

TATE'S LINEAR ALGEBRA

1.1 LINEAR TOPOLOGIES

Fix a ground field k . From now on, a vector space will always mean a k -vector space.

Definition 1.1. A **linear topology** on a vector space V is a separated (Hausdorff) topology invariant under translations that admits an open local base around zero of vector subspaces. A vector space equipped with a linear topology will be referred as **linearly topologized**.

If we endow k with the discrete topology then V will become a topological vector space. From now on, endow k with the discrete topology.

Linear topologies behave nicely under basic topological operations.

Proposition 1.2. *Let V be a linearly topologized vector space. Then*

- (a) *Any vector subspace of V is linearly topologized under its subspace topology.*
- (b) *If $W \subseteq V$ is a closed vector subspace then V/W is linearly topologized under its quotient topology.*
- (c) *If $\{V_\alpha\}_\alpha$ is a collection of linearly topologized vector spaces its product $\prod_\alpha V_\alpha$ and its direct sum $\bigoplus_\alpha V_\alpha$ is linearly topologized under its product topology.*
- (d) *If W is a vector subspace of V , then its topological closure \overline{W} also is a vector subspace of V .*

Proof. Since intersection of vector subspaces is a vector subspace, (a) follows intersecting the fundamental system of neighborhoods in V by the vector subspace. For (b), let $\pi: V \rightarrow V/W$ be the quotient map. Since π is open and surjective the image of a local base is a local base; moreover, the image of a vector subspace under π is a vector subspace. In addition, since W is closed then V/W is Hausdorff. Now, for (c) let $\{U_{\alpha,\beta}\}_\beta$ be a local base of zero in V_α of vector subspaces, the

products $U_{\alpha_1, \beta_1} \times \dots \times U_{\alpha_n, \beta_n} \times \prod_{\gamma} V_{\gamma}$, where γ ranges over $\alpha \neq \alpha_1, \dots, \alpha_n$, for any set $\{(\alpha_1, \beta_1, \dots, \alpha_n, \beta_n)\}$ form a fundamental system of neighborhoods around zero in $\prod_{\alpha} V_{\alpha}$ of open vector subspaces. Note that since $\bigoplus_{\alpha} V_{\alpha} \subseteq \prod_{\alpha} V_{\alpha}$ is a vector subspace (c) follows from (a). Finally, for (d), suppose $x, y \in \overline{W}$, then, for every open vector subspace U , $(x + U) \cap W \neq \emptyset$ and $(y + U) \cap W \neq \emptyset$, therefore for every $\alpha, \beta \in k$ we have $(\alpha x + U) \cap W \neq \emptyset$ and $(\beta y + U) \cap W \neq \emptyset$. Hence, $(\alpha x + \beta y + U) \cap W \neq \emptyset$ for every open vector subspace U and every pair $\alpha, \beta \in k$. It follows (d). \square

Remark 1.3. If we let LinTop_k denote the category of linearly topologized vector spaces over k (where morphisms are given by continuous linear homomorphisms) the previous proposition implies existence of kernels and arbitrary products; whence, existence of limits.

Linear topologies are discrete over a finite dimensional vector space.

Proposition 1.4. *A finite dimensional linearly topologized vector space V is discrete.*

Proof. Let U be an open vector subspace and $0 \neq x \in U$, since V is separated there exists an open vector subspace U_x such that $x \notin U_x$. Thus, $\dim U_x \cap U < \dim U$. Since V is finite dimensional this process can be repeated only a finite amount of times; that is $\{0\}$ is open. It follows that V is discrete. \square

Commensurability

We introduce a partial order in the set of vector subspaces of a vector space V .

Definition 1.5. For vector subspaces A and B of a vector space V we say that $A \prec B$ if the quotient $A/(A \cap B) \cong (A + B)/B$ is finite dimensional (or equivalently $A \subseteq B + W$ for some finite dimensional W). In addition, we say that A and B are **commensurable** (denoted $A \sim B$) if $A \prec B$ and $B \prec A$.

Observe that $A \sim B$ if and only if $(A + B)/(A \cap B) \cong A/(A \cap B) \oplus B/(A \cap B)$ is finite dimensional. We will constantly refer to a vector space V being finite dimensional as $V \sim 0$.

Proposition 1.6. *Let V be a vector spaces and A, B and C be vector subspaces, then:*

(a) If $A \sim B$ and $B \sim C$ then

$$\frac{A+B+C}{A \cap B \cap C} \sim 0$$

(b) If $A \prec B$ and $B \prec C$ then $A \prec C$. Moreover, commensurability is an equivalence relation.

Proof. Consider the following exact sequences

$$0 \rightarrow \frac{A \cap B}{A \cap B \cap C} \rightarrow \frac{B}{B \cap C},$$

and,

$$0 \rightarrow \frac{A \cap B}{A \cap B \cap C} \rightarrow \frac{A+B}{A \cap B \cap C} \rightarrow \frac{A+B}{A \cap B} \rightarrow 0$$

induced by inclusions. The first inclusion plus the fact that $B \sim C$ imply that $(A \cap B)/(A \cap B \cap C)$ is finite dimensional. Now, since $A \sim B$ it follows that $(A+B)/(A \cap B)$ is finite dimensional. Hence, the second exact sequence concludes that $(A+B)/(A \cap B \cap C) \sim 0$. A symmetrical argument shows that $(B+C)/(A \cap B \cap C) \sim 0$. These prove (a). For (b), the inclusion

$$0 \rightarrow \frac{A+C}{A \cap C} \rightarrow \frac{A+B+C}{A \cap B \cap C}$$

plus (a) implies transitivity. \square

Now, we state and prove some useful properties on the relation \prec .

Lemma 1.7. (a) If $A \subseteq B$ then $A \prec B$.

(b) If $A \prec B$ then $f(A) \prec f(B)$ for any k -linear map f

(c) It holds that

$$\sum_{i=1}^m A_i \prec \bigcap_{j=1}^n B_j \iff A_i \prec B_j \text{ for all } i \text{ and } j.$$

Proof. First, (a) is immediate from the definition of \prec . Second, for (b) the map f factors as

$$A/(A \cap B) \rightarrow f(A)/(f(A) \cap f(B)) \rightarrow 0$$

Finally, for (c), if $\sum_{i=1}^m A_i \prec \bigcap_{j=1}^n B_j$ holds then by (a) above, for all i and j we have

$$A_i \prec \sum_{i=1}^m A_i \prec \bigcap_{j=1}^n B_j \prec B_j$$

On the other hand, if $A_i \prec B_j$ for all i and j then there exist finite dimensional subspaces W_{ij} such that $A_i \subseteq B_j + W_{ij}$ for all i and j . Therefore,

$$\sum_{i=1}^m A_i \subseteq \bigcap_{j=1}^n B_j + \sum_{i=1}^m \sum_{j=1}^n W_{ij}. \quad \square$$

Next, we consider another useful lemma.

Lemma 1.8. *Let A, B, A', B' be vector subspaces of a vector space V and suppose that $A \sim A'$ and $B \sim B'$. Then $A + B \sim A' + B'$ and $A \cap B \sim A' \cap B'$.*

Proof. The following exact sequence

$$0 \rightarrow \frac{A + A' + B + B'}{A \cap A' \cap B \cap B'} \rightarrow \frac{A + A'}{A \cap A'} \oplus \frac{B + B'}{B \cap B'} \rightarrow \frac{A + A' + B + B'}{(A \cap A') + (B \cap B')} \rightarrow 0$$

plus $A \sim A'$ and $B \sim B'$ imply that both spaces

$$\frac{A + A' + B + B'}{A \cap A' \cap B \cap B'} \quad \text{and} \quad \frac{A + A' + B + B'}{(A \cap A') + (B \cap B')}$$

are finite dimensional. Since, $(A + A' + B + B') / (A + A') \cap (B + B')$ is a quotient of the second space and $((A \cap A') + (B \cap B')) / ((A \cap A') \cap (B \cap B'))$ is a subspace of the first space we can conclude $A + B \sim A' + B'$ and $A \cap B \sim A' \cap B'$. \square

If we consider the set of equivalence classes of \sim on a vector space V then \prec is a partial order on it and by [Lemma 1.8](#) above it inherits operations \cap and $+$.

Linear compactness

Definition 1.9. Let V be a linearly topologized vector space. A closed subset $L \subseteq V$ is **linearly compact** (respectively **linearly cocompact**) if for every open vector subspace U we have $L \prec U$ (respectively $V / (L + U) \sim 0$).

Linear compactness behaves just as compactness if one uses the correct words.

Proposition 1.10. *Let V be a linearly compact vector space, then*

- (a) *If $A \subseteq V$ is a vector subspace such that for every open vector subspace U of V it holds $A \prec U$ then \bar{A} is linearly compact.*

- (b) If $f: V \rightarrow W$ is a continuous linear homomorphism then $\overline{f(V)}$ is linearly compact.
- (c) If V is discrete then $V \sim 0$.
- (d) Every closed vector subspace of V is linearly compact.
- (e) (Tychonov) If $\{V_\alpha\}_\alpha$ is a collection of linearly compact vector spaces then its product $\prod_\alpha V_\alpha$ and its direct sum $\bigoplus_\alpha V_\alpha$ are linearly compact.

Proof. Let U be any open vector subspace of V , then $A + U$ is closed, that is $A + U = \overline{A + U} \supseteq \overline{A} + U \supseteq A + U$, thus, $\overline{A} + U = A + U$. Since, $(A + U)/U \sim 0$ then $(\overline{A} + U)/U \sim 0$. We get (a).

For (b), since f is a continuous linear map $V \prec f^{-1}(U)$ for all U open vector subspace of W , hence by [Lemma 1.7](#) $f(V) \prec U$ for all open vector subspaces U of W . By the previous observation and (a) we get (b). If V is discrete, then $\{0\}$ is an open vector subspace of E , thus $V \prec U$, we get (c).

For (d), if $A \subseteq V$ is a closed vector subspace, and $V \prec U$ for all open vector subspaces U by [Lemma 1.7](#) we get $A \prec U$.

Finally, for (d), it is enough proving for open vector subspaces $U = \prod_\beta U_\beta \times \prod_\gamma V_\gamma$ where β ranges over a finite set, γ ranges over $\alpha \neq \beta$ and U_β is an open vector subspace of V_β . Then, the quotient

$$\prod_\alpha V_\alpha / U \cong \prod_\beta V_\beta / U_\beta$$

where \cong is a topological and algebraic isomorphism. Since V_α is linearly compact for all α and β ranges over a finite set we conclude that $\prod_\alpha V_\alpha / U$ is finite dimensional; therefore, $\prod_\alpha V_\alpha$ is linearly compact. The proof is analogous for $\bigoplus_\alpha V_\alpha$. \square

Completion

Definition 1.11. Let V be a linearly topologized vector space. Given a local base of open vector subspaces \mathcal{U} we will say that a collection $(x_U)_{U \in \mathcal{U}}$ is a **net indexed by \mathcal{U}** . We will say that $(x_U)_{U \in \mathcal{U}}$ converges to x if for any $U \in \mathcal{U}$ there exists a $W \in \mathcal{U}$ such that for every $U' \subseteq W$ and $U' \in \mathcal{U}$

$$x_{U'} - x \in U.$$

Remark 1.12. The previous definition can be generalized to an arbitrary topological space and nets indexed by any directed set. In our particular case, the directed set $\mathcal{D} = \mathcal{U}^{\text{op}}$.

We now introduce Cauchy nets and completion.

Definition 1.13. A net $(x_U)_{U \in \mathcal{U}}$ in a linearly topologized vector space indexed by an open local base \mathcal{U} of open vector subspaces is said to be **Cauchy** if for every U open vector subspace of zero there exists $W \in \mathcal{U}$ such that for every $U_1, U_2 \in \mathcal{U}$ and $U_1, U_2 \subseteq W$ we have $x_{U_1} - x_{U_2} \in U$.

Definition 1.14. If V be a linearly topologized vector space, we say that V is said to be **complete** if

$$V \cong \varprojlim_{U \in \mathcal{U}} V/U$$

for an open local base of vector subspaces \mathcal{U} . We leave unproven that this definition is independent of the choice of \mathcal{U} . Details can be found in [include reference](#). Moreover, V is complete if and only if every Cauchy net is convergent. One can see this by identifying the inverse limit as a subspace of the product.

Complete this! (no pun intended)

1.2 TATE SPACES

Lattices

Definition 1.15. If V is a linearly topologized vector space we say that a **c-lattice** is an open linearly compact subspace of V , *dually* a discrete linearly cocompact subspace is a **d-lattice**.

First, we prove that existence of a c-lattice in a linearly topologized vector space is equivalent to existence of a d-lattice.

Proposition 1.16. *A linearly topologized vector space V has a c-lattice if and only if it has a d-lattice.*

Proof. Suppose L is a c-lattice in V , choose any direct complement D of L , that is, $V = L \oplus D$. Since L is open, then D is discrete as $D \cap L = 0$, thus 0 is open in D . Moreover, D is closed as it is the fiber of 0 under the projection $V \rightarrow L$ (which is continuous because L is open). Finally, we check that D is

linearly cocompact: let U be any open vector subspace of V , the composition $L \hookrightarrow V \twoheadrightarrow V/(D+U)$ induces a surjection

$$L/(L \cap U) \twoheadrightarrow V/(D+U)$$

thus, since $\dim L/(L \cap U) < \infty$ we conclude $\dim V/(D+U) < \infty$.

Now, suppose D is a d-lattice. Thus, there exists an open vector subspace U such that $U \cap D = 0$. This time, choose L a direct complement for D containing U . Then, the projection $V \rightarrow D$ is continuous because U goes to zero. Therefore, L is open. Now we check linear compactness. Let U be any open vector subspace, the composition $V \twoheadrightarrow L \twoheadrightarrow L/(L \cap U)$ induces a surjection

$$V/(D + (L \cap U)) \twoheadrightarrow L/(L \cap U)$$

since both L and U are open, also $L \cap U$, thus $\dim V/(D + (L \cap U)) < \infty$. It follows, $\dim L/(L \cap U) < \infty$ and L linearly compact. \square

Remark 1.17. Note that in the proof of [Proposition 1.16](#) it is not strictly necessary to choose a direct complement, one can choose a direct complement up to finite dimension; that is, $L + D \sim V$ and $L \cap D \sim 0$.

We now give a characterization of lattices in terms of \prec .

Proposition 1.18. *If V admits a c-lattice, then the set of c-lattices constitutes a base of zero of mutually commensurable vector subspaces.*

Proof. If L and L' are two c-lattices in V then $L \prec L'$ and $L' \prec L$ because both are open; therefore, all c-lattices are commensurable. Moreover, if U is any open vector subspace and L is a c-lattice we claim that $L \cap U$ is a c-lattice. Indeed, let U' be any open vector subspace, then $L \cap U \prec L \prec U'$. In addition, since L and U are open, $L \cap U$ is open. Thus $L \cap U \subseteq U$ is a c-lattice, this proves the statement. \square

We're now ready to introduce the definition of a Tate space.

Definition 1.19. A **Tate space** V is a complete linearly topologized vector space that admits a c-lattice. By the previous proposition and the observation in [Definition 1.14](#) we get

$$V \cong \varprojlim_{L \in \mathcal{L}(V)} V/L$$

where $\mathcal{L}(V)$ runs through all c-lattices of V .

Example 1.20. We give some examples of Tate spaces.

- (a) Any vector space endowed with the discrete topology is a Tate space.
- (b) If $\{V_\alpha\}_\alpha$ is any pro-system of finite dimensional vector spaces (thus, each one endowed with the discrete topology by [Proposition 1.4](#)), let V be their inverse limit endowed with the inverse limit topology. We claim that this is a linearly compact space. Indeed, if we realize V as a subspace of the product $\prod_\alpha V_\alpha$, then basic open vector subspaces are just restriction of finite coordinates. Hence, the quotient of V by any basic open vector subspace is a finite product of V_α , since all V_α are finite dimensional we conclude that V is linearly compact and therefore a Tate space.
- (c) Let $V = k((t))$ with the topology generated by letting $t^n k[[t]]$ for $n \in \mathbb{Z}$ be a system of neighborhoods of zero. Then, $V = k[[t]] \oplus tk[t^{-1}]$ where $k[[t]]$ is the completion of $k[x]$ in the $\langle x \rangle$ -adic topology, hence by the previous item linearly compact and, since it is open is a c-lattice. By the argument given in [Proposition 1.16](#) $tk[t^{-1}]$ is a d-lattice. Therefore, V is a Tate space that is not linearly compact nor discrete.

Duality

If V is a Tate space we consider the following topology on the dual space V^* (where by dual space we mean topological dual). Open vector subspaces are given by

$$L^\perp = \{\phi \in E^* : \phi|_L = 0\}$$

where L is a linearly compact subspace. Equivalently, one can define open vector subspaces in E^* to be D^* where D a direct complement of a linearly compact vector subspace L in E (in this case $D^* \hookrightarrow E^*$ using the decomposition $L \oplus D$).

First, we prove that the word *dually* in [Definition 1.9](#) actually makes sense.

Lemma 1.21. *Duality interchanges linearly compact with discrete spaces and vice-versa.*

Proof. If L is a linearly compact vector space, then L^\perp is open in L^* , thus L^* is discrete. If D is discrete, then $D \cong k^{\oplus \Lambda}$ for some Λ and endowing $k^{\oplus \Lambda}$ with the discrete topology the previous isomorphism is a homeomorphism too. Moreover, since D is discrete every linear functional is continuous. Using [Remark 1.3](#) and the well known identity (where maps are isomorphisms in LinTop_k)

$$(k^{\oplus \Lambda})^* = \text{Hom}_k(k^{\oplus \Lambda}, k) \cong \prod_{\Lambda} \text{Hom}_k(k, k) \cong \prod_{\Lambda} k$$

we get the desired result by Tychonov's theorem in [Proposition 1.10](#). \square

Remark 1.22. Note that in the proof of the previous lemma the dual space of a discrete space is complete, as it is an arbitrary product of k endowed with the product topology.

Proposition 1.23. *If V is a Tate space then V^* with the topology previously introduced is also a Tate space.*

Proof. If we decompose $V = L \oplus D$ where L is a c-lattice and D a d-lattice then $V^* \cong L^* \oplus D^*$ and by [Lemma 1.21](#) L^* is discrete and D^* is linearly compact. Observe that D^* is open in V^* since it is the kernel of the projection $V^* \rightarrow V^*/L^\perp$ and V^*/L^\perp is discrete by the description of our topology in the dual V^* . Since L^* is discrete, then it is complete. Moreover, by the previous remark, D^* is complete, hence V^* is complete too. \square

We're ready to prove the analog of Pontryagin's duality for Tate spaces.

Theorem 1.24. *For a Tate space V the canonical map $V \rightarrow V^{**}$ is an isomorphism.*

Proof. It is enough to prove it for complete linearly compact spaces and discrete spaces, as every Tate space can be decomposed into a direct sum of a c-lattice and a d-lattice. First, we do it for discrete spaces. Suppose D is a discrete vector space. Then, the canonical map

$$\text{ev}: D \rightarrow D^{**}$$

is open and continuous because D and D^{**} are both discrete by [Lemma 1.21](#). Moreover, it is injective, because for every nonzero $v \in D$ there exists a linear continuous functional $\phi \in D^*$ such

that $\phi(v) \neq 0$. Finally, we prove surjectivity. Let $\psi \in D^{**}$. Since $\ker \psi$ is open it contains a basic open vector subspace A^\perp such that $A \subseteq D$ is a linearly compact subspace. Therefore, since D^* is linearly compact it follows that $D^* \sim A^\perp$, that is, the quotient D^*/A^\perp is finite dimensional. Recall that the inclusion $\iota: A \rightarrow D$ induces an isomorphism $D^*/A^\perp \rightarrow A^*$ which is a homeomorphism since both spaces are discrete. We can factor ψ so that the following diagram commutes

$$\begin{array}{ccc}
 D^* & \xrightarrow{\psi} & k \\
 \downarrow & \nearrow \bar{\psi} & \uparrow \\
 D^*/A^\perp & & \\
 \cong \downarrow & \nearrow \bar{\psi} & \\
 A^* & &
 \end{array}$$

However, A^* is finite dimensional, therefore, there exists $a \in A$ such that $\bar{\psi} = \text{ev}_a$ as maps from $A^* \rightarrow k$. Moreover, since $A^\perp \subseteq \ker \psi$ we conclude that $\psi = \text{ev}_a$ as maps $D^* \rightarrow k$. This implies surjectivity. Thus $D \rightarrow D^{**}$ is an isomorphism of topological vector spaces.

Now, suppose L is a complete linearly compact space. We check first that the map

$$\text{ev}: L \rightarrow L^{**}$$

is continuous. Let A^\perp be an open vector subspace in L^{**} where $A \subseteq L^*$ is a linearly compact subspace. By [Lemma 1.21](#) L^* is discrete, hence A is finite dimensional. Suppose that $A = \text{span}(\phi_1, \dots, \phi_n)$ for some $\phi_1, \dots, \phi_n \in A$. Then, $\text{ev}^{-1}(A^\perp) = \ker \phi_1 \cap \dots \cap \ker \phi_n$ which is open in L . Now, we check that ev is injective. Let $v \in L$ be a nonzero vector. Choose a decomposition of $L = U \oplus F$ where U is open and F is finite dimensional containing v (this can be done because L is separated and linearly compact). Let ϕ be a linear functional such that restricted to U is zero and $\phi(v) \neq 0$. Since U is open and F discrete such ϕ exists and it is continuous. This implies injectivity of ev . Now we check that ev is surjective. Let \mathcal{U} is an open local base of vector subspaces. Let $\psi: L^* \rightarrow k$ be a continuous linear functional. By pulling back $\pi_U: L \rightarrow L/U$ we get an injection $\pi_U^*: (L/U)^* \hookrightarrow L^*$ for every $U \in \mathcal{U}$. Since L is linearly compact, then L/U is finite dimensional, thus, there exists $v_U \in L$

such that $\psi \circ \pi_U^* = \text{ev}_{x_U}$ where $\text{ev}: L/U \rightarrow (L/U)^*$. In particular, observe that if $U, U' \in \mathcal{U}$ and $U' \subseteq U$ we get an induced surjection $(L/U)^* \hookrightarrow (L/U')^*$ such that the following diagram

$$\begin{array}{ccc}
 & L^* & \xrightarrow{\psi} k \\
 & \uparrow & \nearrow \text{ev}_{x_{U'}} \\
 & (L/U')^* & \\
 & \uparrow & \nwarrow \text{ev}_{x_U} \\
 & (L/U)^* &
 \end{array}$$

commutes. Observe that this implies that $(v_U)_{U \in \mathcal{U}}$ is a Cauchy net. Indeed, let $U \in \mathcal{U}$ and $W = U$ as in [Definition 1.13](#). Let $U_1, U_2 \subseteq U$ and $U_1, U_2 \in \mathcal{U}$. Notice that commutativity of the following diagram

$$\begin{array}{ccccc}
 & & & & \text{ev}_{x_U} \\
 & & & & \curvearrowright \\
 & & (L/U_1)^* & \xrightarrow{\text{ev}_{x_{U_1}}} & L^* \\
 & \nearrow & & \searrow & \downarrow \psi \\
 (L/U)^* & \xrightarrow{\quad} & L^* & \xrightarrow{\quad} & k \\
 & \searrow & & \nearrow & \uparrow \\
 & & (L/U_2)^* & \xrightarrow{\text{ev}_{x_{U_2}}} & L^* \\
 & & & & \curvearrowleft \text{ev}_{x_U}
 \end{array}$$

implies that $(x_U)_{U \in \mathcal{U}}$ is Cauchy, since $\text{ev}_{x_{U_1}} = \text{ev}_{x_{U_2}}$ coincide in $(L/U)^*$. By completeness there exists $x \in L$ limit of $(x_U)_{U \in \mathcal{U}}$. We claim that $\psi = \text{ev}_v$. Let $\phi \in L^*$. Then, $\ker \phi$ is open and there exists $U \in \mathcal{U}$ such that $U \subseteq \ker \phi$ and by linear compactness $U \sim L$. Hence, if we factor ϕ as follows

$$\begin{array}{ccc}
 L & \xrightarrow{\phi} & k \\
 \pi_U \downarrow & \nearrow \phi_U & \\
 L/U & &
 \end{array}$$

since L/U is discrete we conclude that ϕ_U is continuous. In other words, the image of ϕ_U under the inclusion $(L/U)^* \hookrightarrow L^*$ is ϕ . Thus, $\psi(\phi) = \phi_U(x_U + U) = \phi(x_U)$. We claim that $\psi(\phi) = \phi(x)$. Indeed, convergence of (x_U) implies that there exists a

$W \in \mathcal{U}$ such that $W \subseteq U$ and $x - x_W \in U$. Thus, $\psi(\phi) = \phi_W(x_W + W) = \phi(x_W) = \phi(x)$, since $U \subseteq \ker \phi$. This implies surjectivity of $\text{ev}: L \rightarrow L^{**}$. To conclude, we prove that ev is open. Let U be any open vector subspace in L , thus $L = U \oplus F$ for some F finite dimensional. We claim that $\text{ev}(U) = (F^*)^\perp$. First, the inclusion $\text{ev}(U) \subseteq (F^*)^\perp$ is immediate. Let $\psi \in (F^*)^\perp$. Let $x \in L$ such that $\text{ev}_x = \psi$. Write $x = u + f$ where $u \in U$ and $f \in F$. Hence, $\text{ev}_x = \text{ev}_u + \text{ev}_f$. Since ev is injective, it follows that there exists some $\phi \in F^*$ such that $\phi(f) \neq 0$ if f is nonzero. Therefore, $f = 0$ and $\psi \in \text{ev}(U)$. This concludes the proof. \square

Remark 1.25. Observe that completeness cannot be dropped in the definition of a Tate space while preserving duality. Indeed, if V is linearly compact but not complete its dual is discrete by [Lemma 1.21](#) and by [Remark 1.22](#) its double dual is complete, hence $V \rightarrow V^{**}$ cannot be an isomorphism. In fact, during the proof of the duality theorem we checked that V^{**} is the completion of V .

Morphisms

A **morphism** of Tate spaces is a continuous linear homomorphism between Tate spaces.

Definition 1.26. A morphism $f: V \rightarrow W$ of Tate spaces is said to be **linearly compact** if the closure of $f(V)$ is linearly compact in W . Dually, it is **discrete** if $\ker f$ is open in V .

First, we check the duality natural property for morphisms of Tate spaces.

Proposition 1.27. *A morphism $f: V \rightarrow W$ of Tate spaces is linearly compact if and only if f^* is discrete.*

Proof. Suppose f^* is linearly compact, then $\ker f^* = f(V)^\perp$. However, if $\phi \in W^*$ and $\phi(f(V)) = 0$ then $\phi(\overline{f(V)}) = 0$ by continuity of ϕ . Therefore, $\ker f^* = \overline{f(V)}^\perp$ which is open because $\overline{f(V)}$ is linearly compact. Now, suppose f^* is discrete. Thus, $\ker f^*$ contains a basic open subspace A^\perp such that A is linearly compact in W . Therefore, $\overline{f(V)} \subseteq A$ then $\overline{f(V)} \subseteq A$ and by item (c) in [Proposition 1.10](#) $\overline{f(V)}$ is linearly compact. \square

Discrete and linearly compact operators form a 2-sided ideal in Hom ; that is

Proposition 1.28. *If f is a linearly compact operator (respectively discrete) then its composition (from any side) with an arbitrary morphism of Tate spaces is also linearly compact (respectively discrete).*

Proof. Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ be morphisms of Tate spaces such that g is linearly compact. Then, $\overline{g \circ f(A)} \subseteq \overline{g(B)}$ which is linearly compact, thus $\overline{g \circ f(A)}$ is linearly compact too. On the other hand, note that $\overline{h(g(B))} \subseteq \overline{h \circ g(B)}$; therefore $\overline{h(g(B))} = \overline{h \circ g(B)}$. However, $\overline{g(B)}$ is linearly compact and by item(b) in [Proposition 1.10](#) $\overline{h(g(B))}$ is linearly compact. In addition, the statement for discrete operators follows from the previous proposition. \square

Remark 1.29. If f is a compact operator and g is a discrete operator, then gf is **finite**; that is, $\dim gf(V) < \infty$.

Proof. We have $\overline{f(V)} \prec \ker g$, therefore, $\overline{f(V)} / (\overline{f(V)} \cap \ker g)$ is finite dimensional. We have a surjection

$$\frac{\overline{f(V)}}{\overline{f(V)} \cap \ker g} \rightarrow gf(V)$$

which implies that gf is finite. \square

Definition 1.30. Let V and W be Tate spaces. We denote $\text{Hom}_+(V, W)$ to be the set of linearly compact morphisms and $\text{Hom}_-(V, W)$ the set of discrete ones. Also, set $\text{Hom}_0(V, W)$ to be $\text{Hom}_+(V, W) \cap \text{Hom}_-(V, W)$.

Proposition 1.31. *The sets $\text{Hom}_-(V, W), \text{Hom}_+(V, W)$ and $\text{Hom}_0(V, W)$ are vector subspaces of $\text{Hom}(V, W)$. Moreover,*

$$\text{Hom}_-(V, W) + \text{Hom}_+(V, W) = \text{Hom}(V, W).$$

Proof. Let L be a c-lattice in V and consider $\pi: V \rightarrow L$ be a continuous linear projection. Then π realized as an element of $\text{End}(V)$ satisfies $\pi \in \text{End}_+(V)$ and $1 - \pi \in \text{End}_-(V)$. Hence, by [Proposition 1.28](#) for every $f \in \text{Hom}(V, W)$ $f \circ \pi$ and $f \circ (1 - \pi)$ are linearly compact and discrete respectively. It follows

$$\text{Hom}_-(V, W) + \text{Hom}_+(V, W) = \text{Hom}(V, W).$$

The other statements are immediate. \square

I'll include further theory if necessary.

TRACE AND RESIDUE

2.1 FINITEPOTENT MAPS AND THEIR TRACE

Let k be a fixed ground field and V a vector space over k . In this section we will expand the notion of trace of a linear endomorphism to include certain operators even when V is infinite dimensional.

Finitepotent maps

Definition 2.1. We will say a linear map $f: V \rightarrow V$ is **finitepotent** if

$$\dim f^n(V) < \infty$$

for sufficiently large n .

We characterize finitepotent maps as follows.

Lemma 2.2. *A linear map $f: V \rightarrow V$ is finitepotent if and only if there exists a subspace $W \subseteq V$ such that*

- (i) $\dim f(W) < \infty$,
- (ii) $f(W) \subseteq W$,
- (iii) *the induced map $\bar{f}: V/W \rightarrow V/W$ is nilpotent.*

Proof. If f is finitepotent choose $W = f^n(V)$ for sufficiently large n . The first condition follows from definition. Also, $f(W) = f^{n+1}(V) \subseteq f^n(V) = W$. In addition, $\bar{f}^n = 0$. On the other hand, if such W exists, note that condition (ii) assures that \bar{f} is well defined. Moreover, as \bar{f} is nilpotent, $f^n V \subseteq W$ for sufficiently large n and by condition (i) above $\dim f^n(V) < \infty$. \square

Trace

If f is a finitepotent map and W is as above, $\text{tr}_V(f) \in k$ may be defined as $\text{tr}_W(f)$ where $\text{tr}_W(f)$ is the ordinary trace of f viewed as an endomorphism of W . First, we will check that this definition does not depend on the choice of W . Suppose

$W_1, W_2 \subseteq V$ suffice the properties on [Lemma 2.2](#), then $W = W_1 + W_2$ suffices them too. Hence, as the induced maps on W/W_1 and W/W_2 are nilpotent, they have zero ordinary trace and since

$$\begin{aligned}\mathrm{tr}_W(f) &= \mathrm{tr}_{W_1}(f) + \mathrm{tr}_{W/W_1}(f) \\ \mathrm{tr}_W(f) &= \mathrm{tr}_{W_2}(f) + \mathrm{tr}_{W/W_2}(f),\end{aligned}$$

we obtain $\mathrm{tr}_{W_1}(f) = \mathrm{tr}_{W_2}(f)$, our desired result.

This definition extends some of the properties of the ordinary trace.

Lemma 2.3. (a) *If $\dim V < \infty$, any endomorphism f is finitepotent and $\mathrm{tr}_V(f)$ coincides with the ordinary trace.*

(b) *If f is nilpotent, then it is finitepotent and $\mathrm{tr}_V(f) = 0$.*

(c) *If f is finitepotent and U is a subspace such that $fU \subseteq U$ then the induced maps on U and V/U are finitepotent and satisfy*

$$\mathrm{tr}_V(f) = \mathrm{tr}_U(f) + \mathrm{tr}_{V/U}(f)$$

Proof. Both (a) and (b) are immediate. For (c) if W suffices the properties in [Lemma 2.2](#) for f then $W \cap U$ and $(W + U)/U$ suffice them for the induced maps, that is, they're finitepotent. Since $W/(W \cap U) \cong W + U/U$, the diagram

$$\begin{array}{ccc} W/(W \cap U) & \xrightarrow{\cong} & (W + U)/U \\ \downarrow f & & \downarrow f \\ W/(W \cap U) & \xrightarrow{\cong} & (W + U)/U \end{array}$$

commutes and trace is invariant under conjugation, we get $\mathrm{tr}_{W/(W \cap U)}(f) = \mathrm{tr}_{(W + U)/U}(f)$. Hence

$$\mathrm{tr}_V(f) = \mathrm{tr}_W(f) = \mathrm{tr}_{W \cap U}(f) + \mathrm{tr}_{(W + U)/U}(f) = \mathrm{tr}_U(f) + \mathrm{tr}_{V/U}(f)$$

□

Definition 2.4. A subspace F of $\mathrm{End}_k(V)$ is said to be a **finitepotent subspace** if there exists an n such that for any family of n elements $f_1, \dots, f_n \in F$, the space $f_1 f_2 \cdots f_n V$ is finite dimensional.

The following is the natural linearity property for tr .

Proposition 2.5. *If F is a finitepotent subspace then $\text{tr}_V: F \rightarrow k$ is k -linear*

Proof. It is enough to prove it in the case that F is finite dimensional. Let $W = F^n V$ for n as in the definition of finitepotent subspace, thus $\dim W < \infty$. Hence, for all $f \in F$, W suffices the conditions in Lemma 2.2. It follows that $\text{tr}_V(f) = \text{tr}_W(f)$ which is linear. \square

add note in “general” linearity of trace when .bib is ready

Proposition 2.6. *If $f, g \in \text{End}_k(V)$ and fg is finitepotent then gf is also finitepotent and*

$$\text{tr}_V(fg) = \text{tr}_V(gf).$$

Proof. Since fg is finitepotent let $W = (fg)^n V$ for sufficiently large n has finite dimension. On the other hand, $(gf)^{n+1} V = g(fg)^n f(V) \subseteq g(W)$, therefore, gf is also finitepotent. Let $W' = (gf)^n V$, then $g(W') \subseteq W$ and $f(W) \subseteq W'$. Thus,

$$\dim W' \leq \dim g(W) \leq \dim W,$$

and,

$$\dim W \leq \dim f(W) \leq \dim W',$$

which implies that $W \cong W'$ and that g and f induce mutually inverse isomorphisms between W and W' . Moreover, the diagram

$$\begin{array}{ccc} W & \xrightarrow{fg} & W \\ \downarrow g & & \downarrow g \\ W' & \xrightarrow{gf} & W' \end{array}$$

commutes. We conclude $\text{tr}_W(fg) = \text{tr}_{W'}(gf)$ and it follows $\text{tr}_V(fg) = \text{tr}_V(gf)$. \square

2.2 DIFFERENTIAL CALCULUS

In this section we introduce the theory of derivations and differentials over an arbitrary commutative ring R . Let M be a R -module.

Introduce bibliography from Matsumura

Definition 2.7. A **derivation** from R to M is a map $D: R \rightarrow M$ satisfying properties

- (i) $D(x + y) = D(x) + D(y)$ and,
- (ii) (*Leibniz's Rule*) $D(xy) = xD(y) + yD(x)$

for all $x, y \in R$. The set of derivations from R to M becomes a R -module in a natural way. We will write it as $\text{Der}(R, M)$. Moreover if R is a k -algebra through a map $f: k \rightarrow R$ we say that D is a **k-derivation** if $D \circ f = 0$. In this case, the set of all k -derivations is denoted $\text{Der}_k(R, M)$.

If $M = R$, we will denote $\text{Der}_k(R, R)$ simply by $\text{Der}_k(R)$. In particular, if D and D' are two k -derivations then its bracket $[D, D'] = DD' - D'D$ under composition as $R \rightarrow R$ maps is also a k -derivation. Therefore, $\text{Der}_k(R)$ under this structure is a Lie Algebra.

Proposition 2.8. *If R is a k -algebra, consider the covariant functor from the category Mod_R to itself given by $M \mapsto \text{Der}_k(R, M)$. This functor is representable.*

Proof.

Trace and Tate Spaces

Suppose that V is a Tate space and consider $E = \text{End}_k(V)$ the space of continuous endomorphisms of V . Thus, by [Proposition 1.28](#) and [Proposition 1.31](#) we have 2-sided ideals E_0, E_+ and E_- of E such that $E_+ + E_- = E$ and $E_0 = E_+ \cap E_-$. Moreover, [Remark 1.29](#) implies that E_0 is a finitepotent subspace.