Math Camp 2025 – Problem Set 9

Read the following problems carefully and justify all your work. Avoid using calculators or computers. (This was adapted almost verbatim from a problem set by Christopher Lucas.)

1. There are 10 first-year students in the Department of Political Science at Bear University. The Director of Graduate Studies (DGS) asks each student to select one from four different lunch seminars to attend. The DGS records how many students choose each lunch seminar. She obtains four numbers x_1, x_2, x_3, x_4, e.g., she could obtain $1, 2, 0, 7$. How many different possibilities exist? Explain your answer.

Comment. The trick is to view the possibilities as distinct permutations of the word `X|XX||XXXXXXX`. Convince yourself that this is the case and count them.

2. A box contains three coins. One coin has heads on both sides, one has tails on both sides, and one has one side with head and the other side with tail. One coin is randomly selected from the box, and you observe that one random side of the selected coin is head. What is the probability that the other side of the selected coin is also head? Explain your answer.

3. Suppose that a district contains 100 likely voters, of which 60% support candidate 1. You randomly sample a subset of k likely voters without replacement (this means taking a random subset of k voters), and ask about their preferences.

(a) If $k = 20$, what is the probability that exactly 60% of your sample are supporters of candidate 1?

(b) If $k = 20$, what is the probability that you correctly identify the leading candidate (i.e., that at least 50% of your sample are supporters of candidate 1)? Show how this probability can be calculated.

4. The United States Senate contains two senators from each of the 50 states.

(a) If a committee of 50 senators are selected at random, what is the probability that the group will contain one senator from each state?

(b) If a committee of k senators is selected at random, what is the probability that it will contain at least one of the two senators from New Jersey?

(c) What are the possible sizes (number of members) of the committee such that there is at least 50% probability that the committee will contain at least one of two senators from New Jersey?

5. The *odds* of an event with probability p are defined to be $\frac{p}{1-p}$, e.g., an event with probability $3/4$ is said to have odds of 3 to 1 in favor (or 1 to 3 against). We are interested in a hypothesis H

(which we think of as an event), and we gather new data as evidence (expressed as an event D) to study the hypothesis. The *prior* probability of H is our belief of the probability for H being true before we gather the new data; the *posterior* probability of H is our belief of the probability for H being true after we gather the new data. The *likelihood ratio* is defined as $\dfrac{\Pr(D|H)}{\Pr(D|H^c)}$.

Show that Bayes' rule can be expressed in terms of odds as follows: *the posterior odds of a hypothesis H are the prior odds of H times the likelihood ratio*.

6. Prove the following statements. For each statement, explain what it means in practice with your own example.

(a) $\Pr(A \cup B) \leqslant \Pr(A) + \Pr(B)$.

(b) If $\Pr(A) > 0, \Pr(B) > 0$, and $\Pr(A) < \Pr(A|B)$, then $\Pr(B) < \Pr(B|A)$.

7. Suppose that the PDF of a random variable X is as follows:

$$f(x) = \begin{cases} \frac{4}{3}(1 - x^3) & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Determine the values of the following probabilities (Hint: you should be evaluating a definite integral):

(a) $\Pr(X < \frac{1}{2})$,

(b) $\Pr(\frac{1}{4} < X < \frac{3}{4})$,

(c) $\Pr(X > \frac{1}{3})$.

8. Suppose Bear University and Lion University play a sequence of basketball games against each other, and the first team to win four games wins the series. Let p be the probability that Bear University wins an individual game, and assume that the games are independent. Answer the following questions

(a) What is the probability that Bear University wins the series?

(b) Does the answer to (a) depend on whether the teams always play 7 games (and the winner is the team that one the majority of games), or the teams stop playing more games as soon as one team has won 4 games (as is actually the case in practice: once the series is decided, the two teams do not keep playing more games)?

9. Does social environment affect the political development of preadults? Some work[1] shows that there exists a transmission of political values from parents to children, as manifested in their views during late adolescence. We will use this idea to solve the following questions.

Take that certain political attitudes could be shared by a parent or not. Specifically, if the parent holds the political attitude, there is $3/5$ probability that the child will also share the same attitude. On the other hand, if the parent does not have the political attitude, the child will not have it either. We exclude the possibility of any transmission of political values among children. Assume that a parent, who has probability $1/3$ of having the political attitude, has 2 children (assume that to be a son and a daughter), and answer the following questions:

(a) Conditional on the parent not holding the political attitude, is whether the son has the political attitude independent of whether the daughter has the political attitude?

(b) Conditional on the parent holding the political attitude, is whether the son has the political attitude independent of whether the daughter has the political attitude?

(c) Is whether the son has the political attitude independent of whether the daughter has the political attitude?

(d) What's the probability that neither of the 2 children have the political attitude?

(e) If the son does not have the political attitude, what's the probability that the parent has the political attitude?

(f) Bayes theorem can also be used for three events as shown below. Explain each equation below in your own words.

$$\Pr(X \mid Y \cap Z) = \frac{\Pr(X \cap Y \cap Z)}{\Pr(Y \cap Z)} \tag{1}$$

$$= \frac{\Pr(Y \cap Z \mid X)\Pr(X)}{\Pr(Y \cap Z)} \tag{2}$$

$$= \frac{\Pr(Y \cap Z \mid X)\Pr(X)}{\Pr(Y \cap Z \mid X)\Pr(X) + \Pr(Y \cap Z \mid X^c)\Pr(X^c)} \tag{3}$$

(g) We found that the daughter does not have the political attitude either. Using the equations in part (f), "update" your belief about whether the parent has the attitude.

[1] Jennings, M.K., and Niemi, R.G. (1968). "The transmission of political values from parents to child", *American Political Science Review* 62(1), 169-184.