

# One-Shot Computation of Reachable Sets for Differential Games

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## ABSTRACT

We present a numerical method for computing backward reachable sets in differential games. A backward reachable set for time  $t$  is captured by the  $t$  sublevel set of the lower value function of the game, which coincides with the viscosity solution of a stationary Hamilton-Jacobi-Isaacs (HJI) equation. We solve the stationary HJI equation in a computationally efficient way that does not involve any numerical integration over time, which would otherwise be required for time-dependent HJI equations. Backward reachable sets for all time points can simultaneously be extracted from the solution. The performance of the method is demonstrated by investigating the growth of multicellular structures of non-malignant and malignant breast cells as a proof of principle.

## Categories and Subject Descriptors

G.1.0 [Numerical Analysis]: General

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Reachability Analysis, Differential Games, Hamilton-Jacobi-Isaacs Equations, Cancer Cells Systems Biology

## 1. INTRODUCTION

Reachable set computation is a core component of reachability analysis, which is an important research problem in the formal verification of, and controller synthesis for discrete, continuous and hybrid systems. A number of reachability

algorithms have been developed and extensively studied for several classes of dynamical systems. One broad class of the methods uses polyhedral over-approximations of reachable sets, such as polytopes [4, 11, 13, 5], oriented rectangular hulls [37], zonotopes [19], or ellipsoidal over-approximations [25]. Recent progress in this approach has demonstrated their scalability, accuracy and capability for handling parameter uncertainties [21, 20, 2, 3]. A second approach utilizes interval numerical method-based algorithms to track multi-dimensional axis-aligned intervals of a state to over-approximate reachable sets for systems with (or without) uncertain parameters [22, 33]. A third class of reachability algorithms is based on partitions of the (uncountable) continuous state space [39, 24]. This approach then utilizes an abstraction function that maps the continuous state space into the countable elements of the partition to simplify reachability analysis.

Most of the aforementioned algorithms in the three classes are designed to compute forward reachable sets for systems with (or without) control inputs. However, we often want to compute *backward reachable sets* for systems with adversarial disturbances as well as the control inputs e.g. safety guaranteed-design of controllers. To be more precise, we consider the situation in which we want to control a safety-critical system under the existence of a disturbance. Let  $\Gamma$  be the target set representing the set of unsafe states that we want to avoid. Then the  $t$ -backward reachable set consists of initial states from which the system can reach the unsafe set  $\Gamma$  in time  $t$ . Therefore, the  $t$ -backward reachable set information is useful for predictive threat assessment because it provides us with the set of states that will cause the system to be potentially unsafe at time  $t$ .

A natural approach for computing backward reachable sets with two competing inputs in continuous systems is to formulate a differential game. One general method for this approach uses techniques based on set-valued analysis and viability theory [12]. Another method is based on a Hamilton-Jacobi-Isaacs (HJI) partial differential equation (PDE). By applying the dynamic programming principle, we

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can derive an HJI equation whose viscosity solution is the (lower or upper) value function of the differential game of interest. The time-dependent formulation of the HJI equation [29] is a method for computing backward reachable sets in the presence of competing inputs and is widely applicable to both continuous and hybrid systems [40, 26, 41]. Well-studied numerical techniques [27, 29, 28] based on level set methods [31, 35, 32] have been developed to solve for the backward reachable set. In this time-dependent method, the backward reachable set for time  $t$  is the zero sublevel set of the viscosity solution of the HJI equation at time  $t$ . In other words, we need to integrate the HJI equation backward in time to obtain the solution, and the solution only contains the information of the reachable set for a single time  $t$ .

In this paper, we focus a zero-sum differential game problem with the value function whose  $t$  sublevel set contains the information of the backward reachable set for time  $t$ . That is, we save the information of the reachable sets for all time  $t$  in one value function, which represents the time to reach a target of interest when two inputs are competing each other. This value function is indeed the viscosity solution of a stationary HJI equation. Therefore, once we numerically solve the stationary HJI equation to obtain the solution, we can extract the backward reachable set for any time  $t$  by capturing the  $t$  sublevel set. Unlike the time-dependent method, the method is highly practical when we want to obtain the reachable set for large time  $t$  because the computational time is independent of  $t$ , unlike the time-dependent method. Furthermore, the method uses only  $O(N)$  memory to save the reachable sets for all time, where  $N$  is the number of nodes at which we evaluate the solution. This method is also flexible enough to handle nonlinear time-invariant dynamical systems. We apply our method to formation of structure of non-malignant and malignant mammary epithelial cells in 3D cultures: we assume a dynamic game of cellular growth potential and cell-cell adhesion potential.

## 2. REACHABILITY PROBLEM

Consider the following dynamical system in  $\mathbb{R}^n$  with two competing inputs

$$\dot{y}(t) = f(y(t), u(t), d(t)) \quad (1a)$$

$$y(0) = \mathbf{x}, \quad (1b)$$

where  $u(\cdot)$  and  $d(\cdot)$  are control and disturbance injected into the system, respectively, and  $U \subseteq \mathbb{R}^{nu}$  and  $D \subseteq \mathbb{R}^{nd}$  are feasible sets of control and disturbance values, respectively, and assumed to be compact. The two competing inputs  $u(\cdot)$  and  $d(\cdot)$  are considered as controls for Player I and Player II, respectively. We also introduce sets of admissible controls  $\mathcal{U} = \{u : [0, +\infty) \rightarrow U \mid u(\cdot) \text{ measurable}\}$  and  $\mathcal{D} = \{d : [0, +\infty) \rightarrow D \mid d(\cdot) \text{ measurable}\}$ . We assume that  $f : \mathbb{R}^n \times U \times D \rightarrow \mathbb{R}^n$  is continuous and that there exists  $L > 0$  such that  $\|f(x_1, a, b) - f(x_2, a, b)\| \leq L\|x_1 - x_2\|$  for all  $x_1, x_2 \in \mathbb{R}^n$ ,  $a \in U$ ,  $b \in D$ . Under these assumptions, the dynamical system (1) has a unique solution.

To clarify the objectives of Player I and Player II in a reachability problem that we are interested in, we introduce the following two subproblems:

- (A) (*Backward reachable sets*) Find the set  $\mathcal{B}(t)$  of initial states that can be driven to the target by Player II no matter how Player I chooses her control input, under

the assumption that Player II knows Player I's current and past decisions.

- (B) (*Time-to-reach*) Find the time to reach the target from a starting point  $\mathbf{x}$  when Player I wants to maximize the time, while Player II uses a strategy to minimize the time with knowledge of Player I's current and past decisions.

Note that the *Time-to-reach* problem (B) is a *two-player zero-sum differential game*, in which Player II uses a *nonanticipating strategy*. Once we obtain the time-to-reach function  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  in problem (B), we can compute the backward reachable set  $\mathcal{B}(t)$  as  $\{\mathbf{x} \mid \phi(\mathbf{x}) \leq t\}$  as we will see later. Extracting the reachable sets from the time-to-reach function,  $\phi(\cdot)$ , provides advantages in terms of computational memory and complexity because we save all of the information of reachable sets in one time-to-reach function and the complexity of the algorithm to compute  $\phi(\cdot)$  is invariant in time. We first address problem (B), which is valuable in itself, and then solve problem (A). To this end, we begin by introducing the mathematical notion of the time to reach a closed target  $\Gamma \in \mathbb{R}^n$  with a compact boundary, given  $u(\cdot)$  and  $d(\cdot)$ :

$$T_{\mathbf{x}}[u, d] = \min \{t \mid y(t) \in \Gamma\},$$

which is the *payoff* functional of the game (B). Following Varaiya [43] and Elliott-Kalton [16], we define the set of nonanticipatory strategies for Player II by

$$\begin{aligned} \Theta := \{\theta : \mathcal{U} \rightarrow \mathcal{D} \mid u(\tau) = \hat{u}(\tau) \forall \tau \leq t \text{ implies} \\ \theta[u](\tau) = \theta[\hat{u}](\tau) \forall \tau \leq t\}. \end{aligned}$$

Here,  $\theta[u] \in \mathcal{D}$  denotes the control of Player II with strategy  $\theta$  for the control  $u$  of Player I. Then, problem (B) is equivalent to the differential game problem

$$\phi(\mathbf{x}) := \min_{\theta \in \Theta} \max_{u \in \mathcal{U}} T_{\mathbf{x}}[u, \theta[u]]. \quad (2)$$

We call  $\phi(\cdot)$  the *lower value function* of the differential game problem (B). In this paper, we assume that the minimum and the maximum in (2) exist. Let us introduce the (lower) capturability set  $\mathcal{R}^* = \{\mathbf{x} \in \mathbb{R}^n \mid \phi(\mathbf{x}) < +\infty\}$ . Under the assumption that the minimum and the maximum in (2) exist, the capturability set  $\mathcal{R}^*$  coincides with the state space  $\mathbb{R}^n$ . Applying the dynamic programming principle, we can obtain the following stationary HJI equation with viscosity solution  $\phi$ :

$$H(\mathbf{x}, \nabla \phi(\mathbf{x})) = 0, \quad \text{in } \mathcal{R}^* \setminus \Gamma \quad (3a)$$

$$\phi(\mathbf{x}) = 0, \quad \text{on } \Gamma \quad (3b)$$

where  $H$  denotes the Hamiltonian defined by

$$H(\mathbf{x}, \mathbf{p}) = \min_{a \in U} \max_{b \in D} \{-\mathbf{p}^\top f(\mathbf{x}, a, b) - 1\}. \quad (4)$$

Detailed derivations and discussions are presented in [10, 8, 6] and [17] describes the time-dependent version. The viscosity solution of (3) is a weak solution that is consistent and unique in the domain where an associated free boundary problem yields a viscosity solution bounded below [14, 36]. Having the lower value function  $\phi$  as the viscosity solution of an HJI equation offers practical advantages. Once we have a numerical solution that approximates the viscosity solution, then it is automatically guaranteed that the solution approximates the lower value function due to the

uniqueness and consistency of the viscosity solution. Before presenting a numerical method for approximating the viscosity solution of (3), we need to determine how to extract the  $t$ -backward reachable set from the lower value function  $\phi$ , which represents the time-to-reach, in the differential game setting.

## 2.1 Backward reachable sets from time-to-reach

The backward reachable set for a certain time  $t$  is the collection of all initial states that can be driven to the target set  $\Gamma$  in time  $t$  with some disturbance  $d(\cdot) = \theta[u(\cdot)]$  no matter how we choose the control input  $u(\cdot)$ . A mathematical definition of the backward reachable set for time  $s$  is as follows:

*Definition 1.* An  $s$ -backward reachable set  $\mathcal{B}(s)$  of (1a) is defined by

$$\mathcal{B}(s) = \{\mathbf{x} \mid \exists \theta \in \Theta \text{ such that } \forall u \in \mathcal{U}, \exists \tau \in [0, s] \text{ satisfying } y(0) = \mathbf{x}, y(\tau) \in \Gamma\},$$

where  $y(\cdot)$  evolves with (1a).

Note that this definition is consistent with the description of the reachable set  $\mathcal{B}(t)$  in problem (A). The following proposition suggests a connection between the backward reachable set and lower value function  $\phi$  of the differential game (B).

**PROPOSITION 1.** *Given the lower value function  $\phi$  in (2), we can obtain the  $t$ -reachable set as*

$$\mathcal{B}(t) = \{\mathbf{x} \mid \phi(\mathbf{x}) \leq t\}. \quad (5)$$

**PROOF.** We first claim that  $\mathcal{B}(t) \subseteq \{\mathbf{x} \mid \phi(\mathbf{x}) \leq t\}$ . To see this, we fix a point  $\mathbf{x} \in \mathcal{B}(t)$ . By definition, there exists  $\theta^* \in \Theta$  such that for each  $u \in \mathcal{U}$ , there exists  $\tau \in [0, t]$  with  $y(\tau) \in \Gamma$  and  $y(0) = \mathbf{x}$ . Let  $\tau^*$  be the supremum of these  $\tau$ 's. Then we have

$$\max_{u \in \mathcal{U}} T_{\mathbf{x}}[u, \theta^*[u]] \leq \tau^* \leq t.$$

Due to the definition of the lower value function in (2), we have  $\phi(\mathbf{x}) \leq \max_{u \in \mathcal{U}} T_{\mathbf{x}}[u, \theta^*[u]] \leq t$ , which implies that  $\mathcal{B}(t) \subseteq \{\mathbf{x} \mid \phi(\mathbf{x}) \leq t\}$ , as desired.

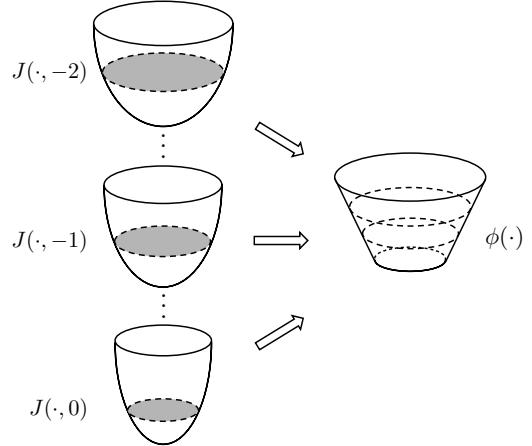
Now, we prove that  $\{\mathbf{x} \mid \phi(\mathbf{x}) \leq t\} \subseteq \mathcal{B}(t)$ . Fix a point  $\mathbf{x}$  such that  $\phi(\mathbf{x}) \leq t$ . Then,

$$\min_{\theta \in \Theta} \max_{u \in \mathcal{U}} T_{\mathbf{x}}[u, \theta[u]] \leq t.$$

This inequality means that there exists a nonanticipatory strategy  $\hat{\theta} \in \Theta$  such that, for any  $\hat{u} \in \mathcal{U}$ ,  $T_{\mathbf{x}}[\hat{u}, \hat{\theta}[\hat{u}]] \leq t$ . Hence, for any control  $\hat{u} \in \mathcal{U}$ , the system starting at  $\mathbf{x}$  can always be steered to target  $\Gamma$  in time  $t$  with the control  $\hat{d} = \hat{\theta}[\hat{u}]$  of Player II. Therefore,  $\mathbf{x} \in \mathcal{B}(t)$ . This completes the proof.  $\square$

Proposition 1 is the key motivation for designing a *one-shot method* for computing  $t$ -reachable sets for all  $t$  at once. This consists of the following two steps:

- (i) Compute  $\phi(\cdot)$  by solving the HJI equation (3).
- (ii) Extract  $\mathcal{B}(t)$  for any  $t$  by capturing the  $t$  sublevel set of  $\phi(\cdot)$ .



**Figure 1:** Compression of zero sublevel sets: the zero sublevel set (gray) of  $J(\cdot, -t)$  corresponds to the set  $\{\mathbf{x} \mid \phi(\mathbf{x}) \leq t\}$ . In this way, we compress the information of reachable sets contained in multiple value functions  $J(\cdot, -t)$ 's into a single *time-to-reach* function  $\phi(\cdot)$ .

Note that a single evaluation of  $\phi(\cdot)$  on the state space is enough to obtain all  $t$ -reachable sets. In practice, a subset of the state space is chosen as the computational domain  $\Omega$  in which we evaluate  $\phi$ . In this case, the  $t$  sublevel set of  $\phi$  coincides with the intersection of the  $t$ -backward reachable set and the computational domain, i.e.,  $\mathcal{B}(t) \cap \Omega$ . The computational domain  $\Omega$  should be chosen large enough in the event that one wants to retrieve a whole region of the  $t$ -backward reachable set. Even if  $\Omega$  is chosen to not include the  $t$ -backward reachable set, because we do not know the size and shape of the reachable set *a priori*, the set  $\{\mathbf{x} \mid \phi(\mathbf{x}) \leq t\}$  is equal to  $\mathcal{B}(t) \cap \Omega$ , which is still valuable information.

## 2.2 Comparison with a time-dependent HJI method

It is important to note that our one-shot method is *time-independent* in the sense that the HJI equation (3) does not depend explicitly on time. In other words, (3) is a *boundary value* problem, whereas an HJI equation in a time-dependent formulation for computing reachable set is an *initial (or final) value* problem of the form

$$\frac{\partial J(\mathbf{x}, t)}{\partial t} + \min[0, H(\mathbf{x}, \nabla J(\mathbf{x}, t))] = 0, \quad \text{in } \Omega \times [-T, 0] \quad (6)$$

$$J(\mathbf{x}, 0) = l(\mathbf{x}), \quad \text{on } \Omega \times \{t = 0\},$$

where the zero sublevel set of the initial value  $l$  coincides with the target  $\Gamma$ . Here, the Hamiltonian is given by  $H(\mathbf{x}, \mathbf{p}) = \max_{a \in U} \min_{b \in D} \mathbf{p}^\top f(\mathbf{x}, a, b)$ . In this formulation [40, 29], we compute the  $t$ -backward reachable set as the zero sublevel set of  $J(\cdot, -t)$ . The time-dependent method is somewhat more flexible than our stationary formulation because it is capable of dealing with time-varying dynamics.

The time-dependent formulation is limited in scalability because the memory and computational cost grow exponentially as the system dimension increases. Our stationary formulation also intrinsically exhibits the scalability problem because we need to evaluate the solution of the sta-

tionary HJI equation at all nodes that appropriately discretize the state space. However, the one-shot method alleviates the cost on computational complexity and memory costs in the following sense. Suppose that we discretize the domain  $\Omega$  with  $N$  nodes and time interval  $[0, T]$  with  $N_T$  number of points. To solve the stationary HJI (3), we use the Lax-Friedrichs sweeping method [23] with complexity independent of time. However, the complexity of any grid-based numerical method for the time-dependent HJI (6) linearly increases as  $N_T$  increases. Hence, we can reduce the computational complexity by using the one-shot method when  $N_T$  is large. This complexity result is linked to the improvement of computational time when we use the one-shot method to compute reachable sets for a long time horizon. Once we solve the stationary HJI equation, then the  $t$ -reachable set (intersected with the computational domain) can be automatically extracted from the solution no matter how large  $t$  is. However, the time-dependent method requires us to integrate the non-stationary HJI equation (6) over time with a sufficiently small time step to satisfy the Courant-Friedrichs-Lowy (CFL) condition for the stability of the method. Hence, it is quite computationally intensive to determine a reachable set for a long time horizon using the time-dependent method. The comparison between the computational times of the two methods is presented in an example in Section 4.

Another important advantage of the one-shot method over the time-dependent method is the improvement in computational memory when we are in a situation that requires holding all  $t$ -reachable sets in a given time interval. As depicted in Figure 1, we need  $N_T$  value functions to hold the reachable sets at  $N_T$  time points, because only the zero sublevel set of each value function contains the reachability information. However, in the one-shot method, every  $t$  sublevel set of one value function  $\phi(\cdot)$  provides the  $t$  reachable set. That is, the one-shot method uses only  $O(N)$  memory since it saves the information of reachable sets for all time in one value function  $\phi(\cdot)$ , while the time-dependent method requires  $O(N \times N_T)$  memory when we want to save the information of  $t$ -reachable sets for  $t \in \{t_1, t_2, \dots, t_{N_T}\}$ . One can reformulate the stationary HJI into a time-dependent HJI (with solution  $\tilde{J}$ ) if the Hamiltonian satisfies the *non-characteristic criterion* [30]. When this reformulation is valid, one may want to use a thresholding formula  $\phi(\mathbf{x}) = \{t \mid \tilde{J}(\mathbf{x}, t) = 0\}$  at each time step to construct the time-to-reach function  $\phi(\cdot)$  from the time-dependent HJI in order to achieve  $O(N)$  memory. However, the resolution of  $\phi$  obtained by this approach can be poor unless high-order interpolations in both time and space are employed. High-order interpolations are not desirable in terms of the memory usage because we must save solution copies at multiple time points. Another downside of this reformulation is that it precludes gaining the advantages of the one-shot method in computational time and complexity.

The information on the time-to-reach (2) that we obtain in the stationary formulation is another benefit. In the time-dependent HJI formulation, the solution rather works as an indicator for distinguishing the reachable set from another region in the computational domain. However, the information on the time-to-reach can be used for synthesizing optimal controllers under adversarial disturbances. Ideally, if the value function in the time-dependent method is a signed-distance function, then its value is related to the distance to

the reachable set. To obtain this information, we need to frequently reinitialize the value function as a signed distance function. However, the reinitialization procedure imposes additional computational burden.

Various numerical techniques and supporting theory allow us to compute solutions of time-dependent Hamilton-Jacobi equations with high accuracy and convergence rate [31, 35, 32, 29]. A MATLAB toolbox for solving (6) is also publicly available [1]. The numerical algorithms for the time-dependent methods enjoy the continuity of the value function [29]. In the stationary HJI, however, the viscosity solution is likely to have the set of discontinuities, which has zero measure. To approximate the discontinuous viscosity solution, one can use a method that directly discretizes the system dynamics and formulate the discrete dynamic programming equation [7, 9]. The numerical solution obtained by this method converges to the discontinuous viscosity solution. In this article, we use the *Lax-Friedrichs sweeping method* [23], which directly discretizes (3) to compute the solution. This method, which will be discussed in the next section, is very simple, easy to implement and handles any number of dimensions. Although the convergence of the Lax-Friedrichs sweeping method for a solution with discontinuities has not been shown, the method seems to work well in practice. When  $\phi$  has severe discontinuities, we recommend that readers use the convergent fully discrete method that can handle the discontinuities [7, 9]. We also note that the fully discrete method for approximating the time-to-reach function using the viability theory [12] is a good alternative to the stationary HJI approach because the viability theory-based method does not assume the continuity of the value function.

### 3. COMPUTATION OF REACHABLE SETS

Let us define a uniform grid on an  $n$ -dimensional computational domain  $\Omega$  with grid spacing  $h$ . Let  $N_i$  be the number of nodes on the  $i$ th axis. If  $\Omega$  is two-dimensional, for example, we let  $\mathbf{x}_{i,j}$  denote the position of node  $(i, j)$ , and let  $\phi_{i,j}$  denote  $\phi(\mathbf{x}_{i,j})$  for  $i = 1, \dots, N_1$  and  $j = 1, \dots, N_2$ . A similar notation can be naturally extended for any higher-dimensional  $\Omega$ .

#### 3.1 Lax-Friedrichs scheme

As we have discussed, the time-to-reach function  $\phi(\cdot)$  can be computed as the lower value function (2) of the differential game problem (B), which also corresponds to the viscosity solution of (3). Therefore, our goal is to solve (3) with an appropriate numerical method, which directly approximates (3) without any time-dependent reformulation. We first consider the one-dimensional case for the sake of exposition. One way to discretize (3) is by using the Lax-Friedrichs scheme for (3), i.e.,

$$H\left(\mathbf{x}_i, \frac{(\phi_x)_i^- + (\phi_x)_i^+}{2}\right) - \sigma_1 \frac{(\phi_x)_i^- - (\phi_x)_i^+}{2} = 0, \quad (7)$$

where  $(\phi_x)_i^-$  and  $(\phi_x)_i^+$  are backward (left-sided) and forward (right-sided) discretizations for  $\phi_x$  at the  $i$ th node, respectively. The Lax-Friedrichs numerical Hamiltonian is consistent. Here the bound on a (numerical) artificial viscosity  $\sigma_1$ ,

$$\sigma_1 \geq \max_{p \in [p_L, p_U]} \left| \frac{\partial H(\mathbf{x}, p)}{\partial p} \right|,$$

is analogous to the monotonicity condition on the Lax-Friedrichs scheme for time-dependent Hamilton-Jacobi equations, where  $p_L$  and  $p_U$  are the lower and upper bounds of  $\phi_x$ , respectively [15]. If we use the first-order approximation of  $(\phi_x)^\pm$ , then the Lax-Friedrichs Hamiltonian can be written as

$$H\left(\mathbf{x}_i, \frac{\phi_{i+1} - \phi_{i-1}}{2h}\right) - \sigma_1 \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{2h} = 0, \quad (8)$$

Alternatively, one can use higher-order approximations such as the third-order *essentially non-oscillatory interpolation* (ENO) or the fifth-order *weighted ENO* (WENO). The discretization provides a system of nonlinear equations for the  $\phi_i$ 's because the Hamiltonian is, in general, nonlinear. We observe that the  $i$ th equation only depends on  $\phi_{i-1}$ ,  $\phi_i$  and  $\phi_{i+1}$  in (8). This motivates us to use a *Gauss-Seidel*-type iterative method even for non-convex Hamiltonians as in the fast sweeping method for convex Hamilton-Jacobi equations [42].

### 3.2 A sweeping algorithm

The Lax-Friedrichs sweeping method [23] approximates the viscosity solution of (3) with the Lax-Friedrichs scheme using the Gauss-Seidel type sweeping idea. In other words, the method uses the following update formula based on (7) for Gauss-Seidel iteration [23]:

$$\begin{aligned} \phi_i^{\text{new}} = & c \left[ -H\left(\mathbf{x}_i, \frac{(\phi_x)_i^- + (\phi_x)_i^+}{2}\right) - \sigma_1 \frac{(\phi_x)_i^- - (\phi_x)_i^+}{2h} \right] \\ & + \phi_i. \end{aligned}$$

If we use the first-order approximations of spatial derivatives, we can rewrite the update formula as

$$\phi_i^{\text{new}} = c \left[ -H\left(\mathbf{x}_i, \frac{\phi_{i+1} - \phi_{i-1}}{2h}\right) + \sigma_1 \frac{\phi_{i+1} + \phi_{i-1}}{2h} \right].$$

For a two-dimensional problem, we can simply extend it to

$$\begin{aligned} \phi_{i,j}^{\text{new}} = & c \left[ -H\left(\mathbf{x}_{i,j}, \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2h}, \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2h}\right) \right. \\ & \left. + \sigma_1 \frac{\phi_{i+1,j} + \phi_{i-1,j}}{2h} + \sigma_2 \frac{\phi_{i,j+1} + \phi_{i,j-1}}{2h} \right], \end{aligned} \quad (9)$$

where  $c = h/(\sigma_1 + \sigma_2)$  and

$$\sigma_1 \geq \max_{\substack{p \in [p_L, p_U] \\ q \in [q_L, q_U]}} \left| \frac{\partial H(\mathbf{x}, p, q)}{\partial p} \right|, \quad \sigma_2 \geq \max_{\substack{p \in [p_L, p_U] \\ q \in [q_L, q_U]}} \left| \frac{\partial H(\mathbf{x}, p, q)}{\partial q} \right|.$$

The essence of the Gauss-Seidel method is its successive displacement, which accelerates the convergence compared to the *Jacobi* method. For example, if we sweep a two-dimensional grid from *SW* to *NE*, then  $\phi_{i,j}^{\text{new}}$  is updated with (9) using updated  $\phi_{i-1,j}$ ,  $\phi_{i,j-1}$  and un-updated  $\phi_{i+1,j}$ ,  $\phi_{i,j+1}$ . To cover all characteristic directions of (3), we alternate the sweeping direction as *SW*, *SE*, *NW* and *NE*. As presented in Algorithm 1, we initialize  $\phi$  with some prescribed value  $\phi_U$  for all  $\mathbf{x} \in \Omega$ . Here,  $\phi_U$  is supposed to be greater than  $\max_{\mathbf{x} \in \Omega} \phi(\mathbf{x})$ . Then,  $\phi_{i,j}$  is updated only if  $\phi_{i,j}^{\text{new}}$  is less than or equal to its previous value as imposed in lines 8, 13, 18, 23, so as to capture the correct characteristic direction that results in the shortest time to reach the target from  $\mathbf{x}_{i,j}$ . Once the solution at a node reaches the minimal value it can achieve, it remains unchanged at the correct solution value. Algorithm 1 can be extended to any dimension. If the state space is  $n$  dimensional, however, we then need to consider  $2^n$  alternating sweeping directions.

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**Algorithm 1:** Pseudocode of the Lax-Friedrichs sweeping-based reachable set computation.

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**1 Initialization:**

2 Set  $\phi(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in \Gamma \\ \phi_U & \text{otherwise;} \end{cases}$

**3 Solving the HJI PDE (3):**

4 **while**  $|\phi - \phi^{\text{old}}| > \epsilon$  **do**  
5      $\phi^{\text{old}} \leftarrow \phi$   
6     **for**  $i = 2 : N_1 - 1; j = 2 : N_2 - 1$  **do**  
7         Compute  $\phi_{i,j}^{\text{new}}$  using (9).  
8          $\phi_{i,j} \leftarrow \min(\phi_{i,j}^{\text{new}}, \phi_{i,j})$ .  
9     **end**  
10     Impose boundary conditions using (10).  
11     **for**  $i = N_1 - 1 : 2; j = 2 : N_2 - 1$  **do**  
12         Compute  $\phi_{i,j}^{\text{new}}$  using (9).  
13          $\phi_{i,j} \leftarrow \min(\phi_{i,j}^{\text{new}}, \phi_{i,j})$ .  
14     **end**  
15     Impose boundary conditions using (10).  
16     **for**  $i = 2 : N_1 - 1; j = N_2 - 1 : 2$  **do**  
17         Compute  $\phi_{i,j}^{\text{new}}$  using (9).  
18          $\phi_{i,j} \leftarrow \min(\phi_{i,j}^{\text{new}}, \phi_{i,j})$ .  
19     **end**  
20     Impose boundary conditions using (10).  
21     **for**  $i = N_1 - 1 : 2; j = N_2 - 1 : 2$  **do**  
22         Compute  $\phi_{i,j}^{\text{new}}$  using (9).  
23          $\phi_{i,j} \leftarrow \min(\phi_{i,j}^{\text{new}}, \phi_{i,j})$ .  
24     **end**  
25     Impose boundary conditions using (10).  
26 **end**

**27 Reachable set extraction:**

28  $\mathcal{B}(t) \leftarrow \{\mathbf{x} \mid \phi(\mathbf{x}) \leq t\}$ .

---

### 3.3 Conditions on computational boundaries

The Lax-Friedrichs scheme requires conditions on computational boundaries, unlike an upwind scheme that automatically assigns a solution value on boundary nodes using the characteristic information. We proceed along the line of linear extrapolation-based conditions [23], which is analogous to the boundary conditions for time-dependent HJI equations with Lax-Friedrichs Hamiltonians [29, 1]. The conditions are as follows:

$$\begin{aligned} \phi_{1,j}^{\text{new}} &= \min[\max(2\phi_{2,j} - \phi_{3,j}, \phi_{3,j}), \phi_{1,j}] \\ \phi_{N_1,j}^{\text{new}} &= \min[\max(2\phi_{N_1-1,j} - \phi_{N_1-2,j}, \phi_{N_1-2,j}), \phi_{N_1,j}] \\ \phi_{i,1}^{\text{new}} &= \min[\max(2\phi_{i,2} - \phi_{i,3}, \phi_{i,3}), \phi_{i,1}] \\ \phi_{i,N_2}^{\text{new}} &= \min[\max(2\phi_{i,N_2-1} - \phi_{i,N_2-2}, \phi_{i,N_2-2}), \phi_{i,N_2}] \end{aligned} \quad (10)$$

These computational boundary conditions prevent inflow characteristics that can introduce spurious data into the numerical solution. To see this, let us consider the case  $\phi_{2,j} < \phi_{3,j}$ , in which characteristic passes  $(2, j)$  in advance to  $(3, j)$ , i.e., the characteristics are locally inflows. Then the rules (10) choose  $\phi_{1,j} = \phi_{3,j} > \phi_{2,j}$  to enforce characteristics outflowing on the computational boundary. If  $\phi_{3,j} \leq \phi_{2,j}$ , on the other hand, we already have outflowing characteristics near the computational boundary. Hence it is reasonable to approximate  $\phi_{0,j}$  as  $2\phi_{2,j} - \phi_{3,j}$ , which is a linear extrapolation, as suggested by the rules (10). The

**Table 1: Errors, convergence rates and computational times (unit: sec) of the test problem.**

$h$	CPU	$\ \phi - \phi^A\ _1$	$r$	$\ \phi - \phi^A\ _\infty$	$r$
0.02	0.12	$5.548 \times 10^{-2}$	0.83	$6.160 \times 10^{-2}$	0.82
0.01	0.65	$3.096 \times 10^{-2}$	0.84	$3.434 \times 10^{-2}$	0.84
0.005	3.91	$1.715 \times 10^{-2}$	0.85	$1.892 \times 10^{-2}$	0.86
0.0025	25.03	$9.420 \times 10^{-3}$	0.86	$1.033 \times 10^{-2}$	0.87

outer minimizations in (10) are to ensure the numerical solution can only decrease in the sweeping algorithm.

## 4. EXAMPLES

### 4.1 Test problem

To test the convergence rate of our algorithm, we consider a simple pursuit-evasion game problem, where Player I is an evader and Player II is a pursuer. Let  $V_1$  and  $V_2$  be the velocities of Player I and Player II, respectively. Then the dynamics of the relative coordinate  $y$  of Player II with respect to Player I can be described by

$$\dot{y} = f(y, u, d) = V_1 u - V_2 d,$$

where  $u$  and  $d$  are controls of Player I and Player II, respectively. If we set the target  $\Gamma$  at the origin, then the  $t$ -backward reachable set  $\mathcal{B}(t)$  corresponds to the set of relative initial configurations of Player II with respect to Player I, with which Player II can capture Player I in time  $t$  using the nonanticipatory strategy. In other words,  $\mathcal{B}(t) = \{\mathbf{x} \mid \phi(\mathbf{x}) \leq t\}$ , where  $\phi(\cdot)$  is defined as (2) with  $\Gamma = \{0\}$ . Let  $U = [-\alpha, +\alpha]$  and  $D = [-\beta, +\beta]$ , then  $\phi(\cdot)$  solves the HJI equation (3) with the Hamiltonian (4) given by

$$H(\mathbf{p}) = (-V_1 \alpha + V_2 \beta) \|p\| - 1.$$

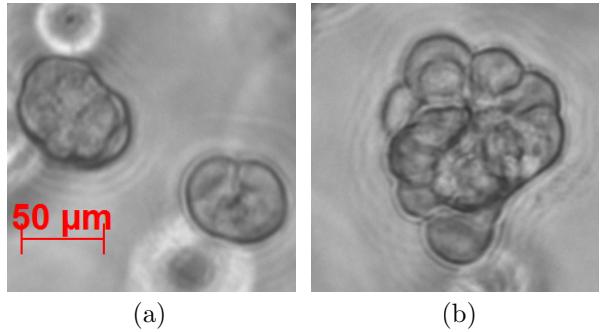
Hence, the HJI equation (3) is indeed an Eikonal equation with analytic solution

$$\phi^A(\mathbf{x}) = \frac{\|\mathbf{x}\|}{-V_1 \alpha + V_2 \beta}.$$

We choose  $(\alpha, \beta, V_1, V_2) = (1, 2, 1, 1)$ . The computational domain  $\Omega$  is chosen as  $[-1, 1]^2$ . The Lax-Friedrichs sweeping method was implemented in C++. All tests in this and the following examples were performed on a 2.53GHz Intel Core i5 processor with 4 GB RAM. The errors, convergence rates and computational times are presented in Table 1. These convergence rates are consistent with the results reported in [23].

### 4.2 Structure formation by non-malignant and malignant cells

To illustrate our method, we applied it to the different growth patterns of non-malignant and malignant cells for identifying the characteristics of growth and adhesion potentials. The non-malignant S1 and malignant T4-2 cell lines of the HMT-3522 breast cancer progression series can be distinguished by their growth pattern in a three-dimensional laminin-rich gel: S1 cells form spherical hollow structures ('acini') with a homogeneous layer of basement membrane formed around them (Figure 2 (a)), while T4-2 cells grow in a disorganized pattern, forming large multicellular clusters



**Figure 2: Multicellular structures formed by (a) non-malignant S1 cells ('acini') and (b) malignant T4-2 cells when grown in a laminin-rich gel.**

and patchy basement membranes surrounding the structure (Figure 2 (b)) [45, 34]. In S1 cells, the cell-cell adhesion protein E-Cadherin is localized at the lateral cell membrane together with its binding partner  $\beta$ -catenin; as a complex they mediate cell-cell adhesion. T4-2 cells display decreased cell-cell adhesion with E-Cadherin not being co-localized with  $\beta$ -catenin [45]. Modifications of E-Cadherin expression not only influence cell adhesion but also modulate cellular growth: inhibition of E-Cadherin in S1 cells leads to higher cell numbers per structure [18]; however, overexpression of E-Cadherin in the breast cancer cell line MDA-MB-231 reduces the structure size to 50% [44]. In this example, we propose a hypothesis of a dynamic game between adhesion and growth potentials and use our technique to predict the size of the initial and intermediate structures, where the structures at a final time are given as data. We want to illustrate how comparing the structure size and shape at the final and initial times can help to draw conclusions about the underlying mechanism.

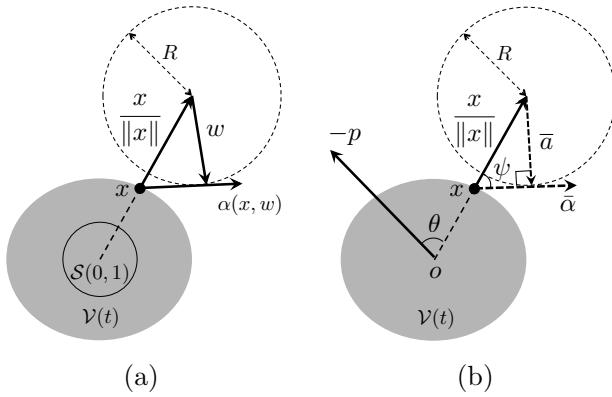
Let  $\mathcal{R}(t)$  and  $\mathcal{V}(t)$  be the outer boundary and the inner volume of the multicellular structure formed by HMT-3522 cells at time  $t$  in a 3D laminin 111-rich gel (l-ECM), respectively. We assume that  $\mathcal{R}(T)$  and  $\mathcal{V}(t)$  are given as data, where  $T > 0$  is the final time and  $\mathcal{V}(t)$  is to be open for all  $t$ . We describe the trajectory starting from a point on  $\mathcal{R}(s)$  for any  $s \leq T$  by the dynamical system  $\dot{y} = f(y, u, d)$ , where  $u$  represents adhesion and  $d$  represents the growth potential. More specifically, we set

$$\dot{y}(t) = f(y(t), u(t), d(t)) = \alpha(y(t), u(t)) \beta(y(t)) d(t),$$

where  $f : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3$ . The first term  $\alpha : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , which determines the direction of the growth, represents the effect of adhesion. We assume that cells with strong adhesion try to maximize cell-cell contact. Therefore, to model the effect of adhesion, we define  $\alpha$  by

$$\alpha(x, w) = \frac{x/\|x\| + w}{\|x/\|x\| + w\|},$$

with  $\|w\| \leq R$ . Note that  $\alpha$  only determines the direction, and that  $\|\alpha(x, w)\| \equiv 1$ . As shown in Figure 3 (a), adhesion changes the growth direction from the normal line of the surface of  $\mathcal{S}(0, 1)$  at  $x$ , where  $\mathcal{S}(0, 1)$  is the unit sphere centered at the origin. As the effect of adhesion gets stronger, the growth direction deviates further from the normal line, i.e.,  $R$  is large in this case.



**Figure 3:** (a) Modeling the effect of adhesion. (b) Angles  $\theta$  and  $\psi$ .

The second term of  $f$  models the effect of the growth response (determined by surrounding growth stimulating agents as well as the cellular sensitivity towards those agents). The growth response leads to cell proliferation in the direction determined by  $\alpha$ . The given function  $\beta : \mathbb{R}^3 \rightarrow \mathbb{R}$  represents the inhomogeneity of the cellular microenvironment. To model the inhomogeneity, we synthesize  $\beta^*$  as a noisy map, the value of which is uniformly distributed at all nodes. At  $\mathbf{x}$ , which is not a node, we define  $\beta(\mathbf{x})$  by the linear interpolation of values at the neighboring nodes. By this definition,  $\beta^*$  is continuous.

We propose a hypothesis that the growth potential minimizes the time-to-reach the boundary of the given final multicellular structure competing with adhesion. We also assume that the growth potential (Player II) has the knowledge of the current and past decision of the adhesion potential (Player I). Then, the  $t$ -backward reachable set  $\mathcal{B}(t)$  corresponds to the set of initial states that can reach the final structure in time  $t$ , no matter how the adhesion potential tries to maximize the time-to-reach. Therefore, the complement of the  $t$ -reachable set,  $\mathcal{B}(t)^c$ , is the set of all initial states that cannot reach the final structure in time  $t$  with some controls of adhesion potential. From this observation we have  $\mathcal{B}(0) = \mathcal{V}(T)^c$ ,  $\mathcal{R}(T-t) \subseteq \mathcal{B}(t)$  and  $\mathcal{B}(t)^c \subseteq \mathcal{V}(T-t)$ . We define the bounds of the adhesion and growth potentials as  $U = \{u | \|u\| \leq R\}$  and  $D = [\underline{d}, \bar{d}]$  with  $\underline{d} \geq 0$ , respectively. To compute the backward reachable set, we proceed along the lines of Section 2. We formulate the HJI equation (3) with the Hamiltonian (4),

$$\begin{aligned} H(\mathbf{x}, \mathbf{p}) &= \min_{a \in U} \max_{b \in D} \{-\mathbf{p}^\top \alpha(\mathbf{x}, a) \beta(\mathbf{x}) b - 1\} \\ &= \|\mathbf{p}\| \cos(\theta + \psi) \beta(\mathbf{x}) \bar{d} \mathbf{1}_{\{\theta + \psi < \pi/2\}} \\ &\quad + \|\mathbf{p}\| \cos(\theta + \psi) \beta(\mathbf{x}) \underline{d} \mathbf{1}_{\{\pi/2 \leq \theta + \psi < \pi\}} \\ &\quad - \|\mathbf{p}\| \beta(\mathbf{x}) \underline{d} \mathbf{1}_{\{\pi \leq \theta + \psi\}} - 1, \end{aligned}$$

where  $\mathbf{1}$  is the indicator function. As depicted in Figure 3 (b),  $\theta \leq \pi$  is the angle between  $-\mathbf{p}$  and  $\mathbf{x}$  and  $\psi$  is the angle between  $\mathbf{x}$  and  $\bar{\alpha}$ , where  $\alpha$  is a unit vector starting from  $\mathbf{x}$  and tangential to the sphere. We choose  $\bar{\alpha}$  so that  $-\mathbf{p}$ ,  $\mathbf{x}$  and  $\bar{\alpha}$  lie in the same plane. (There are two candidates for such  $\bar{\alpha}$ ; we choose the one outward from the (convex) conic hulls of  $\{o, -\mathbf{p}, \mathbf{x}\}$ , as shown in Figure 3 (b).) Then the angle between  $-\mathbf{p}$  and  $\bar{\alpha}$  is equal to

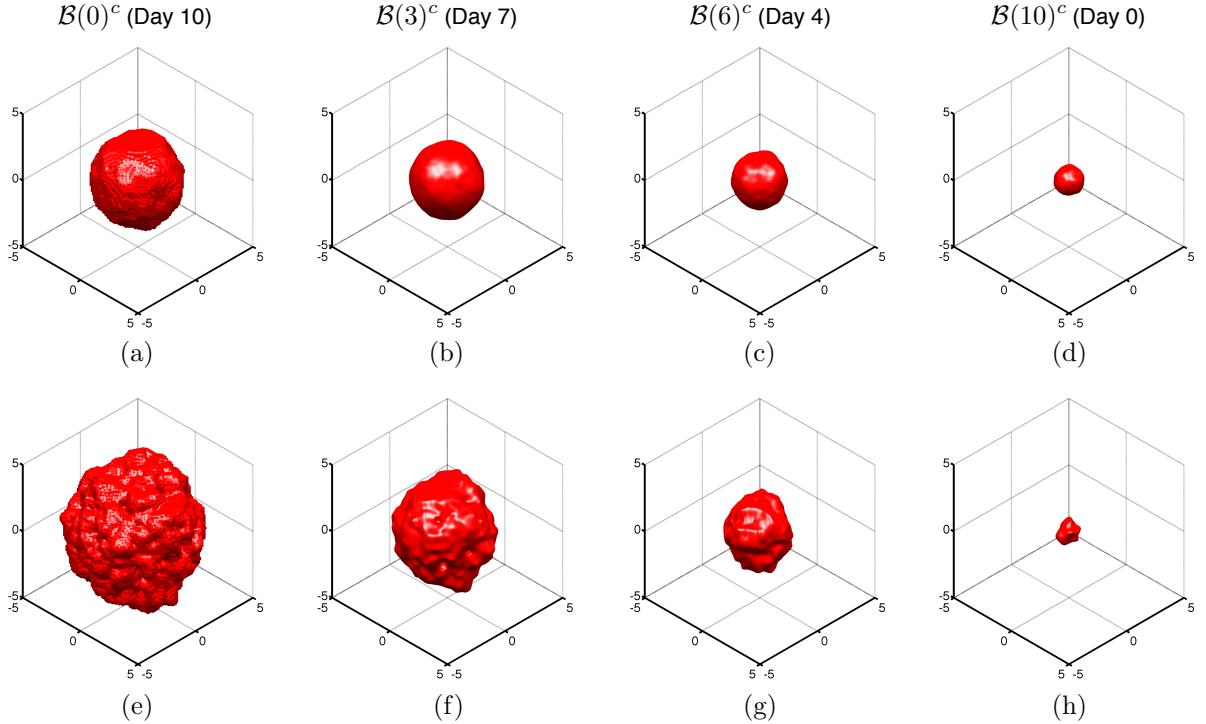
$\theta + \psi = \cos^{-1} \frac{\|\mathbf{p}\|^2 + \|\mathbf{x}\|^2 - \|-\mathbf{p} - \mathbf{x}\|^2}{2\|\mathbf{p}\|\|\mathbf{x}\|} + \sin^{-1} R$ . If  $\theta + \psi < \pi/2$ , then  $\cos(\theta + \psi) \geq 0$  and  $\cos(\cdot)$  is decreasing. Hence, an optimal  $a^*$  can be chosen as  $\bar{a}$  so that  $\alpha(\mathbf{x}, a^*) = \bar{\alpha}$ . Furthermore, an optimal  $b^*$  must be  $\bar{d}$  because  $\beta \geq 0$  and  $\cos(\theta + \psi) \geq 0$ . If  $\pi/2 \leq \theta + \psi < \pi$ , then  $\cos(\theta + \psi) < 0$  and  $\cos(\cdot)$  is decreasing. In this case, an optimal  $a^*$  is  $\bar{a}$  as before so that  $\alpha(\mathbf{x}, a^*) = \bar{\alpha}$ . Due to the fact that  $\beta \geq 0$  and  $\cos(\theta + \psi) < 0$ , an optimal  $b^*$  should be  $\underline{d}$ . If  $\theta + \psi \geq \pi$ , then an optimal  $\psi^*$ , which is determined by an optimal  $a^*$ , is  $\pi - \theta$  so that  $\cos(\theta + \psi^*) = -1$ . Then an optimal  $b^*$  is  $\underline{d}$  because  $\beta \geq 0$  and  $\cos(\theta + \psi) < 0$ .

We compute the backward reachable sets of both non-malignant and malignant cells, where the target is  $\Gamma = \mathcal{V}(T)^c$ , i.e., the complement of the final structure. For non-malignant S1 cells, we assume strong cell adhesion and therefore a high value of  $R$  ( $R = 0.5$ ). For malignant T4-2 cells, we assume weak cell adhesion, which is represented by a low value of  $R$  ( $R = 0.01$ ). As malignant cells have a higher proliferation rate than non-malignant cells, we choose the bound on the growth rate per hour for malignant cells to be  $(\underline{d}, \bar{d}) = (0.001/2, 0.01/2)$  and  $(\underline{d}, \bar{d}) = (0.001/3, 0.01/3)$  for non-malignant cells. Last, the variance of the map of microenvironmental inhomogeneity for the malignant T4-2 cells is chosen to be twice as high as that for non-malignant cells because the formation of the basement membrane surrounding the structures is disturbed in the malignant case.

The complement of the  $t$ -backward reachable set  $\{\mathbf{x} | \phi(\mathbf{x}) \leq t\}$  computed by Algorithm 1 is shown in Figure 4. We chose the computational domain  $\Omega$  as  $[-5, 5]^3$  and evaluated the solution at  $10^3$  nodes with grid spacing  $h = 0.1$ . Recall that  $\mathcal{B}(t)^c \subseteq \mathcal{V}(10-t) \subseteq \mathcal{B}(0)^c$ , we observe that the structure of non-malignant S1 cells with adversarial adhesion has much lower variation than that of malignant T4-2 cells. It was found that multicellular structures formed by S1 cells rotate within the matrigel with a coherent angular motion (CaMo) and display low variation in the structure shapes. However, T4-2 cells show more random lateral movement in the early multicellular stage and thus result in rather high variations in the structures [38].

Although the final structures of non-malignant and malignant cells are widely different in terms of size and shape (Figures 4 (a) and (f)), the smallest possible size of their initial structures are comparable to each other (Figures 4 (e) and (j)), which is significant. This agrees with the experimental data: at an early stage the size of S1 and T4-2 structures is similar *in vitro*. Being able to compare the final, intermediate and initial structures *in vitro* with the final structure and backward reachable sets of the simulation gives us a tool with which to validate hypotheses related to how the structures evolve and which processes are involved. In this particular simulation we find that the competition of adhesion and growth indeed allows for the structure formations observed *in vitro*. In the future, further analysis using our method can clarify the detailed roles of cell-adhesion, growth and the microenvironment for the size and shape of biological multicellular structures.

To compare the one-shot method and the time-dependent method, we also compute the same reachable sets using the C++ implementation of the level set method for time-dependent HJI equations with Lax-Friedrichs Hamiltonians. When solving the time-dependent HJI equation (6), we chose the CFL constant as 0.75 and used the first-order approximations for spatial derivatives and forward Euler method



**Figure 4:** Backward reachable sets obtained by the one-shot method. Structure formation of (a)–(e) non-malignant cells; (f)–(j) malignant cells.

**Table 2: Computational times (unit: sec) for the one-shot method and the time-dependent method.**

$h$	One-shot			Time-dependent		
	$\mathcal{B}(5)$	$\mathcal{B}(10)$	$\mathcal{B}(20)$	$\mathcal{B}(5)$	$\mathcal{B}(10)$	$\mathcal{B}(20)$
0.2	2.74	2.74	2.74	1.49	3.29	6.32
0.1	34.22	34.22	34.22	23.78	46.31	93.00
0.05	458.30	458.30	458.30	373.00	729.16	1466.7

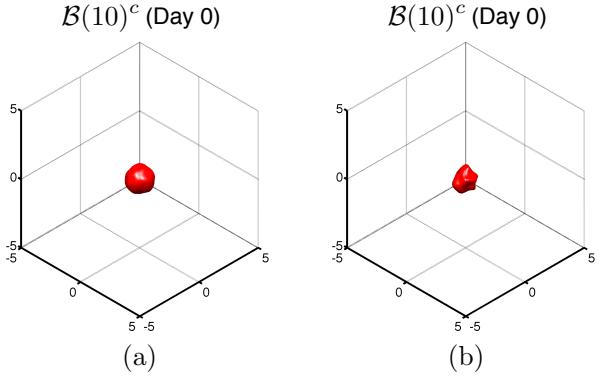
for integration over time, which are the fastest among other discretization options. As shown in Figures 4 and 5, the reachable sets generated by the two methods are very similar. Although the Lax-Friedrichs sweeping method is known to introduce somewhat higher numerical diffusion into the solution, we do not observe these effects in this example due to small artificial viscosities  $\sigma_i$  for  $i = 1, 2, 3$ .

The Lax-Friedrichs sweeping-based one-shot method saves computational time if we compute reachable sets for long time horizons. As presented in Table 2, the one-shot method is faster than the time-dependent method for computing  $\mathcal{B}(10)$  and  $\mathcal{B}(20)$ , although it is slower for  $\mathcal{B}(5)$ . The computational time gap between the two methods increases as we compute the reachable set for a longer time horizon because the one-shot method takes the same time to compute  $\mathcal{B}(t)$  for any  $t$  while the computational time for the time-dependent method increases linearly with respect to  $t$ . For example, when using  $h = 0.05$ , the one-shot method is approximately 3.2 times faster than the time-dependent method for computing  $\mathcal{B}(20)$ . This computational time behavior shows one of the advantages of the one-shot method: because we save

the information of the time-to-reach in one level-set function in the one-shot method, it takes the same time to extract the  $t$ -reachable set for any  $t$ . We also observe that, when we refine the grid twice, the one-shot method takes approximately 12–13 times longer, while the time-dependent method takes approximately 16 times longer. Let  $h^*$  and  $k^*$  be the nominal grid spacing and time step sizes, respectively. The time-dependent method requires the CFL condition, which restricts the time step size by the grid spacing size. If we decrease the grid size by half, i.e.,  $h = h^*/2$ , for example, then the time step size should be  $k = k^*/2$  when using the same CFL constant. Therefore, in theory, the computational complexity of the time-dependent method increases  $2^4 = 16$  times when we use the grid with  $h = h^*/2$ . However, the Lax-Friedrichs sweeping-based one-shot method does not require such a restriction because we do not need to integrate the HJI PDE over time. Recall that the HJI equation (3) is stationary in our formulation. Although we need more sweeping iterations as we refine the grid with  $h = h^*/2$ , the increase in the number of iterations is much less than a factor of two. Therefore, the computational time of the one-shot method with  $h = h^*/2$  is only 12–13 times higher than that with  $h = h^*$ .

## 5. CONCLUSION

We have presented a numerical method for computing the (lower) value function of a dynamic game from which we can extract multiple reachable sets at once. The method is very useful when holding all reachable sets in a given time interval or computing reachable sets for a long time horizon. We applied the method to computing backward reachable



**Figure 5: Backward reachable sets obtained by solving the time-dependent HJI equation (6): (a) non-malignant cells; (b) malignant cells.**

sets when growth and adhesion potentials compete with each other in structure formation of non-malignant and malignant cells.

A weakness of the Lax-Fredrichs sweeping method is that it introduces an undesirable numerical diffusion. It would be interesting to minimize this diffusiveness by using the Roe-Fix schemes or the semi-Lagrangian schemes. The most critical disadvantage of the method is its scalability: the complexity of Algorithm 1 increases exponentially as the system dimension increases because we must evaluate the value function at all nodes that fully discretize the state space. To resolve this scalability issue, we are very interested in utilizing approximate dynamic programming with carefully designed basis functions of the value function. We also plan to extend the one-method for computing reachable sets for stochastic differential games.

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