5. Exercise sheet to Experimental Physics I (WS 20/21)

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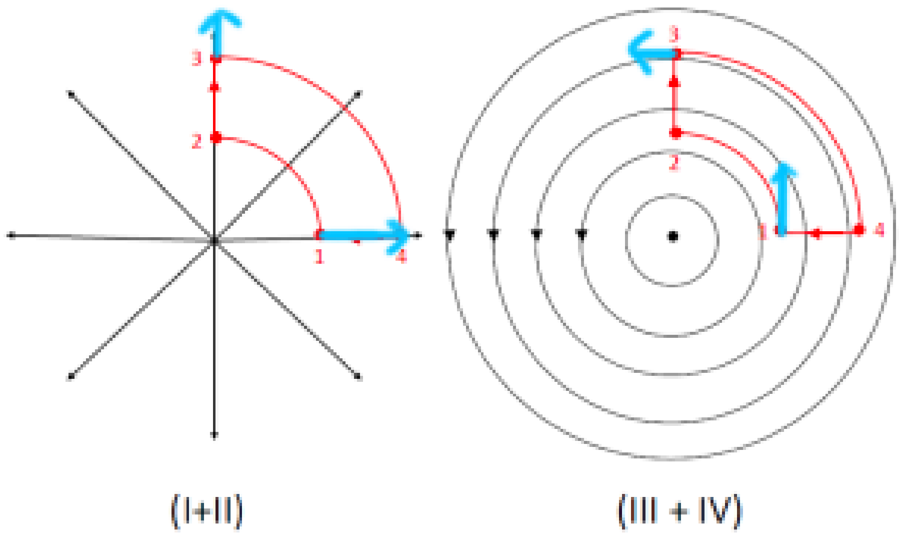
Points: ___/___/___   Σ___

5.1 Exercise 1: Work in force fields

a) Sketch:



b) The work can be determined using the following formula:

$$W_{AB} = \int_{AB} \vec{F}(\vec{r})d\vec{r}$$

i. Specific for the force field i.:

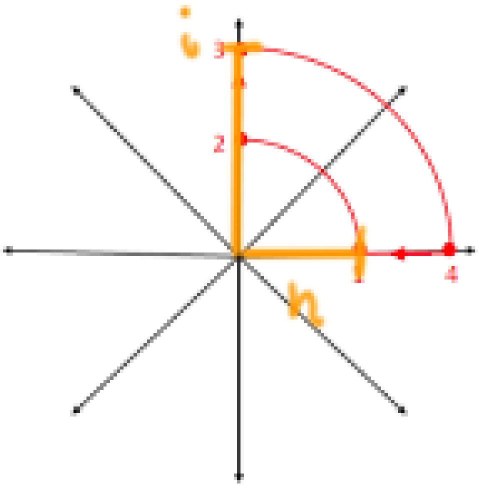
$$W_{AB} = \int_{AB} \frac{a}{r}\vec{e_r}d\vec{r}$$

We know that $\vec{e_r}d\vec{r} = dr$, so

$$W_{AB} = \int_{AB} \frac{a}{r}dr$$

Since we are integrating in respect to the radius, we need to input the respective radii as the interval borders, as per the sketch below, we define the borders to be h for the inner section of the circle and i for the outer section.

Sketch:



(I+II)

With a formula we can work with, we just need to calculate the work on the individual paths.

$$
\begin{aligned}
W_{12} &= \int_h^h \frac{a}{r} dr \\
&= 0
\end{aligned}
$$

Besides, during this path, the movement is constantly perpendicular to the force field, so the dot product $\vec{F}d\vec{r} = 0$

$$
\begin{aligned}
W_{23} &= \int_h^i \frac{a}{r} dr \\
&= a \cdot \ln r|_h^i \\
&= a \ln i - a \ln h
\end{aligned}
$$

$$
W_{34} = 0
$$

For the same reasons that apply to W_{12}

$$
\begin{aligned}
W_{41} &= \int_i^h \frac{a}{r} dr \\
&= a \ln r|_i^h \\
&= a \ln h - a \ln i
\end{aligned}
$$

The total work done is

$$
\begin{aligned}
W_{total} &= 0 + a\ln i - a\ln h + 0 + a\ln h - a\ln i \\
&= 0
\end{aligned}
$$

ii. Following the same logic as in i. we can define the work done between two points to be

$$
W_{AB} = \int_{AB} \frac{a}{r^2} dr
$$

And calculating the individual amounts of work during each path we get:

$$
\begin{aligned}
W_{12} &= \int_h^h \frac{a}{r^2} dr \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
W_{23} &= \int_h^i \frac{a}{r^2} dr \\
&= -\frac{a}{r}|_h^i \\
&= -\frac{a}{i} + \frac{a}{h}
\end{aligned}
$$

$$
W_{34} = 0
$$

$$
\begin{aligned}
W_{41} &= \int_i^h \frac{a}{r^2} dr \\
&= -\frac{a}{r^2}|_i^h \\
&= -\frac{a}{h} + \frac{a}{i}
\end{aligned}
$$

The total work done along this circuit is

$$
\begin{aligned}
W_{total} &= 0 - \frac{a}{i} + \frac{a}{h} + 0 - \frac{a}{h} + \frac{a}{i} \\
&= 0
\end{aligned}
$$

iii. To be able to use the above mentioned formula for the work, we need to first determine how we can describe $d\vec{r}$ and $\vec{e_\phi}$

First: We set up the trajectory of our radius vector

$$\vec{r} = \begin{pmatrix} r \cdot \cos\phi \\ r \cdot \sin\phi \end{pmatrix}$$

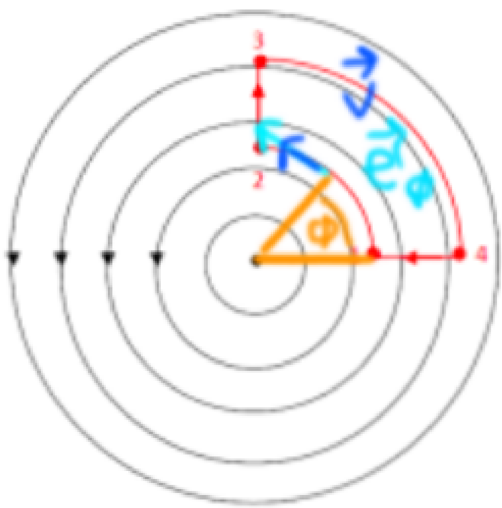
From that, we can conclude that

$$\begin{aligned} \frac{d\vec{r}}{d\phi} &= \begin{pmatrix} -r \cdot \sin\phi \\ r \cdot \sin\phi \end{pmatrix} \\ &= r \cdot \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} \\ &\to d\vec{r} = r \cdot \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} \cdot d\phi \end{aligned}$$

Now we have something which we can input into our original formula so that it looks like the following:

$$W_{AB} = \int_{AB} \frac{a}{r} \vec{e_\phi} r \cdot \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} \cdot d\phi$$

However, the work needs a scalar unit, so we need to get rid of the vectors inside the integral. As shown in the graphic below



(III + IV)

the vector of the angle and the velocity are moving in the same direction, which means that up to a multiplication factor, they are the same. This means that after grooming our formula a little bit, we get:

$$\begin{aligned} W_{AB} &= \int_{AB} a\vec{e_\phi} \cdot \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} \cdot d\phi \\ &= \int_{AB} a \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} \cdot \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} d\phi \end{aligned}$$

and now we can use the dot product to get

$$\begin{aligned}
&= \int_{AB} a(\sin^2\phi + \cos^2\phi) d\phi \\
&= \int_{AB} a d\phi
\end{aligned}$$

Now we are integration in respect to the angle, so we need to input the angle differences in our integration borders to calculate the work done.

With the specific values, we can now calculate the total work done along this path:

$$\begin{aligned}
W_{12} &= \int_0^{\frac{\pi}{2}} a d\phi \\
&= a\phi|_0^{\frac{\pi}{2}} \\
&= a\frac{\pi}{2} - a \cdot 0 \\
&= a\frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
W_{23} &= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} a d\phi \\
&= 0
\end{aligned}$$

We could also argue here, that since the direction of our movement and the direction of the force field are perpendicular to each other, the total work done needs to be 0.

$$\begin{aligned}
W_{34} &= \int_{\frac{\pi}{2}}^{0} a d\phi \\
&= a\phi|_{\frac{\pi}{2}}^{0} \\
&= a \cdot 0 - a\frac{\pi}{2} \\
&= -a\frac{\pi}{2}
\end{aligned}$$

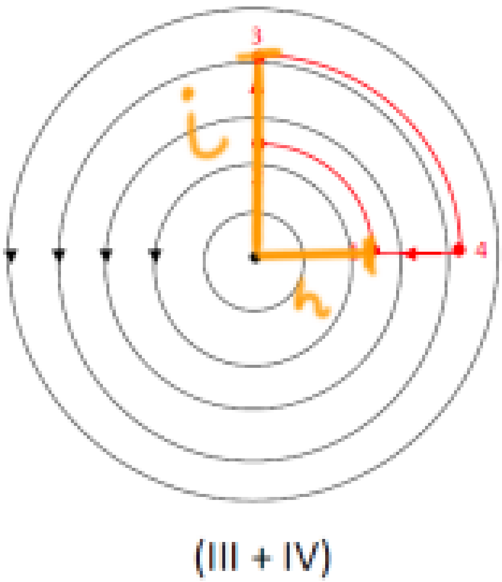
$$\begin{aligned}
W_{total} &= W_{12} + W_{23} + W_{34} + W_{41} \\
&= \frac{a\pi}{2} + 0 - \frac{a\pi}{2} + 0 \\
&= 0
\end{aligned}$$

iv. Using the same logic we applied on example iii., we can set up the integral for the work done in iv. to be

$$
\begin{aligned}
W_{AB} &= \int_{AB} \frac{a}{r^2} \vec{e_\phi} d\vec{r} \\
&= ... \\
&= \int_{AB} \frac{a}{r} d\phi
\end{aligned}
$$

Here, opposite to example iii., the radius plays a role in the total work done. So we need to account for the difference in our calculations. As per shown in the sketch below, we will define h to be the radius of the inner circle section and i to be the radius of the outer section

Sketch:



$$
\begin{aligned}
W_{12} &= \int_0^{\frac{\pi}{2}} \frac{a}{h} d\phi \\
&= \frac{a}{h} \phi|_0^{\frac{\pi}{2}} \\
&= \frac{a}{h} \cdot \frac{\pi}{2} - \frac{a}{h} \cdot 0 \\
&= \frac{\pi a}{2h}
\end{aligned}
$$

$$
W_{23} = 0
$$

Because movement is perpendicular to the force.

$$
\begin{aligned}
W_{34} &= \int_{\frac{\pi}{2}}^{0} \frac{a}{i} d\phi \\
&= \frac{a}{i}\phi|_{\frac{\pi}{2}}^{0} \\
&= \frac{a}{i}\cdot 0 - \frac{a}{i}\cdot\frac{\pi}{2} \\
&= -\frac{\pi a}{2i}
\end{aligned}
$$

The total work done during this circuit is

$$
\begin{aligned}
W_{total} &= 0 + \frac{\pi a}{2h} + 0 - \frac{\pi a}{2i} \\
&= \frac{\pi a}{2h} - \frac{\pi a}{2i} \\
&= \frac{\pi a}{2}(h - i)
\end{aligned}
$$

c) Based on the observations from the previous four examples, we can say that radial force fields are conservative and circular force fields non conservative. Even though example iii. shows characteristics similar to the ones of conservative forces, it is safe to assume that it is the exception and not the norm.

5.2 Exercise 2: Gravity

Given:

- $r_e = 6370\ km = 6370000\ m$
- $\rho = 5.5\ gcm^{-3}$

a) Given.:

  - $F(r) = \gamma\frac{4}{3}\pi\rho mr$
  - $F_G = \dfrac{Gm_1m_2}{r^2}$

  The mass of the earth m_2 can be described using its density ρ and its volume as such:

$$m_2 = \rho V$$

  The volume of a sphere is

$$V = \frac{4}{3}\pi r^3$$

  By inputting

$$m_2 = \rho\frac{4}{3}\pi r^3$$

into the formula for the gravitational force we get:

$$
\begin{aligned}
F_G &= \frac{Gm_1\left(\rho\frac{4}{3}\pi r^3\right)}{r^2} \\
&= \frac{Gm_1\rho 4\pi r}{3}
\end{aligned}
$$

And by redefining our constant G as γ we get

$$F(r) = \frac{4}{3}\pi\gamma\rho m r$$

b) The journey of a test mass traveling through the hole can be described as a harmonic oscillation. The time it takes the mass to go from one end to another is equivalent to half a period length of the oscillation.

We know that the acceleration of the mass in dependance of the radius is

$$
\begin{aligned}
a(r) &= \frac{F(r)}{m} \\
&= \frac{\frac{4}{3}\pi\gamma\rho m r}{m} \\
&= \frac{4}{3}\pi\gamma\rho r
\end{aligned}
$$

whereas

$$\gamma = \frac{r_e^2}{M_e} g$$

and

$$\rho = \frac{M_e}{\frac{4}{3}\pi r_e^3}$$

By inputting these values into $a(r)$ we get

$$
\begin{aligned}
a(r) &= \frac{4\pi r_e^2 g M_e r}{3M_e\frac{4}{3}\pi r_e^3} \\
&= \frac{g}{r_e} r
\end{aligned}
$$

As previously stated, the journey of the mass can be described by a harmonic oscillation, therefore:

$$a = -\omega^2 A \sin \omega t$$

and

$$r = A \sin \omega t$$

$$\rightarrow a(r) = -\omega^2 r$$

If we set both formulas for the acceleration equal we get:

$$\frac{g}{r_e} r = -\omega^2 r$$

If we solve for the angular velocity ω

$$\omega = \sqrt{\frac{g}{r_e}}$$

The period length is

$$T = \frac{2\pi}{\omega}$$

and if we halve that value to figure out how long it takes to go from one end of the earth to the other we get

$$\frac{T}{2} = \frac{2\pi}{\sqrt{\frac{g}{r_e}}}$$
$$\approx 42\ min$$

c) Assuming that a satellite orbiting nearthe surface of the Earth is close enough to it that we can use $g = 9.81\ ms^{-2}$. Even at 10 km above the Earth's surface, the orbit of the satellite would only be $\approx$ 63 km or $\approx$ 0.002 % longer than the Earth's circumference. We can then safely approximate, that the satellite is orbiting the Earth along its surface.

We now need to calculate the necessary centripetal force for the satellite to overcome the gravitational force pulling it down.

$$F_c = F_g$$

Given:

- $F_c = \frac{mv^2}{r}$
- $F_g = mg$

- $r = 6370\ km$

And specifically we are interested in the velocity of the satellite:

$$
\begin{aligned}
\frac{mv^2}{r} &= mg \\
\frac{v^2}{r} &= g \\
v &= \sqrt{rg} \\
v &\approx 7905\ ms^{-1}
\end{aligned}
$$

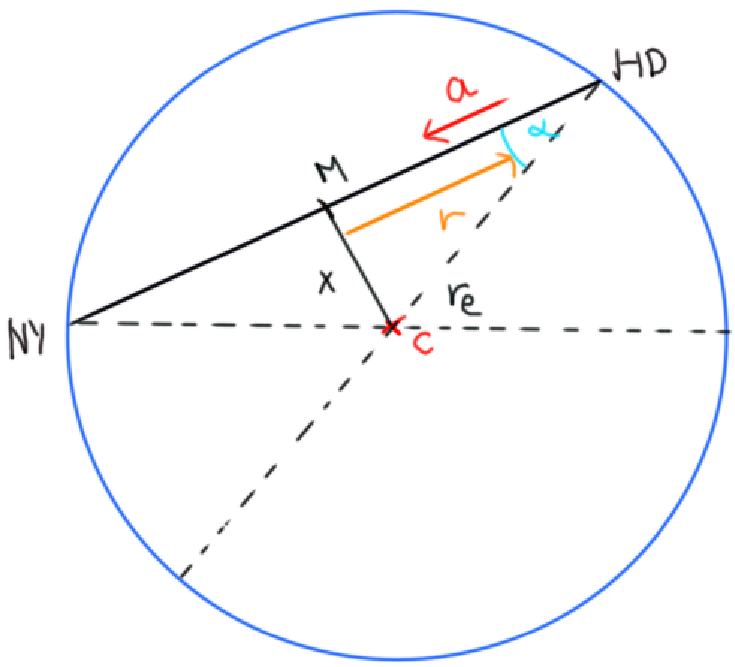
Flying at this speed, it would take the satellite

$$
\begin{aligned}
t &= \frac{s}{v} \\
&= \frac{2\pi r}{v} \\
&\approx 5063\ ms^{-1} \\
&\approx 84\ min
\end{aligned}
$$

It takes the same amount of time for a satellite near the Earth's surface to complete one full orbit, as it would take a mass falling through the Earth to go from one end to another and back to its origin.

d) Contrary to the previous exercise, we are not falling freely through the Earth now, but at a certain angle. We can now picture this as a problem in an inclined plane.

Sketch:



The acceleration in this example would depend on the distance d to the Earth's center, and that distance is

$$d = \sqrt{r^2 + x^2}$$

First we will define the acceleration towards the center of the Earth as

$$\begin{aligned} g_C &= \frac{g}{r_e} d \\ &= \frac{g}{r_e} \sqrt{r^2 + x^2} \end{aligned}$$

as determined in exercise b).

And the effective acceleration towards our goal, due to the inclined plane is

$$\begin{aligned} a(r) &= g_C \cos\alpha \\ &= \frac{g}{r_e} \sqrt{r^2 + x^2} \cos\alpha \end{aligned}$$

and

$$\begin{aligned} \cos\alpha &= \frac{r}{d} \\ &= \frac{r}{\sqrt{r^2 + x^2}} \end{aligned}$$

If we insert this value in $a(r)$ we get

$$\begin{aligned} a(r) &= \frac{g\sqrt{r^2 + x^2}}{r_e} \frac{r}{\sqrt{r^2 + x^2}} \\ &= \frac{g}{r_e} r \end{aligned}$$

At this point, the logic applied in exercise b) to determine the angular velocity will result in the same calculation, so we can again define

$$\omega = \sqrt{\frac{g}{r_e}}$$

None of the remaining necessary calculations are dependant on the distance between two points or the angle of the path relative to a tunnel straight down the Earth, so the time it takes to go from one extreme of the Earth to the other and from Heidelberg to New York would be exactly the same: 42 minutes[1].

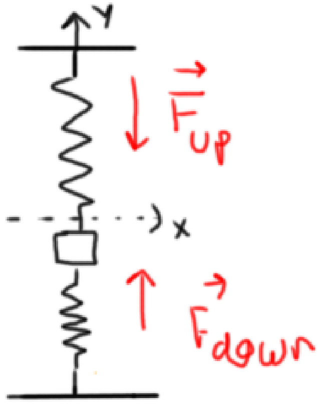
[1] It is unlikely that this is the question that darn machine was trying to formulate before all of Earth was obliterated to build another god darn interstellar highway.

5.3 Exercise 3: Two springs

a) Given.:

- $F = -ky$

Observing the one dimensional case, we can say that the forces from the two springs are equal and opposite to each other.



If one springs expands a distance y along the y-Axis, then it must mean that the opposite spring contracted the same distance y.

The total force is naturally

$$F_{total} = F_{up} + F_{down}$$

whereas

$$F_{up} = -F_{down}$$

and

$$y_{up} = -y_{down}$$

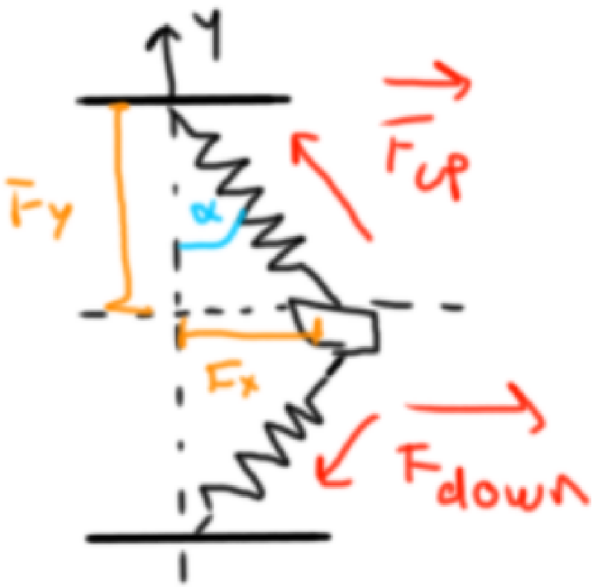
So:

$$\begin{aligned} F_{total} &= -ky_{up} + (ky_{down}) \\ &= -ky_{up} - ky_{up} \\ &= -2ky \end{aligned}$$

b) Given:

- $F_x = -2kx\left(1 - \frac{l_o}{S}\right)$
- $F_y = -2ky$

Sketch:



First, we can write the force of the spring $\vec{F}_{up}$ as

$$\vec{F}_{up} = \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

And we can describe F_x and F_y as follows:

$$F_x = F_{up} \cdot \sin\alpha$$

$$F_y = F_{up} \cdot \cos\alpha$$

and thus

$$\vec{F}_{up} = F_{up} \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}$$

The force of the upper spring wihtout displacement in the x-direction is:

$$\begin{aligned} \vec{F}_{up} &= F_{up} \begin{pmatrix} \sin 0 \\ \cos 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ F_{up} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -ky \end{pmatrix} \end{aligned}$$

And thus

$$F_y = -ky$$

From other trigonometric operations we can determine that:

$$\begin{aligned} F_x &= F_y \cdot \tan\alpha \\ &= -ky \tan\alpha \end{aligned}$$

We also know that

$$\tan\alpha = \frac{x}{S} = \frac{x}{l_0 + y}$$

and

$$y = S - l_0$$

So

$$\begin{aligned} F_x &= -ky \tan\alpha \\ &= -k(S - l_0)\frac{x}{S} \\ &= \frac{kl_0x}{S} - kx \\ &= -kx(1 - \frac{l_0}{S}) \end{aligned}$$

And finally:

$$\vec{F}_{up} = \begin{pmatrix} -kx\left(1 - \dfrac{l_0}{S}\right) \\ -ky \end{pmatrix}$$

If we proceed analogous for the force exerted by the spring below we will come to the same conclusion that

$$\vec{F}_{down} = \begin{pmatrix} -kx\left(1 - \frac{l_0}{S}\right) \\ -ky \end{pmatrix}$$

And by adding up both Forces to get the total force

$$\begin{aligned} \vec{F}_{total} &= \vec{F}_{up} + \vec{F}_{down} \\ &= \begin{pmatrix} -kx\left(1 - \frac{l_0}{S}\right) \\ -ky \end{pmatrix} + \begin{pmatrix} -kx\left(1 - \frac{l_0}{S}\right) \\ -ky \end{pmatrix} \\ &= \begin{pmatrix} -2kx\left(1 - \frac{l_0}{S}\right) \\ -2ky \end{pmatrix} \end{aligned}$$