Orbital structures and Collisionlesss systems

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1. Oort Constants

a) What are the Oort constants A and B?

The Oort constants are two parameters derived by Jan Oort that characterize the local rotational properties of our galaxy, like this:

$$A = \frac{1}{2}\left(\frac{V_0}{R_0} - \frac{dv}{dr}|_{R_0}\right)$$

$$B = -\frac{1}{2}\left(\frac{V_0}{R_0} + \frac{dv}{dr}|_{R_0}\right)$$

Where V_0 is the rotational velocity and R_0 is the distance to the Galactitc center of the Sun respectively.

Some accurate values for this constants are: $A = 14.82 \pm 0.84\, km/s * kpc$ and $B = -12.37 \pm 0.64\, km/s * kpc$

What's great about these constants is that from them, it is possible to determine the orbital properties of the Sun, such as the orbital velocity and period, and infer local properties of the Galactic disk, such as the mass density and how the rotational velocity changes as a function of radius from the Galactic center.

These constants come from more general expressions in terms of circular speed (v_c) at radius R for galaxies:

$$A(R) = \frac{1}{2}\left(\frac{v_c}{R} - \frac{dv_c}{dR}\right) = -\frac{1}{2}R\frac{d\Omega}{dR}$$

$$B(R) = -\frac{1}{2}\left(\frac{v_c}{R} + \frac{dv_c}{dR}\right) = -\left(\Omega + \frac{1}{2}R\frac{d\Omega}{dR}\right)$$

b) How do they relate to the quantities associated with the epicycle approximation?

As the constants are related to the circular velocity of the particles in the system (by definition), the are also indirectly associated with the epicycle approximation. In c) we find that the epicycle frequency can be written in terms of the Oort constants, the mathematical treatment for this to be made is done in the next question.

c) Show that the constants A and B are related to the structure of the stellar orbits through the frequency of the epicycle and the vertical frequency. Find the relationships.

In the epicycle aproximation we found that the frequency in the radial direction is:

$$\kappa = \frac{\partial^2 \phi_{eff}}{\partial R^2}$$

We know that the effective potential is $\phi_{eff} = \phi + \frac{L^2}{2R^2}$ and therefore:

$$\kappa^2 = \frac{\partial^2 \phi_{eff}}{\partial R^2} = \frac{\partial^2 \phi}{\partial R^2} + \frac{\partial^2}{\partial R^2}\left(\frac{L^2}{2R^2}\right)$$

In terms of the angular velocity we have that:

$$\frac{\partial \phi}{\partial R} = R\Omega^2(R) \quad and \quad \Omega^2(R) = \frac{L^2}{R^4}$$

So, if we solve for κ^2 we get:

$$\kappa^2 = \frac{\partial^2 \phi_{eff}}{\partial R^2} = \frac{\partial}{\partial R}\left(\Omega^2(R)R\right) + \frac{3L^2}{R^4}$$

$$\kappa^2 = \Omega^2(R) + R\frac{\partial \Omega^2(R)}{\partial R} + 3\Omega^2(R)$$

$$\kappa^2 = 2R\frac{\partial \Omega(R)}{\partial R}\Omega(R) + 4\Omega^2(R)$$

We take out $\Omega(R)$ as a common factor and get:

$$\kappa^2 = \Omega(R)\left(2R\frac{\partial \Omega(R)}{\partial R} + 4\Omega(R)\right)$$

We use the Oort constant:

$$B(R) = -\left(\Omega(R) + \frac{1}{2}R\frac{d\Omega(R)}{dR}\right)$$

And substitute it in the equation of κ^2:

$$\kappa^2 = -4B\Omega(R)$$

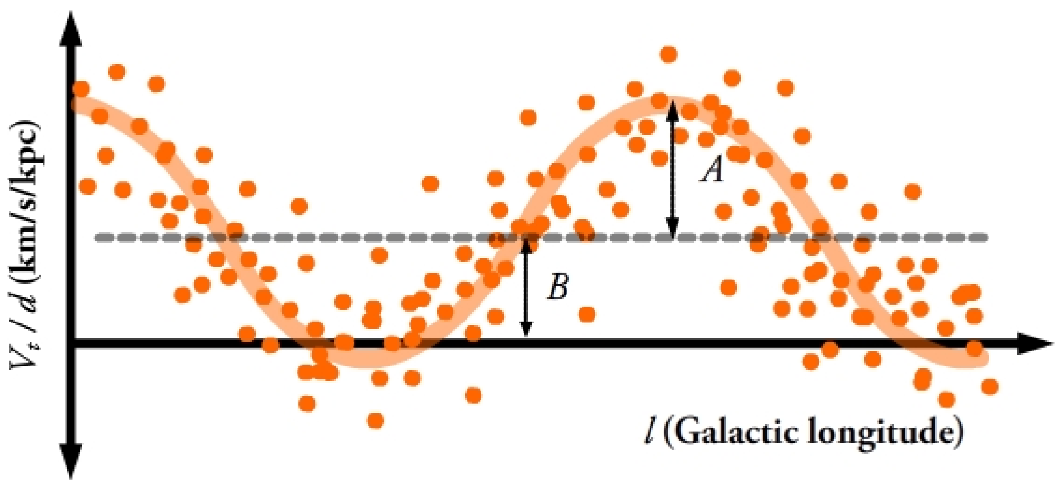
Finally, using the fact that the angular velocity is related to the Oort constants by $\Omega(R) = A - B$, we can put the radial frequency in terms of both constants like this:

$$\kappa^2 = -4B(A - B)$$

The constants do not depend on the vertical direction so it is not possible to relate them to the vertical frequency of the motion.

d) What phyical interpretation can you make about the constants?

First of all it is highly important to mention that these constant are very usefull in determining physical properties of galaxies since they can be easily found by plotting a large sample of the observed stars like this:



As we do not know the non-circular components of the velocity of each star we observe, we plot transverse velocity divided by distance against galactic longitude, this will follow a sine function as we can see in the last figure. With a large sample the function can be well fit and the Oort constants can be measured this way. A will be the amplitude of the sinusoid and B will be the vertical offset from zero.

These constants are both functions of the Sun's orbital velocity and the first derivative of its velocity, for this reason **A** describes the shearing motion in the disk in the neighborhood of the Sun by the local stars, we could understand this shearing motion as a tension of the fluid that surrounds the Sun. **B** describes the change in the angular momentum as a gradient in the solar neighborhood, in the literature this is also understood as the vorticity.

2. In class, it was mentioned that the anisotropy parameter is a quantity that allows us to quantify how much the tensor of velocity dispersion deflects from isotropy. Although in observations we only get information of the projections of the tensor of velocity dispersion along the line of view.

As an example of application of the Jeans equations, study the velocity dispersion (and do some examples) of spherical systems in different usefull scenarios and close to reality. Galactic Dynamics, Page 203 (section d)

For studying the velocity dispersion in spherical systems (just like a Globular Cluster) let's star by first considering that the density and velocity structures of the Galaxy are invariant under rotations about the galactic center so that $\overline{v_\theta^2} = \overline{v_\phi^2}$, this means that the velocity ellipsoids are spheroids with their symmetry axes pointing tot he galaxy center, then:

$$\beta = 1 - \frac{\overline{v_\theta^2}}{\overline{v_r^2}}$$

The various numerical models sugest that $\overline{v_r^2} \geq \overline{v_\theta^2}$ and thus $\beta \geq 0$.

If we take these assumpltions to be true, then the Jeans equation:

$$\frac{d\left(\nu\overline{v_r^2}\right)}{dr} + \frac{\nu}{r}\left[2\overline{v_r^2} - \left(\overline{v_\theta^2} + \overline{v_\phi^2}\right)\right] = -\nu\frac{d\phi}{dr}$$

Becomes:

$$\frac{1}{\nu}\frac{d\left(\nu\overline{v_r^2}\right)}{dr} + 2\frac{\beta\overline{v_r^2}}{r} = -\frac{d\phi}{dr} \quad (1)$$

Let's suppose we measure $\overline{v_r^2}$, β y ν as functions of R for a stellar population in a spherical galaxy, then if $\frac{d\phi}{dr} = \frac{GM(r)}{r^2}$, equation (1) will allow us to determine the mass M(r) and the circular velocity:

$$v_c^2 = \frac{GM(r)}{r} = -\overline{v_r^2}\left(\frac{dln\nu}{dlnr} + \frac{dln\overline{v_r^2}}{dlnr} + 2\beta\right) \quad (2)$$

In order to build a unique mass model for the spherical system, we asume that the ellipsoids of velocity are spherical through the galaxy. Therefore $\beta = 0$. Equation (2) then becomes:

$$M(r) = \frac{-r\overline{v_r^2}}{G}\left(\frac{dln\nu}{dlnr} + \frac{dln\overline{v_r^2}}{dlnr}\right) \quad (3)$$

ν is the luminosity density and $I(R)$ is the surface brightness projected at radius R. We see that ν and $\overline{v_r^2}$ are related with $I(R)$ and $\sigma_v^2(R)$ in this way:

$$I(R) = 2\int_R^\infty \frac{\nu r dr}{\sqrt{r^2 - R^2}} \quad and \quad I(R)\sigma_\rho^2(R) = 2\int_R^\infty \frac{\nu\overline{v_r^2} r dr}{\sqrt{r^2 - R^2}}$$

These equations are Abel integral equations and their solutions are:

$$\nu(r) = -\frac{1}{\pi}\int_r^\infty \frac{dI}{dR}\frac{dR}{\sqrt{R^2 - r^2}} \quad and \quad \nu(r)\overline{v_r^2}(r) = -\frac{1}{\pi}\int_r^\infty \frac{d\left(I\sigma_p^2\right)}{dR}\frac{dR}{\sqrt{R^2 - r^2}}$$

So if $\nu(r)$ and $\overline{v_r^2}$ are determined it is straightforward to find $M(R)$ from equation (3). Now if we define the mass per unit of luminosity in a radius r we have:

$$\gamma(r) = \left[\frac{M(r)}{4\pi \int_0^r \nu r^2 dr} \right]$$

Although equation (3) is very useful for determining the mass, we found that equation by assuming that $\beta = 0$ but we do not have a good theoretical or observational reason to assume that, si let's look an alternative way. Let's assume that the mass density is proportional to the luminosity density like this:

$$\rho(r) = \gamma\nu(r)$$

Through the virial theorem we can find γ that is the parameter we call mass-to-light ratio. For this model to work, we must make sure that for all possible values of r $\beta(r) \leq 1$ and $\overline{v_\theta^2} \geq 0$.

For $\beta \neq 0$, the observed dispersion σ_v is related to $\overline{v_r^2}$ and β like this:

$$I(R)\sigma_p^2(R) = 2\int_R^\infty \left(1 - \beta\frac{R^2}{r^2}\right) \frac{\nu\overline{v_r^2} r dr}{\sqrt{r^2 - R^2}}$$

Now if we use (2) to eliminate β we get:

$$I\sigma_p^2 - R^2 \int_R^\infty \frac{\nu\overline{v_r^2} r dr}{r^2\sqrt{r^2 - R^2}} = \int_R^\infty \left[2\nu\overline{v_r^2} + \frac{R^2}{r}\frac{d\left(\nu\overline{v_r^2}\right)}{dr} \right] \frac{rdr}{\sqrt{r^2 - R^2}}$$

In the last equation, the left hand side is completely determined by observations and the right hand side is linear in $\nu\overline{v_r^2}$ so this equation can be solved analytically, thus giving us a complete model from the observational variables we can measure.

In conclusion, it is indeed possible to derive good mass models for spherical systems but this is only possible when high quality data are available. If the data are very noisy then the functions $\beta(r)$ and $\overline{v_r^2}$ that we calculate from those data become very uncertain because the right hand side of the last equation involves derivatives of the observed quantities thus making the error greater. Therefore, it is much more useful to compare poor quality data with a few simple model galaxies than to build particular models around the poor data.

There are some exact solutions to the collisionless Bolztmann equations and that yields many models that involve a rigurous mathematical treatment. But without taking the problem to the exact solutions there is one alternative approach to build models that is done by choosing $\beta(r)$, $\nu(r)$ and $\rho(r)$ in some sensible and precise way and then trying to solve the problem of solving (2) as a differential equation for $\overline{v_r^2}$, again taking into account the facilities given by the spherical symmetry of the system.