9. Übungsblatt zur Linearen Algebra I (WS 20/21)

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Gruppe: F

Punkte: ___/___/___ Σ___

9.1 Aufgabe 1

Code Leo Knapp: rCJRV8

Code Joshua Detrois: xWYCmR

Code Juan Provencio: QKSYrt

9.2 Aufgabe 2

Geg.:

- $\mathcal{B} = (1, X, X^2, ..., X^n)$
- $\mathcal{B}' = ((1-X)^n, ..., (1-X)^2, (1-X)^1, 1)$

Nennen wir die Elementen aus der alten Basis b_m und aus der neuen Basis b'_n.

Die Basiswechselmatrix besteht aus den Einträgen a_{ij} mit welchen wir die alten Basisvektoren als Linearkombination der neuen Basisvektoren bilden:

$$b_j = a_{1j}b'_1 + a_{2j}b'_2 + a_{3j}b'_3 + ... \tag{1}$$

$$\begin{aligned}
b_1 &= 1 = 1 = 1b'_n && (2)\\
b_2 &= X = -(1-X) + 1 = -1b'_{n-1} + 1b'_n && (3)\\
b_3 &= X^2 = (1-X)^2 + 2X - 1 = 1b'_{n-2} + 2(-b'_{n-1} + b'_n) - 1(b'_n) && (4)\\
&= 1b'_{n-2} - 2b'_{n-1} + 1b'_n && (5)\\
b_4 &= X^3 = -(1-X)^3 + 3X^2 - 3X + 1 = -1b'_{n-3} + 3(b'_{n-2} + 2(-b'_{n-1} + b'_n) - b'_n) - 3(-b'_{n-1} + b'_n) + 1b && (6)\\
&= -1b'_{n-3} + 3b'_{n-2} - 3b'_{n-1} + 1b'_n && (7)\\
b_5 &= X^4 = (1-X)^4 + 4X^3 - 6X^2 + 4X - 1 = 1b'_{n-4} - 4b'_{n-3} + 6b'_{n-2} - 4b'_{n_1} 1b'_n && (8)\\
&... && (9)\\
b_m &= \sum_{k=0}^{m-1} \binom{m-1}{k} (b'_{n-m-1+k})(-1)^{k+1} && (10)
\end{aligned}$$

Wir erkennen sofort den Muster: Die Einträgen der Matrix müssen dem Pascalschen Dreieck folgen.

In Matrixdarstellung erhalten wir also:

$$Mat_{\mathcal{B}\mathcal{B}'}(\mathrm{id}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & 0 & ... & -1 & 1 \\ 0 & 0 & 0 & ... & 1 & -2 & 1 \\ 0 & 0 & ... & -1 & 3 & -3 & 1 \\ 0 & ... & 1 & -4 & 6 & -4 & 1 \end{pmatrix} \tag{11}$$

9.3 Aufgabe 3: Lösung eines linearen Gleichungssystems

Sei das folgende Gleichungssystem mit Einträgen $x_1, ..., x_7 \in \mathbb{F}_2$ gegeben:

$$A = \left(\begin{array}{ccccccc|c} [1] & [1] & [0] & [1] & [0] & [1] & [1] & [0] \\ [1] & [0] & [1] & [1] & [1] & [1] & [1] & [0] \\ [1] & [1] & [0] & [0] & [1] & [1] & [1] & [1] \\ [0] & [1] & [0] & [0] & [1] & [1] & [1] & [1] \\ [1] & [1] & [0] & [0] & [1] & [1] & [0] & [1] \\ [0] & [1] & [1] & [1] & [1] & [0] & [1] & [1] \end{array}\right) \longrightarrow z^{42} \tag{12}$$

$$\to \left(\begin{array}{ccccccc|c} [1] & [1] & [0] & [1] & [0] & [1] & [1] & [0] \\ [0] & [1] & [0] & [0] & [1] & [1] & [1] & [1] \\ [1] & [1] & [0] & [0] & [1] & [1] & [1] & [1] \\ [1] & [0] & [1] & [1] & [1] & [1] & [1] & [0] \\ [1] & [1] & [0] & [0] & [1] & [1] & [0] & [1] \\ [0] & [1] & [1] & [1] & [1] & [0] & [1] & [1] \end{array}\right) \longrightarrow z^{31}(1) \tag{13}$$

$$\to \left(\begin{array}{ccccccc|c} [1] & [1] & [0] & [1] & [0] & [1] & [1] & [0] \\ [0] & [1] & [0] & [0] & [1] & [1] & [1] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [0] & [1] \\ [1] & [0] & [1] & [1] & [1] & [1] & [1] & [0] \\ [1] & [1] & [0] & [0] & [1] & [1] & [0] & [1] \\ [0] & [1] & [1] & [1] & [1] & [0] & [1] & [1] \end{array}\right) \longrightarrow z^{41}(1) \tag{14}$$

$$\to \left(\begin{array}{ccccccc|c} [1] & [1] & [0] & [1] & [0] & [1] & [1] & [0] \\ [0] & [1] & [0] & [0] & [1] & [1] & [1] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [0] & [1] \\ [0] & [1] & [1] & [0] & [1] & [0] & [0] & [0] \\ [1] & [1] & [0] & [0] & [1] & [1] & [0] & [1] \\ [0] & [1] & [1] & [1] & [1] & [0] & [1] & [1] \end{array}\right) \longrightarrow z^{51}(1) \tag{15}$$

$$\to \left(\begin{array}{ccccccc|c} [1] & [1] & [0] & [1] & [0] & [1] & [1] & [0] \\ [0] & [1] & [0] & [0] & [1] & [1] & [1] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [0] & [1] \\ [0] & [1] & [1] & [0] & [1] & [0] & [0] & [0] \\ [0] & [0] & [0] & [1] & [1] & [0] & [1] & [1] \\ [0] & [1] & [1] & [1] & [1] & [0] & [1] & [1] \end{array}\right) \longrightarrow z^{42}(1) \tag{16}$$

$$\to \left(\begin{array}{ccccccc|c} [1] & [1] & [0] & [1] & [0] & [1] & [1] & [0] \\ [0] & [1] & [0] & [0] & [1] & [1] & [1] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [0] & [1] \\ [0] & [0] & [1] & [0] & [0] & [1] & [1] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [1] & [1] \\ [0] & [1] & [1] & [1] & [1] & [0] & [1] & [1] \end{array}\right) \longrightarrow z^{34} \tag{17}$$

$$\to \left(\begin{array}{ccccccc|c} [1] & [1] & [0] & [1] & [0] & [1] & [1] & [0] \\ [0] & [1] & [0] & [0] & [1] & [1] & [1] & [1] \\ [0] & [0] & [1] & [0] & [0] & [1] & [1] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [0] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [1] & [1] \\ [0] & [1] & [1] & [1] & [1] & [0] & [1] & [1] \end{array}\right) \longrightarrow z^{62}(1) \tag{18}$$

$$\to \left(\begin{array}{ccccccc|c} [1] & [1] & [0] & [1] & [0] & [1] & [1] & [0] \\ [0] & [1] & [0] & [0] & [1] & [1] & [1] & [1] \\ [0] & [0] & [1] & [0] & [0] & [1] & [1] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [0] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [1] & [1] \\ [0] & [0] & [1] & [1] & [0] & [1] & [0] & [0] \end{array}\right) \longrightarrow z_{63}(1) \tag{19}$$

$$\to \left(\begin{array}{ccccccc|c} [1] & [1] & [0] & [1] & [0] & [1] & [1] & [0] \\ [0] & [1] & [0] & [0] & [1] & [1] & [1] & [1] \\ [0] & [0] & [1] & [0] & [0] & [1] & [1] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [0] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [1] & [1] \\ [0] & [0] & [0] & [1] & [0] & [0] & [1] & [1] \end{array}\right) \longrightarrow z^{54}(1) \tag{20}$$

$$\to \left(\begin{array}{ccccccc|c} [1] & [1] & [0] & [1] & [0] & [1] & [1] & [0] \\ [0] & [1] & [0] & [0] & [1] & [1] & [1] & [1] \\ [0] & [0] & [1] & [0] & [0] & [1] & [1] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [0] & [1] \\ [0] & [0] & [0] & [0] & [0] & [0] & [1] & [0] \\ [0] & [0] & [0] & [1] & [0] & [0] & [1] & [1] \end{array}\right) \longrightarrow z^{64}(1) \tag{21}$$

$$\to \left(\begin{array}{ccccccc|c} [1] & [1] & [0] & [1] & [0] & [1] & [1] & [0] \\ [0] & [1] & [0] & [0] & [1] & [1] & [1] & [1] \\ [0] & [0] & [1] & [0] & [0] & [1] & [1] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [0] & [1] \\ [0] & [0] & [0] & [0] & [0] & [0] & [1] & [0] \\ [0] & [0] & [0] & [0] & [1] & [0] & [1] & [0] \end{array}\right) \longrightarrow z^{56}(1) \tag{22}$$

$$\to \left(\begin{array}{ccccccc|c} [1] & [1] & [0] & [1] & [0] & [1] & [1] & [0] \\ [0] & [1] & [0] & [0] & [1] & [1] & [1] & [1] \\ [0] & [0] & [1] & [0] & [0] & [1] & [1] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [0] & [1] \\ [0] & [0] & [0] & [0] & [1] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [1] & [0] & [1] & [0] \end{array}\right) \longrightarrow z^{65}(1) \tag{23}$$

$$\to \left(\begin{array}{ccccccc|c} [1] & [1] & [0] & [1] & [0] & [1] & [1] & [0] \\ [0] & [1] & [0] & [0] & [1] & [1] & [1] & [1] \\ [0] & [0] & [1] & [0] & [0] & [1] & [1] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [0] & [1] \\ [0] & [0] & [0] & [0] & [1] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] & [0] & [1] & [0] \end{array}\right) \longrightarrow z_{12}(1) \tag{24}$$

$$\to \left(\begin{array}{ccccccc|c} [1] & [0] & [0] & [1] & [1] & [0] & [0] & [1] \\ [0] & [1] & [0] & [0] & [1] & [1] & [1] & [1] \\ [0] & [0] & [1] & [0] & [0] & [1] & [1] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [0] & [1] \\ [0] & [0] & [0] & [0] & [1] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] & [0] & [1] & [0] \end{array}\right) \longrightarrow z_{14}(1) \tag{25}$$

$$\to \left(\begin{array}{ccccccc|c} [1] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\ [0] & [1] & [0] & [0] & [1] & [1] & [1] & [1] \\ [0] & [0] & [1] & [0] & [0] & [1] & [1] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [0] & [1] \\ [0] & [0] & [0] & [0] & [1] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] & [0] & [1] & [0] \end{array}\right) \longrightarrow z^{25}(1) \tag{26}$$

$$\to \left(\begin{array}{ccccccc|c} [1] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\ [0] & [1] & [0] & [0] & [0] & [1] & [1] & [1] \\ [0] & [0] & [1] & [0] & [0] & [1] & [1] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [0] & [1] \\ [0] & [0] & [0] & [0] & [1] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] & [0] & [1] & [0] \end{array}\right) \longrightarrow z^{26}(1) \tag{27}$$

$$\to \left(\begin{array}{ccccccc|c} [1] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\ [0] & [1] & [0] & [0] & [0] & [1] & [0] & [1] \\ [0] & [0] & [1] & [0] & [0] & [1] & [1] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [0] & [1] \\ [0] & [0] & [0] & [0] & [1] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] & [0] & [1] & [0] \end{array}\right) \longrightarrow z^{36}(1) \tag{28}$$

$$\to \left(\begin{array}{ccccccc|c} [1] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\ [0] & [1] & [0] & [0] & [0] & [1] & [0] & [1] \\ [0] & [0] & [1] & [0] & [0] & [1] & [0] & [1] \\ [0] & [0] & [0] & [1] & [1] & [0] & [0] & [1] \\ [0] & [0] & [0] & [0] & [1] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] & [0] & [1] & [0] \end{array}\right) \longrightarrow z^{45}(1) \tag{29}$$

$$\to \left(\begin{array}{ccccccc|c} [1] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\ [0] & [1] & [0] & [0] & [0] & [1] & [0] & [1] \\ [0] & [0] & [1] & [0] & [0] & [1] & [0] & [1] \\ [0] & [0] & [0] & [1] & [0] & [0] & [0] & [1] \\ [0] & [0] & [0] & [0] & [1] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] & [0] & [1] & [0] \end{array}\right) \tag{30}$$

Daraus folgt, dass

$$x_1 = [0] \tag{31}$$
$$x_2 + x_6 = [1] \tag{32}$$
$$x_3 + x_6 = [1] \tag{33}$$
$$x_4 = [1] \tag{34}$$
$$x_5 = [0] \tag{35}$$
$$x_7 = [0] \tag{36}$$

Das heißt, wir haben zwei verschiedene Lösungen. Enweder $x_6 = [1]$, dann sind $x_2 = [0]$ und $x_3 = [0]$, oder $x_6 = [0]$, dann sind $x_2 = [1]$ und $x_3 = [1]$

Unser Lösungsraum sieht so aus:

$$\text{Lös}(A,b) = \begin{pmatrix}0\\0\\0\\1\\0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\1\\1\\1\\0\\0\\0\end{pmatrix} \tag{37}$$

9.4 Aufgabe 4: Bild und Kern bestimmen mit Matrixdarstellung

a) B und C wähle ich als die jeweilige Standardbasis

$$\begin{aligned} f((1,0,0)) &= (1,0) \\ f((0,1,0)) &= (1,1) \\ f((0,0,1)) &= (-1,-1) \\ \Rightarrow Mat_{BC}(f) &= \begin{pmatrix}1 & 1 & -1\\0 & 1 & -1\end{pmatrix} \end{aligned}$$

b) Ich subtrahiere die zweite Zeile von der Ersten.

$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & -1\end{pmatrix} \tag{38}$$

$$\Rightarrow \text{rang}(f) = 2 \tag{39}$$

c) Lässt sich direkt aus der b) ablesen
$\ker f = \{v \in \mathbb{R}^3 \mid v_1 = 0 \wedge v_2 = -v_3\}$

d) Das Bild von f ist $\mathbb{R}^2$

dim(Bild(f))=2
$\Rightarrow \left\{\binom{1}{0}, \binom{0}{1}\right\}$ ist Basis von Bild(f)