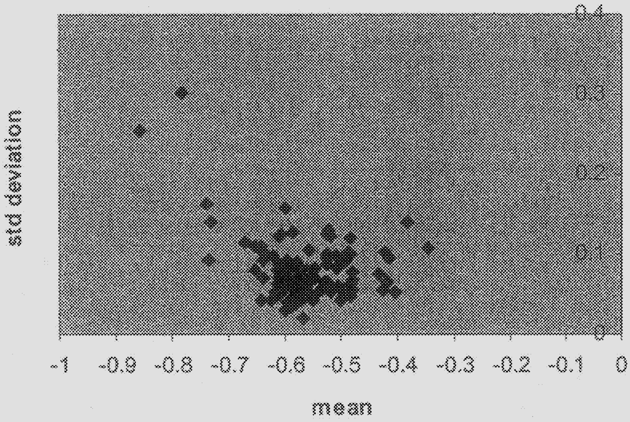

Fig. 5. Standard deviation vs mean of average cumulative payoffs for the case of learning with payoff and memory 1

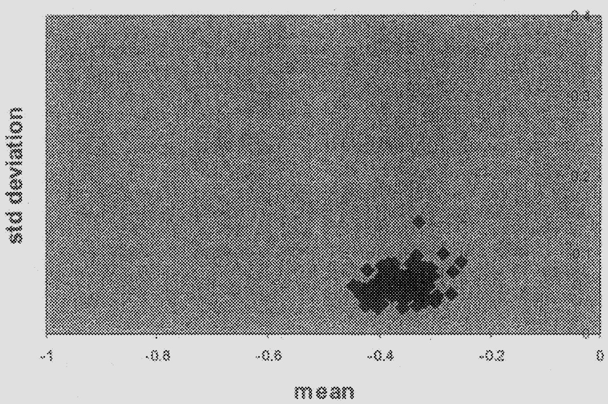

Fig. 6. Standard deviation vs mean of average cumulative payoffs for the case of learning with payoff and memory 7

Learning with information shows similar behavior to learning with payoff, both with means as well as standard deviations, almost quantitatively.

Learning with culture smears the means and makes them vary substantially, but the standard deviations remain more or less in the band defined by no learning. The results of learning with culture with M=3 is shown below. Note the different scale for the mean.

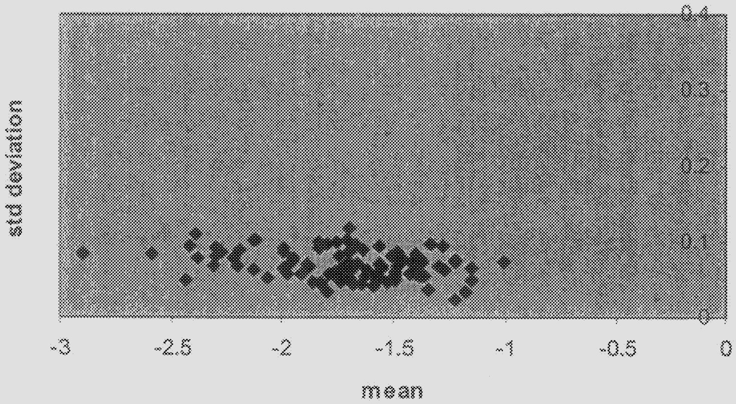

Fig. 7. Standard deviation vs mean of average cumulative payoffs for the case of learning with culture and memory 3

In terms of strategies learnt, one (which is easy to detect) is that of holding on to the same option. We find that some winning agents employ this strategy, though employing this strategy doesn't seem to guarantee winning all the time. The existence or not of less simplistic strategies requires further studies.

4 Conclusion and Discussion

We have introduced a new class of games, where payoffs are delayed. Our studies show how agents can learn to perform better in such games, even without outside information. Giving information on complete payoff vectors seem to be almost equivalent to also giving information on other agents' moves. Information as such increases agents' average performance as well as brings rise to increased variety. Heterogeneity in agents can perhaps be explained through such emergence. Culture as modeled here does not give rise to heterogeneity between agents, but does give rise to an array of heterogeneous overall performances. Because of culture agents behave similarly, but these common behaviors give rise to varied performances between trials.

Evolutionary prisoners' dilemma has brought about the surprising emergence of cooperation, while the minority game has shown the emergence of groups with extreme behaviors [3]. Perhaps the emergence of heterogeneity can be studied using the game proposed in this paper.

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