ELSEVIER

Journal of Economic Behavior & Organization
Vol. 33 (1998) 333-362

JOURNAL OF
Economic Behavior
& Organization

On the consistency of backward-looking expectations: The case of the cobweb

Cars H. Hommes*

Department of Economics and Tinbergen Institute, University of Amsterdam, Roetersstraat 11,NL-1018 WB, Amsterdam, The Netherlands

Received 6 July 1995; accepted 22 January 1996

Abstract

In dynamic models of economic fluctuations backward-looking expectations with systematic forecasting errors are inconsistent with rational behaviour. In non-linear dynamic models exhibiting seemingly unpredictable, chaotic fluctuations, however, simple habitual 'rule of thumb' backward-looking expectation rules may yield non-zero but nevertheless non-systematic forecasting errors. In a chaotic model expectational forecasting errors may have zero autocorrelations at all lags. Even for rational agents patterns in these forecasting errors may be very difficult to detect, especially in the presence of (small) noise. Backward-looking expectations are then not necessarily inconsistent with rational behaviour. We investigate whether simple expectation schemes such as naive or adaptive expectations can be consistent with rational behaviour in the simplest of all non-linear dynamic economic models, the non-linear cobweb model. © 1998 Elsevier Science B.V.

JEL classification: D84; E32

Keywords: Consistent expectations; Non-linear cobweb model; Chaos; Autocorrelations

1. Introduction

Expectations or beliefs play an important role in economic modelling. In making decisions economic agents form expectations about future values of variables. Therefore, in any dynamic economic model, people's beliefs about, say future prices, have to be modelled. There has been a long and still ongoing debate among economists concerning

* Corresponding author.

0167-2681/98/$19.00
PII S0167-2681(97)00062-0

expectation formation. Several decades ago it was common practice to use simple habitual 'rules of thumb' to describe expectations, as a first approximation. Well-known examples include *naive expectations* (tomorrow's expected price is today's price) and *adaptive expectations* (expected price is a weighted average of the previous price and the previous expected price or equivalently a weighted average, with geometrically declining weights, of all past prices). These are only two simple examples of more general *backward-looking expectations* where the expected price is some function of past prices.

It has been argued that there are at least two conceptual problems with simple backward-looking expectations however. Firstly, the actual dynamics of the model depends upon the expectations function. For example, the local stability of a steady state depends on which type of expectation scheme is used. Agents' beliefs thus influence the outcome of economic analysis. Secondly, simple expectational rules may lead to systematic forecasting errors. In particular, the early non-linear business cycle models of Kaldor (1940), Goodwin (1951) and Hicks (1950) have been criticized because expectations were systematically wrong along the business cycles. Rational agents would recognize these cyclic patterns and revise their expectations accordingly.

The Rational Expectations Hypothesis (REH), as introduced by Muth (1961), offers a solution to both problems. In its strongest form, the REH assumes that agents make no systematic forecasting errors and that they have perfect knowledge about the model. Agents use the equations of the model to compute their predictions for the future (e.g. Begg, 1982 or Sheffrin, 1983). Without perfect knowledge about the model, however, it is not clear how agents would reach the RE equilibrium. One possibility would be that agents use some (sophisticated) learning process under which the dynamics might converge to the RE equilibrium (e.g. Marcet and Sargent, 1988). On the other hand, even sophisticated learning processes need not always converge to a RE equilibrium (e.g. Grandmont and Laroque, 1991 or Grandmont, 1994).

The bounded rationality literature (e.g. Sargent, 1993) has recently criticized the REH for the unrealistic assumption that rational agents know even more than the model builder. Instead, it would be more realistic to assume that agents are only *boundedly rational*, not knowing the equations of the model but behaving like econometricians, computing their predictions from econometric estimates (e.g. least squares learning) based upon past observations. Boundedly rational agents thus observe time series and use (sophisticated) econometric techniques to form expectations.

The discovery of deterministic chaos in simple non-linear dynamic models sheds new light on the role of expectations in economic modelling. Chaotic time paths can look very erratic and may be difficult to distinguish from random time series. Indeed chaotic time series can have zero autocorrelations at all lags (Bunow and Weiss, 1979, Sakai and Tokumaru, 1980 and Hall and Wolff, 1993). This fact may have important consequences for the modelling of expectations: in non-linear dynamic economic models expectational errors of simple backward-looking expectation rules may have zero autocorrelations at all lags. We will call such predictors *consistent expectations*. Simple habitual rule of thumb expectational rules which are consistent need not be inconsistent with rational behaviour. From a linear statistical point of view, expectational errors of a simple consistent predictor are indistinguishable from white noise.

In this paper, we investigate whether for the simplest of all non-linear dynamic economic models, the cobweb model, simple backward-looking expectational rules such as naive or adaptive expectations can be consistent, when demand and supply are non-linear and price fluctuations are chaotic. We also investigate the consistency of expectations in the cobweb model with heterogenous beliefs, as introduced in Brock and Hommes (1997), where agents make a rational choice between two different types of price expectations.

The most important difference between our approach and the REH is that agents do not have knowledge about the equations of the model, but base their prediction of future variables only upon observations of past values. The main difference with most of the bounded rationality literature is that our agents do not use (sophisticated) statistical techniques, but a simple 'habitual rule of thumb' expectational rule which does not change over time. If this fixed predictor is consistent and has no systematic forecasting errors there would be no reason to change beliefs.

The paper is organized as follows. Section 2 defines what we mean by consistent expectations. The next sections investigate the consistency of different expectation schemes in the cobweb model. Section 3 considers the case of naive price expectations with a *non-monotonic* supply curve. In all other sections both demand and supply are *non-linear* and *monotonic*. The case with adaptive expectations is investigated in Section 4, whereas Section 5 deals with linear backward-looking or distributed lag expectations, where expected price is a weighted average of past prices. In Section 6 we consider the cobweb model with heterogenous beliefs. Finally, Section 7 concludes and in particular relates the present paper to two related papers by Sorger (1994), Lorenz (1995).[1]

2. Consistent expectations

In this section we define what we mean by *consistent expectations*. Consider a general class of models, where the market equilibrium price is determined by

$$p_t = F(p_t^e), \tag{1}$$

with p_t^e is agents' expected price formed at the beginning of period t. Let price expectations be given by

$$p_t^e = H(\vec{P}_{t-1}), \tag{2}$$

with $\vec{P}_{t-1} = (p_{t-1}, \ldots, p_{t-L})$ a vector of past prices (with a finite number of lags L). H is called the predictor or expectation function or the perceived law of motion. Substituting Eq. (2) into Eq. (1) yields

$$p_t = F(H(\vec{P}_{t-1})) \tag{3}$$

In Sargent's (1993) terminology Eq. (3) is the *actual law of motion* when the perceived law of motion is H. We will define consistency of expectations by means

[1] Both papers were presented at the same workshop as the present paper.

of the autocorrelation function (ACF) of expectational errors. Expectational errors are given by

$$e_t = p_t - H(\vec{P}_{t-1}) = F(H(\vec{P}_{t-1})) - H(\vec{P}_{t-1}). \tag{4}$$

Recall that the (empirical) autocorrelation coefficients ρ_k of expectational errors e_t are defined as (e.g. Box et al., 1994)

$$\rho_k = \frac{c_k}{c_0}, \quad -1 \leq \rho_k \leq +1 \tag{5}$$

with

$$\bar{e} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} e_t \tag{6}$$

$$c_k = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N-k} (e_t - \bar{e})(e_{t+k} - \bar{e}), \quad k \geq 0. \tag{7}$$

We are now ready for the definition of consistent expectations. Assume that all (expected) prices are bounded and contained in some interval $I=[p_{\min}, p_{\max}]$. Let the bounded set $B \subset \Re^L$ be the cartesian product of the interval I.

Definition. *The expectations function H:$B \to \Re$ is called*

1. **consistent** *if for Lebesgue almost all initial states $\vec{P}_0 \in B$ the autocorrelation coefficients ρ_k of the expectational errors are zero for all $k \geq 1$.*
2. **weakly** *consistent if there exists a $K \geq 2$, such that for Lebesgue almost all initial states $\vec{P}_0 \in B$ the autocorrelation coefficients ρ_k of the expectational errors are zero for all $k \geq K$.*
3. **inconsistent** *if it is not weakly consistent.*

In a deterministic model, rational expectations coincide with perfect foresight, with zero expectational errors. In a stochastic model, rational expectations imply zero autocorrelations of expectational errors. Hence, rational expectations are consistent in our terminology. However, the REH assumes that agents have perfect knowledge about the model and solve a fixed point problem to obtain the RE predictor with zero ACF. In our framework agents do not know the equations of the model, but only observe time series comparing actual realizations with predictions. A consistent predictor has the same (empirical) ACF as rational expectations. Agents using linear statistical techniques would not be able to distinguish the errors of a consistent predictor from white noise. If a predictor is consistent, even for rational agents there would thus be no reason to change beliefs. For a weakly consistent predictor, from a (linear) statistical point of view, expectational errors behave like a weakly dependent stochastic process. If a weakly consistent predictor has an empirical ACF with some strongly significant non-zero autocorrelation coefficients, rational agents might learn from this and change beliefs accordingly. However, only strongly significant autocorrelations surviving some (small) amount of noise could be detected from observing time series. An inconsistent predictor has significant autocorrelation coefficients (of arbitrarily high order). Rational agents would discover these from observing time series and would change beliefs.

An important motivation for our definition of consistent expectations is the fact that simple deterministic chaotic processes can have zero autocorrelation coefficients at all positive lags.[2] This observation has important consequences for expectation formation in economic modelling: expectational errors of simple habitual 'rule of thumb' predictors like naive or adaptive expectations need not be inconsistent with rational behaviour. Even for rational agents, the patterns in chaotic expectational errors with (close to) zero ACF might be very difficult to detect, especially in the presence of (small) noise. On the other hand, not all chaotic time series have zero ACF. In fact, many chaotic time series have significant non-zero autocorrelation coefficients. Examples include the quadratic difference equation $x_{t+1} = \lambda x_t(1 - x_t)$, for many parameter values $\lambda<4$, and also for example, the famous Lorenz or Rössler strange attractors. It should be clear that if a strange attractor consists of a finite number of disjointed (closed) sets with time paths jumping in a fixed order between these disjoint pieces of the attractor, the chaotic time paths have non-zero ACF. Hence, having a 'one-piece' strange attractor is a prerequisite for zero ACF. In general, however, ACF of chaotic time series and, in particular, their dependence upon parameters and noise are still an open issue.

In our definition of consistency, we use the notion of "for Lebesgue almost all initial states." The reason for this is the following. In a chaotic process like the quadratic map $x_{t+1} = \lambda x_t(1 - x_t)$, with $\lambda=4$, for Lebesgue almost all initial states $x_0 \in [0,1]$ the corresponding ACF is zero. However, this map also has infinitely many unstable periodic points, which obviously have a non-zero, periodic ACF. Hence, in a typical chaotic system one can only have zero ACF for Lebesgue almost all initial states, with a zero Lebesgue measure set of exceptional initial states with non-zero ACF.

At first sight, it may seem that our definition of consistency suffers from another problem. In a chaotic system, very often small changes of parameters can change the dynamical behaviour from a strange attractor into a stable periodic cycle. In the case of periodic fluctuations a simple backward-looking expectation rule H would be inconsistent (unless it coincides with perfect foresight). Hence, with a small change in the parameters of the model a consistent predictor may change into an inconsistent predictor. Although strictly speaking this is true, we believe the definition is still useful when we take (small) noise into account. Small changes in parameters may destroy a strange attractor and lock the system into a stable cycle, but very often the system remains topologically chaotic, that is, there still exists an unstable Cantor-like set (with Lebegue measure 0) with chaotic dynamics. Although in the long run time paths converge to a stable cycle, in the presence of (small) noise the time paths may become erratic again, because due to the noise they get close to the chaotic orbits in the Cantor set. In a topologically chaotic system a small amount of noise can lead to forecasting errors with small or fast decaying autocorrelation

[2] Sakai and Tokumaru (1980) show that the well-known piecewise linear tent map has zero ACF. Bunow and Weiss (1979) investigate empirical ACF of 1-dimensional chaotic difference equations both with and without noise and present several examples with zero ACF. Hall and Wolff (1993) investigate the strength of statistical dependence of chaotic time series from the quadratic recurrence $x_{t+1}=\lambda x_t(1-x_t)$. Among other things, they show that the quadratic map has zero ACF for $\lambda=4$.

coefficients. In the presence of small noise, our notion of consistent expectations can therefore be *robust* against small perturbations of the system parameters.

Although our approach can be used in a general dynamic model as in Eq. (3), we only investigate consistency of expectations in the context of the simplest of all dynamic economic models: the cobweb model. The cobweb model describes price–quantity fluctuations in a single market with one non-storable good taking one unit of time to produce. Market equilibrium is described by

$$D(p_t) = S(H(\vec{P}_{t-1})) \tag{8}$$

where D is the demand curve, S the supply curve, $\vec{P}_{t-1} = (p_{t-1}, \ldots, p_{t-L})$ a vector of past prices and $H(\vec{P}_{t-1})$ producer's expected price at the beginning of production period t. The actual law of motion for the market clearing price is then

$$p_t = D^{-1}S(H(\vec{P}_{t-1})), \tag{9}$$

which is an example of Eq. (3) with $F{=}D^{-1}S$. In order to keep the model as simple as possible, throughout the paper we assume linear demand given by

$$D(p_t) = a - bp_t. \tag{10}$$

In most cases in the next sections, supply will be an increasing non-linear curve, except in the case of naive expectations in Section 3 where supply will be non-monotonic.

For both, the classical case of naive expectations (Ezekiel, 1938) and for adaptive price expectations (Nerlove, 1958), the 'hog-cycle' in the cobweb model is characterized by persistent up and down price oscillations with systematic expectational forecasting errors, especially when demand and supply are linear. These considerations led Muth (1961) to the introduction of rational expectations (RE) into the cobweb model. In the deterministic cobweb model the assumption of rational expectations or perfect foresight implies that prices will be at their equilibrium steady-state level (the intersection point of demand and supply) for all time. In this paper we focus on the following problem. *Do there exist simple (linear) expectation functions H and non-linear (monotonic) demand and supply curves such that the predictor H is consistent in the cobweb dynamics* (Eq. (9))?

3. Naive expectations

Naive or static price expectations are given by

$$H(\vec{P}_{t-1}) = p_{t-1}, \tag{11}$$

that is, the expected price equals the most recently observed price. The price dynamics in the classical cobweb model with naive expectations is described by

$$p_t = D^{-1}S(p_{t-1}) \tag{12}$$

It is well known that when both demand and supply are *monotonic*, then only three types of price dynamics can occur: (i) convergence to a stable equilibrium steady-state price, (ii) convergence to a stable period 2 up and down price oscillation (the 'hog-cycle') or (iii) unbounded, diverging up and down price oscillations. When the steady-state price is unstable naive expectations are clearly inconsistent, since the ACFs of both prices and expectational errors have period 2, with negative autocorrelation coefficients ρ_k at odd lags and positive autocorrelation coefficients at even lags.

In the case of *non-monotonic* supply and/or demand curves, however, price dynamics can be much more complicated. Artstein (1983), Jensen and Urban (1984), Lichtenberg and Ujihara (1989) and Day and Hanson (1991) have shown that with a decreasing demand curve and a non-monotonic supply curve, the cobweb model with naive expectations can generate chaotic price fluctuations. A non-monotonic supply curve could be justified by an 'income effect.' For example, in an agricultural market at high market prices farmers might decide to work less and consume more leisure instead. Then supply increases at low expected prices but decreases at high expected prices. We consider an example along these lines, since it will lead to chaotic price fluctuations with zero ACF.

Let T be the 'upside down' tent map defined as

$$T(x) = \begin{cases} -2x + 1, & \text{if } 0 \leq x \leq 0.5 \\ 2x - 1, & \text{if } 0.5 < x \leq 1 \end{cases} \tag{13}$$

For Lebesgue almost all initial states $x_0 \in [0,1]$ the time series $\{T^j(x_0)\}_{j=0}^{\infty}$ has zero autocorrelation coefficients ρ_k at all positive lags k. This has been proven by Sakai and Tokumaru (1980), where ACFs of this and other piecewise linear 1-D tent maps have been investigated. The tent map (Eq. (13)) is a well-known example of a chaotic system with zero ACF. It is now easy to find a linear demand curve and a piecewise linear non-monotonic supply curve such that prices in the corresponding cobweb model with naive expectations are generated by the 'upside down' tent map (Eq. (13)). Take, for example, the linear demand curve D and the non-monotonic piecewise linear supply curve S

$$D(p) = 1 - p \tag{14}$$

$$S(p^e) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 0.5 \\ 2 - 2x, & \text{if } 0.5 < x \leq 1 \end{cases} \tag{15}$$

The price dynamics in (Eq. (12)) is then exactly described by $p_{t+1} = T(p_t)$. Fig. 1(a) and (b) show a typical chaotic price time series and its sample ACF at the first 20 lags.[3] All empirical ACFs in this paper are based upon 240 observations after a transient of 100. The dotted lines indicate two times the standard deviation if the data were truly random. Only autocorrelation coefficients outside the dotted lines are significantly different from 0 at the 5% level. Fig. 1(b) therefore indicates that prices are uncorrelated and from a linear statistical point of view indistinguishable from white noise.[4]

Expectational errors are given by $e_t = p_t - p_t^e = p_t - p_{t-1} = T(p_{t-1}) - p_{t-1}$. Fig. 1(c) and (d) show the corresponding time series of expectational errors e_t and its sample ACF at the first 20 lags. Although prices have zero ACF, expectational errors do not have zero

[3] Actually Fig. 1 corresponds to the tent map with slopes 1.99 and −1.99 instead of 2 and −2, since in the latter case all time series will converge to the unstable fixed point 0 on a computer with computations in bits.

[4] There are, of course, other 'nonlinear' techniques to distinguish between (low dimensional) deterministic chaos and a random IID series surveyed, for example, in Brock et al., 1991). In the present case, simply plotting (p_{t+1}, p_t) would reveal that prices are generated by the tent map. In general, however, for higher dimensional systems as in the next sections, this simple method would not work. Moreover, methods to distinguish between chaos and white noise usually require very long time series and are very sensitive to noise.

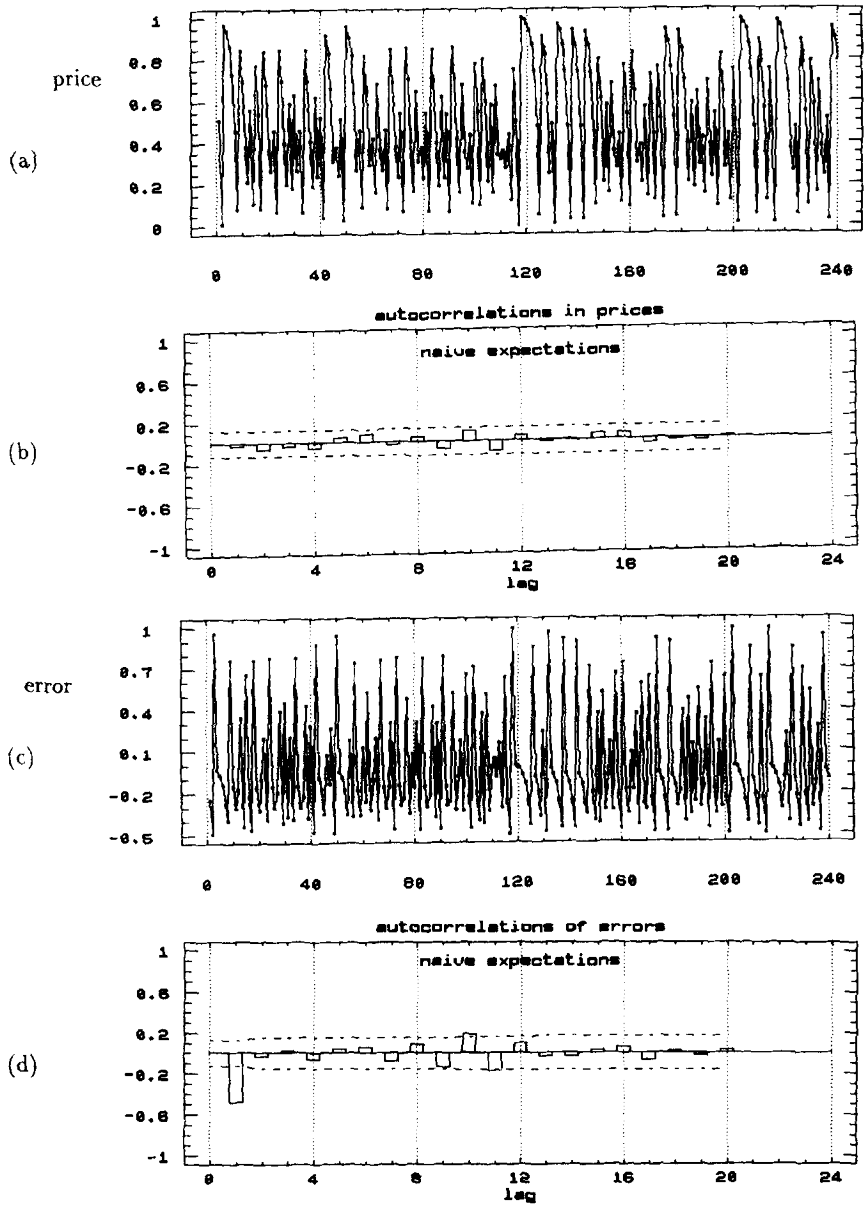


Fig. 1. (a)–(b) Chaotic price fluctuations with zero ACF. (c)–(d) Corresponding expectational errors with ACF.

ACF, but a strongly statistically significant negative autocorrelation coefficient $\rho_1 \approx -0.5$ at the first lag. All higher order autocorrelations are statistically not significantly different from zero, except at lags 10 and 11. Autocorrelations at the latter two lags, however, would disappear in the presence of small noise.

We conclude that, for monotonic demand and supply, price dynamics in an unstable cobweb are simple and naive expectations are inconsistent, since the ACF of expectational errors has period 2. In the presence of a sufficiently strong 'income effect,' leading to a non-monotonic supply curve, price fluctuations can be chaotic with zero ACF. However, naive expectations are not consistent, but only weakly consistent, with strong negative autocorrelation in errors at the first lag.

4. Adaptive expectations

In this section we investigate the consistency of another simple and frequently used expectations scheme, namely, adaptive expectations, which is given by

$$p_t^e = p_{t-1}^e + w(p_{t-1} - p_{t-1}^e), \quad 0 \le w \le 1. \tag{16}$$

The expected price is 'adapted' in the direction of the most recently observed actual price p_{t-1}, with expectations weight factor w. Rearranging Eq. (16) yields the equivalent

$$p_t^e = (1-w)p_{t-1}^e + wp_{t-1}, \quad 0 \le w \le 1, \tag{17}$$

so that expected price is a weighted average of the most recently observed actual price and the most recent expected price. Note that naive expectations are a special case of adaptive expectations with w=1. Using Eq. (17) repeatedly, the adaptive expectations predictor can be written as a weighted sum, with geometrically declining weights, of all past prices, that is,

$$p_t^e = H(\vec{P}_{t-1}) = wp_{t-1} + w(1-w)p_{t-2} + w(1-w)^2 p_{t-3} + \cdots = \sum_{i=1}^{w} w(1-w)^{i-1} p_{t-i}. \tag{18}$$

Recall that in the case of naive expectations and monotonic (non-linear) demand and supply curves, prices either converge to a stable steady state or to a stable period two 'hog-cycle.' However, the case with adaptive expectations and non-linear, monotonic demand and supply is much more complicated. Chiarella (1988), Finkenstädt and Kuhbier (1992) and Hommes (1991a, 1991b, 1994) have shown that with adaptive expectations chaotic price oscillations can arise even when both demand and supply are monotonic.

Using Eq. (17) and the market equilibrium condition (Eq. (8)), one easily derives that the dynamics of expected prices is described by

$$p_t^e = (1-w)p_{t-1}^e + wD^{-1}S(p_{t-1}^e). \tag{19}$$

The one-dimensional map generating the expected price behaviour[5] is

$$f(x) = (1-w)x + wD^{-1}S(x). \tag{20}$$

[5] The dynamics of prices (and quantities) is equivalent to the dynamics of expected prices since $p_t = D^{-1}S(p_t^e)$. However, in order to analyze the dynamical behaviour in the non-linear case, it is more convenient to work with the 1-D difference equation for expected prices.

The main reason for the occurrence of chaotic price fluctuations, even when demand and supply are monotonic, is the following. When demand decreases and supply increases, the composite function $D^{-1}S$ decreases. The graph of the map f in Eq. (20) is a weighted average of the increasing diagonal $y=x$ and the decreasing graph of the (non-linear) map $D^{-1}S$. Hence, for $w=0$ f increases and for $w=1$ (i.e. for naive expectations) the graph of f decreases. For a large set of *non-linear*, monotonic demand and supply curves there typically exists an interval of w-values for which the map f is *non-monotonic*. For such w-values chaotic price oscillations may arise.

Take the linear demand curve (Eq. (10)) as before and following Chiarella (1988) and Hommes (1991a, 1991b, 1994) a non-linear, but monotonic, S-shaped supply curve

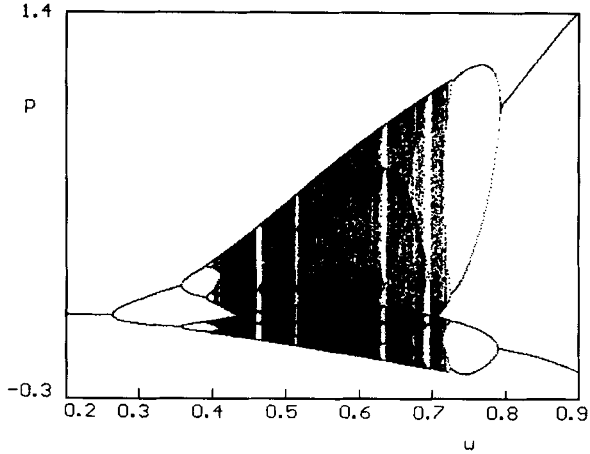
$$S_\lambda(x) = \mathrm{Tanh}(\lambda x) = \frac{e^{\lambda x} - e^{-\lambda x}}{e^{\lambda x} + e^{-\lambda x}}, \tag{21}$$

where the parameter λ tunes the steepness of the S-shape. In this section and in Section 5 we will use this non-linear, monotonic supply curve. We emphasize that our results do not depend upon the specific choice of the supply curve (Eq. (21)); other S-shapes yield very similar results. A similar S-shaped supply curve can be derived from profit maximization, when producers have a fourth degree polynomial convex cost function. Other than S-shaped, monotonic non-linear supply curves can also yield chaotic price dynamics, but with different geometric shapes of the strange attractors than below.

Fig. 2 shows a bifurcation diagram w.r.t. the expectations weight factor w with and without noise. As w decreases from 1 down to 0.7 the price behaviour changes from a stable 2-cycle, after a sequence of period doublings (or period halvings if w increases) through stable 4, 8, 16 etc. cycles into a 2-piece chaotic attractor for $w=0.71$. For many w-values, $0.43 \leq w \leq 0.70$, a one-piece chaotic attractor arises. If w decreases from 0.43 to 0.15, after an infinite sequence of period halvings (period doublings when w increases) finally a stable steady state arises for $w<0.25$. In the noisy bifurcation diagram in Fig. 2(b), with a random supply shock between $\pm 3\%$ in each period, the detailed bifurcation structure has disappeared. For example, the period 5 and period 7 windows have disappeared and only the first bifurcation of the infinite cascade of period doublings remains visible. Roughly speaking, however, the noisy price fluctuations still have the same characteristics as the noise-free case, with a noisy steady state and a noisy 2-cycle for small w-values, a noisy 2- or 4-cycle for large w-values (close to 1) and complicated price fluctuations for intermediate w-values.

For $0.71 \leq w \leq 1$ adaptive expectations are inconsistent, since there is negative correlation in prices at odd lags and positive correlation in prices at even lags. For $w=0.71$ prices converge to a 2-piece chaotic attractor. The chaotic time series jumps back and forth between two intervals. The corresponding ACF of expectational errors in Fig. 3(a) has 'period' 2, with strong negative autocorrelations at odd lags and strong positive autocorrelations at even lags. This represents an example where the price dynamics is chaotic, but adaptive expectations are nevertheless inconsistent. For $w=0.58$ a one-piece chaotic attractor consisting of an entire interval seems to occur. The corresponding ACFs of expectational errors is shown in Fig. 3(b). The ACF exhibits three important features:

(a)



(b)

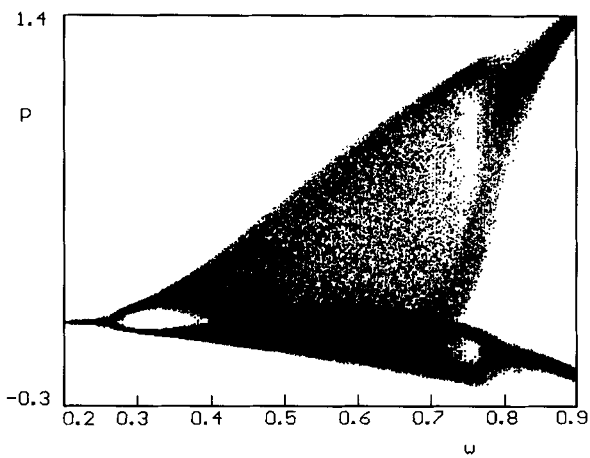
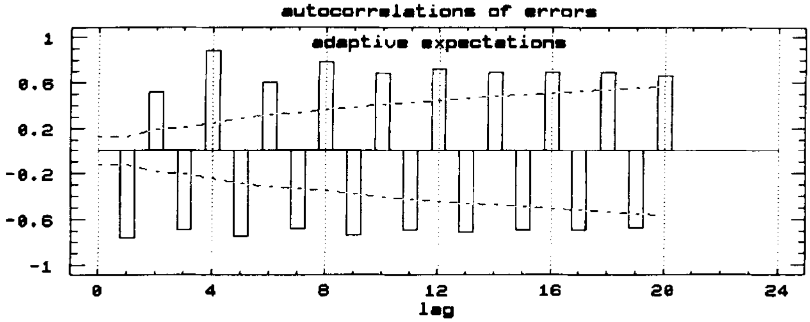


Fig. 2. Bifurcation diagrams w.r.t. expectations weight factor w, with a=0.65, b=1 and λ=10. (a) without noise, (b) with exogenous, random supply shocks of at most ±3% in each period.

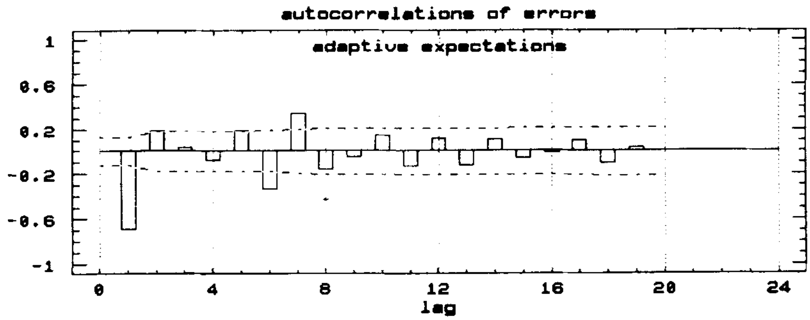
- There is a strong negative correlation at the first lag, $\rho_1 \approx -0.7$. There are also significant autocorrelations at lags 6 and 7.
- Autocorrelation coefficients ρ_k at lags $k \geq 8$ are not statistically significant.
- The -+-+ regularity in autocorrelations (which is so common to the cobweb 'hog cycle') is disturbed, since for example, $\rho_6 < 0$.

The significant autocorrelations at lags 6 and 7 can be explained by the fact that for the close-by parameter value w=0.63 a stable 7-cycle occurs (cf. the bifurcation diagram in Fig. 2(a)). Fig. 3(c) shows the ACF of expectational errors in the presence of small noise (a random supply shock between ±3%). The ACF in the presence of noise still has a strong negative correlation at the first lag, but the autocorrelations at lags 6 and 7 have disappeared. Note that in the bifurcation diagram (Fig. 2(b)) with the same amount of

(a)



(b)



(c)

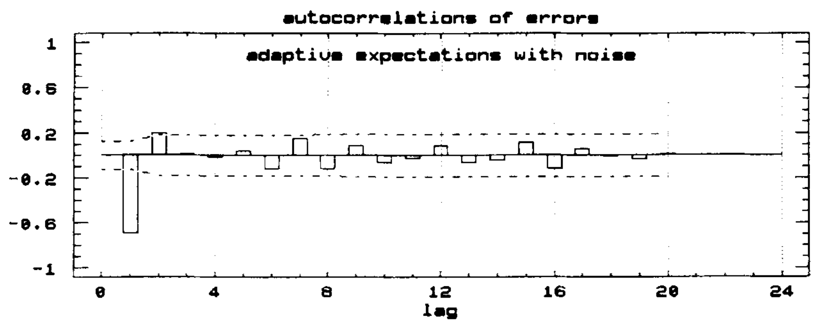


Fig. 3. ACFs of expectational errors, with a=0.65, b=1, λ=10, (a) w=0.71: chaotic price fluctuations on a 2-piece chaotic attractor, (b)–(c) w=0.58; chaotic price fluctuations on a 1-piece chaotic attractor, (b) without noise and (c) with noise.

noise, all stable periodic cycles (including the period 7) inside the chaotic region have also disappeared.

We conclude that for low values of the expectations weight factor, say for $w \leq 0.7$, adaptive expectations can be weakly consistent even when demand and supply are monotonic. Adaptive expectations, however, are not consistent; all (chaotic) examples that we investigated have strong negative autocorrelation at the first lag, even in the presence of some noise.

5. Linear backward-looking expectations

Any forecast of future prices should be some function of past prices. Perhaps it would be more realistic to predict prices not only from past prices, but also from past values of other variables, such as quantities. However, we restrict attention to the case where expectations are a function of past prices only. The simplest expectations scheme then arises when the predictor is a linear function of a finite number of past prices; that is,

$$H(\vec{P}_{t-1}) = w_1 p_{t-1} + w_2 p_{t-2} + \cdots + w_L p_{t-L}, \quad \sum_{i=1}^{L} w_i = 1, \tag{22}$$

so that expected price is a weighted average of L past prices. We will refer to the predictor (Eq. (22)) as *linear backward-looking expectations* (*LBE*). Sometimes this predictor is also referred to as distributed lag expectations. The number of lags L is finite and all coefficients w_i are fixed over time. We will investigate whether LBE can be a consistent predictor when demand and supply are monotonic. Adaptive expectations are a weighted sum, with geometrically declining weights, of all past prices as in Eq. (18). Since adaptive expectations with non-linear, monotonic demand and supply curves can lead to chaotic price fluctuations, similar behaviour is to be expected with LBE.[6]

There are many possibilities for the distribution of weights w_i. In all the cases that we examine, all coefficients w_i of the LBE predictor are positive and sum to 1. Moreover, in all examples more weight is given to more recent observations, that is $w_i > w_{i+1}$, $1 \leq i \leq L-1$. We make this restriction for three reasons. First, this seems to be the easiest case in order to analyze the global dynamics more or less systematically. In fact, opposite signs of the weights or positive non-declining weights can lead to even more complicated dynamics than in our examples below. Second, positive, declining weights are closely related to the adaptive expectations scheme, where all coefficients are also positive and geometrically declining. Hence, we can compare our results for LBE to the case of adaptive expectations. Third, writing $w_1 p_{t-1} + w_2 p_{t-2} = p_{t-1} + (w_1 - 1)(p_{t-1} - p_{t-2}) + (w_1 + w_2 - 1)p_{t-2}$, it is clear that positive weights summing to 1 imply that the coefficient $w_1 - 1 < 0$, meaning that agents expect a reversal of the most recent price change. This seems to be reasonable when looking at the time series of the cobweb model and it should have a stabilizing effect upon the price dynamics.

In all simulations in this section, we use the linear demand curve (Eq. (10)) and the monotonic, S-shaped non-linear supply curve (Eq. (21)). The section is subdivided in two subsections, dealing with two, three or more lags L respectively, for the LBE predictor (Eq. (22)).

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[6] Holmes and Manning (1988) have shown that when expected price is the average of past prices (i.e. all weights w_i are equal) and the number of lags tends to infinity, the steady state is always globally stable. The representation (Eq. (18)) for adaptive expectations and the occurrence of chaos shows that this result cannot be true, even with an infinite number of lags, when more weight is given to more recent prices.

5.1. LBE with two lags

In the simplest case with two lags, linear backward-looking expectations are given by

$$H(\vec{P}_{t-1}) = w_1 p_{t-1} + w_2 p_{t-2}, \quad w_1 + w_2 = 1, \tag{23}$$

that is, expected price is a weighted average of the two most recent past prices. Substituting Eq. (23) into the market equilibrium condition (Eq. (8)) we obtain a second-order difference equation for the price behaviour

$$p_t = D^{-1}S(w_1 p_{t-1} + w_2 p_{t-2}). \tag{24}$$

Writing $p_t{=}y_t$, $p_{t-1} = x_t$ an equivalent 2-dimensional difference equation is obtained:

$$\begin{aligned} x_{t+1} &= y_t \\ y_{t+1} &= D^{-1}S(w_1 y_t + w_2 x_t) \end{aligned} \tag{25}$$

For the linear demand curve (Eq. (10)) the corresponding 2-dimensional map is

$$F(x, y) = (y, [a - S(w_1 y + w_2 x)]/b). \tag{26}$$

It is easily verified that, when both demand and supply are smooth, monotonic curves, the map F is a diffeomorphism.[7] F has a unique steady state (p^{eq}, p^{eq}), where p^{eq} is the price corresponding to the intersection point of demand and supply. The stability of the steady state is determined by the characteristic equation

$$\lambda^2 + \frac{w_1 S'(p^{\mathrm{eq}})}{b}\lambda + \frac{w_2 S'(p^{\mathrm{eq}})}{b} = 0. \tag{27}$$

Fig. 4 shows how the stability of the steady state depends upon w_1 (with $w_2{=}1{-}w_1$) and the ratio $S'(p^{\mathrm{eq}})/b$ of marginal supply and demand at the steady state. Three curves are drawn in the parameter space:

$$\begin{aligned} C_1 &: \frac{S'(p^{\mathrm{eq}})}{b} = \frac{4(1-w_1)}{w_1^2} \quad \text{Discriminant}\, D = 0 \\ C_2 &: \frac{S'(p^{\mathrm{eq}})}{b} = \frac{1}{1-w_1} \quad \text{Hopf-bifurcation}\,(H) \\ C_3 &: \frac{S'(p^{\mathrm{eq}})}{b} = \frac{1}{2w_1 - 1} \quad \text{Period-doubling}\,(PD) \end{aligned} \tag{28}$$

The curve C_1 consists of parameter values for which the eigenvalues of the Jacobian at the steady state change from real to complex. Along the second curve C_2, the eigenvalues of the Jacobian at the steady state are complex and lie on the unit circle. These are parameters for which a Hopf-bifurcation of the steady state occurs. Finally, the curve C_3 consists of parameter values for which the Jacobian matrix at the steady state has an eigenvalue -1 (with the other real eigenvalue between -1 and 0). For these parameters a period doubling bifurcation occurs. The three curves have a common intersection point (w_1, $S'(p^{\mathrm{eq}})/b$)=(2/3,3). We conclude that for low values of $S'(p^{\mathrm{eq}})/b$ the steady-state price

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[7] A diffeomorphism F is a differentiable 1–1 map with a differentiable inverse F^{-1}.

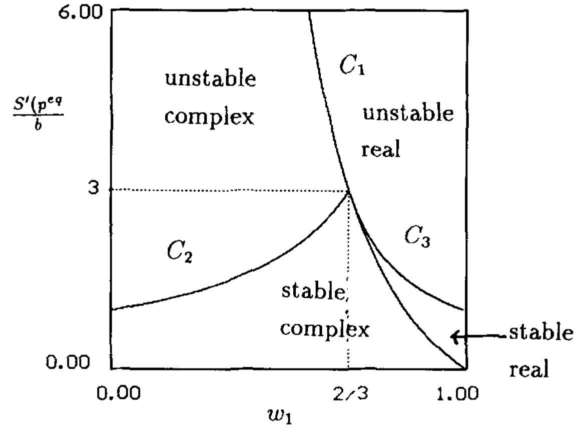

Fig. 4. Stability region of the steady state.

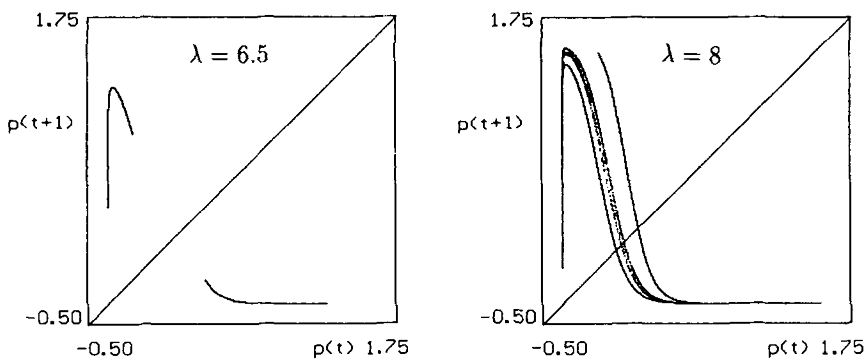

Fig. 5. a=0.65, b=1, w_1=0.75 and w_2=0.25. (a) λ=6.5 two-piece strange attractor and (b) λ=8 one-piece strange attractor.

p^{eq} is stable. As $S'(p^{eq})/b$ increases, the steady state loses stability: for w_1<2/3 through a *Hopf-bifurcation* and for w_1>2/3 through a *period doubling bifurcation.*

For many parameter values the cobweb model with LBE with two lags has a strange attractor. We have numerically observed two different bifurcation routes to chaos when $S'(p^{eq})/b$ increases. The first is the well-known *period doubling route* to chaos. The strange attractors illustrated in the phase diagrams in Fig. 5 for w_1=0.75 and w_2=0.25, arises after an infinite sequence of period-doubling bifurcations. A second route to chaos, the *breaking of an invariant circle,* occurs after the Hopf-bifurcation of the steady state and is illustrated in Fig. 6, for w_1=0.65 and w_2=0.35.[8] Just after the Hopf-bifurcation the

[8] The first simulations for this model were done while visiting the Deparment of Economics, University of Wisconsin. Many of the strange attractors have a very similar geometric structure, for which Dee Dechert, my roommate during this visit, coined the term '*high-heeled shoe*' attractor.

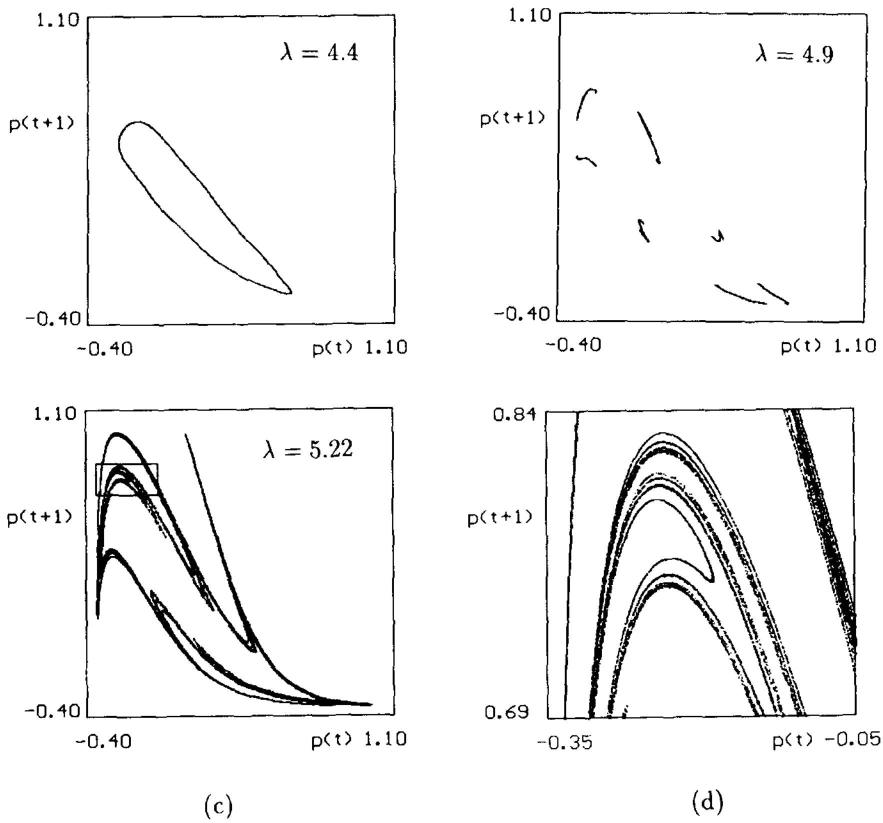

Fig. 6. a=0.65, b=1, w_1=0.65 and w_2=0.35. (a) λ=4.4 quasi-periodic attractor, (b) λ=4.9 seven-piece strange attractor, (c) λ=5.22 one-piece strange attractor and (d) enlargement of (c).

model has an attracting invariant circle with periodic or quasi-periodic price dynamics. As the ratio $S'(p^{eq})/b$ increases the invariant circle grows and becomes more and more complicated turning into a strange attractor through a complex bifurcation sequence of both period doublings and homoclinic bifurcations between the stable and unstable manifolds of periodic saddle points. For mathematical details see for example, Aronson et al. (1982) and Guckenheimer and Holmes (1983). Another economic application is given by de Vilder (1995), who investigates the breaking of an invariant circle and its complicated bifurcation sequences in a 2-D version of the Overlapping Generations model.

For high values of $S'(p^{eq})/b$ the strange attractor disappears and the price dynamics becomes regular again, converging to a stable period 3 cycle. In fact this stable 3-cycle already coexists with the strange attractor in Fig. 6(c). Fig. 7 shows ACFs of expectational errors corresponding to the strange attractors in Fig. 5(b) and Fig. 6(c). The ACFs have the following properties:

- Strong negative autocorrelation at the first lag, $\rho_1 \approx -0.7$.

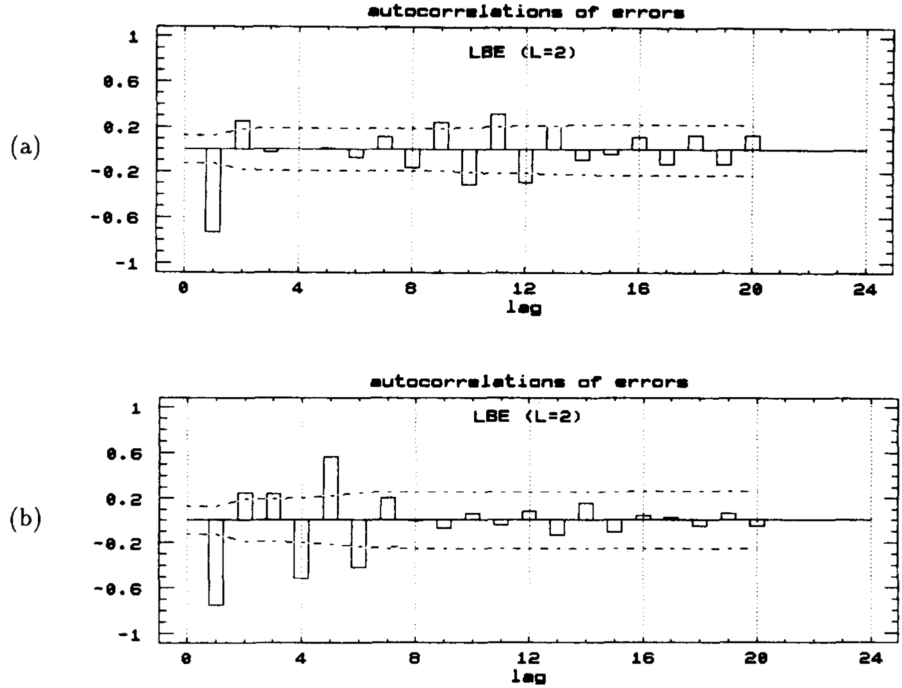

Fig. 7. ACFs of expectational errors corresponding to the one-piece strange attractors in Fig. 5(b) and Fig. 6(c).

- significant autocorrelations at lags 1, 2, 9, 10, 11 and 12 in Fig. 7(a) and at lags 1 to 6 in Fig. 7(b).
- no -+-+ regularity at odd/even lags; it is violated for example, at the ninth lag in Fig. 7(a) and at the third lag in Fig. 7(b).

In summary, when the market is very unstable, that is, for very high values of $S'(p^{eq})/b$, LBE with two lags are inconsistent, since along the stable period 3 cycle the ACF is periodic. For intermediate values of $S'(p^{eq})/b$ LBE with two lags can be weakly consistent when there exists a one-piece strange attractor. LBE with two lags, however, cannot be consistent since in all (chaotic) examples expectational errors have strong negative autocorrelation at the first lag.

5.2. LBE with many lags

Next consider the case of linear backward-looking expectations with many, that is, three or more lags. Expectations are a weighted sum of L past prices as in Eq. (22). Substituting Eq. (22) into the market equilibrium condition (Eq. (8)) the following Lth order difference equation for the price dynamics is obtained:

$$p_t = D^{-1}S(w_1p_{t-1} + w_2p_{t-2} + \cdots + w_Lp_{t-L}). \tag{29}$$

Writing $p_{t-1} = x_{L,t}$, $p_{t-2} = x_{L-1,t}$, $\cdots$ $p_{t-L} = x_{1,t}$the equivalent L-dimensional difference equation is

$$\begin{aligned} x_{1,t+1} &= x_{2,t} \\ x_{2,t+1} &= x_{3,t} \\ &\vdots \\ x_{L-1,t+1} &= x_{L,t} \\ x_{L,t+1} &= D^{-1}S(w_1x_{L,t} + w_2x_{L-1,t} + \ldots + w_Lx_{1,t}) \end{aligned} \tag{30}$$

For the linear demand curve (Eq. (10)) the corresponding L-dimensional map is

$$F(x_1, x_2, \ldots, x_L) = \left(x_2, x_3, \ldots, \frac{a - S(w_1x_L + w_2x_{L-1} + \cdots w_Lx_1)}{b}\right) \tag{31}$$

As in the 2-D case, when both demand and supply are smooth, monotonic curves, the map F is a diffeomorphism. The unique steady state of F is $\vec{P}^{\mathrm{eq}} = (p^{\mathrm{eq}}, p^{\mathrm{eq}}, \ldots, p^{\mathrm{eq}})$, with p^{eq} the price corresponding to the intersection point of demand and supply. The characteristic equation determining the stability of the steady state is

$$\lambda^N + \frac{w_1\,S'(p^{\mathrm{eq}})}{b}\lambda^{N-1} + \cdots + \frac{w_{N-1}S'(p^{\mathrm{eq}})}{b}\lambda + \frac{w_N\,S'(p^{\mathrm{eq}})}{b} = 0. \tag{32}$$

As for the 2-D case, it can be shown that when the ratio $S'(p^{\mathrm{eq}})/b$ of marginal demand and supply at the steady state is sufficiently close to 0, the steady state is stable, whereas for $S'(p^{\mathrm{eq}})/b$ sufficiently large the steady state is unstable.

As a typical example with three lags, we consider the case w_1=0.43, w_2=0.36 and w_3=0.21. For high values of $S'(p^{\mathrm{eq}})/b$ a one-piece strange attractor arises as illustrated in Fig. 8(b). This strange attractor arises after two period-doubling bifurcations of the steady state into a stable 4-cycle, a Hopf bifurcation of the 4-cycle and thereafter the breaking of the four invariant circles into a one-piece strange attractor.

We restrict our analysis of LBE with more than three lags to the case where all weights are positive and declining, so that more weight is given to more recent past prices. As an example consider the weights

$$w_i = \frac{2(L+1-i)}{L(L+1)}, \tag{33}$$

so that the weights form a declining arithmetic sequence summing to 1.

For large L, the stability region of the steady state becomes larger. Nevertheless, for sufficiently high values of the ratio $S'(p^{\mathrm{eq}})/b$ the steady state becomes unstable and complicated price dynamics can arise. Fig. 8(c–d) show strange attractors for the case L=10 and Fig. 8(e–f) for L=100. These strange attractors arise after an infinite sequence of period-doubling bifurcations, when the parameter λ tuning the non-linearity of the supply curve increases. Fig. 8(a) shows the chaotic attractor in the case of adaptive expectations, where $w_i = w(1-w)^{i-1}, i \in \mathcal{N}$, as in (Eq. (18)). From Section 4 it follows that this attractor is (contained in) a 1-dimensional curve, since the cobweb model with adaptive expectations can be reduced to a one-dimensional dynamic system. It is remarkable that the geometric structure of the strange attractors

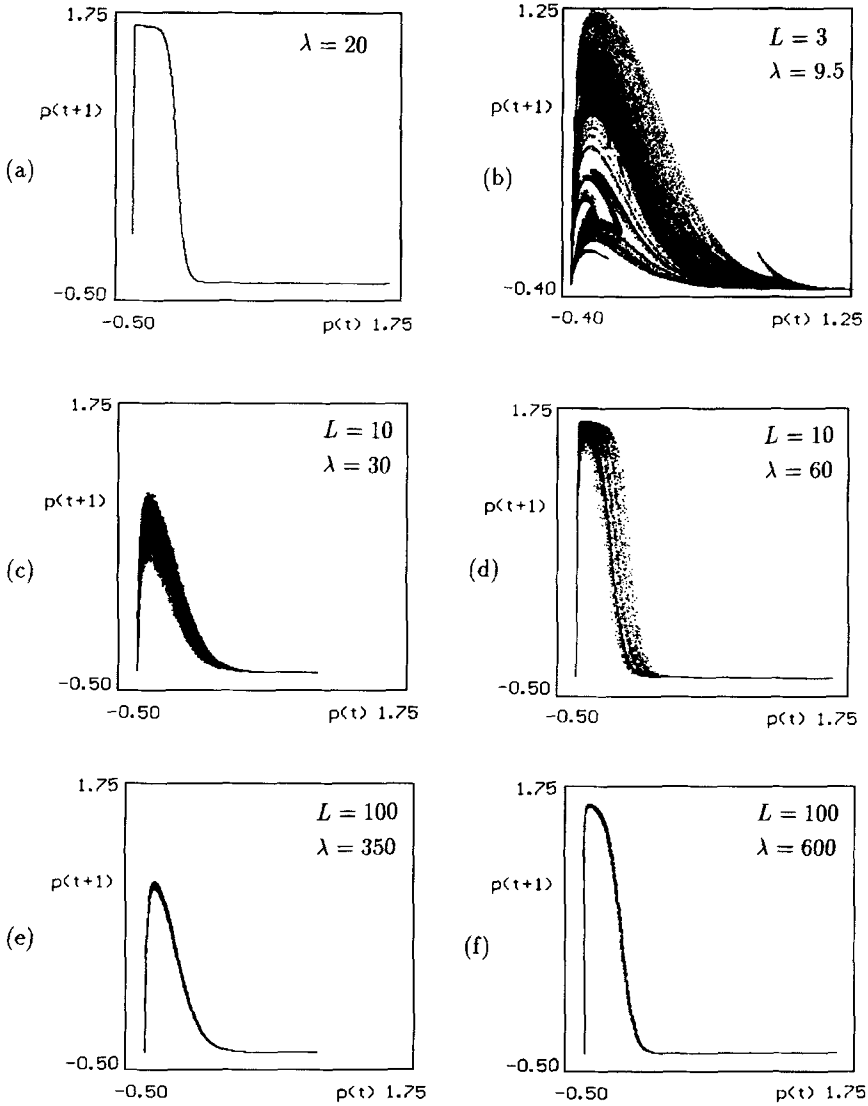

Fig. 8. a=0.65 and b=1. Strange attractors. (a) adaptive expectations: w=0.75 and λ=20, (b) L=3: w_1=0.43, w_2=0.36, w_3=0.21 and λ=9.5. (c–d) L=10, with (c) λ=30 and (d) λ=60, (e–f) L=100 with (e) λ=350 and (f) λ=600.

for LBE with L=3, L=10 and L=100 is very similar to the 1-dimensional chaotic adaptive expectations attractor. In fact they look like a *noisy 'high-heeled-shoe'* adaptive expectations strange attractor. Note, however, that when the number of lags L is large much more non-linearity (i.e. a higher value of λ or the ratio $S'(p^{eq})/b$) is needed before strange attractors arise.

Fig. 9 shows ACFs of expectational errors corresponding to the strange attractors in Fig. 8(b), (c) and (e). Some properties of the ACF are:

- strong negative autocorrelation at the first lag, with $\rho_1 \leq -0.5$ in all cases.

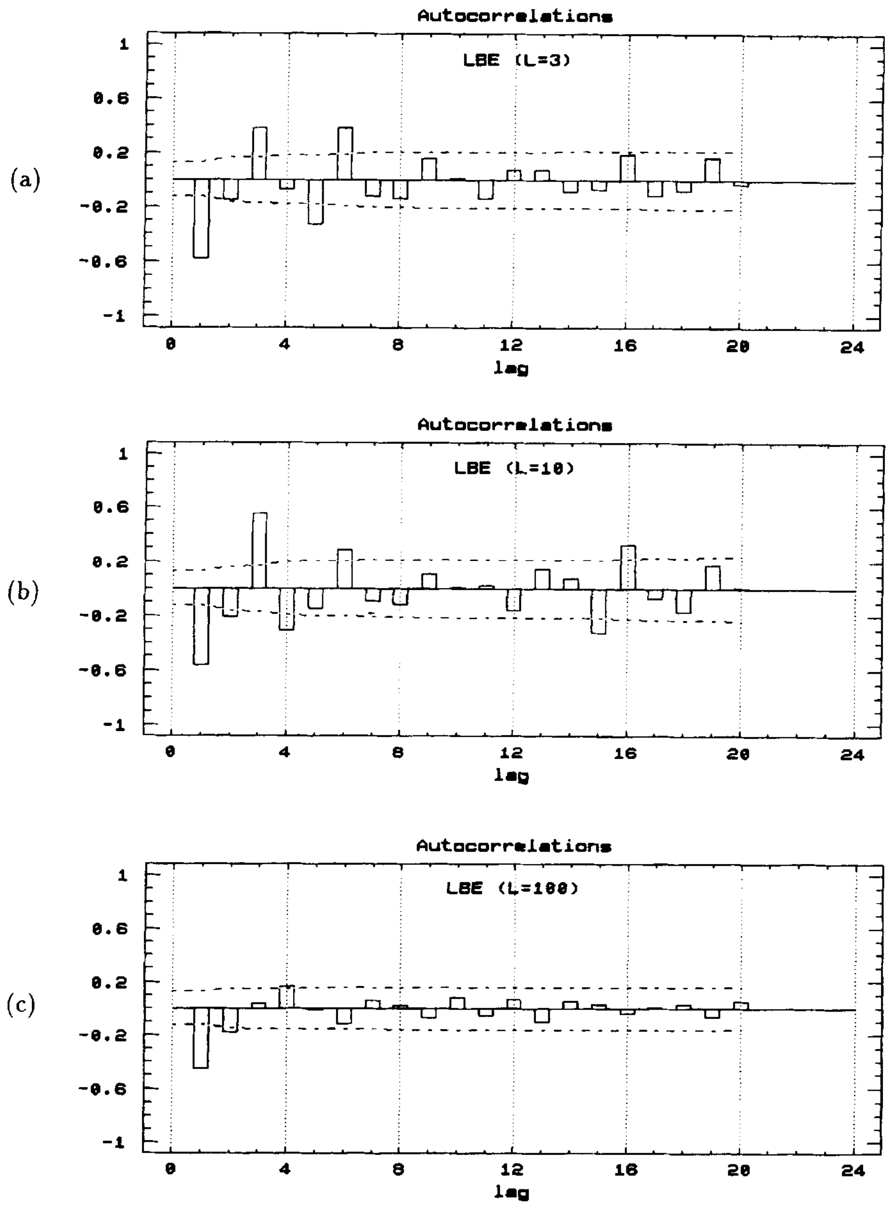

Fig. 9. ACFs of expectational errors corresponding to the one-piece strange attractors in Fig. 8(b), (c), (e).

- L=3: additional significant autocorrelations at lags 2, 3, 5 and 6;
- L=10: additional significant autocorrelations at lags 2, 3, 4, 6, 15 and 16;
- L=100: additional significant autocorrelations at lags 2 and 4 (but with strong non-linearity).
- no -+-+ regularity in autocorrelations at odd/even lags.

Summarizing, in the cobweb model with LBE with three or more lags, one-piece strange attractors arise when the non-linearity of the supply curve is sufficiently high. LBEs are then weakly consistent, but not consistent since in all (chaotic) examples of expectational errors have strong negative autocorrelation at the first lag. In fact the price dynamics is determined by two factors: the amount of non-linearity of the supply curve (the parameter λ) and the distribution of weights, especially the weights given to most recent prices. If price expectations give more weight to more recent prices (like adaptive expectations or LBE with 2 or 3 lags), then moderate non-linearity can yield chaotic price fluctuations with weakly consistent expectations. If price expectations are close to the average price over a long period of time (LBE with many lags and slowly, for example, arithmetically, declining weights) than chaos with weakly consistent expectations can still arise, but only for highly non-linear supply curves. However, we have found no examples of consistent LBE; strong negative autocorrelation at the first lag occurs in all (chaotic) examples.

6. Heterogenous beliefs

In all previous sections, we have assumed *homogenous beliefs* across agents, that is, we have assumed that all agents use the same expectations function. Although this assumption is quite standard in economic modelling, it cannot be an accurate description of reality. Agents or groups of agents often have different beliefs about the future. Recently there have been a number of attempts in modelling *heterogenous beliefs*, where different groups of agents use different expectation functions to predict future market prices or other market characteristics. For example, in recent exchange rate and stock market models two different classes of agents, *fundamentalists* and *chartists*, are distinguished (e.g. Frankel and Froot, 1987, Chiarella, 1992 and DeGrauwe et al., 1993). The fundamentalists gather all available economic information to compute the 'fundamental' market equilibrium as their prediction for the future exchange rate or stock return. On the other hand, chartists extrapolate past trends and patterns to predict future values. In these models either the number or the weight of fundamentalists and chartists, and thus their influence upon the market equilibrium price, changes over time. It has been shown by these authors, mainly by numerical methods, that heterogenous beliefs are a source of instability in the market and may lead to periodic or even chaotic fluctuations in prices.

Brock and Hommes (1997) have recently presented a theoretical foundation of the observation that heterogenous beliefs are a source of instability and chaos. They analyzed the cobweb model with heterogenous beliefs, where in each period agents make a rational choice between a simple habitual 'rule of thumb' predictor (e.g. naive expectations) which is freely available and a more sophisticated predictor (e.g. rational expectations) which can be obtained at certain information costs. This rational choice is based upon predictors' performance in the past; for example, upon past net realized profits or past squared prediction errors. Brock and Hommes (1995) prove the occurrence of strange attractors for several classes of predictors. In this section we investigate whether the predictors used by the two different groups of agents are consistent, by looking at the ACFs of expectational errors in cases of chaotic price fluctuations.

In order to be self-contained we include a brief description of the model. Additional details may be found in Brock and Hommes (1995). In the cobweb model with two groups of agents using different expectations functions H_1 and H_2, market equilibrium is described by

$$D(p_t) = n_{1,t-1}S(H_1(\vec{P}_{t-1})) + n_{2,t-1}S(H_2(\vec{P}_{t-1})), \tag{34}$$

where D and S are demand and supply, $\vec{P}_{t-1} = (p_{t-1}, \ldots, p_{t-L})$ is a vector of past prices and $n_{1,t-1}$ and $n_{2,t-1}$ are the fractions of the two groups of agents at the beginning of period t. H_2 will be a simple, habitual 'rule of thumb' predictor which is freely available to all agents, whereas H_1 will be a more sophisticated predictor which can be obtained at *information costs C*. At the end of period t, after observing the price p_t, agents decide which predictor to use in the next period. The performance measure for predictor selection is squared prediction error in the previous period.[9] The new fractions of agents using each of the two predictors are given by

$$n_{1,t} = e^{-\beta((p_t - H_1(\vec{P}_{t-1}))^2 + C)}/Z_t \tag{35}$$

$$n_{2,t} = e^{-\beta(p_t - H_2(\vec{P}_{t-1}))^2}/Z_t, \tag{36}$$

where Z_t is the sum of the numerators so that $n_{1,t}+n_{2,t}=1$. The parameter β is called the intensity of choice, measuring how fast agents switch predictors. Note that most agents choose H_1 (H_2) when the squared prediction error plus information costs for H_1 are smaller (larger) than the squared prediction error for H_2. Let $m_t = n_{1,t} - n_{2,t}$ be the difference in fractions. Using Eqs. (35) and (36) a simple computation shows that

$$m_t = \mathrm{Tanh}\left(\frac{\beta}{2}[(H_1(\vec{P}_{t-1}) - H_2(\vec{P}_{t-1}))(2p_t - H_1(\vec{P}_{t-1}) - H_2(\vec{P}_{t-1})) - C]\right) \tag{37}$$

To keep the model as simple as possible and to focus on the non-linearity caused by heterogeneity in expectations, both demand and supply are assumed to be linear; therefore,

$$D(p_t) = a - bp_t \tag{38}$$

$$S(H_i(\vec{P}_{t-1})) = dH_i(\vec{P}_{t-1}) \tag{39}$$

Using Eq. (34) it follows that the market equilibrium price is given by

$$p_t = \frac{a - \frac{1+m_{t-1}}{2}\, dH_1(\vec{P}_{t-1}) - \frac{1-m_{t-1}}{2}\, dH_2(\vec{P}_{t-1})}{b}. \tag{40}$$

[9] Using past net realized profits from using predictor H_j yields essentially the same results. To simplify the presentation we concentrate on past squared prediction errors as the performance measure.

The dynamic model determined by Eqs. (37) and (40) is called the *Adaptive Rational Equilibrium Dynamics* (*ARED*), a coupling between the demand–supply equilibrium price dynamics and rational predictor selection. Without loss of generality we fix $a=0$ so that the steady-state equilibrium price $p^{eq}=0$. All 'prices' are then (positive or negative) deviations from their steady-state equilibrium price.

Brock and Hommes (1995) consider several two-predictor cases; for example, rational versus naive expectations and fundamentalists versus naive expectations. In all cases, they show the occurrence of strange attractors, especially when the intensity of choice β is high. Remarkably, this result seems to be independent of the details of the expectation functions. The chaotic price fluctuations are characterized by an irregular switching between cheap 'free riding' and costly stabilizing prediction.

Here we concentrate on the simple example of the fundamentalists or steady-state predictor versus an AR(1) process; that is,

$$H_1(\vec{P}_{t-1}) = 0 \tag{41}$$

$$H_2(\vec{P}_{t-1}) = \sigma p_{t-1}. \tag{42}$$

Hence, one group of agents 'knows' the equilibrium steady-state price $p=0$ and believes that prices will return to their steady state. The other group believes that tomorrow's price will be σ times today's price. In fact, this may be interpreted as a belief that prices follow an AR(1) process with autocorrelation coefficients $\rho_k=\sigma^k$. Substituting Eqs. (41) and (42) into Eqs. (37) and (40) it follows that the ARED is given by

$$p_t = \frac{-b\sigma}{2B}(1 - m_{t-1})p_{t-1} \tag{43}$$

$$m_t = \mathrm{Tanh}\left(\frac{\beta}{2}\left[\sigma^2\left(1 + \frac{b}{B}(1 - m_{t-1})\right)p_{t-1}^2 - C\right]\right). \tag{44}$$

First we consider the case $\sigma=1$, so that H_2 is naive expectations. Fig. 10 shows a typical chaotic time series of prices and the corresponding ACFs of the expectational errors of the two predictors. Both predictors seem to be weakly consistent with only significant autocorrelations at the first two lags. Both predictors have a strongly negative autocorrelation at the first lag, especially the cheap predictor H_2.

In all examples presented so far, price fluctuations exhibit strong negative autocorrelation at the first lag. Therefore, in the next example we consider the case $\sigma=-1$, so the second group of agents believes that a positive (negative) deviation from the steady state will be followed by a negative (positive) deviation of the same amplitude. This means that the second group believes that there is negative autocorrelation in prices. Fig. 11(a) shows a typical chaotic time series of prices. Price fluctuations are now qualitatively quite different from all previous examples. Up and down price oscillations have disappeared. Fig. 11(b) and (c) show the corresponding ACFs for expectational errors. Both predictors seem to be weakly consistent with only significant, this time positive(!), autocorrelation at the first lag. This example clearly illustrates how price expectations affect actual price behaviour in the cobweb model. When agents believe there is positive (negative) autocorrelation in prices at the first lag, actual prices will have negative (positive) autocorrelation at the first lag. In this

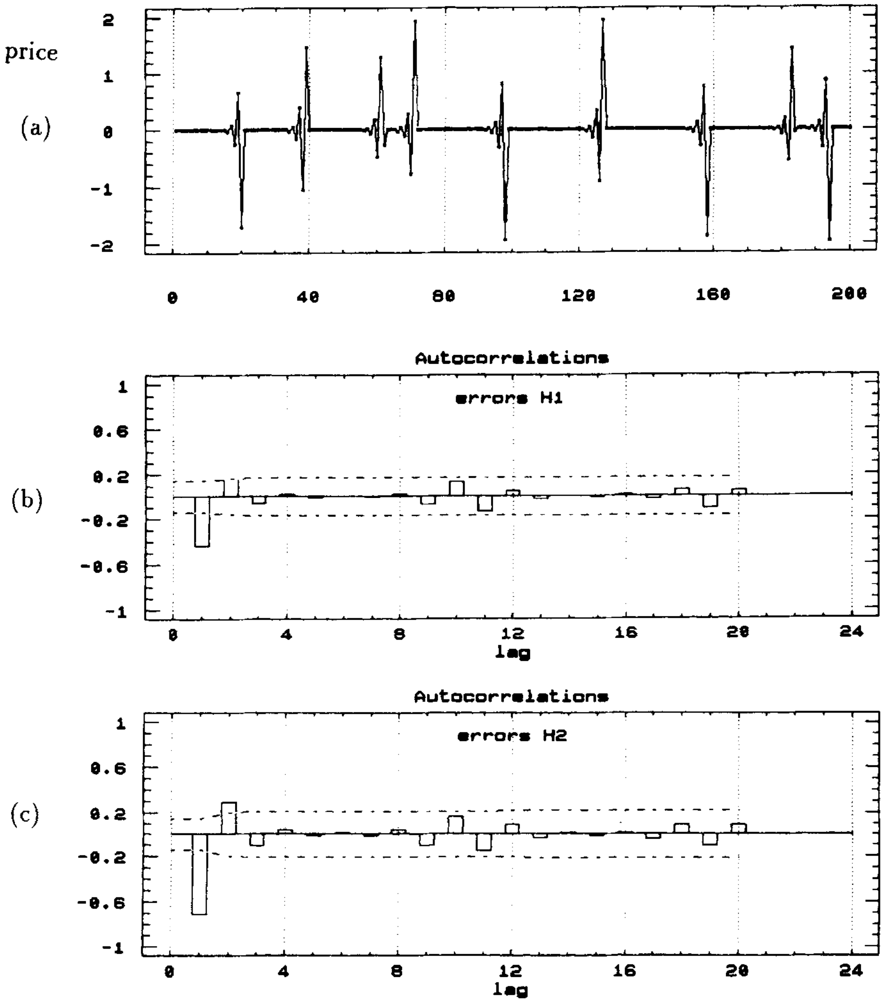

Fig. 10. b=0.5, d=1.35, σ=1 and β=5. (a) Chaotic price time series and corresponding ACFs of expectational errors for (b) fundamentalists predictor H_1 and (c) naive predictor H_2.

respect, it is important to note that for σ<0, the sign in Eq. (43) is positive. One might say that the cobweb model "reverses beliefs": actual price behaviour is "the opposite of agents' beliefs."

In summary, different types of price behaviour are possible in the cobweb model with heterogenous beliefs. When there is one group of agents believing that there is strong negative autocorrelation in prices at the first lag, this may lead to positive autocorrelations in actual prices at the first lag. In the two-predictor case, both predictors can be weakly consistent, but we have found no examples of consistent LBE predictors, since the autocorrelation coefficient at the first lag is either strongly negative or positive.

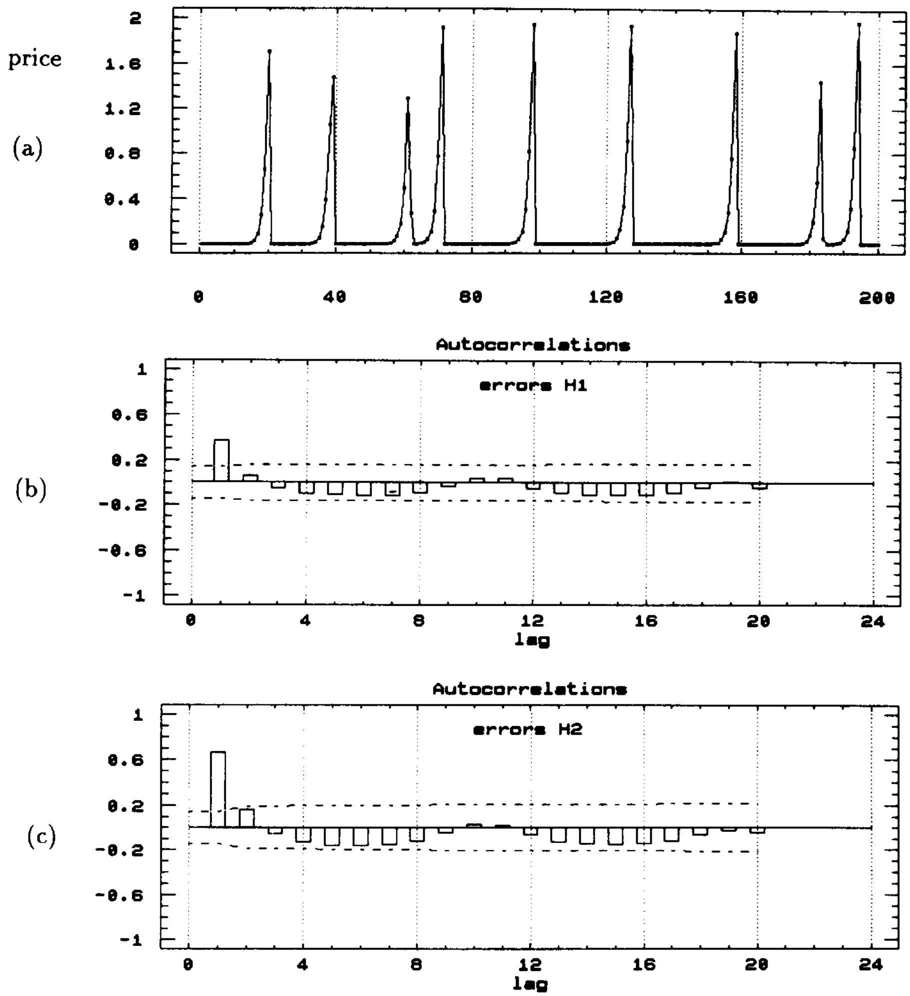

Fig. 11. Same as in Fig. 10 with same parameters, except for $\sigma=-1$, implying that the second group of agents believes in negative autocorrelations in prices.

7. Concluding remarks

7.1. Summary of the results

We have investigated the price dynamics in the cobweb model with linear demand, non-linear supply and simple linear backward-looking expectation schemes, especially focusing on the possibility of chaotic price fluctuations. Naive expectations can only lead to chaos when supply is non-monotonic. All other LBEs that we have considered can lead to chaotic price fluctuations even when both demand and supply are monotonic. We have investigated the consistency of LBE when prices converge to a one-piece strange attractor. Expectations are (weakly) consistent when the autocorrelation coefficients of expectational errors are zero for all (high enough) lags. Whether chaotic price fluctuations and (weakly) consistent expectations occur depends essentially upon two factors: the non-linearity of (demand and) supply and the distribution of the weights given to past prices in the expectations function. For monotonic demand and supply, one might summarize our results as follows:

- When all weight is given to the most recent price (naive price expectation), an unstable cobweb always yields a period 2 cycle with inconsistent expectations.
- When some weight is given to prices in the more distant past, with weights declining exponentially fast into the past (e.g. adaptive expectations or LBE with 2 or 3 lags), a moderate amount of non-linearity may lead to chaotic price fluctuations with weakly consistent expectations.
- When weights decline slowly into the past (LBE with many lags), only strong non-linearity can yield an unstable cobweb. Very strong non-linearity is needed for chaotic price fluctuations with weakly consistent expectations.
- We have found no examples of consistent LBE expectations in the cobweb model. In particular, all chaotic examples have strong negative autocorrelation of expectational errors at the first lag.

In the case of heterogenous beliefs with two groups of agents, fundamentalists versus AR(1) believers, the results are similar. One-piece strange attractors with weakly consistent expectations do occur, but no consistent LBE predictors have been found. Even with heterogenous beliefs, chaotic price fluctuations still exhibit strong negative autocorrelations at the first lag. When there is one group of agents believing that there is strong negative autocorrelation in prices at the first lag, actual prices have positive autocorrelations at the first lag. In this sense, the cobweb model 'reverses' beliefs.

7.2. What have we learned?

Any dynamic economic model is an *expectations feedback system.* Expectations affect actual outcomes, actual outcomes affect expectations through learning, and so on. Rationality requires some kind of consistency between expectations and actual outcomes. The REH in fact imposes this consistency by simply assuming that expectations are a fixed point of this feedback system. Agents 'know' the market equilibrium equations and solve a fixed point problem to obtain their expectations. We have been looking for a different type of consistency between beliefs and actual outcomes, not based upon knowledge of the model, but only based upon observations of time series and especially upon the ACF of expectational errors. Unfortunately, in the non-linear cobweb model, we have found no examples of consistent LBE, not even when the price dynamics is chaotic. One might therefore conclude that our concept of consistent expectations is useless for economic theory.

However, this conclusion seems to be too quick. We believe that the fact that we have found no consistent LBE predictors is due to a number of reasons which make the cobweb model very special. First of all, price expectations and market equilibrium price only depend upon past prices and not upon past values of other variables such as quantities. Secondly, from our extensive analysis, we learned that the cobweb model is in fact the '*wrong*' model for consistent expectations, because the *cobweb model reverses beliefs*. In the cobweb model with monotonic demand and supply, a high (low) expected price always yields a low (high) actual market equilibrium price. Consequently, for LBE predictors giving most weight to the most recently observed past price, strong negative autocorrelation at the first lag occurs, both in prices and expectational errors, even when price fluctuations are chaotic. We expect, however, that in more complicated models where expectations are self-fulfilling and/or depend upon more variables, simple expectational schemes can be consistent.

At this point, we would like to refer to two related papers by Sorger (1994), Lorenz (1995) and make some suggestions for future work. Lorenz (1995) investigates the role of expectations in a Keynesian macroeconomic model. In particular, Lorenz introduces the notion of *perfect cyclic expectations*, where the expected price $p_t^e = p_{t-N}$ and actual prices converge to a period N cycle. Hence, Lorenz concentrates on beliefs which are perfectly consistent with the actual dynamics. Following an idea by Grandmont (1994), Sorger (1994) presents a very nice example of a *self-fulfilling mistake* in an overlapping generations (OLG) economy. In Sorger's example, if agents believe that interest rates are an independently, identically distributed (IID) random series with uniform distribution over the unit interval, then actual dynamics of interest rates is deterministically chaotic, generated by the tent map. Indeed, this example fits our framework of consistent expectations. Not only is the ACF of expectational errors zero, but in addition the ACFs of beliefs equal the empirical ACF of the actual dynamics, since both are zero. Sorger calls this an example of a *consistent expectations equilibrium* (*CEE*), but does not define this notion explicitly. Sorger's example is special, in the sense that it only applies to time series with zero ACF. Many economic and financial time series have non-zero ACF. It would therefore be interesting to investigate whether Sorger's example can be generalized in the following way. Does there exist an OLG-economy where agents believe that interest rates follow a stochastic AR(1) process, while actual interest dynamics is generated by an *asymmetric* tent map (or some other chaotic process) with the same ACF as the AR(1) process?[10] Such asymmetric tent maps exist, as shown by Sakai and Tokumaru (1980). Such OLG-economies would be an interesting class of models fitting our general framework of consistent LBE expectations.

In fact our numerical investigations of the cobweb model suggest another class of models where consistent LBE may exist. Consider the general expectations feedback system

$$p_t = F(H(\vec{P}_{t-1})) \tag{45}$$

where $\vec{P}_{t-1} = (p_{t-1}, \ldots, p_{t-L})$ is a vector of past prices, H is an LBE predictor and F a one-dimensional map. The cobweb model fits this general framework, but is special since $F'(x) < 0$ when demand decreases and supply increases. This is the main reason why the cobweb model 'reverses' beliefs. This class of models (Eq. (45)) with $F'(x)>0$ so that

[10] See Hommes and Sorger (1997) for recent advances in this direction.

beliefs are self-fulfilling in the sense that a higher (lower) expected price also yields a higher (lower) actual price, may be a simple, general class of non-linear models in which LBE expectations can be consistent.

Similarly, consider a general heterogenous belief system, with predictors H_1 and H_2,

$$p_t = n_{1,t-1}F(H_1(\vec{P}_{t-1})) + n_{2,t-1}F(H_2(\vec{P}_{t-1})), \tag{46}$$

where $n_{1,t-1}$ and $n_{2,t-1}$ are the two fractions of agents at the beginning of period t. The cobweb model with heterogenous beliefs (Eq. (40)) is of this form. Our results suggest the following problem. If F is an increasing one-dimensional map and H_1 are fundamentalists beliefs, does there exist an AR(1)-belief, H_2, such that both H_1 and H_2 are consistent, with prices moving on a strange attractor? Financial asset pricing models with heterogenous beliefs might be a suitable class of models for such consistency of expectations.

All expectations functions considered in this paper have constant weights of past prices. Hence, there is no *learning* where agents revise expectations by adapting the weights in accordance to observations. In fact, we have been looking for consistent expectations in the long run, especially on a strange attractor. It would be interesting to introduce a learning mechanism based upon empirical ACFs and investigate whether such a learning scheme can converge to consistent expectations on a strange attractor. Can such *learning to believe in chaos* arise in a simple expectations feedback system like Eq. (45) or Eq. (46)? We leave all these questions for future work.

Acknowledgements

An earlier draft of this paper was presented at the fifth Viennese Workshop on Advances in Nonlinear Economic Dynamics, May 24–26 1995. During the preparation I had stimulating discussions with Buz Brock, Dee Dechert and Jean-Michel Grandmont. I would also like to thank the participants of the workshop, especially Volker Böhm, Hans-Walter Lorenz and Gerhard Sorger, for stimulating comments. Special thanks are due to Manfred Nermuth, Xavier Hof and two anonymous referees for their detailed remarks on an earlier draft. Financial support by the Netherlands Organisation for Scientific Research under grant R46-307 is gratefully acknowledged.

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