# Section 3: Incompressible Fluid Flow

Recall the governing equations:

NSE's: (nonlinear) $\vec{u}_t + \vec{u} \cdot \nabla \vec{u} = \nu \Delta \vec{u} - \frac{1}{\rho} \nabla p$ (Conservation of momentum)

$\nabla \cdot \vec{u} = 0$ (continuity eq'n)

Linearize ($Re = \frac{UL}{\nu} \to 0$): ($\nu = \frac{\mu}{\rho}$)

Stokes' eq'n's $\vec{u}_t = \nu \Delta \vec{u} - \frac{1}{\rho} \nabla p$

Common simplification ($\rho$ constant):

$\frac{1}{\rho} \nabla p \longrightarrow \nabla p$ (rescaled $\tilde{p} = \frac{p}{\rho}$)

# Remember: Pressure Poisson Equation (PPE)

Take divergence of NSE:

$$\Delta p = \nabla \cdot (\mu \Delta \vec{u} - \rho \vec{u} \cdot \nabla \vec{u})$$

elliptic eqn for pressure.

Notes:

- Incompressible NSE's are mixed elliptic-parabolic type.
- When convection dominates ($Re \to \infty$) they take on hyperbolic character.
- Compressible flow has

$$\rho_t + \nabla \cdot (\rho \vec{u}) = 0 \qquad p = p(\rho)$$

which is an evolution equation for $\rho$ .... this is in some ways much easier than the PPE to solve.

# Numerical Issues

① Dimension must be at least 2, since there are no non-trivial 1D incompressible flows.
(exception: 1D hyperbolic problems)

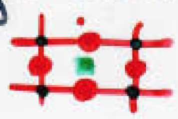
② Choice of unknowns and equations:

- primitive variables: $\vec{u}, p$
- streamfunction-vorticity: $\psi, \vec{\omega}$
- velocity-height for $S^{\text{hallow}}W^{\text{ater}}E^{\text{qns}}$: $\vec{u}, h$
- others ....

③ Boundary conditions for pressure
(derivation and accurate implementation)

④ Choice of FD/FV grid and location of unknowns (node-centred, cell-centered,

edge-centered or a mix)



## 5 Explicit vs. implicit time-marching:

a) advection terms (hyperbolic in nature):

- can devise good high-order explicit methods
- implicit methods overly diffusive.

b) diffusion terms (parabolic in nature):

- need implicit methods for stability and efficiency.

SOLUTION: mixed implicit-explicit method

OR solve fully implicit (too complicated) nonlinear solvers

## 6 Dealing with incompressibility condition and conserving mass as best as possible

REMEMBER: they're related because

$$\left.\begin{array}{ll} \text{incompressible:} & \dfrac{D\rho}{Dt}=0 \\ \text{conservation of mass:} & \dfrac{\partial \rho}{\partial t}+\nabla\cdot(\rho\vec{u})=0 \end{array}\right\} \Rightarrow \nabla\cdot u=0$$

NOTE: solving $\nabla\cdot\vec{u}=0$ directly is problematic
e.g. What are appropriate BC's?

# Outline of this Section

- Stream-function-vorticity formulation for 2D flows (adv: eliminates pressure) (Poz. §13.1)
- Primitive variable methods $(\vec{u}, p)$:
  - artificial compressibility (Poz., p628)
  - projection methods (Poz. §13.4)
  - .... others?...

(APOLOGY)

Many ideas from (numerical) linear algebra will be introduced with minimum detail.

# Stream-Function-Vorticity Methods



**Ref:** Pozrikidis, Ch. 13 (1997).

In 2D, incompressible Navier-Stokes, introduce a streamfunction $\psi$ such that

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

Then $$\nabla \cdot \vec{u} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)$$

$$= 0$$ (automatically satisfied) DROPS OUT!

**Note:** Lines of constant $\psi$ (contours) are streamlines of the flow.

Vorticity is $\vec{\omega} = \nabla \times \vec{u}$ and represents the rotational nature of the flow.

In 2D: $\omega = v_x - u_y$ is a scalar, and

$\omega > 0 \Rightarrow$ clockwise rotation

$\omega < 0 \Rightarrow$ counter-clockwise rotation

Note: $\omega = v_x - u_y = (-\psi_x)_x - (\psi_y)_y = -\Delta\psi.$

$$\boxed{\omega = -\Delta\psi} \qquad (1)$$

elliptic eqn for $\psi$.

Remember 2D NSE's:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (a)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (b)$$

($\rho$ constant) and take $\frac{\partial}{\partial y}(a) - \frac{\partial}{\partial x}(b)$:

$$\boxed{\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu\Delta\omega} \qquad (2)$$

(eliminates pressure)

Equations (1) and (2) (2 eqns, 2 unknowns) replace inc. NSE's (3 eqns, 3 unknowns), with "hidden" conditions $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$, $\omega = v_x - u_y$.

Alternate forms of (2):

$\vec{u}$-$\omega$: $$\frac{\partial \omega}{\partial t} + \vec{u}\cdot\nabla\omega = \nu\Delta\omega$$

$\psi$-$\omega$: $$\frac{\partial \omega}{\partial t} + \underbrace{\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y}}_{J(\omega,\psi)\text{ - Jacobian}} = \nu\Delta\omega$$

$$\begin{cases} \dfrac{\partial \omega}{\partial t} + \dfrac{\partial \psi}{\partial y}\dfrac{\partial \omega}{\partial x} - \dfrac{\partial \psi}{\partial x}\dfrac{\partial \omega}{\partial y} = \nu \Delta \omega \\ \Delta \psi = -\omega \end{cases}$$

Notes:

- Just as the primitive variable eqns have no evolution equation for pressure, there is none for $\psi$ either (elliptic).
- Only 2 unknowns and 2 equations (incompressibility is "built in").
- $\psi$ obeys a fourth order equation:

$$\frac{\partial \Delta\psi}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial \Delta\psi}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \Delta\psi}{\partial y} = \nu \Delta^2 \psi$$

# Boundary conditions

- No flow through solid walls:

$$\vec{u}\cdot\vec{n} = 0 \Rightarrow \psi = c$$ (boundary is a streamline)

and can take $c=0$ since constant is arbitrary

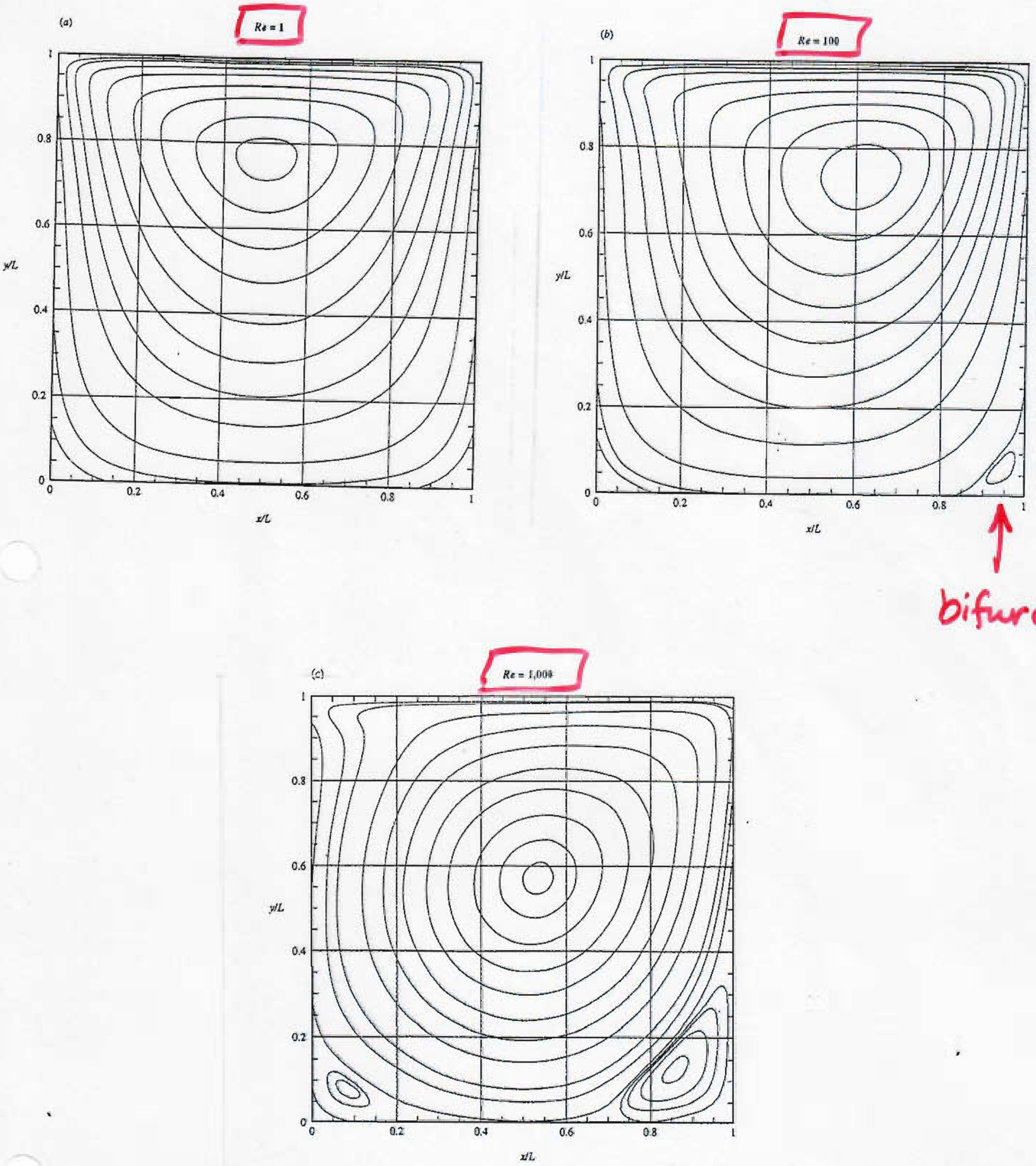
- No tangential slip on solid walls:

$$\vec{u}\cdot\vec{\tau} = 0 \Rightarrow (\psi_y, -\psi_x)\cdot\vec{\tau} = 0$$

On each solid wall, there are two BC's for $\psi$. This makes sense because $\psi$ satisfies a 4th order equation.

COMPARE: to primitive variables

# Driven Cavity Flow with $Re = \frac{VL}{\nu}$




bifurcation

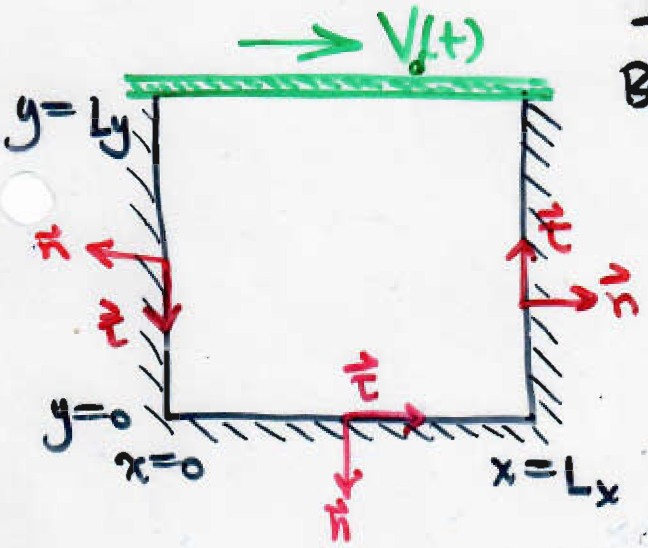
Source: Shankar & Deshpande

# Example:

$$\frac{\partial \omega}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} = \nu \Delta \omega$$
$$\Delta \psi = -\omega$$

Impulsively-started, driven-cavity flows:

- $L_x \times L_y$ rectangular box.
- top lid moves with given velocity $V_d(t)$.
- other three walls are stationary.
- starts with zero velocity – "impulsive".




IC's: $\omega = 0$ at $t=0$

BC's: $\psi = 0$ on all 4 sides.

$\psi_x = 0$ on left – $(\psi_y, -\psi_x)\cdot\vec{t}(0,-1) = 0$

$\psi_x = 0$ on right – $(\psi_y, -\psi_x)\cdot\vec{t}(0,1) = 0$

$\psi_y = 0$ on bottom

$\psi_y = V_d(t)$ on top

Discretize on a regular grid:

$$x_i = i \cdot \Delta x \quad \text{with} \quad \Delta x = \frac{L_x}{N_x}$$

$$y_j = j \cdot \Delta y \quad \text{with} \quad \Delta y = \frac{L_y}{N_y}$$

$$i = 0, 1, \dots, N_x, \quad j = 0, 1, 2, \dots, N_y$$

# Solution Algorithm

EXPLICIT, $O(\Delta t, \Delta x^2)$

(Keeps track of u, v as "intermediate" variables)

① Initialize:

$$u^0_{ij} = v^0_{ij} = 0 \quad \text{(IC/BC)}$$ at all interior and boundary points i, j

except $u^0_{i,N} = \mathbf{V}_d$

Set $n = 0$.

② Differentiate velocity to get vorticity at interior points:

$$\omega^n_{ij} = \frac{v^n_{i+1,j} - v^n_{i-1,j}}{2\Delta x} - \frac{u^n_{i,j+1} - u^n_{i,j-1}}{2\Delta y}$$

$$i = 1, 2, \ldots, N_x - 1 \qquad j = 1, 2, \ldots, N_y - 1$$

At side and bottom boundaries, use second-order, one-sided differences:

- Along $x = 0$: $\omega^n_{0,j} = v_x|_{0,j} - \cancel{u_y|_{0,j}}^{\,0}$

because $u=0$ along entire side & $u(0,y,t)=0$ $\Rightarrow u_y(0,y,t)=0$

$$\approx \frac{\cancel{-3v^n_{0j}}^{\,0} + 4v^n_{1,j} - v^n_{2,j}}{2\Delta x}$$

$$= \frac{4v^n_{1,j} - v^n_{2,j}}{2\Delta x}$$

- **Along $x = L_x$:**

$$\omega^n_{N,j} = v^n_x\big|_{N,j} - \cancelto{0}{u^n_y\big|_{N,j}}$$
$$\approx \frac{V^n_{N-2,j} - 4V^n_{N-1,j} + \cancelto{0}{3V^n_{N,j}}}{2\Delta x}$$
$$= \frac{V^n_{N-2,j} - 4V^n_{N-1,j}}{2\Delta x}$$

- **Along $y = 0$:**

$$\omega^n_{i,0} = \cancelto{0}{v^n_x\big|_{i,0}} - u^n_y\big|_{i,0}$$
$$= -\left(\frac{\cancelto{0}{-3U^n_{i,0}} + 4U^n_{i,1} - U^n_{i,2}}{2\Delta y}\right)$$
$$= -\frac{4U^n_{i,1} + U^n_{i,2}}{2\Delta y}$$

- **Along $y = L_y$:**

$$\omega^n_{i,N} = \cancelto{0}{v^n_x\big|_{i,N}} - u^n_y\big|_{i,N}$$
$$= -\left(\frac{U^n_{i,N-2} - 4U^n_{i,N-1} + 3V_d}{2\Delta y}\right)$$

$\Rightarrow$ all boundary values of $\omega$ are known!

3) Solve vorticity equation using FTCS (explicit) and BC's from ② to set $\omega_{ij}^{n+1}$:

NOTE: at high Re, use upwind differences!

$$\omega_{ij}^{n+1} = \omega_{ij}^{n} - \Delta t\left[U_{ij}^{n} D_0^x \omega_{ij}^{n} + V_{ij}^{n} D_0^y \omega_{ij}^{n}\right] + \nu\Delta t\left[D_2^x \omega_{ij}^{n} + D_2^y \omega_{ij}^{n}\right]$$

for $i = 1, 2, \ldots, N_x - 1$
$j = 1, 2, \ldots, N_y - 1.$

4) Solve Poisson eqn $\Delta\psi^{n+1} = -\omega^{n+1}$:

$$D_2^x \psi_{ij}^{n+1} + D_2^y \psi_{ij}^{n+1} = -\omega_{ij}^{n+1}$$

with $\psi_{ij}^{n+1}\big|_{\partial\Omega} = 0$ ← only use Dirichlet BC.

for $i = 1, 2, \ldots, N_x - 1$
$j = 1, 2, \ldots, N_y - 1.$

5) Difference $\psi_{ij}^{n+1}$ to get velocities:

$$u_{ij}^{n+1} = D_0^y \psi_{ij}^{n+1}$$

$$v_{ij}^{n+1} = -D_0^x \psi_{ij}^{n+1}$$

for $i = 1, 2, \ldots, N_x - 1$
$j = 1, 2, \ldots, N_y - 1$ (only interior values needed)

6) Increment $n$ and goto step 2.

# Notes

- Only Dirichlet BC's for $\psi$ are used, although we could have used Neumann BC's also (from $\vec{u}\cdot\vec{t}=0$) BUT then need a compatibility condition

$$\int_{\partial\Omega} \nabla\psi\cdot\vec{n}\,ds = -\int_{\Omega} \omega\, d\vec{x}.$$

(problematic in solver)

- Use upwinding when Re is large.

- Can easily improve accuracy and stability using a semi-implicit method to update vorticity in step 3 (e.g. ADI).

  - VERSION 1: linearize (constant $U^n, V^n$) in both steps.
  - VERSION 2:
    - take first ADI substep (in x), and update $\psi$.
    - calculate an intermediate velocity.
    - use this for second ADI substep (in y).
    - exchange order in alternate steps.

  $O(\Delta t)$

Getting $O(\Delta t^2)$ is tricky (see Pozrikidis, p.609)

- corner grid points excluded – avoids problems with "corner singularities" in some other methods.

- extension to 3D – see Pozrikidis, Ch. 13.