Experiment 211 **March 9, 2022**

Coupled pendulums

Beginner's Practica for Physicists II

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Contents

1 Goal of the experiment

The motivation for this experiment is to analyze and better understand the system of a coupled oscillator by looking at the example of two coupled pendulums. For that we will determine the eigenfrequencies of symmetrical and asymmetrical oscillation as well as the mixed frequency composed of the former two. We will also determine the coupling strength depending on the position of the spring.

2 Foundations

2.1 Coupled harmonic oscillation

As a simplified model we offer the regular harmonic oscillation (HO), in which we can relate the moment of inertia J and the momentum D through the differential equation of a simple HO

$$J\ddot{\phi} = -D\phi \tag{1}$$

with the frequency $\omega = \sqrt{\frac{D}{J}}$.

Assuming we couple two pendulums with a spring with the momentum $D' = D_F d^2$, whereas D_F is the spring constant and d is the length from which the spring is attached to to the pendulum axis, we will observe an additional torque on the springs which contribute to our initial differential equation

$$M_1 = D'(\phi_2 - \phi_1) \tag{2}$$
$$M_2 = D'(\phi_1 - \phi_2) \tag{3}$$

The result is a system of coupled differential equations

$$J\ddot{\phi}_1 = -D\phi_1 + D'(\phi_2 - \phi_1) \tag{4}$$
$$J\ddot{\phi}_2 = -D\phi_2 + D'(\phi_1 - \phi_2) \tag{5}$$

which after an appropriate substitution can be solved through the following:

$$\phi_1(t) = \frac{1}{2}(A_1 \cos\omega_1 t + B_1 \sin\omega_1 t + A_2 \cos\omega_2 t + B_2 \sin\omega_2 t) \qquad |\omega_1 = \sqrt{\frac{D}{J}} \tag{6}$$
$$\phi_2(t) = \frac{1}{2}(A_1 \cos\omega_1 t + B_1 \sin\omega_1 t - A_2 \cos\omega_2 t - B_2 \sin\omega_2 t) \qquad |\omega_2 = \sqrt{\frac{D+2D'}{J}} \tag{7}$$

(8)

These are the general solutions to every set of complicated motion in the system, but we want to focus on the following special cases for given starting conditions.

2.1.1 Symmetrical oscillation

In this example, both pendulums are displaced parallel to each other and let go simultaneously.

$$\phi_1(0) = \phi_2(0) = \phi_0 \tag{9}$$

$$\dot{\phi}_1(0) = \dot{\phi}_2(0) = 0 \tag{10}$$

and we obtain the following values for the constants A_i, B_i

$$A_1 = 2\phi_0, \qquad A_2 = B_1 = B_2 = 0 \tag{11}$$

and the solution

$$\phi_1 = \phi_2 = \phi_0 \cos \omega_1 t \tag{12}$$

we can appreciate that this solution doesn't depend on the momentum D' exerted by the spring. This motion represents left on Figure 1.

2.1.2 Asymmetrical oscillation

In this case, both pendulums are extended in opposite directions and let go simultaneously:

$$\phi_1(0) = -\phi_2(0) = \phi_0 \tag{13}$$

$$\dot{\phi}_1(0) = \dot{\phi}_2(0) = 0 \tag{14}$$

which gives the constants

$$A_2 = 2\phi_0, \qquad A_1 = B_1 = B_2 = 0 \tag{15}$$

and the solution for the motion

$$\phi_1 = -\phi_2 = \phi_0 \cos \omega_2 t \tag{16}$$

The pendulums oscillate out of phase but with the same frequency.

2.1.3 Beat oscillation

For the beat oscillation we will initially hold one pendulum in place while we extend the other one and let go.

$$\phi_1 = 0 \tag{17}$$
$$\phi_2 = \phi_0 \tag{18}$$
$$\dot{\phi}_1 = \dot{\phi}_2 = 0 \tag{19}$$

with the constants

$$A_1 = -A_2 = \phi_0, \qquad B_1 = B_2 = 0 \tag{20}$$

For the solutions we get a set of more complicated frequency dependancies

$$\phi_1 = \phi_0 \sin\left(\frac{\omega_2 - \omega_1}{2} t\right) \sin\left(\frac{\omega_1 + \omega_2}{2} t\right) \tag{21}$$
$$\phi_2 = \phi_0 \sin\left(\frac{\omega_2 - \omega_1}{2} t\right) \sin\left(\frac{\omega_1 + \omega_2}{2} t\right) \tag{22}$$

The frequencies

$$\omega_I = \frac{1}{2}(\omega_1 + \omega_2) \tag{23}$$
$$\omega_{II} = \frac{1}{2}(\omega_2 - \omega_2) \tag{24}$$

describe the frequency (ω_I) at which each pendulum oscillates and the beat frequency (ω_{II}) at which the energy of the pendulums oscillates.

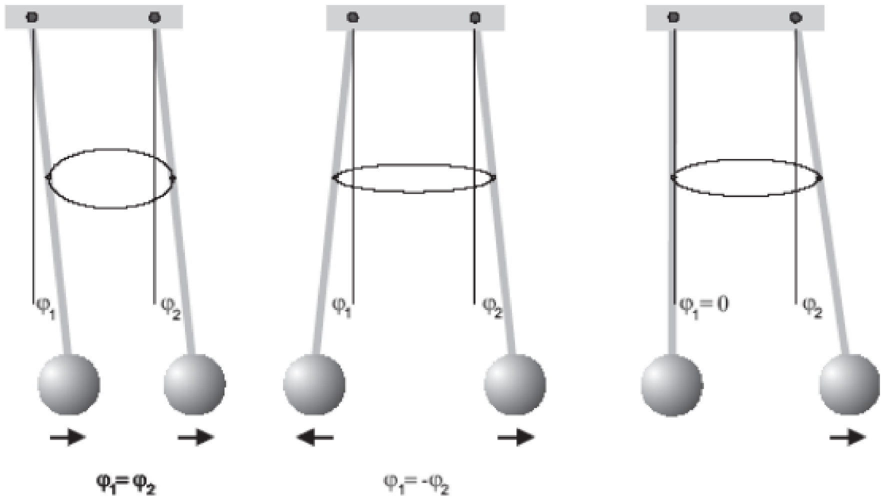


Figure 1: *left*: symmetrical oscillation, *center*: asymmetrical oscillation, *right*: beat oscillation

2.2 Coupling strength

To quantisize the strength of the coupling we introduce the appropriately named coupling strength κ, which can be calculated using both momentums D and D' as

$$\kappa = \frac{D'}{D + D'} \tag{25}$$

and using the definitions for ω_1 and ω_2 we get

$$\kappa = \frac{\omega_2^2 - \omega_1^2}{\omega_1^2 + \omega_2^2} \tag{26}$$

Under the assumption that we are dealing with a weak coupling $D \gg D'$ we can simplify (25) to

$$\kappa \approx \frac{D'}{D} = \frac{\omega_2^2 - \omega_1^2}{2\omega_1^2} = \frac{\omega_2^2}{\omega_1^2} - \frac{1}{2} \tag{27}$$

and (**??**)

For the different distance depending D', we have a proportionality to the square of the distance, so we can also determine the relationship between two coupling strengths using our measured frequencies and compare them to the expected relationship according to the distances.

$$\frac{\kappa_i}{\kappa_j} = \frac{d_i^2}{d_j^2} \tag{28}$$

3 Experimental setup

3.1 Materials and equipment

- Two brass pendulums
- Coupling spring
- Magnetic angle-measurement recorder
- Analog-digital converter
- PC

3.2 Setup

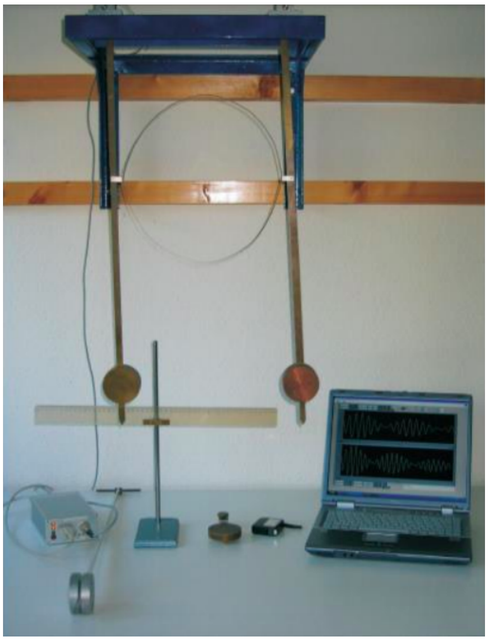
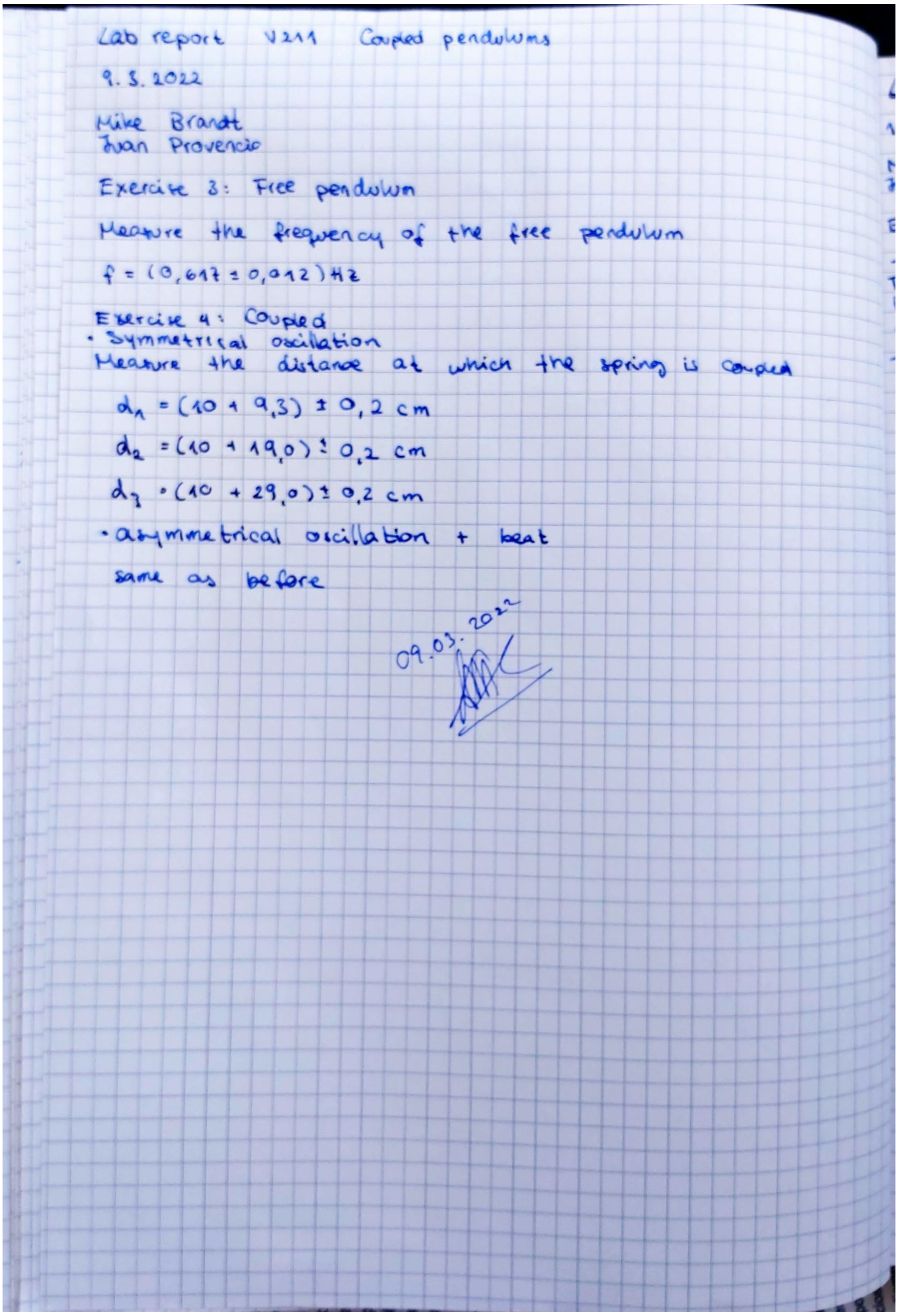


Figure 2: Experimental setup

4 Measurement and evaluation

4.1 Lab report



Lab report V211 Coupled pendulums

9.3.2022

Mike Brandt
Juan Provencio

Exercise 3: Free pendulum

Measure the frequency of the free pendulum

$f = (0{,}617 \pm 0{,}012)\,\mathrm{Hz}$

Exercise 4: Coupled

- Symmetrical oscillation

Measure the distance at which the spring is coupled

$d_1 = (10 + 9{,}3) \pm 0{,}2\,\mathrm{cm}$

$d_2 = (10 + 19{,}0) \pm 0{,}2\,\mathrm{cm}$

$d_3 = (10 + 29{,}0) \pm 0{,}2\,\mathrm{cm}$

- asymmetrical oscillation + beat

same as before

09.03.2022

Figure 3: Lab report

In addition to this, we printed out for each configuration of distance and coupling modus (d_1, d_2, d_3 and symmetrical, asymmetrical and beat) the corresponding frequency spectrum and gaussian fit.

4.1.1 Symmetrical oscillation

In this case, the pendulums remain at a constant distance to each other and swing in unison so that no extra force is exerted by the coupling spring.

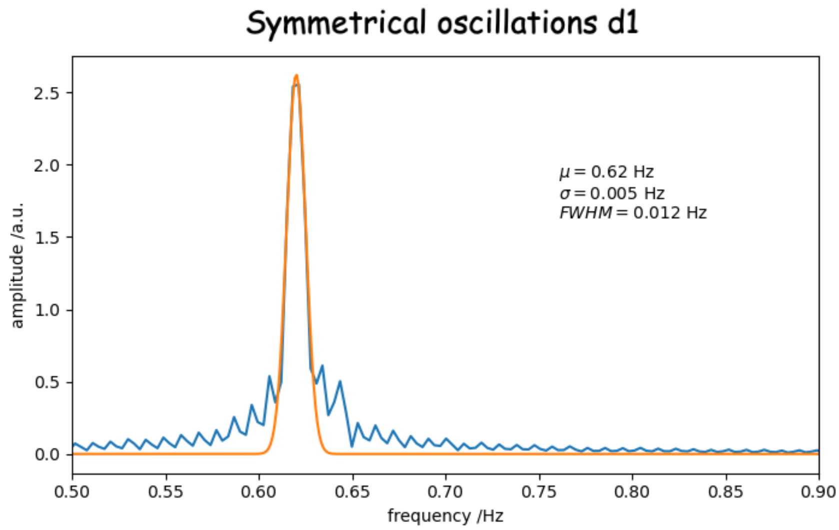


Diagramm 1: Symmetrical oscillations at distance d_1

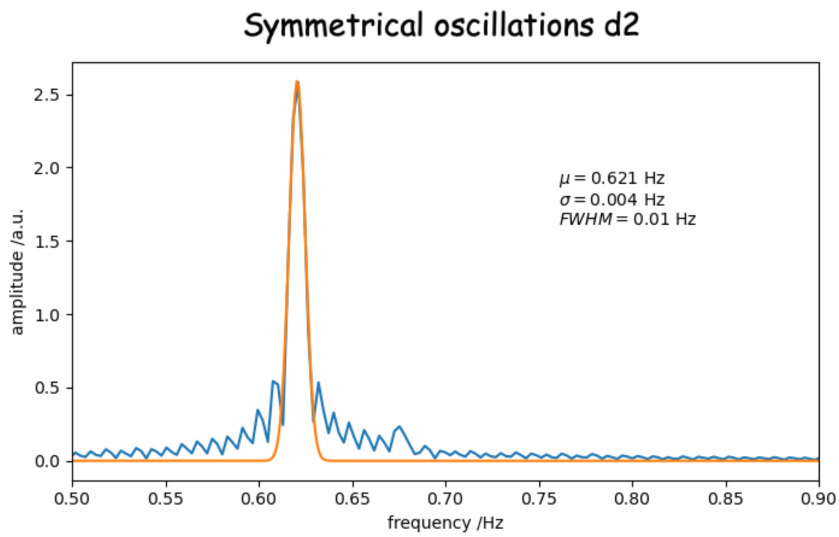


Diagramm 2: Symmetrical oscillations at distance d_2

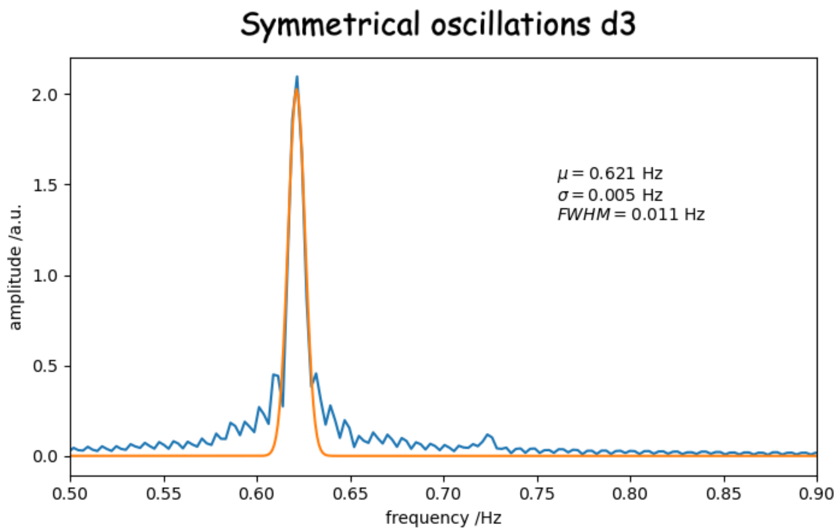


Diagramm 3: Symmetrical oscillations at distance d_3

4.1.2 Asymmetrical oscillation

The pendulums are extended in opposite directions and let go simultaneously.

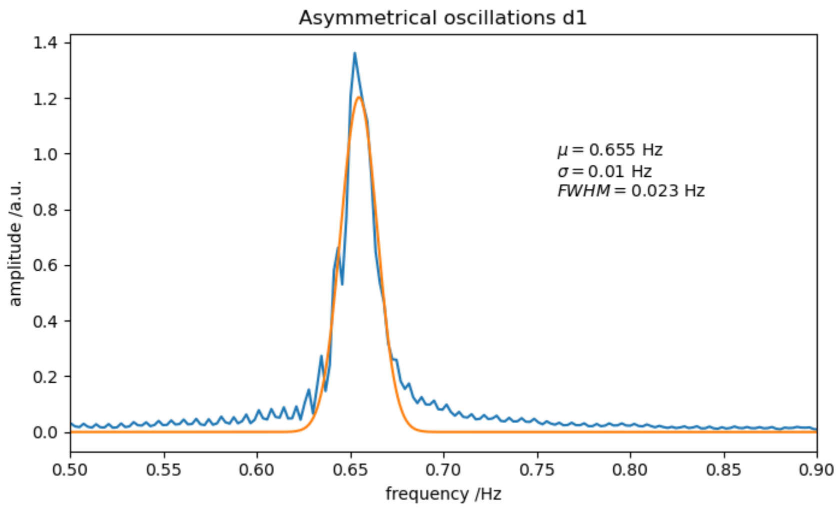


Diagramm 4: Asymmetrical oscillations at distance d_1

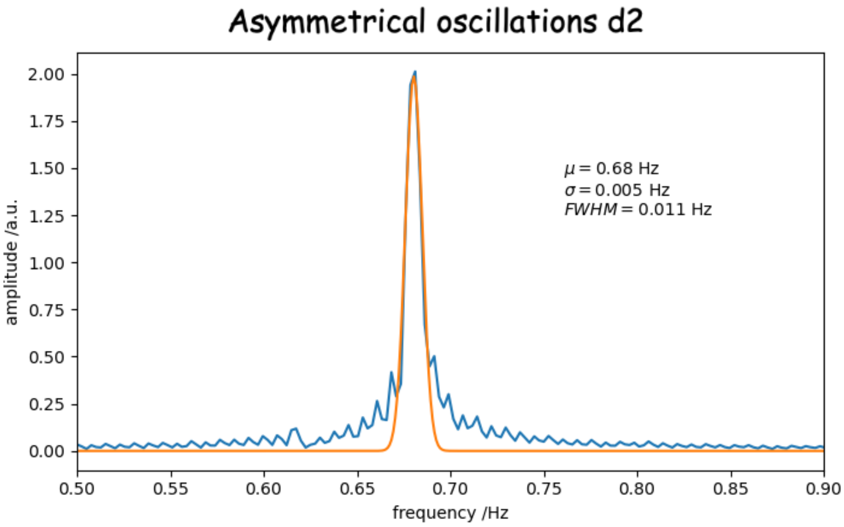


Diagramm 5: Asymmetrical oscillations at distance d_2

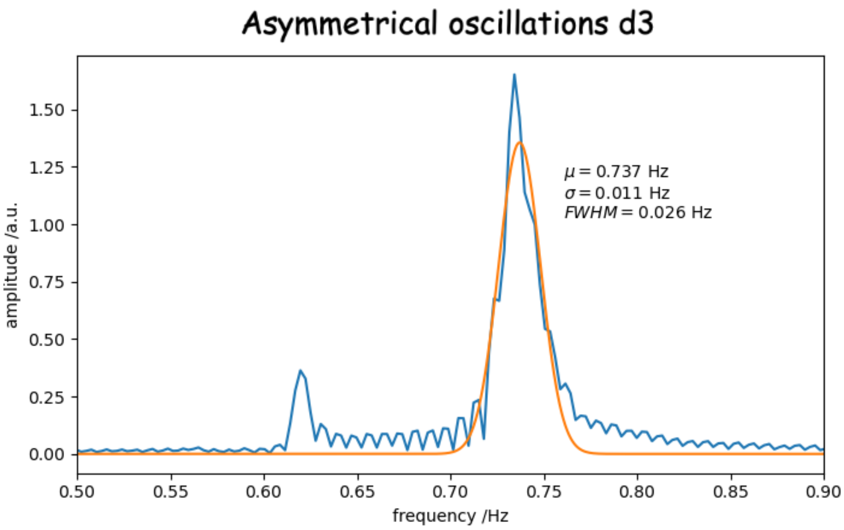


Diagramm 6: Asymmetrical oscillations at distance d_3

4.1.3 Beat oscillation

One pendulum is held in position and the other is extended. As the initially swinging pendulum is let go, kinetic energy is transfered to the standing pendulum so that it too comes to a swinging while the original one slows down and comes to a stop as the process begins again.

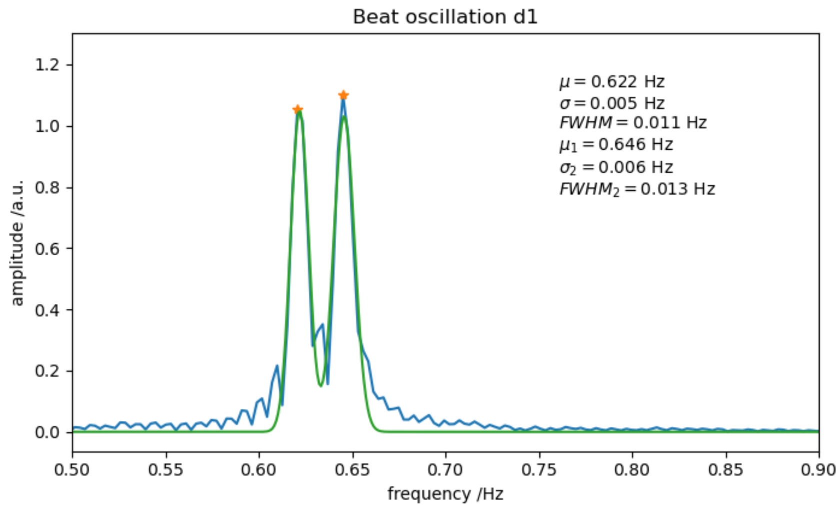


Diagramm 7: Beatoscillations at distance d_1

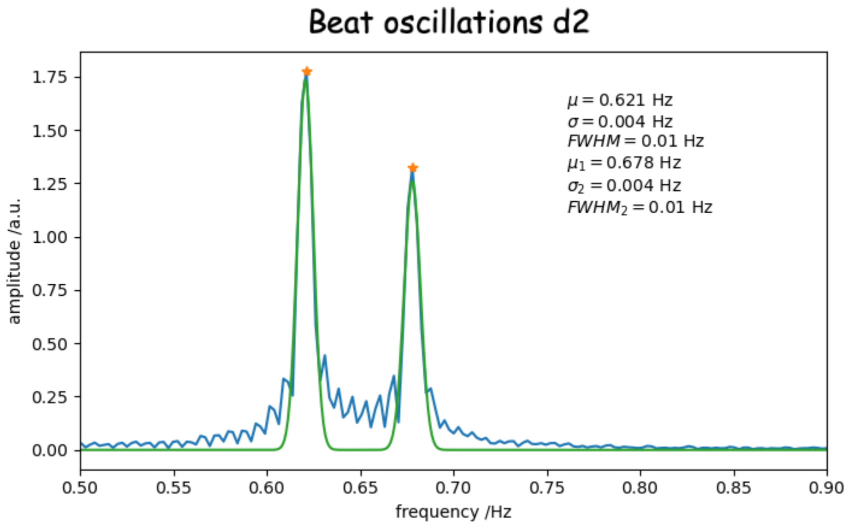


Diagramm 8: Beatoscillations at distance d_2

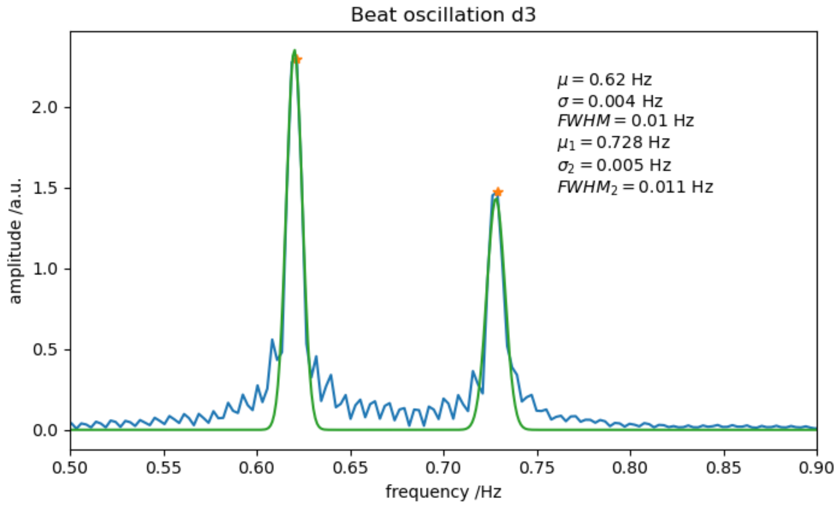


Diagramm 9: Beatoscillations at distance d_3

4.1.4 Inductive coupling

Not successfully downloaded would also be pictures of the oscilloscope by different positions of inductive coupling. We could observe differences in the oscillogram dependent on the distance between the two induction coils. The graph apparead to be that of a dampened harmonic oscillator. For a larger separation the intensity was dampened greatly.

4.2 Evaluation

In the following we will use gaussian error propagation to calculate the uncertainty of our measurements. This bases on the idea that the errors are relatively small and the factors in a formula are independent of each other. For trivial cases this will not be done explicitly in the evaluation, but the exact formulas can still be consulted in the Annex where the Python code is.

4.2.1 Frequency

Given that the output of the gaussian fit is the frequency f, we will proceed to first determine the wanted frequencies as opposed to the angular frequencies according to the introduction. For this we will take into account the proportionality factor $\omega = 2\pi f$.

First of all, we will determine the mixed frequency $f_{I,II}$ depending on the pure frequencies f_{12} of the symmetrical and asymmetrical oscillation respetively according to equations

(23) and (24) of the introduction:

$$f_I = \frac{1}{2}(f_1 + f_2) \tag{29}$$

$$f_{II} = \frac{1}{2}(f_2 - f_1) \tag{30}$$

Results are displayed in the following table.

Table 1: Frequencies of different couplings

		d_1	d_2	d_2
Symmetrical:	f_1 [Hz]	0,620(5)	0,621(5)	0,621(5)
Asymmetrical:	f_2 [Hz]	0,655(10)	0,680(5)	0,737(11)
Beat:	f_{1b} [Hz]	0,622(5)	0,621(4)	0,620(4)
	f_{2b} [Hz]	0,646(6)	0,678(4)	0,728(6)
Mixed:	f_I [Hz]	0,634(4)	0,650(3)	0,674(4)
	f_{II} [Hz]	0,012(4)	0,029(3)	0,054(4)
	$f_{\mathrm{I,theo}}$ [Hz]	0,638(6)	0,651(4)	0,679(6)
	$f_{\mathrm{II,theo}}$ [Hz]	0,018(6)	0,030(4)	0,058(6)

4.2.2 Coupling strength

Up next we will determine the coupling strength according to (27). Similarly, we can determine it using the frequency, since the factors of $(2\pi)^2$ cancel each other out.

$$\kappa = \frac{f_2^2 - f_1^2}{2f_1^2} = \frac{f_2^2}{2f_1^2} - \frac{1}{2} \tag{31}$$

$$\Delta\kappa = \sqrt{\left(\frac{f_2 \cdot \Delta f_2}{f_1^2}\right)^2 + \left(\frac{f_2^2 \cdot \Delta f_1}{f_1^3}\right)^2} \tag{32}$$

Analogously, we will calculate the coupling strength using the frequencies of the symmetrical and asymmetrical oscillations, as well as the symmetrical and asymmetrical components of the beat oscillation (indexed with $f_{s,1/2}$). We get following results

Table 2: Coupling strength at different positions of the spring

	d_1	d_2	d_3
κ	0,058(19)	0,100(13)	0,204(24)
κ_s	0,039(13)	0,096(10)	0,189(14)

As per the assumption used to derive the formula for the coupling strength, we obtain a proportionality relation to the distance at which the spring was situated. In this case is the following relationship valid:

$$\frac{\kappa_i}{\kappa_j} = \frac{d_i^2}{d_j^2} \tag{33}$$

$$\Delta\left(\frac{\kappa_i}{\kappa_j}\right) = \sqrt{\left(\frac{\Delta\kappa_i}{\kappa_j}\right)^2 + \left(\frac{\kappa_i \cdot \Delta\kappa_j}{\kappa_j^2}\right)} \tag{34}$$

$$\Delta\left(\frac{d_i^2}{d_j^2}\right) = \sqrt{\left(\frac{2d_i \cdot \Delta d_i}{d_j^2}\right)^2 + \left(\frac{2d_i^2 \cdot \Delta d_j}{d_j^3}\right)^2} \tag{35}$$

In the discussion we will take a closer look at the discrepancies of both of the coupling strengths, but as a short spoiler, the σ-deviation between them is smaller than one, so for the further comparison we will only take a look at the not indexed κ.

Table 3: Quotient of coupling strengths and squared distances

i / j	$\frac{\kappa_i}{\kappa_j}$	$\frac{d_i^2}{d_j^2}$
1 / 2	0,58(21)	0,443(11)
1 / 3	0,28(10)	0,245(6)
2 / 3	0,49(9)	0,553(10)

5 Recap and discussion

5.1 Recap

In this experiment we took a look at a coupled system of pendulums swinging in different setups. For this task, we initially measured the free oscillation of the pendulum to get acquainted with the measuring and fitting programm. Despite that, we still forgot to click refresh on the "title-setting-button" for the gaussian fit a few times.

On the pendulums were three small holes at specific heights to adjust the brass spring and coupling strength. At each of the positions, we measured the frequency for a symmetrical, an asymmetrical and a beat coupling.

5.2 Discussion

To begin we want to check very quickly, that the free frequency of the uncpupled pendulum identical is with the one of the coupled symmetrical oscillation and that we had no greater influence on the measurement. To calculate this and the following σ-deviations we will use the following formula for a value G and comparison value G_{Lit}:

$$\frac{|G - G_{\text{Lit}}|}{\sqrt{(\Delta G)^2 + (\Delta G_{\text{Lit}})^2}} \tag{36}$$

Table 4: Comparison between free pendulum and symmetrical oscillation

Position	f_1 (sym.) [Hz]	$f_{s,1}$ (sym., beat) [Hz]	f_{free} [Hz]	$\sigma_{f_1,f_{\text{free}}}$	$\sigma_{f_1,f_{\text{free}}}$
d_1	0,620(5)	0,622(5)	0,620(5)	0	0,28
d_2	0,621(5)	0,621(4)		0,14	0,16
d_3	0,621(5)	0,620(4)		0,14	0

We are pleased to check that all of the values lie within a negligible σ-deviation and conclude that no systematic error was performed during the process. Now we move on to the more interesting comparisons.

Up next we measured with both symmetric frequencies the mixed frequencies f_I and f_{II}, which describe the frequency with which each pendulum oscillates and the frequency at which the energy between oscillates.

Table 5: Comparison between mixed frequencies

		d_1	d_2	d_2
Mixed:	f_I [Hz]	0,634(4)	0,650(3)	0,674(4)
	f_{II} [Hz]	0,012(4)	0,029(3)	0,054(4)
	$f_{\text{I, theo}}$ [Hz]	0,638(6)	0,651(4)	0,679(6)
	$f_{\text{II, theo}}$ [Hz]	0,018(6)	0,030(4)	0,058(6)
$f_I - f_{\text{I, theo}}$ [Hz]		-0,0035	-0,001	-0,005
$f_{II} - f_{\text{II, theo}}$ [Hz]		-0,0055	-0,001	-0,004
σ_I		0,5	0,22	0,7
σ_{II}		0,8	0,22	0,6

With these results, we can observe the rather unspectacular σ-deviation of under 1 for each measurement, which strengthens our confidence in a well executed experiment. The small deviation also doesn't lie in an exaggerated error margin, because the relative error is under 1% for all the measurements of the frequency f_I and while it is rather large for the frequency f_{II}, this lies solely in the fact that it itself lies in the range of the error calculated by Python. What is interesting or maybe slightly concerning is the signed absolute difference between the experimental and theoretical values. We can observe that the experimental frequencies f_I and f_{II} are in all cases slightly smaller than their theoretical counterparts so the difference is negative. Although this isn't a large enough sample to make a clear judgement about it, we can in fact conclude that there is a small systematic error. In this case f_{II} describes the frequency at which the energy between the two pendulums oscillates, it could be that given that there are small losses to friction and air resistance for which we did not account in the evaluation, the theoretical frequency f_{II} (calculated using both of the beat peak frequencies) would be smaller than expected. A similar thought can be expressed about the frequency f_I. Still, this effect appears to be negligible enough to not have a significant effect on the results.

Next we calculated the coupling strength, likewise for the (a)symmetrical and for the beat oscillation.

Table 6: Comparison of coupling strength

	d_1	d_2	d_3
κ	0,058(19)	0,100(13)	0,204(24)
κ_s	0,039(13)	0,096(10)	0,189(14)
σ	0,8	0,21	0,5

The σ-deviation is similarly under the 1 σ-distance for each measurement, so we can be satisfied with these results as well.

Finally we'll compare how our assumption for the sake of simplicity holds up and if the quotient of the coupling strengths and that of the distances squared are compatible with each other. In this case we have the following results

Table 7: Quotient of coupling strengths and squared distances

i / j	$\frac{\kappa_i}{\kappa_j}$	$\frac{d_i^2}{d_j^2}$	σ
1 / 2	0,58(21)	0,443(11)	0,7
1 / 3	0,28(10)	0,245(6)	0,4
2 / 3	0,59(9)	0,553(10)	0,7

This again proves to be satisfactory with small σ-deviations, however here it's more obvious that the large error margins are responsible for that. To reduce this error there's little to be done, since at least the more significant part was entirely calculated by the python programm.

A further source for errors during this experiment could probably be the small oscillations in the perpendicular plane. It wasn't possible to keep the motion of the pendulums stricyl on the $x-z$ plane, if z is the vertical and x the horiontal coordinate. A small perturbation in the y-direction was unavoidable and the effect that this could have on the measurements is worth mentioning, but probably negligible.

In short, the results of this experiment show no significant deviations from the expectations

6 Sources

Wagner, J., Universität Heidelberg (2021). Physikalisches Praktikum PAP 2.1 für Studierende der Physik B.Sc..

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March 22, 2022

7 Annex

[5]:
```
import matplotlib.pyplot as plt
import matplotlib.mlab as mlab
%matplotlib inline
import numpy as np
from numpy import exp, sqrt, log, pi
from scipy.optimize import curve_fit
from scipy.stats import chi2
from scipy import odr
from scipy.integrate import quad
from scipy.signal import find_peaks
from scipy.signal import argrelextrema, argrelmin, argrelmax


def fehler(name, G, sig_G, G_lit, sig_G_lit):
    print(name)
    print('Relativer Fehler: ', sig_G / G * 100)
    print('Rel. Fehler (Vergleich):', sig_G_lit / G_lit * 100)
    print('Absoluter Fehler: ', G - G_lit)
#    print('Verhältnis:', G / G_lit)
    print('Sigma-Abweichung: ', np.abs(G - G_lit) / sqrt(sig_G ** 2
                                                            + sig_G_lit **␣
 ↪2),'\n')


def fehler_small(name, G, sig_G):
```

```
    print(name)
    print('Relativer Fehler: ', sig_G / G * 100)


def ergebnis(name, G, sig_G, komma, einheit):
    print(name + ' =', np.round(G, komma), '+/-', np.round(sig_G,␣
 ↪komma), einheit)


def ergebnis_large(name, G, sig_G, komma, einheit):
    print(name + ' =', np.round(G, komma))
    print('+/-'.rjust(len(name) + 2), np.round(sig_G, komma), einheit)


def fitparameter(name, G, sig_G, komma, einheit):
    return name + ' =' + str(np.round(G, komma)) + '$\pm$' + str(np.
 ↪round(sig_G, komma)) + einheit
```

Frequencies

[6]:
```
# Experimental values
f_free = 0.62 # Hz
sig_f_free = 0.005

f_1 = np.array([0.620, 0.621, 0.621])
sig_f_1 = np.ones(3) * 0.005

f_2 = np.array([0.655, 0.680, 0.737])
sig_f_2 = np.array([0.010, 0.005, 0.011])

f_s1 = np.array([0.622, 0.621, 0.620])
sig_f_s1 = np.array([0.005, 0.004, 0.004])

f_s2 = np.array([0.646, 0.678, 0.728])
sig_f_s2 = np.array([0.006, 0.004, 0.006])

f_I = 1 / 2 * (f_s1 + f_s2)
```

```
f_II = 1 / 2 * (f_s2 - f_s1)
sig_f = 1 / 2 * sqrt( (sig_f_s1) ** 2 + (sig_f_s2) ** 2)

# Theoretical values
f_I_theo = 1 / 2 * (f_1 + f_2)
f_II_theo = 1 / 2 * (f_2 - f_1)
sig_f_theo = 1 / 2 * sqrt( (sig_f_1) ** 2 + (sig_f_2) ** 2 )

print('Experimentell:')
ergebnis_large('f_I', f_I, sig_f, 3, '[Hz]')
ergebnis_large('f_II', f_II, sig_f, 3, '[Hz]')

print('Theoretisch:')
ergebnis_large('f_I_theo', f_I_theo, sig_f_theo, 3, '[Hz]')
ergebnis_large('f_II_theo', f_II_theo, sig_f_theo, 3, '[Hz]')
```

```
Experimentell:
f_I = [0.634 0.65  0.674]
  +/- [0.004 0.003 0.004] [Hz]
f_II = [0.012 0.029 0.054]
   +/- [0.004 0.003 0.004] [Hz]
Theoretisch:
f_I_theo = [0.638 0.651 0.679]
       +/- [0.006 0.004 0.006] [Hz]
f_II_theo = [0.018 0.03  0.058]
        +/- [0.006 0.004 0.006] [Hz]
```

Coupling strengths

[7]:
```
kappa = f_2 ** 2 / (2 * f_1 ** 2) - 1 / 2
sig_kappa = sqrt( (f_2 * sig_f_2 / (f_1 ** 2)) ** 2
                 + (f_2 ** 2 * sig_f_1 / (f_1 ** 3)) ** 2 )

kappa_s = f_s2 ** 2 / (2 * f_s1 ** 2) - 1 / 2
sig_kappa_s = sqrt( (f_s2 * sig_f_s2 / (f_s1 ** 2)) ** 2
                 + (f_s2 ** 2 * sig_f_s1 / (f_s1 ** 3)) ** 2 )
```

```
ergebnis_large('kappa', kappa, sig_kappa, 3, '')
ergebnis_large('kappa_s', kappa_s, sig_kappa_s, 3, '')
```

```
kappa = [0.058 0.1   0.204]
    +/- [0.019 0.013 0.024]
kappa_s = [0.039 0.096 0.189]
      +/- [0.013 0.01  0.014]
```

Coupling quotients

[8]:
```
d = 10 + np.array([9.3, 19, 29])
sig_d = 0.2 * np.ones(3)

v_kappa = np.array([])
sig_v_kappa = np.array([])
v_d = np.array([])
sig_v_d = np.array([])
#order = np.array([])
#alist = np.array([1, 2, 3])

for i in range(len(kappa)):
    for j in range(len(kappa)):
        if i != j and i < j:
#             order = np.append(order, alist[i] / alist[j])
            v_kappa = np.append(v_kappa, kappa[i] / kappa[j])
            sig_v_kappa = np.append(sig_v_kappa,
                                    sqrt( (sig_kappa[i] / kappa[j]) ** 2
                                        + (kappa[i] * sig_kappa[j]
                                           / kappa[j] ** 2) ** 2))





for i in range(len(d)):
    for j in range(len(d)):
        if i != j and i < j:
            v_d = np.append(v_d, d[i] ** 2 / d[j] ** 2)
```

```
            sig_v_d = np.append(sig_v_d, sqrt( (2 * d[i] * sig_d[i] /␣
↪d[j] ** 2) ** 2
                                                    + (d[i] ** 2 * 2 *␣
↪sig_d[j]
                                                       / d[j] ** 3) ** 2) )



ergebnis_large('v_kappa', v_kappa, sig_v_kappa, 3, '')
ergebnis_large('v_d', v_d, sig_v_d, 3, '')
```

```
v_kappa = [0.583 0.284 0.487]
      +/- [0.208 0.1   0.086]
v_d = [0.443 0.245 0.553]
  +/- [0.011 0.006 0.01 ]
```

Sigmas

[9]:
```
# Sanity check symmetrical oscillation
fehler('Symmetrical osc.', f_1, sig_f_1, f_free, sig_f_free)
fehler('Symmetrical osc. (beat)', f_s1, sig_f_s1, f_free, sig_f_free)
```

```
Symmetrical osc.
Relativer Fehler:  [0.80645161 0.80515298 0.80515298]
Rel. Fehler (Vergleich): 0.8064516129032258
Absoluter Fehler:  [0.    0.001 0.001]
Sigma-Abweichung:  [0.         0.14142136 0.14142136]

Symmetrical osc. (beat)
Relativer Fehler:  [0.80385852 0.64412238 0.64516129]
Rel. Fehler (Vergleich): 0.8064516129032258
Absoluter Fehler:  [0.002 0.001 0.   ]
Sigma-Abweichung:  [0.28284271 0.15617376 0.        ]
```

[10]:
```
# Frequenzen f_I, II
fehler('Frequenz f_I', f_I, sig_f, f_I_theo, sig_f_theo)
fehler('Frequenz f_II', f_II, sig_f, f_II_theo, sig_f_theo)
```

```
Frequenz f_I
```

```
Relativer Fehler:  [0.61595029 0.43547762 0.53494826]
Rel. Fehler (Vergleich): [0.8768894  0.54351021 0.88976774]
Absoluter Fehler:  [-0.0035 -0.001  -0.005 ]
Sigma-Abweichung:  [0.5132649  0.22086305 0.71066905]

Frequenz f_II
Relativer Fehler:  [32.54270698  9.9243057   6.67694681]
Rel. Fehler (Vergleich): [31.94382825 11.9848607  10.41641894]
Absoluter Fehler:  [-0.0055 -0.001  -0.004 ]
Sigma-Abweichung:  [0.80655913 0.22086305 0.56853524]
```

[11]:
```
# Kappas
fehler('kappa', kappa, sig_kappa, kappa_s, sig_kappa_s)
```

```
kappa
Relativer Fehler:  [33.1995471  13.13700693 11.6949143 ]
Rel. Fehler (Vergleich): [33.68889214 10.84565862  7.62052822]
Absoluter Fehler:  [0.01871539 0.00352141 0.01487652]
Sigma-Abweichung:  [0.80027512 0.21069344 0.5330823 ]
```

[12]:
```
# Verhältnisse
fehler('v_kappa_d', v_kappa, sig_v_kappa, v_d, sig_v_d)
```

```
v_kappa_d
Relativer Fehler:  [35.70421374 35.19916119 17.58840446]
Rel. Fehler (Vergleich): [2.48956108 2.31243526 1.71884745]
Absoluter Fehler:  [ 0.14032874  0.03929943 -0.0656536 ]
Sigma-Abweichung:  [0.67293081 0.39222824 0.76138886]
```

[ ]:
```

```