COA exercises

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1. How do you define a compact object? What is the compactness parameter and what is its physical interpretation? Say what is the value of the compactness parameter for the typical WD, NS and BH.

A compact object is a very dense object characterized by having a large "compactness" which is proportional to its mass and inversely proportional to its size. Their binding energy is so large that it's a fraction of their rest mass and their escape velocty is a fraction of c. The compactness parameter is:

$$\Xi = \frac{GM}{Rc^2}$$

Which is basically a measure of the gravitational potential.:

$$\Phi(R) = -\Xi c^2$$

Compact objects are defined to have $\Xi \gtrsim 10^{-4}$.

Object	**Ξ (compactness)**
WD	$\sim 10^{-4} \rightarrow 10^{-3}$
NS	$\sim 0.2 \rightarrow 0.4$
BH	1

2. What is the nature of compact objects' main energy reservoir and how can it be detected?

The main energy reservoir of compact objects has the form of *gravitational energy*. The gravitational binding energy of a compact object is a fraction of its rest mass energy, and is huge for compact objects (meaning it would take a lot of energy to disassemble it). The energy reservoir can be detected in the form of released energy. Energy can be released in two ways:

(a) Gravitational collapse: if an object, e.g. a star, with radius R_* collapses to an object of radius R, a fraction of the *rest mass energy* is released. (The released energy can be calculated as follows: $\Delta E_{collapse} = E_{grav}(R_*) - E_{grav}(R) \approx -E_{grav}(R) \sim \Theta Mc^2$.)

(b) Accretion: in this case, a massive particle is accreted onto the surface of the compact object, and a fraction of the *gravitational energy* of this particle is converted into radiation which we can detect. (If a particle of mass m is accreted from infinity, it gains a kinetic energy ΔE_k = -$\phi(R)m = \Theta mc^2$, again a fraction of the particle's rest mass energy.)

Another note: the potential energy of compact objects could also be observed via the motion of a companion satellite.

3. What is the formation scenario for a neutron star (be exhaustive, progenitor mass, impact of metallicity, etc...)?

A neutron star is the last evolutionary stage of massive stars that have a mass of approximately 8 $M_\odot < M_* < 30\ M_\odot$. High mass stars like this are able to fuse elements up to iron in their core. Once the core of the star is composed of iron, nuclear fusion can no longer occur. The core of the star contracts due to the lack of support, until the temperature is high enough to dissociate iron again, taking away energy from the radiation field in the star, leading to an acceleration of the core collapse. The temperature keeps increasing until even helium can be photodissociated. The core of the star now consists mainly of protons, neutrons and free electrons. Core collapse continues, and the density can become so high that neutronization/inverse β-decay occurs, the fusion of protons and electrons to give neutrons:

$$\begin{aligned} p^+ + e^- + 1.36\text{MeV} &\to n + \overline{\nu_e} \\ {}^{56}Fe + 3.7MeV &\to^{56} Mn + \overline{\nu_e} \end{aligned} \tag{1}$$

The core is now mainly composed of neutrons. Neutrons are fermions, and obey the Pauli exclusion principle. This means that there is a maximum density that the core can attain. Therefore, the core collapse is halted by the strong force (the interaction force between individual nucleons), and a neutron star is formed.

During this process, powerful stellar winds are occurring, and moreover, as the density increases due to core collapse, the matter eventually rebounds (as the density is getting too high due to the Pauli exclusion principle), and this produces a shock which causes the outer layers of the star to be ejected: a supernova.

The final mass of a neutron star depends on the progenitor mass and metallicity. Mass is lost due to stellar winds in the core collapse stage, and during to the supernova explosion following core collapse. The amount of mass that is lost due to stellar winds depends on the metallicity: if this is higher, the winds are stronger, as heavier elements have a bigger cross section, resulting in a bigger opacity. For stars with a mass of $\lesssim 8\ M_\odot$ this mass loss is not important, but for high mass stars it is. The resulting mass before core collapse will be lower if the stellar winds are strong, leading to a lower mass neutron star. After the supernova explosion, the resulting neutron star will have a mass ranging from approximately 1 - 3 $M_\odot$. Currently, neutron stars have been observed with masses between 1 - 2 $M_\odot$, from the dynamics of binary systems (either double NSs or NS + WD systems, the first is more accurate as GR effects help constraining mass more precisely).

4. What is the formation scenario for a white dwarf (be exhaustive, progenitor mass, impact of metallicity, composition, etc. . .)?

White dwarfs originate from stars with a mass of $M_* \lesssim 8$ $M_\odot$. Depending on the mass of the star, various burning phases occur in the core: Stars with $M_* < 0.8$ $M_\odot$ have a very long lifetime on the main sequence and are difficult to produce. These stars are not massive enough to burn helium, resulting in a helium rich core at the end of the nuclear burning phase. Stars with 0.8 $M_\odot \lesssim M_* \lesssim 2$ $M_\odot$ develop a degenerate helium core after the main sequence, and the burning of helium occurs quickly in a helium flash. The core of the star is now carbon rich (C-core), but these stars are not massive enough to increase the central temperature enough to fuse carbon. For 2 $M_\odot \lesssim M_* \lesssim 8$ $M_\odot$, elements up to oxygen can be fused. Their helium core is not degenerate and the helium burning phase is stable (no helium flash). After the helium burning phase, carbon is fused, and, the core will be carbon and oxygen rich (C-O core). Stars with $M_* \sim$ 8-9 $M_\odot$ will fuse elements up to magnesium, leading to an oxygen, magnesium and neon rich core (O-Mg-Ne core).
In any case, the (He/C/C-O/Ne-O) core becomes degenerate as the star climbs high enough in the asymptotic giant branch (AGB) after the He burning phase. In this degenerate core, the pressure from free electrons will eventually stop its contraction, and a white dwarf composed mainly of He/C/C-O/Ne-O-Mg forms in the centre of the star. AGB stars suffer from strong mass loss due to strong stellar winds: the star will enter the planetary nebula phase, where the outer layers are shed due to the stellar winds. In the end, all that is left will be the white dwarf, composed mainly of helium, carbon, carbon and oxygen or neon and oxygen.
For low mass stars like these, mass loss due to stellar winds, which increases with increasing metallicity as explained in Q3, is not important before the planetary nebula phase. Therefore, the metallicity of the progenitor does not affect the mass of the white dwarf. White dwarfs form with masses ranging from $\sim 0.1 - 1.4$ $M_\odot$.

5. What is the formation scenario for a solar mass black hole?

I think *stellar* mass black hole is meant here (not solar), maybe ask Elena? Solar mass black holes don't exist, so maybe this is a trick question :p Yeah, I think she meant stellar mass black hole, she has typos everywhere in the assignment so this must be one of them.
A black hole is the last evolutionary stage of a massive star with $M_* \gtrsim 30$ $M_\odot$. The final evolutionary stages of massive stars ($M_* > 8$ $M_\odot$) are explained in Q3. In the case of the NS, the strong force in the core after neutronisation was able to halt the core collapse. However, if the mass of the remnant at this point is larger than approximately 3 $M_\odot$, the neutron star will become a black hole. In this case, the gravitational force is too large and cannot be counterbalanced by the strong interaction between the nucleons. The object will collapse into a singularity.
For a black hole, the resulting mass depends on the strength of the stellar winds and thus on the metallicity (see Q3): massive stars lose more mass if they are of higher metallicity. If there are no metals at all, the mass of the core at core collapse is equal to the initial mass of the progenitor star. This also means that for zero metallicity, the maximum black hole mass only depends on the maximum possible mass for stars. For solar metallicity stars, the maximum black hole mass turns out to be ~ 25 $M_\odot$. The minimum mass for a stellar mass black hole is ~ 3 $M_\odot$, which can only form if the metallicity is sufficiently high so that enough of the initial mass is lost.
From measurements of X-ray binaries and black hole mergers (gravitational waves), black holes have been found with masses ranging from $\sim 5 - 80$ $M_\odot$.

6. Exercise:

(a) Given the formation mechanism of NS and WD, estimate their magnetic field if the progenitor star has a field of 155 Gauss.

(b) Comment on the result.

(a) To get the resulting magnetic field we must think of the conservation of magnetic flux $\phi_B \propto BR^2$ from the progenitor to the resulting compact object:

$$BR^2 = B_p R_p^2 \Rightarrow \frac{B}{B_p} = \left(\frac{R_p}{R}\right)^2$$

For a WD $(R_p/R)^2 \approx 5 \times 10^3$ and for a NS $(R_p/R)^2 \approx 5 \times 10^{11}$

So, using a progenitor magnetic field of 155 Gauss we get:

For a White Dwarf: $B = 7.75 \times 10^5$ Gauss

For a Neutron Star: $B = 7.75 \times 10^{13}$ Gauss

(b) We note that the resulting magnetic field of the compact object is very large, specially for the Neutron Star. The smaller the object, the bigger its magnetic field. These numbers are very big compared to for example the magnetic field of Earth which is only $\sim$0.25 Gauss.

7. Exercise:

(a) The Sun rotation period is $\sim$24.5 day, assuming no losses of any kind, what would be the rotation period of the resulting WD ?

(b) Comment on the result.

(a) As in the previous exercise, this one can be done using the conservation of a quantity, in this case, the Angular momentum is conserved. $\Omega_\odot R_\odot^2 = \Omega R^2$, thus:

$$\Omega_\odot R_\odot^2 = \Omega_{\rm WD} R_{\rm WD}^2 \Rightarrow \frac{\Omega_{\rm WD}}{\Omega_\odot} = \left(\frac{R_\odot}{R_{\rm WD}}\right)^2 \approx 5 \times 10^3$$

So, if the Sun rotation period is $\sim$24.5 days, then the WD's resulting WD angular frequency would be:

$$\Omega_{\rm WD} = 5 \times 10^3 \times \frac{2\pi}{24.5} = 1.3 \times 10^3 \text{ rev/day} \approx 54 \text{ rev/hour}$$

Which corresponds to an orbital period of

$$T_{\rm WD} = 0.12 \text{ hour} \approx 7.2 \text{ min}$$

(b) Our sun will have a very short orbital period (compared to the current one) once it becomes a WD, although this orbital period is still very large compared to NS which can be of the order or milli seconds.

8. Describe the main steps in the history of discovering, theoretically describing and observing WDs

The first white dwarf to be detected was 40 Eri. In the first HR diagram published by Russell in 1914 a WD is present: 40 Eridanus B, he was the first to determine its spectral type (actually done by Williamina Flemming) and realised that it was too dim for its temperature $\rightarrow$ must have a small size. Sirius B was very important to our understanding of WDs. In 1914 its spectrum was measured and an imprecise temperature of 8000 K obtained. Via the Stefan-Boltzmann law, a sub solar radius of 18×10^3 km was inferred, implying ultra dense matter of $\sim 7 \times 10^4$ g/cm^3 (Sun ~ 1 kg/m^3) !
In 1926 Eddington says he does not expect these objects to behave as an ideal gas, but he foresaw that they would be common objects. In the same year, Dirac and Fermi independently introduce the quantum mechanical statistical framework for fermions (Fermi-Dirac distribution) and Fowler understands that because of this, the electron degeneracy pressure can compensate gravity in white dwarfs and ensure equilibrium.
In 1930, Chandrasekhar introduces the relativistic corrections to the equation of state and derives a maximum mass. Later he understands that this means that only low mass stars can become WDs (stellar evolution)
In 1949, Samuil A. Kaplan introduces general relativistic corrections to describe the structure. After that, efforts have been made to improve the equation of state, to account for, e.g., interaction between electrons and ions, and the effect of a non-zero temperature. In 1956-1958, Schtzman, Harrison and Wheeler publish the first realistic equation of state, including inverse beta decay.

9.

(a) Define what is an "equation of state"

(b) Exhaustively describe the simplest possible one for WDs.

(c) Comment on its validity: does it allow us to match observations?

(a) En equation of state is a equation that defines the relation between thermodynamical properties of an element of matter. For a given composition, it allows to derive pressure, internal energy, entropy, etc... as a function of temperature and density (or any two "state variables"). when used alongside the conservation laws, it provides a way to derive the WD internal structure. Definition internal structure: thermodynamical quantities as a function of radius in the star

(b) An 'isolated' WD ultimately cools to very low temperatures so one can assume that the pressure of degenerate electrons at $T = 0$ supports the star against gravitational collapse.
With this in mind, the simplest EoS will be that of a cold, degenerate, non interacting fermion gas given by Fermi-Dirac statistics:

$$\frac{dn}{dp} = \frac{4\pi g}{h^3} p^2 \frac{1}{exp(\frac{E-u}{kT}) - 1}$$

In the case of $T = 0$ the exponential part can be written as:
1 if $E < u$
0 if $E > u$

(c) This EoS is an excellent approximation and allows us to build WD structure in accordance to data since the total pressure is dominated by electron degeneracy pressure.

10. Exercise:

(a) For a cold degenerate electron gas derive how the normalised Fermi momentum p_F/mc depends on the total matter density ρ and the electron fraction Y_e.

(b) Are electrons relativistic for a typical WDs with $Y_e = 0.5$ and average matter density $\rho \sim 10^6 g/cm^3$?

(a) In the limit $T \to 0$:

$$\frac{dn}{dp} = \frac{4\pi g}{h^3}p^2 \Rightarrow \int dn = \int_0^{p_F} \frac{4\pi g}{h^3}p^2 dp$$

$$n = \frac{4\pi g}{3h^3}p_F^3 \Rightarrow p_F = \left(\frac{2}{g}3\pi^2\hbar^3 n\right)^{1/3}$$

So the normalized Fermi momentum χ_F is:

$$\chi_F = \frac{p_F}{mc} = \left(\frac{2}{g}3\pi^2\left(\frac{\hbar}{mc}\right)^3 n\right)^{1/3}$$

taking into account that $n = \frac{Y_e\rho}{m_H}$, we get:

$$\chi_F = \left(\frac{2}{g}3\pi^2\left(\frac{\hbar}{mc}\right)^3 \frac{Y_e\rho}{m_H}\right)^{1/3}$$

(b) For $Y_e = 0.5$, $\rho \sim 10^6 g/cm^3$, $m = m_e$ and g=2:

$\chi_F \approx 0.8$ so electrons are midly relativistic! (the denser, the more relativistic)

11. Exercise: what does it mean that a gas of particle is degenerate? Show that electrons in WDs are fully degenerate but protons aren't.

We say that a gas of particles is degenerate when its energy is smaller than the Fermi (or degeneracy) Temperature.
The Fermi temperature is:

$$k_B T_F = E_F - mc^2 \quad \rightarrow \quad E_F = k_B T_F + mc^2$$

And the Fermi momentum is:

$$p_F = mc\sqrt{\left(\frac{E_F}{mc^2}\right)^2 - 1}$$

So replacing E_F in the Fermi momentum equation:

$$p_F = mc\sqrt{\left(\frac{k_B T_F + mc^2}{mc^2}\right)^2 - 1}$$

$$p_F^2 = m^2c^2\left[\left(\frac{k_B T_F + mc^2}{mc^2}\right)^2 - 1\right]$$

$$p_F^2 + m^2c^2 = \left(\frac{k_B T_F + mc^2}{mc^2}\right)^2 m^2c^2$$

$$\sqrt{\frac{p_F^2}{m^2c^2} + 1} = \frac{k_B T_F + mc^2}{mc^2}$$

$$mc^2\sqrt{\frac{p_F^2}{m^2c^2} + 1} - mc^2 = k_B T_F$$

with $\chi = \frac{p_F}{mc}$ we get:

$$T_F = \frac{mc^2}{k_B}\left(\sqrt{\chi^2 + 1} - 1\right)$$

For $\chi = 0.8$, $m_e = 9.1 \times 10^{-28}$ g, $c = 3 \times 10^8$ cm/s, $k_B = 1.4 \times 10^{-16}$ and noting that a typical central temperature of a WD is $T_{\rm WD} \approx 10^7 K$, we get

$$T_F \approx 1.66 \times 10^9 K \gg T_{\rm WD}$$

We see that electrons are fully degenerate, but in the case of protons, the Fermi temperature is inversely proportional to the mass:

$$T_F \propto n^{2/3}/m$$

$$T_{F,e} \approx 2000 \times T_{F,p/n}$$

$$T_{F,n/p} \approx 10^6 K < T_{\rm WD} \ll T_{F,e}$$

So protons are not degenerate.

12. **Exercise:** **The typical electron pressure in WDs is of the order of 5×10^{22} Ba, how does it compare with**

(a) the gas thermal pressure and

(b) the average pressure needed to ensure hydrostatic equilibrium?

(c) Comment on the result.

$$P = \frac{1}{3}\int_0^\infty p v_p n(p) dp \approx 5 \times 10^{22}\text{bar} = P_{\text{deg,e}}$$

(a) For thermal pressure we have:

$$P_{\text{th}} = k_B T \frac{\rho}{m_H} \approx 10^{20}\text{bar}$$

so $P_{\text{th}} \ll P_{\text{deg,e}}$

(b) To see what is the pressure needed to ensure hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{Gm(r)\rho}{r^2}$$

$$P = -\int_0^R \frac{Gm(r)\rho}{r^2} dr$$

$$P_{\text{grav}} \simeq \alpha \frac{GM}{3R}\bar{\rho} \approx 10^{22}\ P_{\text{deg,e}}$$

(c) We note that the electron degeneracy pressure is enough to ensure hydrostatic equilibrium in a WD, thus makes thermal pressure unimportant in the stability of the compact object. We can also say that WD's pressure does not depend on the temperature, so the WD does not need to be "hot" to ensure hydrostatic equilibrium because $P_{\text{deg,e}}$ is enough.

13. **

(a) Define what is the stellar structure of a stellar object;

(b) Given an EoS for WDs, how does one derive their stellar structure?

(c) Complete the answer to b) by writing the main relevant equations in spherical symmetry, ignoring rotation and magnetic field

(d) Use a compactness argument to support the assumption that a Newtonian framework (equations) is sufficient.

(a) The stellar structure describes the thermodynamical quantities of the star as a function of the **radius**. Examples are $\rho = \rho(r)$, $P = P(r)$, etc.

(b) The strategy is to find $\rho = \rho(r)$. For that we need three equations: mass conservation (relates $m(r)$ to $\rho(r)$, hydrostatic equilibrium (relates $P(r)$ to $m(r)$ and $\rho(r)$) and the equation of state (relates P to ρ). Given the third, you have three equations and three unknown variables ($P(r)$, $\rho(r)$, $m(r)$), so you can solve the system.

(c) The equations are:

i. *Mass conservation*

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

ii. *Hydrostatic equilibrium*

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}$$

iii. Equation of state

$$P = K\rho^{\gamma}$$

(d) Qualitatively, "low" compactness means that we are in the weak gravity regime, so a newtonian framework is enough to describe gravity. I think this answer needs to be completed better.

14. **What is the Chandrasekhar's mass ?

(a) Explain the assumptions needed to derive the Lane-Emden equations.

(b) State and use for your discussion the results from the relevant Lane-Emden equations and draw by hand a mass versus matter density plot: be quantitative and precise about the mass and density ranges.

The Chandrasekhar's mass is the maximum mass for WD ($\sim 1.44 M_{\odot}$), if the star exceeds this mass it will become a NS or a BH. It's basically a limit mass given by the equation of state for degenerate electrons. This implies that only low mass stars can become WDs.

(a) To derive the the Lane-Emden equation you need the three equations stated in *13)*, namely mass conservation, hydrostatic equilibrium and equation of state. For these you assume a newtonian framework, spherical symmetry, a barotropic fluid (pressure just depends on density), no rotation and no magnetic field.

Taking the hydrostatic equilibrium equation, deriving by r on both sides, and using the mass conservation for dm/dr you get:

$$\frac{1}{r^2 \rho} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G$$

Using the equation of state, you obtain the *Lane-Emden* equation:

$$\frac{1}{r^2 \rho} \frac{d}{dr} \left(\frac{r^2}{\rho} K \gamma \rho^{\gamma - 1} \frac{d\rho}{dr} \right) = -4\pi G$$

To solve this equation, one further needs the boundary conditions:

i. $\rho(r = 0) = \rho_c$, where ρ_c is the central density;

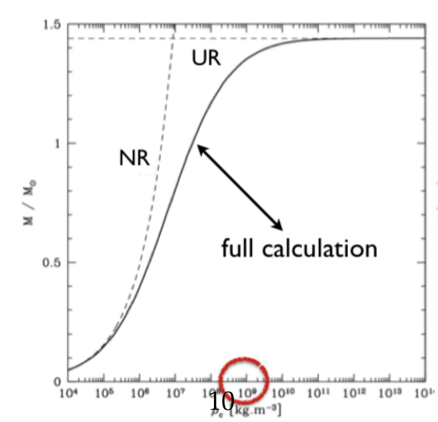
ii. $\frac{d\rho}{dr}(r = 0) = \frac{dP}{dr}(r = 0) = 0$, because near the center $m(r) \approx \frac{4\pi}{3} \rho_c r^3$, then taking the hydrostatic equilibrium equation, $\frac{dP}{dr} \propto r \rightarrow \frac{dP}{dr}(r = 0) = 0$.

(b) Since the equation of state depends on γ, you will get different solutions for different values of γ. The density profile $\rho(r)$ can be integrated from 0 to R, where R is the size of the star (using the mass conservation equation) to understand how the mass depends on the size. You then get the following results:

i. *Non-relativistic electrons limit:* $\gamma = 5/3$. In this case, $M \propto R^{-3}$.

ii. *Ultra-relativistic electrons limit:* $\gamma = 4/3$. In this case, $M = constant$.

This second case means that there is a maximum mass, the Chandrasekhar mass.



The figure can be found in slide 48 of the White Dwarf lecture.

15. **What is the Chandrasekhar's mass? give a simple argument based on equilibrium of forces.

The Chandrasekhar's mass is the maximum mass for WDs ($\sim 1.44 M_\odot$), if the star exceeds this mass it will become a NS or a BH. It's basically a limit mass given by the equation of state for degenerate electrons. This implies that only low mass stars can become WDs.

We start by noting that in order to have our stable compact object we need $f_{\text{press}} = f_{\text{grav}}$
The dependency of f_{press} on mass and radius is:

$$f_{\text{press}} \propto \frac{dP}{dr} \propto \frac{P}{R} \propto \kappa \frac{\rho^\gamma}{R} \propto \kappa \frac{M^\gamma}{R^{3\gamma+1}}$$

The dependency of f_{grav} on mass and radius is:

$$f_{\text{grav}} \propto \frac{GM(r)\rho}{r^2} \propto \frac{GM^2}{R^5}$$

So if $f_{\text{press}} = f_{\text{grav}}$, then:

$$\kappa \frac{M^\gamma}{R^{3\gamma+1}} \propto \frac{GM^2}{R^5} \quad \Rightarrow \quad M^{\gamma-2} \propto \frac{G}{\kappa} R^{3\gamma-4}$$

In the case of $\gamma = 5/3$ (non-relativistic) we get

$$M^{-1/3} \propto \frac{G}{\kappa} R \quad \Rightarrow \quad M \propto R^-3$$

In the case of $\gamma = 4/3$ (relativistic) we get

$$M^{-2/3} \propto \frac{G}{\kappa} R^0 \quad \Rightarrow \quad M_{CH} \equiv \left(\alpha \frac{\kappa}{G}\right)^{2/3}$$

So we see that if $M > M_{CH}$ then $f_{\text{grav}} > f_{\text{press}}$ always and the star collapses.

16. Describe the main steps in the history of discovering, theoretically describing and observing BHs.

In the 1700s there was already the intuition that there must be some "dark" stars that wouldn't allow light to escape from them. Michell (1784) and Laplace (1796) considered a particle of light of mass "m" with mechanical energy

$$E = m\left(\frac{1}{2}v_0^2 - \frac{GM}{r}\right)$$

Light escapes if $E > 0$, equivalently if $r > R_g = \frac{2GM}{c^2}$

So an object with size $R < R_g$ cannot emit light, and therefore it is a "black hole". This was the first time that the idea of an object with size R could be dark and not necessarily a point mass with $R = 0$.

It was the advent of GR that brought this idea into a more elaborate context. In 1915, two months after Einstein published GR, Karl Schwarzschild calculated the exact solution of Einstein equation in spherical symmetry, in vacuum. It has only one parameter: mass M. It applies to any spherical object and it can be used to describe space-time around (r>0) a black hole placed at r=0.

In 1930 Chandrasekhar calculates a maximum mass for WDs, extended to all degenerate objects by Landau. Eddington (who did not believe in Chandrasekhar's limit) envisages black holes (together with Landau) but he feels that nature must somehow prevent this total collapse to a point.

In 1939 Oppenheimer and Snyder showed that a consequence of the collapse of an homogeneous sphere without pressure in GR, the sphere eventually is not able to exchange information with the rest of the world. It is the first rigorous calculation of BH formation.

End 1950s - beginnning 1960s: BH was an object of study for physicists, with little interest showed by astrophysicists. At the end of the 50s Wheeler and co. reconsider the issue of collapse for a massive object (Wheeler in 1968 came up with the name "black hole").

In 1963 Kerr found a family of solutions of the Einstein equations in vacuum, with no charge but relaxing the assumption of asymmetry (spin). In 1963 Newman extends the solution to charged BH (not really expected in nature) (The Kerr-Newman solution is the most complete description of the space-time outside a stationay BH).

Ealry 60s: The beginning of X-ray astronomy and of the astrophysics of compact objects. In 1962, Riccardo Giacconi and co-authors discovered the first X=ray source beyond the solar system, following the launch of the USA Arobee rocket (Nobel prize in 2002).

1960s: BH becomes of astrophysical interest only after the discovery of the compact X-ray sources (1962), quasars (1963), and pulsars (1968). The compact object field is of growing importance.

The first BH candidate was observed in 1970 with the observation of the X-ray binary Cygnus X-I. The compact object mass is at least $6M_\odot$, that excludes a NS for any equation of state.
The indirect observation of the supermassive BH in the centre of the MW and the suppermassive BH in other galaxies
In 2015 we had the first detection of a BH in GW (by LIGO) and of a binary black holes candidate.

17. **

(a) What is the Schwarzchild radius?

(b) what are its properties (physical interpretation) for a non-rotating black holes (i.e in the classical Schwarzschild's solution).

(a) The Schwarzschild is a physical parameter that shows up in the Schwarzschild solution to Einstein's field equations, corresponding to the radius defining the event horizon of a Schwarzschild (non-rotating) BH. It is a characteristic radius associated with every object that has mass and it's given by:

$$R_s = \frac{2GM}{c^2}$$

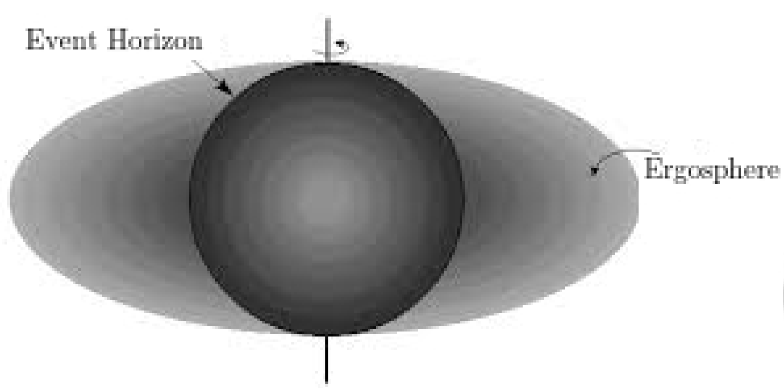
(b) **i)** It is a "static" limit because for $r < R_s$ no static solution is possible.
ii) R_s is a dynamical boundary ($dr < 0$ always) which means that any mass that passes through R_s inevitably falls towards $r = 0$. Also, no Hydrostatic Equilibrium object with size $R < R_s$ is possible, it would collapse.
iii) It's an event horizon, because it divides two regions with no casual connection (signals cannot be exchanged between $r < R_s$ and $r > R_s$).

18. What is the Ergosphere of a black hole?

The Ergosphere of a Kerr (spinning) BH is a region just outside the event horizon in which any particle cannot be at rest and it must rotate as dictated by the BH spin. Basically the gravity field of the BH rotates with him dragging space-time which makes it impossible for any object to be at rest as it must follow the geometry of this rotating space time.

$$R_H \leq r \leq R_{\text{static}}(\theta) = R_g(1 + \sqrt{1 - a^2 \cos\theta^2})$$

Theoretically it's possible to extract energy and mass from this region.



19. Give a simple description in words of what are gravitational waves.

Gravitational waves are ripples of space time produced by accelerations of mass distributions. These disturbances of space time propagate outward at the speed of light in the form of waves. They transport energy and angular momentum that was lost by the source that produced them.
They were first proposed by Henri Poincaré and later predicted by Albert Einstein on the basis of his general theory of relativity. Detecting GW allows us to study the physics of their progenitors in different ways than with their electromagnetic radiation so they are of great importance in astronomical research.

20. **Given an equal mass black holes binary, describe

(a) the main dependence of the wave amplitude

(b) the frequency at which the object emits

(c) gravitational wave luminosity

Feel free to add more to this as it is a double points answer and I dont think it seems like a lot for the answer

(a) Given equal Mass BH binary $M_1 = M_2$ –> $M_1 + M_2 = M$ the main dependence of the gravitational wave (GW) amplitude (h) is the change in separation $\dot{R}$ as the distance to merger will be the same through out the coalescence of the BH merger. This can be simplified as $\dot{R} = R f_{GW}$. So the amplitude will depend on the separation at a given instant as well as the f_{GW}, which also depends on R and Mass. $h(R) \propto \frac{1}{R^2}$

(b) The f_{GW} produced by a BH binary is given as 2 twice the orbital frequency for the a mass of M at a separation R, $f_{GW} \propto \frac{1}{R^{\frac{2}{3}}}$. What this tells us is as the BH spiral inward the frequency of the GW that is emitted increases (in turn this leads to a larger amplitude). This is signified by the 'chirp' we see in the detections of the GW. when looking at the change in frequency we can see that the $\dot{f}_{GW} \propto f^{\frac{11}{3}}$ which in turn is related to R.

(c) The Gw Luminosity (L_{GW}) is definied as the amount of energy carried away per second by the waves and is described as $L \approx L_{GW} \Xi^5$ since dealing with BHs Ξ is 1 and L_{GW} is 10^{59} ergs/s at coalescence. This energy if in optical would out shine the entire visible universe.

21. **Exercise:**

(a) Calculate the frequency of a black hole binary at merger as a function of its total mass. Take an equal mass black hole binary.

(b) What is the typical merger frequency for a binary composed of two million solar mass black holes?

(c) What is the typical merger frequency for a binary with two ten solar mass black holes?

(d) Which one between binary (b) and binary (c) will be detected by the Laser Interferometer Space Antenna (LISA)?

(a)

$$f_{GW} = 2f_0 \quad \text{where} \quad f_0 = \frac{1}{P} \quad \text{and} \quad P = 2\pi\sqrt{\frac{R^3}{G(M_1+M_2)}}$$

Noting that $M_1 = M_2$, we obtain:

$$f_{GW} = \frac{2}{P} = \frac{1}{\pi}\sqrt{\frac{G(M_1+M_2)}{R^3}} = \frac{1}{\pi}\sqrt{\frac{2GM}{R^3}} \approx \frac{1}{\pi}\sqrt{\frac{GM}{R^3}}$$

We want to know the frequency at merger, therefore $R = R_s = \frac{2GM}{c^2}$, which yields

$$f_{GW} \approx \frac{1}{\pi}\sqrt{\frac{GM}{\left(\frac{2GM}{c^2}\right)^3}} \approx \frac{1}{\sqrt{8}\pi}\frac{c^3}{GM}$$

(b) For $M_1 = M_2 = 10^6 M_\odot$ we get:

$$f_{GW} \approx 2.28 \times 10^{-2} \approx 0.2 \text{ mHz}$$

(c) For $M_1 = M_2 = 10M_\odot$ we get:

$$f_{GW} \approx 2.28 \times 10^3 \approx 2 \text{ kHz}$$

(d) Only (b) will be detected by LISA.

22. Exercise: Derive the a-dimensional strain magnitude "h" at merger for GW150914: i.e use a distance $d = 410$ Mpc and equal black hole masses $M_1 = 30M_\odot = M_2$.

So I did this the dimensional way cos I do not understand how the relations of c=G=1 come about, but for this it is important you consider all pre-factors and constants otherwise it does not work. Using relation $\dot{R} = f_{GW}R$ we know:

$$h \approx \frac{G}{c^4}\frac{M\dot{R}^2}{r} \approx \frac{G}{c^4}\frac{Mf_{GW}^2R^2}{r} \tag{2}$$

$$R = \left(\frac{GM}{\pi^2 f_{GW}^2}\right)^{\frac{1}{3}} \tag{3}$$

subbing in to h and re-arranging we arrive at final equation

$$h \approx \frac{G}{c^4}\frac{M\left(\frac{GM}{\pi^2 f_{GW}^2}\right)^{\frac{2}{3}} f_{GW}^2}{r} \approx \frac{G^{\frac{5}{3}}}{c^4}\frac{M^{\frac{5}{3}} f_{GW}^{\frac{2}{3}}}{\pi^{\frac{4}{3}} r} \tag{4}$$

subbing in the all the corresponding valeus in SI units arrive at a h of 1.44×10^{-21}

23. **Exercise: Derive the dependence of the rate of change of the frequency (=df/dt) as a function of frequency (f) and total mass.

We want to find the rate of change of f, $df/dt = \dot{f}$. We know that

$$f \propto 1/P,\ P = 2\pi\sqrt{\frac{R^3}{GM}}$$

Where M is the total mass and R the distance between the two merging objects. Therefore

$$f \propto M^{1/2}R^{-3/2}$$

$$\dot{f} \propto -M^{1/2}R^{-5/2}\dot{R}$$

Note that $\dot{M} = 0$ as the total mass is conserved. $\dot{R}$ is the coalescence rate, the rate of change of the distance between the two objects. To find it, we first need to realise that the radius is shrinking because the orbit loses energy due to the fact that gravitational waves are emitted. The gravitational wave luminosity is equal to this change in orbital energy:

$$L = -\dot{E}_{orb}$$

We already have an expression for the gravitational wave luminosity: $L \sim 4\pi r^2 f_{GW}^2 h^2$, so

$$L \propto f^2h^2$$

Using $f \propto M^{1/2}R^{-3/2}$ and $h \propto Mf^2R^2$ (This is the result from (22), but In order to derive it you need to assume that $\dot{R} \sim fR$ which is true for circular orbits, however later in this question you derive a different expression for $\dot{R}$, so this is kind of a sketchy step. Without this step, I cannot arrive to the result in the slides though...)

$$L \propto f^2M^2f^4R^4 \propto f^6M^2R^4 \propto M^3R^{-9}M^2R^4 \propto M^5R^{-5}$$

Maybe you can skip this step entirely, as the relation $L \propto M^5R^{-5}$ is also in the slides...
The orbital energy is given by

$$E_{orb} = -\frac{GM_1M_2}{R} \propto \frac{M_1M_2}{R} \propto \frac{M^2}{R}$$

Where we use the fact that M_1 and M_2 are both fractions of the total mass M. The change in orbital energy is

$$\dot{E}_{orb} \propto -\frac{M^2}{R^2}\dot{R}$$

So we get

$$L = -\dot{E}_{orb} \rightarrow M^5R^{-5} \propto M^2R^{-2}\dot{R} \rightarrow \dot{R} \propto -(M/R)^3$$

Therefore

$$\dot{f} \propto M^{1/2}R^{-5/2}M^3R^{-3} \propto M^{7/2}R^{-11/2} \propto M^{5/3}f^{11/3}$$

Where in the last step, $R^{-11/2} \propto f^{11/3}M^{-11/6}$ was used.

$$\dot{R} \propto -(M/R)^3$$

24. Describe the main steps in the history of discovering, theoretically describing and observing NSs

Neutrons were discovered by James Chadwick in 1932 in Cambridge (Nobel prize in 1935)

In 1930 S. Chandrasekhar introduces the relativistic corrections needed to describe the equation of state for degenerate electrons in high density environment. He calculated the WD structure in this regime and derive that only one mass is possible. Thus there is a limiting mass for WDs

In 1931/1932 Landau proposed that the fate of star more massive than the limit would be an object "where the atomic nuclei were so close to each other to form a gigantic nucleus"

1934 Baade and Zwicky proposed the existence of NSs. They had discovered that some "Novea" were extragalactic and therefore much more intrinsically luminous than Galactic Novea. They called them "Supernovae".

1939 first calculations of NS structure by Oppenheimer and Volkoff. They use general relativity and an equation of state for a perfect gas of degenerate neutrons (WWII stopped momentarily further advancements...)

At the end of 1950s theoretical works on NSs restarted, in particular around the question of what is the equation of state for ultra-dense matter. Notable works by Harrison, Wikano and Wheeler (1958), Cameron (1959), Ambartsumyan and Saakyan (1960) and Hamada and Salpeter (1961). • These more realistic equations of state showed that NS must have very small radii 10 km. Despite their large surface temperature, the small size makes direct detection of surface emission challenging. • NSs during these years remained a study subject for (a small group of) theoreticians.

In 1962, Giacconi and co-authors discovered the first X-ray source beyond the solar system (Sco X-1, Lx $6 \times 10^4 L_\odot$), following the launch of the USA Arobee rocket (Nobel prize in 2002). It is interpreted as a young and hot neutron star (Te $\sim 2 \times 10^{10}$ K). The question is then how fast NSs cool down. We now know that Sco X-I is an accreting neutron star (not an isolated one)

At the same time, there is a debate on whether QSOs (discovered in 1963) are neutron stars. This was ruled out by studying the gravitational redshift it was soon realised that the observed redshift in Quasars spectra could be larger than that: universe expansion

In 1964, Hoyle, Narlikar and Wheeler proposed that a highly magnetised NS ($\sim 10^{10}$ Gauss) is at the centre of the Crab nebula.In 1967, Pacini proposed that the source of energy for the Crab nebula is a highly magnetised and rotating NS.

In 1967, Hewish and his student Bell discover the first radio pulsar: a period radio source, with an extremely short (modern value P = 1.337301 s) and stable period. (Nobel prize to Hewish in 1974). • In 1968, they published their discovery after much double checking. Name: CP 1919 (Cambridge Pulsar, right ascension 19, declination 19. The Modern name is "PSR B 1919+21") • the short duration period indicated an object with < 1000 km radio, white dwarf ??

In 1968, discovery of the radio Pulsar Vela (PSR B 0833-45) and Crab (PSR B 0531+21). They are both within supernova remnants. These discovery validate Baade, Zwicky and Pacini's predictions.

In 1968, Gold proposes a pulsar model as magnetised, rapidly rotating NS, with which he predicts with precision the Crab period and its derivative, that were only measured in 1969

25. Exercise: Show that there is little alternative to a NS to explain a pulsar: in particular consider a pulsar rotating with angular frequency $\Omega = 0.1$ ms^{-1} (ms = millisecond).

To proof this, we must check what is the maximum angular frequency allowed to keep the object from not falling apart due to its rapid rotation. So we must check the equilibrium of the centrifugal and gravitational forces $F_{\text{cent}} = F_{\text{grav}}$

$$\Omega^2 R = \frac{GM}{R^2} \Rightarrow \Omega_{\text{max}} = \sqrt{\frac{GM}{R^3}} = \sqrt{\frac{4\pi G\bar{\rho}}{3}} \text{ where } \bar{\rho} = \frac{3M}{4\pi R^3}$$

This implies that $\Omega < \Omega_{\text{max}}$ thus $\bar{\rho} > \bar{\rho}_{\text{min}} = \frac{3\Omega^2}{4\pi G}$

In the case of $\Omega = 0.1$ ms^{-1} we get a mean density of 3.6×10^{10} g/cm^3 but no system is known to be this dense, even a compact object as a WD has a density of about 10^6 g/cm^3 so this object must be a neutron start.

26. **

(a) Define what is the stellar structure of a stellar object;

(b) Given barotropic EoS for NSs, how does one derive their stellar structure?

(c) Complete the answer to (b) by writting the main relevant equations in spherical symmetry, ignoring rotation and magnetic field

(d) Use a compactness argument to support the assumption that a Newtoninan framework (equations) is not sufficient.

(a) The stellar structure describes the thermodynamical quantities of the star as a function of the **radius**. Examples are $\rho = \rho(r)$, $P = P(r)$, etc.

(b) Using a Barotopric EoS for the NSs use relation that $P(\rho) \propto \rho^\gamma$. Using this and taking a general relativistic approach to the a set of equations with general relativistic corrections can describe the EoS for a NS. This was first derived by Tolman, Oppenheimer and Volkoff and is known as the TOV EoS for NSs. This assumes that in Einstiens field equations the diagonal of $T_{\mu\nu}$ are non-zero.

(c) These equations are then worked out as such:

$$\frac{dm}{dr} = 4\pi r^2 \rho(r) \tag{5}$$

$$\frac{dP}{dr} = -\left(\rho(r) + \frac{P(r)}{c^2}\right)\frac{d\phi}{dr} \tag{6}$$

$$\frac{d\phi}{dr} = \frac{Gm(r)}{r}\left(1 - 2\frac{Gm(r)}{rc^2}\right)^{-1}\left(1 + 4\pi\frac{P(r)r^3}{m(r)c^2}\right) \tag{7}$$

(d) In limit of small compactness, $\Xi << 1$, and for $P << \rho c^2$ the equations return to the Newtonian framework of mass conservation, hydrostatic equilibrium and gravitational potential. However we know for NS $\Xi = 0.2 - 0.4$, hence not $<< 1$. So for NS stars using a newtonian frame work is not enough to calculate the EoS.

27. **

(a) What are the theoretically expected electromagnetic counterparts to a NS-NS merger and

(b) on which timescales?

(c) at which wavelengths?

(a) We expect four or more EM counterparts for a NS-NS merger: 1. Prompt Radio Emission, 2. Short gamma ray burst (SGRB), 3. Optical-near IR kilo/macronova, 4. Slow radio remnant.

(b) 1. Prompt Radio Emission: t_{merger} + seconds
2. Short gamma ray burst (SGRB): t_{merger} + 2 seconds
3. Optical-near IR kilo/macronova: t_{merger} + hours/days
4. Slow radio remnant: t_{merger} + months/years

(c) I don't know what she wants here, because the names already say what kind of wavelength range, but I don't see any specific numbers or ranges in the slides

28. Describe the steps that brought to the localization of the host galaxy of GW170817: you should mention both the role of the GW detectors as that of the gamma ray detectors.

The signal from a single detector at LIGO was measured and alerted a GW detection , however due to a glitch in the second detector it was not registered there. After some post-processing the glitch was removed and the signal could be seen. This was fit by the filter matching process for a NS-NS merger, giving a very high SNR (highest recorded) this prompted looking at the Virgo data as it should have been easily visible on that detector, however it wasn't! This lead the teams to believe that the merger was in Virgos blind spot. Correlating regions in the sky were LIGO could observe and where VIRGOs blind spot was, the location of the merger was narrowed down to a $28\,\mathrm{deg}^2$ patch of sky. This didn't end the narrowing down of the location as seconds after the merger a short gamma ray burst (SGRB) was detected by the Fermi space telescope and the INTEGRAL telescope (through manual processing). Using these telescopes area of observations the location of the merger was refined for more EM observation follow ups.

29. **Discuss the importance of the historical detection of GW170817 and its electromagnetic counterparts**

Could probably be a bit more added to this, or at least more detail given the weighting of question.
The discovery of GW170817 and its EM counterparts was of great Historical importance as it confirmed and allowed the testing of the following:
-Test GR in the strong regime.
-measure the Hubble Constant by combining the luminosity distance of teh host galaxy and the redshift.
-study, validate and constrain the NS EoS.
-unvail the origin of SGRBs
-settle the origin of elements heavier than Fe by r-processes.

The origin of the emission of SGRBs have puzzled scientists for years, with some believing that they are due to advanced civilisations. but the measurement and correlation with the merger of two NSs confirms that the merger of NSs is the origin of SGRBs due to fitting expected models and time scales.
The large neutron flux due to the kilonova explosion allows for the capture by heavier 'seed' elements such as Fe causing larger and larger nuclei and hence explaining the enrichment of heavier elements by rapid processes (r-process). This helps confirm the origin of these heavier elements in the universe. Spectroscopic measurements showed that $10M_{\oplus}$ of heavy elements due to the GW170817 merger.
While currently only a small number of measurements, with more and more observations of NS-NS mergers statistical methods can be used to constraint the Hubble constant further.
The masses and frequencies inferred by the measurements of the GW will allow for a better constraining of the EoS of states based on the masses observed.

30. ** Exercise: **Given the equation of enery conservation below, the "one zone model" and radiative diffusion derive:**

(a) The light curve of a kilonova.

(b) The peak luminosity

(c) The peak time

(d) Evaluate (b) and (c) for GW170817 i.e ejecta velocity $v = 0.3c$; gas opacity $k = 0.1$ cm^2/g; neutron star radius $R_0 = 10$ km; eject mass $M_{\mathbf{eje}} = 0.01M_{\odot}$

$$\frac{dE(t)}{dt} = -p\frac{dV}{dt} + \dot{\epsilon} - L(t)$$

Answer below

(a) so using the following relations $V(t) = V_0(\frac{t}{t_0})^3$, $p = \frac{u}{3}$, E =uV and $\chi = uR^4$ the equation above is derived as such:

$$\dot{\mathrm{E}} = -p\frac{dV}{dt} - L(t) \tag{8}$$

$$\dot{\mathrm{E}} = -\frac{u}{t}\frac{dV}{dt} - L(t) \tag{9}$$

note:

$$\frac{1}{V(t)}\frac{dV}{dt} = \frac{3}{t} \tag{10}$$

dividing across by V

$$\frac{\dot{E}}{V} = -\frac{u}{Vt}\frac{dV}{dt} - \frac{L}{V} \tag{11}$$

note:

$$\dot{E} = \dot{u}V + u\dot{V} \tag{12}$$

$$\frac{\dot{u}V + u\dot{V}}{V} = -\frac{u}{3} - \frac{L}{V} \tag{13}$$

$$\dot{u} = -\frac{u}{t} - \frac{3u}{t} - \frac{L}{V} \tag{14}$$

subbing above in to

$$\dot{\chi} = \dot{u}R^4 + u\dot{R}^4 \tag{15}$$

and using relation

$$\dot{R}^4 = 4uR^4 \tag{16}$$

arrive at equ.

$$\dot{\chi} = -\frac{LR^4}{V} \tag{17}$$

$$\frac{\dot{\chi}}{\chi} = -\frac{L}{uV} \tag{18}$$

$$L = 4\pi R^2 \frac{c}{3\kappa\rho}\frac{u}{R} \tag{19}$$

$$L = 4\pi R^2 \frac{cV}{3\kappa M}\frac{u}{R} \tag{20}$$

$$\frac{\dot{\chi}}{\chi} = -\frac{4\pi c R_0}{3\kappa M}\frac{R}{R_0} \tag{21}$$

using $\frac{R}{R_0} = \frac{t}{t_0}$ and $t_{dyn} = \frac{3\kappa M}{4\pi c R_0}$(lecture notes)

$$\frac{\dot{\chi}}{\chi} = -\frac{t}{t_{dyn}t_0} \tag{22}$$

Using relation of $\chi \propto L$ ans solving the differential.

$$L(t) = L_{peak}e^{\frac{t^2}{2t_{dyn}t_0}} \tag{23}$$

(b) using the Luminosity equation $L = 4\pi R\frac{cV}{3\kappa M}u$ and the equation $v = \sqrt{(2E_0/M)}$ we arrive at the peak luminosity using the initial values of R_0,

$$L_{peak} = 4\pi R_0 \frac{cv^2}{3\kappa} \tag{24}$$

(c) This one is pretty easy $t_p = \sqrt{t_{dyn}t_0}$

(d) Using the values given and making sure everything is in cgs $L_{peak} \approx 5.10^{36} \approx L_{Edd}$. similarly but also using $t_0 = R_0/v$ and $t_{dyn} = \frac{3\kappa M}{4\pi c R_0}$, $t_p = 128736s$ ≈1.5 days

31. Derive the accretion luminosity L_{acc} onto

(a) a generic object of radius R

(b) a non-spinning BH

(c) a maximally spinning BH

utterly stuck on this question, in particular b) as for c) we just need to add the following term $R_h = \frac{R_s}{2}(1+\sqrt{1-a^2}$

(a) L_{acc} = -dE_{grav}/dt = $\frac{GMdm}{rdt}$ = $\frac{GM\dot{M}}{r}$ at R —> $\frac{GM\dot{M}}{R}$

(b) $L_{acc} = 0.06\dot{M}c^2$ its deriving the efficiency (0.06) I have no idea how to.

(c)

32. ** For a geometrically accretion disc in hydrostatic equilibrium in the vertical direction use the momentum equation to

(a) derive the azimuthal and vertical velocity and

(b) to answer the following: Is the azimuthal velocity subsonic? Consider now the radial motion of matter towards the accreting object

(c) what is this flow due to?

(d) write explicitly an expression for the radial velocity and

(e) compare it with the local sound speed.

(a) if disc is in hydrostatic equilibrium then the vertical velocity is 0 this implies that the orbits are keplerian hence can use $v^2/R = GM/R$ to calculate the azimuthal velocity $v_\phi = \sqrt{\frac{GM}{R}}$

(b)

(c)

(d)

(e)

33. What is the "trapping radius"?

(a) Derive it

(b) Explain its physical meaning

The trapping radius is radius within which the local optical depth $\tau(r) \sim \kappa\rho(r)r$ makes photon diffusion outward slower than accretion inward. In other words, it is a radius" within which an accretion flow would advect photons inward faster than they could diffuse outward.

(a) To find R_{tr}, we set the photon diffusion timescale equal to the dynamical timescale (timescale of inward accretion), $t_{\rm diff} = t_{\rm dyn}$:

$$\frac{R}{c\tau} = \frac{R}{v} \rightarrow \tau(R_{tr}) = \frac{c}{v} \approx \kappa\rho(R_{tr})R_{tr} \rightarrow \kappa\rho(R_{tr})vR_{tr}/c \approx 1$$

We can use that the mass accretion rate is given by the velocity of that mass multiplied by the surface at radius R_{tr} (why?): $\dot{M} = 4\pi\rho v R_{tr}^2$, then

$$\frac{\kappa\dot{M}}{4\pi R_{tr}c} \approx 1 \rightarrow \frac{\kappa\dot{M}}{4\pi c} \approx R_{tr}$$

Also, the Eddington luminosity is given by $L_{Edd} = \frac{4\pi cGM}{\kappa} \rightarrow \frac{\kappa}{4\pi c} = GM/L_{edd}$:

$$\frac{GM\dot{M}}{L_{Edd}} \approx R_{tr}$$

Now we plug in for the Eddington luminosity the critical accretion rate: $L_{Edd} = \dot{M}_{cr}\eta c^2$:

$$\frac{GM\dot{M}}{\dot{M}_{cr}\eta c^2} \approx R_{tr}$$

Now lastly, we use the definition of the Schwarzschild radius $R_S = 2GM/c^2 \rightarrow GM/c^2 = R_S/2$:

$$R_{tr} \approx \frac{\dot{M}R_S}{\dot{M}_{cr}2\eta} \approx \frac{\dot{M}}{\dot{M}_{cr}}R_S$$

Where in the last step, we used that $\eta \approx 0.06 - 0.4$, and taking he 0.4 value, $2\eta \approx 1$.

(b) Physically, inside this radius, accretion flow would advect photons inward faster than they can diffuse outward, so they are "trapped". A consequence of the this is that if the trapping radius is larger than the Schwarzschild radius, we can have $\dot{M} \ll \dot{M}_{cr}$, the accretion flow can be bigger than its "critical" value derived from the Eddington luminosity limit in a very optically thick medium. As photons are advected inward, even though $\dot{M} \ll \dot{M}_{cr}$, the observed luminosity will still be limited by L_{Edd}, as the created photons inside the trapping radius are advected inward and cannot be observed.

34. In an optically thick, geometrically thin disc

(a) what is the predicted temperature radial profile?

(b) using this prediction, derive in which frequency range we expect i) a WD; ii) a NS; and iii) a one-million-solar mass BH to emit.

(a) In an OTGT disc, the Luminosity accretion can be given as such $L_{acc} = \frac{GM\dot{M}}{2R}$ and temperature dependent Luminosity is given as $L = 4\pi R^2 T^4$ relating the two equations, find that:

$$T = \sqrt[4]{\frac{GM\dot{M}}{8\pi R^3}} \tag{25}$$

(b) Using typical values for WD, accretion efficiency and a critical mass accretion rate subbing in to the equation get a T of roughly 36000k the observed λ is then calculated as $\lambda = \frac{hc}{kT}$
This is then repeated for a typical NS and SMBH values giving the following regimes:
-WD = optical
-NS = X-ray
-SMBH = UV

35. Derive the Eddington limit

The Eddington's limit is the luminosity limit in which $f_{\text{rad}} = f_{\text{grav}}$. We start by getting the radiation force which can be calculated assuming that we deal with a point mass that radiates isotropically $F = L/4\pi r^2$ from where we get the radiation force per unit mass:

$$f_{\text{rad}} = \frac{L\kappa_T}{4\pi r^2 c}$$

where κ_T is the Thompson opacity $\kappa_T = \sigma_T/m_p$. Now, for the gravitational force we have

$$f_{\text{grav}} = \frac{GM}{r^2}$$

Then for $f_{\text{rad}} = f_{\text{grav}}$ we get:

$$\frac{L\kappa_T}{4\pi r^2 c} = \frac{GM}{r^2}$$

that solving for L yields

$$L_{edd} = \frac{4\pi GMc}{\kappa_T} \approx 10^{38} \left(\frac{M}{M_\odot}\right) \frac{erg}{s}$$