Sheet 0 **July 6, 2023**

General Relativity

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Exercise 1 *Black stars*

Newtonian gravity is the field theory defined by the following equation:

$$\nabla^2\phi = 4\pi G\rho \tag{1}$$

where ϕ is the potential of the gravitational field, G is Newton's constant and ρ is the density of source mass.

1.1 We show that the work done by the field on test particles is independent of the path, i.e. the field is conservative.

For this we take two approaches. For the first approach we take the definition of a conservative field and an irrotational field. An irrotational field $\mathbf{V}$ is one, for which applies

$$\boldsymbol{\nabla} \times \mathbf{V} = 0. \tag{2}$$

A conservative (vector) field is one which can be defined as the gradient of a scalar field

$$\mathbf{V} = \boldsymbol{\nabla}\phi. \tag{3}$$

Per definition, the line integral of a conservative field is independent of the path. We can demonstrate this by integrating the field over a closed path, defined by the boundary of a surface ∂A and applying Stoke's Theorem:

$$\begin{align} \oint_{\partial A} \mathbf{V}\,\mathrm{d}\mathbf{x} &= \int_A \boldsymbol{\nabla} \times \mathbf{V}\,\mathrm{d}\mathbf{A} \tag{4}\\ &= \int_A \boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi\,\mathrm{d}\mathbf{A} \tag{5}\\ &= 0. \tag{6} \end{align}$$

This follows pretty straightforwardly.

1.2 We find the Equations of Motion (EoM) for a particle of mass m moving in the gravitational potential ϕ with the Lagrangian

$$\begin{align} \mathcal{L} &= \frac{1}{2}m\|\dot{\mathbf{x}}\|^2 - m\phi(\mathbf{x}) \tag{7}\\ &= \frac{1}{2}m\sqrt{\dot{x}_i\dot{x}^i}^2 - m\phi(\mathbf{x}). \tag{8} \end{align}$$

We use the Euler-Lagrange-Equations (ELE) for it:

$$\frac{\partial\mathcal{L}}{\partial x^a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial \dot{x^a}}\right) = 0 \tag{9}$$

$$= -m\partial_a\phi(\mathbf{x}) - \frac{\mathrm{d}}{\mathrm{d}t}\dot{x}_a \tag{10}$$

$$\ddot{x}_a = -m\partial_a\phi. \tag{11}$$

1.3 We show that the potential outside a spherically symmetric body of mass M and radius R is

$$\phi(\mathbf{x}(r,\theta,\varphi)) = -\frac{GM}{r}. \tag{12}$$

For that we want to solve (1)

$$\nabla^2\phi = 4\pi G\rho(r). \tag{13}$$

We can do this by integrating twice over a volume enclosing the whole body, with $r > R$:

$$\begin{aligned}
\int_V \nabla^2\phi \,\mathrm{d}V &= \int_V 4\pi G\rho(r)\,\mathrm{d}V && (14)\\
4\pi\int_0^r \nabla'^2\phi\; r'^2\,\mathrm{d}r' &= 4\pi G\int\rho(r)\,\mathrm{d}V && (15)\\
4\pi\int_0^r \frac{1}{r'^2}\partial_{r'}\left(r'^2\partial_{r'}\phi\right)r'^2\,\mathrm{d}r' &= 4\pi GM && (16)\\
\int_0^r \partial_{r'}\left(r'^2\partial_{r'}\phi\right)\mathrm{d}r' &= GMr^2\partial_r\phi \qquad\qquad = GM && (17)\\
\partial_r\phi &= \frac{GM}{r^2} \qquad\qquad\qquad \int && (18)\\
\phi &= -\frac{GM}{r}. && (19)
\end{aligned}$$

1.4 We now compute the escape velocity needed for an object to escape the gravitational attraction of the massive body. For this, the body requires enough kinetic energy to escape to infinity:

$$\begin{aligned}
E_{\text{kin, req.}} &= \int_R^\infty m\boldsymbol{\nabla}\phi\,\mathrm{d}\mathbf{r} && (20)\\
&= \int_R^\infty m\partial_r\phi\;\mathbf{e_r}\cdot\mathrm{d}\mathbf{r} && (21)\\
&= 0 - m\phi(R). && (22)
\end{aligned}$$

We now set the kinetic energy equal to this and get

$$\begin{aligned}
\frac{1}{2}mv^2 &= -m\phi(R) && (23)\\
\frac{m}{2}v^2 &= m\frac{GM}{R}v \qquad\qquad = \sqrt{\frac{2GM}{R}}. && (24)
\end{aligned}$$

If we set the velocity to c, then the resulting radius of the object is

$$R = \frac{2GM}{c^2}. \tag{25}$$

Exercise 2 *Differential geometry, oh my!*

A curve on the plane is a function $\gamma(\tau)$ from (a subset of) the real line into $\mathbb{R}^2$, $\gamma : \mathbb{R} \to \mathbb{R}^2$. The length of the curve is defined as

$$L[\gamma] = \int \mathrm{d}\tau\,\|\gamma'(\tau)\| \tag{26}$$

2.1 We compute the length of the following curves:

(a) an ellipse, whose equation is $a^{-2}c^2+b^{-2}y^2 = 1$, in Cartesian coordinates as a function of a, b;

We first have to parametrise γ, which is pretty straightforward as an ellipse is nothing else than a squashed circle:

$$\gamma(t) = \begin{pmatrix} a\cos t \\ b\sin t \end{pmatrix} \qquad \gamma'(t) = \begin{pmatrix} -a\sin t \\ b\cos t \end{pmatrix} \tag{27}$$

$$\|\gamma'(t)\| = \sqrt{a^2\sin^2 t + b^2\cos^2 t} \tag{28}$$

and we have then

$$L[\gamma] = \int \mathrm{d}t\,\sqrt{a^2\sin^2 t + b^2\sin^2 t}. \tag{29}$$

This is what's known as an elliptic integral, particularly an incomplete elliptic integral of the second class. They don't have nice analyical solutions so our job here is done.

(b) a logarithmic spiral with the coordinates $r = e^\theta$ in polar coordinates as a function of the turns n.

$$\gamma(t) = \begin{pmatrix} \mathrm{e}^t\cos t \\ \mathrm{e}^t\sin t \end{pmatrix} \qquad \gamma'(t) = \begin{pmatrix} \mathrm{e}^t\cos t - \mathrm{e}^t\sin t \\ \mathrm{e}^t\sin t + \mathrm{e}^t\cos t \end{pmatrix} \tag{30}$$

$$\|\gamma'(t)\| = \sqrt{2\mathrm{e}^{2t}\cos^2 t + 2\mathrm{e}^{2t}\sin^2 t} \tag{31}$$

$$= \sqrt{2}\mathrm{e}^t. \tag{32}$$

From which follows:

$$L[\gamma](n) = \sqrt{2}\int_0^{2\pi n} \mathrm{e}^t\,\mathrm{d}t \tag{33}$$

$$= \sqrt{2}\Big(\mathrm{e}^{2\pi n} - 1\Big). \tag{34}$$

2.2 We show that the length of the curve is unaffected by a change in parameter $\tau \to t$ with $t = f(\tau) : \mathbb{R} \to \mathbb{R}$ differentiable.

$$L[\gamma(\tau)] = \int \mathrm{d}\tau\,\|\gamma'(\tau)\|. \tag{35}$$

For this to be valid, $\mathrm{d}\tau$ and $\|\gamma'(\tau)\|$ should transform inversely to each other after applying the substitution. We check if that's the case:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}f^{-1}(t) \tag{36}$$

$$= \frac{1}{f'(f^{-1}(t))} \tag{37}$$

$$= \frac{1}{f'(\tau)} \tag{38}$$

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{f'(\tau)}. \tag{39}$$

For the norm we also check:

$$\|\gamma'(t(\tau))\| = \left\|\gamma'(t)\cdot\frac{\mathrm{d}t}{\mathrm{d}\tau}\right\| \tag{40}$$

$$= \|\gamma'(t)\| \cdot f'(\tau). \tag{41}$$

This coincides with what we expect:

$$L[\gamma] = \int \frac{\mathrm{d}t}{f'(\tau)} \|\gamma'(t)\| \cdot f'(\tau) \tag{42}$$
$$= \int \mathrm{d}t \, \|\gamma'(t)\| \tag{43}$$

2.3 An important reparametrization is the arc length s:

$$s(\tau) = \int_0^\tau \mathrm{d}\rho \, \|\gamma'(\rho)\| \tag{44}$$

with

$$\mathbf{u} = \frac{\mathrm{d}\gamma(s)}{\mathrm{d}s}. \tag{45}$$

We show that $\|u\| = 1$:

$$\frac{\mathrm{d}\gamma(s)}{\mathrm{d}s} = \frac{\mathrm{d}\gamma(s(\tau))}{\mathrm{d}\tau} \left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^{-1} \tag{46}$$
$$= \frac{\mathrm{d}\gamma}{\mathrm{d}\tau} \left\{ \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\int_0^\tau \mathrm{d}\rho \, \|\gamma'(\rho)\| \right) \right\}^{-1} \tag{47}$$
$$= \frac{\mathrm{d}\gamma}{\mathrm{d}\tau} \left\{ \int_0^\tau \mathrm{d}\rho \, \partial_\rho \|\gamma'(\rho)\| \right\}^{-1} \tag{48}$$
$$= \frac{\mathrm{d}\gamma}{\mathrm{d}\tau} \|\gamma'(\tau)\|^{-1} \tag{49}$$

From this it's clear, that $\mathbf{u} = \dfrac{\gamma'(\tau)}{\|\gamma'(\tau)\|}$ is a unit vector.

2.4 We define the curvature as $\kappa = \left\|\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s}\right\|$ and we show that the acceleration of a body moving along the curve has an orthogonal component proportional to the curvature.

The acceleration of a body along the curve is given by

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \tag{50}$$

and since we learned previously that $\mathbf{u}$ is the unit tangent vector, then the velocity is given by $\mathbf{v} = v\mathbf{u}$.

$$\mathbf{a} = \frac{\mathrm{d}v\mathbf{u}}{\mathrm{d}t} \tag{51}$$
$$= \frac{\mathrm{d}v}{\mathrm{d}t}\mathbf{u} + v\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \tag{52}$$
$$= \frac{\mathrm{d}v}{\mathrm{d}t}\mathbf{u} + v\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} \tag{53}$$
$$= \frac{\mathrm{d}v}{\mathrm{d}t}\mathbf{u} + v\kappa\frac{\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s}}{\kappa}\frac{\mathrm{d}s}{\mathrm{d}t} \tag{54}$$
$$= \frac{\mathrm{d}v}{\mathrm{d}t}\mathbf{u} + v\kappa\mathbf{k}. \tag{55}$$

Left is to check, that also $\mathbf{k}$ is orthogonal to $\mathbf{u}$:

$$\mathbf{k}\cdot\mathbf{u} = \frac{1}{\kappa}\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s}\cdot\mathbf{u} \tag{56}$$

$$= \frac{1}{\kappa}\frac{\mathrm{d}s}{\mathrm{d}t}\frac{1}{2}\frac{\mathrm{d}\mathbf{u}\cdot\mathbf{u}}{\mathrm{d}s} \tag{57}$$

$$= \frac{1}{\kappa}\frac{\mathrm{d}s}{\mathrm{d}t}\frac{1}{2}\frac{\mathrm{d}\,1}{\mathrm{d}s} = 0. \tag{58}$$

2.5 Inertial bodies in $\mathbb{R}^2$ must move along straght lines because the curvature κ of the trajectory makes a first appearance when we consider accelerated objects. Inertial bodies are not accelerating and thus their trajectories have no curvature