Sheet 1 **July 6, 2023**

General Relativity

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Exercise 1 The Lorentz group...

In this first exercise you will get acquainted with the group of Lorentz transformations. The components of a 4-vector $x \in \mathbb{M}$ with respect to some basis are denoted as are denoted as x^μ. The standard (Cartesian) basis of $\mathbb{M}$ is $\{e_0, \dots, E_3\}$ with $(e_\mu)^\nu = \delta^\nu_\mu$. The metric η is the bilinear form $\mathbb{M} \times \mathbb{M} \to \mathbb{R}$ defined as

$$\eta(x, y) = -x_0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3 \tag{1}$$

1.1 The map $\underline{x^* = \eta(x, \cdot)} : y \mapsto \eta(x, y)$ is linear and per definition the following holds true:

$$x^*(y) = \eta(x, y) = \eta_{\mu\nu} x^\mu y^\nu. \tag{2}$$

Also, per definition one writes the dual of x^μ as x^ν, so that it's clear that

$$(x^\mu)^* = x_\mu = \eta_{\mu\nu} x^\nu \tag{3}$$

1.2 We verify explicitly that $\eta^{\mu\nu}$ is the inverse of $\eta_{\mu\nu}$. We can do this by using the metric itself to appropriately raise and lower indices:

$$\begin{aligned} \eta^{\mu\nu} &= \eta^{\mu\alpha}\eta^{\nu\beta}\eta_{\alpha\beta} & \Big| \cdot (\eta^{\nu\beta})^{-1} \quad (4) \\ \eta^{\mu\nu}(\eta^{\nu\beta})^{-1} &= \eta^{\mu\alpha}\eta_{\alpha\beta} & (5) \\ \delta^\mu_\beta &= \eta^{\mu\alpha}\eta_{\alpha\beta}. & (6) \end{aligned}$$

This proves our statement.

1.3 We compute the trace of the matrix

$$\eta^\mu_\mu = \delta^\mu_\mu = 4. \tag{7}$$

1.4 We verify explicitly that the matrices

$$\mathbf{R}^3(\theta) = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix} \qquad \mathbf{B}^1(\phi) = \begin{pmatrix} \cosh\phi & -\sinh\phi & & \\ -\sinh\phi & \cosh\phi & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{8}$$

satisfy

$$\eta = \mathbf{\Lambda}^T \eta \mathbf{\Lambda}. \tag{9}$$

While it's great practice to do this by hand to hone the matrix multiplication skills, it also presents the opportunity to start getting warmed up on Mathematica, which will help us immensely later:

In[1]:=
```
R3[theta_] := {{1, 0, 0, 0}, {0, Cos[theta],
    Sin[theta], 0}, {0, -Sin[theta], Cos[theta], 0},
    {0, 0, 0, 1}}
eta = DiagonalMatrix[{-1, 1, 1, 1}];
Simplify[Transpose[R3[theta]].eta.R3[theta]]
```

The last command outputs η again so we can see that it holds. Similarly for $\mathbf{B}^1(\phi)$. Necessary for this calculations are the trigonometric identities $\cos^2 x + \sin^2 x = 1$ and $\cosh^2 x - \sinh^2 x = 1$.

1.5 We show that the previous matrices also satisfy the group properties, as they belong to the Lorentz group $O(1,3)$. We check that the application of two of these matrices after each other is also a member of the group and that there's an identity operator and and inverse for every member.

In[2]:=
```
B1[phi_] := {{Cosh[phi], -Sinh[phi], 0, 0},
    {-Sinh[phi], Cosh[phi], 0, 0},
    {0, 0, 1, 0}, {0, 0, 0, 1}}
Simplify[B1[phi] . B1[psi]] // MatrixForm
```

This outputs

$$\begin{aligned}\mathbf{B}^1(\phi)\cdot\mathbf{B}^1(\psi) &= \begin{pmatrix} \cosh\phi+\psi & -\sinh\phi+\psi & & \\ -\sinh\phi+\psi & \cosh\phi+\psi & & \\ & & 1 & \\ & & & 1 \end{pmatrix} && (10)\\ &= \mathbf{B}^1(\phi+\psi) && (11)\end{aligned}$$

The identity element is just $\mathbf{B}^1(0)$ and the inverse to an element $\mathbf{B}^1(\phi)$ is $\mathbf{B}^1(-\phi)$.

1.6 We take a look at a restricted group of Lorentz transformations, called the *proper ortochronous group*. One could in theory permit the following matrices

$$\mathbf{P} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad \mathbf{T} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{12}$$

into the group to make it even more general.

To remain within the *proper ortochronous* the first thing that comes to mind is setting the condition, that $\det\{\mathbf{\Lambda}\} = 0$. The *orthochronous* aspect also gives us a hint regarding the time component, i.e. the ${\Lambda^0}_0$ component. For it to move forward with time we demand that ${\Lambda^0}_0 > 0$.

Exercise 2 ... and the Lorentz algebra

2.1 We define an infinitesimal Lorentz transformation as

$${\Lambda^\mu}_\nu = \delta^\mu_\nu + {\omega^\mu}_\nu. \tag{13}$$

We show that

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{14}$$

We start from (9):

$$\begin{aligned}
\eta_{\mu\nu} &= \Lambda^{\alpha}{}_{\mu}\Lambda^{\alpha}{}_{\nu}\eta_{\alpha\beta} && (15)\\
&= \left(\delta^{\alpha}_{\mu} + \omega^{\alpha}{}_{\mu}\right)\left(\delta^{\beta}_{\nu} + \omega^{\beta}{}_{\nu}\right)\eta_{\alpha\beta} && (16)\\
&= \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}\eta_{\alpha\beta} + \delta^{\alpha}_{\mu}\omega^{\beta}{}_{\nu}\eta_{\alpha\beta} + \delta^{\beta}_{\nu}\omega^{\alpha}{}_{\mu}\eta_{\alpha\beta} + \mathcal{O}(\omega^2) && (17)\\
&= \eta_{\mu\nu} + \omega^{\beta}{}_{\nu}\eta_{\mu\beta} + \omega^{\alpha}{}_{\mu}\eta_{\alpha\nu} && (18)\\
0 &= \omega^{\beta}{}_{\nu}\eta_{\mu\beta} + \omega^{\alpha}{}_{\mu}\eta_{\alpha\nu} && (19)\\
&= \omega_{\mu\nu} + \omega_{\nu\mu}. && (20)
\end{aligned}$$

2.2 We define the following matrices

$$\left(\mathbf{J}^3\right)^{\mu}{}_{\nu} = \begin{pmatrix} 0 & & & \\ & & i & \\ & -i & & \\ & & & 0 \end{pmatrix} \qquad \left(\mathbf{K}^1\right)^{\mu}{}_{\nu} = \begin{pmatrix} & i & & \\ i & & & \\ & & 0 & \\ & & & 0 \end{pmatrix}. \tag{21}$$

We verify that $(\mathbf{J}^3)_{\mu\nu}$ and $(\mathbf{K}^1)_{\mu\nu}$ are antisymmetric

$$\begin{aligned}
\left(\mathbf{J}^3\right)_{\mu\nu} &= \eta_{\mu\alpha}\left(\mathbf{J}^3\right)^{\alpha}{}_{\nu} = \eta\cdot\mathbf{J}^3 && (22)\\
&= \begin{pmatrix} 0 & & & \\ & & i & \\ & -i & & \\ & & & 0 \end{pmatrix} && (23)\\
\left(\mathbf{K}^1\right)_{\mu\nu} &= \eta_{\mu\alpha}\left(\mathbf{K}^1\right)^{\alpha}{}_{\nu} = \eta\cdot\mathbf{K}^1 && (24)\\
&= \begin{pmatrix} & i & & \\ -i & & & \\ & & 0 & \\ & & & 0 \end{pmatrix}. && (25)
\end{aligned}$$

Jup. Checks out. We also check that

$$\exp\left\{i\theta\mathbf{J}^3\right\} = \mathbf{R}^3(\theta) \qquad \exp\left\{i\phi\mathbf{K}^1\right\} = \mathbf{B}^1(\phi). \tag{26}$$

Although we could infer this from the first few terms by comparing the entries with the taylor series for cos and sin, we again use our good ol' friend Mathematica:

```
In[3]:= J3Munu = {{0, 0, 0, 0}, {0, 0, -I, 0},
            {0, I, 0, 0}, {0, 0, 0, 0}};
        K1Munu = {{0, I, 0, 0}, {I, 0, 0, 0},
            {0, 0, 0, 0}, {0, 0, 0, 0}};
        MatrixExp[I phi K1Munu] // MatrixForm
        MatrixExp[I theta J3Munu] // MatrixForm
```

This readily gives us

$$\exp\left\{i\theta\mathbf{J}^3\right\} = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix} = \mathbf{R}^3(\theta) \tag{27}$$

$$\exp\left\{i\phi\mathbf{K}^1\right\} = \begin{pmatrix} \cosh\phi & -\sinh\phi & & \\ -\sinh\phi & \cosh\phi & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \mathbf{B}^1. \tag{28}$$

2.3 We also verify that the two matrices in (21) satisfy the commutation relation

$$[\mathbf{J}^i, \mathbf{K}^j] = i\varepsilon^{ijk}\mathbf{K}^k \tag{29}$$

with $\mathbf{J}^i = \frac{1}{2}\varepsilon^{ijk}J^{jk}$ and $\mathbf{K}^i = J^{i0}$ with

$$\left(J^{ab}\right)^{\mu}_{\ \nu} = -\left(J^{ba}\right)^{\mu}_{\ \nu} = i\left(\eta^{a\mu}\delta^{a}_{\nu} - \eta^{b\mu}\delta^{a}_{\nu}\right) \tag{30}$$

It's only necessary to check this for $\mathbf{J}^3$ and $\mathbf{K}^1$, so we write

$$[\mathbf{J}^3, \mathbf{K}^1] = \mathbf{J}^3\mathbf{K}^1 - \mathbf{K}^1\mathbf{J}^3 \tag{31}$$

$$= \begin{pmatrix} & & -1 & \\ & 0 & & \\ -1 & & & \\ & & & 0 \end{pmatrix} \tag{32}$$

This we could determine simply by doing the multiplication of the known matrices. It's left to prove if

$$[\mathbf{J}^3, \mathbf{K}^1] = i\mathbf{K}^2 \tag{33}$$

as per (29). We can do this by analyzing the form of $\mathbf{K}^2$:

$$\left(\mathbf{K}^2\right)^{\mu}_{\ \nu} = \left(J^{20}\right)^{\mu}_{\ \nu} \tag{34}$$

$$= i\left(\eta^{2\mu}\delta^{0}_{\nu} - \eta^{0\mu}\delta^{2}_{\nu}\right) \tag{35}$$

$$= \begin{cases} i & (\mu = 2 \wedge \nu = 0) \vee (\mu = 0 \wedge \nu = 2) \\ 0 & \text{else.} \end{cases} \tag{36}$$

From this we can clearly see, that $\mathbf{K}^2$ fulfills (32).

Exercise 3 Relativistic fields

Consider a Lagrangian with Lagrangian density

$$L = \int \mathrm{d}t\, \mathcal{L}\left(\phi^i, \partial_\mu\phi^i\right) \tag{37}$$

depending on N fields ϕ^i. The ELE are

$$\frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi^i\right)}\right) = 0 \tag{38}$$

3.1 We determine the eom for a scalar field with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu\phi\right)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \tag{39}$$

with ELE:

$$-m^2\phi + \partial_\mu(\partial^\mu\phi) = 0 \tag{40}$$

$$(\partial^2 - m^2)\phi = 0 \tag{41}$$

3.2 We define the electromagnetic field strength as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{42}$$

and the current vector as

$$J^\mu = \partial_\nu F^{\mu\nu}. \tag{43}$$

We derive (43) from the following Lagrangian as the corresponding eom:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu \tag{44}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu)} \right) &= J^\mu - \frac{1}{4} \partial_\nu \left(\frac{\partial}{\partial \big(\partial_\mu A_\nu\big)} \big(F_{\alpha\beta} F^{\alpha\beta} \big) \right) && (45)\\
&= J^\mu - \frac{1}{4} \partial_\nu \left(\frac{\partial F_{\alpha\beta}}{\partial \big(\partial_\mu A_\nu\big)} F^{\alpha\beta} + \frac{\partial F_{\rho\sigma}}{\partial \big(\partial_\mu A_\nu\big)} F_{\alpha\beta} \eta^{\alpha\rho} \eta^{\sigma\beta} \right) && (46)\\
&= J^\mu - \frac{1}{4} \partial_\nu \left(2 \frac{\partial F_{\alpha\beta}}{\partial \big(\partial_\mu A_\nu\big)} F^{\alpha\beta} \right) && (47)\\
&= J^\mu - \frac{1}{2} \partial_\nu \left\{ \left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\nu_\alpha \delta^\mu_\beta \right) F^{\alpha\beta} \right\} && (48)\\
&= J^\mu - \frac{1}{2} \partial_\nu \{ F^{\mu\nu} - F^{\nu\mu} \} && (49)\\
&= J^\mu - \partial_\nu F^{\mu\nu} = 0 && (50)
\end{aligned}$$

which gives us the expected result.

3.3 Given the electric field $\mathbf{E} = \boldsymbol{\nabla} A_0$ and the magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, we can write the components of the electromagnetic field tensor as

$$(F)_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{51}$$

Using Mathematica, again, we show that a rotation of the matrix preserves the separation of the electric and magnetic components, but not a boost. For this we'll use the $\mathbf{R}^3$ and $\mathbf{B}^1$ matrices from before:

In[4]:=

```
Fmunu = {{0, -Subscript[e, 1],
    -Subscript[e, 2], -Subscript[e, 3]},
   {Subscript[e, 1], 0,
    Subscript[b, 3], -Subscript[b, 2]},
   {Subscript[e, 2], -Subscript[b, 3],
    0, Subscript[b, 1]},
   {Subscript[e, 3], Subscript[b, 2],
    -Subscript[b, 1], 0}};
TeXForm[Simplify[Transpose[R3[theta]].Fmunu.R3[theta]]
// MatrixForm]
```

which gives us[1]

$$F_{\mu\alpha}\Big(\mathbf{R}^3(\theta)\Big)^{\alpha}_{\ \nu} = \tag{52}$$

$$\begin{pmatrix} 0 & e_2\sin(\theta)-e_1\cos(\theta) & e_1(-\sin(\theta))-e_2\cos(\theta) & -e_3 \\ e_1\cos(\theta)-e_2\sin(\theta) & 0 & b_3 & b_1(-\sin(\theta))-b_2\cos(\theta) \\ e_1\sin(\theta)+e_2\cos(\theta) & -b_3 & 0 & b_1\cos(\theta)-b_2\sin(\theta) \\ e_3 & b_1\sin(\theta)+b_2\cos(\theta) & b_2\sin(\theta)-b_1\cos(\theta) & 0 \end{pmatrix} \tag{53}$$

On the other hand, a boost will give us

$$F_{\mu\alpha}\Big(\mathbf{K}^1(\phi)\Big)^{\alpha}_{\ \nu} = \tag{54}$$

$$\begin{pmatrix} 0 & -e_1 & b_3(-\,\mathrm{sh}(\phi))-e_2\,\mathrm{ch}(\phi) & b_2\,\mathrm{sh}(\phi)-e_3\,\mathrm{ch}(\phi) \\ e_1 & 0 & b_3\,\mathrm{ch}(\phi)+e_2\,\mathrm{sh}(\phi) & e_3\,\mathrm{sh}(\phi)-b_2\,\mathrm{ch}(\phi) \\ b_3\,\mathrm{sh}(\phi)+e_2\,\mathrm{ch}(\phi) & b_3(-\,\mathrm{ch}(\phi))-e_2\,\mathrm{sh}(\phi) & 0 & b_1 \\ e_3\,\mathrm{ch}(\phi)-b_2\,\mathrm{sh}(\phi) & b_2\,\mathrm{ch}(\phi)-e_3\,\mathrm{sh}(\phi) & -b_1 & 0 \end{pmatrix} \tag{55}$$

which has the electric and magnetic field all mixed up

[1] I manually changed the e's to E's, because the E in Mathematica is already taken by the Exponential function. But otherwise, it delivers us perfectly what we're looking for. The TeXForm command even changed the theta automatically to θ. Crazy. Actually I decided to leave the e_i, b_i because I had to correct it a lot and it was giving me a headache. Also, I turned the sinh and cosh to sh and ch for it to fit on the page