The chaotic physical pendulum

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Problems in Theoretical Physics I - Mechanics

Chaotic behaviour of the damped, harmonically driven pendulum

The description given here is based on the book by Gregory L. Baker and J. P. Gollub. *Chaotic dynamics: an introduction.* Cambridge University Press, 1990.
Even more on the pendulum can be found in a book recently published at the other place: Gregory L. Baker and James A. Blackburn. *The Pendulum - A Case Study in Physics.* Oxford University Press, 2005.

See also publication on a Demonstration Experiment developed at the University of Heidelberg by H. Hauptfleisch, T. Gasenzer, K. Meier, O. Nachtmann, and J. Schemmel: arXiv:**0905.2015 [physics.ed-ph]** [Abstract, PDF]

Theory

A damped, sinusoidally driven pendulum of mass m (or weight W) and length l is described by the following equation of motion:

$$m l \mathrm{d}^2\theta/\mathrm{d}t^2 + \gamma \mathrm{d}\theta/\mathrm{d}t + W\sin\theta = A\cos(\omega_D t)$$

where the left hand terms represent acceleration, damping and gravitation, and the right hand side is the forcing term. In dimensionless form the equation can be rewritten as:

$$\mathrm{d}^2\theta/\mathrm{d}t^2 + (1/q)\mathrm{d}\theta/\mathrm{d}t + \sin\theta = g\cos(\omega_D t)$$

where q is the damping factor, g is the forcing amplitude and ω_D is the angular forcing frequency. One may rewrite this differential equation as a set of first order coupled equations:

$$\mathrm{d}\omega/\mathrm{d}t = -(1/q)\omega - \sin\theta + g\cos\phi$$

$$\mathrm{d}\theta/\mathrm{d}t = \omega$$

$$\mathrm{d}\phi/\mathrm{d}t = \omega_D$$

where ω is the angular frequency, θ is the angular position, and ϕ is the phase of the forcing term. (ω,θ,ϕ) are the three dynamic variables necessary for chaos and the $\sin\theta$ and $\cos\phi$ terms provide the nonlinearity. The pendulum displays a variety of ordered and chaotic behaviour depending on the values of the parameters g,ω_D and q.

For small oscillations the undamped pendulum has a natural angular frequency of unity in the units employed here. Chaotic behaviour arises partly from the interaction between the driving frequency and the natural frequency so we can expect to find interesting behaviour for driving frequencies close to 1.

Phase plots

The pendulum has a 3-dimensional phase space with ω,θ,ϕ as the coordinate axes. In the following phase portraits the phase trajectory is projected onto the (θ,ω) plane. In this plane periodic motion appears as closed loops.

The green part of the trajectories indicates the initial oscillation before the motion becomes periodic. In the chaotic cases, the whole trajectory is plotted in black. The red points represent the *Poincare section*, which indicates the state of the pendulum after each period of the driving force. It therefore corresponds to a stroboscopic view at the pendulum in the phase plane. For the chaotic cases only a limited number of phase and Poincare points have been calculated. With the time approaching infinity, the corresponding phase plots would become uniformly black within certain regions and the Poincare sections would loose their discreteness.

LINKS
SynQS
Synthetic Quantum Systems

Kirchhoff Institute for Physics

Institute for Theoretical Physics

Department of Physics and Astronomy

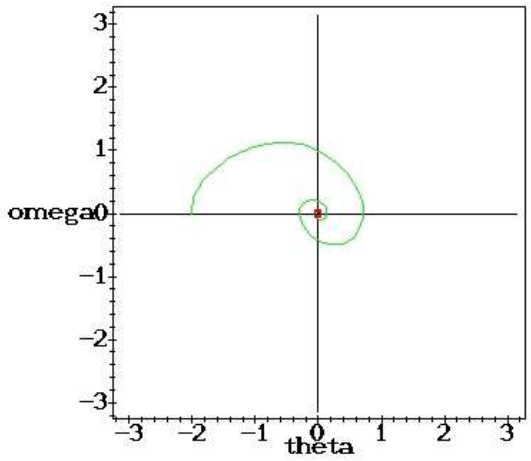
CQD
Heidelberg Center for Quantum Dynamics

AQuS
Analog Quantum Simulators for Many-Body Dynamics

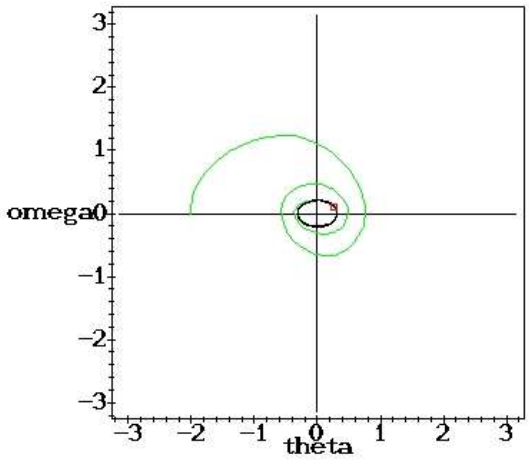
SFB 1225
ISOQUANT

EMMI
Helmholtz Alliance ExtreMe Matter Institute

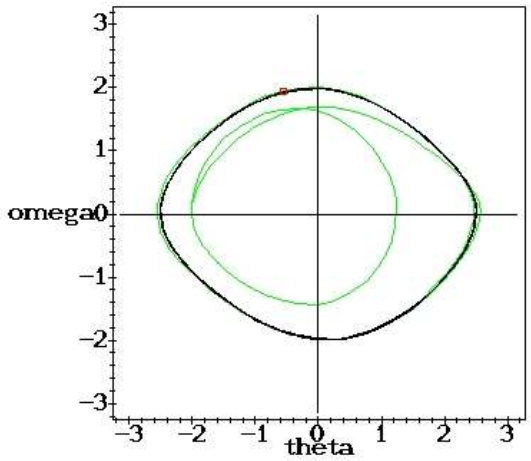
HGSFP
Heidelberg Graduate School of Fundamental Physics



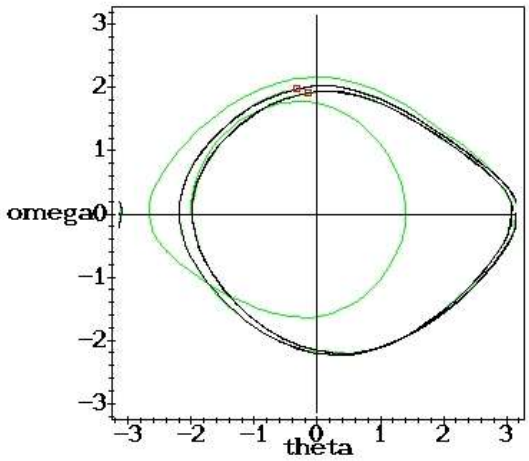
Without drivin force: Damped oscillator. g=0, q=2, ω_D=0.6667



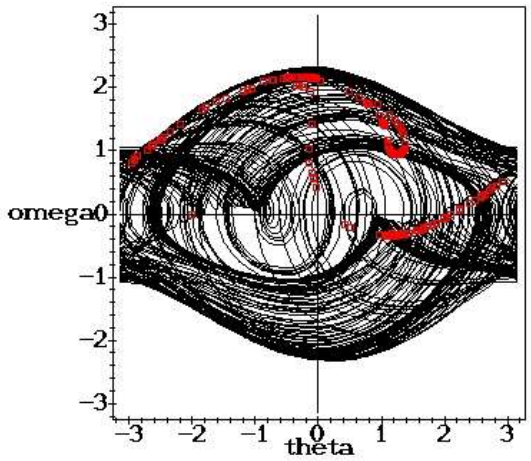
Low driving force: Periodic motion.g=0.2, q=2, ω_D=0.6667



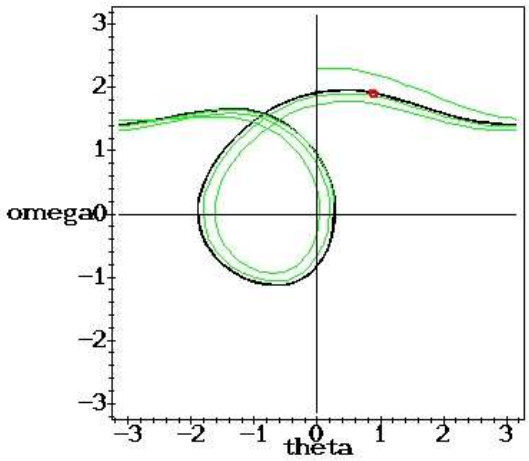
Medium driving force: Single Poincare point.g=0.9, q=2, ω_D=0.6667



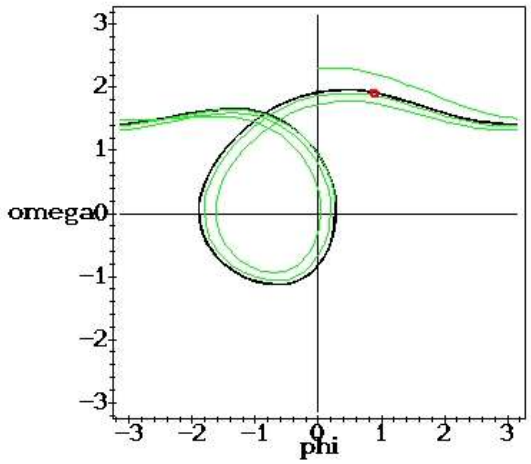
Attractor with bifurcation: Two Poincare points.g=1.07, q=2, ω_D=0.6667



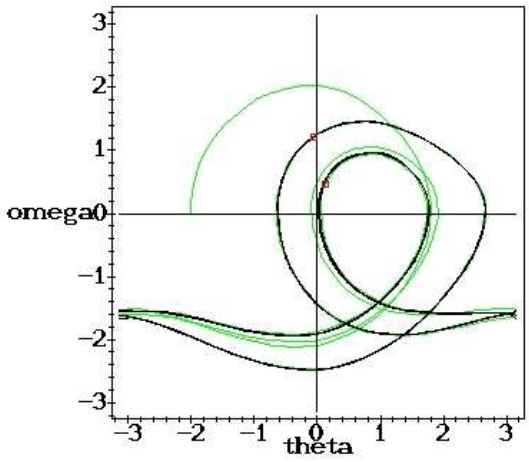
Chaotic state: g=1.15, q=2, ω_D=0.6667



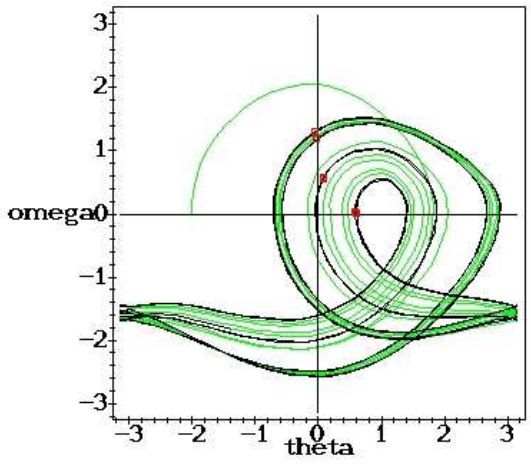
Again periodic, single Poincare point: g=1.35, q=2, ω_D=0.6667



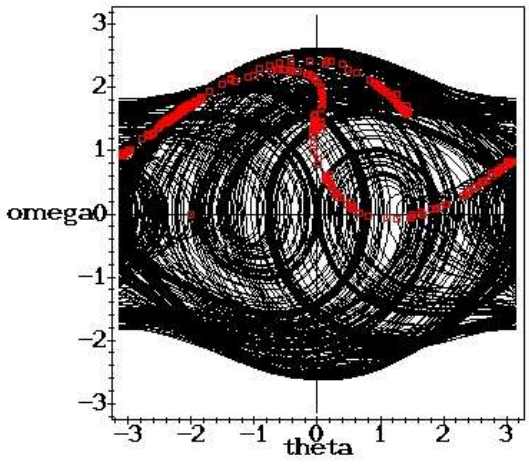
Same force, different initial cond.: g=1.35, q=2, ω_D=0.6667



Again a bifurcation: g=1.45, q=2, ω_D=0.6667



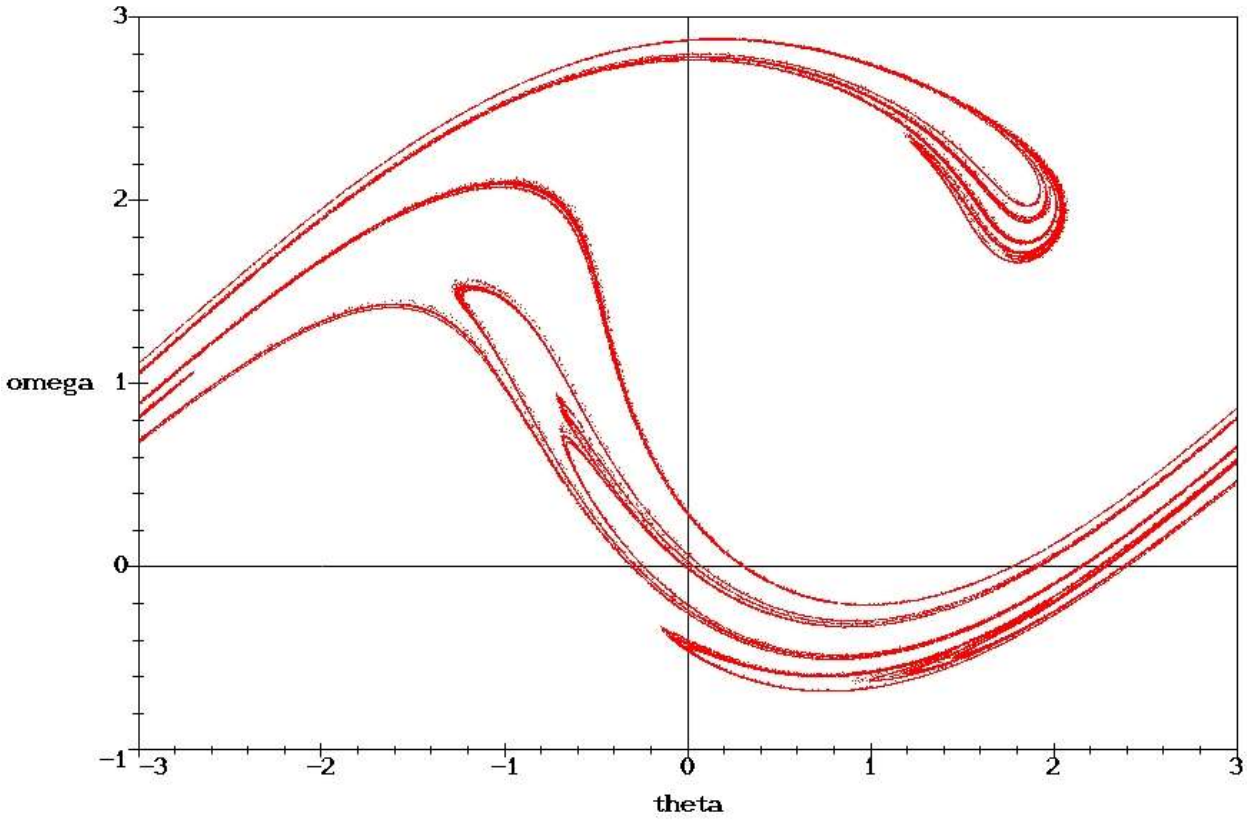
Another bifurcation, four Poincare points: g=1.47, q=2, ω_D=0.6667

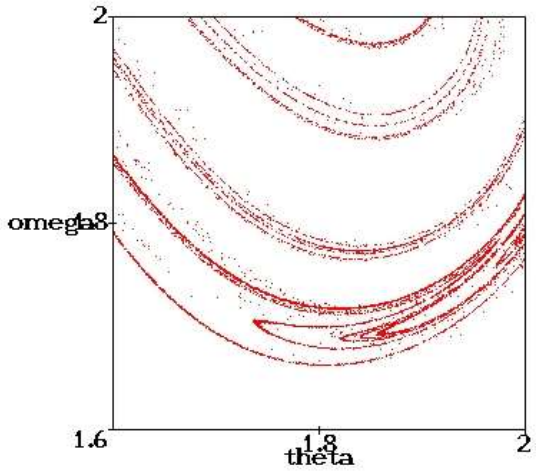


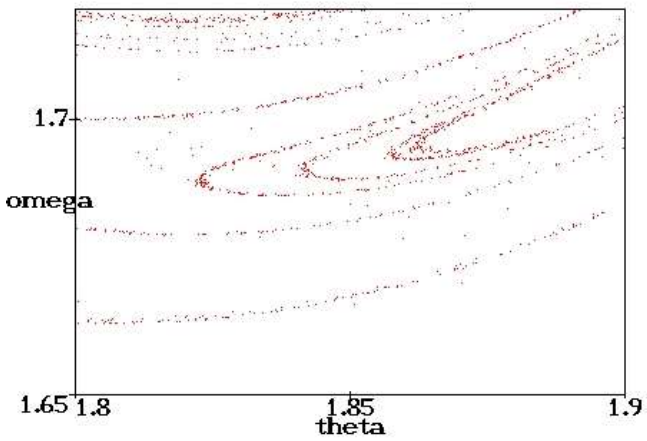
Again chaotic: g=1.5, q=2, ω_D=0.6667

The Poincare Section

In the following plots (parts of) the Poincare section for the chaotic case with g=1.5, q=4,ω_D=0.6667 (i.e., with a weaker damping than above) are shown. 100000 cycles have been computed. They nicely illustrate the self similarity of the Poincare section.

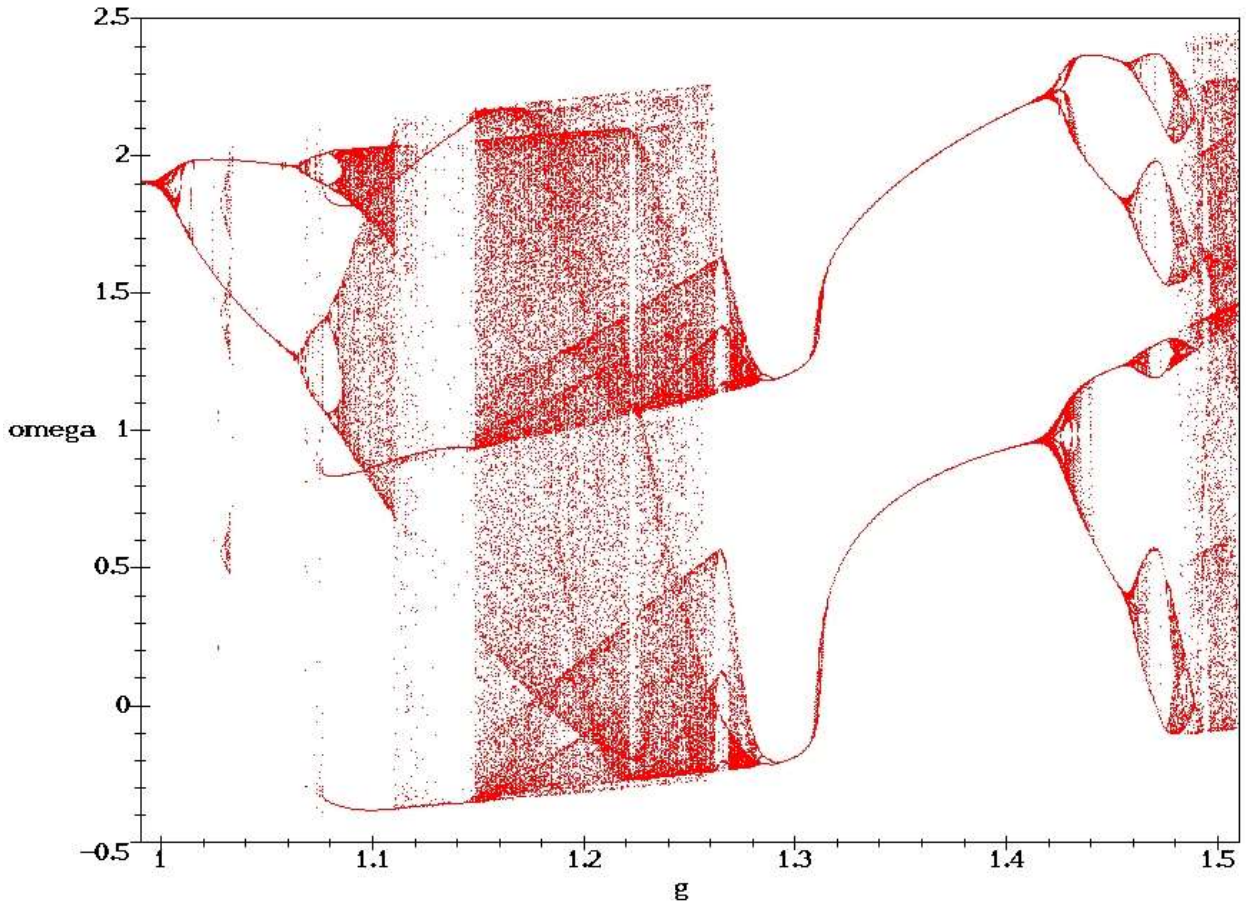






The Bifurcation Diagram

The above plots may be better related to each other using the bifurcation diagram, which shows the Poincare section projected onto the omega axes for varying values of g between 1.0 and 1.5. The regions with many irregularly sputtered points correspond to chaotic behaviour of the pendulum. For each g, 50 cycles, i.e. Poincare points have been calculated after an initial number of 100 cycles. This limited number of data points causes the lines to widen around the bifurcations.


2.5
2
1.5
omega 1
0.5
0
-0.5
1
1.1
1.2
1.3
1.4
1.5
g