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A loop test for bending length and rigidity

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Abstract. The theory of a proposed new test for bending length is presented. It consists in laying down on a horizontal surface a strip of the material in the form of a loop. The bending length is then proportional to the height of the loop with a factor of proportionality of 1·103. This factor has been found by numerical integration of the equations of equilibrium of a strip bending under its own weight.

1. Introduction

The bending length of a strip of material is a measure of the ratio of its flexural rigidity to its weight. For a material in linear form (for example a strip of cloth, sheet rubber, paper, wire mesh, or a length of rope, string, yarn or fibre) the bending length c is defined as

$$c = (B/\omega)^{1/3}$$

where B is the flexural rigidity of the strip, and ω is its weight per unit length.

Peirce (1930) has discussed a number of methods for the measurement of bending length, and an instrument, the Shirley bending length tester (Shirley Institute Test Leaflet DFI A, Jan. 1957), has been made and marketed on the basis of one of his methods. This tester, which uses a cantilever principle, is suitable for the measurement of the bending length of materials in strip form (one inch wide) up to about 10 cm.

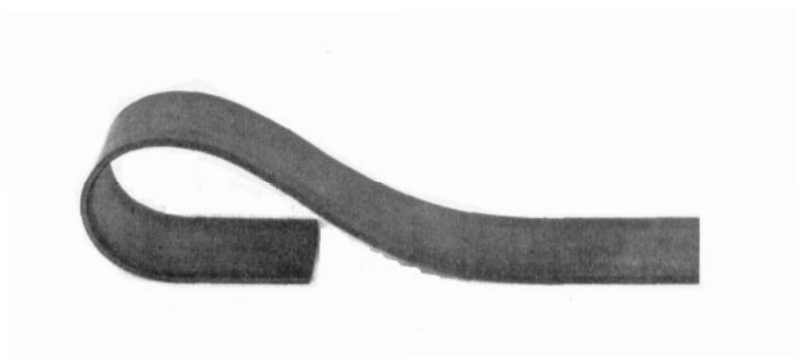

Figure 1. A 'loop' of neoprene.

In this paper we describe a new test for bending length. This test consists in laying down a loop of the strip on a horizontal surface, as in figure 1. This strip must at least five bending lengths long. Further practical details have been described elsewhere (Stuart and Baird 1966).

It will be shown that the form of the loop is invariant, and that the bending length is proportional to the height of the loop. This invariance follows from the boundary conditions at the ends of the loop.

It will be shown that the curvature of the strip at the ends of the loop is zero, and that the tension in the strip at these points is zero. It then follows that the system is sufficiently determined to allow the equations of equilibrium to be integrated. The factor of proportionality of bending length to loop height Y_M has been calculated to be 1·103, that is $c = 1{\cdot}103\ Y_M$.

2. Theory

2.1. *Equations of equilibrium in dimensionless coordinates*

We consider the equilibrium of an interval of strip of uniform flexural rigidity B and weight per unit length ω. The distance along the strip is s, measured from a point on it, $s = 0$, where the inclination θ of the strip to the horizontal is zero. The projections of the interval $(0, s)$ on to the horizontal OX and the vertical OY are X and Y respectively.

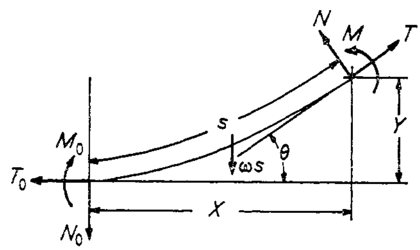

Figure 2. The interval $(0, s)$ of a strip bending under its own weight. Forces and moments are drawn in to indicate the senses taken.

The tension, shear force, and bending moment at s are T, N and M, with senses as indicated in figure 2. Simple theory then gives

$$N = -\frac{dM}{ds}. \tag{1}$$

The bending moment and curvature are related such that

$$M = B\frac{d\theta}{ds} \tag{2}$$

where B is the flexural rigidity. Thus if B is uniform

$$N = -B\frac{d^2\theta}{ds^2}. \tag{3}$$

Equating forces on the interval $(0, s)$ first normal to, then parallel to, the tangent at s, we obtain

$$N = (\omega s + N_0)\cos\theta - T_0 \sin\theta \tag{4}$$

and

$$T = (\omega s + N_0)\sin\theta + T_0 \cos\theta \tag{5}$$

where the suffix 0 indicates values at $s = 0$.

We now introduce the bending length $c = (B/\omega)^{1/3}$ and define the dimensionless variables

$$(\sigma, x, y) = \frac{1}{c}(s, X, Y)$$

and the dimensionless forces

$$(\tau, \nu) = \frac{1}{\omega c}(T, N).$$

Then, combining equations (3) and (4), we have in terms of our new variables the relationships

$$\nu = -\frac{d^2\theta}{d\sigma^2} = (\sigma + \nu_0)\cos\theta - \tau_0 \sin\theta \tag{6}$$

and

$$\tau = (\sigma + \nu_0)\sin\theta + \tau_0 \cos\theta. \tag{7}$$

These equations admit a simple integral. Thus, differentiating equation (7) with respect to σ

$$\frac{d\tau}{d\sigma} = \{(\sigma + \nu_0)\cos\theta - \tau_0 \sin\theta\}\frac{d\theta}{d\sigma} + \sin\theta$$

$$= \frac{d}{d\sigma}\left\{-\frac{1}{2}\left(\frac{d\theta}{d\sigma}\right)^2 + y\right\}$$

since $\sin\theta = dy/d\sigma$, and

$$\frac{d^2\theta}{d\sigma^2}\frac{d\theta}{d\sigma} = \frac{d}{d\sigma}\left\{\frac{1}{2}\left(\frac{d\theta}{d\sigma}\right)^2\right\}.$$

Thus

$$\tau + \frac{1}{2}\left(\frac{d\theta}{d\sigma}\right)^2 - y = \text{constant}. \tag{8}$$

2.2. *Boundary conditions for the loop*

The part of the loop free of the supporting surface is the interval $0 < \sigma < \sigma_1$. The total weight, $\omega\sigma_1$, of the loop is supported by the shear forces in the strip at $\sigma = 0(+)$ and $\sigma = \sigma_1(-)$ (see figure 3). This is indicated by equation (6) if we put $\theta = \theta_1 = -\pi$. That is

$$\nu_1 = -(\sigma + \nu_0)$$

or

$$\sigma_1 = -\nu_0 - \nu_1.$$

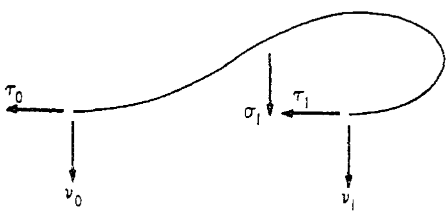

Figure 3. Forces acting on the interval of loop free of the supporting surface $0<\sigma<\sigma_1$.

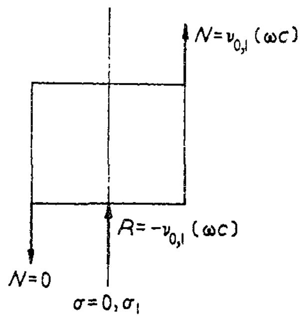

Figure 4. A small element of strip spanning $\sigma = 0$ or $0 = \sigma_1$.

The parts of the strip resting on the supporting surface, $\sigma < 0$ and $\sigma > \sigma_1$, should lie flat, so that the curvature there, and hence the bending moment and shear force, should be zero. The supporting surface will provide a distributed reaction ω per unit length to balance the distributed weight of the strip.

Thus there is a discontinuity in the shear force as we pass through both $\sigma = 0$ and $\sigma = \sigma_1$, and it can be seen on consideration (see figure 4) of the equilibrium of a small element of strip spanning either $\sigma = 0$ or $\sigma = \sigma_1$ that the supporting surface provides at these points concentrated reactions $R_0 = -\nu_0(\omega c)$ and $R_1 = -\nu_1(\omega c)$, which thus support the loop. Thus

$$R_0 + R_1 = (-\nu_0 - \nu_1)\,\omega c = \sigma_1\,\omega c.$$

There is thus at $\sigma = 0$ and $\sigma = \sigma_1$ a simple discontinuity in the shear force and hence in the second derivative of θ, $d^2\theta/d\sigma^2$. Consequently the first derivative of θ, $d\theta/d\sigma$, should be continuous at $\sigma = 0$ and $\sigma = \sigma_1$. Thus because $d\theta/d\sigma = 0$ for $\sigma < 0$ and $\sigma > \sigma_1$, $d\theta/d\sigma$ will be zero at $\sigma = 0$ and $\sigma = \sigma_1$. So the boundary conditions are at

$$\sigma = 0: \quad \theta = 0, \qquad \frac{d\theta}{d\sigma} = 0, \quad y = 0$$

and at

$$\sigma = \sigma_1: \quad \theta = -\pi, \quad \frac{d\theta}{d\sigma} = 0, \quad y = 0.$$

On applying the boundary conditions at $\sigma = 0$ to equation (8) we have

$$\tau - \tau_0 = -\frac{1}{2}\left(\frac{d\theta}{d\sigma}\right)^2 + y \tag{9}$$

whence, at $\sigma = \sigma_1$,

$$\tau_1 - \tau_0 = 0. \tag{10}$$

But from equation (7) at $\sigma = \sigma_1$

$$\tau_1 = -\tau_0 \tag{11}$$

So from equations (10) and (11) $\tau_0 = \tau_1 = 0$. Equation (9) can now be written

$$y = \tau + \frac{1}{2}\left(\frac{d\theta}{d\sigma}\right)^2 \tag{12}$$

and the equations of equilibrium become

$$-\nu = \frac{d^2\theta}{d\sigma^2} = -(\sigma + \nu_0)\cos\theta \tag{13}$$

and

$$\tau = (\sigma + \nu_0)\sin\theta$$

with boundary conditions at

$$\sigma = 0: \quad \theta = 0, \qquad \frac{d\theta}{d\sigma} = 0, \quad y = 0, \quad \tau = 0$$

and at

$$\sigma = \sigma_1: \quad \theta = -\pi, \quad \frac{d\theta}{d\sigma} = 0, \quad y = 0, \quad \tau = 0.$$

Note 1

It might be argued that frictional forces associated with the reactions R_0 and R_1 might allow a bending moment to be supported at each end of the loop. The elements of the couples would be the frictional forces acting on the bottom surface of the strip and balancing tensions acting along the neutral axis of the strip. The couple arm would be half the strip thickness at each end. In dimensionless coordinates

$$\left|\frac{d\theta}{d\sigma}\right|_{\sigma=0} = |\tau_0|\,\frac{t}{2c}$$

and

$$\left|\frac{d\theta}{d\sigma}\right|_{\sigma=\sigma_1} = |\tau_1|\,\frac{t}{2c}$$

where t is the thickness of the strip.

But from equation (7) at $\sigma = \sigma_1$, $\tau_1 = -\tau_0$ (equation (11)); so

$$\left(\frac{d\theta}{d\sigma}\right)^2_{\sigma=0} = \left(\frac{d\theta}{\sigma d}\right)^2_{\sigma=\sigma_1}$$

and, substituting this with $y_0 = y_1 = 0$ into equation (8),

$$\tau_0 + \left(\frac{d\theta}{d\sigma}\right)^2_{\sigma=0} - y_0 = \tau_1 + \left(\frac{d\theta}{d\sigma}\right)^2_{\sigma=\sigma_1} - y_1.$$

We obtain again $\tau_0 = \tau_1$, hence we find again that $\tau_0 = \tau_1 = 0$ and

$$\left|\frac{d\theta}{d\sigma}\right|_{\sigma=0} = \left|\frac{d\theta}{d\sigma}\right|_{\sigma=\sigma_1} = 0.$$

Note 2

At the point of maximum height (denoted by suffix M) $\theta_M = 0$, so from equation (14)

$$\tau_M = 0 \tag{15}$$

and hence the height y_M and the curvature $(d\theta/d\sigma)_M$ are related (from equation (12)):

$$y_M = \frac{1}{2}\left(\frac{d\theta}{d\sigma}\right)^2_M. \tag{16}$$

Note 3

By a simple procedure we can obtain an expansion of $\theta = \theta(\sigma)$ near $\sigma = 0(+)$ to obtain

$$\theta = -\tfrac{1}{2}\nu_0\sigma^2 - \tfrac{1}{6}\sigma^3 - O(\sigma^6). \tag{17}$$

3. Calculations

The differential equation (13) is in a form suitable for numerical integration. ν_0 is unknown, but we can integrate using an estimated value, ν_0' say, and find out how far the solution then departs from the required boundary conditions.

We introduce a discrete parameter i so that, with h as the interval size, $\sigma_i = ih$.

The second-order differential coefficient in (13) is replaced using the equation

$$\theta_{i+1} - 2\theta_i + \theta_{i-1} = h^2\theta_1'' + \tfrac{1}{12}h^4\theta_2^{iv} + O(h^6)$$

to obtain the recurrence relationship

$$\theta_{i+1} = 2\theta_i - \theta_{i-1} - h^2(ih + \nu_0)\cos\theta_i + O(h^4). \tag{18}$$

The calculation is begun by using the series expansion of θ near $\sigma = 0$ (equation (17)). Thus, with an estimated value ν_0' of ν_0,

$$\theta_0' = 0$$
$$\theta_1' = -\tfrac{1}{2}\nu_0'h^2 - \tfrac{1}{6}h^3$$

and for $i \geqslant 1$

$$\theta_{i+1}' = 2\theta_i' - \theta_{i-1}' - h^2(ih + \nu_0')\cos\theta_i'.$$

Here the primes indicate that an estimated value of ν_0 is being used.

Now the boundary conditions at $\sigma = \sigma_1$ are $\theta = -\pi$ and $d\theta/d\sigma = 0$. Since $d^2\theta/d\sigma^2 = -\nu_1$ at $\sigma = \sigma_1$, which should be positive as ν_1 should be negative, θ has a minimum there, i.e. $\theta_1 = -\pi$.

So we proceed to evaluate successive values of θ_i' until $i = j + 1$ where

$$\theta_{j-1}' > \theta_j' \text{ but } \theta_j' \leqslant \theta_{j+1}'.$$

(Some care must be exercised here since θ also has a minimum at $\sigma = 0$. θ_i increases near $i = 0$, and there is no point in looking for a minimum until σ exceeds σ_M where the height is a maximum and θ_M is zero again.)

Having obtained the three successive values of θ', θ_{j-1}', θ_j' and θ_{j+1}', the value of the minimum is estimated using quadratic interpolation, which gives

$$\theta_1' = \theta_j' - \frac{\tfrac{1}{2}(\theta_{j+1}' - \theta_{j-1}')^2}{\theta_{j+1}' - 2\theta_j' + \theta_{j-1}'}. \tag{18}$$

With the correct value of ν_0, θ_1' should be $-\pi$, so

$$\epsilon' = \theta_1' + \pi$$

can be used as an indicator of the error in ν_0'.

When several pairs of the values of (ν_0', ϵ') have been determined we can interpolate (or extrapolate) to obtain a new value of ν_0' for which ϵ' should be zero.

The value of y_M is found in a similar way to θ_1', in that successive values of

$$y_i = \int_0^{\sigma_i} \sin\theta \, d\sigma = \sum_{k=0}^{k=i} \sin\theta_k \, h$$

are examined until

$$y_{j-1} < y_j \text{ but } y_j \geqslant y_{j+1}.$$

The value of y_M is then found by quadratic interpolation using a formula similar to equation (18). The values obtained for successive approximations are shown in the table.

h	ν_0'	ϵ'	Y_M'	σ_M'	ν_1'
0·02	−1·000 000	−0·097 765			
	−0·950 000	−0·017 237			
	†−0·939 297	−0·001 300			
				4·683 259	
	‡−0·938 412	−0·000 001 (5)	0·906 396		−3·776 863
0·01	−1·000 000	−0·097 705			
	−0·950 000	−0·017 178			
	†−0·939 334	−0·001 296			
				4·683 419	
	‡−0·938 452	−0·000 001 (2)	0·906 549		−3·776 870
0·005	−1·000 000	−0·097 690			
	−0·950 000	−0·017 163			
	†−0·939 431	−0·001 294			
				4·683 459	
	‡−0·938 462	−0·000 001 (3)	0·906 586		−3·776 873

† When two pairs of values (ν_0', ϵ') are obtained a third value of ν_0' is found using linear extrapolation, setting $\nu_0 = a + b\epsilon$.

‡ When three values of (ν_0', ϵ') are available the next value of ν_0' is obtained using a quadratic extrapolation based on setting $\nu_0 = a + b\epsilon + c\epsilon^2$.

The calculations were programmed by R. Mann for the C.D.C. 3600 of the C.S.I.R.O. Scientific Computing Network. The form of the loop shown in figure 5 is that plotted (with $h = 0{\cdot}005$) by the Calcomp 12 in. plotter peripheral to this C.D.C. 3600.

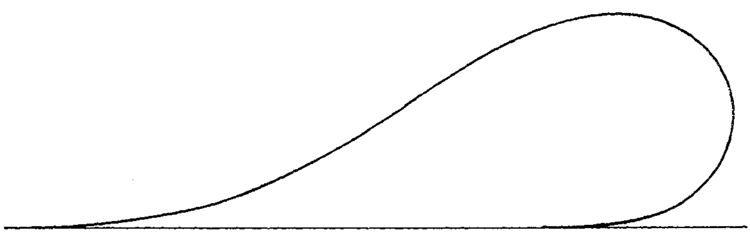

Figure 5. Loop shape as plotted by the C.D.C. 3600 computer.

These results can be extrapolated to zero interval size. Thus, if we suppose

$$y_M(h) = y_M + \alpha h^2$$

we obtain $y_M = 0{\cdot}906\,600$ and $\alpha = -0{\cdot}51$. Thus we take as the true value $y_M = 0{\cdot}9066$ and so for the bending length

$$c = \frac{Y_M}{0{\cdot}9066} = 1{\cdot}103 \; Y_M$$

where Y_M is the height of the loop.

References

Peirce, F. T., 1930, *J. Text. Inst.*, **21**, T377.
Stuart, I. M., and Baird, K., 1966, *Text. Res. J.*, **36**, 91.