CHAPTER 2

The Solow Growth Model

The previous chapter introduced a number of basic facts and posed the main questions concerning the sources of economic growth over time and the causes of differences in economic performance across countries. These questions are central not only for growth theory but also for macroeconomics and social sciences more generally. Our next task is to develop a simple framework that can help us think about the *proximate* causes and the mechanics of the process of economic growth and cross-country income differences. We will use this framework both to study potential sources of economic growth and also to perform simple comparative statics to gain an understanding of what features of societies are conducive to higher levels of income per capita and more rapid economic growth.

Our starting point will be the so-called Solow-Swan model named after Robert (Bob) Solow and Trevor Swan, or simply the *Solow model* for the more famous of the two economists. These two economists published two pathbreaking articles in the same year, 1956 (Solow, 1956, and Swan, 1956) introducing the Solow model. Bob Solow later developed many implications and applications of this model and was awarded the Nobel prize in economics for these contributions. This model has shaped the way we approach not only economic growth but the entire field of macroeconomics. Consequently, a byproduct of our analysis of this chapter will be a detailed exposition of the workhorse model of much of macroeconomics.

The Solow model is remarkable in its simplicity. Looking at it today, one may fail to appreciate how much of an intellectual breakthrough it was relative to what came before. Before the advent of the Solow growth model, the most common approach to economic growth built on the model developed by Roy Harrod and Evsey Domar (Harrod, 1939, Domar, 1946). The Harrod-Domar model emphasized potential dysfunctional aspects of economic growth, for example, how economic growth could go hand-in-hand with increasing unemployment (see Exercise 2.13 on this model). The Solow model demonstrated why the Harrod-Domar model was not an attractive place to start. At the center of the Solow growth model, distinguishing it from the Harrod-Domar model, is the *neoclassical* aggregate production function. This function not only enables the Solow model to make contact with microeconomics, but it also serves as a bridge between the model and the data as we will see in the next chapter.

An important feature of the Solow model, which will be shared by many models we will see in this book, is that it is a simple and *abstract* representation of a complex economy. At first, it may appear too simple or too abstract. After all, to do justice to the process of growth or macroeconomic equilibrium, we have to think of many different individuals with different tastes, abilities, incomes and roles in society, many different sectors and multiple social interactions. Instead, the Solow model cuts through these complications by constructing a simple one-good economy, with little reference to individual decisions. Therefore, for us the Solow model will be both a starting point and a springboard for richer models.

Despite its mathematical simplicity, the Solow model can be best appreciated by going back to the microeconomic foundations of general equilibrium theory, and this is where we begin. Since the Solow model is the workhorse model of macroeconomics in general, a good grasp of its workings and foundations is not only useful in our investigations of economic growth, but also essential for modern macroeconomic analysis. We now study the Solow model and return to the neoclassical growth model in Chapter 8.

2.1. The Economic Environment of the Basic Solow Model

Economic growth and development are dynamic processes, focusing on how and why output, capital, consumption and population change over time. The study of economic growth and development therefore necessitates dynamic models. Despite its simplicity, the Solow growth model is a dynamic general equilibrium model.

The Solow model can be formulated either in discrete or in continuous time. We start with the discrete time version, both because it is conceptually simpler and it is more commonly used in macroeconomic applications. However, many growth models are formulated in continuous time and we will also provide a detailed exposition of the continuous-time version of the Solow model and show that it is often more tractable.

2.1.1. Households and Production. Consider a closed economy, with a unique final good. The economy is in discrete time running to an infinite horizon, so that time is indexed by $t = 0, 1, 2, ...$ Time periods here can correspond to days, weeks, or years. So far we do not need to take a position on this.

The economy is inhabited by a large number of households, and for now we are going to make relatively few assumptions on households because in this baseline model, they will not be optimizing. This is the main difference between the Solow model and the *neoclassical growth model*. The latter is the Solow model plus dynamic consumer (household) optimization. To fix ideas, you may want to assume that all households are identical, so that the economy admits *a representative consumer*—meaning that the demand and labor supply side of the economy can be represented as if it resulted from the behavior of a single household. We will return to what the representative consumer assumption entails in Chapter 5 and see that it is not totally innocuous. But that is for later.

What do we need to know about households in this economy? The answer is: not much. We do not yet endow households with preferences (utility functions). Instead, for now, we simply assume that they save a constant exogenous fraction s of their disposable income—irrespective of what else is happening in the economy. This is the same assumption used in basic Keynesian models and in the Harrod-Domar model mentioned above. It is also at odds with reality. Individuals do not save a constant fraction of their incomes; for example, if they did, then the announcement by the government that there will be a large tax increase next year should have no effect on their saving decisions, which seems both unreasonable and empirically incorrect. Nevertheless, the exogenous constant saving rate is a convenient starting point and we will spend a lot of time in the rest of the book analyzing how consumers behave and make intertemporal choices.

The other key agents in the economy are firms. Firms, like consumers, are highly heterogeneous in practice. Even within a narrowly-defined sector of an economy (such as sports shoes manufacturing), no two firms are identical. But again for simplicity, we start with an assumption similar to the representative consumer assumption, but now applied to firms. We assume that all firms in this economy have access to the same production function for the final good, or in other words, we assume that the economy admits a *representative firm*, with a representative (or aggregate) production function. Moreover, we also assume that this aggregate production function exhibits *constant returns to scale* (see below for a definition). More explicitly, the aggregate production function for the unique final good is

$$Y(t) = F[K(t), L(t), A(t)] \tag{2.1}$$

where $Y(t)$ is the total amount of production of the final good at time t, $K(t)$ is the capital stock, $L(t)$ is total employment, and $A(t)$ is technology at time t. Employment can be measured in different ways. For example, we may want to think of $L(t)$ as corresponding to hours of employment or number of employees. The capital stock $K(t)$ corresponds to the quantity of "machines" (or more explicitly, equipment and structures) used in production, and it is typically measured in terms of the value of the machines. There are multiple ways of thinking of capital (and equally many ways of specifying how capital comes into existence). Since our objective here is to start out with a simple workable model, we make the rather sharp simplifying assumption that capital is the same as the final good of the economy. However, instead of being consumed, capital is used in the production process of more goods. To take a concrete example, think of the final good as "corn". Corn can be used both for consumption and as an input, as "seed", for the production of more corn tomorrow. Capital then corresponds to the amount of corn used as seeds for further production.

Technology, on the other hand, has no natural unit. This means that $A(t)$, for us, is a *shifter* of the production function (2.1). For mathematical convenience, we will often represent $A(t)$ in terms of a number, but it is useful to bear in mind that, at the end of the day, it is a representation of a more abstract concept. Later we will discuss models in which $A(t)$ can be multidimensional, so that we can analyze economies with different types of technologies. As noted in Chapter 1, we may often want to think of a broad notion of technology, incorporating the effects of the organization of production and of markets on the efficiency with which the factors of production are utilized. In the current model, $A(t)$ represents all these effects.

A major assumption of the Solow growth model (and of the neoclassical growth model we will study in Chapter 8) is that technology is *free*; it is publicly available as a non-excludable, non-rival good. Recall that a good is *non-rival* if its consumption or use by others does not preclude my consumption or use. It is *non-excludable*, if it is impossible to prevent the person from using it or from consuming it. Technology is a good candidate for a non-excludable, non–rival good, since once the society has some knowledge useful for increasing the efficiency of production, this knowledge can be used by any firm without impinging on the use of it by others. Moreover, it is typically difficult to prevent firms from using this knowledge (at least once it is in the public domain and it is not protected by patents). For example, once the society knows how to make wheels, everybody can use that knowledge to make wheels without diminishing the ability of others to do the same (making the knowledge to produce wheels non-rival). Moreover, unless somebody has a well-enforced patent on wheels, anybody can decide to produce wheels (making the know-how to produce wheels non-excludable). The implication of the assumptions that technology is non-rival and non-excludable is that $A(t)$ is freely available to all potential firms in the economy and firms do not have to pay for making use of this technology. Departing from models in which technology is freely available will be a major step towards developing models of endogenous technological progress in Part 4 and towards understanding why there may be significant technology differences across countries in Part 6 below.

As an aside, you might want to note that some authors use x_t or K_t when working with discrete time and reserve the notation $x(t)$ or $K(t)$ for continuous time. Since we will go back and forth between continuous time and discrete time, we use the latter notation throughout. When there is no risk of confusion, we will drop time arguments, but whenever there is the slightest risk of confusion, we will err on the side of caution and include the time arguments.

Now we impose some standard assumptions on the production function.

Assumption 1. ***(Continuity, Differentiability, Positive and Diminishing Marginal Products, and Constant Returns to Scale)*** *The production function* $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ *is twice continuously differentiable in* K *and* L*, and satisfies*

$$F_K(K,L,A) \equiv \frac{\partial F(K,L,A)}{\partial K} > 0, \quad F_L(K,L,A) \equiv \frac{\partial F(K,L,A)}{\partial L} > 0,$$
$$F_{KK}(K,L,A) \equiv \frac{\partial^2 F(K,L,A)}{\partial K^2} < 0, \quad F_{LL}(K,L,A) \equiv \frac{\partial^2 F(K,L,A)}{\partial L^2} < 0.$$

Moreover, F *exhibits constant returns to scale in* K *and* L*.*

All of the components of Assumption 1 are important. First, the notation $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ implies that the production function takes nonnegative arguments (i.e., $K, L \in \mathbb{R}_+$) and maps to nonnegative levels of output ($Y \in \mathbb{R}_+$). It is natural that the level of capital and the level of employment should be positive. Since A has no natural units, it could have been negative. But there is no loss of generality in restricting it to be positive. The second important aspect of Assumption 1 is that F is a continuous function in its arguments and is also differentiable. There are many interesting production functions which are not differentiable and some interesting ones that are not even continuous. But working with continuously differentiable functions makes it possible for us to use differential calculus, and the loss of some generality is a small price to pay for this convenience. Assumption 1 also specifies that marginal products are positive (so that the level of production increases with the amount of inputs); this also rules out some potential production functions and can be relaxed without much complication (see Exercise 2.4). More importantly, Assumption 1 imposes that the marginal product of both capital and labor are diminishing, i.e., $F_{KK} < 0$ and $F_{LL} < 0$, so that more capital, holding everything else constant, increases output by less and less, and the same applies to labor. This property is sometimes also referred to as "diminishing returns" to capital and labor. We will see below that the degree of diminishing returns to capital will play a very important role in many of the results of the basic growth model. In fact, these features distinguish the Solow growth model from its antecedent, the Harrod-Domar model (see Exercise 2.13).

The other important assumption is that of constant returns to scale. Recall that F exhibits *constant returns to scale* in K and L if it is *linearly homogeneous* (homogeneous of degree 1) in these two variables. More specifically:

Definition 2.1. *Let $z \in \mathbb{R}^K$ for some $K \geq 1$. The function $g(x, y, z)$ is homogeneous of degree m in $x \in \mathbb{R}$ and $y \in \mathbb{R}$ if and only if*

$$g(\lambda x, \lambda y, z) = \lambda^m g(x, y, z) \text{ for all } \lambda \in \mathbb{R}_+ \text{ and } z \in \mathbb{R}^K.$$

It can be easily verified that linear homogeneity implies that the production function F is concave, though not strictly so (see Exercise 2.1).

Linearly homogeneous (constant returns to scale) production functions are particularly useful because of the following theorem:

Theorem 2.1. ***(Euler's Theorem)*** *Suppose that $g : \mathbb{R}^{K+2} \to \mathbb{R}$ is continuously differentiable in $x \in \mathbb{R}$ and $y \in \mathbb{R}$, with partial derivatives denoted by g_x and g_y and is homogeneous of degree m in x and y. Then*

$$mg(x, y, z) = g_x(x, y, z)\, x + g_y(x, y, z)\, y \text{ for all } x \in \mathbb{R},\ y \in \mathbb{R} \text{ and } z \in \mathbb{R}^K.$$

Moreover, $g_x(x, y, z)$ and $g_y(x, y, z)$ are themselves homogeneous of degree $m - 1$ in x and y.

Proof. We have that g is continuously differentiable and

$$\lambda^m g(x, y, z) = g(\lambda x, \lambda y, z). \tag{2.2}$$

Differentiate both sides of equation (2.2) with respect to λ, which gives

$$m\lambda^{m-1} g(x, y, z) = g_x(\lambda x, \lambda y, z)\, x + g_y(\lambda x, \lambda y, z)\, y$$

for any λ. Setting $\lambda = 1$ yields the first result. To obtain the second result, differentiate both sides of equation (2.2) with respect to x:

$$\lambda g_x(\lambda x, \lambda y, z) = \lambda^m g_x(x, y, z).$$

Dividing both sides by λ establishes the desired result. □

2.1.2. Market Structure, Endowments and Market Clearing. For most of the book, we will assume that all factor markets are competitive. This is yet another assumption that is not totally innocuous. Both labor markets and capital markets have imperfections that have important implications for economic growth. But it is only by starting out with the competitive benchmark that we can best appreciate the implications of these imperfections for economic growth. Furthermore, until we come to models of endogenous technological change, we will assume that product markets are also competitive, so ours will be a prototypical *competitive general equilibrium model*.

As in standard competitive general equilibrium models, the next step is to specify endowments, that is, what the economy starts with in terms of labor and capital and who owns these endowments. Let us imagine that all factors of production are owned by households. In particular, households own all of the labor, which they supply inelastically. Inelastic supply means that there is some endowment of labor in the economy, for example equal to the population, $\bar{L}(t)$, and all of this will be supplied regardless of the price (as long as it is nonnegative). The *labor market clearing* condition can then be expressed as:

$$L(t) = \bar{L}(t) \tag{2.3}$$

for all t, where $L(t)$ denotes the demand for labor (and also the level of employment). More generally, this equation should be written in complementary slackness form. In particular, let the *wage rate* (or the rental price of labor) at time t be $w(t)$, then the labor market clearing condition takes the form $L(t) \leq \bar{L}(t)$, $w(t) \geq 0$ and $\left(L(t) - \bar{L}(t)\right) w(t) = 0$. The complementary slackness formulation makes sure that labor market clearing does not happen at a negative wage—or that if labor demand happens to be low enough, employment could be below $\bar{L}(t)$ at zero wage. However, this will not be an issue in most of the models studied in this book (in particular, Assumption 1 and competitive labor markets make sure that wages have to be strictly positive), thus we will use the simpler condition (2.3) throughout.

The households also own the capital stock of the economy and rent it to firms. We denote the *rental price of capital* at time t be $R(t)$. The capital market clearing condition is similar to (2.3) and requires the demand for capital by firms to be equal to the supply of capital by households: $K^s(t) = K^d(t)$, where $K^s(t)$ is the supply of capital by households and $K^d(t)$ is the demand by firms. Capital market clearing is straightforward to impose in the class of models analyzed in this book by imposing that the amount of capital $K(t)$ used in production at time t is consistent with household behavior and firms' optimization.

We take households' initial holdings of capital, $K(0)$, as given (as part of the description of the environment), and this will determine the initial condition of the dynamical system we will be analyzing. For now how this initial capital stock is distributed among the households is not important, since households optimization decisions are not modeled explicitly and the economy is simply assumed to save a fraction s of its income. When we turn to models with household optimization below, an important part of the description of the environment will be to specify the preferences and the budget constraints of households.

At this point, we could also introduce $P(t)$ as the price of the final good at time t. But we do not need to do this, since we have a choice of a numeraire commodity in this economy, whose price will be normalized to 1. In particular, you will remember from basic general equilibrium theory that Walras' Law implies that we should choose the price of one of the commodities as numeraire. In fact, throughout we will do something stronger. We will normalize the price of the final good to 1 *in all periods*. Ordinarily, one cannot choose more than one numeraire—otherwise, one would be fixing the relative price between the two numeraires. But in dynamic economies, we can build on an insight by Kenneth Arrow (Arrow, 1964) that it is sufficient to price *securities* (assets) that transfer one unit of consumption from one date (or state of the world) to another. In the context of dynamic economies, this implies that we need to keep track of an *interest rate* across periods, denoted by $r(t)$, and this will enable us to normalize the price of the final good to 1 in every period (and naturally, we will keep track of the wage rate $w(t)$, which will determine the intertemporal price of labor relative to final goods at any date t).

This discussion should already alert you to a central fact: you should think of all of the models we discuss in this book as *general equilibrium economies*, where different commodities correspond to the same good at different dates. Recall from basic general equilibrium theory that the same good at different dates (or in different states or localities) is a different commodity. Therefore, in almost all of the models that we will study in this book, there will be *an infinite number of commodities*, since time runs to infinity. This raises a number of special issues, which we will discuss as we go along.

Now returning to our treatment of the basic model, the next assumption is that capital depreciates, meaning that machines that are used in production lose some of their value because of wear and tear. In terms of our corn example above, some of the corn that is used as seeds is no longer available for consumption or for use as seeds in the following period. We assume that this depreciation takes an "exponential form," which is mathematically very tractable. This means that capital depreciates (exponentially) at the rate δ, so that out of 1 unit of capital this period, only $1-\delta$ is left for next period. As noted above, depreciation here stands for the wear and tear of the machinery, but it can also represent the replacement of old machines by new machines in more realistic models (see Chapter 14). For now it is treated as a black box, and it is another one of the black boxes that will be opened later in the book.

The loss of part of the capital stock affects the interest rate (rate of return to savings) faced by the household. Given the assumption of exponential depreciation at the rate δ and the normalization of the price of the final goods to 1, this implies that the *interest rate* faced by the household will be $r(t) = R(t) - \delta$. Recall that a unit of final good can be consumed now or used as capital and rented to firms. In the latter case, the household will receive $R(t)$ units of good in the next period as the rental price, but will lose δ units of the capital, since δ fraction of capital depreciates over time. This implies that the individual has given up one unit of commodity dated $t-1$ for $r(t)$ units of commodity dated t. The relationship between $r(t)$ and $R(t)$ explains the similarity between the symbols for the interest rate and the rental rate of capital. The interest rate faced by households will play a central role when we model the dynamic optimization decisions of households. In the Solow model, this interest rate does not directly affect the allocation of resources.

2.1.3. Firm Optimization. We are now in a position to look at the optimization problem of firms. Throughout the book we assume that the only objective of firms is to maximize profits. Since we have assumed the existence of an aggregate production function, we only need to consider the problem of a *representative firm*. Therefore, the (representative) firm maximization problem can be written as

$$\max_{L(t),K(t)} F[K(t), L(t), A(t)] - w(t) L(t) - R(t) K(t). \tag{2.4}$$

A couple of features are worth noting:

(1) The maximization problem is set up in terms of aggregate variables. This is without loss of any generality given the representative firm.
(2) There is nothing multiplying the F term, since the price of the final good has been normalized to 1. Thus the first term in (2.4) is the revenues of the representative firm (or the revenues of all of the firms in the economy).
(3) This way of writing the problem already imposes competitive factor markets, since the firm is taking as given the rental prices of labor and capital, $w(t)$ and $R(t)$ (which are in terms of the numeraire, the final good).
(4) This is a concave problem, since F is concave (see Exercise 2.1).

Since F is differentiable from Assumption 1, the first-order necessary conditions of the maximization problem (2.4) imply the important and well-known result that the competitive rental rates are equal to marginal products:

$$w(t) = F_L[K(t), L(t), A(t)]. \tag{2.5}$$

and

$$R(t) = F_K[K(t), L(t), A(t)]. \tag{2.6}$$

Note also that in (2.5) and (2.6), we used the symbols $K(t)$ and $L(t)$. These represent the amount of capital and labor used by firms. In fact, solving for $K(t)$ and $L(t)$, we can derive the capital and labor demands of firms in this economy at rental prices $w(t)$ and $R(t)$—thus we could have used $K^d(t)$ instead of $K(t)$, but this additional notation is not necessary.

This is where Euler's Theorem, Theorem 2.1, becomes useful. Combined with competitive factor markets, this theorem implies:

Proposition 2.1. *Suppose Assumption 1 holds. Then in the equilibrium of the Solow growth model, firms make no profits, and in particular,*

$$Y(t) = w(t)L(t) + R(t)K(t).$$

Proof. This follows immediately from Theorem 2.1 for the case of $m = 1$, i.e., constant returns to scale. □

This result is both important and convenient; it implies that firms make no profits, so in contrast to the basic general equilibrium theory with strictly convex production sets, the ownership of firms does not need to be specified. All we need to know is that firms are profit-maximizing entities.

In addition to these standard assumptions on the production function, in macroeconomics and growth theory we often impose the following additional boundary conditions, referred to as Inada conditions.

Assumption 2. ***(Inada conditions)*** *F satisfies the Inada conditions*

$$\begin{aligned} \lim_{K\to 0} F_K(K,L,A) &= \infty \text{ and } \lim_{K\to\infty} F_K(K,L,A) = 0 \text{ for all } L > 0 \text{ and all } A \\ \lim_{L\to 0} F_L(K,L,A) &= \infty \text{ and } \lim_{L\to\infty} F_L(K,L,A) = 0 \text{ for all } K > 0 \text{ and all } A. \end{aligned}$$

The role of these conditions—especially in ensuring the existence of *interior equilibria*—will become clear in a little. They imply that the "first units" of capital and labor are highly productive and that when capital or labor are sufficiently abundant, their marginal products are close to zero. Figure 2.1 draws the production function $F(K,L,A)$ as a function of K, for given L and A, in two different cases; in Panel A, the Inada conditions are satisfied, while in Panel B, they are not.

We will refer to Assumptions 1 and 2 throughout much of the book.

2.2. The Solow Model in Discrete Time

We now start with the analysis of the dynamics of economic growth in the discrete time Solow model.

2.2.1. Fundamental Law of Motion of the Solow Model. Recall that K depreciates exponentially at the rate δ, so that the law of motion of the capital stock

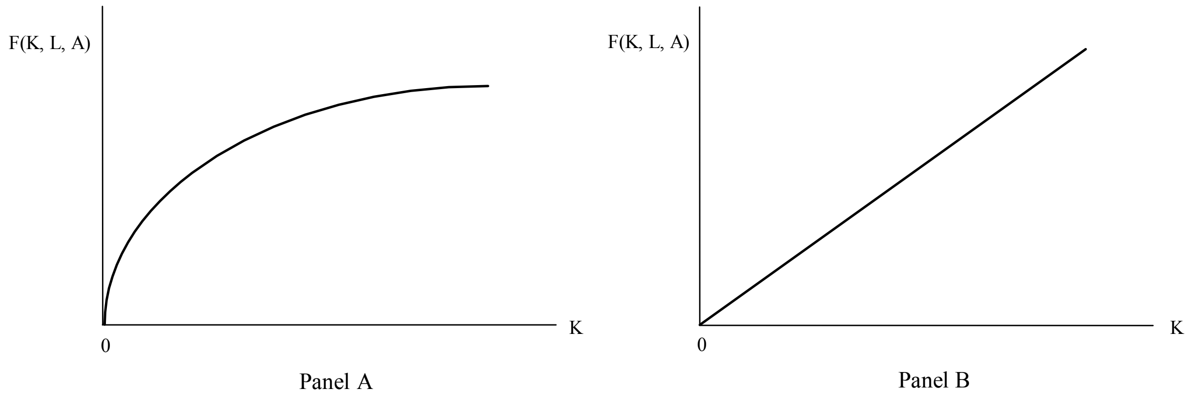

Figure 2.1. Production functions and the marginal product of capital. The example in Panel A satisfies the Inada conditions in Assumption 2, while the example in Panel B does not.

is given by

$$K(t+1) = (1-\delta) K(t) + I(t), \tag{2.7}$$

where $I(t)$ is investment at time t.

From national income accounting for a closed economy, we have that the total amount of final goods in the economy must be either consumed or invested, thus

$$Y(t) = C(t) + I(t), \tag{2.8}$$

where $C(t)$ is consumption.[1] Using (2.1), (2.7) and (2.8), any *feasible* dynamic allocation in this economy must satisfy

$$K(t+1) \leq F[K(t), L(t), A(t)] + (1-\delta) K(t) - C(t)$$

for $t = 0, 1, ...$ The question now is to determine the equilibrium dynamic allocation among the set of feasible dynamic allocations. Here the *behavioral rule* of the constant saving rate simplifies the structure of equilibrium considerably. It is important to notice that the constant saving rate is a behavioral rule—it is not derived from the maximization of a well-defined utility function. This means that any welfare

[1]In addition, we can introduce government spending $G(t)$ on the right-hand side of (2.8). Government spending does not play a major role in the Solow growth model, thus we set it equal to 0 (see Exercise 2.3).

comparisons based on the Solow model have to be taken with a grain of salt, since we do not know what the preferences of the individuals are.

Since the economy is closed (and there is no government spending), aggregate investment is equal to savings,

$$S(t) = I(t) = Y(t) - C(t).$$

Individuals are assumed to save a constant fraction s of their income,

$$S(t) = sY(t), \tag{2.9}$$

while they consume the remaining $1-s$ fraction of their income:

$$C(t) = (1-s)Y(t) \tag{2.10}$$

In terms of capital market clearing, this implies that the supply of capital resulting from households' behavior can be expressed as $K^s(t) = (1-\delta)K(t) + S(t) = (1-\delta)K(t) + sY(t)$. Setting supply and demand equal to each other, this implies $K(t) = K^s(t)$. Moreover, from (2.3), we have $L(t) = \bar{L}(t)$. Combining these market clearing conditions with (2.1) and (2.7), we obtain *the fundamental law of motion* the Solow growth model:

$$K(t+1) = sF[K(t), L(t), A(t)] + (1-\delta)K(t). \tag{2.11}$$

This is a nonlinear *difference equation*. The equilibrium of the Solow growth model is described by this equation together with laws of motion for $L(t)$ (or $\bar{L}(t)$) and $A(t)$.

2.2.2. Definition of Equilibrium. The Solow model is a mixture of an old-style Keynesian model and a modern dynamic macroeconomic model. Households do not optimize when it comes to their savings/consumption decisions. Instead, their behavior is captured by a *behavioral rule*. Nevertheless, firms still maximize and factor markets clear. Thus it is useful to start defining equilibria in the way that is customary in modern dynamic macro models. Since $L(t) = \bar{L}(t)$ from (2.3), throughout we write the exogenous evolution of labor endowments in terms of $L(t)$ to simplify notation.

Definition 2.2. *In the basic Solow model for a given sequence of* $\{L(t), A(t)\}_{t=0}^{\infty}$ *and an initial capital stock* $K(0)$*, an equilibrium path is a sequence of capital stocks,* *output levels, consumption levels, wages and rental rates* $\{K(t), Y(t), C(t), w(t), R(t)\}_{t=0}^{\infty}$ *such that* $K(t)$ *satisfies (2.11),* $Y(t)$ *is given by (2.1),* $C(t)$ *is given by (2.10), and* $w(t)$ *and* $R(t)$ *are given by (2.5) and (2.6).*

The most important point to note about Definition 2.2 is that an equilibrium is defined as an entire path of allocations and prices. An economic equilibrium does *not* refer to a static object; it specifies the entire path of behavior of the economy.

2.2.3. Equilibrium Without Population Growth and Technological Progress. We can make more progress towards characterizing the equilibria by exploiting the constant returns to scale nature of the production function. To do this, let us make some further assumptions, which will be relaxed later in this chapter:

(1) There is no population growth; total population is constant at some level $L > 0$. Moreover, since individuals supply labor inelastically, this implies $L(t) = L$.
(2) There is no technological progress, so that $A(t) = A$.

Let us define the capital-labor ratio of the economy as

$$k(t) \equiv \frac{K(t)}{L}, \tag{2.12}$$

which is a key object for the analysis. Now using the constant returns to scale assumption, we can express output (income) per capita, $y(t) \equiv Y(t)/L$, as

$$\begin{aligned} y(t) &= F\left[\frac{K(t)}{L}, 1, A\right] \\ &\equiv f(k(t)). \end{aligned} \tag{2.13}$$

In other words, with constant returns to scale output per capita is simply a function of the capital-labor ratio. From Theorem 2.1, we can also express the marginal products of capital and labor (and thus their rental prices) as

$$\begin{aligned} R(t) &= f'(k(t)) > 0 \text{ and} \\ w(t) &= f(k(t)) - k(t) f'(k(t)) > 0. \end{aligned} \tag{2.14}$$

The fact that both of these factor prices are positive follows from Assumption 1, which imposed that the first derivatives of F with respect to capital and labor are always positive.

Example 2.1. **(The Cobb-Douglas Production Function)** Let us consider the most common example of production function used in macroeconomics, the Cobb-Douglas production function—and already add the caveat that even though the Cobb-Douglas production function is convenient and widely used, it is a very special production function and many interesting phenomena are ruled out by this production function as we will discuss later in this book. The Cobb-Douglas production function can be written as

$$\begin{aligned} Y(t) &= F[K(t), L(t)] \\ &= AK(t)^{\alpha} L(t)^{1-\alpha}, \; 0 < \alpha < 1. \end{aligned} \tag{2.15}$$

It can easily be verified that this production function satisfies Assumptions 1 and 2, including the constant returns to scale feature imposed in Assumption 1. Dividing both sides by $L(t)$, we have the representation of the production function in per capita terms as in (2.13):

$$y(t) = Ak(t)^{\alpha},$$

with $y(t)$ as output per worker and $k(t)$ capital-labor ratio as defined in (2.12). The representation of factor prices as in (2.14) can also be verified. From the per capita production function representation, in particular equation (2.14), the rental price of capital can be expressed as

$$\begin{aligned} R(t) &= \frac{\partial Ak(t)^{\alpha}}{\partial k(t)}, \\ &= \alpha Ak(t)^{-(1-\alpha)}. \end{aligned}$$

Alternatively, in terms of the original production function (2.15), the rental price of capital in (2.6) is given by

$$\begin{aligned} R(t) &= \alpha AK(t)^{\alpha-1} L(t)^{1-\alpha} \\ &= \alpha Ak(t)^{-(1-\alpha)}, \end{aligned}$$

which is equal to the previous expression and thus verifies the form of the marginal product given in equation (2.14). Similarly, from (2.14),

$$\begin{aligned} w(t) &= Ak(t)^{\alpha} - \alpha Ak(t)^{-(1-\alpha)} \times k(t) \\ &= (1-\alpha) AK(t)^{\alpha} L(t)^{-\alpha}, \end{aligned}$$

which verifies the alternative expression for the wage rate in (2.5).

Returning to the analysis with the general production function, the per capita representation of the aggregate production function enables us to divide both sides of (2.11) by L to obtain the following simple difference equation for the evolution of the capital-labor ratio:

$$k(t+1) = sf(k(t)) + (1-\delta)k(t). \tag{2.16}$$

Since this difference equation is derived from (2.11), it also can be referred to as the *equilibrium difference equation* of the Solow model, in that it describes the equilibrium behavior of the key object of the model, the capital-labor ratio. The other equilibrium quantities can be obtained from the capital-labor ratio $k(t)$.

At this point, we can also define a *steady-state equilibrium* for this model.

Definition 2.3. *A steady-state equilibrium without technological progress and population growth is an equilibrium path in which* $k(t) = k^*$ *for all* t.

In a steady-state equilibrium the capital-labor ratio remains constant. Since there is no population growth, this implies that the level of the capital stock will also remain constant. Mathematically, a "steady-state equilibrium" corresponds to a "stationary point" of the equilibrium difference equation (2.16). Most of the models we will analyze in this book will admit a steady-state equilibrium, and typically the economy will tend to this steady state equilibrium over time (but often never reach it in finite time). This is also the case for this simple model.

This can be seen by plotting the difference equation that governs the equilibrium behavior of this economy, (2.16), which is done in Figure 2.2. The thick curve represents (2.16) and the dashed line corresponds to the 45° line. Their (positive) intersection gives the steady-state value of the capital-labor ratio k^*, which satisfies

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}. \tag{2.17}$$

Notice that in Figure 2.2 there is another intersection between (2.16) and the 45^0 line at $k = 0$. This is because the figure assumes that $f(0) = 0$, thus there is no production without capital, and if there is no production, there is no savings, and the system remains at $k = 0$, making $k = 0$ a steady-state equilibrium. We will

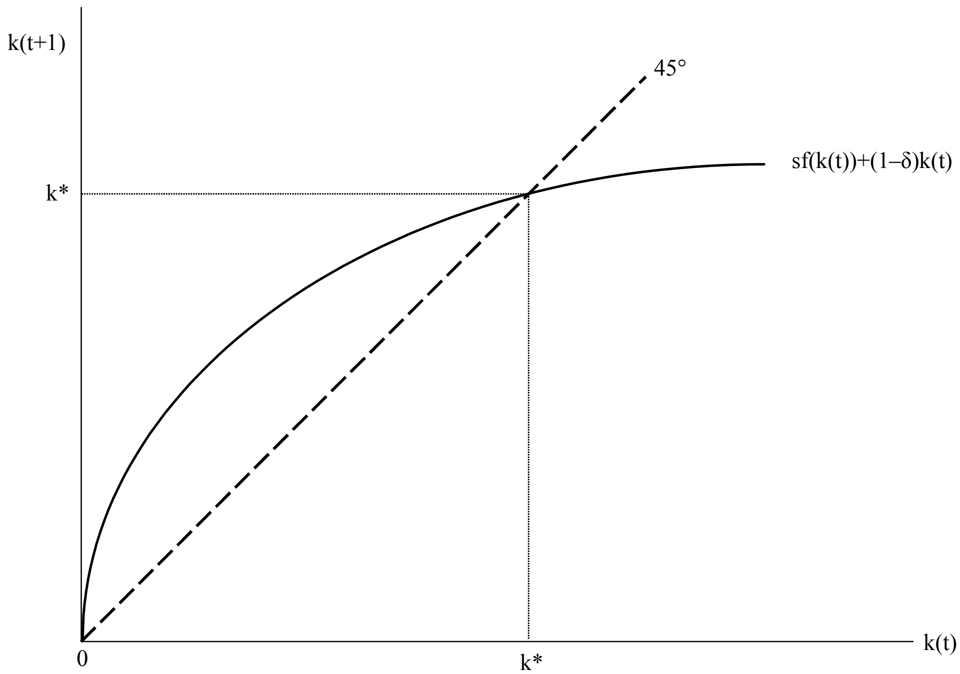

Figure 2.2. Determination of the steady-state capital-labor ratio in the Solow model without population growth and technological change.

ignore this intersection throughout. This is for a number of reasons. First, $k = 0$ is a steady-state equilibrium only when $f(0) = 0$, which corresponds to the case where capital is an essential factor, meaning that if $K(t) = 0$, then output is equal to zero irrespective of the amount of labor and the level of technology. However, if capital is not essential, $f(0)$ will be positive and $k = 0$ will cease to be a steady state equilibrium (a stationary point of the difference equation (2.16)). This is illustrated in Figure 2.3, which draws (2.16) for the case where $f(0) = \varepsilon$ for any $\varepsilon > 0$. Second, as we will see below, this intersection, even when it exists, is an *unstable point*, thus the economy would never travel towards this point starting with $K(0) > 0$. Third, this intersection has no economic interest for us.

An alternative visual representation of the steady state is to view it as the intersection between a ray through the origin with slope δ (representing the function δk) and the function $sf(k)$. Figure 2.4 shows this picture, which is also useful for

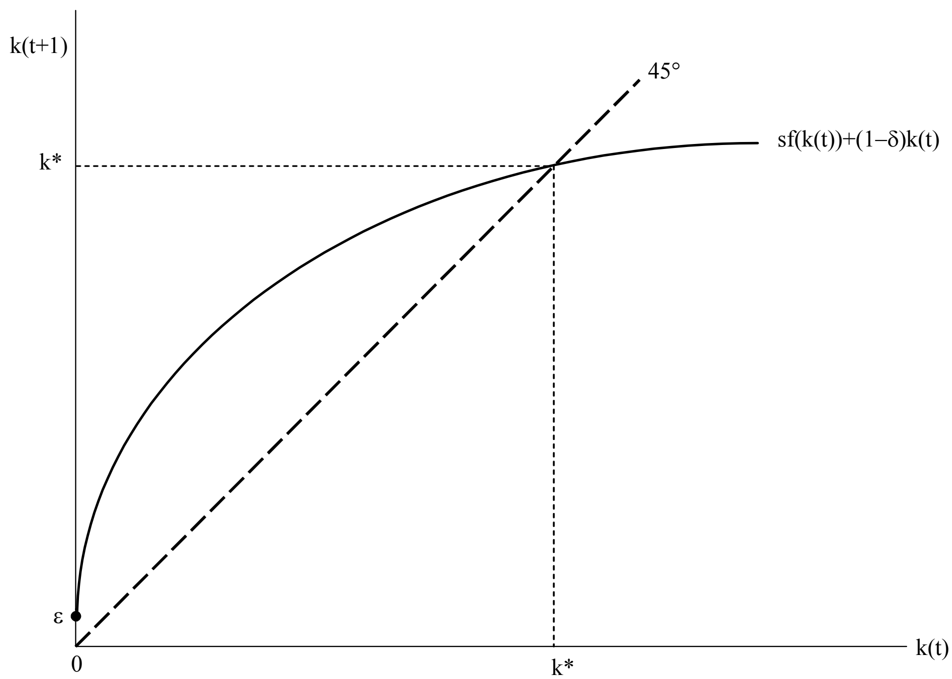

Figure 2.3. Unique steady state in the basic Solow model when $f(0) = \varepsilon > 0$.

two other purposes. First, it depicts the levels of consumption and investment in a single figure. The vertical distance between the horizontal axis and the δk line at the steady-state equilibrium gives the amount of investment per capita (equal to δk^*), while the vertical distance between the function $f(k)$ and the δk line at k^* gives the level of consumption per capita. Clearly, the sum of these two terms make up $f(k^*)$. Second, Figure 2.4 also emphasizes that the steady-state equilibrium in the Solow model essentially sets investment, $sf(k)$, equal to the amount of capital that needs to be "replenished", δk. This interpretation will be particularly useful when we incorporate population growth and technological change below.

This analysis therefore leads to the following proposition (with the convention that the intersection at $k = 0$ is being ignored even when $f(0) = 0$):

Proposition 2.2. *Consider the basic Solow growth model and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where*

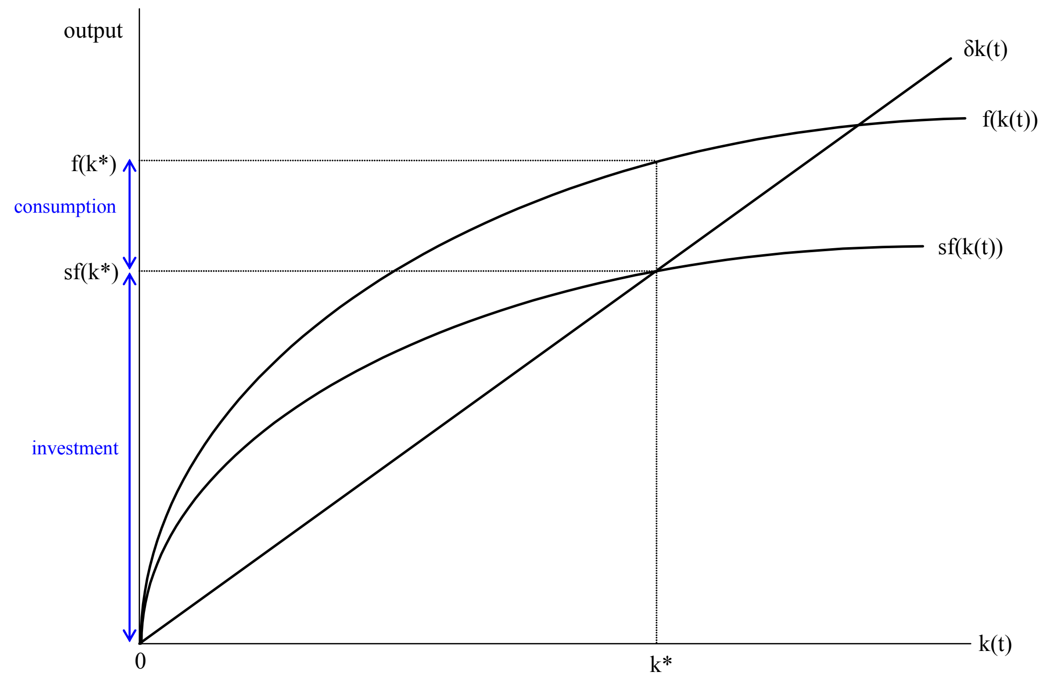

Figure 2.4. Investment and consumption in the steady-state equilibrium.

the capital-labor ratio $k^* \in (0, \infty)$ is given by (2.17), per capita output is given by

$$y^* = f(k^*) \tag{2.18}$$

and per capita consumption is given by

$$c^* = (1 - s) f(k^*) . \tag{2.19}$$

Proof. The preceding argument establishes that (2.17) any k^* that satisfies (2.16) is a steady state. To establish existence, note that from Assumption 2 (and from L'Hopital's rule), $\lim_{k \to 0} f(k)/k = \infty$ and $\lim_{k \to \infty} f(k)/k = 0$. Moreover, $f(k)/k$ is continuous from Assumption 1, so by the intermediate value theorem (see Mathematical Appendix) there exists k^* such that (2.17) is satisfied. To see uniqueness, differentiate $f(k)/k$ with respect to k, which gives

$$\frac{\partial [f(k)/k]}{\partial k} = \frac{f'(k)k - f(k)}{k^2} = -\frac{w}{k^2} < 0, \tag{2.20}$$

where the last equality uses (2.14). Since $f(k)/k$ is everywhere (strictly) decreasing, there can only exist a unique value k^* that satisfies (2.17).

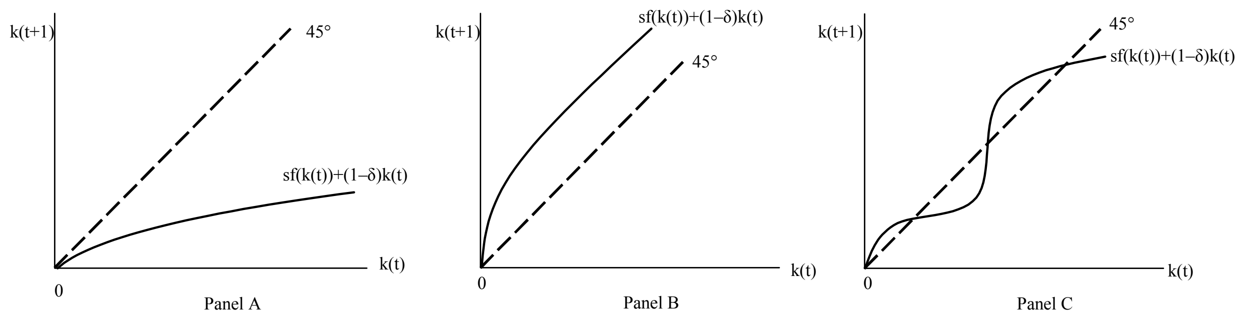

Figure 2.5. Examples of nonexistence and nonuniqueness of steady states when Assumptions 1 and 2 are not satisfied.

Equation (2.18) and (2.19) then follow by definition. □

Figure 2.5 shows through a series of examples why Assumptions 1 and 2 cannot be dispensed with for the existence and uniqueness results in Proposition 2.2. In the first two panels, the failure of Assumption 2 leads to a situation in which there is no steady state equilibrium with positive activity, while in the third panel, the failure of Assumption 1 leads to non-uniqueness of steady states.

So far the model is very parsimonious: it does not have many parameters and abstracts from many features of the real world in order to focus on the question of interest. Recall that an understanding of how cross-country differences in certain parameters translate into differences in growth rates or output levels is essential for our focus. This will be done in the next proposition. But before doing so, let us generalize the production function in one simple way, and assume that

$$f(k) = a\tilde{f}(k),$$

where $a > 0$, so that a is a shift parameter, with greater values corresponding to greater productivity of factors. This type of productivity is referred to as "Hicks-neutral" as we will see below, but for now it is just a convenient way of looking at the impact of productivity differences across countries. Since $f(k)$ satisfies the regularity conditions imposed above, so does $\tilde{f}(k)$.

PROPOSITION 2.3. *Suppose Assumptions 1 and 2 hold and $f(k) = a\tilde{f}(k)$. Denote the steady-state level of the capital-labor ratio by $k^*(a, s, \delta)$ and the steady-state level of output by $y^*(a, s, \delta)$ when the underlying parameters are a, s and δ. Then we have*

$$\begin{aligned} \frac{\partial k^*(a,s,\delta)}{\partial a} &> 0, \ \frac{\partial k^*(a,s,\delta)}{\partial s} > 0 \ \text{and} \ \frac{\partial k^*(a,s,\delta)}{\partial \delta} < 0 \\ \frac{\partial y^*(a,s,\delta)}{\partial a} &> 0, \ \frac{\partial y^*(a,s,\delta)}{\partial s} > 0 \ \text{and} \ \frac{\partial y^*(a,s,\delta)}{\partial \delta} < 0. \end{aligned}$$

PROOF. The proof follows immediately by writing

$$\frac{\tilde{f}(k^*)}{k^*} = \frac{\delta}{as},$$

which holds for an open set of values of k^*. Now apply the implicit function theorem to obtain the results. For example,

$$\frac{\partial k^*}{\partial s} = \frac{\delta (k^*)^2}{as^2 w^*} > 0$$

where $w^* = f(k^*) - k^* f'(k^*) > 0$. The other results follow similarly. □

Therefore, countries with higher saving rates and better technologies will have higher capital-labor ratios and will be richer. Those with greater (technological) depreciation, will tend to have lower capital-labor ratios and will be poorer. All of the results in Proposition 2.3 are intuitive, and start giving us a sense of some of the potential determinants of the capital-labor ratios and output levels across countries.

The same comparative statics with respect to a and δ immediately apply to c^* as well. However, it is straightforward to see that c^* will not be monotone in the saving rate (think, for example, of the extreme case where $s = 1$), and in fact, there will exist a specific level of the saving rate, s_{gold}, referred to as the "golden rule" saving rate, which maximizes the steady-state level of consumption. Since we are treating the saving rate as an exogenous parameter and have not specified the objective function of households yet, we cannot say whether the golden rule saving rate is "better" than some other saving rate. It is nevertheless interesting to characterize what this golden rule saving rate corresponds to.

To do this, let us first write the steady state relationship between c^* and s and suppress the other parameters:

$$\begin{aligned} c^*(s) &= (1-s) f(k^*(s)), \\ &= f(k^*(s)) - \delta k^*(s), \end{aligned}$$

where the second equality exploits the fact that in steady state $sf(k) = \delta k$. Now differentiating this second line with respect to s (again using the implicit function theorem), we have

$$\frac{\partial c^*(s)}{\partial s} = \left[f'(k^*(s)) - \delta\right] \frac{\partial k^*}{\partial s}. \tag{2.21}$$

We define the golden rule saving rate s_{gold} to be such that $\partial c^*(s_{gold})/\partial s = 0$. The corresponding steady-state golden rule capital stock is defined as k^*_{gold}. These quantities and the relationship between consumption and the saving rate are plotted in Figure 2.6.

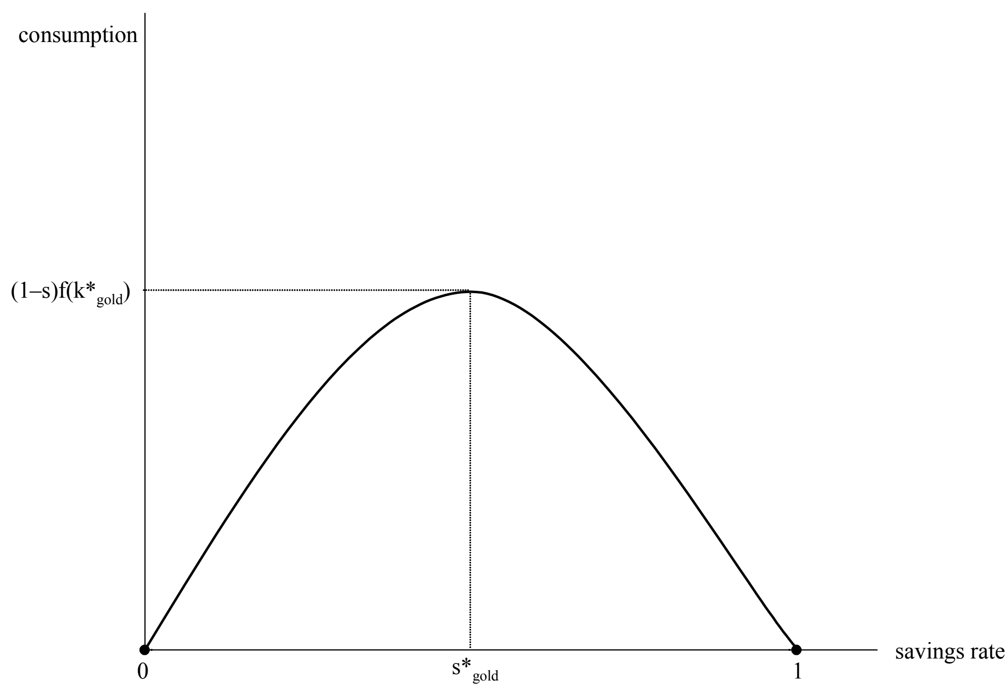

Figure 2.6. The "golden rule" level of savings rate, which maximizes steady-state consumption.

The next proposition shows that s_{gold} and k^*_{gold} are uniquely defined and the latter satisfies (2.22).

PROPOSITION 2.4. *In the basic Solow growth model, the highest level of consumption is reached for s_{gold}, with the corresponding steady state capital level k^*_{gold} such that*

$$f'\left(k^*_{gold}\right) = \delta. \tag{2.22}$$

PROOF. By definition $\partial c^*\left(s_{gold}\right)/\partial s = 0$. From Proposition 2.3, $\partial k^*/\partial s > 0$, thus (2.21) can be equal to zero only when $f'\left(k^*\left(s_{gold}\right)\right) = \delta$. Moreover, when $f'\left(k^*\left(s_{gold}\right)\right) = \delta$, it can be verified that $\partial^2 c^*\left(s_{gold}\right)/\partial s^2 < 0$, so $f'\left(k^*\left(s_{gold}\right)\right) = \delta$ is indeed a local maximum. That $f'\left(k^*\left(s_{gold}\right)\right) = \delta$ is also the global maximum is a consequence of the following observations: $\forall\ s \in [0,1]$ we have $\partial k^*/\partial s > 0$ and moreover, when $s < s_{gold}$, $f'\left(k^*\left(s\right)\right) - \delta > 0$ by the concavity of f, so $\partial c^*\left(s\right)/\partial s > 0$ for all $s < s_{gold}$, and by the converse argument, $\partial c^*\left(s\right)/\partial s < 0$ for all $s > s_{gold}$. Therefore, only s_{gold} satisfies $f'\left(k^*\left(s\right)\right) = \delta$ and gives the unique global maximum of consumption per capita. □

In other words, there exists a unique saving rate, s_{gold}, and also unique corresponding capital-labor ratio, k^*_{gold}, which maximize the level of steady-state consumption. When the economy is below k^*_{gold}, the higher saving rate will increase consumption, whereas when the economy is above k^*_{gold}, steady-state consumption can be increased by saving less. In the latter case, lower savings translate into higher consumption because the capital-labor ratio of the economy is too high so that individuals are investing too much and not consuming enough. This is the essence of what is referred to as *dynamic inefficiency*, which we will encounter in greater detail in models of overlapping generations in Chapter 9. However, recall that there is no explicit utility function here, so statements about "inefficiency" have to be considered with caution. In fact, the reason why such dynamic inefficiency will not arise once we endogenize consumption-saving decisions of individuals will be apparent to many of you already.

2.3. Transitional Dynamics in the Discrete Time Solow Model

Proposition 2.2 establishes the existence of a unique steady-state equilibrium (with positive activity). Recall, however, that an *equilibrium path* does not refer simply to the steady state, but to the entire path of capital stock, output, consumption and factor prices. This is an important point to bear in mind, especially since the term "equilibrium" is used differently in economics than in physical sciences. Typically, in engineering and physical sciences, an equilibrium refers to a point of rest of a dynamical system, thus to what we have so far referred to as *the steady state equilibrium*. One may then be tempted to say that the system is in "disequilibrium" when it is away from the steady state. However, in economics, the non-steady-state behavior of an economy is also governed by optimizing behavior of households and firms and market clearing. Most economies spend much of their time in non-steady-state situations. Thus we are typically interested in the entire dynamic equilibrium path of the economy, not just its steady state.

To determine what the equilibrium path of our simple economy looks like we need to study the "transitional dynamics" of the equilibrium difference equation (2.16) starting from an arbitrary capital-labor ratio, $k(0) > 0$. Of special interest is the answer to the question of whether the economy will tend to this steady state starting from an arbitrary capital-labor ratio, and how it will behave along the transition path. It is important to consider an arbitrary capital-labor ratio, since, as noted above, the total amount of capital at the beginning of the economy, $K(0)$, is taken as a state variable, while for now, the supply of labor L is fixed. Therefore, at time $t = 0$, the economy starts with $k(0) = K(0)/L$ as its initial value and then follows the law of motion given by the difference equation (2.16). Thus the question is whether the difference equation (2.16) will take us to the unique steady state starting from an arbitrary initial capital-labor ratio.

Before doing this, recall some definitions and key results from the theory of dynamical systems. Consider the nonlinear system of autonomous difference equations,

$$\mathbf{x}(t+1) = \mathbf{G}(\mathbf{x}(t)), \tag{2.23}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$. Let $\mathbf{x}^*$ be a fixed point of the mapping $\mathbf{G}(\cdot)$, i.e.,

$$\mathbf{x}^* = \mathbf{G}(\mathbf{x}^*) .$$

Such a $\mathbf{x}^*$ is sometimes referred to as "an equilibrium point" of the difference equation (2.23). Since in economics, equilibrium has a different meaning, we will refer to $\mathbf{x}^*$ as a stationary point or a *steady state* of (2.23). We will often make use of the stability properties of the steady states of systems of difference equations. The relevant notion of stability is introduced in the next definition.

Definition 2.4. *A steady state* $\mathbf{x}^*$ *is (locally)* asymptotically stable *if there exists an open set* $B(\mathbf{x}^*) \ni \mathbf{x}^*$ *such that for any solution* $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ *to (2.23) with* $\mathbf{x}(0) \in B(\mathbf{x}^*)$*, we have* $\mathbf{x}(t) \to \mathbf{x}^*$*. Moreover,* $\mathbf{x}^*$ *is* globally asymptotically stable *if for all* $\mathbf{x}(0) \in \mathbb{R}^n$*, for any solution* $\{\mathbf{x}(t)\}_{t=0}^{\infty}$*, we have* $\mathbf{x}(t) \to \mathbf{x}^*$*.*

The next theorem provides the main results on the stability properties of systems of linear difference equations. The Mathematical Appendix contains an overview of eigenvalues and some other properties of difference equations.

Theorem 2.2. *Consider the following linear difference equation system*

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{b} \tag{2.24}$$

with initial value $\mathbf{x}(0)$*, where* $\mathbf{x}(t) \in \mathbb{R}^n$ *for all* t*,* $\mathbf{A}$ *is an* $n \times n$ *matrix and* $\mathbf{b}$ *is a* $n \times 1$ *column vector. Let* $\mathbf{x}^*$ *be the steady state of the difference equation given by* $\mathbf{A}\mathbf{x}^* + \mathbf{b} = \mathbf{x}^*$*. Suppose that all of the eigenvalues of* $\mathbf{A}$ *are strictly inside the unit circle in the complex plane. Then the steady state of the difference equation (2.24),* $\mathbf{x}^*$*, is* globally asymptotically stable*, in the sense that starting from any* $\mathbf{x}(0) \in \mathbb{R}^n$*, the unique solution* $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ *satisfies* $\mathbf{x}(t) \to \mathbf{x}^*$*.*

Proof. See Luenberger (1979, Chapter 5, Theorem 1). □

Next let us return to the nonlinear autonomous system (2.23). Unfortunately, much less can be said about nonlinear systems, but the following is a standard *local* stability result.

Theorem 2.3. *Consider the following nonlinear autonomous system*

$$\mathbf{x}(t+1) = \mathbf{G}[\mathbf{x}(t)] \tag{2.25}$$

with initial value $\mathbf{x}(0)$*, where* $\mathbf{G}:\mathbb{R}^n \to \mathbb{R}^n$*. Let* $\mathbf{x}^*$ *be a steady state of this system, i.e.,* $\mathbf{G}(\mathbf{x}^*) = \mathbf{x}^*$*, and suppose that* $\mathbf{G}$ *is continuously differentiable at* $\mathbf{x}^*$*. Define*

$$\mathbf{A} \equiv \nabla \mathbf{G}(\mathbf{x}^*),$$

and suppose that all of the eigenvalues of $\mathbf{A}$ *are strictly inside the unit circle. Then the steady state of the difference equation (2.25)* $\mathbf{x}^*$ *is locally* asymptotically stable, *in the sense that there exists an open neighborhood of* $\mathbf{x}^*$*,* $\mathbf{B}(\mathbf{x}^*) \subset \mathbb{R}^n$ *such that starting from any* $\mathbf{x}(0) \in \mathbf{B}(\mathbf{x}^*)$*, we have* $\mathbf{x}(t) \to \mathbf{x}^*$*.*

Proof. See Luenberger (1979, Chapter 9). □

An immediate corollary of Theorem 2.3 the following useful result:

Corollary 2.1. *Let* $x(t), a, b \in \mathbb{R}$*, then the unique steady state of the linear difference equation* $x(t+1) = ax(t) + b$ *is globally asymptotically stable (in the sense that* $x(t) \to x^* = b/(1-a)$*) if* $|a| < 1$*.*

Let $g: \mathbb{R} \to \mathbb{R}$ *be a continuous function, differentiable at the steady state* x^**, defined by* $g(x^*) = x^*$*. Then, the steady state of the nonlinear difference equation* $x(t+1) = g(x(t))$*,* x^**, is locally asymptotically stable if* $|g'(x^*)| < 1$*. Moreover, if* $|g'(x)| < 1$ *for all* $x \in \mathbb{R}$*, then* x^* *is globally asymptotically stable.*

Proof. The first part follows immediately from Theorem 2.2. The local stability of g in the second part follows from Theorem 2.3. Global stability follows since

$$\begin{aligned} |x(t+1) - x^*| &= |g(x(t)) - g(x^*)| \\ &= \left| \int_{x^*}^{x(t)} g'(x)dx \right| \\ &< |x(t) - x^*|, \end{aligned}$$

where the last inequality follows from the hypothesis that $|g'(x)| < 1$ for all $x \in \mathbb{R}$. □

We can now apply Corollary 2.1 to the equilibrium difference equation of the Solow model, (2.16):

Proposition 2.5. *Suppose that Assumptions 1 and 2 hold, then the steady-state equilibrium of the Solow growth model described by the difference equation (2.16) is globally asymptotically stable, and starting from any $k(0) > 0$, $k(t)$ monotonically converges to k^*.*

Proof. Let $g(k) \equiv sf(k) + (1-\delta)k$. First observe that $g'(k)$ exists and is always strictly positive, i.e., $g'(k) > 0$ for all k. Next, from (2.16), we have

$$k(t+1) = g(k(t)), \tag{2.26}$$

with a unique steady state at k^*. From (2.17), the steady-state capital k^* satisfies $\delta k^* = sf(k^*)$, or

$$k^* = g(k^*). \tag{2.27}$$

Now recall that $f(\cdot)$ is concave and differentiable from Assumption 1 and satisfies $f(0) \geq 0$ from Assumption 2. For any strictly concave differentiable function, we have

$$f(k) > f(0) + kf'(k) \geq kf'(k), \tag{2.28}$$

where the second inequality uses the fact that $f(0) \geq 0$. Since (2.28) implies that $\delta = sf(k^*)/k^* > sf'(k^*)$, we have $g'(k^*) = sf'(k^*) + 1 - \delta < 1$. Therefore,

$$g'(k^*) \in (0,1).$$

Corollary 2.1 then establishes local asymptotic stability.

To prove global stability, note that for all $k(t) \in (0, k^*)$,

$$\begin{aligned} k(t+1) - k^* &= g(k(t)) - g(k^*) \\ &= -\int_{k(t)}^{k^*} g'(k)\,dk, \\ &< 0 \end{aligned}$$

where the first line follows by subtracting (2.27) from (2.26), the second line uses the fundamental theorem of calculus, and the third line follows from the observation that $g'(k) > 0$ for all k. Next, (2.16) also implies

$$\begin{aligned}\frac{k(t+1)-k(t)}{k(t)} &= s\frac{f(k(t))}{k(t)}-\delta\\ &> s\frac{f(k^*)}{k^*}-\delta\\ &= 0,\end{aligned}$$

where the second line uses the fact that $f(k)/k$ is decreasing in k (from (2.28) above) and the last line uses the definition of k^*. These two arguments together establish that for all $k(t) \in (0, k^*)$, $k(t+1) \in (k(t), k^*)$. An identical argument implies that for all $k(t) > k^*$, $k(t+1) \in (k^*, k(t))$. Therefore, $\{k(t)\}_{t=0}^{\infty}$ monotonically converges to k^* and is globally stable. □

This stability result can be seen diagrammatically in Figure 2.7. Starting from initial capital stock $k(0)$, which is below the steady-state level k^*, the economy grows towards k^* and the economy experiences *capital deepening*—meaning that the capital-labor ratio will increase. Together with capital deepening comes growth of per capita income. If, instead, the economy were to start with $k'(0) > k^*$, it would reach the steady state by decumulating capital and contracting (i.e., negative growth).

The following proposition is an immediate corollary of Proposition 2.5:

PROPOSITION 2.6. *Suppose that Assumptions 1 and 2 hold, and $k(0) < k^*$, then $\{w(t)\}_{t=0}^{\infty}$ is an increasing sequence and $\{R(t)\}_{t=0}^{\infty}$ is a decreasing sequence. If $k(0) > k^*$, the opposite results apply.*

PROOF. See Exercise 2.5. □

Recall that when the economy starts with too little capital relative to its labor supply, the capital-labor ratio will increase. This implies that the marginal product of capital will fall due to diminishing returns to capital, and the wage rate will increase. Conversely, if it starts with too much capital, it will decumulate capital, and in the process the wage rate will decline and the rate of return to capital will increase.

The analysis has established that the Solow growth model has a number of nice properties; unique steady state, asymptotic stability, and finally, simple and intuitive

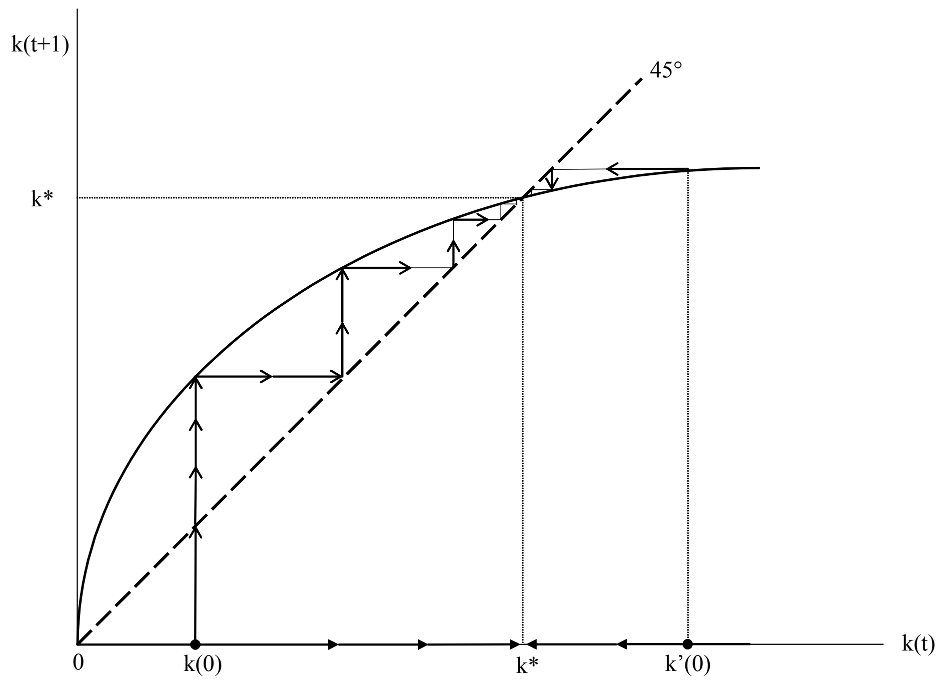

Figure 2.7. Transitional dynamics in the basic Solow model.

comparative statics. Yet, so far, it has no growth. The steady state is the point at which there is no growth in the capital-labor ratio, no more capital deepening and no growth in output per capita. Consequently, the basic Solow model (without technological progress) can only generate economic growth along the transition path when the economy starts with $k(0) < k^*$. This growth is not sustained, however: it slows down over time and eventually comes to an end. We will see in Section 2.6 that the Solow model can incorporate economic growth by allowing *exogenous* technological change. Before doing this, it is useful to look at the relationship between the discrete-time and continuous-time formulations.

2.4. The Solow Model in Continuous Time

2.4.1. From Difference to Differential Equations. Recall that the time periods could refer to days, weeks, months or years. In some sense, the time unit is not important. This suggests that perhaps it may be more convenient to look at