1

TATE'S LINEAR ALGEBRA

1.1 LINEAR TOPOLOGIES

Fix a ground field k. From now on, a vector space will always mean a k-vector space.

Definition 1.1. A **linear topology** on a vector space V is a separated (Hausdorff) topology invariant under translations that admits an open local base around zero of vector subspaces. A vector space equipped with a linear topology will be referred as **linearly topologized**.

If we endow k with the discrete topology then V will become a topological vector space. From now on, endow k with the discrete topology.
Linear topologies behave nicely under basic topological operations.

Proposition 1.2. *Let V be a linearly topologized vector space. Then*

(a) Any vector subspace of V is linearly topologized under its subspace topology.

(b) If $W \subseteq V$ is a closed vector subspace then V/W is linearly topologized under its quotient topology.

(c) If $\{V_\alpha\}_\alpha$ is a collection of linearly topologized vector spaces its product $\prod_\alpha V_\alpha$ and its direct sum $\bigoplus_\alpha V_\alpha$ is linearly topologized under its product topology.

(d) If W is a vector subspace of V, then its topological closure $\overline{W}$ also is a vector subspace of V.

Proof. Since intersection of vector subspaces is a vector subspace, (a) follows intersecting the fundamental system of neighborhoods in V by the vector subspace. For (b), let $\pi\colon V \to V/W$ be the quotient map. Since π is open and surjective the image of a local base is a local base; moreover, the image of a vector subspace under π is a vector subspace. In addition, since Finally, for (c) let $\{U_{\alpha,\beta}\}_\beta$ be a local base of zero in V_α of vector subspaces, the products $U_{\alpha_1,\beta_1} \times \ldots \times U_{\alpha_n,\beta_n} \times \prod_\gamma V_\gamma$, where γ ranges over $\alpha \neq \alpha_1, \ldots, \alpha_n$, for any set $\{(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)\}$ form a fundamental system of neighborhoods around zero in $\prod_\alpha V_\alpha$ of open vector subspaces. Note that since $\bigoplus_\alpha V_\alpha \subseteq \prod_\alpha V_\alpha$ is a vector subspace (c) follows from (a). Finally, for (d), suppose $x, y \in \overline{W}$, then, for every open vector subspace U, $(x+U) \cap W \neq \emptyset$ and $(y+U) \cap W \neq \emptyset$, therefore for every $\alpha, \beta \in k$ we have $(\alpha x + U) \cap W \neq \emptyset$ and $(\beta y + U) \cap W \neq \emptyset$. Hence, $(\alpha x + \beta y + U) \cap W \neq \emptyset$ for every open vector subspace U and every pair $\alpha, \beta \in k$. It follows (d). □

Remark 1.3. Using an argument similar to the previous proposition one can check that in the category LinTop_k of linearly topologized vector spaces limits and colimits indexed by small categories exist.

Finite dimensional vector spaces are meaningless for linear topologies.

Proposition 1.4. *A finite dimensional linearly topologized vector space V is discrete.*

Proof. Let U be an open vector subspace and $0 \neq x \in U$, since V is separated and linearly topologized there exists an open vector subspace U_x such that $x \notin U_x$ then $\dim U_x \cap U < \dim U$, since V is finite dimensional this process can be repeated only a finite amount of times; that is $\{0\}$ is open. It follows that V is discrete. □

Commensurability

We introduce a partial order in the set of vector subspaces of a vector space V.

Definition 1.5. For vector subspaces A and B of a vector space V we say that $A \prec B$ if the quotient $A/(A \cap B) \cong (A+B)/B$ is finite dimensional (or equivalently $A \subseteq B + W$ for some finite dimensional W). In addition, we say that A and B are **commensurable** (denoted $A \sim B$) if $A \prec B$ and $B \prec A$.

Observe that $A \sim B$ if and only if $(A+B)/(A\cap B) \cong A/(A \cap B) \oplus B/(A \cap B)$ is finite dimensional. We will constantly refer to a vector space V being finite dimensional as $V \sim 0$.

Proposition 1.6. *Let V be a vector spaces and A, B and L be vector subspaces, then:*

(a) If $A \sim B$ and $B \sim L$ then

$$(A+B+L)/(A \cap B \cap L) \sim 0$$

(b) If $A \prec B$ and $B \prec L$ then $A \prec L$. Moreover, commensurability is an equivalence relation.

Proof. Consider the following exact sequences

$$0 \to (A \cap B)/(A \cap B \cap L) \to B/(B \cap L),$$

and,

$$0 \to (A \cap B)/(A \cap B \cap L) \to (A+B)/(A \cap B \cap L) \to (A+B)/(A \cap B) \to 0$$

induced by inclusions. The first inclusion plus the fact that $B \sim L$ imply that $(A \cap B)/(A \cap B \cap L)$ is finite dimensional. Now, since $A \sim B$ it follows that $(A+B)/(A \cap B)$ is finite dimensional. Hence, the second exact sequence concludes that $(A+B)/(A \cap B \cap L)$. A symmetrical argument shows that $(B+L)/(A \cap B \cap L) \sim 0$. These prove (a). For (b), the inclusion

$$0 \to (A+L)/(A \cap L) \to (A+B+L)/(A \cap B \cap L)$$

plus (a) implies transitivity. □

Now, we state and prove some useful properties on the relation $\prec$.

Lemma 1.7. *(a) If $A \subseteq B$ then $A \prec B$.*

(b) If $A \prec B$ then $f(A) \prec f(B)$ for any k-linear map f

(c) It holds that

$$\sum_{i=1}^{m} A_i \prec \bigcap_{j=1}^{n} B_j \iff A_i \prec B_j \text{ for all } i \text{ and } j.$$

Proof. First, (a) is immediate from the definition of $\prec$. Second, for (b) the map f factors as

$$A/(A \cap B) \to f(A)/(f(A) \cap f(B)) \to 0$$

Finally, for (c), if $\sum_{i=1}^{m} A_i \prec \bigcap_{j=1}^{n} B_j$ holds then by (a) above, for all i and j we have

$$A_i \prec \sum_{i=1}^{m} A_i \prec \bigcap_{j=1}^{n} B_j \prec B_j$$

On the other hand, if $A_i \prec B_j$ for all i and j then there exists finite dimensional subspaces W_{ij} such that $A_i \subseteq B_j + W_{ij}$ for all i and j. Therefore,

$$\sum_{i=1}^{m} A_i \subseteq \bigcap_{j=1}^{n} B_j + \sum_{i=1}^{m} \sum_{j=1}^{n} W_{ij}.$$

□

Next, we consider another useful lemma.

Lemma 1.8. *Let A, B, A', B' be vector subspaces of a vector space V and suppose that $A \sim A'$ and $B \sim B'$. Then $A + B \sim A' + B'$ and $A \cap B \sim A' \cap B'$.*

Proof. The following exact sequence

$$\begin{aligned} 0 \to (A + A' + B + B')/(A \cap A') \cap (B \cap B') \to \\ (A + A')/(A \cap A') \oplus (B + B')/(B \cap B') \to \\ (A + A' + B + B')/(A \cap A') + (B \cap B') \to 0 \end{aligned}$$

plus $A \sim A'$ and $B \sim B'$ imply that both spaces

$$(A + A' + B + B')/(A \cap A') \cap (B \cap B') \quad \text{and,}$$
$$(A + A' + B + B')/((A \cap A') + (B \cap B'))$$

are finite dimensional. Since, $(A + A' + B + B')/(A + A') \cap (B + B')$ is a quotient of the second space and $((A \cap A') + (B \cap B'))/((A \cap A') \cap (B \cap B'))$ is a subspace of the first space we can conclude $A + B \sim A' + B'$ and $A \cap B \sim A' \cap B'$. □

If we consider the set of equivalence classes $\mathscr{L}(V)$ of $\sim$ on a vector space V then $\prec$ is a partial order on it and by Lemma 1.8 above $\mathscr{L}(V)$ inherits operations $\cap$ and $+$.

Linear compactness

Definition 1.9. Let V be a linearly topologized vector space. A closed subset $L \subseteq V$ is **linearly compact** (respectively **linearly cocompact**) if for every open vector subspace U we have $L \prec U$ (respectively $V/(L + U) \sim 0$).

Linear compactness behaves just as compactness if one uses the correct words.

Proposition 1.10. *Let V be a linearly compact vector space, then*

(a) If $A \subseteq V$ is a vector subspace such that for every open vector subspace U of V it holds $A \prec U$ then $\overline{A}$ is linearly compact.

(b) If $f \colon V \to W$ is a continuous linear homomorphism then $\overline{f(V)}$ is linearly compact.

(c) *If V is discrete then $V \sim 0$.*

(d) *Every closed vector subspace of V is linearly compact.*

(e) *(Tychonov) If $\{V_\alpha\}_\alpha$ is a collection of linearly compact vector spaces then its product $\prod_\alpha V_\alpha$ and its direct sum $\bigoplus_\alpha V_\alpha$ are linearly compact.*

Proof. Let U be any open vector subspace of V, then $A + U$ is closed, that is $A + U = \overline{A + U} \supseteq \overline{A} + U \supseteq A + U$, thus, $\overline{A} + U = A + U$. Since, $(A + U)/U \sim 0$ then $(\overline{A} + U)/U \sim 0$. We get (a).

For (b), since f is a continuous linear map $V \prec f^{-1}(U)$ for all U open vector subspace of W, hence by Lemma 1.7 $f(V) \prec U$ for all open vector subspaces U of W. By the previous observation and (a) we get (b). If V is discrete, then $\{0\}$ is an open vector subspace of E, thus $V \prec U$, we get (c).

For (d), if $A \subseteq V$ is a closed vector subspace, and $V \prec U$ for all open vector subspaces U by Lemma 1.7 we get $A \prec U$.

Finally, for (d), it is enough proving for open vector subspaces $U = \prod_\beta U_\beta \times \prod_\gamma V_\gamma$ where β ranges over a finite set, γ ranges over $\alpha \neq \beta$ and U_β is an open vector subspace of V_β. Then, the quotient

$$\prod_\alpha V_\alpha / U \cong \prod_\beta V_\beta / U_\beta$$

where $\cong$ is a topological and algebraic isomorphism. Since V_α is linearly compact for all α and β ranges over a finite set we conclude that $\prod_\alpha V_\alpha / U$ is finite dimensional; therefore, $\prod_\alpha V_\alpha$ is linearly compact. The proof is analogous for $\bigoplus_\alpha V_\alpha$. $\square$

Completion

Definition 1.11. If V be a linearly topologized vector space, recall that V is said to be **complete** if

$$V \cong \varprojlim_{U \in \mathrm{Op}(V)} V/U$$

where $\mathrm{Op}(V)$ runs through all open vector subspaces of V. In particular, this implies that for every base $\mathscr{U}$ of zero made from open vector subspaces of V we have

$$V \cong \varprojlim_{U \in \mathscr{U}} V/U$$

1.2 TATE SPACES

Lattices

Definition 1.12. If V is a linearly topologized vector space we say that a **c-lattice** is an open linearly compact subspace of V, *dually* a discrete linearly cocompact subspace is a **d-lattice**.

First, we prove that existence of a c-lattice in a linearly topologized vector space is equivalent to existence of a d-lattice.

Proposition 1.13. *A linearly topologized vector space V has a c-lattice if and only if it has a d-lattice.*

Proof. Suppose L is a c-lattice in V, choose any direct complement D of L, that is, $V = L \oplus D$. Since L is open, then D is discrete as $D \cap L = 0$, thus 0 is open in D. Moreover, D is closed as it is the fiber of 0 under the projection $V \to L$. Finally, we check that D is linearly cocompact: let U be any open vector subspace of V, the composition $L \hookrightarrow V \twoheadrightarrow V/(D+U)$ induces a surjection

$$L/(L \cap U) \twoheadrightarrow V/(D+U)$$

thus, since $\dim L/(L \cap U) < \infty$ we conclude $\dim V/(D+U) < \infty$.
Now, suppose D is a d-lattice, again, choose L a direct complement for D. Analogous as the proof for D being discrete and closed in the previous paragraph it follows the one for L being open. We just check that L is linearly compact. Let U be any open vector subspace, the composition $V \twoheadrightarrow L \twoheadrightarrow L/(L \cap U)$ induces a surjection

$$V/(D + (L \cap U)) \twoheadrightarrow L/(L \cap U)$$

since both L and U are open, also $L \cap U$, thus $\dim V/(D + (L \cap U)) < \infty$. It follows, $\dim L/(L \cap U) < \infty$ and L linearly compact. □

Remark 1.14. Note that in the proof of Proposition 1.13 it is not strictly necessary to choose a direct complement, one can choose a direct complement up to finite dimension; that is, $L + D \sim V$ and $L \cap D \sim 0$.

We now give a characterization of lattices in terms of $\prec$.

Proposition 1.15. *A linearly compact subspace is a c-lattice if and only if it is maximal among the set of linearly compact sets ordered by $\prec$.*

Proof. this one needs some thinking □

Proposition 1.16. *If V admits a c-lattice, then the set of c-lattices constitutes a base of zero of mutually commensurable vector subspaces.*

Proof. By the previous proposition all c-lattices must be commensurable. Moreover, if U is any vector subspace and L is a c-lattice we claim that $L \cap U$ is a c-lattice. Indeed, let U' be any open vector subspace, then $L \cap U \prec L \prec U'$. In addition, since L and U are open, $L \cap U$ is open. Thus $L \cap U \subseteq U$ is a c-lattice, this proves the statement. □

We're now ready to introduce the definition of a Tate space.

Definition 1.17. A **Tate space** V is a complete linearly topologized vector space that admits a c-lattice. By the previous proposition and the observation in Definition 1.11 we get

$$V \cong \varprojlim_{L \in \mathscr{L}(V)} V/L$$

where $\mathscr{L}(V)$ runs through all c-lattices of V.

Example 1.18. We give some examples of Tate spaces.

(a) Any vector space endowed with the discrete topology is a Tate space.

(b) If $\{V_\alpha\}_\alpha$ is any pro-system of finite dimensional vector spaces (thus, each one endowed with the discrete topology by Proposition 1.4), let V be their inverse limit endowed with the inverse limit topology. We claim that this is a linearly compact space. Indeed, if we realize V

as a subspace of the product $\prod_\alpha V_\alpha$, then basic open vector subspaces are just restriction of finite coordinates. Hence, the quotient of V by any basic open vector subspace is a finite product of V_α, since all V_α are finite dimensional we conclude that V is linearly compact and therefore a Tate space.

(c) Let $V = k((t))$ with the topology generated by letting $t^n k[[t]]$ for $n \in \mathbb{Z}$ be a system of neighborhoods of zero. Then, $V = k[[t]] \oplus tk[t^{-1}]$ where $k[[t]]$ is the completion of $k[x]$ in the $\langle x \rangle$-adic topology, hence by the previous item linearly compact and, since it is open a c-lattice. By Proposition 1.13 $tk[t^{-1}]$ is a d-lattice. Therefore, V is a Tate space that is not linearly compact nor discrete.

Duality

If V is a Tate space we consider the following topology on the dual space V^* (where by dual space we mean topological dual). Open vector subspaces are given by

$$L^\perp = \{\phi \in E^* : \phi|_L = 0\}$$

where L is a linearly compact subspace. Equivalently, one can define open vector subspaces in E^* to be D^* where D a direct complement of a linearly compact vector subspace L in E (in this case $D^* \hookrightarrow E^*$ using the decomposition $L \oplus D$).

First, we prove that the word *dually* in Definition 1.9 actually makes sense.

Lemma 1.19. *Duality interchanges discrete and linearly compact spaces.*

Proof. If L is a linearly compact vector space, then $L^\perp$ is open in L^*, thus L^* is discrete. If D is discrete, then $D \cong k^{\oplus \Lambda}$ for some Λ and endowing $k^{\oplus \Lambda}$ with the discrete topology the previous isomorphism is a homeomorphism too. Moreover, since D is discrete every linear functional is continuous. Using Remark 1.3 and the well known identity (where maps are isomorphisms in LinTop_k)

$$(k^{\oplus \Lambda})^* = \mathrm{Hom}_k(k^{\oplus \Lambda}, k) \cong \prod_\Lambda \mathrm{Hom}_k(k,k) \cong \prod_\Lambda k$$

we get the desired result by Tychonov's theorem in Proposition 1.10. $\square$

Remark 1.20. Note that in the proof of the previous lemma the dual space of a discrete space is complete, as it is an arbitrary product of k.

Proposition 1.21. *If V is a Tate space then V^* with the topology previously introduced is also a Tate space.*

Proof. If we decompose $V = L \oplus D$ where L is a c-lattice and D a d-lattice then $V^* \cong L^* \oplus D^*$ and by Lemma 1.19 L^* is discrete and D^* is linearly compact. Observe that D^* is open in V^* since it is the kernel of the projection $V^* \to V^*/L^\perp$ and $V^*/L^\perp$ is discrete by the description of our topology in the dual V^*. Since L^* is discrete, then it is complete. Moreover, by the previous remark, D^* is complete, hence V^* is complete too. $\square$

We're ready to prove the analog of Pontryagin's duality for locally compact groups in our context.

Theorem 1.22. *For a Tate space V the canonical map $V \to V^{**}$ is an isomorphism.*

Proof. It is enough to prove it for complete linearly compact spaces and discrete spaces, as every Tate space can be decomposed into a direct sum of a c-lattice and a d-lattice. First, we do it for discrete spaces. Suppose D is a discrete vector space. Then, the canonical map

$$\mathrm{ev}\colon D \to D^{**}$$

is open and continuous because D and D^{**} are both discrete by Lemma 1.19. Moreover, it is injective, because for every nonzero $v \in D$ there exists a linear continuous functional $\phi \in D^*$ such that $\phi(v) \neq 0$. Finally, we prove surjectivity. Let $\psi \in D^{**}$. Since $\ker\psi$ is open it contains a basic open vector subspace $A^\perp$ such that $A \subseteq D$ is a linearly compact subspace. Therefore, since D^* is linearly compact then $D^* \sim A^\perp$, that is, the quotient $D^*/A^\perp$ is finite dimensional. Recall that the inclusion $\iota\colon A \to D$ induces an isomorphism $D^*/A^\perp \to A^*$ which is a homeomorphism since both spaces are discrete. We can factor ψ such as the following diagram is commutative

$$\begin{tikzcd}
D^* \arrow[r, "\psi"] \arrow[d] & k \\
D^*/A^\perp \arrow[ur, dashed, "\tilde{\psi}"] \arrow[d, "\cong"'] & \\
A^* \arrow[uur, dashed, bend right, "\overline{\psi}"'] &
\end{tikzcd}$$

However, A^* is finite dimensional, therefore, there exists some $a \in A$ such that $\overline{\psi} = \mathrm{ev}_a$ as maps from $A^* \to k$. Moreover, since $A^\perp \subseteq \ker\psi$ we conclude that $\psi = \mathrm{ev}_a$ as maps $D^* \to k$. This implies surjectivity. Thus $D \to D^{**}$ is an isomorphism of topological vector spaces.

Now, suppose L is a complete linearly compact space. We check first that the map

$$\mathrm{ev}\colon L \to L^{**}$$

is continuous. Let $A^\perp$ be an open vector subspace in L^{**} where $A \subseteq L^*$ is a linearly compact subspace. By Lemma 1.19 L^* is discrete, hence A is finite dimensional. Suppose $A = \mathrm{span}(\phi_1, \dots, \phi_n)$ for some $\phi_1, \dots, \phi_n \in A$. Then, $\mathrm{ev}^{-1}(A^\perp) = \ker\phi_1 \cap \dots \cap \ker\phi_n$ which is open in L. Now, we check that ev is injective. Let $v \in L$ be a nonzero vector. Choose a decomposition of $L = U \oplus F$ where U is open and F is the span of v (this can be done because L is linearly compact). Define ϕ a linear functional to be zero in U and $\phi(v) \neq 0$. Since U is open and F discrete then ϕ is continuous. This implies injectivity of ev. Now we check that ev is surjective. Since L is complete

$$L \cong \varprojlim_{U \in \mathscr{U}} L/U$$

where $\mathscr{U}$ runs over open vector subspaces of U. Let $\psi\colon L^* \to k$ be a continuous linear functional. By pulling back $\pi_U\colon L \to L/U$ we get an injection $\pi_U^*\colon (L/U)^* \hookrightarrow L^*$ for every $U \in \mathscr{U}$. Since L is linearly compact, then L/U is finite dimensional, thus, there exists some $v_U \in L$ such that $\psi \circ \pi_U^* = \mathrm{ev}_{v_U}$ where $\mathrm{ev}\colon L/U \to (L/U)^{**}$. In particular, observe that if $U, U' \in \mathscr{U}$ and

$U' \subseteq U$ we get an induced injection $(L/U)^* \hookrightarrow (L/U')^*$ such that the following diagram

$$\begin{tikzcd}
L^* \arrow[r, "\psi"] & k \\
(L/U')^* \arrow[u, hook] \arrow[ur, "\mathrm{ev}_{v_{U'}}"'] & \\
(L/U)^* \arrow[u, hook] \arrow[uur, bend right, "\mathrm{ev}_{v_U}"'] \arrow[uu, bend left] &
\end{tikzcd}$$

commutes. Note that this implies that $(v_U)_{U \in \mathscr{U}}$ is a Cauchy net and by completeness of V it follows that it is convergent. Therefore, there exists some $v \in L$ limit of $(v_U)_{U \in \mathscr{U}}$. We claim that $\psi = \mathrm{ev}_v$. Let $\phi \in L^*$. Then, $\ker \phi$ is open and since L is linearly compact then $\ker \phi \sim L$. Hence the quotient $L/\ker \phi$ is discrete and the factor $\overline{\phi}$ such that the following diagram

$$\begin{tikzcd}
L \arrow[r, "\phi"] \arrow[d, "\pi_{\ker \phi}"'] & k \\
L/\ker \phi \arrow[ur, "\overline{\phi}"'] &
\end{tikzcd}$$

commutes is continuous. In other words, the image of $\overline{\phi}$ under the inclusion $(L/\ker \phi)^* \hookrightarrow L^*$ is ϕ. Thus, $\psi(\phi) = \mathrm{ev}_{v_{\ker \phi}}(\overline{\phi})$ and by convergence $\psi(\phi) = \mathrm{ev}_v(\phi)$. This implies surjectivity of $\mathrm{ev}\colon L \to L^{**}$. To conclude, we prove that ev is open. □

Morphisms

A **morphism** of Tate spaces is a continuous linear homomorphism between Tate spaces.

Definition 1.23. A morphism $f\colon A \to B$ of Tate spaces is said to be **linearly compact** if the closure of fA is linearly compact in B. Dually, it is **discrete** if $\ker f$ is open in A.

Proposition 1.24. *A morphism $f\colon A \to B$ of Tate spaces is linearly compact if and only if f^* is discrete.*

Proof. If its linearly compact then A □

2

TRACE AND RESIDUE

2.1 FINITEPOTENT MAPS AND THEIR TRACE

Let k be a fixed ground field and V a vector space over k. In this section we will expand the notion of trace of a linear endomorphism to include certain operators even when V is infinite dimensional.

Finitepotent maps

Definition 2.1. We will say a linear map $f\colon V \to V$ is **finitepotent** if

$$\dim f^n(V) < \infty$$

for sufficiently large n.

We characterize finitepotent maps as follows.

Lemma 2.2. *A linear map $f\colon V \to V$ is finitepotent if and only if there exists a subspace $W \subseteq V$ such that*

(i) $\dim f(W) < \infty$,

(ii) $f(W) \subseteq W$,

(iii) the induced map $\bar{f}\colon V/W \to V/W$ is nilpotent.

Proof. If f is finitepotent choose $W = f^n(V)$ for sufficiently large n. The first condition follows from definition. Also, $f(W) = f^{n+1}(V) \subseteq f^n(V) = W$. In addition, $\bar{f}^n = 0$. On the other hand, if such W exists, note that condition (ii) assures that $\bar{f}$ is well defined. Moreover, as $\bar{f}$ is nilpotent, $f^n V \subseteq W$ for sufficiently large n and by condition (i) above $\dim f^n(V) < \infty$. □

Trace

If f is a finitepotent map and W is as above, $\mathrm{tr}_V(f) \in k$ may be defined as $\mathrm{tr}_W(f)$ where $\mathrm{tr}_W(f)$ is the ordinary trace of f viewed as a endomorphism of W. First, we will check that this definition does not depend on the choice of W. Suppose $W_1, W_2 \subseteq V$ suffice the properties on Lemma 2.2, then $W = W_1 + W_2$ suffices them too. Hence, as the induced maps on W/W_1 and W/W_2 are nilpotent, they have have zero ordinary trace and since

$$\begin{aligned}\mathrm{tr}_W(f) &= \mathrm{tr}_{W_1}(f) + \mathrm{tr}_{W/W_1}(f)\\ \mathrm{tr}_W(f) &= \mathrm{tr}_{W_2}(f) + \mathrm{tr}_{W/W_2}(f),\end{aligned}$$

we obtain $\mathrm{tr}_{W_1}(f) = \mathrm{tr}_{W_2}(f)$, our desired result.

This definition extends some of the properties of the ordinary trace.

Lemma 2.3. *(a) If $\dim V < \infty$, any endomorphism f is finitepotent and $\mathrm{tr}_V(f)$ coincides with the ordinary trace.*

(b) If f is nilpotent, then it is finitepotent and $\mathrm{tr}_V(f) = 0$.

(c) *If f is finitepotent and U is a subspace such that $fU \subseteq U$ then the induced maps on U and V/U are finitepotent and satisfy*

$$\mathrm{tr}_V(f) = \mathrm{tr}_U(f) + \mathrm{tr}_{V/U}(f)$$

Proof. Both (a) and (b) are immediate. For (c) if W suffices the properties in Lemma 2.2 for f then $W \cap U$ and $(W+U)/U$ suffice them for the induced maps, that is, they're finitepotent. Since $W/(W\cap U) \cong W + U/U$, the diagram

$$\begin{tikzcd} W/(W\cap U) \arrow[r, "\cong"] \arrow[d, "f"] & (W+U)/U \arrow[d, "f"] \\ W/(W\cap U) \arrow[r, "\cong"] & (W+U)/U \end{tikzcd}$$

commutes and trace is invariant under conjugation, we get $\mathrm{tr}_{W/(W\cap U)}(f) = \mathrm{tr}_{(W+U)/U}(f)$. Hence

$$\mathrm{tr}_V(f) = \mathrm{tr}_W(f) = \mathrm{tr}_{W\cap U}(f) + \mathrm{tr}_{(W+U)/U}(f) = \mathrm{tr}_U(f) + \mathrm{tr}_{V/U}(f) \qquad \square$$

Definition 2.4. A subspace F of $\mathrm{End}_k(V)$ is said to be a **finitepotent subspace** if there exists an n such that for any family of n elements $f_1, \dots, f_n \in F$, the space $f_1 f_2 \cdots f_n V$ is finite dimensional.

The following is the natural linearity property for tr.

Proposition 2.5. *If F is a finitepotent subspace then $\mathrm{tr}_V \colon F \to k$ is k-linear*

Proof. It is enough to prove it in the case that F is finite dimensional. Let $W = F^n V$ for n as in the definition of finitepotent subspace, thus $\dim W < \infty$. Hence, for all $f \in F$, W suffices the conditions in Lemma 2.2. It follows that $\mathrm{tr}_V(f) = \mathrm{tr}_W(f)$ which is linear. $\square$

add note in "general" linearity of trace when .bib is ready

Proposition 2.6. *If $f, g \in \mathrm{End}_k(V)$ and fg is finitepotent then gf is also finitepotent and*

$$\mathrm{tr}_V(fg) = \mathrm{tr}_V(gf).$$

Proof. Since fg is finitepotent let $W = (fg)^n V$ for sufficiently large n has finite dimension. On the other hand, $(gf)^{n+1}V = g(fg)^n f(V) \subseteq g(W)$, therefore, gf is also finitepotent. Let $W' = (gf)^n V$, then $g(W') \subseteq W$ and $f(W) \subseteq W'$. Thus,

$$\dim W' \leq \dim g(W) \leq \dim W \quad \text{and,} \quad \dim W \leq \dim f(W) \leq \dim W',$$

which implies that $W \cong W'$ and that g and f induce mutually inverse isomorphism between W and W'. Moreover, the diagram

$$\begin{tikzcd} W \arrow[r, "fg"] \arrow[d, "g"] & W \arrow[d, "g"] \\ W' \arrow[r, "gf"] & W' \end{tikzcd}$$

commutes. We conclude $\mathrm{tr}_W(fg) = \mathrm{tr}_{W'}(gf)$ and it follows $\mathrm{tr}_V(fg) = \mathrm{tr}_V(gf)$. $\square$