# $p$-OBOP models of *Optimizing Weak Orders via MILP*

Juan A. Aledo, Concepción Domínguez, Juan de Dios Jaime-Alcántara, Mercedes Landete

**Abstract**

This supplementary material analyses the computational behaviour of two mixed-integer programming models for the $p$-OBOP introduced in the article *Optimizing Weak Orders via MILP*. We report detailed results for assignment-based and representative-based formulations, focusing on solving times and optimality gaps for different instances and values of $p$. In addition, we provide a counterexample showing that the sequence of optimal values of the $p$-OBOP as a function of $p$ is not necessarily unimodal.

## 1 Comparison of $p$-OBOP models

This section analyses the computational behaviour of two families of equivalent mixed-integer programming formulations for the $p$-OBOP. On the one hand, we consider the formulations derived from model (2), which are based on assigning each item to exactly one of the $p$ buckets. On the other hand, we study the formulations obtained from model (3), which rely on selecting representatives to define the bucket structure. Although all variants model the same optimisation problem, they differ in the structural constraints they incorporate and in whether certain binary variables are relaxed to the continuous domain $[0, 1]$.

Before presenting the results, we recall that the base ordering inequalities used throughout the analysis are:

$$x_{rs} + x_{sr} \geq 1 \qquad \forall\, r, s \in [[n]] \,:\, r < s, \tag{1a}$$

$$x_{rs} + x_{st} \leq 1 + x_{rt} \qquad \forall\, r, s, t \in [[n]] \,:\, r \neq s \neq t \neq r. \tag{1b}$$

### 1.1 Variants of the assignment-based model

The formulation of interest is the following:

$$\min \quad f(\Pi, \boldsymbol{x}) \tag{2a}$$

$$\text{s.t.} \quad \sum_{u=1}^{p} y_{ru} = 1 \qquad \forall\, r \in [[n]], \tag{2b}$$

$$\sum_{r=1}^{n} y_{ru} \geq 1 \qquad \forall\, u \in [[p]], \tag{2c}$$

$$y_{ru} + y_{su} \leq x_{rs} + x_{sr} \qquad \forall\, r < s,\ \forall\, u, \tag{2d}$$

$$x_{sr} + \sum_{v=1}^{u} y_{rv} + \sum_{v=u+1}^{p} y_{sv} \leq 2 \qquad \forall\, r \neq s,\ \forall\, u \in [[p-1]], \tag{2e}$$

$$x_{rs} \in \{0, 1\} \qquad \forall\, r \neq s, \tag{2f}$$

$$y_{ru} \in \{0, 1\} \qquad \forall\, r, u. \tag{2g}$$

Table 1 summarises the computational results for the variants obtained by augmenting model (2) with the base inequalities (1a) and (1b), as well as by optionally relaxing the binary variables $\boldsymbol{x}$ to the continuous domain $[0, 1]$.

| Instance | $n$ | $m$ | $p$ | (2) | | (2) + (1a) | | (2) + (1b) | | (2) + (1a) + (1b) | | (2) + Relax $\boldsymbol{x}$ | | (2) + (1a) + Relax $\boldsymbol{x}$ | | (2) + (1b) + Relax $\boldsymbol{x}$ | | (2) + (1a) + (1b) + Relax $\boldsymbol{x}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | Time | Gap | Time | Gap | Time | Gap | Time | Gap | Time | Gap | Time | Gap | Time | Gap | Time | Gap |
| ED-10-02 | 51 | 9 | 2 | **4.59** | 0.00 | 10.01 | 0.00 | 1124.23 | 0.00 | 38.99 | 0.00 | 9.35 | 0.00 | 11.34 | 0.00 | 2680.83 | 0.00 | 3066.90 | 0.00 |
| ED-10-03 | 54 | 10 | 2 | **5.19** | 0.00 | 7.83 | 0.00 | 1054.20 | 0.00 | 14.57 | 0.00 | 9.91 | 0.00 | 10.97 | 0.00 | 3202.51 | 0.00 | 3600 | 3.53 |
| ED-10-11 | 50 | 11 | 2 | **7.39** | 0.00 | 9.85 | 0.00 | 1051.74 | 0.00 | 22.82 | 0.00 | 13.58 | 0.00 | 12.63 | 0.00 | 3486.87 | 0.00 | 3600 | 4.17 |
| ED-10-14 | 62 | 15 | 2 | **8.90** | 0.00 | 9.85 | 0.00 | 3161.16 | 0.00 | 22.62 | 0.00 | 22.10 | 0.00 | 21.25 | 0.00 | 3600 | 117.32 | 3600 | 116.52 |
| ED-10-15 | 52 | 14 | 2 | **8.23** | 0.00 | 10.66 | 0.00 | 1578.47 | 0.00 | 33.63 | 0.00 | 11.31 | 0.00 | 15.43 | 0.00 | 3600 | 14.23 | 3600 | 14.68 |
| ED-15-20 | 122 | 4 | 2 | 25.31 | 0.00 | **22.16** | 0.00 | 3600 | 249.02 | 3600 | 25.44 | 2417.31 | 0.00 | 3600 | 4.97 | 3600 | – | 3600 | – |
| ED-15-21 | 96 | 4 | 2 | 3.76 | 0.00 | **2.38** | 0.00 | 3600 | 278.45 | 3212.55 | 0.00 | 147.83 | 0.00 | 124.00 | 0.00 | 3600 | – | 3600 | – |
| ED-15-22 | 112 | 4 | 2 | 6.06 | 0.00 | **4.28** | 0.00 | 3600 | 248.51 | 3600 | 44.75 | 325.79 | 0.00 | 976.62 | 0.00 | 3600 | – | 3600 | – |
| ED-15-23 | 142 | 4 | 2 | 9.98 | 0.00 | **6.73** | 0.00 | 3600 | 285.59 | 3600 | 42.92 | 1398.02 | 0.00 | 1114.80 | 0.00 | 3600 | – | 3600 | – |
| ED-15-24 | 91 | 4 | 2 | 3.16 | 0.00 | **1.83** | 0.00 | 3600 | 363.25 | 2169.20 | 0.00 | 63.17 | 0.00 | 142.08 | 0.00 | 3600 | – | 3600 | – |
| ED-10-02 | 51 | 9 | 5 | 3600 | 269.43 | 3600 | 127.69 | 3600 | 84.64 | **42.90** | 0.00 | 3600 | 273.35 | 3600 | 198.51 | 3600 | 404.79 | 3600 | 229.71 |
| ED-10-03 | 54 | 10 | 5 | 3600 | 208.06 | 3600 | 107.92 | 3600 | 175.98 | **160.25** | 0.00 | 3600 | 208.06 | 3600 | 182.37 | 3600 | 336.39 | 3600 | 368.91 |
| ED-10-11 | 50 | 11 | 5 | 3600 | 198.60 | 3600 | 126.83 | 3600 | 70.60 | **11.42** | 0.00 | 3600 | 197.32 | 3600 | 225.44 | 3600 | 221.34 | 3600 | 272.88 |
| ED-10-14 | 62 | 15 | 5 | 3600 | 286.95 | 3600 | 132.28 | 3600 | 87.52 | 3600 | **0.15** | 3600 | 266.93 | 3600 | 268.48 | 3600 | 457.14 | 3600 | 315.48 |
| ED-10-15 | 52 | 14 | 5 | 3600 | 199.46 | 3600 | 90.75 | 3600 | 74.91 | **3015.40** | 0.00 | 3600 | 200.31 | 3600 | 187.88 | 3600 | 412.97 | 3600 | 279.67 |
| ED-15-20 | 122 | 4 | 5 | 3600 | 763.11 | 3600 | 119.42 | 3600 | 84.80 | 3600 | **0.59** | 3600 | 763.11 | 3600 | 795.55 | 3600 | – | 3600 | – |
| ED-15-21 | 96 | 4 | 5 | 3600 | 356.73 | 3600 | 26.36 | 3600 | 87.69 | 3600 | **8.64** | 3600 | 356.73 | 3600 | 337.02 | 3600 | – | 3600 | 213017.65 |
| ED-15-22 | 112 | 4 | 5 | 3600 | 430.48 | 3600 | 23.40 | 3600 | – | 3600 | **11.16** | 3600 | 429.15 | 3600 | 394.22 | 3600 | – | 3600 | – |
| ED-15-23 | 142 | 4 | 5 | 3600 | 741.91 | 3600 | 91.94 | 3600 | 88.04 | 3600 | **16.56** | 3600 | 741.91 | 3600 | 621.63 | 3600 | – | 3600 | – |
| ED-15-24 | 91 | 4 | 5 | 3600 | 259.77 | 3600 | 28.21 | 3600 | 114.81 | **1342.87** | 0.00 | 3600 | 277.02 | 3600 | 280.64 | 3600 | – | 3600 | – |
| ED-10-02 | 51 | 9 | 20 | 3600 | 806.42 | 3600 | – | 3600 | **295.61** | 3600 | – | 3600 | 2479.97 | 3600 | 2231.35 | 3600 | 352.49 | 3600 | 316.13 |
| ED-10-03 | 54 | 10 | 20 | 3600 | 1026.68 | 3600 | – | 3600 | 390.53 | 3600 | **20.05** | 3600 | 3193.27 | 3600 | 2623.30 | 3600 | 322.45 | 3600 | 325.34 |
| ED-10-11 | 50 | 11 | 20 | 3600 | 646.02 | 3600 | 642.73 | 3600 | 270.56 | 3600 | **17.19** | 3600 | 2799.74 | 3600 | 2917.96 | 3600 | 289.24 | 3600 | 226.44 |
| ED-10-14 | 62 | 15 | 20 | 3600 | 482.54 | 3600 | 702.66 | 3600 | 248.15 | 3600 | **19.68** | 3600 | 3765.04 | 3600 | 5848.68 | 3600 | 301.58 | 3600 | 289.21 |
| ED-10-15 | 52 | 14 | 20 | 3600 | 439.71 | 3600 | 478.08 | 3600 | 222.94 | 3600 | **19.81** | 3600 | 2926.52 | 3600 | 3961.12 | 3600 | 325.02 | 3600 | 264.03 |
| ED-15-20 | 122 | 4 | 20 | 3600 | 100.37 | 3600 | 121.47 | 3600 | 282.43 | 3600 | **2.69** | 3600 | 691270.79 | 3600 | 395850.89 | 3600 | – | 3600 | – |
| ED-15-21 | 96 | 4 | 20 | 3600 | 76.04 | 3600 | 137.67 | 3600 | 443.44 | 3600 | **3.45** | 3600 | 36400.84 | 3600 | 29764.27 | 3600 | – | 3600 | – |
| ED-15-22 | 112 | 4 | 20 | 3600 | 90.03 | 3600 | 398.01 | 3600 | 275.86 | 3600 | **2.85** | 3600 | 105678.72 | 3600 | 152830.47 | 3600 | – | 3600 | – |
| ED-15-23 | 142 | 4 | 20 | 3600 | 116.76 | 3600 | 125.10 | 3600 | 483.92 | 3600 | **34.99** | 3600 | – | 3600 | – | 3600 | – | 3600 | – |
| ED-15-24 | 91 | 4 | 20 | 3600 | 108.03 | 3600 | 134.34 | 3600 | 171.75 | 3600 | **2.82** | 3600 | 34007.68 | 3600 | 7958.28 | 3600 | – | 3600 | 330337.37 |

Table 1: Comparison of model (2) variants for the $p$-OBOP. The term "Relax $\boldsymbol{x}$" denotes the relaxation of binary variables $\boldsymbol{x}$ to the continuous domain $[0, 1]$.

As shown in Table 1, although all formulations are equivalent from a modeling perspective, their computational behavior varies considerably. The inclusion of additional structural constraints strongly influences both solving time and solution quality. In particular, the formulation that combines the transitivity constraints with the before-or-after constraints without relaxing the binary variables $\boldsymbol{x}$ achieves the best overall performance. This variant consistently provides exact solutions within reasonable computation times, demonstrating a favorable balance between model strength and numerical efficiency in the resolution of the $p$-OBOP.

### 1.2 Variants of the representative-based model

Recall that the model is the following:

$$\min \quad f(\Pi, \boldsymbol{x}) \tag{3a}$$

$$\text{s.t.} \quad (1a) - (1b) \tag{3b}$$

$$\sum_{r=1}^{n} \alpha_r = p \tag{3c}$$

$$\beta_{rs} \leq \alpha_s \qquad \forall\, r, s \in [[n]] \;:\; r < s, \tag{3d}$$

$$\sum_{s=r+1}^{n} \beta_{rs} + \alpha_r \;=\; 1 \qquad \forall\, r \in [[n]], \tag{3e}$$

$$x_{rs} \geq \beta_{rs} \qquad \forall\, r, s \in [[n]] \;:\; r < s, \tag{3f}$$

$$x_{sr} \geq \beta_{rs} \qquad \forall\, r, s \in [[n]] \;:\; r < s, \tag{3g}$$

$$x_{rs} + x_{sr} + \alpha_r \leq 2 \qquad \forall\, r, s \in [[n]] \ : \ r < s, \tag{3h}$$

$$x_{rs} \in \{0, 1\} \qquad \forall\, r, s \in [[n]] \ : \ r \neq s, \tag{3i}$$

$$\alpha_r \in \{0, 1\} \qquad \forall\, r \in [[n]], \tag{3j}$$

$$\beta_{rs} \in \{0, 1\} \qquad \forall\, r, s \in [[n]] \ : \ r < s. \tag{3k}$$

We also considered the following valid inequality:

$$\beta_{rs} + \alpha_s \leq x_{rs} + x_{sr} \qquad \forall\, r < s. \tag{4}$$

Including inequality (4) and removing constraints (3f) and (3g) yields an alternative formulation of the problem that is equivalent to the original one.

Building on these results, a second set of experiments was conducted to evaluate the impact of specific inequalities on the performance of model (3). In this case, the objective is to determine which combinations or substitutions of inequalities lead to stronger formulations and faster convergence.

| Instance | $n$ | $m$ | $p$ | (3) | | (3) − (3f) − (3g) + (4) | | (3) + (4) | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | Time | Gap | Time | Gap | Time | Gap |
| ED-10-02 | 51 | 9 | 2 | 995.41 | 0.00 | 1016.43 | 0.00 | **493.93** | 0.00 |
| ED-10-03 | 54 | 10 | 2 | **196.27** | 0.00 | 869.89 | 0.00 | 366.32 | 0.00 |
| ED-10-11 | 50 | 11 | 2 | **10.56** | 0.01 | 182.56 | 0.00 | 11.66 | 0.00 |
| ED-10-14 | 62 | 15 | 2 | **11.27** | 0.00 | 21.31 | 0.00 | 12.07 | 0.00 |
| ED-10-15 | 52 | 14 | 2 | **781.46** | 0.00 | 1172.56 | 0.00 | 889.97 | 0.00 |
| ED-15-20 | 122 | 4 | 2 | 3600 | **23.02** | 3600 | 87.40 | 3600 | 23.02 |
| ED-15-21 | 96 | 4 | 2 | 3600 | 32.54 | 3600 | 69.36 | 3600 | **29.61** |
| ED-15-22 | 112 | 4 | 2 | 3600 | **37.40** | 3600 | 41.68 | 3600 | 37.41 |
| ED-15-23 | 142 | 4 | 2 | 3600 | – | 3600 | 111.18 | 3600 | **23.15** |
| ED-15-24 | 91 | 4 | 2 | 3600 | **32.42** | 3600 | 62.11 | 3600 | 61.43 |
| ED-10-02 | 51 | 9 | 5 | **51.86** | 0.01 | 323.51 | 0.01 | 302.15 | 0.01 |
| ED-10-03 | 54 | 10 | 5 | **6.01** | 0.00 | 47.08 | 0.00 | 34.54 | 0.00 |
| ED-10-11 | 50 | 11 | 5 | **5.93** | 0.00 | 27.07 | 0.00 | 34.78 | 0.00 |
| ED-10-14 | 62 | 15 | 5 | **11.00** | 0.00 | 71.60 | 0.00 | 66.62 | 0.00 |
| ED-10-15 | 52 | 14 | 5 | **7.70** | 0.01 | 43.33 | 0.01 | 35.95 | 0.01 |
| ED-15-20 | 122 | 4 | 5 | 3600 | 0.11 | 3600 | 3.40 | 3600 | **0.07** |
| ED-15-21 | 96 | 4 | 5 | 3600 | **1.26** | 3600 | 2.75 | 3600 | 2.13 |
| ED-15-22 | 112 | 4 | 5 | 3600 | 9.12 | 3600 | 98.29 | 3600 | **7.91** |
| ED-15-23 | 142 | 4 | 5 | 3600 | 23.25 | 3600 | 46.12 | 3600 | **22.27** |
| ED-15-24 | 91 | 4 | 5 | **95.49** | 0.00 | 3600 | 2.63 | 230.99 | 0.00 |
| ED-10-02 | 51 | 9 | 20 | 3600 | 15.46 | 1015.58 | 0.00 | **433.73** | 0.00 |
| ED-10-03 | 54 | 10 | 20 | 3600 | 4.59 | 860.85 | 0.00 | **342.96** | 0.00 |
| ED-10-11 | 50 | 11 | 20 | 3600 | 15.78 | 181.26 | 0.00 | **21.13** | 0.00 |
| ED-10-14 | 62 | 15 | 20 | 3600 | 0.60 | 21.28 | 0.00 | **12.96** | 0.00 |
| ED-10-15 | 52 | 14 | 20 | 3600 | 10.19 | 1400.03 | 0.00 | **883.35** | 0.00 |
| ED-15-20 | 122 | 4 | 20 | 3600 | 52.28 | 3600 | 87.40 | 3600 | **23.02** |
| ED-15-21 | 96 | 4 | 20 | **2145.80** | 0.04 | 3600 | 69.36 | 3600 | 29.61 |
| ED-15-22 | 112 | 4 | 20 | 3600 | **37.40** | 3600 | 41.68 | 3600 | 37.41 |
| ED-15-23 | 142 | 4 | 20 | 3600 | – | 3600 | 111.18 | 3600 | **23.15** |
| ED-15-24 | 91 | 4 | 20 | 3600 | **32.42** | 3600 | 62.11 | 3600 | 61.43 |

Table 2: Comparison of model (3) variants for the $p$-OBOP.

From Table 2, we observe that the inclusion of additional inequalities can substantially enhance the model's performance, particularly for smaller values of $p$. In general, the strengthened formulations improve convergence and reduce the optimality gap, confirming the positive impact of incorporating well-chosen valid inequalities. Together with the previous analysis, these findings provide valuable insight into which modelling strategies, both in terms of constraint structure and variable relaxation, yield the most effective formulations for the $p$-OBOP.

## 2 Behaviour of the $p$-OBOP model

A natural question arises as to whether the existence of optimal solutions for two distinct numbers of buckets implies the existence of an optimal solution for every intermediate number of buckets. The following example demonstrates that this property does not necessarily hold.

**Example 1.** *Consider the following instance of the* OBOP *with* 4 *items, whose pairwise order matrix is shown in Figure 1.*

$$\frac{1}{100}\begin{pmatrix} 50 & 55 & 90 & 100 \\ 45 & 50 & 20 & 80 \\ 10 & 80 & 50 & 65 \\ 0 & 20 & 35 & 50 \end{pmatrix}$$

Figure 1: Pairwise order matrix used in Example 1.

*Solving this instance with model* (2) *for values of $p$ ranging from 1 to 4 yields the optimal values shown in Figure 2.*

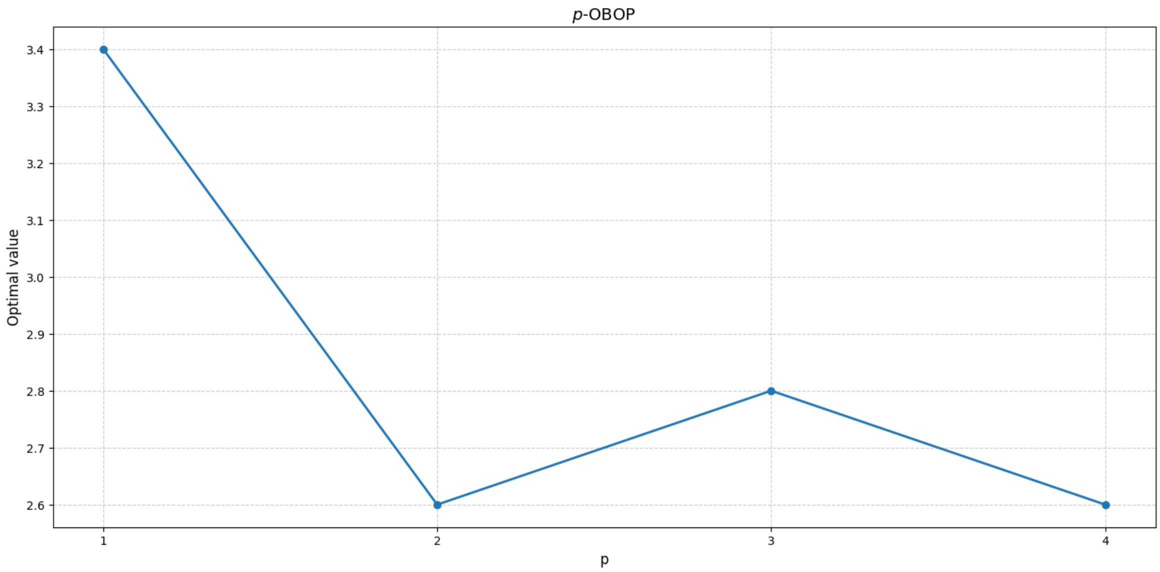


Figure 2: Optimal values of the $p$-OBOP for different values of $p$

This example illustrates that the curve of optimal values for the $p$-OBOP as a function of $p$ is not necessarily unimodal. In particular, the existence of optimal solutions for $p = n - 1$ and $p = n + 1$ does not imply the existence of one for $p = n$.