# 1 TATES LINEAR ALGEBRA

## 1.1 LINEAR TOPOLOGIES

Fix a ground field $k$. From now on, a vector space will always mean a $k$-vector space.

**Definition 1.1.** A **linear topology** on a vector space $E$ is a separated (Hausdorff) topology invariant under translations that admits an open local base around zero of vector subspaces. A vector space equipped with a linear topology will be referred as **linearly topologized**.

If we endow $k$ with the discrete topology then $E$ will become a topological vector space. From now on, endow $k$ to have a discrete topology.
Linear topologies behave nicely under basic topological operations.

**Proposition 1.2.** *Let $E$ be a linearly topologized vector space. Then*

*(a) Any vector subspace of $E$ is linearly topologized under its subspace topology.*

*(b) If $F \subseteq E$ is a closed vector subspace then $E/F$ is linearly topologized under its quotient topology.*

*(c) If $\{E_\alpha\}_\alpha$ is a collection of linearly topologized vector spaces its product $\prod_\alpha E_\alpha$ and its direct sum $\bigoplus_\alpha E_\alpha$ is linearly topologized under its product topology.*

*Proof.* Since intersection of vector subspaces is a vector subspace, (a) follows intersecting the fundamental system of neighborhoods in $E$ by the vector subspace. For (b), let $\pi\colon E \to E/F$ be the quotient map. Since $\pi$ is open and surjective the image of a local base is a local base; moreover, the image of a vector subspace under $\pi$ is a vector subspace, then (b) follows. Finally, for (c) let $\{U_{\alpha,\beta}\}_\beta$ be a local base of zero in $E_\alpha$ of vector subspaces, the products $U_{\alpha_1,\beta_1} \times \ldots \times U_{\alpha_n,\beta_n} \times \prod_\gamma E_\gamma$, where $\gamma$ ranges over $\alpha \neq \alpha_1, \ldots, \alpha_n$, for any set $\{(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)\}$ form a fundamental system of neighborhoods around zero in $\prod_\alpha E_\alpha$ of open vector subspaces. Note that since $\bigoplus_\alpha E_\alpha \subseteq \prod_\alpha E_\alpha$ is a vector subspace (c) follows from (a). $\square$

Finite dimensional vector spaces are meaningless for linear topologies.

**Proposition 1.3.** *A finite dimensional linearly topologized vector space $E$ is discrete.*

*Proof.* Let $U$ be an open vector subspace and $0 \neq x \in U$, since $E$ is separated and linearly topologized there exists an open vector subspace $U_x$ such that $x \notin U_x$ then $\dim U_x \cap U < \dim U$, since $E$ is finite dimensional this process can be repeted only a finite amount of times; that is $\{0\}$ is open. It follows that $E$ is discrete. $\square$

### *Linear compactness*

**Definition 1.4.** Let $E$ be a linearly topologized vector space. A closed subset $C \subseteq E$ is **linearly compact** (respectively **linearly cocompact**) if for every open vector subspace $U$ we have $\dim C/(C \cap U) < \infty$ (respectively $\dim E/(C + U) < \infty$).

Linear compactness behaves just as compactness if one uses the correct words.

**Proposition 1.5.** *Let E be a linearly compact vector space and F a linearly topologized vector space. Then*

*(a) If $\varphi\colon E \to F$ is a continuous linear homomorphism then $\varphi(E)$ is linearly compact.*

*(b) If E is discrete then E must be finite dimensional.*

*(c) Every closed vector subspace of E is linearly compact.*

*(d) (Tychonov) If $\{E_\alpha\}_\alpha$ is a collection of linearly compact vector spaces then its product $\prod_\alpha E_\alpha$ and its direct sum $\bigoplus_\alpha E_\alpha$ are linearly compact.*

*Proof.* Let $U \subseteq F$ be an open vector subspace, then since $\varphi$ is linear a continuous $\varphi^{-1}(U)$ is an open vector subspace of $E$. Consider the surjective induced map

$$E/\varphi^{-1}(U) \twoheadrightarrow \varphi(E)/\varphi(E) \cap U$$

as $E$ is linearly compact it follows that $\dim \varphi(E)/\varphi(E) \cap U < \infty$. We get (a). If $E$ is discrete, then $\{0\}$ is an open vector subspace of $E$, (b) follows. For (c), let $G \subseteq E$ be a closed vector subspace and take any open vector subspace $U$ of $E$, then the inclusion $G \hookrightarrow E$ induces

$$G/G \cap U \hookrightarrow E/U$$

where the latter is finite dimensional ($E$ is linearly compact). Finally, for (d), it is enough proving for open vector subspaces $U = \prod_\beta U_\beta \times \prod_\gamma E_\gamma$ where $\beta$ ranges over a finite set, $\gamma$ ranges over $\alpha \neq \beta$ and $U_\beta$ is an open vector subspace of $E_\beta$. Then, the quotient

$$\prod_\alpha E_\alpha/U \cong \prod_\beta E_\beta/U_\beta$$

where $\cong$ is a topological and algebraic isomorphism. Since $E_\alpha$ is linearly compact for all $\alpha$ and $\beta$ ranges over a finite set we conclude that $\prod_\alpha E_\alpha/U$ is finite dimensional; therefore, $\prod_\alpha E_\alpha$ is linearly compact. The proof is analogous for $\bigoplus_\alpha E_\alpha$. □

*Completeness*

If $E$ is linearly topologized it admits a fundamental system of neighborhoods consisting of open vector subspaces

$$E \supseteq U_0 \supseteq U_1 \supseteq \ldots \supseteq U_\alpha \supseteq \ldots$$

**Definition 1.6.** In the previous context, we say that $E$ is **complete** if

$$E \cong \hat{E} := \varprojlim_\alpha E/U_\alpha$$

where $\cong$ is an isomorphism of topological vector spaces.

## 1.2 TATE SPACES

**Definition 1.7.** Let $E$ be a linearly topologized vector space. An open linearly compact subspace of $E$ is called a **c-lattice** if it is open; dually, a **d-lattice** is a discrete linearly cocompact subspace of $E$. We say that $E$ is a **Tate space** or **Tate vector space** if it contains a c-lattice.

**Proposition 1.8.** *A linearly topologized vector space $E$ has a c-lattice if and only if it has a d-lattice.*

*Proof.* Suppose $C$ is a c-lattice in $E$, choose any direct complement $D$ of $C$, that is, $E = C \oplus D$. Since $C$ is open, then $D$ is discrete as $D \cap C = 0$, thus $0$ is open in $D$. Moreover, $D$ is closed as it is the fiber of $0$ under the projection $E \to C$. Finally, we check that $D$ is linearly cocompact: let $U$ be any open vector subspace of $E$, the composition $C \hookrightarrow E \twoheadrightarrow E/(D+U)$ induces a surjection

$$C/(C \cap U) \twoheadrightarrow E/(D+U)$$

thus, since $\dim C/(C \cap U) < \infty$ we conclude $\dim E/(D+U) < \infty$.
Now, suppose $D$ is a d-lattice, again, choose $C$ a direct complement for $D$. Analogous as the proof for $D$ being discrete and closed in the previous paragraph it follows the one for $C$ being open and closed. We just check that $C$ is linearly compact. Let $U$ be any open vector subspace, the composition $E \twoheadrightarrow C \twoheadrightarrow C/(C \cap U)$ induces a surjection

$$E/(D + (C \cap U)) \twoheadrightarrow C/(C \cap U)$$

since both $C$ and $U$ are open, also $C \cap U$, thus $\dim E/(D + (C \cap U)) < \infty$. It follows, $\dim C/(C \cap U) < \infty$ and $C$ linearly compact. □

*Remark 1.9.* Note that in the proof of Proposition 1.8 it is not strictly necessary to choose a direct complement, one can choose a direct complement up to finite dimension; that is, $C + D = E$ and $\dim C + D < \infty$. We used a direct complement to facilitate the proof.

### *Duality*

If $E$ is a Tate space we consider the following topology on the dual space $E^*$ (where by dual space we mean topological dual). Open vector subspaces are given by

$$N(C) = \{\phi \in E^* \colon \phi|_C = 0\}$$

where $C$ is linearly compact subspace. Equivalently, one can define open vector subspaces in $E^*$ to be $D^*$ where $D$ a direct complement of a linearly compact vector subspace $C$ in $E$ (in this case $D^* \hookrightarrow E^*$ using the decomposition $C \oplus D$).
First, we prove that the word *dually* in Definition 1.4 actually makes sense.

**Proposition 1.10.** *Duality interchanges discrete and linearly compact spaces.*

*Proof.*