1 TATE'S LINEAR ALGEBRA

1.1 LINEAR TOPOLOGIES

Fix a ground field k. From now on, a vector space will always mean a k-vector space.

Definition 1.1. A **linear topology** on a vector space E is a separated (Hausdorff) topology invariant under translations that admits an open local base around zero of vector subspaces. A vector space equipped with a linear topology will be referred as **linearly topologized**.

If we endow k with the discrete topology then E will become a topological vector space. From now on, endow k to have a discrete topology.
Linear topologies behave nicely under basic topological operations.

Proposition 1.2. *Let E be a linearly topologized vector space. Then*

(a) Any vector subspace of E is linearly topologized under its subspace topology.

(b) If $F \subseteq E$ is a closed vector subspace then E/F is linearly topologized under its quotient topology.

(c) If $\{E_\alpha\}_\alpha$ is a collection of linearly topologized vector spaces its product $\prod_\alpha E_\alpha$ and its direct sum $\bigoplus_\alpha E_\alpha$ is linearly topologized under its product topology.

Proof. Since intersection of vector subspaces is a vector subspace, (a) follows intersecting the fundamental system of neighborhoods in E by the vector subspace. For (b), let $\pi\colon E \to E/F$ be the quotient map. Since π is open and surjective the image of a local base is a local base; moreover, the image of a vector subspace under π is a vector subspace, then (b) follows. Finally, for (c) let $\{U_{\alpha,\beta}\}_\beta$ be a local base of zero in E_α of vector subspaces, the products $U_{\alpha_1,\beta_1} \times \ldots \times U_{\alpha_n,\beta_n} \times \prod_\gamma E_\gamma$, where γ ranges over $\alpha \neq \alpha_1, \ldots, \alpha_n$, for any set $\{(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)\}$ form a fundamental system of neighborhoods around zero in $\prod_\alpha E_\alpha$ of open vector subspaces. Note that since $\bigoplus_\alpha E_\alpha \subseteq \prod_\alpha E_\alpha$ is a vector subspace (c) follows from (a). $\square$

Finite dimensional vector spaces are meaningless for linear topologies.

Proposition 1.3. *A finite dimensional linearly topologized vector space E is discrete.*

Proof. Let U be an open vector subspace and $0 \neq x \in U$, since E is separated and linearly topologized there exists an open vector subspace U_x such that $x \notin U_x$ then $\dim U_x \cap U < \dim U$, since E is finite dimensional this process can be repeted only a finite amount of times; that is $\{0\}$ is open. It follows that E is discrete. $\square$

Linear compactness

Definition 1.4. Let E be a linearly topologized vector space. A closed subset $C \subseteq E$ is **linearly compact** (respectively **linearly cocompact**) if for every open vector subspace U we have $\dim C/(C \cap U) < \infty$ (respectively $\dim E/(C + U) < \infty$).

Linear compactness behaves just as compactness if one uses the correct words.

Proposition 1.5. *Let E be a linearly compact vector space and F a linearly topologized vector space. Then*

(a) If $\varphi \colon E \to F$ is a continuous linear homomorphism then $\varphi(E)$ is linearly compact.

(b) If E is discrete then E must be finite dimensional.

(c) Every closed vector subspace of E is linearly compact.

(d) (Tychonov) If $\{E_\alpha\}_\alpha$ is a collection of linearly compact vector spaces then its product $\prod_\alpha E_\alpha$ and its direct sum $\bigoplus_\alpha E_\alpha$ are linearly compact.

Proof. Let $U \subseteq F$ be an open vector subspace, then since φ is linear a continuous $\varphi^{-1}(U)$ is an open vector subspace of E. Consider the surjective induced map

$$E/\varphi^{-1}(U) \twoheadrightarrow \varphi(E)/\varphi(E) \cap U$$

as E is linearly compact it follows that $\dim \varphi(E)/\varphi(E) \cap U < \infty$. We get (a). If E is discrete, then $\{0\}$ is an open vector subspace of E, (b) follows. For (c), let $G \subseteq E$ be a closed vector subspace and take any open vector subspace U of E, then the inclusion $G \hookrightarrow E$ induces

$$G/G \cap U \hookrightarrow E/U$$

where the latter is finite dimensional (E is linearly compact). Finally, for (d), it is enough proving for open vector subspaces $U = \prod_\beta U_\beta \times \prod_\gamma E_\gamma$ where β ranges over a finite set, γ ranges over $\alpha \neq \beta$ and U_β is an open vector subspace of E_β. Then, the quotient

$$\prod_\alpha E_\alpha/U \cong \prod_\beta E_\beta/U_\beta$$

where $\cong$ is a topological and algebraic isomorphism. Since E_α is linearly compact for all α and β ranges over a finite set we conclude that $\prod_\alpha E_\alpha/U$ is finite dimensional; therefore, $\prod_\alpha E_\alpha$ is linearly compact. The proof is analogous for $\bigoplus_\alpha E_\alpha$. $\square$

Completeness

If E is linearly topologized it admits a fundamental system of neighborhoods consisting of open vector subspaces

$$E \supseteq U_0 \supseteq U_1 \supseteq \ldots \supseteq U_\alpha \supseteq \ldots$$

Definition 1.6. In the previous context, we say that E is **complete** if

$$E \cong \hat{E} := \varprojlim_\alpha E/U_\alpha$$

where $\cong$ is an isomorphism of topological vector spaces.

1.2 TATE SPACES

Definition 1.7. Let E be a linearly topologized vector space. An open linearly compact subspace of E is called a **c-lattice** if it is open; dually, a **d-lattice** is a discrete linearly cocompact subspace of E. We say that E is a **Tate space** or **Tate vector space** if it contains a c-lattice.

Proposition 1.8. *A linearly topologized vector space E has a c-lattice if and only if it has a d-lattice.*

Proof. Suppose C is a c-lattice in E, choose any direct complement D of C, that is, $E = C \oplus D$. Since C is open, then D is discrete as $D \cap C = 0$, thus 0 is open in D. Moreover, D is closed as it is the fiber of 0 under the projection $E \to C$. Finally, we check that D is linearly cocompact: let U be any open vector subspace of E, the composition $C \hookrightarrow E \twoheadrightarrow E/(D+U)$ induces a surjection

$$C/(C \cap U) \twoheadrightarrow E/(D+U)$$

thus, since $\dim C/(C \cap U) < \infty$ we conclude $\dim E/(D+U) < \infty$.
Now, suppose D is a d-lattice, again, choose C a direct complement for D. Analogous as the proof for D being discrete and closed in the previous paragraph it follows the one for C being open and closed. We just check that C is linearly compact. Let U be any open vector subspace, the composition $E \twoheadrightarrow C \twoheadrightarrow C/(C \cap U)$ induces a surjection

$$E/(D + (C \cap U)) \twoheadrightarrow C/(C \cap U)$$

since both C and U are open, also $C \cap U$, thus $\dim E/(D + (C \cap U)) < \infty$. It follows, $\dim C/(C \cap U) < \infty$ and C linearly compact. □

Remark 1.9. Note that in the proof of Proposition 1.8 it is not strictly necessary to choose a direct complement, one can choose a direct complement up to finite dimension; that is, $C + D = E$ and $\dim C + D < \infty$. We used a direct complement to facilitate the proof.

Duality

If E is a Tate space we consider the following topology on the dual space E^* (where by dual space we mean topological dual). Open vector subspaces are given by

$$N(C) = \{\phi \in E^* : \phi|_C = 0\}$$

where C is linearly compact subspace. Equivalently, one can define open vector subspaces in E^* to be D^* where D a direct complement of a linearly compact vector subspace C in E (in this case $D^* \hookrightarrow E^*$ using the decomposition $C \oplus D$).
First, we prove that the word *dually* in Definition 1.4 actually makes sense.

Proposition 1.10. *Duality interchanges discrete and linearly compact spaces.*

Proof.

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Morphisms

A **morphism** of Tate spaces is a continuous linear homomorphism between Tate spaces.

Definition 1.11. A morphism $f\colon A \to B$ of Tate spaces is said to be **linearly compact** if fA is linearly compact in B. Dually, it is discrete if $\ker f$ is open in A.

2

TRACE AND RESIDUE

2.1 FINITEPOTENT MAPS AND THEIR TRACE

Let k be a fixed ground field and V a vector space over k. In this section we will expand the notion of trace of a linear endomorphism to include certain operators even when V is infinite dimensional.

Finitepotent maps

Definition 2.1. We will say a linear map $f\colon V \to V$ is **finitepotent** if

$$\dim f^n V < \infty$$

for sufficiently large n.

We characterize finitepotent maps as follows.

Lemma 2.2. *A linear map $f\colon V \to V$ is finitepotent if and only if there exists a subspace $W \subseteq V$ such that*

(i) $\dim fW < \infty$,

(ii) $fW \subseteq W$,

(iii) the induced map $\bar{f}\colon V/W \to V/W$ is nilpotent.

Proof. If f is finitepotent choose $W = f^n V$ for sufficiently large n. The first condition follows from definition. Also, $fW = f^{n+1}V \subseteq f^n V = W$. In addition, $\bar{f}^n = 0$. On the other hand, if such W exists, note that condition (ii) assures that $\bar{f}$ is well defined. Moreover, as $\bar{f}$ is nilpotent, $f^n V \subseteq W$ for sufficiently large n and by condition (i) above $\dim f^n V < \infty$. $\square$

Trace

If f is a finitepotent map and W is as above, $\mathrm{tr}_V(f) \in k$ may be defined as $\mathrm{tr}_W(f)$ where $\mathrm{tr}_W(f)$ is the ordinary trace of f viewed as a endomorphism of W. First, we will check that this definition does not depend on the choice of W. Suppose $W_1, W_2 \subseteq V$ suffice the properties on Lemma 2.2, then $W = W_1 + W_2$ suffices them too. Hence, as the induced maps on W/W_1 and W/W_2 are nilpotent, they have have zero ordinary trace and since

$$\begin{aligned} \mathrm{tr}_W(f) &= \mathrm{tr}_{W_1}(f) + \mathrm{tr}_{W/W_1}(f) \\ \mathrm{tr}_W(f) &= \mathrm{tr}_{W_2}(f) + \mathrm{tr}_{W/W_2}(f), \end{aligned}$$

we obtain $\mathrm{tr}_{W_1}(f) = \mathrm{tr}_{W_2}(f)$, our desired result.

This definition extends some of the properties of the ordinary trace.

Lemma 2.3. *(a) If $\dim V < \infty$, any endomorphism f is finitepotent and $\mathrm{tr}_V(f)$ coincides with the ordinary trace.*

(b) If f is nilpotent, then it is finitepotent and $\mathrm{tr}_V(f) = 0$.

(c) *If f is finitepotent and U is a subspace such that $fU \subseteq U$ then the induced maps on U and V/U are finitepotent and satisfy*

$$\mathrm{tr}_V(f) = \mathrm{tr}_U(f) + \mathrm{tr}_{V/U}(f)$$

Proof. Both (a) and (b) are immediate. For (c) if W suffices the properties in Lemma 2.2 for f then $W \cap U$ and $(W+U)/U$ suffice them for the induced maps, that is, they're finitepotent. Since $W/(W \cap U) \cong W + U/U$, the diagram

$$\begin{tikzcd} W/(W\cap U) \arrow[r, "\cong"] \arrow[d, "f"] & (W+U)/U \arrow[d, "f"] \\ W/(W\cap U) \arrow[r, "\cong"] & (W+U)/U \end{tikzcd}$$

commutes and trace is invariant under conjugation, we get $\mathrm{tr}_{W/(W\cap U)}(f) = \mathrm{tr}_{(W+U)/U}(f)$. Hence

$$\mathrm{tr}_V(f) = \mathrm{tr}_W(f) = \mathrm{tr}_{W\cap U}(f) + \mathrm{tr}_{(W+U)/U}(f) = \mathrm{tr}_U(f) + \mathrm{tr}_{V/U}(f) \qquad \square$$

Definition 2.4. A subspace F of $\mathrm{End}_k)V$ is said to be a **finitepotent subspace** if there exists an n such that for any family of n elements $f_1, \ldots, f_n \in F$, the space $f_1 f_2 \cdots f_n V$ is finite dimensional.

The following is the natural linearity property for tr.

Proposition 2.5. *If F is a finitepotent subspace then $\mathrm{tr}_V : F \to k$ is k-linear*

Proof. It is enough to prove it in the case that F is finite dimensional. Let $W = F^n V$ for n as in the definition of finitepotent subspace, thus $\dim W < \infty$. Hence, for all $f \in F$, W suffices the conditions in Lemma 2.2. It follows that $\mathrm{tr}_V(f) = \mathrm{tr}_W(f)$ which is linear. $\square$

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Proposition 2.6. *If $f, g \in \mathrm{End}_k(V)$ and fg is finitepotent then gf is also finitepotent and*

$$\mathrm{tr}_V(fg) = \mathrm{tr}_V(gf).$$

Proof. Since fg is finitepotent let $W = (fg)^n V$ for sufficiently large n has finite dimension. On the other hand, $(gf)^{n+1}V = g(fg)^n fV \subseteq gW$, therefore, gf is also finitepotent. Let $W' = (gf)^n V$, then $gW' \subseteq W$ and $fW \subseteq W'$. Thus,

$$\dim W' \leq \dim gW \leq \dim W \quad \text{and,} \quad \dim W \leq \dim fW \leq \dim W',$$

which implies that $W \cong W'$ and that g and f induce mutually inverse isomorphism between W and W'. Moreover, the diagram

$$\begin{tikzcd} W \arrow[r, "fg"] \arrow[d, "g"] & W \arrow[d, "g"] \\ W' \arrow[r, "gf"] & W' \end{tikzcd}$$

commutes. We conclude $\mathrm{tr}_W(fg) = \mathrm{tr}_{W'}(gf)$ and it follows $\mathrm{tr}_V(fg) = \mathrm{tr}_V(gf)$. $\square$