6. Übungsblatt zur Theoretischen Physik (WS 20/21)

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Gruppe: F

Punkte: ___/___/___ Σ___

6.1 Aufgabe 1

Geg.:

- $V(x) = \frac{-\alpha}{x} + \frac{\beta}{x^2}$
- $\alpha,\ \beta > 0$
- Für $x > 0$

a) ges.: Min

Lsg.: $V'(x) = 0$

$$\begin{aligned}
V'(x) &= \frac{\alpha}{x^2} - \frac{2\beta}{x^3} && (1)\\
0 &= \frac{\alpha}{x^2} - \frac{2\beta}{x^3} && (2)\\
&= \frac{1}{x^2}\left(\alpha - \frac{2\beta}{x}\right) && (3)\\
&= x^2\left(\frac{1}{\alpha} - \frac{x}{2\beta}\right) \qquad x > 0 && (4)\\
\rightarrow 0 &= \frac{1}{\alpha} - \frac{x}{2\beta} && (5)\\
\frac{x}{2\beta} &= \frac{1}{\alpha} && (6)\\
x_0 &= \frac{2\beta}{\alpha}\Big|\Big| && (7)
\end{aligned}$$

b) Ges.: Taylor-Entwicklung bis zur 2. Ordnung:

$$V(x) = \sum_{n=0}^{\infty} \frac{V^{(n)}(x_0)}{n!}(x - x_0)^n \tag{8}$$

Ableitungen bestimmen:

$$V'(x) = \frac{\alpha}{x^2} - \frac{2\beta}{x^3} \qquad V'(x_0) = \frac{\alpha^3}{(2\beta)^2} - \frac{\alpha^3}{8\beta^2} \tag{9}$$

$$V''(x) = \frac{-2\alpha}{x^3} + \frac{3 \cdot 2\beta}{x^4} \qquad V''(x_0) = \frac{-\alpha^4}{4\beta^3} + \frac{3\alpha^4}{(2\beta)^3} \tag{10}$$

$$V'''(x) = \frac{3 \cdot 2\alpha}{x^4} + \frac{-4 \cdot 3 \cdot 2\beta}{x^5} \qquad V'''(x_0) = \frac{3\alpha^5}{8\beta^4} - \frac{-3\alpha^5}{4\beta^4} \tag{11}$$

$$\ldots \tag{12}$$

$$V(x) \approx \frac{\frac{-\alpha^2}{2\beta} + \frac{\alpha^2}{4\beta}}{0!}\left(x - \frac{2\beta}{\alpha}\right)^0 + \frac{\frac{\alpha^3}{(2\beta)^2} - \frac{\alpha^3}{8\beta^2}}{1!}\left(x - \frac{2\beta}{\alpha}\right)^1 + \frac{\frac{-\alpha^4}{4\beta^3} + \frac{3\alpha^4}{(2\beta)^3}}{2!}\left(x - \frac{2\beta}{\alpha}\right)^2 \tag{13}$$

$$= -\frac{\alpha^2}{4\beta} + \frac{\alpha^3}{8\beta^2}\left(x - \frac{2\beta}{\alpha}\right) + \frac{\alpha^4}{16\beta^3}\left(x - \frac{2\beta}{\alpha}\right)^2 + O(x^3)\Big|\Big| \tag{14}$$

c) Bestimme ω:

$$F(x) = -\frac{dV(x)}{dt} \qquad F(x) = m\ddot{x} \tag{15}$$

$$= -\frac{\alpha^3}{8\beta^2} - \frac{\alpha^3}{8\beta^2} - \frac{2\alpha^4}{16\beta^3}x + O(x^2) \tag{16}$$

$$= -\frac{3\alpha^3}{8\beta^2} - \frac{\alpha^3}{8\beta^3}x + O(x^2) \tag{17}$$

$$\rightarrow m\ddot{x} = -\frac{3\alpha^3}{8\beta^2} - \frac{\alpha^3}{8\beta^3}x \qquad x = A\sin\omega t + B\cos\omega t + C \tag{18}$$

$$\ddot{x} = -A\omega^2\sin\omega t - B\omega^2\cos\omega t \tag{19}$$

$$\tag{20}$$

6.2 Aufgabe 2

a) Geg.:

- $F(x) = m\ddot{x}$
- $V(x) = -\int_{x_0}^{x} F(x')\, dx'$

Nun setzen wir beide Definitionen gleich und gucken, wohin uns das bringt:

$$m\ddot{x} = F(x) = -\frac{dV(x)}{dx} \tag{21}$$

$$m\ddot{x}\dot{x} = \frac{dV(x}{dx}\dot{x} \tag{22}$$

$$m\ddot{x}\dot{x} = \frac{dV(x)}{dx}\frac{dx}{dt} \tag{23}$$

$$m\ddot{x}\dot{x} = \frac{dV(x)}{dt} \tag{24}$$

Man kann merken, dass $\ddot{x}\dot{x} = \frac{d\dot{x}^2}{dt}\frac{1}{2}$. Die Herleitung dafür ist folgende:

$$\text{Nach der Kettenregel: } \frac{d\dot{x}^2}{dt} = 2(\dot{x})\ddot{x} \tag{25}$$

Also können wir das zurück in unsere Gleichung einsetzen:

$$\frac{m}{2}\frac{d\dot{x}^2}{dt} = -\frac{dV(x)}{dt} \tag{26}$$

$$\int \frac{m}{2}\frac{d\dot{x}^2}{dt} = -\int \frac{dV(x)}{dt} \tag{27}$$

$$\frac{m\dot{x}^2}{2} = -V(x) + C \tag{28}$$

Nun Sei C die Gesamtenergie, V die potentielle Energie und $\frac{m\dot{x}^2}{2}$ die kinetische Energie, so haben wir den Energieerhaltungssatz hergeleitet:

$$E_{ges} = E_{kin} + E_{pot} \tag{29}$$

b) Geg.:

- $\frac{mv^2}{2} + V(x) = E$

Ges.:

- $t(x) := t - t_0 = \pm\sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx'}{\sqrt{E - V(x')}}$

Zur gesuchcten Formel kommen wir folgenderweise:

$$\frac{mv^2}{2} + V(x) = E \tag{30}$$

$$v^2 = \frac{2(E - V(x))}{m} \tag{31}$$

$$v = \sqrt{\frac{2(E - V(x))}{m}} \tag{32}$$

$$\frac{dx}{dt} = \sqrt{\frac{2(E - V(x))}{m}} \tag{33}$$

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}}\sqrt{E - V(x)} \tag{34}$$

$$\frac{dt}{dx} = \sqrt{\frac{m}{2}}\frac{1}{\sqrt{E - V(x)}} \tag{35}$$

$$dt = \sqrt{\frac{m}{2}}\frac{1}{\sqrt{E - V(x)}}\ dx \tag{36}$$

$$\int_{t_0}^{t} dt = \sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx'}{\sqrt{E - V(x)}} \tag{37}$$

$$t - t_0 = \sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx'}{\sqrt{E - V(x)}} \tag{38}$$

c) Geg.:

- $F(x) = -kx$
- $x(0) = x_0$
- $\dot{x} = 0$
- $t_0 = 0$

i. Ges.:

- $t(x)$ für die angegebene Kraft

Bestimme das Potential $V(x)$

$$-kx = -\frac{dV(x)}{dt} \tag{39}$$

$$\frac{kx^2}{2} + C = V(x) \tag{40}$$

Setze in das Integral ein

$$t - t_0 = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx'}{\sqrt{E - \frac{kx^2}{2} + C_1}} \qquad C_1 := E + C \tag{41}$$

$$t = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx'}{\sqrt{C_1 - \frac{kx^2}{2}}} \qquad a := \frac{k}{2} \tag{42}$$

$$t = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx'}{\sqrt{C_1 - ax^2}} \tag{43}$$

Aus den vorigen Übungsblätter wissen wir, dass

$$\int_{x_0}^{x} \frac{dx'}{\sqrt{C_1 - ax^2}} = \left[\frac{\arcsin \frac{\sqrt{a}x}{\sqrt{C_1}}}{\sqrt{a}}\right]_{x_0}^{x} \tag{44}$$

$$\rightarrow t = \pm\sqrt{\frac{m}{2}}\left[\frac{\arcsin \frac{\sqrt{a}x}{\sqrt{C_1}}}{\sqrt{a}}\right]_{x_0}^{x} \tag{45}$$

$$= \pm\sqrt{\frac{m}{2}}\left[\frac{\arcsin \frac{\sqrt{a}x}{\sqrt{C_1}}}{\sqrt{a}} - \frac{\arcsin \frac{\sqrt{a}x_0}{\sqrt{C_1}}}{\sqrt{a}}\right] \quad \text{Einsetzen von a, } C_1 \tag{46}$$

$$= \pm\sqrt{\frac{m}{2}}\left[\frac{\arcsin \frac{\sqrt{\frac{k}{2}}x}{\sqrt{E+C}}}{\sqrt{\frac{k}{2}}} - \frac{\arcsin \frac{\sqrt{\frac{k}{2}}x_0}{\sqrt{E+C}}}{\sqrt{\frac{k}{2}}}\right] \tag{47}$$

$$= \pm\sqrt{\frac{m}{2}}\frac{1}{\sqrt{\frac{k}{2}}}\left[\arcsin \frac{\sqrt{\frac{k}{2}}x}{\sqrt{E+C}} - \arcsin \frac{\sqrt{\frac{k}{2}}x_0}{\sqrt{E+C}}\right] \tag{48}$$

$$= \pm\sqrt{\frac{m}{k}}\left[\arcsin \frac{\sqrt{\frac{k}{2}}x}{\sqrt{E+C}} - \arcsin \frac{\sqrt{\frac{k}{2}}x_0}{\sqrt{E+C}}\right] \tag{49}$$

$$= \pm\sqrt{\frac{m}{k}}\left[\arcsin \frac{\frac{\sqrt{k}}{\sqrt{2}}x}{\sqrt{E+C}} - \arcsin \frac{\frac{\sqrt{k}}{\sqrt{2}}x_0}{\sqrt{E+C}}\right] \tag{50}$$

$$t(x) = \pm\sqrt{\frac{m}{k}}\left[\arcsin \frac{\sqrt{k}x}{\sqrt{2(E+C)}} - \arcsin \frac{\sqrt{k}x_0}{\sqrt{2(E+C)}}\right] \Bigg|\Bigg| \tag{51}$$

Parametriesere $x(t)$:

$$\frac{t}{\pm\sqrt{\frac{m}{k}}} = \left[\arcsin\frac{\sqrt{k}x}{\sqrt{2(E+C)}} - \arcsin\frac{\sqrt{k}x_0}{\sqrt{2(E+C)}}\right] \tag{52}$$

$$\mp t\sqrt{\frac{k}{m}} = \left[\arcsin\frac{\sqrt{k}x}{\sqrt{2(E+C)}} - \arcsin\frac{\sqrt{k}x_0}{\sqrt{2(E+C)}}\right] \tag{53}$$

$$\left[\arcsin\frac{\sqrt{k}x}{\sqrt{2(E+C)}}\right] = \mp t\sqrt{\frac{k}{m}} + \arcsin\frac{\sqrt{k}x_0}{\sqrt{2(E+C)}} \tag{54}$$

$$\frac{\sqrt{k}x}{\sqrt{2(E+C)}} = \sin\left(\mp t\sqrt{\frac{k}{m}} + \arcsin\frac{\sqrt{k}x_0}{\sqrt{2(E+C)}}\right) \tag{55}$$

$$x(t) = \sin\left(\mp t\sqrt{\frac{k}{m}} + \arcsin\frac{\sqrt{k}x_0}{\sqrt{2(E+C)}}\right)\frac{\sqrt{2(E+C)}}{\sqrt{k}}\bigg\| \tag{56}$$

Bestimme x_0:

$$\dot{x} = \cos\left(\mp t\sqrt{\frac{k}{m}} + \arcsin\frac{\sqrt{k}x_0}{\sqrt{2(E+C)}}\right)\sqrt{\frac{k}{m}}\frac{\sqrt{2(E+C)}}{\sqrt{k}} \tag{57}$$

$$= \cos\left(\mp t\sqrt{\frac{k}{m}} + \arcsin\frac{\sqrt{k}x_0}{\sqrt{2(E+C)}}\right)\frac{\sqrt{2(E+C)}}{\sqrt{m}} \tag{58}$$

$\dot{x}(0) = 0$

$$0 = \cos\left(\mp 0\sqrt{\frac{k}{m}} + \arcsin\frac{\sqrt{k}x_0}{\sqrt{2(E+C)}}\right)\frac{\sqrt{2(E+C)}}{\sqrt{m}} \tag{59}$$

$$\to 0 = \cos\left(\arcsin\frac{\sqrt{k}x_0}{\sqrt{2(E+C)}}\right) \tag{60}$$

$$\to \frac{\pi}{2} = \arcsin\frac{\sqrt{k}x_0}{\sqrt{2(E+C)}} \tag{61}$$

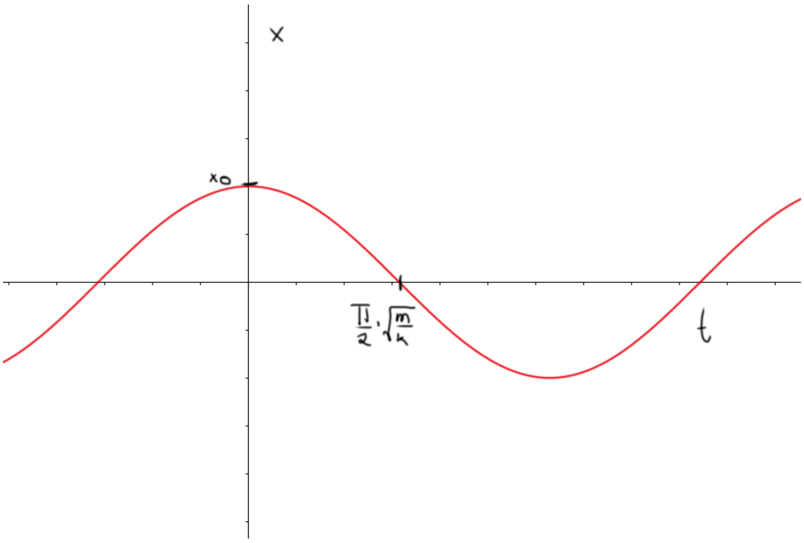
$$\sin\frac{\pi}{2} = \frac{\sqrt{k}x_0}{\sqrt{2(E+C)}} \tag{62}$$

$$\frac{\sqrt{2(E+C)}}{\sqrt{k}} = x_0 \tag{63}$$

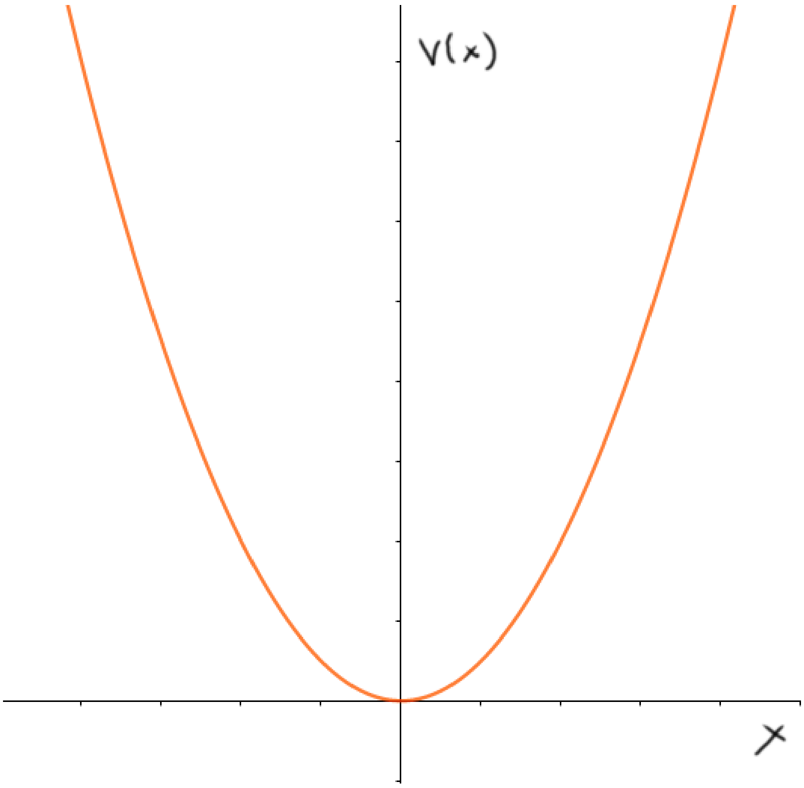
Setze x_0 in $x(t)$ ein:

$$
\begin{aligned}
x(t) &= \sin\left(\mp t\sqrt{\frac{k}{m}} + \arcsin\frac{\sqrt{k}}{\sqrt{2(E+C)}}\frac{\sqrt{2(E+C)}}{\sqrt{k}}\right)\frac{\sqrt{2(E+C)}}{\sqrt{k}} && (64)\\
&= \sin\left(\mp t\sqrt{\frac{k}{m}} + \arcsin 1\right)\frac{\sqrt{2(E+C)}}{\sqrt{k}} && (65)\\
&= \sin\left(\mp t\sqrt{\frac{k}{m}} + \frac{\pi}{2}\right)\frac{\sqrt{2(E+C)}}{\sqrt{k}} && (66)\\
&= \cos\left(\mp t\sqrt{\frac{k}{m}}\right)\frac{\sqrt{2(E+C)}}{\sqrt{k}}\Bigg\| && (67)
\end{aligned}
$$

ii. Skizze: Bahnkurve



iii. Skizze: Potential



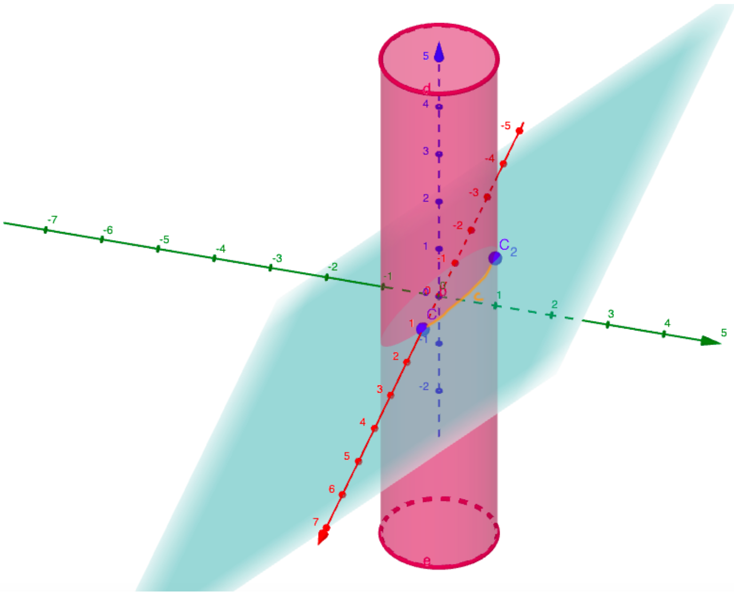
Die Umkehrpunkte der Bewegung gibt es wenn der Abstand zum Bezugspunkt maximal ist, also an den Stellen $\pm x_0$

6.3 Aufgabe 3

Geg.:

- $\vec{F}(\vec{x}) = \begin{pmatrix} x \\ x^2 + y^2 + z^2 \\ ye^z \end{pmatrix}$
- $P_1 = (1, 0, 0)$
- $P_2 = (0, 1, 1)$

a) Skizze:



Geg.:

- $\vec{x}(\phi, z) = \begin{pmatrix} \cos\phi \\ \sin\phi \\ z \end{pmatrix}$

Parametrisiere C:

Da im Weg C gilt: $y = z$,

$$\vec{x}(\phi) = \begin{pmatrix} \cos\phi \\ \sin\phi \\ \sin\phi \end{pmatrix} \tag{68}$$

Bestimme $\phi \in [a, b]$:

Anfangspunkt:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\phi \\ \sin\phi \\ \sin\phi \end{pmatrix} \to \phi_0 = a = 0 \tag{69}$$

Endpunkt:

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\phi \\ \sin\phi \\ \sin\phi \end{pmatrix} \rightarrow \phi_1 = b = \frac{\pi}{2} \tag{70}$$

Also $\phi \in \left[0, \frac{\pi}{2}\right]$

b) Ges.:

- $\int_C \vec{F}(\vec{x})d\vec{x} = \int_a^b \vec{F}(\vec{x}(\phi))\frac{d\vec{x}(\phi)}{d\phi}d\phi$

Bestimme $\frac{d\vec{x}(\phi)}{d\phi}$:

$$\vec{x}(\phi) = \begin{pmatrix} \cos\phi \\ \sin\phi \\ \sin\phi \end{pmatrix} \tag{71}$$

$$\rightarrow \frac{d\vec{x}(\phi)}{d\phi} = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ \cos\phi \end{pmatrix} \tag{72}$$

Setze in $\vec{F}(\vec{x})$ ein:

$$\vec{F}(\vec{x}) = \begin{pmatrix} \cos\phi \\ \cos^2\phi + \sin^2\phi + \sin^2\phi \\ \sin\phi e^{\sin\phi} \end{pmatrix} \tag{73}$$

Setze in Integral ein:

$$\int_C \vec{F}(\vec{x})d\vec{x} = \int_a^b \vec{F}(\vec{x}(\phi))\frac{d\vec{x}(\phi)}{d\phi}d\phi \tag{74}$$

$$= \int_a^b \begin{pmatrix} \cos\phi \\ \cos^2\phi + \sin^2\phi + \sin^2\phi \\ \sin\phi e^{\sin\phi} \end{pmatrix} \cdot \begin{pmatrix} -\sin\phi \\ \cos\phi \\ \cos\phi \end{pmatrix} \cdot d\phi \tag{75}$$

$$= \int_a^b [(-\sin\phi\cos\phi) + (\cos\phi + \cos\phi\sin^2\phi) + (\cos\phi\sin\phi e^{\sin\phi})]d\phi \tag{76}$$

$$= \int_a^b (-\sin\phi\cos\phi)d\phi + \int_a^b (\cos\phi)d\phi + \int_a^b (\cos\phi\sin^2\phi)d\phi + \int_a^b (\cos\phi\sin\phi e^{\sin\phi})d\phi \tag{77}$$

i.

$$\int_a^b (-\sin\phi\cos\phi)d\phi \qquad \text{u-Substitution} \tag{78}$$

$$u = \sin\phi \tag{79}$$

$$\frac{du}{d\phi} = \cos\phi \tag{80}$$

$$= -\int_a^b udu \tag{81}$$

$$= -\frac{u^2}{2}\bigg|_a^b \tag{82}$$

$$= -\frac{\sin^2\phi}{2}\bigg|_0^{\frac{\pi}{2}} \tag{83}$$

$$= -\frac{1}{2} \tag{84}$$

ii.

$$\int_a^b (\cos\phi)d\phi \tag{85}$$

$$= \sin\phi|_0^{\frac{\pi}{2}} \tag{86}$$

$$= 1 \tag{87}$$

iii.

$$\int_a^b (\cos\phi\sin^2\phi)d\phi \qquad \text{u-Substitution} \tag{88}$$

$$u = \sin\phi \tag{89}$$

$$\frac{du}{d\phi} = \cos\phi \tag{90}$$

$$= \int_a^b u^2du \tag{91}$$

$$= \frac{u^3}{3}\bigg|_a^b \tag{92}$$

$$= \frac{\sin^3\phi}{3}\bigg|_0^{\frac{\pi}{2}} \tag{93}$$

$$= \frac{1}{3} \tag{94}$$

iv.

$$\int_a^b (\cos\phi \sin\phi e^{\sin\phi}) d\phi \qquad \text{u-Substitution} \tag{95}$$

$$u = \sin\phi \tag{96}$$

$$\frac{du}{d\phi} = \cos\phi \tag{97}$$

$$= \int_a^b u e^u du \tag{98}$$

$$= (u-1)e^u|_a^b \tag{99}$$

$$= (\sin\phi - 1)e^{\sin\phi}|_0^{\frac{\pi}{2}} \tag{100}$$

$$= 1 \tag{101}$$

Die Gesamtsumme der einzelnen Integralen ist:

$$\int_C \vec{F}(\vec{x}) d\vec{x} = -\frac{1}{2} + 1 + \frac{1}{3} + 1 \tag{102}$$

$$= \frac{11}{6} \tag{103}$$

6.4 Aufgabe 4

a) Geg.:

- $y' = \frac{-2y}{x} + \frac{\ln x}{x^2}$

$$\frac{dy}{dx} = \frac{-2y}{x} + \frac{\ln x}{x^2} \qquad | + \frac{2y}{x} \tag{104}$$

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{\ln x}{x^2} \qquad | \cdot x^2 \tag{105}$$

$$x^2\frac{dy}{dx} + 2xy = \ln x \qquad |2x = \frac{dx^2}{dx} \tag{106}$$

$$x^2\frac{dy}{dx} + \frac{dx^2}{dx}y = \ln x \qquad |(u \cdot v)' = uv' + u'v \text{ mit} \tag{107}$$

$$u = x^2, v = y \tag{108}$$

$$\frac{d(x^2y)}{dx} = \ln x \qquad | \int \tag{109}$$

$$x^2 y = x\ln x - x + C \tag{110}$$

$$y = \frac{x\ln x - x + C}{x^2} \Bigg\| \tag{111}$$

b) Geg.:

- $y' = -2xy + 2xe^{-x^2}$
- $y(0) = 2$

$$\frac{dy}{dx} = -2xy + 2xe^{-x^2} \qquad | + 2xy \tag{112}$$

$$\frac{dy}{dx} + 2xy = 2xe^{-x^2} \qquad | \cdot e^{x^2} \tag{113}$$

$$e^{x^2}\frac{dy}{dx} + 2xye^{x^2} = 2x \qquad |2e^{x^2} = \frac{de^{x^2}}{dx} \tag{114}$$

$$e^{x^2}\frac{dy}{dx} + \frac{de^{x^2}}{dx}y = 2x \qquad |(u \cdot v)' = uv' + u'v \text{ mit} \tag{115}$$

$$u = e^{x^2}, v = y \tag{116}$$

$$\frac{d(e^{x^2}y)}{dx} = 2x \qquad | \int \tag{117}$$

$$e^{x^2}y = x^2 + C \qquad | : e^{x^2} \tag{118}$$

$$y = e^{-x^2}(x^2 + C) \ || \tag{119}$$

Mit Anfangsbedingungen:

$$2 = e^0 \cdot C \tag{120}$$

$$C = 2 \tag{121}$$

Also:

$$y = e^{-x^2}(x^2 + 2) \tag{122}$$