1 TATE'S LINEAR ALGEBRA

1.1 LINEAR TOPOLOGIES

Fix a ground field k. From now on, a vector space will always mean a k-vector space.

Definition 1.1. A **linear topology** on a vector space V is a separated (Hausdorff) topology invariant under translations that admits an open local base around zero of vector subspaces. A vector space equipped with a linear topology will be referred as **linearly topologized**.

If we endow k with the discrete topology then V will become a topological vector space. From now on, endow k with the a discrete topology.
Linear topologies behave nicely under basic topological operations.

Proposition 1.2. *Let V be a linearly topologized vector space. Then*

(a) Any vector subspace of V is linearly topologized under its subspace topology.

(b) If $W \subseteq V$ is a closed vector subspace then V/W is linearly topologized under its quotient topology.

(c) If $\{V_\alpha\}_\alpha$ is a collection of linearly topologized vector spaces its product $\prod_\alpha V_\alpha$ and its direct sum $\bigoplus_\alpha V_\alpha$ is linearly topologized under its product topology.

(d) If W is a vector subspace of V, then its topological closure $\overline{W}$ also is a vector subspace of V.

Proof. Since intersection of vector subspaces is a vector subspace, (a) follows intersecting the fundamental system of neighborhoods in V by the vector subspace. For (b), let $\pi\colon V \to V/W$ be the quotient map. Since π is open and surjective the image of a local base is a local base; moreover, the image of a vector subspace under π is a vector subspace. In addition, since Finally, for (c) let $\{U_{\alpha,\beta}\}_\beta$ be a local base of zero in V_α of vector subspaces, the products $U_{\alpha_1,\beta_1} \times \ldots \times U_{\alpha_n,\beta_n} \times \prod_\gamma V_\gamma$, where γ ranges over $\alpha \neq \alpha_1, \ldots, \alpha_n$, for any set $\{(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)\}$ form a fundamental system of neighborhoods around zero in $\prod_\alpha V_\alpha$ of open vector subspaces. Note that since $\bigoplus_\alpha V_\alpha \subseteq \prod_\alpha V_\alpha$ is a vector subspace (c) follows from (a). Finally, for (d), suppose $x, y \in \overline{W}$, then, for every open vector subspace U, $(x+U) \cap W \neq \emptyset$ and $(y+U) \cap W \neq \emptyset$, therefore for every $\alpha, \beta \in k$ we have $(\alpha x + U) \cap W \neq \emptyset$ and $(\beta y + U) \cap W \neq \emptyset$. Hence, $(\alpha x + \beta y + U) \cap W \neq \emptyset$ for every open vector subspace U and every pair $\alpha, \beta \in k$. It follows (d). □

Finite dimensional vector spaces are meaningless for linear topologies.

Proposition 1.3. *A finite dimensional linearly topologized vector space V is discrete.*

Proof. Let U be an open vector subspace and $0 \neq x \in U$, since V is separated and linearly topologized there exists an open vector subspace U_x such that $x \notin U_x$ then $\dim U_x \cap U < \dim U$, since V is finite dimensional this process can be repeated only a finite amount of times; that is $\{0\}$ is open. It follows that V is discrete. □

Commensurability

We introduce a partial order in the set of vector subspaces of a vector space V.

Definition 1.4. For vector subspaces A and B of a vector space V we say that $A \prec B$ if the quotient $A/(A \cap B) \cong (A+B)/B$ is finite dimensional (or equivalently $A \subseteq B + W$ where W is some finite dimensional subspace). In addition, we say that A and B are commensurable (denoted $A \sim B$) if $A \prec B$ and $B \prec A$.

Observe that $A \sim B$ if and only if $(A+B)/(A \cap B) \cong A/(A \cap B) \oplus B/(A \cap B)$ is finite dimensional. We will constantly refer to a vector space V being finite dimensional as $V \sim 0$.

Proposition 1.5. *Let V be a vector spaces and A, B and C be vector subspaces, then:*

(a) If $A \sim B$ and $B \sim C$ then

$$(A+B+C)/(A \cap B \cap C) \sim 0$$

(b) If $A \prec B$ and $B \prec C$ then $A \prec C$. Moreover, commensurability is an equivalence relation.

Proof. Consider the following exact sequences

$$0 \to (A \cap B)/(A \cap B \cap C) \to B/(B \cap C),$$

and,

$$0 \to (A \cap B)/(A \cap B \cap C) \to (A+B)/(A \cap B \cap C) \to (A+B)/(A \cap B) \to 0$$

induced by inclusions. The first inclusion plus the fact that $B \sim C$ imply that $(A \cap B)/(A \cap B \cap C)$ is finite dimensional. Now, since $A \sim B$ it follows that $(A+B)/(A \cap B)$ is finite dimensional. Hence, the second exact sequence concludes that $(A+B)/(A \cap B \cap C)$. A symmetrical argument shows that $(B+C)/(A \cap B \cap C) \sim 0$. These prove (a). For (b), the inclusion

$$0 \to (A+C)/(A \cap C) \to (A+B+C)/(A \cap B \cap C)$$

plus (a) implies transitivity. □

Now, we state and prove some useful properties on the relation $\prec$.

Lemma 1.6. *(a) If $A \subseteq B$ then $A \prec B$.*

(b) If $A \prec B$ then $f(A) \prec f(B)$ for any k-linear map f

(c) It holds that

$$\sum_{i=1}^{m} A_i \prec \bigcap_{j=1}^{n} B_j \iff A_i \prec B_j \text{ for all } i \text{ and } j.$$

Proof. First, (a) is immediate from the definition of $\prec$. Second, for (b) the map f factors

$$A/(A \cap B) \to f(A)/(f(A) \cap f(B)) \to 0$$

Finally, for (c), if $\sum_{i=1}^m A_i \prec \bigcap_{j=1}^n B_j$ holds then by (a) above, for all i and j we have

$$A_i \prec \sum_{i=1}^m A_i \prec \bigcap_{j=1}^n B_j \prec B_j$$

On the other hand, if $A_i \prec B_j$ for all i and j then there exists finite dimensional subspaces W_{ij} such that $A_i \subseteq B_j + W_{ij}$ for all i and j. Therefore,

$$\sum_{i=1}^m A_i \subseteq \bigcap_{j=1}^n B_j + \sum_{i=1}^m \sum_{j=1}^n W_{ij}.$$

□

Next, we consider another useful lemma.

Lemma 1.7. *Let A, B, A', B' be vector subspaces of a vector space V and suppose that $A \sim A'$ and $B \sim B'$. Then $A + B \sim A' + B'$ and $A \cap B \sim A' \cap B'$.*

Proof. The following exact sequence

$$\begin{aligned}0 \to (A+A'+B+B')/(A\cap A')\cap(B\cap B') \to \\ (A+A')/(A\cap A') \oplus (B+B')/(B\cap B') \to \\ (A+A'+B+B')/(A\cap A')+(B\cap B') \to 0\end{aligned}$$

plus $A \sim A'$ and $B \sim B'$ imply that both spaces

$$\begin{aligned}(A+A'+B+B')/(A\cap A')\cap(B\cap B') \quad \text{and,} \\ (A+A'+B+B')/((A\cap A')+(B\cap B'))\end{aligned}$$

are finite dimensional. Since, $(A+A'+B+B')/(A+A')\cap(B+B')$ is a quotient of the second space and $((A\cap A')+(B\cap B'))/((A\cap A')\cap(B\cap B'))$ is a subspace of the first space we can conclude $A+B \sim A'+B'$ and $A\cap B \sim A'\cap B'$. □

If we consider the set of equivalence classes $\mathscr{L}(V)$ of $\sim$ on a vector space V then $\prec$ is a partial order on it and by Lemma 1.7 above $\mathscr{L}(V)$ inherits operations $\cap$ and $+$.

Linear compactness

Definition 1.8. Let V be a linearly topologized vector space. A closed subset $C \subseteq V$ is **linearly compact** (respectively **linearly cocompact**) if for every open vector subspace U we have $\dim C/(C\cap U) < \infty$ (respectively $\dim V/(C+U) < \infty$).

Linear compactness behaves just as compactness if one uses the correct words.

Proposition 1.9. *Let V be a linearly compact vector space, then*

(a) If $A \subseteq V$ is a vector subspace such that for every open U vector subspace of V we have $\dim W/(W\cap U) < \infty$ then $\overline{A}$ is linearly compact.

(b) If $\varphi\colon V \to W$ is a continuous linear homomorphism then $\overline{\varphi(V)}$ is linearly compact.

(c) If E is discrete then E must be finite dimensional.

(d) *Every closed vector subspace of E is linearly compact.*

(e) *(Tychonov) If $\{E_\alpha\}_\alpha$ is a collection of linearly compact vector spaces then its product $\prod_\alpha E_\alpha$ and its direct sum $\bigoplus_\alpha E_\alpha$ are linearly compact.*

Proof. Let $U \subseteq F$ be an open vector subspace, then since φ is linear a continuous $\varphi^{-1}(U)$ is an open vector subspace of E. Consider the surjective induced map

$$E/\varphi^{-1}(U) \twoheadrightarrow \varphi(E)/\varphi(E) \cap U$$

as E is linearly compact it follows that $\dim \varphi(E)/\varphi(E) \cap U < \infty$. We get (a). If E is discrete, then $\{0\}$ is an open vector subspace of E, (b) follows. For (c), let $G \subseteq E$ be a closed vector subspace and take any open vector subspace U of E, then the inclusion $G \hookrightarrow E$ induces

$$G/G \cap U \hookrightarrow E/U$$

where the latter is finite dimensional (E is linearly compact). Finally, for (d), it is enough proving for open vector subspaces $U = \prod_\beta U_\beta \times \prod_\gamma E_\gamma$ where β ranges over a finite set, γ ranges over $\alpha \neq \beta$ and U_β is an open vector subspace of E_β. Then, the quotient

$$\prod_\alpha E_\alpha/U \cong \prod_\beta E_\beta/U_\beta$$

where $\cong$ is a topological and algebraic isomorphism. Since E_α is linearly compact for all α and β ranges over a finite set we conclude that $\prod_\alpha E_\alpha/U$ is finite dimensional; therefore, $\prod_\alpha E_\alpha$ is linearly compact. The proof is analogous for $\bigoplus_\alpha E_\alpha$. □

Completeness

If E is linearly topologized it admits a fundamental system of neighborhoods consisting of open vector subspaces

$$E \supseteq U_0 \supseteq U_1 \supseteq \ldots \supseteq U_\alpha \supseteq \ldots$$

Definition 1.10. In the previous context, we say that E is **complete** if

$$E \cong \hat{E} := \varprojlim_\alpha E/U_\alpha$$

where $\cong$ is an isomorphism of topological vector spaces.

1.2 TATE SPACES

Definition 1.11. Let E be a linearly topologized vector space. An open linearly compact subspace of E is called a **c-lattice** if it is open; dually, a **d-lattice** is a discrete linearly cocompact subspace of E. We say that E is a **Tate space** or **Tate vector space** if it contains a c-lattice.

Proposition 1.12. *A linearly topologized vector space E has a c-lattice if and only if it has a d-lattice.*

Proof. Suppose C is a c-lattice in E, choose any direct complement D of C, that is, $E = C \oplus D$. Since C is open, then D is discrete as $D \cap C = 0$, thus 0 is open in D. Moreover, D is closed as it is the fiber of 0 under the projection $E \to C$. Finally, we check that D is linearly cocompact: let U be any open vector subspace of E, the composition $C \hookrightarrow E \twoheadrightarrow E/(D+U)$ induces a surjection

$$C/(C \cap U) \twoheadrightarrow E/(D+U)$$

thus, since $\dim C/(C \cap U) < \infty$ we conclude $\dim E/(D+U) < \infty$.
Now, suppose D is a d-lattice, again, choose C a direct complement for D. Analogous as the proof for D being discrete and closed in the previous paragraph it follows the one for C being open and closed. We just check that C is linearly compact. Let U be any open vector subspace, the composition $E \twoheadrightarrow C \twoheadrightarrow C/(C \cap U)$ induces a surjection

$$E/(D + (C \cap U)) \twoheadrightarrow C/(C \cap U)$$

since both C and U are open, also $C \cap U$, thus $\dim E/(D + (C \cap U)) < \infty$. It follows, $\dim C/(C \cap U) < \infty$ and C linearly compact. □

Remark 1.13. Note that in the proof of Proposition 1.12 it is not strictly necessary to choose a direct complement, one can choose a direct complement up to finite dimension; that is, $C + D = E$ and $\dim C + D < \infty$. We used a direct complement to facilitate the proof.

Duality

If E is a Tate space we consider the following topology on the dual space E^* (where by dual space we mean topological dual). Open vector subspaces are given by

$$N(C) = \{\phi \in E^* : \phi|_C = 0\}$$

where C is linearly compact subspace. Equivalently, one can define open vector subspaces in E^* to be D^* where D a direct complement of a linearly compact vector subspace C in E (in this case $D^* \hookrightarrow E^*$ using the decomposition $C \oplus D$).
First, we prove that the word *dually* in Definition 1.8 actually makes sense.

Proposition 1.14. *Duality interchanges discrete and linearly compact spaces.*

Proof.

Theorem 1.15. *For a Tate space A the canonical map $A \to A^{**}$ is an isomorphism.*

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Morphisms

A **morphism** of Tate spaces is a continuous linear homomorphism between Tate spaces.

Definition 1.16. A morphism $f: A \to B$ of Tate spaces is said to be **linearly compact** if the closure of fA is linearly compact in B. Dually, it is **discrete** if $\ker f$ is open in A.

Proposition 1.17. *A morphism $f: A \to B$ of Tate spaces is linearly compact if and only if f^* is discrete.*

Proof. If its linearly compact then □

2

TRACE AND RESIDUE

2.1 FINITEPOTENT MAPS AND THEIR TRACE

Let k be a fixed ground field and V a vector space over k. In this section we will expand the notion of trace of a linear endomorphism to include certain operators even when V is infinite dimensional.

Finitepotent maps

Definition 2.1. We will say a linear map $f\colon V \to V$ is **finitepotent** if

$$\dim f^n V < \infty$$

for sufficiently large n.

We characterize finitepotent maps as follows.

Lemma 2.2. *A linear map $f\colon V \to V$ is finitepotent if and only if there exists a subspace $W \subseteq V$ such that*

(i) $\dim fW < \infty$,

(ii) $fW \subseteq W$,

(iii) the induced map $\bar{f}\colon V/W \to V/W$ is nilpotent.

Proof. If f is finitepotent choose $W = f^n V$ for sufficiently large n. The first condition follows from definition. Also, $fW = f^{n+1}V \subseteq f^n V = W$. In addition, $\bar{f}^n = 0$. On the other hand, if such W exists, note that condition (ii) assures that $\bar{f}$ is well defined. Moreover, as $\bar{f}$ is nilpotent, $f^n V \subseteq W$ for sufficiently large n and by condition (i) above $\dim f^n V < \infty$. □

Trace

If f is a finitepotent map and W is as above, $\operatorname{tr}_V(f) \in k$ may be defined as $\operatorname{tr}_W(f)$ where $\operatorname{tr}_W(f)$ is the ordinary trace of f viewed as a endomorphism of W. First, we will check that this definition does not depend on the choice of W. Suppose $W_1, W_2 \subseteq V$ suffice the properties on Lemma 2.2, then $W = W_1 + W_2$ suffices them too. Hence, as the induced maps on W/W_1 and W/W_2 are nilpotent, they have have zero ordinary trace and since

$$\operatorname{tr}_W(f) = \operatorname{tr}_{W_1}(f) + \operatorname{tr}_{W/W_1}(f)$$
$$\operatorname{tr}_W(f) = \operatorname{tr}_{W_2}(f) + \operatorname{tr}_{W/W_2}(f),$$

we obtain $\operatorname{tr}_{W_1}(f) = \operatorname{tr}_{W_2}(f)$, our desired result.

This definition extends some of the properties of the ordinary trace.

Lemma 2.3. *(a) If $\dim V < \infty$, any endomorphism f is finitepotent and $\operatorname{tr}_V(f)$ coincides with the ordinary trace.*

(b) If f is nilpotent, then it is finitepotent and $\operatorname{tr}_V(f) = 0$.

(c) *If f is finitepotent and U is a subspace such that $fU \subseteq U$ then the induced maps on U and V/U are finitepotent and satisfy*

$$\mathrm{tr}_V(f) = \mathrm{tr}_U(f) + \mathrm{tr}_{V/U}(f)$$

Proof. Both (a) and (b) are immediate. For (c) if W suffices the properties in Lemma 2.2 for f then $W \cap U$ and $(W+U)/U$ suffice them for the induced maps, that is, they're finitepotent. Since $W/(W\cap U) \cong W + U/U$, the diagram

$$\begin{tikzcd} W/(W\cap U) \arrow[r, "\cong"] \arrow[d, "f"] & (W+U)/U \arrow[d, "f"] \\ W/(W\cap U) \arrow[r, "\cong"] & (W+U)/U \end{tikzcd}$$

commutes and trace is invariant under conjugation, we get $\mathrm{tr}_{W/(W\cap U)}(f) = \mathrm{tr}_{(W+U)/U}(f)$. Hence

$$\mathrm{tr}_V(f) = \mathrm{tr}_W(f) = \mathrm{tr}_{W\cap U}(f) + \mathrm{tr}_{(W+U)/U}(f) = \mathrm{tr}_U(f) + \mathrm{tr}_{V/U}(f) \qquad \square$$

Definition 2.4. A subspace F of $\mathrm{End}_k(V)$ is said to be a **finitepotent subspace** if there exists an n such that for any family of n elements $f_1, \dots, f_n \in F$, the space $f_1 f_2 \cdots f_n V$ is finite dimensional.

The following is the natural linearity property for tr.

Proposition 2.5. *If F is a finitepotent subspace then $\mathrm{tr}_V \colon F \to k$ is k-linear*

Proof. It is enough to prove it in the case that F is finite dimensional. Let $W = F^n V$ for n as in the definition of finitepotent subspace, thus $\dim W < \infty$. Hence, for all $f \in F$, W suffices the conditions in Lemma 2.2. It follows that $\mathrm{tr}_V(f) = \mathrm{tr}_W(f)$ which is linear. $\square$

add note in "general" linearity of trace when .bib is ready

Proposition 2.6. *If $f, g \in \mathrm{End}_k(V)$ and fg is finitepotent then gf is also finitepotent and*

$$\mathrm{tr}_V(fg) = \mathrm{tr}_V(gf).$$

Proof. Since fg is finitepotent let $W = (fg)^n V$ for sufficiently large n has finite dimension. On the other hand, $(gf)^{n+1}V = g(fg)^n fV \subseteq gW$, therefore, gf is also finitepotent. Let $W' = (gf)^n V$, then $gW' \subseteq W$ and $fW \subseteq W'$. Thus,

$$\dim W' \leq \dim gW \leq \dim W \quad \text{and,} \quad \dim W \leq \dim fW \leq \dim W',$$

which implies that $W \cong W'$ and that g and f induce mutually inverse isomorphism between W and W'. Moreover, the diagram

$$\begin{tikzcd} W \arrow[r, "fg"] \arrow[d, "g"] & W \arrow[d, "g"] \\ W' \arrow[r, "gf"] & W' \end{tikzcd}$$

commutes. We conclude $\mathrm{tr}_W(fg) = \mathrm{tr}_{W'}(gf)$ and it follows $\mathrm{tr}_V(fg) = \mathrm{tr}_V(gf)$. $\square$