Tutorial 5

1. $\max\limits_{\|\vec{x}\|\neq 0} \dfrac{\|A\vec{x}\|}{\|\vec{x}\|} = \max\limits_{\|\vec{x}\|\neq 0} \left\| \dfrac{A\vec{x}}{\|\vec{x}\|} \right\|$ since $a\|\vec{y}\| = \|a\vec{y}\|$

note that $\dfrac{\vec{x}}{\|\vec{x}\|}$ is a unit vector $\hat{x}$.

$= \max\limits_{\|\vec{x}\|\neq 0} \|A\hat{x}\|$ ← $\hat{x}$ is a unit vector so $\|\hat{x}\|=1$

$= \max\limits_{\|\hat{x}\|=1} \|A\hat{x}\|$

$\therefore \max\limits_{\|x\|\neq 0} \dfrac{\|Ax\|}{\|x\|} = \max\limits_{\|x\|=1} \|Ax\| \quad \blacksquare$

2 Start with our induced matrix norm definition:

(1) $\|A\|_1 = \max\limits_{\|\vec{x}\|=1} \underbrace{\|A\vec{x}\|_1}$ ← now let's look at this part:

(2) $\underbrace{\|A\vec{x}\|_1}_{n\times 1 \text{ vector}} = \left\| \sum\limits_{k=1}^{n} x_k \vec{A}_k \right\|_1$

x_k: kth element in $\vec{x}$; $\vec{A}_k$: kth column of A

(note that x_k is a number, not a vector)

$\leq \sum\limits_{k=1}^{n} \|x_k \vec{A}_k\|_1$ ← this ensures no subtraction in the summation, so we get "$\leq$"

$= \sum\limits_{k=1}^{n} |x_k| \|\vec{A}_k\|_1$ ← since $a\|\vec{y}\| = \|a\vec{y}\|$

(3) Putting (1) & (2) together:

$$\|A\|_1 = \max_{\|\vec{x}\|=1} \|A\vec{x}\|_1 \leq \max_{\|\vec{x}\|=1} \sum_{k=1}^{n} |x_k| \|\vec{A}_k\|_1$$

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(4) If we modify the last result of (3) so that rather than use $\|\vec{A}_k\|_1$ in the summation, it uses $\max_k \|\vec{A}_k\|_1$ (the column with the largest 1-norm), then we get:

$$\max_{\|\vec{x}\|=1} \sum_{k=1}^{n} |x_k| \|A_k\| \leq \left(\max_{k\in[1,n]} \|A_k\|_1\right) \underbrace{\sum_{k=1}^{n} |x_k|}_{\text{this } = 1 \text{ since } \|x\|=1}$$

Notice that $\max_k \|A_k\|_1 = \max_k \sum_{j=1}^{n} |A_{jk}|$.

(both are the maximum absolute column sum)

$\therefore$ we have shown that $\|A\|_1 \leq \max_k \sum_{j=1}^{n} |A_{jk}|$, but we have to show the converse as well

(5) As with the previous case, we can begin with:

$$\|A\vec{x}\|_1 = \left\|\sum_{k=1}^{n} x_k \vec{A}_k\right\|_1$$

but now let's pick some $\vec{x}'$ such that $x'_k = \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases}$ where l is the column in A with the largest absolute column sum.

It follows that: $\|A\vec{x}'\|_1 = \left\|\sum_{k=1}^{n} x'_k \vec{A}_k\right\|_1$

$$= \|\vec{A}_l\|_1$$

$$= \sum_j |A_{jl}|$$ ← by definition of the 1-norm for vectors

$$= \max_k \sum_j |A_{jk}|$$

(6) $\|A\|_1 = \max_{\|x\|=1} \|Ax\|_1 \geq \|A\vec{x}'\| = \max_k \sum_j |A_{jk}|$ $\quad \therefore \|A\|_1 \geq \max_k \sum_j |A_{jk}|$

(7) Since we have shown the inequality in both directions (in (4) and (6)), we conclude that $\|A\|_1 = \max_k \sum_{j=1}^{n} |A_{jk}|$ ∎

3. Recall the following definitions

1) symmetric positive definite: A is positive deffinite if $x^T A x > 0 \;\; \forall \vec{x} \neq 0$

where A is an $m \times n$ matrix and x is an $n \times 1$ vector

2) symmetric positive semi-definite: same as 1) but $x^T A x \geq 0 \;\; \forall \vec{x} \neq 0$

We want to show that $A = M^T M$ is symmetric positive definite if M is an $m \times n$ matrix of full-rank and $m > n$.

① $x^T A x = x^T M^T M x = (Mx)^T Mx = \|Mx\|^2 \geq 0$

$\therefore$ A is symmetric positive semi-definite.

Q: Why, if $\vec{x} \neq 0$, is $\|Mx\|^2$ not strictly greater than zero?

A: $\vec{x}$ could be in null(M) (ex. $\left\| \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\|^2 = 0$)

② Now we will use the fact that M is full-rank to show that this is not possible:

since $m > n$

$$\text{full-rank} \Rightarrow \text{rank}(M) = \min(m, n) = n$$

Recall from linear algebra that:

$$\text{null}(M) = n - \text{rank}(M)$$

$$\therefore \text{null}(M) = n - n = 0$$

other than the zero vector

$\therefore$ since M is full-rank, its null space is empty, so $M\vec{x} = 0$ only if $\vec{x} = 0$

Using this with our result from ① we get that

$$x^T A x = \|Mx\|^2 > 0 \;\; \forall \vec{x} \neq 0$$

$\therefore$ A is symmetric positive definite $\blacksquare$

Why is $null(M) = 0$ if M is full-rank?

In our case, $m > n$, so full-rank implies that all the ~~columns~~ rows in M are linearly independent. This means that if we row-redure M, the result is an upper triangular matrix with no empty rows:

$$M = \begin{bmatrix} \sim & \sim & \sim & \sim \\ & \sim & \sim & \sim \\ 0 & & \sim & \sim \\ & & & \sim \end{bmatrix}$$

Now multiply this to some vector x $\begin{bmatrix} \sim \\ \sim \\ \sim \\ \sim \end{bmatrix}$

Call the resulting vector y: $y = Mx = \begin{bmatrix} \sim & \sim & \sim & \sim \\ & \sim & \sim & \sim \\ 0 & & \sim & \sim \\ & & & \sim \end{bmatrix} \begin{bmatrix} \sim \\ \sim \\ \sim \\ \sim \end{bmatrix} \neq 0$ if $x \neq 0$

Since the last row of M is zero everywhere but the last column, the last element in y can't be zero unless the last element in x is zero. If that is the case, then the second last element in y will only be zero if the second last element in x is zero.

From this we can see that the only way for all the elements in y to be zero is if all the elements in x are zero

$\therefore$ If M is full rank, $M\vec{x} \neq 0 \quad \forall \vec{x} \neq 0$

4. Multiplying the left side (LS) by $(A-\vec{u}\vec{v}^T)$ gives

$$LS = (A-\vec{u}\vec{v}^T)(A-\vec{u}\vec{v}^T)^{-1} = \mathbb{I}$$

Multiplying by the same thing on the right side:

$$RS = (A-\vec{u}\vec{v}^T)\left(A^{-1} + \frac{A^{-1}\vec{u}\vec{v}^T A^{-1}}{1-\vec{v}^T A^{-1}\vec{u}}\right)$$

$$= AA^{-1} - \vec{u}\vec{v}^T A^{-1} + \frac{AA^{-1}\vec{u}\vec{v}^T A^{-1}}{1-v^T A^{-1}\vec{u}} - \frac{\vec{u}\vec{v}^T A^{-1}\vec{u}\vec{v}^T A^{-1}}{1-\vec{v}^T A^{-1}\vec{u}}$$

Consider the dimensions of the vectors/matrices involved:

A is $n \times n$ $\quad$ $\vec{u}$ is $n \times 1$ $\quad$ $\vec{v}^T$ is $1 \times n$

$\therefore \vec{v}^T A^{-1}\vec{u}$ has dimensions $(1\times n)\cdot(n\times n)\cdot(n\times 1) = (1\times n)\cdot(n\times 1) = 1\times 1$

ie. it is a number, not a vector or matrix

Replacing $\vec{v}^T A^{-1}\vec{u} = c$ and $AA^{-1} = \mathbb{I}$ we get:

$$RS = \mathbb{I} - \vec{u}\vec{v}^T A^{-1} + \frac{\vec{u}\vec{v}^T A^{-1}}{1-c} - \frac{c\vec{u}\vec{v}^T A^{-1}}{1-c}$$

note that c is a number so $\vec{u}c\vec{v}^T A^{-1}$ was rewritten as $c\vec{u}\vec{v}^T A^{-1}$

Factoring out $\frac{\vec{u}\vec{v}^T A^{-1}}{1-c}$ from the last two terms gives:

$$RS = \mathbb{I} - \vec{u}\vec{v}^T A^{-1} + \frac{\vec{u}\vec{v}^T A^{-1}}{1-c}(1-c)$$

$$= \mathbb{I} - \vec{u}\vec{v}^T A^{-1} + \vec{u}\vec{v}^T A^{-1}$$

$$= \mathbb{I}$$

$\therefore LS = RS$ ∎