Lecture 4: Derivatives

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Plan

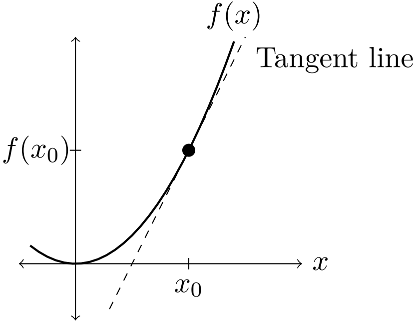
Derivatives

1. Definition
2. Calculation

Tangent Line

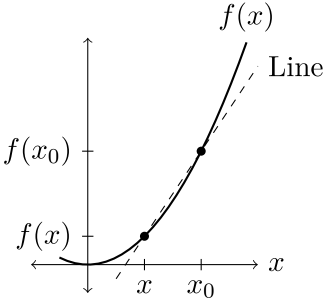
Consider the function $f(x) = x^2$.

We want to calculate the slope of the line that "touches" it at $x_0 = 1$.



We can approximate the tangent by lines that pass through $(x_0, f(x_0))$ and $(x, f(x))$ with $x \to x_0$.

The slope is $\dfrac{f(x) - f(x_0)}{x - x_0}$.



Derivatives

It makes sense to define the **derivative** of f at x_0 as the limit

$$\boldsymbol{f'(x_0)} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

if it exists. If it exists and is not infinite, we say that f is **differentiable** at x_0.

Note that $x \to x_0$ is the same as $h \equiv x - x_0 \to 0$, and $x = x_0 + (x - x_0) = x_0 + h$, so we can write

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Example

If $f(x) = ax + b$ we have

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{a(x+h) + b - (ax+b)}{h} = \lim_{h\to 0} \frac{ah}{h} \\
&= \lim_{h\to 0} a = a.
\end{aligned}
$$

If $f(x) = x^2$ then

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} (2x + h) = 2x.
\end{aligned}
$$

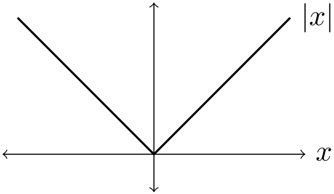
Differentiable $\Rightarrow$ Continuous

We should think of the derivative as the rate of change of $f(x)$ locally.

Notice that if $f'(x_0)$ exists then f has to be continuous at x_0, because

$$\begin{aligned}
\lim_{x\to x_0} f(x) &= \lim_{x\to x_0}(f(x)-f(x_0))+f(x_0) \\
&= \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}(x-x_0)\right]+f(x_0) \\
&= f'(x)\cdot 0+f(x_0)=f(x_0).
\end{aligned}$$

But not every continuous function is differentiable. For example, $f(x)=|x|$ is not differentiable at 0.



Calculating Derivatives

If f and g are differentiable at x, $(f \pm g)'(x) = f'(x) \pm g'(x)$. Why?

$$\begin{aligned}(f \pm g)'(x) &= \lim_{h\to 0} \frac{(f(x+h) \pm g(x+h)) - (f(x) \pm g(x))}{h} \\ &= \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} \pm \frac{g(x+h) - g(x)}{h} \right) = f'(x) \pm g'(x).\end{aligned}$$

Calculating Derivatives

If f and g are differentiable at x, $\boldsymbol{(fg)'(x) = f'(x)g(x) + f(x)g'(x)}$:

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{(f(x+h) - f(x))g(x+h) - f(x)(g(x+h) - g(x))}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h}\right] \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

Example. $(x^3)' = (x \cdot x^2)' = x' \cdot x^2 + x \cdot (x^2)' = x^2 + x \cdot (2x) = 3x^2.$

In general, $\boldsymbol{(x^n)'} = nx^{n-1}$.

Exercise

Question 1

Calculate the derivative of the following functions:

1. $f(t) = 14t - 7$
2. $f(y) = y^3 + 3y^2 - 12$
3. $f(x) = (x^2 + 1)(x^3 - 1)$
4. $f(x) = x^6 + 5x^5 - 2x^2 + 8$

Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, define $\boldsymbol{(f \circ g)(x)} = f(g(x))$, and

$$
\begin{aligned}
\boldsymbol{(f \circ g)'(x)} &= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\
&= \lim_{h \to 0} \left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \frac{g(x+h) - g(x)}{h} \right] \\
&= f'(g(x))g'(x).
\end{aligned}
$$

For example, if $f(x) = \frac{1}{x}$ we have

$$
f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{\frac{x-(x+h)}{(x+h)x}}{h} = \lim_{h \to 0} \frac{-h}{(x+h)xh} = \lim_{h \to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2}.
$$

Therefore, $\boldsymbol{\left(\frac{1}{g(x)}\right)'} = (f \circ g)'(x) = f'(g(x))g'(x) = -\frac{1}{g(x)^2} g'(x) = -\frac{g'(x)}{g(x)^2}.$

Calculating Derivatives

Notice that if a is constant then $a' = 0$.

Hence if a is constant then $\boldsymbol{(af(x))'} = a'f(x) + af'(x) = af'(x)$.

And

$$\left(\frac{\boldsymbol{f(x)}}{\boldsymbol{g(x)}}\right)' = \left(f(x)\cdot\frac{1}{g(x)}\right)' = f'(x)\frac{1}{g(x)} + f(x)\left(\frac{1}{g(x)}\right)'$$
$$= \frac{f'(x)}{g(x)} + f(x)\frac{-g'(x)}{g(x)^2} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

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Exercises

Question 2

Calculate the derivative of the following functions:

1. $f(x) = \dfrac{x}{x+1}$
2. $f(x) = \dfrac{1}{x^2+1}$.

Exp and Log

We have $\boldsymbol{\exp'(x)} = \exp(x)$:

$$\begin{aligned}\exp'(x) &= \lim_{h\to 0}\frac{\exp(x+h)-\exp(x)}{h} = \lim_{h\to 0}\frac{\exp(x)\exp(h)-\exp(x)}{h}\\ &= \lim_{h\to 0}\exp(x)\frac{\exp(h)-1}{h} = \exp(x)\underbrace{\lim_{h\to 0}\frac{\exp(h)-1}{h}}_{=1} = \exp(x).\end{aligned}$$

The fact that $\lim_{h\to 0}\frac{\exp(h)-1}{h} = 1$ follows from the definition of e but is not obvious.

log is differentiable, although I won't prove it now. Let's assume it's true.

Notice that $\exp(\log(x)) = x$.

Differentiating both sides we get $\exp'(\log(x))\log'(x) = 1$.

Using $\exp' = \exp$, this is $\exp(\log(x))\log'(x) = 1$, i.e., $\boldsymbol{\log'(x)} = \frac{1}{x}$.

General Power Rule

If $k \in \mathbb{R}$ and $f(x) = x^k$ for $x > 0$ we have

$$\begin{aligned} f'(x) &= (x^k)' \\ &= (\exp(\log(x^k)))' \\ &= (\exp(k \log(x)))' \\ &= \exp'(k \log(x)) k \log'(x) \\ &= \exp(\log(x^k)) k x^{-1} \\ &= x^k k x^{-1} = k x^{k-1}. \end{aligned}$$

Question 3

Calculate the derivative of $f(x) = a^x$ for every $a > 0$.

Exercises

Question 4

Calculate the derivative of the following functions:

1. $f(x) = 3x^{1/3}$
2. $f(x) = \frac{1}{100}x^{25} - \frac{1}{10}x^{0.25}$
3. $f(y) = 1 - 1/y^2$
4. $f(x) = \ln(2\pi x^2)$
5. $f(x) = e^{-x^2}$

Higher Order Derivatives

If $f : U \to \mathbb{R}$ is **differentiable** (i.e., differentiable at every $x \in U$) then $f' : U \to \mathbb{R}$ is another function, and we can ask if it is itself differentiable.

Example. Take $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 & \text{if } x \geqslant 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

Notice that it is differentiable. This is only non-obvious at 0. We have

$$\begin{aligned} \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^+} \frac{h^2}{h} = \lim_{h\to 0^+} h = 0, \quad \text{and} \\ \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^-} \frac{-h^2}{h} = \lim_{h\to 0^-} -h = 0, \end{aligned}$$

so $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = 0$.

Higher Order Derivatives

We obtain

$$f'(x) = \begin{cases} 2x & \text{if } x \geqslant 0, \\ -2x & \text{if } x < 0. \end{cases}$$

In other words, $f'(x) = 2|x|$. This is not differentiable at 0. So f' is not differentiable.

But if f' is differentiable at x we can define the **second derivative** $f''(x)$ of f as the derivative of f' at x.

Question 5
If $f(x) = x^3$ calculate $f''(x)$ for all $x \in \mathbb{R}$.

Optimization

If $f : D \to \mathbb{R}$ is maximized or minimized at x in the interior of D and $f'(x)$ exists then $f'(x) = 0$.

> The reason is that if $f'(x) > 0$ then when $h > 0$ is sufficiently small we have $\frac{1}{h}(f(x+h) - f(x)) \approx f'(x) > 0$, so $f(x+h) > f(x)$, and $f(x)$ is not the maximum value of f. If $f'(x) < 0$ then when $h < 0$ is small we have $\frac{1}{h}(f(x+h) - f(x)) \approx f'(x) < 0$, so multiplying by h we get $f(x+h) - f(x) > 0$, and again $f(x)$ is not the maximum.

If $f : (a, b) \to \mathbb{R}$ is differentiable and $f' > 0$ then f is increasing. (Intuition: $f'(x) > 0$ means that f has a “positive slope” at x.)

If $f' < 0$ then f is decreasing.

This is very useful! Example: what is the maximum of $f(x) = x - x^2$? We have $f'(x) = 1 - 2x$, so $f' > 0$ if $x < \frac{1}{2}$ and $f' < 0$ if $x > \frac{1}{2}$. Hence f increases and then decreases. Therefore the max is at $x = \frac{1}{2}$.

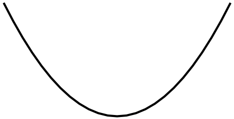
Example

Question 6

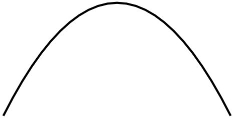
Find the minimum and maximum of $f : [0, 1] \to \mathbb{R}$ given by $f(x) = 2x^2 - x + 1$.

Convexity

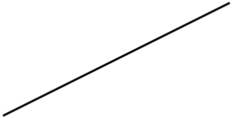
If f is twice differentiable then we say it's **convex** if $f'' \geqslant 0$, and **concave** if $f'' \leqslant 0$.



Convex



Concave



Both

If f is concave and $f'(x) = 0$ then f is maximized at x.

If f is convex and $f'(x) = 0$ then f is minimized at x.

If f'' exists around x, is continuous at x, and $f''(x) > 0$ then $f'' > 0$ around x, so f is concave around x, and, so, if $f'(x) = 0$ then f has a **local maximum** at x.

If f'' exists around x, is continuous at x, $f''(x) > 0$, and $f'(x) = 0$ then f has a **local minimum** at x.

Example

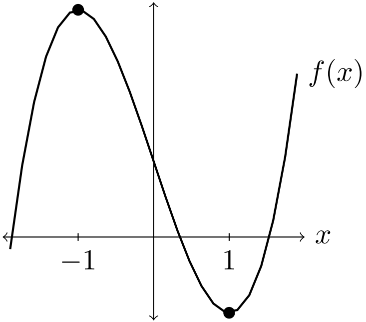
Take $f(x) = x^3 - 3x + 1$.

We have $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$.

$f'(x) = 0$ is $3x^2 = 3$, i.e., $x = -1$ or $x = 1$.

$f''(-1) = -6 < 0$, so f has a local maximum at -1.

$f''(1) = 6 > 0$, so f has a local minimum at -1.

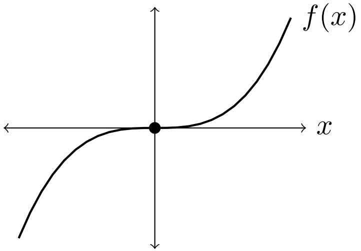


What if $f''(x) = 0$?

Take $f(x) = x^3$.

We have $f'(x) = 3x^2$ and $f''(x) = 6x$.

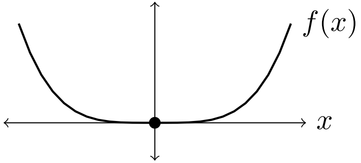
We have $f'(0) = 0$ but $f''(0) = 0$, so it's not obvious if it's a local minimum or maximum. It's neither.



Take $f(x) = x^4$.

We have $f'(x) = 4x^3$ and $f''(x) = 12x^2 \geqslant 0$, so it's convex.

We have $f'(0) = 0$ but $f''(0) = 0$. But f is convex, so it must be minimized at 0.



Exercise

Question 7

Find the minimum and maximum of the following functions:

1. $f : [0,1] \to \mathbb{R},\ f(x) = x(1-x).$
2. $f : [0,1] \to \mathbb{R},\ f(x) = x - \sqrt{x}.$
3. $f : [0,6] \to \mathbb{R},\ f(x) = x^3 - \frac{15}{2}x^2 + 12x + 8.$
4. $f : [-4,4] \to \mathbb{R},\ f(x) = 3x^4 - 4x^3 - 36x^2.$

L'Hôpital's Rule

Theorem

Suppose $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x)$, both are 0 or ∞, and f, g are differentiable near x_0. Then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)},$$

provided the latter limit exists.

Examples.

1. $\displaystyle\lim_{x\to 2} \frac{x^2-4}{x^3-8} = \lim_{x\to 2} \frac{2x}{3x^2} = \frac{1}{3}.$
2. $\displaystyle\lim_{x\to +\infty} \frac{e^x}{x^2} = \lim_{x\to +\infty} \frac{e^x}{2x} = \lim_{x\to +\infty} \frac{e^x}{2} = +\infty.$
3. $\displaystyle\lim_{x\to 0} x\log(x) = \lim_{x\to 0} \frac{\log(x)}{1/x} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0}(-x) = 0.$

Exercise

Question 9

Calculate the following limits:

1. $\lim\limits_{x \to +\infty} \dfrac{\log(x)}{x}$.
2. $\lim\limits_{x \to 1} \dfrac{\sqrt{x} - 1}{x - 1}$.
3. $\lim\limits_{x \to 0^+} x^x$.

Question 10

Find the minimum and maximum value of $f : (0, 1] \to \mathbb{R}$ given by $f(x) = x \log(x)$.