1

TATE'S LINEAR ALGEBRA

In this chapter we explore linear topologies on vector spaces in order to introduce Tate spaces and their structure. Tate spaces will be central in the definition of abstract residues in Chapter 2 and the study of algebraic curves in Chapter 3. We follow definitions in [BD04] closely but not religiously.

1.1 LINEAR TOPOLOGIES

Let k be a field. From now on, a vector space will always mean a k-vector space.

Definition 1.1. A **linear topology** on a vector space V is a separated (Hausdorff) topology which is invariant under translations and which admits a base of open neighborhoods of zero consisting of vector subspaces. A vector space equipped with a linear topology will be referred as **linearly topologized**.

If we endow k with the discrete topology then V will become a topological vector space. From now on, endow k with the discrete topology. Linear topologies behave nicely under basic topological operations.

Theorem 1.2. *Let V be a linearly topologized vector space. Then*

(a) If $W \subseteq V$ is a vector subspace then W is linearly topologized as well.

(b) If $W \subseteq V$ is a closed vector subspace then V/W is linearly topologized under its quotient topology.

(c) If $\{V_\alpha\}_\alpha$ is a collection of linearly topologized vector spaces its product $\prod_\alpha V_\alpha$ (in its product topology) and its direct sum $\bigoplus_\alpha V_\alpha$ (as a subspace of the product) are linearly topologized.

(d) If W is a vector subspace of V, then its topological closure $\overline{W}$ also is a vector subspace of V.

Proof. If $\mathscr{U}$ is a system of neighborhoods around zero consisting of vector subspaces in V then $\{U \cap W \mid U \in \mathscr{U}\}$ is a system of neighborhoods around zero consisting of vector subspaces in W. For (b), let $\pi\colon V \to V/W$ be the quotient map. Since π is open and surjective the image of a local base is a local base; moreover, the image of a vector subspace under π is a vector subspace. In addition, since W is closed it follows that V/W is Hausdorff. Now, for (c) let $\{U_{\alpha,\beta}\}_\beta$ be a local base of zero in V_α of vector subspaces, the products $U_{\alpha_1,\beta_1} \times \ldots \times U_{\alpha_n,\beta_n} \times \prod_\gamma V_\gamma$, where γ ranges over $\alpha \neq \alpha_1, \ldots, \alpha_n$, for any set $\{(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)\}$ form a fundamental system of neighborhoods around zero in $\prod_\alpha V_\alpha$ of open vector subspaces. Note that since $\bigoplus_\alpha V_\alpha \subseteq \prod_\alpha V_\alpha$ is a vector subspace (c) follows from (a). Finally, for (d), suppose $x, y \in \overline{W}$, then, for every open vector subspace U, $(x+U) \cap W \neq \emptyset$ and $(y+U) \cap W \neq \emptyset$, therefore for every $\alpha, \beta \in k$ we have $(\alpha x + U) \cap W \neq \emptyset$ and $(\beta y + U) \cap W \neq \emptyset$. Hence, $(\alpha x + \beta y + U) \cap W \neq \emptyset$ for every open vector subspace U and every pair $\alpha, \beta \in k$. It follows (d). □

Corollary 1.3. *Let* LinTop_k *denote the category of linearly topologized vector spaces over k, where morphisms are given by continuous linear homomorphisms. Then, limits exist in* LinTop_k.

Proof. By Theorem 1.2 it follows that kernels and arbitrary products exist in LinTop_k; therefore, limits exist in LinTop_k. □

Proposition 1.4. *A finite dimensional linearly topologized vector space V is discrete.*

Proof. Since V is Hausdorff, it follows that

$$\bigcap_{U \in \mathscr{U}} U = \{0\},$$

for $\mathscr{U}$ a system of neighborhoods of zero consisting of vector subspaces of V. Since V is finite dimensional there exist $U_1, \ldots, U_n \in \mathscr{U}$ such that

$$\bigcap_{U \in \mathscr{U}} U = U_1 \cap \ldots \cap U_n = \{0\}.$$

Therefore, $\{0\}$ is open. This implies that V is separated. □

Proposition 1.5. *Let V be a linearly topologized vector space and W a vector subspace of V. Then*

$$\overline{W} = \bigcap_{U \in \mathscr{U}} (W + U).$$

for a system of neighborhoods of zero $\mathscr{U}$ *consisting of vector subspaces of V.*

Proof. Since every $W + U$ is open, it is also closed. Hence the intersection is closed. If $x \in V - \overline{W}$, then $(x + U) \cap W = \{0\}$ for some $U \in \mathscr{U}$. Hence, $x \notin W + U$. This proves the proposition. □

Commensurability

We introduce a partial order on the set of vector subspaces of a vector space V.

Definition 1.6. For vector subspaces A and B of a vector space V we say that $A \prec B$ if the quotient $A/(A \cap B) \cong (A+B)/B$ is finite dimensional (or equivalently $A \subseteq B + W$ for some finite dimensional W). In addition, we say that A and B are **commensurable** (denoted $A \sim B$) if $A \prec B$ and $B \prec A$.

Observe that $A \sim B$ if and only if $(A+B)/(A \cap B) \cong A/(A \cap B) \oplus B/(A \cap B)$ is finite dimensional. We will constantly refer to a vector space V being finite dimensional as $V \sim 0$.

Proposition 1.7. *Let V be a vector spaces and A, B and C be vector subspaces, then:*

(a) If $A \sim B$ and $B \sim C$ then

$$\frac{A+B+C}{A \cap B \cap C} \sim 0$$

(b) If $A \prec B$ and $B \prec C$ then $A \prec C$. Moreover, commensurability is an equivalence relation.

Proof. Consider the following exact sequences

$$0 \to \frac{A \cap B}{A \cap B \cap C} \to \frac{B}{B \cap C},$$

and,

$$0 \to \frac{A \cap B}{A \cap B \cap C} \to \frac{A+B}{A \cap B \cap C} \to \frac{A+B}{A \cap B} \to 0$$

induced by inclusions. The first inclusion plus the fact that $B \sim C$ imply that $(A \cap B)/(A \cap B \cap C)$ is finite dimensional. Now, since $A \sim B$ it follows that $(A+B)/(A \cap B)$ is finite dimensional. Therefore, using the second exact sequence we conclude that $(A+B)/(A \cap B \cap C)$ is finite dimensional. A symmetrical argument shows that $(B+C)/(A \cap B \cap C) \sim 0$. These prove (a). For (b), the inclusion

$$0 \to \frac{A+C}{A \cap C} \to \frac{A+B+C}{A \cap B \cap C}$$

plus (a) implies transitivity. $\square$

Now, we state and prove some useful properties on the relation $\prec$.

Lemma 1.8. *Let V be a vector space and A, B vector subspaces of V. Then*

(a) If $A \subseteq B$ it follows $A \prec B$.

(b) If $A \prec B$ then $f(A) \prec f(B)$ for any k-linear map f

(c) Let $\{A_i\}_{i=1}^m$ and $\{B_j\}_{j=1}^n$ be two collections of vector subspaces of V. Then,

$$\sum_{i=1}^m A_i \prec \bigcap_{j=1}^n B_j \iff A_i \prec B_j \text{ for all } i \text{ and } j.$$

Proof. First, (a) is immediate from the definition of $\prec$. Second, for (b) the map f factors as

$$A/(A \cap B) \to f(A)/(f(A) \cap f(B)) \to 0$$

Finally, for (c), if $\sum_{i=1}^m A_i \prec \bigcap_{j=1}^n B_j$ holds then by (a) above, for all i and j we have

$$A_i \prec \sum_{i=1}^m A_i \prec \bigcap_{j=1}^n B_j \prec B_j$$

On the other hand, if $A_i \prec B_j$ for all i and j then there exist finite dimensional subspaces W_{ij} such that $A_i \subseteq B_j + W_{ij}$ for all i and j. Therefore,

$$\sum_{i=1}^m A_i \subseteq \bigcap_{j=1}^n B_j + \sum_{i=1}^m \sum_{j=1}^n W_{ij}.$$

□

Next, we consider another useful lemma.

Lemma 1.9. *Let A, B, A', B' be vector subspaces of a vector space V and suppose that $A \sim A'$ and $B \sim B'$. Then $A + B \sim A' + B'$ and $A \cap B \sim A' \cap B'$.*

Proof. The following exact sequence

$$0 \to \frac{A + A' + B + B'}{A \cap A' \cap B \cap B'} \to \frac{A + A'}{A \cap A'} \oplus \frac{B + B'}{B \cap B'} \to \frac{A + A' + B + B'}{(A \cap A') + (B \cap B')} \to 0$$

plus $A \sim A'$ and $B \sim B'$ imply that both spaces

$$\frac{A + A' + B + B'}{A \cap A' \cap B \cap B'} \quad \text{and,} \quad \frac{A + A' + B + B'}{(A \cap A') + (B \cap B')}$$

are finite dimensional. Since, $(A + A' + B + B')/(A + A') \cap (B + B')$ is a quotient of the second space and $((A \cap A') + (B \cap B'))/((A \cap A') \cap (B \cap B'))$ is a subspace of the first space we can conclude $A + B \sim A' + B'$ and $A \cap B \sim A' \cap B'$. □

If we consider the set of equivalence classes of $\sim$ on a vector space V then $\prec$ is a partial order on it and by Lemma 1.9 above it inherits operations $\cap$ and $+$.

Linear compactness

Definition 1.10. Let V be a linearly topologized vector space. A closed vector subspace $L \subseteq V$ is **linearly compact** (respectively **linearly cocompact**) if for every open vector subspace U we have $L \prec U$ (respectively $V/(L+U) \sim 0$).

Remark 1.11. Linear compactness was introduced by S. Lefschetz in his influential [Lef42] using different terms. He defined a coset of V (also called linear variety) to be a set $x + W$ where $x \in V$ and W is a subspace V. Then, he defined a linearly topologized vector space V to be linearly compact if for every collection of closed cosets X_α having the finite intersection property follows that $\bigcap_\alpha X_\alpha \neq \varnothing$. In these terms, linear compactness seems like a natural generalization of compactness for linearly topologized vector spaces. We extend this discussion in Remark 1.26.

Linear compactness behaves just as compactness if one uses the correct words.

Theorem 1.12. *Let V be a linearly compact vector space, then*

(a) If $A \subseteq V$ is a vector subspace satisfying $A \prec U$ for all open vector subspaces U of V then $\overline{A}$ is linearly compact.

(b) If $f \colon V \to W$ is a continuous linear homomorphism then $\overline{f(V)}$ is linearly compact.

(c) If V is discrete then $V \sim 0$.

(d) Every closed vector subspace of V is linearly compact.

(e) (Tychonov) If $\{V_\alpha\}_\alpha$ is a collection of linearly compact vector spaces then its product $\prod_\alpha V_\alpha$ and its direct sum $\bigoplus_\alpha V_\alpha$ are linearly compact.

Proof. Let U be any open vector subspace of V. Then, $A + U$ is closed, that is $A + U = \overline{A+U} \supseteq \overline{A} + U \supseteq A + U$, thus, $\overline{A} + U = A + U$. Since, $(A+U)/U \sim 0$ it follows that $(\overline{A}+U)/U \sim 0$.

For (b), since f is a continuous linear map $V \prec f^{-1}(U)$ for all U open vector subspaces of W. Hence by Lemma 1.8 $f(V) \prec U$ for all open vector subspaces U of W. The previous observation and (b) yield (a). If V is discrete, then $\{0\}$ is an open vector subspace of V; therefore, V is finite dimensional.

For (d), if $A \subseteq V$ is a closed vector subspace, and $V \prec U$ for all open vector subspaces U by Lemma 1.8 we get $A \prec U$.

Finally, for (d), it is enough proving for open vector subspaces $U = \prod_\beta U_\beta \times \prod_\gamma V_\gamma$ where β ranges over a finite set, γ ranges over $\alpha \neq \beta$ and U_β is an open vector subspace of V_β. Then, the quotient

$$\prod_\alpha V_\alpha / U \cong \prod_\beta V_\beta / U_\beta$$

where $\cong$ is a topological and algebraic isomorphism. Since V_α is linearly compact for all α and β ranges over a finite set we conclude that $\prod_\alpha V_\alpha / U$ is finite dimensional; therefore, $\prod_\alpha V_\alpha$ is linearly compact. The proof is analogous for $\bigoplus_\alpha V_\alpha$. □

Completion

In this subsection we expose with little detail the properties of completion. A general reference for completion of arbitrary modules over a commutative ring is [Mat86] Section 8.

Definition 1.13. Let V a linearly topologized vector space. Let $\mathscr{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ be a system of neighborhoods of zero consisting of open vector subspaces of V indexed by Λ, so that $\lambda < \mu \iff U_\lambda \supseteq U_\mu$. Then, the inverse limit

$$\widehat{V} = \varprojlim_{U \in \mathscr{U}} V/U = \varprojlim_{\lambda \in \Lambda} V/U_\lambda$$

is the **completion** of V. Observe that existence of such limit is guaranteed by Corollary 1.3. Recall that $\widehat{V}$ can be described as a subspace of the product

$$\widehat{V} = \{(\overline{x_\lambda})_\lambda \in \prod_{\lambda \in \Lambda} V/U_\lambda \colon \overline{x_\mu} = \pi^\lambda_\mu(\overline{x_\lambda}) \text{ for all } \mu \leq \lambda\},$$

where $\pi^\lambda_\mu \colon V/U_\lambda \to V/U_\mu$ is the induced map by the projection $\pi_\mu \colon V \to V/U_\mu$. In this way $\widehat{V}$ carries a natural topology as a subspace of the product $\prod_{U \in \mathscr{U}} V/U$. There is always a natural map

$$V \to \widehat{V}$$

induced by the quotient maps π_μ. This map is continuous and its image is dense in $\widehat{V}$. If the map $V \to \widehat{V}$ is a topological isomorphism we say that V is **complete**. Let $\mathscr{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ and $\mathscr{U}' = \{U'_{\lambda'}\}_{\lambda' \in \Lambda}$ be two different systems of neighborhoods of consisting of vector subspaces in V. Then for all $\lambda \in \Lambda$ there exists $\lambda' \in \Lambda'$ such that $U_{\lambda'} \subseteq U_\lambda$ and for all $\mu' \in \Lambda'$ there exists $\mu \in \Lambda$ such that $U_\mu \subseteq U_{\mu'}$. This implies that $\varprojlim_{U \in \mathscr{U}}$ and $\varprojlim_{U' \in \mathscr{U}'}$ satisfy the same universal property. Therefore, they are canonically isomorphic. In other words, completion does not depend on the choice of filtration $\mathscr{U}$.

1.2 TATE SPACES

Lattices

Definition 1.14. If V is a linearly topologized vector space we say that a **c-lattice** is an open linearly compact subspace of V, *dually* a discrete linearly cocompact subspace is a **d-lattice**.

First, we prove that existence of a c-lattice in a linearly topologized vector space is equivalent to existence of a d-lattice.

Proposition 1.15. *A linearly topologized vector space V contains a c-lattice if and only if it contains a d-lattice.*

Proof. Suppose L is a c-lattice in V. Choose any direct complement D of L, that is, $V = L \oplus D$. Since L is open, then D is discrete as $D \cap L = 0$, thus 0 is open in D. Moreover, D is closed it is the kernel the projection $V \to L$ (which is continuous because L is open). Finally, we check that D is linearly cocompact: let U be any open vector subspace of V, the composition $L \hookrightarrow V \twoheadrightarrow V/(D+U)$ induces a surjection

$$L/(L \cap U) \twoheadrightarrow V/(D+U).$$

Since $\dim L/(L \cap U) < \infty$ we conclude that $\dim V/(D+U) < \infty$.
Now, suppose D is a d-lattice. Thus, there exists an open vector subspace U such that $U \cap D = 0$. This time, choose L a direct complement for D containing U. Then, the projection $V \to D$ is continuous because U is mapped to zero. Therefore, L is open. Now, we prove that L is linearly compact. Let U be any open vector subspace. Then, the composition $V \twoheadrightarrow L \twoheadrightarrow L/(L \cap U)$ induces a surjection

$$V/(D + (L \cap U)) \twoheadrightarrow L/(L \cap U).$$

Since both L and U are open, so is $L \cap U$. Therefore, $\dim V/(D + (L \cap U)) < \infty$. It follows that $\dim L/(L \cap U) < \infty$ and L is linearly compact. □

Remark 1.16. Note that in the proof of Proposition 1.15 it is not strictly necessary to choose a direct complement, one can choose a direct complement up to finite dimension; that is, $L + D \sim V$ and $L \cap D \sim 0$.

Proposition 1.17. *If V admits a c-lattice, then the set of c-lattices constitutes a system of neighborhoods of zero consisting of mutually commensurable vector subspaces.*

Proof. If L and L' are two c-lattices in V then $L \prec L'$ and $L' \prec L$ because both are open. Therefore, all c-lattices are commensurable. Moreover, if U is any open vector subspace and L is a c-lattice we claim that $L \cap U$ is a c-lattice as well. Indeed, let U' be any open vector subspace, then $L \cap U \prec L \prec U'$. In addition, since L and U are open, so is $L \cap U$. Hence, $L \cap U \subseteq U$ is a c-lattice. This proves the statement. □

We are now ready to introduce the definition of a Tate space.

Definition 1.18. A linearly topologized vector space V is a **Tate space** if it is complete and admits a c-lattice. By Proposition 1.17 and the observation in Definition 1.13 it follows that

$$V \cong \varprojlim_{L \in \mathscr{U}} V/L$$

where $\mathscr{U}$ runs through all c-lattices of V.

Example 1.19. We give some examples of Tate spaces.

(a) Any vector space endowed with the discrete topology is a Tate space.

(b) If $\{V_\alpha\}_\alpha$ is any projective system of finite dimensional vector spaces (thus, each one endowed with the discrete topology by Proposition 1.4), let V be their inverse limit endowed with the inverse limit topology. We claim that this is a linearly compact space. Indeed, if we realize V as a subspace of the product $\prod_\alpha V_\alpha$, a basic open vector subspace in V is given by

$$\{(x_\alpha)_\alpha \in V \colon x_{\alpha_1} = x_{\alpha_2} = \cdots = x_{\alpha_n} = 0\}$$

for some finite collection of indices $\alpha_1, \dots, \alpha_n$. Hence, the quotient of V by any basic open vector subspace is a vector subspace a finite product of V_α, since all V_α are finite dimensional we conclude that V is linearly compact. Observe that a system of neighborhoods consisting of vector subspaces of V is the collection of kernels of the projections $V \to V_\alpha$. Therefore, V is complete.

(c) Let $V = k((t))$ with the topology generated by letting $t^n k[[t]]$ for $n \in \mathbb{Z}$ be a system of neighborhoods of zero. Then, $V = k[[t]] \oplus tk[t^{-1}]$ where $k[[t]]$ is the completion of $k[x]$ in the $\langle x \rangle$-adic topology, hence by the previous item linearly compact and, since it is open is a c-lattice. By the argument given in Proposition 1.15 $tk[t^{-1}]$ is a d-lattice. Therefore, V is a Tate space that is neither linearly compact nor discrete.

A closer look at the previous examples motivates the following proposition:

Proposition 1.20. *A linearly topologized vector space V is a Tate space if and only if it there exists a collection $\mathscr{U}$ in V of mutually commensurable open vector subspaces in V such that the natural map*

$$V \to \varprojlim_{U \in \mathscr{U}} V/U$$

is a topological isomorphism. In particular, every Tate space arises in the following way: Let V be a k-vector space endowed with a collection $\mathscr{U}$ of vector subspaces satisfying the following conditions:

(i) $\mathscr{U}$ filters down to 0 and up to V.

(ii) Every two subspaces in $\mathscr{U}$ are mutually commensurable.

(iii) The natural map

$$V \to \varprojlim_{U \in \mathscr{U}} V/U$$

is an isomorphism. Then, $(V, \mathscr{U})$ becomes a Tate space by imposing (iii) to be a topological isomorphism letting the quotient V/U be discrete for all $U \in \mathscr{U}$.

Proof. By the observation in Definition 1.18 such collection is simply the collection of c-lattices in V. Now, suppose that such collection exists in V. Then, $\mathscr{U}$ is a system of neighborhoods of zero consisting of mutually commensurable open vector subspaces in V, any $U \in \mathscr{U}$ is a c-lattice and V is complete. $\square$

Duality

If V is a Tate space we consider the following topology on the dual space V^* (where by dual space we mean topological dual). Open vector subspaces are given by

$$L^{\perp} = \{\phi \in E^* \colon \phi|_L = 0\}$$

where L is a linearly compact subspace. Equivalently, one can define open vector subspaces in E^* to be D^* where D a direct complement of a linearly compact vector subspace L in E (in this case $D^* \hookrightarrow E^*$ using the decomposition $L \oplus D$).
First, we prove that the word *dually* in Definition 1.10 actually makes sense.

Lemma 1.21. *Duality interchanges linearly compact with discrete spaces and vice-versa.*

Proof. If L is a linearly compact vector space, then $L^{\perp}$ is open in L^*, thus L^* is discrete. If D is discrete, then $D \cong k^{\oplus \Lambda}$ for some Λ and endowing $k^{\oplus \Lambda}$ with the discrete topology the previous isomorphism is a homeomorphism as well. Moreover, since D is discrete every linear functional is continuous. Using Corollary 1.3 and the well known identity (where maps are isomorphisms in LinTop_k)

$$(k^{\oplus \Lambda})^* = \mathrm{Hom}_k(k^{\oplus \Lambda}, k) \cong \prod_{\Lambda} \mathrm{Hom}_k(k,k) \cong \prod_{\Lambda} k$$

we get the desired result by Tychonov's theorem in Theorem 1.12. □

Remark 1.22. A closer look at the proof of the previous lemma indicates that the dual space of a discrete space is always complete.

Proposition 1.23. *If V is a Tate space then V^* is also a Tate space.*

Proof. If we decompose $V = L \oplus D$ where L is a c-lattice and D a d-lattice then $V^* \cong L^* \oplus D^*$ and by Lemma 1.21 L^* is discrete and D^* is linearly compact. Observe that D^* is open in V^* since it is the kernel of the projection $V^* \to V^*/L^{\perp}$ and $V^*/L^{\perp}$ is discrete by the description of our topology in the dual V^*. Since L^* is discrete, then it is complete. Moreover, by Remark 1.22 D^* is complete. Hence, so is V^*. □

We are now ready to prove the duality theorem for Tate spaces.

Theorem 1.24. *For a Tate space V the canonical map $V \to V^{**}$ is an isomorphism.*

Proof. It is enough to prove it for complete linearly compact spaces and discrete spaces, as every Tate space can be decomposed into a direct sum of a c-lattice and a d-lattice. First, suppose D is a discrete vector space. Then, the canonical map

$$\mathrm{ev}\colon D \to D^{**}$$

is open and continuous because D and D^{**} are both discrete by Lemma 1.21. Moreover, it is injective, because for every nonzero $x \in D$ there exists a linear continuous functional $\phi \in D^*$ such that $\phi(x) \neq 0$. Finally, we prove surjectivity. Let $\psi \in D^{**}$. Since $\ker \psi$ is open it contains a basic open vector subspace $A^\perp$ such that $A \subseteq D$ is a linearly compact subspace. Since D^* is linearly compact it follows that $D^* \sim A^\perp$, that is, the quotient $D^*/A^\perp$ is finite dimensional. Recall that the inclusion $\iota\colon A \to D$ induces an isomorphism $D^*/A^\perp \to A^*$ which is a homeomorphism since both spaces are discrete. We can factor ψ so that the following diagram commutes

$$\begin{tikzcd}
D^* \arrow[r, "\psi"] \arrow[d] & k \\
D^*/A^\perp \arrow[ur, dashed, "\tilde{\psi}"] \arrow[d, "\cong"'] & \\
A^* \arrow[uur, dashed, bend right, "\overline{\psi}"'] &
\end{tikzcd}$$

However, A^* is finite dimensional, therefore, there exists $a \in A$ such that $\overline{\psi} = \mathrm{ev}_a$ as maps $A^* \to k$. Moreover, since $A^\perp \subseteq \ker \psi$ we conclude that $\psi = \mathrm{ev}_a$ as maps $D^* \to k$. This implies surjectivity. Thus $D \to D^{**}$ is an isomorphism of topological vector spaces.

Now, suppose L is a complete linearly compact space. We check first that the map

$$\mathrm{ev}\colon L \to L^{**}$$

is continuous. Let $A^\perp$ be an open vector subspace in L^{**} where $A \subseteq L^*$ is a linearly compact subspace. By Lemma 1.21 L^* is discrete, hence A is finite dimensional. Suppose that $A = \mathrm{span}(\phi_1, \ldots, \phi_n)$ for some $\phi_1, \ldots, \phi_n \in A$. Then, $\mathrm{ev}^{-1}(A^\perp) = \ker \phi_1 \cap \ldots \cap \ker \phi_n$ which is open in L. Now, we check that ev is injective. Let $v \in L$ be a nonzero vector. Choose a decomposition of $L = U \oplus F$ where U is open and F is finite dimensional containing v (this can be done because L is separated and linearly compact). Let ϕ be a linear functional such that $\phi|_U = 0$ and $\phi(v) \neq 0$. Since U is open and F discrete such ϕ exists and it is continuous. Now, we check that ev is surjective. Let $\mathscr{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ be a system of neighborhoods consisting of open vector subspaces. Let $\psi\colon L^* \to k$ be a continuous linear functional. The dual map of $\pi_\lambda\colon L \to L/U_\lambda$ yields an injection $\pi_\lambda^*\colon (L/U)^* \hookrightarrow L^*$ for every $U_\lambda \in \mathscr{U}$. Since L is linearly compact, the vector space L/U_λ is finite dimensional. Thus, there exists a unique $\overline{x_\lambda} \in L$ such that $\psi \circ \pi_\lambda^* = \mathrm{ev}_{\overline{x_\lambda}}$ where $\mathrm{ev}\colon L/U_\lambda \to (L/U_\lambda)^{**}$. In addition, observe that if $\mu \leq \lambda$ there is an induced injection $\pi_\mu^{\lambda *}\colon (L/U_\mu)^* \hookrightarrow (L/U_\lambda)^*$ such that the following diagram

$$\begin{tikzcd}
L^* \arrow[r, "\psi"] & k \\
(L/U_\lambda)^* \arrow[u, hook, "\pi_\lambda^*"] \arrow[ur, "\mathrm{ev}_{\overline{x_\lambda}}"'] & \\
(L/U_\mu)^* \arrow[u, hook, "\pi_\mu^{\lambda *}"] \arrow[uu, hook, bend left=60, "\pi_\mu^*"] \arrow[uur, bend right, "\mathrm{ev}_{\overline{x_\mu}}"'] &
\end{tikzcd}$$

commutes. Observe that uniqueness of $\overline{x_\lambda} \in L/U_\lambda$ implies $\overline{x_\mu} = \pi_\mu^\lambda(\overline{x_\lambda})$ for all $\mu \leq \lambda$. Indeed, for all $\phi_\mu \in (L/U_\mu)^*$ the equality

$$\begin{aligned}
\phi_\mu(\overline{x_\mu}) &= \psi(\pi_\mu^*(\phi_\mu)) \\
&= \psi(\pi_\lambda^* \circ \pi_\mu^{\lambda *}(\phi_\mu)) \\
&= \psi(\pi_\lambda^*(\phi_\mu \circ \pi_\mu^\lambda)) \\
&= (\phi_\mu \circ \pi_\mu^\lambda)(\overline{x_\lambda}) \\
&= \phi_\mu(\pi_\mu^\lambda(\overline{x_\lambda}))
\end{aligned}$$

holds. Therefore, $\overline{x_\mu} = \pi_\mu^\lambda(\overline{x_\lambda})$ for all $\mu \leq \lambda$. Then, $(x_\lambda)_{\lambda \in \Lambda}$ is contained in the completion $\widehat{V}$ as described in Definition 1.13. Since $V \to \widehat{V}$ is an isomorphism, there exists $x \in L$ such that $\pi_\lambda(x) = \overline{x_\lambda}$ for all $\lambda \in \Lambda$. We claim that $\psi = \mathrm{ev}_x$. Let $\phi \in L^*$. Then, $\ker\phi$ is open and there exists $\lambda \in \Lambda$ such that $U_\lambda \subseteq \ker\phi$. Hence, we can factor ϕ as follows

$$\begin{tikzcd} L \arrow[r, "\phi"] \arrow[d, "\pi_\lambda"'] & k \\ L/U_\lambda \arrow[ur, "\phi_\lambda"'] & \end{tikzcd}$$

Now, since L/U is discrete it follows that ϕ_λ is continuous. Moreover, $\pi_\lambda^*(\phi_\lambda) = \phi$. Hence

$$\psi(\phi) = \psi(\pi_\lambda^*(\phi_\lambda)) = \phi_\lambda(\overline{x_\lambda}) = \phi_\lambda(\pi_\lambda(x) = \phi(x).$$

This implies surjectivity of $\mathrm{ev}\colon L \to L^{**}$. Finally, we prove that ev is open. Let U be any open vector subspace in L, thus $L = U \oplus F$ for some finite dimensional F. We claim that $\mathrm{ev}(U) = (F^*)^\perp$. First, the inclusion $\mathrm{ev}(U) \subseteq (F^*)^\perp$ is immediate. Let $\psi \in (F^*)^\perp$. Let $x \in L$ such that $\mathrm{ev}_x = \psi$. Write $x = u + f$ where $u \in U$ and $f \in F$. Hence, $\mathrm{ev}_x = \mathrm{ev}_u + \mathrm{ev}_f$. Since ev is injective, it follows that there exists some $\phi \in F^*$ such that $\phi(f) \neq 0$ if f is nonzero. Therefore, $f = 0$ and $\psi \in \mathrm{ev}(U)$. This concludes the proof. $\square$

Remark 1.25. Observe that completeness cannot be dropped in the definition of a Tate space while preserving duality. Indeed, if V is linearly compact but not complete its dual is discrete by Lemma 1.21 and by Remark 1.22 its double dual is complete, hence $V \to V^{**}$ cannot be an isomorphism. In fact, during the proof of the duality theorem we checked that V^{**} is the completion of V.

Remark 1.26. We now discuss definitions of linearly compact spaces as given in [Lef42] and [BD04]. In [Lef42] it is proven that a linearly compact vector space must be complete while our definition does not imply it necessarily. However, when V is a complete space both definitions coincide. Indeed, Lefschetz proves that every linearly compact space is the dual of a discrete space which coincides with our definition of a complete linearly compact vector space by Theorem 1.24. Therefore, his definition of a **locally linearly compact vector space** (that is, a linearly topologized vector space admitting an open linearly compact vector subspace) coincides with our notion of Tate space.

Morphisms

Definition 1.27. A morphism $f\colon V \to W$ of Tate spaces is said to be **linearly compact** if the closure of $f(V)$ is linearly compact in W. Dually, it is **discrete** if $\ker f$ is open in V.

First, we check the duality natural property for morphisms of Tate spaces.

Proposition 1.28. *A morphism $f\colon V \to W$ of Tate spaces is linearly compact if and only if f^* is discrete.*

Proof. Suppose f^* is linearly compact, then $\ker f^* = f(V)^\perp$. However, if $\phi \in W^*$ and $\phi(f(V)) = 0$ then $\phi(\overline{f(V)}) = 0$ by continuity of ϕ. Therefore, $\ker f^* = \overline{f(V)}^\perp$ which is open because $\overline{f(V)}$ is linearly compact. Now, suppose f^* is discrete. Thus, $\ker f^*$ contains a basic open subspace $A^\perp$ such that A is linearly compact in W. Therefore, $f(V) \subseteq A$ then $\overline{f(V)} \subseteq A$ and by item (c) in Theorem 1.12 $\overline{f(V)}$ is linearly compact. □

Discrete and linearly compact operators form a 2-sided ideal in Hom; that is

Proposition 1.29. *If f is a linearly compact morphism (respectively discrete) then its composition with an arbitrary morphism of Tate spaces is also linearly compact (respectively discrete).*

Proof. Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ be morphisms of Tate spaces such that g is linearly compact. Then, $\overline{g \circ f(A)} \subseteq \overline{g(B)}$ which is linearly compact, thus $\overline{g \circ f(A)}$ is linearly compact as well. On the other hand, note that $h(\overline{g(B)}) \subseteq \overline{h \circ g(B)}$; therefore $\overline{h(\overline{g(B)})} = \overline{h \circ g(B)}$. However, $\overline{g(B)}$ is linearly compact and by item (b) of Theorem 1.12 $h(\overline{g(B)})$ is linearly compact. In addition, the statement for discrete operators follows from Proposition 1.28. □

Remark 1.30. If f is a compact operator and g is a discrete operator, then gf is of **finite-rank**; that is, $\dim gf(V) < \infty$.

Proof. We have $\overline{f(V)} \prec \ker g$, therefore, $\overline{f(V)}/(\overline{f(V)} \cap \ker g)$ is finite dimensional. We have a surjection

$$\frac{\overline{f(V)}}{\overline{f(V)} \cap \ker g} \to gf(V)$$

which implies that gf is of finite-rank. □

Definition 1.31. Let V and W be Tate spaces. We denote $\mathrm{Hom}_+(V,W)$ to be the set of linearly compact morphisms and $\mathrm{Hom}_-(V,W)$ the set of discrete ones. Let

$$\mathrm{Hom}_0(V,W) := \mathrm{Hom}_+(V,W) \cap \mathrm{Hom}_-(V,W).$$

Proposition 1.32. *The sets* $\mathrm{Hom}_-(V,W), \mathrm{Hom}_+(V,W)$ *and* $\mathrm{Hom}_0(V,W)$ *are vector subspaces of* $\mathrm{Hom}(V,W)$. *Moreover,*

$$\mathrm{Hom}_-(V,W) + \mathrm{Hom}_+(V,W) = \mathrm{Hom}(V,W).$$

Proof. Let L be a c-lattice in V and consider $\pi \colon V \to L$ be a continuous linear projection. Then π realized as an element of $\mathrm{End}(V)$ satisfies $\pi \in \mathrm{End}_+(V)$ and $1-\pi \in \mathrm{End}_-(V)$. Hence, by Proposition 1.29 for every $f \in \mathrm{Hom}(V,W)$ $f \circ \pi$ and $f \circ (1-\pi)$ are linearly compact and discrete respectively. It follows

$$\mathrm{Hom}_-(V,W) + \mathrm{Hom}_+(V,W) = \mathrm{Hom}(V,W).$$

The other statements are immediate. □

I'll include further theory if necessary.

2

TRACE AND RESIDUE

We extend the definition of trace to a certain class of infinite rank endomorphisms in order to define an abstract residue. We follow the original structure of Tate's elegant article [Tat68] while translating his statements in the language of Tate's Linear Algebra.

2.1 FINITE-POTENT MAPS AND THEIR TRACE

Let k be a fixed field and V a vector space over k. In this section we will extend the notion of trace of a linear endomorphism to include certain operators even when V is infinite dimensional.

Finite-potent maps

Definition 2.1. We will say a linear map $f\colon V \to V$ is **finite-potent** if

$$\dim f^n(V) < \infty$$

for sufficiently large n.

The following is characterization of finite-potent endomorphisms.

Lemma 2.2. *A linear map $f\colon V \to V$ is finite-potent if and only if there exists a subspace $W \subseteq V$ such that*

(i) $\dim f(W) < \infty$,

(ii) $f(W) \subseteq W$,

(iii) the induced map $\bar{f}\colon V/W \to V/W$ is nilpotent.

*A subspace W is a **trace-subspace** for f if satisfies the previous previous properties.*

Proof. If f is finite-potent choose $W = f^n(V)$ for sufficiently large n. The first condition follows from definition. Also, $f(W) = f^{n+1}(V) \subseteq f^n(V) = W$. In addition, $\bar{f}^n = 0$. On the other hand, if such W exists, note that condition (ii) assures that $\bar{f}$ is well defined. Moreover, as $\bar{f}$ is nilpotent, $f^n V \subseteq W$ for sufficiently large n and by condition (i) above $\dim f^n(V) < \infty$. □

Observe that a trace-subspace for a finite-potent map f is not unique. In particular, if W is trace-subspace for f then $f^n(W)$ is trace-subspace for f for all n.

Notation 2.3. If f is a finite-rank endomorphism in a vector space V we will denote its ordinary trace by $\mathrm{tr}_V(f)$. Moreover, if W is a subspace of V invariant under f, that is, $f(W) \subseteq W$ then $\mathrm{tr}_W(f) := \mathrm{tr}_W(f|_W)$. In addition, if $\overline{f}$ is the induced map such that the following diagram commutes

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\pi_W"] & V \arrow[d, "\pi_W"] \\ V/W \arrow[r, "\overline{f}"] & V/W \end{tikzcd}$$

then $\mathrm{tr}_{V/W}(f) := \mathrm{tr}_{V/W}(\overline{f})$. The use of this notation is consistent throughout the document.

Trace

If f is a finite-potent map and W is a trace-subspace for f the **trace** $\mathrm{tr}_V(f) \in k$ of f may be defined as

$$\mathrm{tr}_V(f) = \mathrm{tr}_W(f)$$

Observe that $\mathrm{tr}_W(f)$ is well-defined because $f|_W$ is of finite-rank.

Proposition 2.4. *The definition of* tr_V *does not depend on the choice of trace-subspace for* f*.*

Proof. Suppose $W_1, W_2 \subseteq V$ are two trace-subspaces for f then $W = W_1 + W_2$ is trace-subspace for f as well. Hence, the induced maps on W/W_1 and W/W_2 are nilpotent. Therefore, $\mathrm{tr}_{W/W_1}(f) = \mathrm{tr}_{W/W_2}(f) = 0$ and using the well-known identify of the ordinary trace

$$\mathrm{tr}_W(f) = \mathrm{tr}_{W_1}(f) + \mathrm{tr}_{W/W_1}(f)$$
$$\mathrm{tr}_W(f) = \mathrm{tr}_{W_2}(f) + \mathrm{tr}_{W/W_2}(f),$$

we obtain $\mathrm{tr}_{W_1}(f) = \mathrm{tr}_{W_2}(f)$, our desired result. □

This definition extends some of the properties of the ordinary trace.

Lemma 2.5. *(a) If* $\dim V < \infty$*, any endomorphism* f *is finite-potent and* $\mathrm{tr}_V(f)$ *coincides with the ordinary trace.*

(b) If f *is nilpotent, then it is finite-potent and* $\mathrm{tr}_V(f) = 0$*.*

(c) If f *is finite-potent and* U *is a subspace such that* $f(U) \subseteq U$ *then the induced maps on* U *and* V/U *are finite-potent and satisfy*

$$\mathrm{tr}_V(f) = \mathrm{tr}_U(f) + \mathrm{tr}_{V/U}(f)$$

Proof. Both (a) and (b) are immediate. For (c) if W is a trace-subspace for f then $W \cap U$ and $(W+U)/U$ are trace-subspaces for the induced maps respectively. Hence, by Lemma 2.2 both induced maps are finite-potent. Since $W/(W \cap U) \cong (W+U)/U$, the diagram

$$\begin{tikzcd} W/(W \cap U) \arrow[r, "\cong"] \arrow[d, "f"] & (W+U)/U \arrow[d, "f"] \\ W/(W \cap U) \arrow[r, "\cong"] & (W+U)/U \end{tikzcd}$$

commutes. Moreover, recall that the ordinary trace is invariant under conjugation, that is, $\mathrm{tr}_W(\varphi \circ f \circ \varphi^{-1}) = \mathrm{tr}_W(f)$ for every automorphism φ of W. Therefore, it follows that $\mathrm{tr}_{W/(W\cap U)}(f) = \mathrm{tr}_{(W+U)/U}(f)$. We conclude that

$$\mathrm{tr}_V(f) = \mathrm{tr}_W(f) = \mathrm{tr}_{W\cap U}(f) + \mathrm{tr}_{(W+U)/U}(f) = \mathrm{tr}_U(f) + \mathrm{tr}_{V/U}(f). \qquad \square$$

Definition 2.6. A subspace F of $\mathrm{End}_k(V)$ is said to be a **finite-potent subspace** if there exists an n such that for any family of n elements $f_1, \ldots, f_n \in F$, the space $f_1 f_2 \cdots f_n V$ is finite dimensional.

Observe that if F is a finite-potent subspace of $\mathrm{End}_k(V)$ then every $f \in F$ is finite-potent.

Proposition 2.7. *If* F *is a finite-potent subspace then* $\mathrm{tr}_V \colon F \to k$ *is* k*-linear*

Proof. It is enough to prove it in the case that F is finite dimensional. Let $W = F^n V$ for n as in the definition of finite-potent subspace, thus $\dim W < \infty$. Hence, W is a trace-subspace for all $f \in F$. It follows that $\mathrm{tr}_V(f) = \mathrm{tr}_W(f)$ for all f. Since $\mathrm{tr}_W \colon \mathrm{End}_k(V) \to k$ is k-linear, so is $\mathrm{tr}_V \colon F \to k$. $\square$

Remark 2.8. In his paper, Tate asked if general linearity for finite-potent maps followed. His question was answered negatively in [AST07] where general linearity is reduced to the following: if the sum of two nilpotent endomorphisms is finite-potent, is the sum necessarily traceless?

Proposition 2.9. *If $f, g \in \mathrm{End}_k(V)$ and fg is finite-potent then gf is also finite-potent and*

$$\mathrm{tr}_V(fg) = \mathrm{tr}_V(gf).$$

Proof. Since fg is finite-potent then $W = (fg)^n V$ is finite-dimensional for a sufficiently large n. On the other hand, $(gf)^{n+1}V = g(fg)^n f(V) \subseteq g(W)$, therefore, gf is also finite-potent. Let $W' = (gf)^n V$, then $g(W') \subseteq W$ and $f(W) \subseteq W'$. Thus,

$$\dim W' \leq \dim g(W) \leq \dim W,$$

and,

$$\dim W \leq \dim f(W) \leq \dim W',$$

which implies that $W \cong W'$ and that g and f induce mutually inverse isomorphisms between W and W'. Moreover, the diagram

$$\begin{tikzcd} W \arrow[r, "fg"] \arrow[d, "g"] & W \arrow[d, "g"] \\ W' \arrow[r, "gf"] & W' \end{tikzcd}$$

commutes. We conclude $\mathrm{tr}_W(fg) = \mathrm{tr}_{W'}(gf)$ and it follows $\mathrm{tr}_V(fg) = \mathrm{tr}_V(gf)$. □

Trace and Tate Spaces

Suppose that V is a Tate space and consider $\mathrm{End}_k(V)$ the space of continuous endomorphisms of V. By Proposition 1.29 and Proposition 1.32 we have 2-sided ideals $\mathrm{End}_0(V), \mathrm{End}_+(V)$ and $\mathrm{End}_-(V)$ of E such that $\mathrm{End}_+(V) + \mathrm{End}_-(V) = E$ and $\mathrm{End}_0(V) = \mathrm{End}_+(V) \cap \mathrm{End}_-(V)$. Moreover, Remark 1.30 implies that $\mathrm{End}_0(V)$ is a finite-potent subspace.

Lemma 2.10. *Suppose $f \in \mathrm{End}_+(V)$ and $g \in \mathrm{End}_-(V)$ or $f \in \mathrm{End}_-(V)$ and $g \in \mathrm{End}_+(V)$. Then the commutator $[f, g] = fg - gf$ belongs to $\mathrm{End}_0(V)$ and it is traceless.*

Proof. This immediate from the previous discussion and Proposition 2.9. □

2.2 DIFFERENTIAL CALCULUS

In this section we introduce the theory of derivations and differentials over an arbitrary commutative k-algebra A. Let M be an A-module. We follow [GD64] Section 20 and [Mat86] Section 25.

Definition 2.11. A **derivation** from A to M is a map $D\colon A \to M$ satisfying properties

(i) $D(a+b) = D(a) + D(b)$ and,

(ii) (*Leibniz Rule*) $D(ab) = aD(b) + bD(a)$

for all $a, b \in A$.

The set of derivations from A to M is an A-module in the natural way. We will denote it by $\mathrm{Der}(A, M)$. Moreover if A is a k-algebra through a map $\varphi\colon k \to A$ we say that D is a **k-derivation** if D is a derivation and $D \circ \varphi = 0$. In this case, the set of all k-derivations is denoted $\mathrm{Der}_k(A, M)$. If $M = A$, we will denote $\mathrm{Der}_k(A, A)$ simply by $\mathrm{Der}_k(A)$.

Definition 2.12. Let B be a k-algebra and C an ideal in B with $C^2 = 0$; set $A = B/C$. In this way, C can be viewed as an A-module. In this situation we say that B is an **extension** of the k-algebra A by the A-module C. Usually, we simply write the exact sequence

$$0 \to C \to B \xrightarrow{\pi} A \to 0$$

As usual, we will say that such sequence **splits** if there exists a retraction; that is, a k-algebra homomorphism $\rho\colon A \to B$ such that $\pi \circ \rho = 1_A$. In this case we can identify $B = C \oplus A$. Conversely, starting from any k-algebra A and any A-module C, one can always define a structure on $A \oplus C$ such that $A \oplus C$ is an extension of A by C. Namely,

$$(a, c)(a', c') = (aa', ac' + a'c)$$

for $a, a' \in A$ and $c, c' \in C$. Common notations for this algebra are $D_A(C)$ or $A * C$.

Definition 2.13. Given a commutative diagram of k-algebras

$$\begin{tikzcd} B \arrow[r, "f"] & A \\ & C \arrow[u, "g"] \arrow[ul, "h"] \end{tikzcd}$$

thinking of f as a fixed map; we say that h is a **lifting** of g to B.

Lemma 2.14. *Let h and $h'\colon C \to B$ be two liftings of g to B. Let $K = \ker f$ and suppose $K^2 = 0$. Then, it follows that $h - h'\colon C \to K$ is a k-derivation. Conversely, if $D \in \mathrm{Der}_k(C, K)$ then $h + D$ is another lifting of g to B.*

Proof. First, observe that $(h - h')(C)$ lies in K because both h and h' are liftings of g to B. Since $K^2 = 0$, then K can be considered as $f(B)$-module and by means of g as a C-module. Then, $h - h'\colon C \to K$ is a map of C-modules. Now, let $c, c' \in C$ then

$$\begin{aligned}(h - h')(cc') &= h(c)h(c') - h'(c)h'(c') \\ &= h(c)h(c') - h'(c)h'(c') - h(c)h'(c') + h'(c')h(c)\end{aligned}$$

since $c \cdot k = h(c)k = h'(c)k$ for all $k \in K$ it follows that

$$\begin{aligned}(h - h')(cc') &= c \cdot h(c') - c' \cdot h'(c') - c \cdot h'(c') + c' \cdot h(c) \\ &= c \cdot (h - h')(c') + c' \cdot (h - h')(c)\end{aligned}$$

which implies that $h - h'$ is a k-derivation. Observe that $h + D$ is a lifting because $D(C)$ lies in K. □

Theorem 2.15. *If A is a k-algebra, consider the covariant functor from the category Mod_A to itself given by $M \mapsto \mathrm{Der}_k(A, M)$. This functor is representable.*

Proof. Let $\mu\colon A \otimes_k A \to A$ be the k-algebra homomorphism given by $f \otimes g \to fg$. Set

$$I = \ker \mu, \quad \Omega_{A/k} = I/I^2, \quad \text{and,} \quad B = (A \otimes_k A)/I^2.$$

Thus, μ induces $\mu'\colon B \to A$ such that

$$0 \to \Omega_{A/k} \to B \to A \to 0$$

is an extension of A by $\Omega_{A/k}$. We claim that this extension splits. Moreover it has two splittings, by considering retractions

$$j_1\colon A \to B \quad \text{and,} \quad j_2\colon A \to B,$$

defined by $a \mapsto a \otimes 1 \bmod I^2$ and $a \mapsto 1 \otimes a \bmod I^2$. By Lemma 2.14 $d := j_2 - j_1$ is a k-derivation of A to $\Omega_{A/k}$. Now, we prove that

$$\mathrm{Der}_k(A, M) \cong \mathrm{Hom}_A(\Omega_{A/k}, M). \tag{1}$$

Let $D \in \mathrm{Der}_k(A, M)$ and define $\varphi\colon A \otimes_k A \to A * M$ by $\varphi(x \otimes y) = (xy, xD(y))$ then φ is a k-algebra homomorphism since it is compatible with the operation in $A * M$ defined in Definition 2.12. In addition, if $\sum x_i \otimes y_i$ lies in I then

$$\mu\left(\sum x_i \otimes y_i\right) = \sum x_i y_i = 0 \implies \varphi\left(\sum x_i \otimes y_i\right) = \left(0, \sum x_i D(y_i)\right)$$

whence $\varphi(I)$ lies in M. Moreover, by Leibniz's Rule φ factors through I^2 yielding a map $f\colon \Omega_{A/k} \to M$. For $a \in A$ it follows that

$$\begin{aligned} f(da) &= f(1 \otimes a - a \otimes 1 \mod I^2) = \varphi(1 \otimes a) - \varphi(a \otimes 1) \\ &= D(a) - aD(1) = D(a). \end{aligned}$$

Therefore, $D = f \circ d$. Now, we prove that such f is unique. First, observe that $\Omega_{A/k}$ has the A-module structure induced by multiplication by $a \otimes 1$ (or $1 \otimes a$ since $1 \otimes a - a \otimes 1 \in I$). Therefore, if $\xi = \sum x_i \otimes y_i \mod I^2 \in \Omega_{A/k}$ then $a\xi = \sum a x_i \otimes y_i \mod I^2$, and $f(a\xi) = \sum a x_i D(y_i) = a f(\xi)$, so that f is A-linear. We have

$$a \otimes a' = (a \otimes 1)(1 \otimes a' - a' \otimes 1) + aa' \otimes 1$$

so that if $\omega = \sum x_i \otimes y_i \in I$ then $\omega \mod I^2 = \sum x_i dy_i$ since $\sum x_i y_i = 0$. We conclude that $\{da \mid a \in A\}$ is a set of generators for the A-module $\Omega_{A/k}$. This implies uniqueness of f. Therefore, (1) holds. □

Definition 2.16. The module $\Omega_{A/k}$ introduced in the proof of the previous theorem is called **module of differentials** of A over k or **module of Kähler differentials**, and for $a \in A$ the element $da \in \Omega_{A/k}$ is called the **differential** of a.

Example 2.17. If A is generated as k-algebra by a subset $S \subseteq A$ then $\Omega_{A/k}$ is generated by $\{ds \mid s \in S\}$. Indeed, if $a \in A$ then there exist $a_i \in S$ and a polynomial $f(X) \in k[X_1, \dots, X_n]$ such that $a = f(a_1, \dots, a_n)$. Thus,

$$da = \sum_{i=1}^{n} f_i(a_1, \dots, a_n) da_i \quad \text{where} \quad f_i = \frac{\partial f}{\partial x_i}.$$

In particular, if $A = k[X_1, \dots, X_n]$ then $\Omega_{A/k} = AdX_1 + \dots AdX_n$ since $X_1, \dots, X_n$ are linearity independent; this follows from the fact that $\partial_i X_j = \delta_{ij}$.

Lemma 2.18. *Let K be a k-commutative algebra. The map $c\colon K \otimes_k K \to \Omega_{K/k}$ defined by $c(f \otimes g) = f dg$ satisfies:*

(i) c *is surjective.*

(ii) $\ker c$ *is generated over* k *by the elements of the form*

$$f \otimes gh - fg \otimes h - fh \otimes g$$

Proof. The k-bilinear map $(f,g) \mapsto f dg$ induces c. Since $\{df \mid f \in K\}$ is a generating set for $\Omega_{K/k}$ as a K-module it follows that c is surjective. For (b), observe that it is equivalent showing that $\ker(c)$ is generated over K by the elements of the form $1 \otimes gh - g \otimes h - h \otimes g$. Let A be the K-module generated by those elements. We wish to prove that

$$A \to K \otimes_k K \to \Omega_{K/k} \to 0$$

is exact. By left-exactness of Hom it is equivalent to prove that for all K-modules M the induced sequence

$$0 \to \mathrm{Hom}_K(\Omega_{K/k}, M) \to \mathrm{Hom}_K(K \otimes_k K, M) \to \mathrm{Hom}_K(A, M)$$

is exact. By Theorem 2.15 there is a canonical isomorphism $\mathrm{Hom}_K(\Omega_{K/k}, M) \cong \mathrm{Der}_k(K, M)$. Under this identification, we wish to prove that

$$0 \to \mathrm{Der}_k(K, M) \to \mathrm{Hom}_K(K \otimes_k K, M) \to \mathrm{Hom}_K(A, M)$$

is exact. Observe that the first map is given by $D \mapsto \varphi_D$ where $\varphi_D(f \otimes g) = fD(g)$. Note that the restriction $\varphi_D \colon A \to M$ is trivial. Indeed,

$$\varphi_D(1 \otimes gh - g \otimes h - h \otimes g) = D(gh) - gD(h) - hD(g) = 0$$

by the Leibniz rule. Now, let $\psi \in \mathrm{Hom}_K(K \otimes_k K, M)$ so that $\psi(A) = 0$. Let $D_\psi \colon K \to M$ be the k-derivation defined by $f \mapsto \psi(f \otimes 1)$. First, we prove that ψ_D is a k-derivation. Observe that k-linearity is obvious. Now, we prove the Leibniz rule for D_ψ. Consider

$$\begin{aligned} D_\psi(fg) = \psi(fg \otimes 1) &= \psi(f \otimes g + g \otimes f) \\ &= f\psi(1 \otimes g) + g\psi(1 \otimes f) \\ &= fD_\psi(f) + gD_\psi(g), \end{aligned}$$

where the third equality follows from the fact that ψ vanishes in A. Finally, it is clear that $\varphi_{D_\psi} = \psi$. □

Further theory to be included if necessary.

2.3 ABSTRACT RESIDUE AND ITS PROPERTIES

Throughout this section let k be a field, K a commutative k-algebra with 1, and V a K-module so that when viewed as a k-vector space it is a Tate space and K acts continuously on V. Namely, for all $f \in K$ the map

$$\begin{aligned} f\colon V &\to V \\ x &\mapsto fx \end{aligned}$$

is continuous. In this way, K operates on V through $\mathrm{End}_k(V)$ (maintaining notation from Section 2.1). We will not notationally distinguish $f \in K$ from its induced map in $\mathrm{End}_k(V)$.

Lemma 2.19. *Let $f, g \in K$. Then, there are $f_+, g_+ \in \mathrm{End}_+(V)$ so that*

$$f = f_+ \mod \mathrm{End}_-(V), \quad g = g_+ \mod \mathrm{End}_-(V)$$

and, the equality

$$\mathrm{tr}([f_+, g_+]) = \mathrm{tr}([f, g_+]) = \mathrm{tr}([f_+, g])$$

holds.

Proof. The existence of f_+ and g_+ is immediate from the fact that $\mathrm{End}_k(V) = \mathrm{End}_+(V) + \mathrm{End}_-(V)$. Clearly $[f_+, g_+] \in \mathrm{End}_+(V)$ and the fact that K is commutative implies that $[f, g] = 0$. Therefore,

$$[f_+, g_+] = [f, g] \mod \mathrm{End}_-(V).$$

Hence, $[f_+, g_+] \in E_0$. Similarly, $[f, g_+]$ and $[f_+, g]$ belong to $\mathrm{End}_0(V)$. Whence, one can consider their trace. Furthermore, $f_+ \in \mathrm{End}_+(V)$ and $g_+ - g \in \mathrm{End}_-(V)$ thus $\mathrm{tr}([f_+, g_+ - g]) = 0$ by Lemma 2.10; we conclude $\mathrm{tr}([f_+, g_+]) = \mathrm{tr}([f_+, g])$. The other equality follows similarly. $\square$

Notation 2.20. Lemma 2.19 implies that common values of traces $[f_+, g_+]$, $[f_+, g]$ and $[f, g_+]$ depend only on f and g and not in the choice of f_+ and g_+. Therefore, we will always denote $f_\pm$ to be elements in $\mathrm{End}_\pm(V)$ such that

$$f = f_+ \mod \mathrm{End}_-(V), \text{ and } f = f_- \mod \mathrm{End}_+(V).$$

Choices of $f_\pm$ are not unique, but for practical reasons we will not worry about those issues.

Lemma 2.19 implies that the assignment $(f, g) \mapsto \operatorname{tr}([f_+, g_+])$ is well-defined. Observe that this assignment k-bilinear by Proposition 2.7. Thus, there exists a map

$$\begin{aligned} r \colon K \otimes_k K &\to k \\ f \otimes g &\mapsto \operatorname{tr}([f_+, g_+]). \end{aligned}$$

With these tools at our hands we are ready to prove the existence of residue.

Theorem 2.21. *There exists a unique k-linear map*

$$\operatorname{res}_V \colon \Omega_{K/k} \to k$$

such that for each pair of elements $f, g \in K$ we have

$$\operatorname{res}_V(f dg) = \operatorname{tr}([f_+, g_+]).$$

Proof. Let $c \colon K \otimes_k K \to \Omega_{K/k}$ be as in Lemma 2.18. Then, since c is surjective, res_V if it exists it is uniquely determined by the commutativity of the following diagram

$$\begin{tikzcd} K \otimes_k K \arrow[r, "r"] \arrow[d, two heads, "c"] & k \\ \Omega_{K/k} \arrow[ur, dotted, "\operatorname{res}_V"'] & \end{tikzcd}$$

Therefore, such map exists if and only if it vanishes on $\ker c$. To see this, let f, g and h in K and choose f_+, g_+ and h_+ in $\operatorname{End}_+(V)$ coinciding with f, g and h modulo $\operatorname{End}_-(V)$ respectively. Then,

$$fg = f_+g_+ + (f_+g_- + f_-g_+ + f_-g_-),$$

and $f_+g_- + f_-g_+ + f_-g_- \in \operatorname{End}_-(V)$. Whence, $(fg)_+ = f_+g_+$. Analogously $(gh)_+ = g_+h_+$ and $(fh)_+ = f_+h_+$. This fact and the identify

$$[f_+, g_+h_+] - [f_+g_+, h_+] - [f_+h_+, g_+] = 0$$

imply the desired conclusion. □

Properties of residue

We prove some of the main properties of res.

Proposition 2.22. *For all $f, g \in K$ it follows that*

(a) $\operatorname{res}_V(fdg) + \operatorname{res}_V(gdf) = 0$, *and*

(b) $\operatorname{res}_V(df) = 0$.

Proof. Since $[f_+, g_+] + [g_+, f_+] = 0$ we get (a). For (b) use (a) with $g = 1$. □

Proposition 2.23. *Let W be a closed K-submodule of V. Then, for $\omega \in \Omega_{K/k}$ the identity*

$$\operatorname{res}_V(\omega) = \operatorname{res}_W(\omega) + \operatorname{res}_{V/W}(\omega)$$

holds.

Proof. It is enough to prove the claim for $\omega = fdg$. By Lemma 2.5 item (c) we only need to check that for all $f \in K$ the induced map $\overline{f}\colon V/W \to V/W$ and $f \circ \iota$, where ι denotes the inclusion $W \to V$, satisfy

$$\begin{aligned} \overline{f} &= \overline{f_+} \mod \operatorname{End}_-(V/W), \\ f \circ \iota &= f_+ \circ \iota \mod \operatorname{End}_-(W), \\ \overline{f_+} &\in \operatorname{End}_+(V/W), \quad \text{and} \\ f_+ \circ \iota &\in \operatorname{End}_+(W), \end{aligned}$$

These statements are straightforward to prove and we leave them as an exercise to the reader. □

Proposition 2.24. *If V is the direct sum of two closed submodules W_1 and W_2 then*

$$\operatorname{res}_V(\omega) = \operatorname{res}_{W_1}(\omega) + \operatorname{res}_{W_2}(\omega)$$

holds for all $\omega \in \Omega_{K/k}$.

Proof. Immediate from Proposition 2.23. □

If our Tate space is trivial so its residue.

Proposition 2.25. *If V is either linearly compact or discrete then $\operatorname{res}_V(\Omega_{K/k}) = 0$.*

Proof. If V is linearity compact then $\mathrm{End}_+(V) = E$ and $f_+ = f$ for all $f \in K$. Since $[f, g] = 0$ then

$$\mathrm{res}_V(fdg) = 0. \tag{2}$$

On the other hand, if V is discrete then $E = \mathrm{End}_-(V)$. Thus, $f = 0 \mod \mathrm{End}_-(V)$ for all $f \in K$. Thus, (2) holds. □

Proposition 2.26. *Let f and g belong to K. Then, if there exists a c-lattice L in V so that $fL + fgL + fg^2L \subseteq L$ it holds $\mathrm{res}_V(fdg) = 0$. In particular, when there exists L a c-lattice so that $fL \subseteq L$ and $gL \subseteq L$ then $\mathrm{res}_V(fdg) = 0$.*

Proof. Let π be a continuous projection from V to L. Then $\pi f \in \mathrm{End}_+(V)$ and $\pi f = f \mod \mathrm{End}_-(V)$. Thus, it follows that

$$\mathrm{res}_V(fdg) = \mathrm{tr}([\pi f, g])$$

by Lemma 2.19. Let $h = [\pi f, g]$ and $W = L + gL$. Let $h_{V/W}$ and h_W be the induced maps on V/W and W respectively. Then, the relation $fL + fgL + fg^2L \subseteq L$ implies that $h_{V/W} = 0$ and $h_W = 0$. By Lemma 2.5 item (c) we conclude

$$\mathrm{res}_V(fdg) = \mathrm{tr}_V(h) = \mathrm{tr}_W(h) + \mathrm{tr}_{V/W}(h) = 0. \qquad \square$$

In the following two propositions we examine the residue of a power.

Proposition 2.27. *Let $f \in K$, then $\mathrm{res}_V(f^n df) = 0$ for all $n \geq 0$. Moreover, if f is invertible the same holds for $n \leq -2$.*

Proof. First, if $f_+ = f \mod \mathrm{End}_-(V)$ then $f_+^n = f^n \mod \mathrm{End}_-(V)$. Therefore,

$$\mathrm{res}_V(f^n df) = \mathrm{tr}([f_+, f_+^n]) = 0.$$

Second, if f is invertible then

$$fd(f^{-1}) + f^{-1}df = d(ff^{-1}) = d(1) = 0.$$

which implies

$$f^{-2}df = -d(f^{-1}),$$

and multiplying by f^{-n} both sides, where $n \geq 0$, implies

$$f^{-2-n}df = -(f^{-1})^n d(f^{-1}).$$

By the preceding statement, $(f^{-1})^n d(f^{-1})$ has zero residue. □

Proposition 2.28. *If f is invertible, so that $fL \subseteq L$ for some c-lattice L, then*

$$\operatorname{res}_V(f^{-1}df) = \dim_k(L/fL).$$

Proof. If π is a continuous projection of V into L then

$$\operatorname{res}_V(f^{-1}df) = \operatorname{tr}([\pi f^{-1}, f]).$$

Let $g = [\pi f^{-1}, f]$. Since $fL \subseteq L$ we obtain

$$g_{V/L} = 0, \quad g_{L/fL} = 1 \quad \text{and,} \quad g_{fL} = 0,$$

where $g_{V/L}, g_{L/fL}$ and g_{fL} denote the induced maps in V/L, L/fL and, fL respectively. Then, by Lemma 2.5 item (c) it follows that

$$\operatorname{tr}_V(g) = \operatorname{tr}_L(g) + \operatorname{tr}_{V/L}(g) = \operatorname{tr}_{fL}(g) + \operatorname{tr}_{L/fL}(g) + \operatorname{tr}_{V/L}(g).$$

Observe that $\dim L/fL < \infty$ since fL is open and L is linearly compact. □

Relationship of residues under extensions

Finally, we explore the case where K' is a commutative k-algebra containing K. We will examine $\Omega_{K'/k}$ and $\Omega_{K/k}$ and the relationship between their residues. In this case the injection $K \to K'$ induces a map between $\Omega_{K/k} \to \Omega_{K'/k}$ which may not be injective.

Proposition 2.29. *Let V be a Tate space such that multiplication by any $f \in K'$ induces a continuous endomorphism in $\operatorname{End}_k(V)$. Therefore, for all $g \in K$ multiplication by g is continuous as well. Hence, we can define*

$$\operatorname{res}_V \colon \Omega_{K/k} \to k, \quad \textit{and} \quad \operatorname{res}'_V \colon \Omega_{K'/k} \to k.$$

In this situation, the diagram

$$\begin{tikzcd} \Omega_{K/k} \arrow[r] \arrow[dr, "\operatorname{res}_V"'] & \Omega_{K'/k} \arrow[d, "\operatorname{res}'_V"] \\ & k \end{tikzcd}$$

commutes.

Proof. For $f, g \in K$ their residue symbol is independent whether $f\,dg$ is thought as an element in $\Omega_{K'/k}$ or $\Omega_{K/k}$. This observation implies the commutativity of the diagram. □

Now, assume that K' is free K-module of finite rank n and consider the tensor product $V' = K' \otimes_K V$. Since the tensor product and direct sum commute, it follows that $V' \cong K^n \otimes_K V \cong (K \otimes_K V)^n \cong V^n$. In coordinates, if (x_i) is a K-base for K' then the map $(v_1, \ldots, v_n) \mapsto x_1 \otimes v_1 + \ldots + x_n \otimes v_n$ is an isomorphism. With the topology induced by this isomorphism V' is a Tate space.

Proposition 2.30. *The space* $\mathrm{End}(V')$ *is isomorphic to the space of* $n \times n$ *matrices with entries in* $\mathrm{End}(V)$ *denoted* $\mathrm{M}_n(\mathrm{End}_0(V))$*. Moreover, if K acts continuously on V so does K′ on V′.*

Proof. Let φ be a continuous k-endomorphism of V', then there exists a unique set $\{\varphi_{ij}\}_{i,j=1}^n$ contained in $\mathrm{End}(V)$ such that

$$\varphi\left(\sum_i x_i \otimes v_i\right) = \sum_{i,j} x_i \otimes \varphi_{ij}(v_j)$$

for all $v_1, \ldots, v_n \in V$. Now, let $f' \in K'$, then

$$f' x_i = \sum f_{ij} x_j$$

where $f_{ij} \in K$. Since $f_{ij} \in \mathrm{End}(V)$ it follows that $f' \in \mathrm{End}(V')$ by the description of our topology in V'. □

Let $\mathrm{End}_0'(V')$ be the inverse image of $\mathrm{M}_n(\mathrm{End}_0(V))$ under the isomorphism in Proposition 2.30. Note that $\mathrm{End}_0'(V') \subseteq \mathrm{End}_0(V')$. Therefore, the map

$$\mathrm{tr}_{V'} \colon \mathrm{End}_0'(V') \to k$$

is well-defined.

Proposition 2.31. *For* $\varphi \in \mathrm{End}_0'(V')$ *the identity*

$$\mathrm{tr}_{V'}(\varphi) = \sum_i \mathrm{tr}_V(\varphi_{ii})$$

holds.

Proof. Write (φ_{ij}) as a sum of a strictly lower triangular, strictly upper triangular and diagonal matrix. Namely,

$$\varphi = \varphi_{LT} + \varphi_{UT} + \varphi_D,$$

where φ_{LT}, φ_{UT} and φ_D have a matrix representation of a strictly lower, strictly upper and diagonal matrix respectively. Observe that φ_{LT}, φ_{UT},

φ_D belong to $\mathrm{End}_0'(V')$ and φ_{LT} and φ_{UT} are nilpotent. By Lemma 2.5 it follows that

$$\mathrm{tr}_{V'}(\varphi) = \mathrm{tr}_{V'}(\varphi_D).$$

On the other hand, by definition

$$\mathrm{tr}_{V'}(\varphi_D) = \sum \mathrm{tr}_V(\varphi_{ii}).$$

□

Theorem 2.32. *For all $f' \in K'$ and $g \in K$ the equality*

$$\mathrm{res}_V'(f'dg) = \mathrm{res}_V((\mathrm{tr}_{K'/K}(f')dg))$$

holds.

Proof. Let L be a c-lattice in V then $L' = x_1 \otimes L + \ldots + x_n \otimes L$ is a c-lattice in V'. Let $\pi\colon V \to L$ be a linear continuous projection and π' be the corresponding element to $(\delta_{ij}\pi)$ under the isomorphism $\mathrm{End}(V') \cong \mathrm{M}_n(\mathrm{End}(V))$. Therefore, $\pi'\colon V' \to L'$ is a linear continuous projection. On the other hand, let $f' \in K'$ and $g \in K$. Then, f' corresponds to $(f_{ij}) \in \mathrm{M}_n(K)$ and let g' be the corresponding element to $(\delta_{ij}g)$ in $\mathrm{End}(V')$. Hence, the commutator $[\pi'f', g']$ is mapped to $[\pi f_{ij}, g]$ by the map $\mathrm{End}(V') \to \mathrm{M}_n(\mathrm{End}(V))$. By Proposition 2.31, it follows that

$$\begin{aligned}
\mathrm{res}_{V'}(f'dg) &= \mathrm{tr}_{V'}([\pi'f', g']) \\
&= \sum \mathrm{tr}_V([\pi f_{ii}, g]) \\
&= \sum \mathrm{res}_V(f_{ii}dg) \\
&= \mathrm{res}_V\left(\left(\sum f_{ii}\right)dg\right) \\
&= \mathrm{res}_V\left(\mathrm{tr}_{K'/K}(f')dg\right).
\end{aligned}$$

3 ALGEBRAIC CURVES

In the preceding chapter we presented the “residue map” in an abstract context. In this chapter we explore residues on algebraic curves using Tate’s construction. First, we recall briefly the basic theory of algebraic projective curves. We reference the reader to [BP02].

3.1 BASIC THEORY OF ALGEBRAIC CURVES

Let k be an algebraically closed field.

Definition 3.1. An **algebraic curve** is a one-dimensional non-singular projective variety.

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