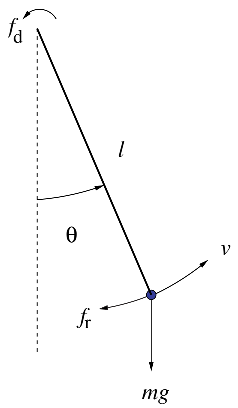

Fig. 4.3 A sketch of a driven pendulum under damping: f_d is the driving force and f_r is the resistive force.

with

$$c_1 = \tau g(y, t), \tag{4.34}$$

$$c_2 = \tau g\left(y + \frac{c_1}{2}, t + \frac{\tau}{2}\right), \tag{4.35}$$

$$c_3 = \tau g\left(y + \frac{c_2}{2}, t + \frac{\tau}{2}\right), \tag{4.36}$$

$$c_4 = \tau g(y + c_3, t + \tau). \tag{4.37}$$

We can easily show that the above selection of parameters does satisfy the required equations. As pointed out earlier, this selection is not unique and can be modified according to the problem under study.

4.5 Chaotic dynamics of a driven pendulum

Before discussing numerical methods for solving boundary-value and eigenvalue problems, let us apply the Runge–Kutta method to the initial-value problem of a dynamical system. Even though we are going to examine only one special system, the approach, as shown below, is quite general and suitable for all other problems.

Consider a pendulum consisting of a light rod of length l and a point mass m attached to the lower end. Assume that the pendulum is confined to a vertical plane, acted upon by a driving force f_d and a resistive force f_r as shown in Fig. 4.3. The motion of the pendulum is described by Newton's equation along the tangential direction of the circular motion of the point mass,

$$ma_t = f_g + f_d + f_r, \tag{4.38}$$

where $f_g = -mg \sin\theta$ is the contribution of gravity along the direction of motion, with θ being the angle made by the rod with respect to the vertical line, and $a_t = l d^2\theta/dt^2$ is the acceleration along the tangential direction. Assume that the time dependency of the driving force is periodic as

$$f_{\mathrm{d}}(t) = f_0 \cos \omega_0 t, \tag{4.39}$$

and the resistive force $f_{\mathrm{r}} = -\kappa v$, where $v = l d\theta/dt$ is the velocity of the mass and κ is a positive damping parameter. This is a reasonable assumption for a pendulum set in a dense medium under a harmonic driving force. If we rewrite Eq. (4.38) in a dimensionless form with $\sqrt{l/g}$ chosen as the unit of time, we have

$$\frac{d^2\theta}{dt^2} + q\frac{d\theta}{dt} + \sin\theta = b\cos\omega_0 t, \tag{4.40}$$

where $q = \kappa/m$ and $b = f_0/ml$ are redefined parameters. As discussed at the beginning of this chapter, we can write the derivatives as variables. We can thus transform higher-order differential equations into a set of first-order differential equations. If we choose $y_1 = \theta$ and $y_2 = \omega = d\theta/dt$, we have

$$\frac{dy_1}{dt} = y_2, \tag{4.41}$$

$$\frac{dy_2}{dt} = -qy_2 - \sin y_1 + b\cos\omega_0 t, \tag{4.42}$$

which are in the form of Eq. (4.1). In principle, we can use any method discussed so far to solve this equation set. However, considering the accuracy required for long-time behavior, we use the fourth-order Runge–Kutta method here.

As we will show later from the numerical solutions of Eqs. (4.41) and (4.42), in different regions of the parameter space (q, b, ω_0) the system has quite different dynamics. Specifically, in some parameter regions the motion of the pendulum is totally chaotic.

If we generalize the fourth-order Runge–Kutta method discussed in the preceding section to multivariable cases, we have

$$\mathbf{y}_{i+1} = \mathbf{y}_i + \frac{1}{6}(\mathbf{c}_1 + 2\mathbf{c}_2 + 2\mathbf{c}_3 + \mathbf{c}_4), \tag{4.43}$$

with

$$\mathbf{c}_1 = \tau\mathbf{g}(\mathbf{y}_i, t_i), \tag{4.44}$$

$$\mathbf{c}_2 = \tau\mathbf{g}\left(\mathbf{y}_i + \frac{\mathbf{c}_1}{2}, t_i + \frac{\tau}{2}\right), \tag{4.45}$$

$$\mathbf{c}_3 = \tau\mathbf{g}\left(\mathbf{y}_i + \frac{\mathbf{c}_2}{2}, t_i + \frac{\tau}{2}\right), \tag{4.46}$$

$$\mathbf{c}_4 = \tau\mathbf{g}(\mathbf{y}_i + \mathbf{c}_3, t_i + \tau), \tag{4.47}$$

where $\mathbf{y}_i$ for any i and $\mathbf{c}_j$ for $j = 1, 2, 3, 4$ are multidimensional vectors. Note that generalizing an algorithm for the initial-value problem from the single-variable case to the multivariable case is straightforward. Other algorithms we have discussed can be generalized in exactly the same fashion.

In principle, the pendulum problem has three dynamical variables: the angle between the rod and the vertical line, θ, its first-order derivative $\omega = d\theta/dt$, and the phase of the driving force $\phi = \omega_0 t$. This is important because a dynamical system cannot be chaotic unless it has three or more dynamical variables. However in practice, we only need to worry about θ and ω because $\phi = \omega_0 t$ is the solution of ϕ.

Any physical quantities that are functions of θ are periodic: for example, $\omega(\theta) = \omega(\theta \pm 2n\pi)$, where n is an integer. Therefore, we can confine θ in the region $[-\pi, \pi]$. If θ is outside this region, it can be transformed back with $\theta' = \theta \pm 2n\pi$ without losing any generality. The following program is an implementation of the fourth-order Runge–Kutta algorithm as applied to the driven pendulum under damping.

```
// A program to study the driven pendulum under damping
// via the fourth-order Runge-Kutta algorithm.

import java.lang.*;
public class Pendulum {
  static final int n = 100, nt = 10, m = 5;
  public static void main(String argv[]) {
    double y1[] = new double[n+1];
    double y2[] = new double[n+1];
    double y[] = new double[2];

 // Set up time step and initial values
    double dt = 3*Math.PI/nt;
    y1[0] = y[0] = 0;
    y2[0] = y[1] = 2;

 // Perform the 4th-order Runge-Kutta integration
    for (int i=0; i<n; ++i) {
      double t = dt*i;
      y = rungeKutta(y, t, dt);
      y1[i+1] = y[0];
      y2[i+1] = y[1];

   // Bring theta back to the region [-pi, pi]
      int np = (int) (y1[i+1]/(2*Math.PI)+0.5);
      y1[i+1] -= 2*Math.PI*np;
    }

 // Output the result in every m time steps
    for (int i=0; i<=n; i+=m) {
      System.out.println("Angle: " + y1[i]);
      System.out.println("Angular velocity: " + y2[i]);
      System.out.println();
    }
  }

// Method to complete one Runge-Kutta step.

  public static double[] rungeKutta(double y[],
    double t, double dt) {
    int l = y.length;
    double c1[] = new double[l];
    double c2[] = new double[l];
    double c3[] = new double[l];
    double c4[] = new double[l];
```

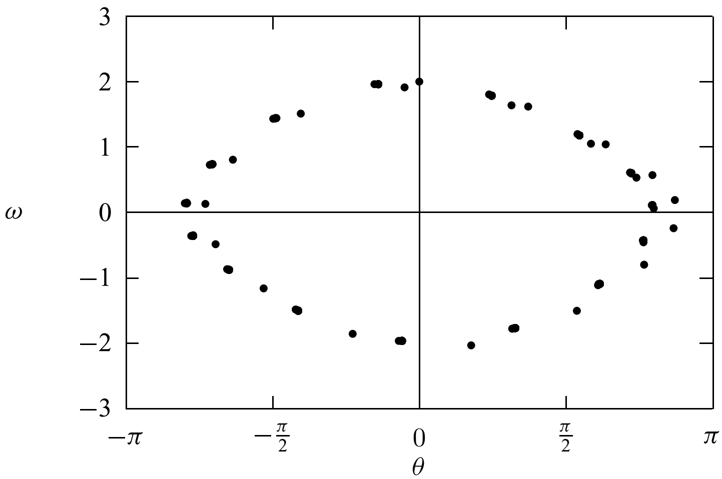

Fig. 4.4 The angular velocity ω versus the angle θ, with parameters $\omega_0 = 2/3$, $q = 0.5$, and $b = 0.9$. Under the given condition the system is apparently periodic. Here 1000 points from 10 000 time steps are shown.

```
    c1 = g(y, t);
    for (int i=0; i<l; ++i) c2[i] = y[i] + dt*c1[i]/2;
    c2 = g(c2, t+dt/2);
    for (int i=0; i<l; ++i) c3[i] = y[i] + dt*c2[i]/2;
    c3 = g(c3, t+dt/2);
    for (int i=0; i<l; ++i) c4[i] = y[i] + dt*c3[i];
    c4 = g(c4, t+dt);
    for (int i=0; i<l; ++i)
      c1[i] = y[i] + dt*(c1[i]+2*(c2[i]+c3[i])+c4[i])/6;
    return c1;
  }

// Method to provide the generalized velocity vector.

  public static double[] g(double y[], double t) {
    int l = y.length;
    double q = 0.5, b = 0.9, omega0 = 2.0/3;
    double v[] = new double[l];
    v[0] = y[1];
    v[1] = -Math.sin(y[0])+b*Math.cos(omega0*t);
    v[1] -= q*y[1];
    return v;
  }
}
```

Depending on the choice of the three parameters, q, b, and ω_0, the system can be periodic or chaotic. In Fig. 4.4 and Fig. 4.5, we show two typical numerical results. The dynamical behavior of the pendulum shown in Fig. 4.4 is periodic in the selected parameter region, and the dynamical behavior shown in Fig. 4.5 is chaotic in another parameter region. We can modify the program developed here to explore the dynamics of the pendulum through the whole parameter space and many important aspects of chaos. Interested readers can find discussions on these aspects in Baker and Gollub (1996).

Several interesting features appear in the results shown in Fig. 4.4 and Fig. 4.5. In Fig. 4.4, the motion of the system is periodic, with a period $T = 2T_0$, where

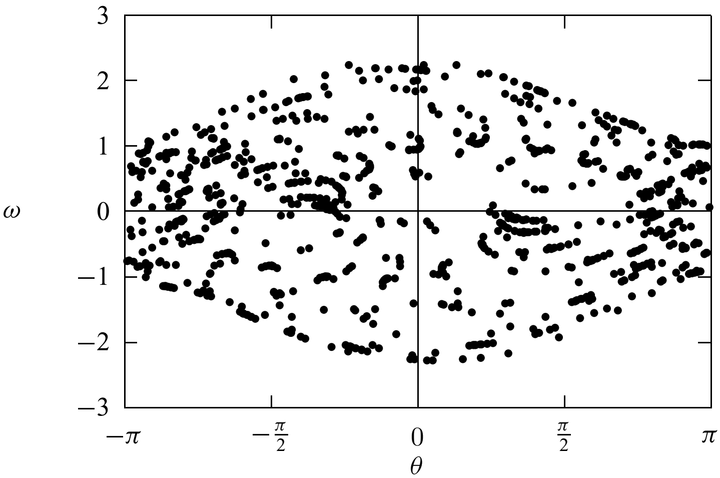

Fig. 4.5 The same plot as in Fig. 4.4, with parameters $\omega_0 = 2/3$, $q = 0.5$, and $b = 1.15$. The system at this point of the parameter space is apparently chaotic. Here 1000 points from 10 000 time steps are shown.

$T_0 = 2\pi/\omega_0$ is the period of the driving force. If we explore other parameter regions, we would find other periodic motions with $T = nT_0$, where n is an even, positive integer. The reason why n is even is that the system is moving away from being periodic to being chaotic; period doubling is one of the routes for a dynaimcal system to develop chaos. The chaotic behavior shown in Fig. 4.5 appears to be totally irregular; however, detailed analysis shows that the phase-space diagram (the ω–θ plot) has self-similarity at all length scales, as indicated by the fractal structure in chaos.

4.6 Boundary-value and eigenvalue problems

Another class of problems in physics requires the solving of differential equations with the values of physical quantities or their derivatives given at the boundaries of a specified region. This applies to the solution of the Poisson equation with a given charge distribution and known boundary values of the electrostatic potential or of the stationary Schrödinger equation with a given potential and boundary conditions.

A typical boundary-value problem in physics is usually given as a second-order differential equation

$$u'' = f(u, u'; x), \tag{4.48}$$

where u is a function of x, u' and u'' are the first-order and second-order derivatives of u with respect to x, and $f(u, u'; x)$ is a function of u, u', and x. Either u or u' is given at each boundary point. Note that we can always choose a coordinate system so that the boundaries of the system are at $x = 0$ and $x = 1$ without losing any generality if the system is finite. For example, if the actual boundaries are at $x = x_1$ and $x = x_2$ for a given problem, we can always bring them back to $x' = 0$