TATE'S LINEAR ALGEBRA AND RESIDUES ON CURVES

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A dissertation submitted in partial satisfaction of the requirements for the degree of Mathematician

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Noviembre 2018

Dedicada a mamá y papá

ACKNOWLEDGMENTS

I express my heart-felt gratitude to Professor Paul Bressler for his guidance not only during my thesis, but during all my undergraduate education. I have learned in many ways from him and I am so happy that I'd chose him as my advisor. I thank all the members of the seminar *Topics in Differential Calculus on Manifolds* for allowing me to share my work during some sessions of the seminar. All their questions, comments and suggestions clarified me many aspects of the theory.

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INTRODUCTION

Let X be a Riemann surface and ω a meromorphic differential on X. For a point $p \in X$ and a choice of a local coordinate z near p we may write $\omega = f(z)\,dz$. The residue $\mathrm{res}_p(\omega)$ of ω at the point p is defined as the coefficient of z^{-1} in the Laurent expansion of f.

One would like to extend this notion for any projective smooth curve $(X, \mathcal{O}_X)$ over a field k. Let $K = k(X)$ denote the function field of X. For a point $p \in X$ let $\widehat{\mathcal{O}}_{X,p}$ denote the completion of the local ring $\mathcal{O}_{X,p}$ (with respect to the $\mathfrak{m}_p$-topology where $\mathfrak{m}_p$ is the maximal ideal of $\mathcal{O}_{X,p}$). Let K_p denote the field of fractions of $\widehat{\mathcal{O}}_{X,p}$. Since X is smooth, there exists a local uniformizing parameter t at p such that $\widehat{\mathcal{O}}_{X,p} \simeq k[[t]]$ and $K_p \simeq k((t))$. Thus, we could imitate the "analytic" approach and define for $\omega = f\,dg$, where expansions of f and g are given by

$$f = \sum_{\nu \gg -\infty} a_\nu t^\nu, \quad \text{and} \quad g = \sum_{\mu \gg -\infty} b_\mu t^\mu,$$

the residue of ω at p to be

$$\mathrm{res}_p(\omega) = \text{coefficient of } t^{-1} \text{ in } f(t)g'(t) = \sum_{\nu+\mu=0} \mu a_\nu b_\mu.$$

In this case, it is not immediate from this is definition that the coefficient in question is independent of the uniformizing parameter t (in particular, when $\mathrm{char}(k) \neq 0$).

In [Tat68], John Tate proposed a new coordinate-free approach to residues in terms of traces of certain operators on infinite dimensional vector spaces which coincides with the construction described above. In this new setting, all classical theorems on residues (e.g. the sum of residues over a complete curve on closed points is zero) follow fairly easily from Tate's construction. Moreover, all constructions and proofs are independent of the characteristic of the ground field.

Tate formulated his theory in terms of "linear algebra modulo finite dimensional vector spaces". The linear algebra in question is that of so called *Tate vector spaces* (terminology most likely due to Alexander

Beilinson). A Tate vector space (shortened Tate Space) is a topological vector space over the base field k with properties abstracting those of (a finite dimensional vector space over) the field of Laurent series $k((t))$ endowed with t-adic topology. In his article, what Tate implicitly discovered is that the Lie algebra (under the commutator bracket) of continuous k-linear endomorphisms $\mathfrak{gl}_{\text{cont}}(V)$ of a Tate space V admits a canonical central extension $\widehat{\mathfrak{gl}_{\text{cont}}(V)}$ by k:

$$0 \to k \to \widehat{\mathfrak{gl}_{\text{cont}}(V)} \to \mathfrak{gl}_{\text{cont}}(V) \to 0. \tag{1}$$

Many well known central extensions of Lie algebras, such as Kac-Moody extensions of loop algebras and the Virasoro extension are obtained by pulling back Tate's extension (1) along a suitable map to $\mathfrak{gl}_{\text{cont}}(V)$.

In this document we explore Tate's construction in the language of Tate spaces. In Chapter 1 we discuss linear topologies over vector spaces and their properties. This leads to the definition of Tate spaces. We discuss duality and morphisms of Tate spaces. In Chapter 2 we define the notion of abstract residue using Tate spaces. In order to do this, we extend the definition of trace to certain infinite rank endomorphisms on Tate spaces. Finally, in Chapter 3 we discuss the many applications of residue on projective smooth curves. In particular, we prove the residue theorem and the Riemann-Roch formula for projective smooth curves over an algebraically closed field.

1

TATE'S LINEAR ALGEBRA

In this chapter we explore linear topologies on vector spaces in order to introduce Tate spaces and their structure. Tate spaces will be central in the definition of abstract residues in Chapter 2 and the study of algebraic curves in Chapter 3. We follow definitions in [BD04] closely but not religiously.

1.1 LINEAR TOPOLOGIES

Let k be a field. From now on, a vector space will always mean a k-vector space.

Definition 1.1. A **linear topology** on a vector space V is a separated (Hausdorff) topology, which is invariant under translations and admits a base of open neighborhoods of zero consisting of vector subspaces. A vector space equipped with a linear topology will be referred as **linearly topologized**.

If we endow k with the discrete topology, then V becomes a topological vector space. From now on, endow k with the discrete topology.
Linear topologies behave nicely under basic topological operations.

Theorem 1.2. *Let V be a linearly topologized vector space.*

(a) If $W \subseteq V$ is a vector subspace, then W is linearly topologized as well.

(b) If $W \subseteq V$ is a closed vector subspace, then V/W is linearly topologized under its quotient topology.

(c) If $\{V_\alpha\}_\alpha$ is a collection of linearly topologized vector spaces, then its product $\prod_\alpha V_\alpha$ (in its product topology) and its direct sum $\bigoplus_\alpha V_\alpha$ (as a subspace of the product) are linearly topologized.

(d) If W is a vector subspace of V, then its topological closure $\overline{W}$ also is a vector subspace of V.

Proof. If $\mathscr{U}$ is a system of neighborhoods around zero consisting of vector subspaces in V, then $\{U \cap W \mid U \in \mathscr{U}\}$ is a system of neighborhoods around zero consisting of vector subspaces of W. For (b), let $\pi\colon V \to V/W$ be the quotient map. Since π is open and surjective, the image of a local base is a local base; moreover, the image of a vector subspace under π is a vector subspace. In addition, since W is closed, it follows that V/W is Hausdorff. Now, for (c), let $\{U_{\alpha,\beta}\}_\beta$ be a local base of zero in V_α of vector subspaces. Then the products $U_{\alpha_1,\beta_1} \times \ldots \times U_{\alpha_n,\beta_n} \times \prod_\gamma V_\gamma$, where γ ranges over $\alpha \neq \alpha_1, \ldots, \alpha_n$, for any set $\{(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)\}$ form a fundamental system of neighborhoods around zero in $\prod_\alpha V_\alpha$ of open vector subspaces. Since $\bigoplus_\alpha V_\alpha \subseteq \prod_\alpha V_\alpha$ is a vector subspace, (c) follows from (a). Finally, for (d), suppose $x, y \in \overline{W}$. Then for every open vector subspace U it follows that $(x + U) \cap W \neq \emptyset$ and $(y + U) \cap W \neq \emptyset$. Therefore, for every $\alpha, \beta \in k$ we have $(\alpha x + U) \cap W \neq \emptyset$ and $(\beta y + U) \cap W \neq \emptyset$. Hence, $(\alpha x + \beta y + U) \cap W \neq \emptyset$ for every open vector subspace U and every pair $\alpha, \beta \in k$. □

Corollary 1.3. *Let* LinTop_k *denote the category of linearly topologized vector spaces over k, where morphisms are given by continuous linear homomorphisms. Then limits exist in* LinTop_k.

Proof. By Theorem 1.2, it follows that kernels and arbitrary products exist in LinTop_k; therefore, limits exist in LinTop_k. □

Proposition 1.4. *Let* $W \subseteq V$ *be an open vector subspace of* V. *Then* W *is closed.*

Proof. Observe that
$$V - W = \bigcup_{x \notin W} W + x.$$
Since W is open, every translation $W + x$ is also open. Then W is closed. □

Proposition 1.5. *A finite dimensional linearly topologized vector space* V *is discrete.*

Proof. Since V is Hausdorff, it follows that
$$\bigcap_{U \in \mathscr{U}} U = \{0\},$$

for $\mathscr{U}$ a system of neighborhoods of zero consisting of vector subspaces of V. Since V is finite dimensional, there exist $U_1, \ldots, U_n \in \mathscr{U}$ such that

$$\bigcap_{U \in \mathscr{U}} U = U_1 \cap \ldots \cap U_n = \{0\}.$$

Therefore, $\{0\}$ is open. This implies that V is separated. $\square$

Proposition 1.6. *Let V be a linearly topologized vector space and W a vector subspace of V. Then*

$$\overline{W} = \bigcap_{U \in \mathscr{U}} (W + U).$$

for a system of neighborhoods of zero $\mathscr{U}$ consisting of vector subspaces of V.

Proof. Since every $W + U$ is open, it is also closed. Hence their intersection is closed. If $x \in V - \overline{W}$, then $(x + U) \cap W = \{0\}$ for some $U \in \mathscr{U}$. Hence, $x \notin W + U$. This proves the proposition. $\square$

1.1.1 *Commensurability*

We introduce a partial order on the set of vector subspaces of a vector space V.

Definition 1.7. For vector subspaces A and B of a vector space V we say that $A \prec B$ if the quotient $A/(A \cap B) \cong (A + B)/B$ is finite dimensional (or equivalently $A \subseteq B + W$ for some finite dimensional W). In addition, we say that A and B are **commensurable** (denoted $A \sim B$) if $A \prec B$ and $B \prec A$.

Observe that $A \sim B$ if and only if $(A + B)/(A \cap B) \cong A/(A \cap B) \oplus B/(A \cap B)$ is finite dimensional. To shorten notation, we will constantly refer to a vector space V being finite dimensional as $V \sim 0$.

Proposition 1.8. *Let V be a vector space and A, B and C be vector subspaces.*

(a) If $A \sim B$ and $B \sim C$, then

$$\frac{A + B + C}{A \cap B \cap C} \sim 0.$$

(b) If $A \sim B$ and $B \sim C$, then $A \sim C$. Moreover, commensurability is an equivalence relation.

Proof. Consider the following exact sequences

$$0 \to \frac{A \cap B}{A \cap B \cap C} \to \frac{B}{B \cap C},$$

and,

$$0 \to \frac{A \cap B}{A \cap B \cap C} \to \frac{A + B}{A \cap B \cap C} \to \frac{A + B}{A \cap B} \to 0$$

induced by inclusions. The first inclusion plus the fact that $B \sim C$ imply that $(A \cap B)/(A \cap B \cap C)$ is finite dimensional. Now, since $A \sim B$, $(A + B)/(A \cap B)$ is finite dimensional. Therefore, using the second exact sequence, we conclude that $(A + B)/(A \cap B \cap C)$ is finite dimensional. A symmetrical argument shows that $(B + C)/(A \cap B \cap C) \sim 0$. These prove (a). For (b), the inclusion

$$0 \to \frac{A + C}{A \cap C} \to \frac{A + B + C}{A \cap B \cap C}$$

plus (a) implies transitivity. □

Now, we state and prove some useful properties on the relation $\prec$.

Lemma 1.9. *Let V be a vector space and A, B vector subspaces of V.*

(a) If $A \subseteq B$, then $A \prec B$.

(b) If $A \prec B$, then $f(A) \prec f(B)$ for any k-linear map f

(c) Let $\{A_i\}_{i=1}^m$ and $\{B_j\}_{j=1}^n$ be two collections of vector subspaces of V. Then,

$$\sum_{i=1}^{m} A_i \prec \bigcap_{j=1}^{n} B_j \iff A_i \prec B_j \text{ for all } i \text{ and } j.$$

Proof. First, (a) is immediate from the definition of $\prec$. Second, for (b, the map f factors as

$$A/(A \cap B) \to f(A)/(f(A) \cap f(B)) \to 0.$$

Finally, for (c), if $\sum_{i=1}^m A_i \prec \bigcap_{j=1}^n B_j$ holds, then by (a) above, for all i and j we have

$$A_i \prec \sum_{i=1}^{m} A_i \prec \bigcap_{j=1}^{n} B_j \prec B_j.$$

On the other hand, if $A_i \prec B_j$ for all i and j, then there exist finite dimensional subspaces W_{ij} such that $A_i \subseteq B_j + W_{ij}$ for all i and j. Therefore,

$$\sum_{i=1}^{m} A_i \subseteq \bigcap_{j=1}^{n} B_j + \sum_{i=1}^{m}\sum_{j=1}^{n} W_{ij}.$$

□

Next, we consider another useful lemma.

Lemma 1.10. *Let A, B, A', B' be vector subspaces of a vector space V and suppose that $A \sim A'$ and $B \sim B'$. Then $A + B \sim A' + B'$ and $A \cap B \sim A' \cap B'$.*

Proof. The following exact sequence

$$0 \to \frac{A + A' + B + B'}{A \cap A' \cap B \cap B'} \to \frac{A + A'}{A \cap A'} \oplus \frac{B + B'}{B \cap B'} \to \frac{A + A' + B + B'}{(A \cap A') + (B \cap B')} \to 0$$

plus $A \sim A'$ and $B \sim B'$ imply that both spaces

$$\frac{A + A' + B + B'}{A \cap A' \cap B \cap B'} \quad \text{and,} \quad \frac{A + A' + B + B'}{(A \cap A') + (B \cap B')}$$

are finite dimensional. Since $(A + A' + B + B')/(A + A') \cap (B + B')$ is a quotient of the second space and $((A \cap A') + (B \cap B'))/((A \cap A') \cap (B \cap B'))$ is a subspace of the first space, we can conclude $A + B \sim A' + B'$ and $A \cap B \sim A' \cap B'$. □

Therefore, the set of equivalence classes of $\sim$ is partially ordered by $\prec$. Moreover, by Lemma 1.10, it inherits operations $\cap$ and $+$.

1.1.2 *Linear compactness*

Definition 1.11. Let V be a linearly topologized vector space. A closed vector subspace $L \subseteq V$ is **linearly compact** (respectively **linearly cocompact**) if for every open vector subspace U we have $L \prec U$ (respectively $V/(L + U) \sim 0$).

Remark 1.12. Linear compactness was introduced by S. Lefschetz in his influential [Lef42] using different terms. He defined a coset of V (also called linear variety) to be a set $x + W$ where $x \in V$ and W is a subspace V. Then, he defined a linearly compact vector space to be a linearly topologized vector space V such that for every collection of closed cosets X_α having the finite intersection property, it follows that $\bigcap_\alpha X_\alpha \neq \emptyset$. In these terms, linear compactness seems like a natural generalization of compactness for linearly topologized vector spaces. We extend this discussion in Remark 1.27.

Linear compactness behaves just as compactness if one uses the correct words.

Theorem 1.13. *Let V be a linearly compact vector space.*

(a) If $A \subseteq V$ is a vector subspace satisfying $A \prec U$ for all open vector subspaces U of V, then $\overline{A}$ is linearly compact.

(b) If $f\colon V \to W$ is a continuous linear homomorphism, then $\overline{f(V)}$ is linearly compact.

(c) If V is discrete, then $V \sim 0$.

(d) Every closed vector subspace of V is linearly compact.

(e) (Tychonov) If $\{V_\alpha\}_\alpha$ is a collection of linearly compact vector spaces, then its product $\prod_\alpha V_\alpha$ and its direct sum $\bigoplus_\alpha V_\alpha$ are linearly compact.

Proof. Let U be any open vector subspace of V. Then $A + U$ is closed, that is $A + U = \overline{A + U} \supseteq \overline{A} + U \supseteq A + U$. Hence, $\overline{A} + U = A + U$. Since $(A + U)/U \sim 0$, it follows that $(\overline{A} + U)/U$ is finite dimensional.

For (b), since f is a continuous linear map, it follows that $V \prec f^{-1}(U)$ for all U open vector subspace of W. Hence, by Lemma 1.9 $f(V) \prec U$ for all open vector subspaces U of W. The previous observation and (b) yield (a). If V is discrete, then $\{0\}$ is an open vector subspace of V; therefore, V is finite dimensional.

For (d), if $A \subseteq V$ is a closed vector subspace, and $V \prec U$ for all open vector subspaces U by Lemma 1.9, we get $A \prec U$.

Finally, for (e), it is enough to prove the statement for open vector subspaces of the form $U = \prod_\beta U_\beta \times \prod_\gamma V_\gamma$, where β ranges over a finite set, γ ranges over $\alpha \neq \beta$ and U_β is an open vector subspace of V_β. In this case, the quotient

$$\prod_\alpha V_\alpha / U \cong \prod_\beta V_\beta / U_\beta,$$

where $\cong$ is a topological and algebraic isomorphism. Since V_α is linearly compact for all α and β ranges over a finite set, we conclude that $\prod_\alpha V_\alpha / U$ is finite dimensional; therefore, $\prod_\alpha V_\alpha$ is linearly compact. The proof is analogous for $\bigoplus_\alpha V_\alpha$. $\square$

1.1.3 *Completion*

In this subsection we expose with little detail the properties of completion. A general reference for completion of arbitrary modules over a commutative ring is [Mat86] Section 8.

Definition 1.14. Let V a linearly topologized vector space. Let $\mathscr{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ be a system of neighborhoods of zero consisting of open vector subspaces of V indexed by Λ, so that $\lambda < \mu \iff U_\lambda \supseteq U_\mu$. Then, the inverse limit

$$\widehat{V} = \varprojlim_{U\in\mathscr{U}} V/U = \varprojlim_{\lambda\in\Lambda} V/U_\lambda$$

is the **completion** of V. Observe that existence of such limit is guaranteed by Corollary 1.3. Recall that $\widehat{V}$ can be described as a subspace of the product by

$$\widehat{V} = \{(\overline{x_\lambda})_\lambda \in \prod_{\lambda\in\Lambda} V/U_\lambda \colon \overline{x_\mu} = \pi^\lambda_\mu(\overline{x_\lambda}) \text{ for all } \mu \leq \lambda\},$$

where $\pi^\lambda_\mu \colon V/U_\lambda \to V/U_\mu$ is the induced map by the projection $\pi_\mu \colon V \to V/U_\mu$. In this way $\widehat{V}$ carries a natural topology as a subspace of the product $\prod_{U\in\mathscr{U}} V/U$. There is always a natural map

$$V \to \widehat{V}$$

induced by the quotient maps π_μ. This map is continuous and its image is dense in $\widehat{V}$. If the map $V \to \widehat{V}$ is a topological isomorphism we say that V is **complete**. Let $\mathscr{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ and $\mathscr{U}' = \{U'_{\lambda'}\}_{\lambda'\in\Lambda}$ be two different systems of neighborhoods consisting of vector subspaces of V. Then for all $\lambda \in \Lambda$ there exists $\lambda' \in \Lambda'$ such that $U_{\lambda'} \subseteq U_\lambda$ and for all $\mu' \in \Lambda'$ there exists $\mu \in \Lambda$ such that $U_\mu \subseteq U_{\mu'}$. This implies that $\varprojlim_{U\in\mathscr{U}} V/U$ and $\varprojlim_{U'\in\mathscr{U}'} V/U'$ satisfy the same universal property. Therefore, they are canonically isomorphic. In other words, the completion does not depend on the choice of filtration $\mathscr{U}$.

1.2 TATE SPACES

1.2.1 *Lattices*

Definition 1.15. If V is a linearly topologized vector space we say that a **c-lattice** is an open linearly compact subspace of V. *Dually*, a discrete linearly cocompact subspace is a **d-lattice**.

First, we prove that existence of a c-lattice in a linearly topologized vector space is equivalent to existence of a d-lattice.

Proposition 1.16. *A linearly topologized vector space V contains a c-lattice if and only if it contains a d-lattice.*

Proof. Suppose L is a c-lattice in V. Choose any direct complement D of L, that is, $V = L \oplus D$. Since L is open, D is discrete. Moreover, D is closed because it is the kernel of the projection $V \to L$ (which is continuous because L is open). Finally, we check that D is linearly co-compact: let U be any open vector subspace of V, then the composition $L \hookrightarrow V \twoheadrightarrow V/(D+U)$ induces a surjection

$$L/(L \cap U) \twoheadrightarrow V/(D+U).$$

Since $\dim L/(L \cap U) < \infty$, we conclude that $\dim V/(D+U) < \infty$. Now, suppose D is a d-lattice. Thus, there exists an open vector subspace U such that $U \cap D = 0$. This time choose L a direct complement for D containing U. Then the projection $V \to D$ is continuous because U is mapped to zero. Therefore, L is open. Now, we prove that L is linearly compact. Let U be any open vector subspace. Then the composition $V \twoheadrightarrow L \twoheadrightarrow L/(L \cap U)$ induces a surjection

$$V/(D + (L \cap U)) \twoheadrightarrow L/(L \cap U).$$

Since both L and U are open, so is $L \cap U$. Therefore, $\dim V/(D + (L \cap U)) < \infty$. It follows that $\dim L/(L \cap U) < \infty$ and L is linearly compact. □

Remark 1.17. Note that in the proof of Proposition 1.16 it is not strictly necessary to choose a direct complement. One can choose a direct complement up to finite dimension; that is, $L + D \sim V$ and $L \cap D \sim 0$.

Proposition 1.18. *If V admits a c-lattice, then the set of c-lattices constitutes a system of neighborhoods of zero consisting of mutually commensurable vector subspaces.*

Proof. If L and L' are two c-lattices in V, then $L \prec L'$ and $L' \prec L$ because both are open. Therefore, all c-lattices are commensurable. Moreover, if U is any open vector subspace and L is a c-lattice, we claim that $L \cap U$ is a c-lattice as well. Indeed, let U' be any open vector subspace, then $L \cap U \prec L \prec U'$. In addition, since L and U are open, so is $L \cap U$. Hence, $L \cap U \subseteq U$ is a c-lattice. This proves the statement. □

We are now ready to introduce the definition of a Tate space.

Definition 1.19. A linearly topologized vector space V is a **Tate space** if it is complete and admits a c-lattice. By Proposition 1.18 and the observation in Definition 1.14, it follows that

$$V \cong \varprojlim_{L \in \mathscr{U}} V/L,$$

where $\mathscr{U}$ runs through all c-lattices of V.

Example 1.20. We give some examples of Tate spaces.

(a) Any vector space endowed with the discrete topology is a Tate space.

(b) Let $\{V_\alpha\}_\alpha$ be any projective system of finite dimensional vector spaces (thus, each one endowed with the discrete topology by Proposition 1.5). Let V be their inverse limit endowed with the inverse limit topology. We claim that this is a linearly compact space. Indeed, if we realize V as a subspace of the product $\prod_\alpha V_\alpha$, then a basic open vector subspace in V is a subspace of the form

$$\{(x_\alpha)_\alpha \in V \colon x_{\alpha_1} = x_{\alpha_2} = \cdots = x_{\alpha_n} = 0\}$$

for some finite collection of indices $\alpha_1, \ldots, \alpha_n$. Hence, the quotient of V by any basic open vector subspace is a vector subspace of a finite product of V_α. Since all V_α are finite dimensional, we conclude that V is linearly compact. Observe that a system of neighborhoods consisting of vector subspaces of V is the collection of kernels of the projections $V \to V_\alpha$. Therefore, V is complete. Moreover, it is straightforward to prove that every complete linearly compact space arises in this way.

(c) Let $V = k((t))$ with the topology generated by letting $t^n k[[t]]$ for $n \in \mathbb{Z}$ be a system of neighborhoods of zero. Then $V = k[[t]] \oplus tk[t^{-1}]$ where $k[[t]]$ is the completion of $k[x]$ in the $\langle x \rangle$-adic topology. Hence, by the previous item, $k[[t]]$ is linearly compact and, since it is open, it is a c-lattice. By the argument given in Proposition 1.16, $tk[t^{-1}]$ is a d-lattice. Therefore, V is a Tate space that is neither linearly compact nor discrete.

A closer look at the previous examples motivates the following proposition:

Proposition 1.21. *A linearly topologized vector space V is a Tate space if and only if it there exists a collection $\mathscr{U}$ in V of mutually commensurable open vector subspaces in V such that the natural map*

$$V \to \varprojlim_{U \in \mathscr{U}} V/U$$

is a topological isomorphism. In particular, every Tate space arises in the following way: Let V be a k-vector space endowed with a collection $\mathscr{U}$ of vector subspaces satisfying the following conditions:

(i) *$\mathscr{U}$ filters down to 0 and up to V.*

(ii) *Every two subspaces in $\mathscr{U}$ are mutually commensurable.*

(iii) *The natural map*

$$V \to \varprojlim_{U \in \mathscr{U}} V/U$$

is an isomorphism. Then, $(V, \mathscr{U})$ becomes a Tate space by imposing (iii) to be a topological isomorphism letting the quotient V/U be discrete for all $U \in \mathscr{U}$.

Proof. By the observation in Definition 1.19, such collection is simply the collection of c-lattices in V. Now, suppose that such collection exists in V. Then $\mathscr{U}$ is a system of neighborhoods of zero consisting of mutually commensurable open vector subspaces in V, any $U \in \mathscr{U}$ is a c-lattice and V is complete. □

1.2.2 *Duality*

Let V be a Tate space and consider the following topology on the dual space V^* (where by dual space we mean topological dual). We let the subspaces of the form

$$L^\perp = \{\phi \in V^* : \phi|_L = 0\},$$

where L is a linearly compact subspace, to be a system of neighborhoods of zero. Equivalently, one can define open vector subspaces in V^* to be sets of the form D^* where D is a direct complement of a linearly compact vector subspace L in V (in this case $D^* \hookrightarrow V^*$ using the decomposition $L \oplus D$).
First, we prove that the word *dually* in Definition 1.11 actually makes sense.

Lemma 1.22. *Duality interchanges linearly compact with discrete spaces and vice-versa.*

Proof. If L is a linearly compact vector space, then $L^{\perp}$ is open in L^*. Thus, L^* is discrete. If D is discrete, then $D \cong k^{\oplus \Lambda}$ for some Λ and endowing $k^{\oplus \Lambda}$ with the discrete topology. Moreover, since D is discrete, every linear functional is continuous. Using Corollary 1.3 and the well known identity (where maps are isomorphisms in LinTop_k)

$$(k^{\oplus \Lambda})^* = \mathrm{Hom}_k(k^{\oplus \Lambda}, k) \cong \prod_{\Lambda} \mathrm{Hom}_k(k, k) \cong \prod_{\Lambda} k,$$

we get the desired result by Tychonov's theorem in Theorem 1.13. □

Remark 1.23. A closer look at the proof of the previous lemma indicates that the dual space of a discrete space is always complete.

Proposition 1.24. *If V is a Tate space, then V^* is also a Tate space.*

Proof. If we decompose $V = L \oplus D$, where L is a c-lattice and D a d-lattice, then $V^* \cong L^* \oplus D^*$. Therefore, by Lemma 1.22, L^* is discrete and D^* is linearly compact. Observe that D^* is open in V^* because it is the kernel of the projection $V^* \to V^*/L^{\perp}$ and $V^*/L^{\perp}$ is discrete by the description of our topology in the dual V^*. Since L^* is discrete, it is complete. Moreover, by Remark 1.23, D^* is complete. Hence, V^* is complete. □

We are now ready to prove the duality theorem for Tate spaces.

Theorem 1.25. *For a Tate space V the canonical map $V \to V^{**}$ is an isomorphism.*

Proof. It is enough to prove it for complete linearly compact spaces and discrete spaces, as every Tate space can be decomposed into a direct sum of a c-lattice and a d-lattice. First, suppose D is a discrete vector space. Then the canonical map

$$\mathrm{ev}\colon D \to D^{**}$$

is open and continuous because D and D^{**} are both discrete by Lemma 1.22. Moreover, it is injective. Indeed, for every nonzero $x \in D$ there exists a linear continuous functional $\phi \in D^*$ such that $\phi(x) \neq 0$. Finally, we prove surjectivity. Let $\psi \in D^{**}$. Since $\ker \psi$ is open, it contains a basic open vector subspace $A^\perp$ such that $A \subseteq D$ is a linearly compact subspace. Since D^* is linearly compact, it follows that $D^* \sim A^\perp$, that is, the quotient $D^*/A^\perp$ is finite dimensional. Recall that the inclusion $\iota\colon A \to D$ induces an isomorphism $D^*/A^\perp \to A^*$ which is a homeomorphism since both spaces are discrete. We can factor ψ so that the following diagram commutes

$$\begin{tikzcd}
D^* \arrow[r, "\psi"] \arrow[d] & k \\
D^*/A^\perp \arrow[ur, dashed, "\tilde{\psi}"] \arrow[d, "\cong"'] & \\
A^* \arrow[uur, dashed, bend right, "\overline{\psi}"'] &
\end{tikzcd}$$

However, A^* is finite dimensional, therefore, there exists $a \in A$ such that $\overline{\psi} = \mathrm{ev}_a$ as maps $A^* \to k$. Moreover, since $A^\perp \subseteq \ker \psi$, we conclude that $\psi = \mathrm{ev}_a$ as maps $D^* \to k$. This implies surjectivity. Thus, $D \to D^{**}$ is an isomorphism of topological vector spaces.

Now, suppose L is a complete linearly compact space. We claim that the map

$$\mathrm{ev}\colon L \to L^{**}$$

is continuous. Indeed, let $A^\perp$ be an open vector subspace in L^{**}, where $A \subseteq L^*$ is a linearly compact subspace. By Lemma 1.22, L^* is discrete. Hence, A is finite dimensional. Suppose that $A = \mathrm{span}(\phi_1, \ldots, \phi_n)$ for some $\phi_1, \ldots, \phi_n \in A$. Then $\mathrm{ev}^{-1}(A^\perp) = \ker \phi_1 \cap \ldots \cap \ker \phi_n$, which is open in L. Now, we check that ev is injective. Let $v \in L$ be a nonzero vector. Choose a decomposition of $L = U \oplus F$ where U is open and F is finite dimensional containing v (this can be done because L is separated and linearly compact). Let ϕ be a linear functional such that $\phi|_U = 0$ and $\phi(v) \neq 0$. Since U is open and F discrete such ϕ exists and it is continuous. Now, we check that ev is surjective. Let $\mathscr{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ be a system of neighborhoods consisting of open vector subspaces. Let $\psi\colon L^* \to k$ be a continuous linear functional. The dual map of $\pi_\lambda\colon L \to L/U_\lambda$ yields an injection $\pi_\lambda^*\colon (L/U)^* \hookrightarrow L^*$ for every $U_\lambda \in \mathscr{U}$. Since L is linearly compact, the vector space L/U_λ is finite dimensional. Thus, there exists a unique $\overline{x_\lambda} \in L$ such that $\psi \circ \pi_\lambda^* = \mathrm{ev}_{\overline{x_\lambda}}$ where $\mathrm{ev}\colon L/U_\lambda \to (L/U_\lambda)^{**}$. In addition, observe that if $\mu \leq \lambda$ there is an induced injection $\pi_\mu^{\lambda *}\colon (L/U_\mu)^* \hookrightarrow (L/U_\lambda)^*$ such that the following diagram

$$\begin{tikzcd}
L^* \arrow[rr, "\psi"] & & k \\
(L/U_\lambda)^* \arrow[u, hook, "\pi_\lambda^*"] \arrow[urr, "\mathrm{ev}_{\overline{x_\lambda}}"'] & & \\
(L/U_\mu)^* \arrow[u, hook, "\pi_\mu^{\lambda *}"] \arrow[uu, bend left, hook, "\pi_\mu^*"] \arrow[uurr, bend right, "\mathrm{ev}_{\overline{x_\mu}}"'] & &
\end{tikzcd}$$

commutes. Observe that uniqueness of $\overline{x_\lambda} \in L/U_\lambda$ implies $\overline{x_\mu} = \pi_\mu^\lambda(\overline{x_\lambda})$ for all $\mu \leq \lambda$. Indeed, for all $\phi_\mu \in (L/U_\mu)^*$ the equality

$$\begin{aligned}
\phi_\mu(\overline{x_\mu}) &= \psi(\pi_\mu^*(\phi_\mu)) \\
&= \psi(\pi_\lambda^* \circ \pi_\mu^{\lambda *}(\phi_\mu)) \\
&= \psi(\pi_\lambda^*(\phi_\mu \circ \pi_\mu^\lambda)) \\
&= (\phi_\mu \circ \pi_\mu^\lambda)(\overline{x_\lambda}) \\
&= \phi_\mu(\pi_\mu^\lambda(\overline{x_\lambda}))
\end{aligned}$$

holds. Therefore, $\overline{x_\mu} = \pi_\mu^\lambda(\overline{x_\lambda})$ for all $\mu \leq \lambda$. Then $(x_\lambda)_{\lambda \in \Lambda}$ belongs to the completion $\widehat{V}$ as described in Definition 1.14. Since $V \to \widehat{V}$ is an isomorphism, there exists $x \in L$ such that $\pi_\lambda(x) = \overline{x_\lambda}$ for all $\lambda \in \Lambda$. We claim that $\psi = \mathrm{ev}_x$. Let $\phi \in L^*$. Then $\ker \phi$ is open and there exists $\lambda \in \Lambda$ such that $U_\lambda \subseteq \ker \phi$. Hence, we can factor ϕ as follows

$$\begin{tikzcd}
L \arrow[r, "\phi"] \arrow[d, "\pi_\lambda"'] & k \\
L/U_\lambda \arrow[ur, "\phi_\lambda"'] &
\end{tikzcd}$$

Now, since L/U is discrete, it follows that ϕ_λ is continuous. Moreover, $\pi_\lambda^*(\phi_\lambda) = \phi$. Hence,

$$\psi(\phi) = \psi(\pi_\lambda^*(\phi_\lambda)) = \phi_\lambda(\overline{x_\lambda}) = \phi_\lambda(\pi_\lambda(x) = \phi(x).$$

This implies surjectivity of $\mathrm{ev}\colon L \to L^{**}$. Finally, we prove that ev is open. Let U be any open vector subspace in L. Then $L = U \oplus F$ for some finite dimensional F. We claim that $\mathrm{ev}(U) = (F^*)^\perp$. First, the inclusion $\mathrm{ev}(U) \subseteq (F^*)^\perp$ is immediate. Let $\psi \in (F^*)^\perp$ and $x \in L$ such that $\mathrm{ev}_x = \psi$. Write $x = u + f$, where $u \in U$ and $f \in F$. Hence,

$\mathrm{ev}_x = \mathrm{ev}_u + \mathrm{ev}_f$. Since ev is injective, there exists some $\phi \in F^*$ such that $\phi(f) \neq 0$ if f is nonzero. Therefore, $f = 0$ and $\psi \in \mathrm{ev}(U)$. This concludes the proof. □

Remark 1.26. Observe that completeness cannot be dropped in the definition of a Tate space while preserving duality. Indeed, if V is linearly compact but not complete, then its dual is discrete by Lemma 1.22. Moreover, by Remark 1.23, V^{**} is complete. Hence, $V \to V^{**}$ cannot be an isomorphism. In fact, by a similar argument in the proof of the duality theorem, one can see that V^{**} is the completion of V.

Remark 1.27. We now discuss definitions of linearly compact spaces as given in [Lef42] and [BD04]. In [Lef42] Lefschetz proves that a linearly compact vector space must be complete space. However, from our definition a linearly compact spaces is not necessarily complete. Nevertheless, when V is a complete space both definitions coincide. Indeed, Lefschetz proves that every linearly compact space is the dual of a discrete space, which coincides with our definition of a complete linearly compact vector space by Theorem 1.25. Therefore, his definition of a **locally linearly compact vector space** (that is, a linearly topologized vector space admitting an open linearly compact vector subspace) coincides with our notion of Tate space.

1.2.3 *Morphisms*

Definition 1.28. A morphism $f\colon V \to W$ of Tate spaces is said to be **linearly compact** if the closure of $f(V)$ is linearly compact in W. Dually, it is **discrete** if $\ker f$ is open in V.

First, we check the natural duality property for morphisms of Tate spaces.

Proposition 1.29. *A morphism $f\colon V \to W$ of Tate spaces is linearly compact if and only if f^* is discrete.*

Proof. Suppose f^* is linearly compact, then $\ker f^* = f(V)^{\perp}$. However, if $\phi \in W^*$ and $\phi(f(V)) = 0$, then $\phi(\overline{f(V)}) = 0$ by continuity of ϕ. Therefore, $\ker f^* = \overline{f(V)}^{\perp}$, which is open because $\overline{f(V)}$ is linearly compact. Now, suppose f^* is discrete. Then $\ker f^*$ contains a basic open subspace $A^{\perp}$ such that A is linearly compact in W. Therefore, $f(V) \subseteq A$ and $\overline{f(V)} \subseteq A$. Moreover, by item (c) in Theorem 1.13, $\overline{f(V)}$ is linearly compact. □

Discrete and linearly compact endomorphisms form a 2-sided ideal in Hom; that is

Proposition 1.30. *If f is a linearly compact morphism (respectively discrete), then its composition with an arbitrary morphism of Tate spaces is also linearly compact (respectively discrete).*

Proof. Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ be morphisms of Tate spaces such that g is linearly compact. Then $\overline{g \circ f(A)} \subseteq \overline{g(B)}$, which is linearly compact. Thus, $\overline{g \circ f(A)}$ is linearly compact as well. On the other hand, note that $h(\overline{g(B)}) \subseteq \overline{h \circ g(B)}$; therefore, $\overline{h(\overline{g(B)})} = \overline{h \circ g(B)}$. However, $\overline{g(B)}$ is linearly compact. Hence, by item (b) of Theorem 1.13, $h(\overline{g(B)})$ is linearly compact. Observe that the statement for discrete endomorphisms follows from Proposition 1.29. □

Remark 1.31. If f is a compact operator and g is a discrete operator, then gf is of **finite-rank**; that is, $\dim gf(V) < \infty$.

Proof. We have $\overline{f(V)} \prec \ker g$; therefore, $\overline{f(V)}/(\overline{f(V)} \cap \ker g)$ is finite dimensional. Observe that the surjection

$$\frac{\overline{f(V)}}{\overline{f(V)} \cap \ker g} \to gf(V)$$

implies that gf is of finite-rank. □

Definition 1.32. Let V and W be Tate spaces. We denote by $\mathrm{Hom}_+(V, W)$ the set of linearly compact morphisms and by $\mathrm{Hom}_-(V, W)$ the set of discrete endomorphisms. Moreover, their intersection is denoted by $\mathrm{Hom}_0(V, W)$.

Proposition 1.33. *The sets*

$$\mathrm{Hom}_-(V, W), \mathrm{Hom}_+(V, W) \text{ and, } \mathrm{Hom}_0(V, W)$$

are vector subspaces of $\mathrm{Hom}(V, W)$. *Moreover,*

$$\mathrm{Hom}_-(V, W) + \mathrm{Hom}_+(V, W) = \mathrm{Hom}(V, W).$$

Proof. Let L be a c-lattice in V and consider $\pi\colon V \to L$ a continuous linear projection. Observe that $\pi \in \mathrm{End}_+(V)$ and $1 - \pi \in \mathrm{End}_-(V)$. Hence, by Proposition 1.30, for every $f \in \mathrm{Hom}(V, W)$ it follows that $f \circ \pi$ and $f \circ (1 - \pi)$ are linearly compact and discrete endomorphisms respectively. Therefore,

$$\mathrm{Hom}_-(V, W) + \mathrm{Hom}_+(V, W) = \mathrm{Hom}(V, W).$$

The other statements are immediate. □

2

TRACE AND RESIDUE

We extend the definition of trace to a certain class of infinite rank endomorphisms in order to define an abstract residue. We follow the original structure of Tate's elegant article [Tat68] while translating his statements in the language of Tate's Linear Algebra.

2.1 FINITE-POTENT MAPS AND THEIR TRACE

Let k be a fixed field and V a vector space over k. In this section we will extend the notion of trace of a linear endomorphism to include certain operators even when V is infinite dimensional.

2.1.1 *Finite-potent maps*

Definition 2.1. We will say a linear map $f\colon V \to V$ is **finite-potent** if

$$\dim f^n(V) < \infty$$

for sufficiently large n.

The following is characterization of finite-potent endomorphisms.

Lemma 2.2. *A linear map $f\colon V \to V$ is finite-potent if and only if there exists a subspace $W \subseteq V$ such that*

(i) $\dim f(W) < \infty$*,*

(ii) $f(W) \subseteq W$ *and*

(iii) the induced map $\bar{f}\colon V/W \to V/W$ is nilpotent.

*A subspace W is a **trace-subspace** for f if satisfies the previous properties.*

Proof. If f is finite-potent, choose $W = f^n(V)$ for sufficiently large n. The first condition follows from definition. Also, $f(W) = f^{n+1}(V) \subseteq f^n(V) = W$. In addition, $\bar{f}^n = 0$. On the other hand, if such W exists, note that condition (ii) assures that $\bar{f}$ is well defined. Moreover, as $\bar{f}$ is nilpotent, $f^n V \subseteq W$ for sufficiently large n and by condition (i) above $\dim f^n(V) < \infty$. $\square$

Observe that a trace-subspace for a finite-potent map f is not unique. In particular, if W is trace-subspace for f then $f^n(W)$ is trace-subspace for f for all n.

Notation 2.3. If f is a finite-rank endomorphism in a vector space V, we will denote its ordinary trace by $\mathrm{tr}_V(f)$. Moreover, if W is a subspace of V invariant under f, that is, $f(W) \subseteq W$, then $\mathrm{tr}_W(f) := \mathrm{tr}_W(f|_W)$. In addition, if $\overline{f}$ is the induced map such that the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{f} & V \\ \big\downarrow{\scriptstyle \pi_W} & & \big\downarrow{\scriptstyle \pi_W} \\ V/W & \xrightarrow{\overline{f}} & V/W, \end{array}$$

then $\mathrm{tr}_{V/W}(f) := \mathrm{tr}_{V/W}(\overline{f})$. The use of this notation is consistent throughout the document.

2.1.2 *Trace*

If f is a finite-potent map and W is a trace-subspace for f, the **trace** $\mathrm{tr}_V(f) \in k$ of f is defined by

$$\mathrm{tr}_V(f) := \mathrm{tr}_W(f).$$

Observe that $\mathrm{tr}_W(f)$ is well-defined because $f|_W$ is of finite-rank.

Proposition 2.4. *The definition of* tr_V *does not depend on the choice of trace-subspace for* f.

Proof. Suppose $W_1, W_2 \subseteq V$ are two trace-subspaces for f, then $W = W_1 + W_2$ is trace-subspace for f as well. Hence, the induced maps on W/W_1 and W/W_2 are nilpotent. Therefore, $\mathrm{tr}_{W/W_1}(f) = \mathrm{tr}_{W/W_2}(f) = 0$ and using the well-known identify of the ordinary trace

$$\mathrm{tr}_W(f) = \mathrm{tr}_{W_1}(f) + \mathrm{tr}_{W/W_1}(f)$$
$$\mathrm{tr}_W(f) = \mathrm{tr}_{W_2}(f) + \mathrm{tr}_{W/W_2}(f),$$

we obtain $\mathrm{tr}_{W_1}(f) = \mathrm{tr}_{W_2}(f)$, our desired result. □

This definition extends some of the properties of the ordinary trace.

Lemma 2.5. *(a) If* $\dim V < \infty$, *then any endomorphism* f *is finite-potent and* $\mathrm{tr}_V(f)$ *coincides with the ordinary trace.*

(b) *If f is nilpotent, then it is finite-potent and* $\mathrm{tr}_V(f) = 0$.

(c) *If f is finite-potent and U is a subspace such that $f(U) \subseteq U$, then the induced maps on U and V/U are finite-potent and satisfy the identity*

$$\mathrm{tr}_V(f) = \mathrm{tr}_U(f) + \mathrm{tr}_{V/U}(f).$$

Proof. Both (a) and (b) are immediate. For (c), if W is a trace-subspace for f, then $W \cap U$ and $(W+U)/U$ are trace-subspaces for the induced maps respectively. Hence, by Lemma 2.2, both induced maps are finite-potent. Since $W/(W \cap U) \cong (W+U)/U$, the diagram

$$\begin{array}{ccc} W/(W\cap U) & \xrightarrow{\cong} & (W+U)/U \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f} \\ W/(W\cap U) & \xrightarrow{\cong} & (W+U)/U \end{array}$$

commutes. Moreover, recall that the ordinary trace is invariant under conjugation, that is, $\mathrm{tr}_W(\varphi \circ f \circ \varphi^{-1}) = \mathrm{tr}_W(f)$ for every automorphism φ of W. Therefore, it follows that $\mathrm{tr}_{W/(W\cap U)}(f) = \mathrm{tr}_{(W+U)/U}(f)$. We conclude that

$$\mathrm{tr}_V(f) = \mathrm{tr}_W(f) = \mathrm{tr}_{W\cap U}(f) + \mathrm{tr}_{(W+U)/U}(f) = \mathrm{tr}_U(f) + \mathrm{tr}_{V/U}(f). \qquad \square$$

Definition 2.6. A subspace F of $\mathrm{End}_k(V)$ is said to be a **finite-potent subspace** if there exists an n such that for any family of n elements $f_1, \dots, f_n \in F$ the space $f_1 f_2 \cdots f_n V$ is finite dimensional.

Observe that if F is a finite-potent subspace of $\mathrm{End}_k(V)$, then every $f \in F$ is finite-potent.

Proposition 2.7. *If F is a finite-potent subspace, then* $\mathrm{tr}_V \colon F \to k$ *is k-linear.*

Proof. It is enough to prove the statement in the case where F is finite dimensional. Let $W = F^n V$ for n as in the definition of finite-potent subspace. Then $\dim W < \infty$. Hence, W is a trace-subspace for all $f \in F$. It follows that $\mathrm{tr}_V(f) = \mathrm{tr}_W(f)$ for all f. Since $\mathrm{tr}_W \colon \mathrm{End}_k(V) \to k$ is k-linear, so is $\mathrm{tr}_V \colon F \to k$. $\square$

Remark 2.8. In his paper, Tate asked if general linearity for finite-potent maps followed. His question was answered negatively in [AST07] where general linearity is reduced to the following question: if the sum of two nilpotent endomorphisms is finite-potent, is the sum necessarily traceless?

Proposition 2.9. *If $f, g \in \mathrm{End}_k(V)$ and fg is finite-potent, then so is gf and*

$$\mathrm{tr}_V(fg) = \mathrm{tr}_V(gf).$$

Proof. Since fg is finite-potent, $W = (fg)^n V$ is finite-dimensional for a sufficiently large n. On the other hand, $(gf)^{n+1}V = g(fg)^n f(V) \subseteq g(W)$; therefore, gf is also finite-potent. Let $W' = (gf)^n V$. Then $g(W') \subseteq W$ and $f(W) \subseteq W'$. Thus,

$$\dim W' \leq \dim g(W) \leq \dim W,$$

and,

$$\dim W \leq \dim f(W) \leq \dim W',$$

which implies that $W \cong W'$ and that g and f induce mutually inverse isomorphisms between W and W'. Moreover, the diagram

$$\begin{array}{ccc} W & \xrightarrow{fg} & W \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ W' & \xrightarrow{gf} & W' \end{array}$$

commutes. We conclude that $\mathrm{tr}_W(fg) = \mathrm{tr}_{W'}(gf)$. Hence, $\mathrm{tr}_V(fg) = \mathrm{tr}_V(gf)$. □

2.1.3 *Trace and Tate Spaces*

Suppose that V is a Tate space and consider $\mathrm{End}_k(V)$ the space of continuous endomorphisms of V. By Proposition 1.30 and Proposition 1.33, there are 2-sided ideals $\mathrm{End}_0(V), \mathrm{End}_+(V)$ and $\mathrm{End}_-(V)$ of $\mathrm{End}_k(V)$ such that $\mathrm{End}_+(V) + \mathrm{End}_-(V) = \mathrm{End}_k(V)$ and $\mathrm{End}_0(V) = \mathrm{End}_+(V) \cap \mathrm{End}_-(V)$. Moreover, Remark 1.31 implies that $\mathrm{End}_0(V)$ is a finite-potent subspace.

Lemma 2.10. *Suppose that $f \in \mathrm{End}_+(V)$, $g \in \mathrm{End}_-(V)$ or $f \in \mathrm{End}_-(V)$ and $g \in \mathrm{End}_+(V)$. Then the commutator $[f, g] = fg - gf$ belongs to $\mathrm{End}_0(V)$ and it is traceless.*

Proof. This immediate from the previous discussion and Proposition 2.9.

2.2 DIFFERENTIAL CALCULUS

In this section we introduce the theory of derivations and differentials over an arbitrary commutative k-algebra A. Let M be an A-module. We follow [Gro64] Section 20 and [Mat86] Section 25.

Definition 2.11. A **derivation** from A to M is a map $D\colon A \to M$ satisfying properties

(i) $D(a+b) = D(a) + D(b)$ and,

(ii) (*Leibniz Rule*) $D(ab) = aD(b) + bD(a)$

for all $a, b \in A$.

The set of derivations from A to M is an A-module in the natural way. We will denote it by $\mathrm{Der}(A, M)$. Moreover, if A is a k-algebra through a map $\varphi\colon k \to A$ we say that D is a **k-derivation** if D is a derivation and $D \circ \varphi = 0$. In this case, the set of all k-derivations is denoted $\mathrm{Der}_k(A, M)$. If $M = A$, we will denote $\mathrm{Der}_k(A, A)$ simply by $\mathrm{Der}_k(A)$.

Definition 2.12. Let B be a k-algebra and C an ideal in B with $C^2 = 0$; set $A = B/C$. In this way, C can be viewed as an A-module. In this situation, we say that B is an **extension** of the k-algebra A by the A-module C. We simply write the exact sequence

$$0 \to C \to B \xrightarrow{\pi} A \to 0.$$

Moreover, we will say that such sequence **splits** if there exists a retraction; that is, a k-algebra homomorphism $\rho\colon A \to B$ such that $\pi \circ \rho = 1_A$. In this case, we can identify $B = C \oplus A$. Conversely, starting from any k-algebra A and any A-module C, one can always define a structure on $A \oplus C$ such that $A \oplus C$ is an extension of A by C. Namely,

$$(a, c)(a', c') = (aa', ac' + a'c)$$

for $a, a' \in A$ and $c, c' \in C$. Common notations for this algebra are $D_A(C)$ or $A * C$.

Definition 2.13. Given a commutative diagram of k-algebras

$$\begin{tikzcd} B \arrow[r, "f"] & A \\ & C \arrow[u, "g"] \arrow[ul, "h"] \end{tikzcd}$$

thinking of f as a fixed map; we say that h is a **lifting** of g to B.

Lemma 2.14. *Let h and $h' \colon C \to B$ be two liftings of g to B. Let $K = \ker f$ and suppose $K^2 = 0$. Then $h - h' \colon C \to K$ is a k-derivation. Conversely, if $D \in \mathrm{Der}_k(C, K)$, then $h + D$ is another lifting of g to B.*

Proof. First, observe that $(h - h')(C)$ lies in K because both h and h' are liftings of g to B. Since $K^2 = 0$, K can be considered as $f(B)$-module and by means of g as a C-module. Then $h - h' \colon C \to K$ is a map of C-modules. Now, let $c, c' \in C$, then

$$\begin{aligned}(h - h')(cc') &= h(c)h(c') - h'(c)h'(c') \\ &= h(c)h(c') - h'(c)h'(c') - h(c)h'(c') + h'(c')h(c).\end{aligned}$$

Since $c \cdot k = h(c)k = h'(c)k$ for all $k \in K$, it follows that

$$\begin{aligned}(h - h')(cc') &= c \cdot h(c') - c' \cdot h'(c') - c \cdot h'(c') + c' \cdot h(c) \\ &= c \cdot (h - h')(c') + c' \cdot (h - h')(c).\end{aligned}$$

This implies that $h - h'$ is a k-derivation. Observe that $h + D$ is a lifting of g to B because $D(C)$ lies in K. □

Theorem 2.15. *If A is a k-algebra, consider the covariant functor from the category Mod_A to itself given by $M \mapsto \mathrm{Der}_k(A, M)$. This functor is representable.*

Proof. Let $\mu \colon A \otimes_k A \to A$ be the k-algebra homomorphism given by $f \otimes g \to fg$. Set

$$I = \ker \mu, \quad \Omega_{A/k} = I/I^2, \quad \text{and,} \quad B = (A \otimes_k A)/I^2.$$

Thus, μ induces $\mu' \colon B \to A$ such that

$$0 \to \Omega_{A/k} \to B \to A \to 0$$

is an extension of A by $\Omega_{A/k}$. We claim that this extension splits. Moreover, it has two splittings. Indeed consider the retractions

$$j_1 \colon A \to B \quad \text{and,} \quad j_2 \colon A \to B$$

defined by $a \mapsto a \otimes 1 \bmod I^2$ and $a \mapsto 1 \otimes a \bmod I^2$. By Lemma 2.14, $d := j_2 - j_1$ is a k-derivation of A to $\Omega_{A/k}$. Now, we prove that

$$\mathrm{Der}_k(A, M) \cong \mathrm{Hom}_A(\Omega_{A/k}, M). \tag{2}$$

Let $D \in \mathrm{Der}_k(A, M)$ and define $\varphi \colon A \otimes_k A \to A * M$ by $\varphi(x \otimes y) = (xy, xD(y))$. Then φ is a k-algebra homomorphism because it is compatible with the operation in $A * M$ defined in Definition 2.12. In addition, if $\sum x_i \otimes y_i$ lies in I, then

$$\mu\left(\sum x_i \otimes y_i\right) = \sum x_i y_i = 0 \implies \varphi\left(\sum x_i \otimes y_i\right) = \left(0, \sum x_i D(y_i)\right)$$

whence $\varphi(I)$ lies in M. Moreover, by Leibniz's Rule φ factors through I^2 yielding a map $f \colon \Omega_{A/k} \to M$. For $a \in A$ it follows that

$$\begin{aligned} f(da) &= f(1 \otimes a - a \otimes 1 \mod I^2) = \varphi(1 \otimes a) - \varphi(a \otimes 1) \\ &= D(a) - aD(1) = D(a). \end{aligned}$$

Therefore, $D = f \circ d$. Now, we prove that such f is unique. First, observe that $\Omega_{A/k}$ has the A-module structure induced by multiplication by $a \otimes 1$ (or $1 \otimes a$ since $1 \otimes a - a \otimes 1 \in I$). Therefore, if $\xi = \sum x_i \otimes y_i \mod I^2 \in \Omega_{A/k}$, then $a\xi = \sum a x_i \otimes y_i \mod I^2$, and $f(a\xi) = \sum a x_i D(y_i) = af(\xi)$, so that f is A-linear. We have

$$a \otimes a' = (a \otimes 1)(1 \otimes a' - a' \otimes 1) + aa' \otimes 1,$$

so that if $\omega = \sum x_i \otimes y_i \in I$, then $\omega \mod I^2 = \sum x_i dy_i$ because $\sum x_i y_i = 0$. We conclude that $\{da \mid a \in A\}$ is a set of generators for the A-module $\Omega_{A/k}$. This implies uniqueness of f. Therefore, (2) holds. □

Definition 2.16. The module $\Omega_{A/k}$ introduced in the proof of the previous theorem is called the **module of differentials** of A over k or **module of Kähler differentials**, and for $a \in A$ the element $da \in \Omega_{A/k}$ is called the **differential** of a.

Example 2.17. If A is generated as k-algebra by a subset $S \subseteq A$, then $\Omega_{A/k}$ is generated by $\{ds \mid s \in S\}$. Indeed, if $a \in A$, then there exist $a_i \in S$ and a polynomial $f(X) \in k[X_1, \ldots, X_n]$ such that $a = f(a_1, \ldots, a_n)$. Thus,

$$da = \sum_{i=1}^{n} f_i(a_1, \ldots, a_n) da_i \quad \text{where} \quad f_i = \frac{\partial f}{\partial x_i}.$$

In particular, if $A = k[X_1, \ldots, X_n]$ then $\Omega_{A/k} = AdX_1 + \ldots AdX_n$ because $X_1, \ldots, X_n$ are linearity independent; this follows from the fact that $\partial_i X_j = \delta_{ij}$.

Lemma 2.18. *Let K be a k-commutative algebra. The map $c \colon K \otimes_k K \to \Omega_{K/k}$ defined by $c(f \otimes g) = fdg$ satisfies:*

(i) c is surjective.

(ii) $\ker c$ is generated over k by the elements of the form

$$f \otimes gh - fg \otimes h - fh \otimes g$$

Proof. The k-bilinear map $(f, g) \mapsto f dg$ induces c. Since $\{df \mid f \in K\}$ is a generating set for $\Omega_{K/k}$ as a K-module, it follows that c is surjective. For (ii), observe that it is equivalent proving that $\ker(c)$ is generated over K by the elements of the form $1 \otimes gh - g \otimes h - h \otimes g$. Let A be the K-module generated by those elements. We wish to prove that

$$A \to K \otimes_k K \to \Omega_{K/k} \to 0$$

is exact. By left-exactness of Hom it is equivalent to prove that for all K-modules M the induced sequence

$$0 \to \mathrm{Hom}_K(\Omega_{K/k}, M) \to \mathrm{Hom}_K(K \otimes_k K, M) \to \mathrm{Hom}_K(A, M)$$

is exact. By Theorem 2.15, there is a canonical isomorphism $\mathrm{Hom}_K(\Omega_{K/k}, M) \cong \mathrm{Der}_k(K, M)$. Under this identification, we wish to prove that

$$0 \to \mathrm{Der}_k(K, M) \to \mathrm{Hom}_K(K \otimes_k K, M) \to \mathrm{Hom}_K(A, M)$$

is exact. Observe that the first map is given by $D \mapsto \varphi_D$ where $\varphi_D(f \otimes g) = fD(g)$. Note that the restriction $\varphi_D \colon A \to M$ is trivial. Indeed,

$$\varphi_D(1 \otimes gh - g \otimes h - h \otimes g) = D(gh) - gD(h) - hD(g) = 0$$

by the Leibniz rule. Now, let $\psi \in \mathrm{Hom}_K(K \otimes_k K, M)$ so that $\psi(A) = 0$. Let $D_\psi \colon K \to M$ be the k-derivation defined by $f \mapsto \psi(1 \otimes f)$. First, we prove that ψ_D is a k-derivation. Observe that k-linearity is obvious. Now, we prove the Leibniz rule for D_ψ. Consider

$$\begin{aligned} D_\psi(fg) = \psi(1 \otimes fg) &= \psi(f \otimes g + g \otimes f) \\ &= f\psi(1 \otimes g) + g\psi(1 \otimes f) \\ &= fD_\psi(f) + gD_\psi(g), \end{aligned}$$

where the third equality follows from the fact that ψ vanishes on A. Finally, it is clear that $\varphi_{D_\psi} = \psi$. □

2.3 ABSTRACT RESIDUE AND ITS PROPERTIES

2.3.1 *Existence of residue map*

Throughout this section let k be a field, K a commutative k-algebra with 1, and V a K-module so that when viewed as a k-vector space it is a Tate space and K acts continuously on V. Namely, for all $f \in K$ the map

$$\begin{aligned} f\colon V &\to V \\ x &\mapsto fx \end{aligned}$$

is continuous. In this way, K operates on V through $\mathrm{End}_k(V)$ (maintaining notation from Section 2.1.3). We will not notationally distinguish $f \in K$ from its induced map in $\mathrm{End}_k(V)$.

Lemma 2.19. *Let $f, g \in K$. Then there are $f_+, g_+ \in \mathrm{End}_+(V)$ so that*

$$f = f_+ \mod \mathrm{End}_-(V), \quad g = g_+ \mod \mathrm{End}_-(V)$$

and, the equality

$$\mathrm{tr}([f_+, g_+]) = \mathrm{tr}([f, g_+]) = \mathrm{tr}([f_+, g])$$

holds.

Proof. The existence of f_+ and g_+ is immediate from the fact that $\mathrm{End}_k(V) = \mathrm{End}_+(V) + \mathrm{End}_-(V)$. Clearly $[f_+, g_+] \in \mathrm{End}_+(V)$. Moreover, the fact that K is commutative implies that $[f, g] = 0$. Therefore,

$$[f_+, g_+] = [f, g] \mod \mathrm{End}_-(V).$$

Hence, $[f_+, g_+] \in E_0$. Similarly, $[f, g_+]$ and $[f_+, g]$ belong to $\mathrm{End}_0(V)$. Whence, one can consider their trace. Furthermore, if $f_+ \in \mathrm{End}_+(V)$ and $g_+ - g \in \mathrm{End}_-(V)$, then $\mathrm{tr}([f_+, g_+ - g]) = 0$ by Lemma 2.10. We conclude that $\mathrm{tr}([f_+, g_+]) = \mathrm{tr}([f_+, g])$. The other equality follows similarly. $\square$

Notation 2.20. Lemma 2.19 implies that common values of traces $[f_+, g_+]$, $[f_+, g]$ and $[f, g_+]$ depend only on f and g and not in the choice of f_+ and g_+. Therefore, we will always denote $f_\pm$ to be elements in $\mathrm{End}_\pm(V)$ such that

$$f = f_+ \mod \mathrm{End}_-(V), \text{ and } f = f_- \mod \mathrm{End}_+(V).$$

Lemma 2.19 implies that the assignment $(f,g) \mapsto \mathrm{tr}([f_+,g_+])$ is well-defined. Observe that this assignment is k-bilinear by Proposition 2.7. Thus, there exists a map

$$\begin{aligned} r\colon K \otimes_k K &\to k \\ f \otimes g &\mapsto \mathrm{tr}([f_+,g_+]). \end{aligned}$$

With these tools at our hands we are ready to prove the existence of residue.

Theorem 2.21. *There exists a unique k-linear map*

$$\mathrm{res}_V\colon \Omega_{K/k} \to k$$

such that for each pair of elements $f,g \in K$ we have

$$\mathrm{res}_V(fdg) = \mathrm{tr}([f_+,g_+]).$$

Proof. Let $c\colon K \otimes_k K \to \Omega_{K/k}$ be as in Lemma 2.18. Since c is surjective, if res_V exists it is uniquely determined by the commutativity of the following diagram

$$\begin{tikzcd} K \otimes_k K \arrow[r, "r"] \arrow[d, two heads, "c"] & k \\ \Omega_{K/k} \arrow[ur, dotted, "\mathrm{res}_V"'] & \end{tikzcd}$$

Therefore, such map exists if and only if it vanishes on $\ker c$. To see this, let f,g and h in K and choose f_+,g_+ and h_+ in $\mathrm{End}_+(V)$ coinciding with f,g and h modulo $\mathrm{End}_-(V)$ respectively. Then,

$$fg = f_+g_+ + (f_+g_- + f_-g_+ + f_-g_-),$$

and $f_+g_- + f_-g_+ + f_-g_- \in \mathrm{End}_-(V)$. Whence, $(fg)_+ = f_+g_+$. Analogously $(gh)_+ = g_+h_+$ and $(fh)_+ = f_+h_+$. This fact and the identify

$$[f_+,g_+h_+] - [f_+g_+,h_+] - [f_+h_+,g_+] = 0$$

imply the desired conclusion. □

2.3.2 *Properties of residue*

We prove some of the main properties of residue.

Proposition 2.22. *For all $f,g \in K$ it follows that* *(a)* $\mathrm{res}_V(fdg) + \mathrm{res}_V(gdf) = 0$, *and*

(b) $\mathrm{res}_V(df) = 0$.

Proof. Since $[f_+, g_+] + [g_+, f_+] = 0$, we get (a). For (b) use (a) with $g = 1$. □

Proposition 2.23. *Let W be a closed K-submodule of V. Then, for $\omega \in \Omega_{K/k}$, the identity*

$$\mathrm{res}_V(\omega) = \mathrm{res}_W(\omega) + \mathrm{res}_{V/W}(\omega)$$

holds.

Proof. It is enough to prove the claim for $\omega = fdg$. By Lemma 2.5 item (c), we only need to check that for all $f \in K$ the induced map $\overline{f}\colon V/W \to V/W$ and $f \circ \iota$, where ι denotes the inclusion $W \to V$, satisfy

$$\begin{aligned} \overline{f} &= \overline{f_+} \mod \mathrm{End}_-(V/W), \\ f \circ \iota &= f_+ \circ \iota \mod \mathrm{End}_-(W), \\ \overline{f_+} &\in \mathrm{End}_+(V/W), \quad \text{and} \\ f_+ \circ \iota &\in \mathrm{End}_+(W). \end{aligned}$$

These statements are straightforward from definitions. □

Proposition 2.24. *If V is the direct sum of two closed submodules W_1 and W_2, then*

$$\mathrm{res}_V(\omega) = \mathrm{res}_{W_1}(\omega) + \mathrm{res}_{W_2}(\omega)$$

holds for all $\omega \in \Omega_{K/k}$.

Proof. Immediate from Proposition 2.23. □

If our Tate space is trivial, so its residue.

Proposition 2.25. *If V is either linearly compact or discrete, then $\mathrm{res}_V(\Omega_{K/k}) = 0$.*

Proof. If V is linearity compact, then $\mathrm{End}_+(V) = \mathrm{End}_k(V)$ and $f_+ = f$ for all $f \in K$. Since $[f, g] = 0$, it follows that

$$\mathrm{res}_V(fdg) = 0. \tag{3}$$

On the other hand, if V is discrete, then $\mathrm{End}_k(V) = \mathrm{End}_-(V)$. Hence, $f = 0 \mod \mathrm{End}_-(V)$ for all $f \in K$. Thus, (3) holds. □

Proposition 2.26. *Let f and g belong to K. Then, if there exists a c-lattice L in V so that $fL + fgL + fg^2L \subseteq L$, it holds $\mathrm{res}_V(fdg) = 0$. In particular, if there exists L a c-lattice so that $fL \subseteq L$ and $gL \subseteq L$, then $\mathrm{res}_V(fdg) = 0$.*

Proof. Let π be a continuous projection from V to L. Then $\pi f \in \mathrm{End}_+(V)$ and $\pi f = f \mod \mathrm{End}_-(V)$. Thus, it follows that

$$\mathrm{res}_V(fdg) = \mathrm{tr}([\pi f, g])$$

by Lemma 2.19. Let $h = [\pi f, g]$ and $W = L + gL$. Let $h_{V/W}$ and h_W be the induced maps on V/W and W respectively. Then the relation $fL + fgL + fg^2L \subseteq L$ implies that $h_{V/W} = 0$ and $h_W = 0$. By Lemma 2.5 item (c), we conclude that

$$\mathrm{res}_V(fdg) = \mathrm{tr}_V(h) = \mathrm{tr}_W(h) + \mathrm{tr}_{V/W}(h) = 0. \qquad \square$$

In the following two propositions we examine the residue of a power.

Proposition 2.27. *Let $f \in K$, then $\mathrm{res}_V(f^n df) = 0$ for all $n \geq 0$. Moreover, if f is invertible the same identity holds for $n \leq -2$.*

Proof. First, if $f_+ = f \mod \mathrm{End}_-(V)$, then $f_+^n = f^n \mod \mathrm{End}_-(V)$. Therefore,

$$\mathrm{res}_V(f^n df) = \mathrm{tr}([f_+, f_+^n]) = 0.$$

Second, if f is invertible, then

$$fd(f^{-1}) + f^{-1}df = d(ff^{-1}) = d(1) = 0.$$

This implies that

$$f^{-2}df = -d(f^{-1}).$$

Hence, multiplying by f^{-n} on both sides, where $n \geq 0$, it follows that

$$f^{-2-n}df = -(f^{-1})^n d(f^{-1}).$$

By the preceding statement, $(f^{-1})^n d(f^{-1})$ has zero residue. $\square$

Proposition 2.28. *If f is invertible, so that $fL \subseteq L$ for some c-lattice L, then*

$$\mathrm{res}_V(f^{-1}df) = \dim_k(L/fL).$$

Proof. If π is a continuous projection of V into L, then

$$\operatorname{res}_V(f^{-1}df) = \operatorname{tr}([\pi f^{-1}, f]).$$

Let $g = [\pi f^{-1}, f]$. Since $fL \subseteq L$, we obtain

$$g_{V/L} = 0, \quad g_{L/fL} = 1 \quad \text{and,} \quad g_{fL} = 0,$$

where $g_{V/L}, g_{L/fL}$ and g_{fL} denote the induced maps in V/L, L/fL and, fL respectively. Then by Lemma 2.5 item (c), it follows that

$$\operatorname{tr}_V(g) = \operatorname{tr}_L(g) + \operatorname{tr}_{V/L}(g) = \operatorname{tr}_{fL}(g) + \operatorname{tr}_{L/fL}(g) + \operatorname{tr}_{V/L}(g).$$

Observe that $\dim L/fL < \infty$ since fL is open and L is linearly compact. □

3 ALGEBRAIC CURVES

In the preceding chapter we presented the "residue map" in an abstract context. In this chapter we explore residues on algebraic curves using Tate's construction.

3.1 BASIC THEORY OF ALGEBRAIC CURVES

In this section we recall briefly and with little detail the basic theory of algebraic projective curves. For a complete exposition we reference the reader to [BP02] and [Har77]. We will borrow many results from commutative algebra, most of them can be found in [Mat86], [AK12] and [AM69].

Let k be an algebraically closed field. Let $(X, \mathscr{O}_X)$ be a projective variety. Then

$$k(X) := \varinjlim_{U \subseteq X} \mathscr{O}_X(U)$$

is the **function field** or **field of rational functions** of X. In addition, consider the stalk

$$\mathscr{O}_{X,p} := \varinjlim_{p \in U \subseteq X} \mathscr{O}_X(U)$$

of regular functions near p. We obtain natural injections

$$\mathscr{O}_X(X) \to \mathscr{O}_{X,p} \to k(X).$$

Proposition 3.1. *The fraction field of $\mathscr{O}_{X,p}$ is $k(X)$.*

Proof. Let $U \subseteq X$ be an affine neighborhood of p. Suppose that A is the coordinate ring of X on U and let $\mathfrak{p}$ be the maximal ideal of A corresponding to p. Therefore, $A_{\mathfrak{p}} = \mathscr{O}_{X,p}$. Since U is affine, $\mathscr{O}_X(U) = A$ and $k(U) = \operatorname{Frac} A$. Moreover, irreducibility of X implies $k(X) = k(U)$. Hence,

$$k(X) = k(U) = \operatorname{Frac} A = \operatorname{Frac} A_{\mathfrak{p}} = \operatorname{Frac} \mathscr{O}_{X,p}. \qquad \square$$

In addition, $\mathscr{O}_{X,p}$ is a noetherian local ring of Krull dimension $\dim X$. Its maximal ideal of regular functions near p that vanish in p is denoted $\mathfrak{m}_p$. Observe that evaluation at p yields the isomorphism $\mathscr{O}_{X,p}/\mathfrak{m}_p \cong k$.

3.1.1 *Smoothness and completeness*

Definition 3.2. A local ring $(A, \mathfrak{m})$, where $\mathfrak{m}$ denotes its maximal ideal, is called **regular** if $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = \dim A$.

Let $(A, \mathfrak{m})$ be a noetherian regular local ring. Let $k = A/\mathfrak{m}$ be its residue field. In this situation, $(A, \mathfrak{m})$ carries a natural topology, called the $\mathfrak{m}$-adic topology. Namely, $\{\mathfrak{m}^n\}_{n\geq 1}$ is a system of neighborhoods around zero and we let the topology to be translation invariant. We already mentioned this topology briefly in Example 1.20 for the polynomial ring. The $\mathfrak{m}$-adic topology is separated. Indeed,

$$\bigcap_{n\geq 1} \mathfrak{m}^n = \{0\}$$

by Krull intersection theorem. See Theorem (18.29) in [AK12]. Just as in Definition 1.14 we define the **completion** $\widehat{A}$ of A to be

$$\widehat{A} := \varprojlim_{n\geq 1} A/\mathfrak{m}^n.$$

There is a natural map $A \to \widehat{A}$. In particular, since A is separated, this map is injective. When this map is an isomorphism we say that A is **complete**. We summarize several properties of completion in the following theorem:

Theorem 3.3. *Let $(A, \mathfrak{m})$ be a noetherian regular local ring. Then*

(a) $\widehat{A}$ is a noetherian regular local ring and $\widehat{\mathfrak{m}}$ is its maximal ideal.

(b) Krull dimension is preserved under completion, that is, $\dim A = \dim \widehat{A}$.

(c) (Cohen structure theorem) If $\dim A = n$, *then*

$$\widehat{A} \cong k\,[[t_1, \ldots, t_n]]\,.$$

Where $t_1, \ldots t_n$ are mapped to generators of $\mathfrak{m}$.

Proof. See Chapter 22 in [AK12]. □

Now, we explore these results in the geometrical setting.

Definition 3.4. If $\mathscr{O}_{X,p}$ is a regular local ring, that is, $\dim_k \mathfrak{m}_p/\mathfrak{m}_p^2 = \dim \mathscr{O}_{X,p} = \dim X$, we say that X is **smooth at** p. Naturally, X is called **smooth** if it is smooth at every point $p \in X$.

We get the following result immediately from Theorem 3.3.

Corollary 3.5. *If X is smooth, then $\widehat{\mathcal{O}_{X,p}} \cong k[[t_1, t_2, \ldots, t_n]]$ where $n = \dim X$.*

Now, we focus in one-dimensional varieties.

Definition 3.6. A **smooth algebraic curve** is a one-dimensional smooth projective variety.

In dimension 1 smoothness can be interpreted in the language of valuations.

3.1.2 *Valuation theory*

Let k be a field.

Definition 3.7. A **discrete valuation** is a surjective group homomorphism $\nu \colon k^\times \to \mathbb{Z}$ such that, for every $x \in k^\times$ and $y \neq -x$ in $k^\times$

$$\nu(x+y) \geq \min\{\nu(x), \nu(y)\}.$$

As a convention, we let $\nu(0) = \infty$. We denote by

$$A_\nu = \{x \in k \colon \nu(x) \geq 0\}$$

the **discrete valuation ring** or **DVR** of ν. Clearly, A is a subring, thus a domain. Consider

$$\mathfrak{m}_\nu = \{x \in k \colon \nu(x) > 0\}.$$

Notice that, if $x \in k$, but $x \notin A_\nu$, then $x^{-1} \in \mathfrak{m}_\nu$. Hence, $\mathrm{Frac}(A_\nu) = K$. Further, observe that

$$A_\nu^\times = A_\nu - \mathfrak{m}_\nu.$$

Therefore, A_ν is a local domain with maximal ideal $\mathfrak{m}_\nu$. An element $t \in \mathfrak{m}_\nu$ with $\nu(t) = 1$ is called a **uniformizing parameter**. Such t is irreducible because if $t = ab$ with $\nu(a) \geq 0$ and $\nu(b) \geq 0$ implies $\nu(a) = 0$ or $\nu(b) = 0$ since $1 = \nu(a) + \nu(b)$. Further, any $x \in k^\times$ has the unique factorization $x = ut^n$ where $u \in A_\nu^\times$ and $n := \nu(x)$. Moreover, A_ν is a principal ideal domain. In fact, any nonzero ideal $\mathfrak{a} \subseteq A_\nu$ has the form

$$\mathfrak{a} = \langle t^m \rangle \quad \text{where} \quad m := \min\{\nu(x) \colon x \in \mathfrak{a}\}.$$

Indeed, given a nonzero $x \in \mathfrak{a}$, say $x = ut^n$ where $u \in A_\nu^\times$. Then $t^n \in \mathfrak{a}$. So $n \geq m$. Set $y := ut^{n-m}$. Then $y \in A_\nu$ and $x = yt^m$, as desired. Finally, $\mathfrak{m} = \langle t \rangle$ and $\dim A_\nu = 1$. Therefore, A is regular local ring of dimension one.

We have the following characterization theorem for DVRs.

Theorem 3.8. *Let A be a noetherian one-dimensional local ring, $\mathfrak{m}$ its maximal ideal and $k = A/\mathfrak{m}$ its residue field. Then these conditions are equivalent:*

(i) *A is a DVR.*

(ii) *A is integrally closed.*

(iii) *$\mathfrak{m}$ is principal.*

(iv) $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$.

(v) *Every non-zero ideal is a power of $\mathfrak{m}$.*

Proof. See Proposition 9.2 in [AM69]. □

Corollary 3.9. *Let X be a one-dimensional projective variety. Then X is smooth if and only if $\mathscr{O}_{X,p}$ is a DVR for all p.*

Example 3.10. Let $(X, \mathscr{O}_X)$ be a smooth algebraic curve. Let $p \in X$ and consider $\mathscr{O}_{X,p}$. In this case $\mathscr{O}_{X,p}$ is a DVR. Let $t_p \in \mathscr{O}_{X,p}$ be an uniformizing parameter. Hence, if $f \in k(X)$. it follows that $f = ut_p^n$ for some unit $u \in \mathscr{O}_{X,p}$ and $n \in \mathbb{Z}$. Then $\nu_p(f) = n$.

3.1.3 *Divisors*

Let X be a smooth algebraic curve.

Definition 3.11. A **divisor** D on X is a finite formal sum of points on X, namely

$$D = \sum_{p \in X} n_p p, \quad n_p \in \mathbb{Z}, \text{ and } n_p = 0 \quad \text{for almost all } p.$$

The **degree** of D is

$$\deg D := \sum_{p \in X} n_p.$$

We say that a divisor D is **effective** and write $D \geq 0$ if $n_p \geq 0$ for all $p \in X$. We write $D \geq D'$ if the difference $D - D'$ is an effective divisor. If $f \in k(X)$ is a non-zero rational function on X, the **principal divisor** (f) is defined by

$$(f) := \sum_{p \in X} \nu_p(f) p.$$

In other words, (f) is the sum of the zeroes of f minus the sum of poles of f, counting multiplicities. Observe that $(f) + (g) = (fg)$.

We say that two divisors D and D' are **linearly equivalent** if the differ by a principal divisor, that is, there exists some rational function f such that $D + (f) = D'$.

Given a divisor $D = \sum_p n_p p$ we define the k-vector space

$$L(D) := \{f \in k(X) \colon f = 0 \text{ or } \nu_p(f) \geq -n_p\}.$$

For instance, if D is effective $L(D)$ consists of rational functions having a pole at $p \in X$ of order at worst n_p for each $p \in X$. We write $\ell(D) := \dim_k(L(D))$.

3.2 TATE SPACES OVER ALGEBRAIC CURVES

Let $(X, \mathscr{O}_X)$ be a smooth algebraic curve over an algebraically closed field k.

3.2.1 *Residue in coordinates*

Let $K := k(X)$. For all $p \in X$ we use the following notation $L_p := \widehat{\mathscr{O}_{X,p}}$ and $K_p := \operatorname{Frac} L_p$. From Theorem 3.3 and *Theorem* 3.8, it follows that L_p is a DVR. Let t_p be an uniformizing parameter in L_p. The topology in K_p is defined by letting $\{t_p^n L_p\}_{n\in\mathbb{Z}}$ be a system of neighborhoods of zero in K_p. This system is compatible with the valuation induced by L_p.

Proposition 3.12. *K_p is a Tate space and L_p is a c-lattice in K_p.*

Proof. Observe that the map $\mathfrak{m}_p^n \mathscr{O}_{X,p}/\mathfrak{m}_p^{n+1}\mathscr{O}_{X,p} \to \mathscr{O}_{X,p}/\mathfrak{m}_p\mathscr{O}_{X,p}$ induced by inclusions is an isomorphism. Then the exactness of

$$0 \to \mathfrak{m}_p^n \mathscr{O}_{X,p}/\mathfrak{m}_p^{n+1}\mathscr{O}_{X,p} \to \mathscr{O}_{X,p}/\mathfrak{m}_p^{n+1}\mathscr{O}_{X,p} \to \mathscr{O}_{X,p}/\mathfrak{m}_p^{n}\mathscr{O}_{X,p} \to 0$$

implies that every quotient $\mathscr{O}_{X,p}/\mathfrak{m}_p^n\mathscr{O}_{X,p}$ is finite-dimensional over k. Therefore,

$$L_p = \varprojlim_{n\geq 1} \mathscr{O}_{X,p}/\mathfrak{m}_p^n\mathscr{O}_{X,p}$$

is an inverse limit of finite-dimensional k-vector spaces. Hence, L_p is complete and linearly compact by Example 1.20. Since L_p is open in K_p, it is a c-lattice and K_p is a Tate space. $\square$

Remark 3.13. (a) Observe that $\{t_p^n L_p\}_{n\in\mathbb{Z}}$ is a mutually commensurable system of neighborhoods around zero of consisting of k-vector subspaces of K_p.

(b) Let $f \in K_p$, observe that $fL_p = t_p^n L_p$ for some $n \in \mathbb{Z}$ and uniformizing parameter $t_p \in L_p$. It follows that multiplication by f in K_p is continuous in K_p, that is, K_p (and particularly K) acts continuously over itself.

Definition 3.14. Let $f, g \in K_p$. We define the residue of the differential fdg at $p \in X$ to be

$$\operatorname{res}_p(fdg) = \operatorname{res}_{K_p}(fdg).$$

where res_{K_p} denotes the abstract residue defined in Theorem 2.21.

Proposition 3.15. *Let $p \in X$. If $\omega \in \Omega_{K_p/k}$ has no poles at p then $\operatorname{res}_p(\omega) = 0$.*

Proof. Clear from Proposition 2.26. □

Theorem 3.16. *Let $f, g \in K_p$. By the structure theorem in Theorem 3.3, it follows that $f = \sum_{\nu\gg-\infty}^{\infty} a_\nu t_p^\nu$ and $g = \sum_{\mu\gg-\infty}^{\infty} b_\mu t_p^\mu$ for some $a_\nu, b_\mu \in k$ and a uniformizing parameter $t_p \in L_p$. Recall that the formal derivative of g is*

$$g' = \sum_{\mu\gg-\infty}^{\infty} \mu b_\mu t_p^{\mu-1}.$$

Then

$$\operatorname{res}_p(fdg) = \textit{coefficient of } t_p^{-1} \textit{ in } fg',$$

which is given by the Cauchy product

$$\operatorname{res}_p(fdg) = \sum_{\mu+\nu=0} \mu a_\nu b_\mu.$$

Proof. Let

$$\tilde{f} = \sum_{\nu\gg-\infty}^{N} a_\nu t_p^\nu, \quad \text{and} \quad \tilde{g} = \sum_{\mu\gg-\infty}^{N} b_\mu t_p^\mu.$$

Then

$$\begin{aligned}\operatorname{res}_p(fdg) &= \operatorname{res}_p((\tilde{f} + (f-\tilde{f}))d(\tilde{g} + (g-\tilde{g}))) \\ &= \operatorname{res}_p(\tilde{f}d\tilde{g}) + \operatorname{res}_p(\tilde{f}d(g-\tilde{g})) + \operatorname{res}_p((f-\tilde{f})d\tilde{g}) \\ &+ \operatorname{res}_p((f-\tilde{f})d(g-\tilde{g})).\end{aligned}$$

If N is sufficiently large, then by Proposition 2.26, it follows that

$$\operatorname{res}_p((f-\tilde{f})d(g-\tilde{g})) = \operatorname{res}_p(\tilde{f}d(g-\tilde{g})) = \operatorname{res}_p((f-\tilde{f})d\tilde{g}) = 0.$$

Therefore, we can assume that only finitely many of the a_ν and b_μ are non-zero. Now, $fdg = fg'dt$ and by Proposition 2.27 only the term of t_p^{-1} can have non-zero residue. Then, by Proposition 2.28, it follows that

$$\operatorname{res}_p(t_p^{-1}dt_p) = \dim_k(L_p/t_pL_p) = \dim_k k = 1.$$

Hence, k-linearity of residue implies the desired conclusion. □

Corollary 3.17. *Let $f \in K_p$. Then the coefficient of t_p^{-1} in the Laurent series expansion of f is independent of the choice of uniformizing parameter t_p.*

Remark 3.18. Before Tate introduced this approach to residues of differentials on algebraic curves, residues were defined by the formula in Theorem 3.16. However, to prove that such formula is well-defined, it is necessary to argue that the coefficient of t_p^{-1} is independent of the choice of uniformizing parameter t_p. If $\operatorname{char} k = 0$, one can realize X as an analytical variety and reduce independence to the invariance of the formula

$$\operatorname{res}_p(\omega) = \frac{1}{2\pi i}\oint_p \omega.$$

Nevertheless, in the general setting it is not obvious why invariance follows. In Corollary 3.17 we gave a clean but theory-demanding proof of such result. We reference the reader to [Ser88] Chapter 2 Section 10 for a direct proof.

3.2.2 *Adèles and the residue theorem*

Our next goal is to prove the residue theorem. In order to prove it, we will take an *adèlic* approach, borrowing many techniques from number theory.

Definition 3.19. Let X be a smooth algebraic curve. Let $Y \subseteq X$ be any subset. Let $\mathscr{F}$ denote the set of all finite subsets of Y. Let $S \in \mathscr{F}$, the **S-adèle** of K indexed by Y is defined as the product

$$\widetilde{K}_{Y,S} := \prod_{p\in Y\setminus S} L_p \times \prod_{p\in S} K_p$$

in its product topology. The **adéle** $\widetilde{K}_Y$ of K indexed by Y is the direct limit of the system indexed by $\mathscr{F}$, namely, if $S \subseteq T$ there exists a injection $\iota_{ST} \colon \widetilde{K}_{Y,S} \hookrightarrow \widetilde{K}_{Y,T}$ given by the inclusion. Endow

$$\widetilde{K}_Y = \varinjlim_{S \in \mathscr{F}} \widetilde{K}_{Y,S}$$

with its direct limit topology. Usually, for $X = Y$ we will simply write $\widetilde{K}$ for $\widetilde{K}_X$.

The adèle of K is a particular case of a **restricted product** of a collection of topological spaces. In the literature, this construction defined as a *set* in terms of the product. We give this characterization in the following proposition.

Proposition 3.20.

$$\widetilde{K}_Y = \{(f_p) \colon f_p \in K_p \textit{ for all } p \in Y \textit{ and } f_p \in L_p \textit{ for almost all } p \in Y\}$$

where almost all means for all but finitely many $p \in Y$, equipped with the following collection as a basis for its topology

$$\left\{ \prod_{p \in Y} U_p \colon U_p \textit{ is open for all } p \in Y \textit{ and } U_p = L_p \textit{ for almost all } p \in Y \right\}.$$

Proof. Let $K^\sharp$ be the topological space defined in the statement of the proposition. Observe that $K^\sharp$ is linearly topologized as a k-vector space. We will prove that $K^\sharp$ satisfies the universal property of $\widetilde{K}_Y$ in the category LinTop_k. First, observe that the inclusion

$$\widetilde{K}_{Y,S} \hookrightarrow K^\sharp$$

is a continuous homomorphism and the diagram

$$\begin{tikzcd} & K^\sharp & \\ \widetilde{K}_{Y,S} \arrow[ur, hook] \arrow[rr, hook, "\iota_{ST}"] & & \widetilde{K}_{Y,T} \arrow[ul, hook] \end{tikzcd}$$

commutes. Moreover, for every P equipped with continuous homomorphisms $\varphi_S \colon \widetilde{K}_{Y,S} \to P$ such that the diagram

$$\begin{tikzcd} & P & \\ \widetilde{K}_{Y,S} \arrow[ur, hook, "\varphi_S"] \arrow[rr, hook, "\iota_{ST}"] & & \widetilde{K}_{Y,T} \arrow[ul, hook, "\varphi_T"'] \end{tikzcd}$$

commutes. Then define $\varphi\colon K^\sharp \to P$ as follows: for $(f_p) \in K^\sharp$ there exists a $S \in \mathscr{F}$ such that $f_p \in L_p$ if and only if $p \in S$. Define $\varphi((f_p)_{p\in Y}) = \varphi_S((f_p)_{p\in Y})$. This is a continuous homomorphism and it is the only one such that the diagram

$$\begin{tikzcd} & P & \\ & K^\sharp \arrow[u, dotted, "\varphi"] & \\ \widetilde{K}_{X,S} \arrow[uur, "\varphi_S"] \arrow[ur, hook] \arrow[rr, "\iota_{ST}"] & & \widetilde{K}_{X,T} \arrow[uul, "\varphi_T"'] \arrow[ul, hook'] \end{tikzcd}$$

commutes. This implies $K^\sharp = \widetilde{K}_Y$ (or canonically isomorphic). $\square$

Remark 3.21. Observe that the topology in $\widetilde{K}_Y$ is finer than the one it inherits as a subspace of the product $\prod_{p\in Y} K_p$. For instance, observe that

$$\widetilde{L}_Y := \prod_{p\in Y} L_p$$

is open in $\widetilde{K}$, but it is open in $\prod_{p\in Y} K_p$ if and only if Y is finite.

If $D = \sum_p n_p p$ is a divisor, let

$$\widetilde{K}(D) = \{(f_p)_p \in \widetilde{K}\colon \nu_p(f_p) \geq -n_p\}$$

a k-vector subspace of $\widetilde{K}$. By the description we gave in Proposition 3.20 $\widetilde{K}(D)$ is open for all divisors D and they form a system of neighborhoods around zero of mutually commensurable vector subspaces in $\widetilde{K}$. In this notation, $\widetilde{K}(0) = \widetilde{L}$.

Proposition 3.22. *$\widetilde{K}_Y$ is a Tate space and $\widetilde{L}_Y$ is a c-lattice in $\widetilde{K}_Y$.*

Proof. First, observe that

$$\prod_{p\in Y} L_p = \varprojlim_{p\in Y} L_p = \varprojlim_{p\in Y} \varprojlim_{n\geq 1} \mathscr{O}_{Y,p}/\mathfrak{m}_p^n\mathscr{O}_{Y,p} = \varprojlim_{(p,n)\in Y\times\mathbb{N}^{\geq 0}} \mathscr{O}_{Y,p}/\mathfrak{m}_p^n\mathscr{O}_{Y,p}$$

for Y realized as trivial category. Hence, $\widetilde{L}_Y$ is the inverse limit of a projective system of finite dimensional k-vector spaces. Therefore, $\widetilde{L}_Y$ is a complete linearly compact vector space over k. Since $\widetilde{L}_Y$ is open in $\widetilde{K}_Y$, it is a c-lattice in $\widetilde{K}_Y$ and $\widetilde{K}_Y$ is a Tate space. $\square$

Proposition 3.23. *K is realized as a discrete vector subspace of $\widetilde{K}$ by means of the diagonal embedding $f \mapsto (f)_{p\in X}$.*

Proof. Observe that

$$K \cap \widetilde{L} = \bigcap_{p\in X} \mathscr{O}_{X,p} = \mathscr{O}_X(X).$$

Indeed, the first equality is obvious and the second follows from the fact that a regular function is globally defined on X if and only if it is regular at every point p. Since X is projective and geometrically irreducible, $\mathscr{O}_X(X) \cong k$ (see, e.g [Har77] Chapter 1 Theorem 3.4). Therefore, $K \cap \widetilde{L}$ is a finite-dimensional k-vector space, thus discrete. Since $\widetilde{L}$ is open, it follows that K is discrete. □

Our next objective is to show that $\widetilde{K}/K$ is linearly compact. To prove this, for a divisor $D = \sum_{p\in X} n_p p$ we study the quotient

$$I(D) := \widetilde{K}/(\widetilde{K}(D) + K).$$

Many techniques here are taken from [Ful89]. By construction we have the following exact sequence

$$0 \to L(D) \to K \to \widetilde{K}/\widetilde{K}(D) \to I(D) \to 0.$$

An element $(\overline{f}_p)_p \in \widetilde{K}/\widetilde{K}(D)$ can be described as follows: at each point $p \in X$, $\overline{f}_p$ is given by a **Laurent tail**

$$\overline{f}_p = a_\nu t_p^\nu + a_{\nu+1} t_p^{\nu+1} + \ldots + a_{-n_p-1} t_p^{-n_p-1}$$

where t_p is a uniformizing parameter at p, $\nu \in \mathbb{Z}$, $\nu \leq n_p$, and $a_i \in k$. Moreover, only finitely many $\overline{f}_p$ are non-zero.

Example 3.24. Let $X = \mathbb{P}^1$. We claim $I(0) = 0$. Indeed, an element $\overline{f} \in \widetilde{K}/\widetilde{K}(0)$ is a finite collection of Laurent tails as above, where $n_p = 0$ for all p. Choose an affine coordinate $x = X_1/X_0$ on $\mathbb{P}^1$ such that the points $p_1, \ldots, p_k$ such that $\overline{f}_p \neq 0$ lie in the affine piece $\mathbb{A}^1_x = (X_0 \neq 0) \subset \mathbb{P}^1$, with x coordinates $\alpha_1, \ldots, \alpha_k$. Then $x - \alpha_i$ is a local parameter at p_i, and we can write

$$\overline{f}_{p_i} = g_i = a_{\nu_i,i}\,(x-\alpha_i)^{\nu_i} + \cdots + a_{-1,i}(x-\alpha_1)^{-1}$$

Define $g = \sum g_i \in k(X)$, then g has Laurent tail $\overline{f}_{p_i}$ at p_i and is regular elsewhere, that is, $g \mapsto \overline{f} \in \widetilde{K}/\widetilde{K}(0)$. Hence $I(0) = 0$ as claimed.

Lemma 3.25. *Suppose $D \leq D'$. Then, there is a natural surjection $I(D) \to I(D')$, and the kernel has dimension*

$$(\deg D' - \ell(D')) - (\deg D - \ell(D)).$$

Proof. By definition $\widetilde{K}(D) \subseteq \widetilde{K}(D')$, so $I(D)$ surjects onto $I(D')$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K/L(D) & \longrightarrow & \widetilde{K}/\widetilde{K}(D) & \longrightarrow & I(D) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K/L(D') & \longrightarrow & \widetilde{K}/\widetilde{K}(D') & \longrightarrow & I(D') & \longrightarrow & 0 \end{array}$$

The vertical arrows are surjective, so the kernels form an exact sequence

$$0 \to L(D')/L(D) \to \widetilde{K}(D')/\widetilde{K}(D) \to M \to 0$$

where $M = \ker(I(D) \to I(D'))$. Finally, observe that $\widetilde{K}(D')/\widetilde{K}(D)$ has dimension $\deg D' - \deg D$. Indeed, writing $D = \sum_p n_p p$ and $D' = \sum_p n'_p p$, an element $(\overline{f}_p)_p \in \widetilde{K}(D')/\widetilde{K}(D)$ is given by Laurent tails

$$\overline{f}_p = a_{-n'_p} t_p^{-n'_p} + a_{-n'_p+1} t_p^{-n'_p+1} + \cdots + a_{-n_p-1} t_p^{-n_p-1}$$

where t_p denotes a uniformizing parameter at p. So,

$$\dim_k \widetilde{K}(D')/\widetilde{K}(D) = \sum_p (n'_p - n_p) = \deg D' - \deg D$$

as claimed. This yields the formula for $\dim_k M$. □

Lemma 3.26 (Riemann's theorem). *There exists $N \in \mathbb{N}$ such that* $\deg D - \ell(D) \leq N$ *for all divisors D on X.*

Proof. Let $f \in K$ be a non-constant rational function and

$$F = (1 : f) \colon X \to \mathbb{P}^1$$

the associated morphism. Let $A = F^*(0 : 1)$, so A is the divisor of degree d given by the sum of the poles of f with multiplicities. Let D be a divisor on X. We claim that there exists a linearly equivalent divisor D' such that $D' \leq nA$ for some $n \in \mathbb{N}$. Indeed, write $D = \sum_p n_p p$, and define

$$h = \prod_{p \in S} (f - f(p))^{n_p}$$

where

$$S = \{p \in X \colon n_p > 0, f(p) \neq \infty\}.$$

Then $(h) \geq D - nA$ for some $n \in \mathbb{N}$, that is, $D' := D - (h) \leq nA$, as desired.

We now establish the result for divisors $D = nA$ for $n \in \mathbb{N}$. The morphism F corresponds to the field extension $k(f) \subseteq k(X)$ of degree d. Pick a basis $g_1, \dots, g_d$ for $k(X)$ over $k(f)$. Then, by the construction in the previous paragraph, there exist polynomials $q_i(t) \in k[t]$ such that $q_i(f)g_i \in L(n_0 A)$ for all $1 \leq i \leq d$ and some $n_0 \in \mathbb{N}$. Indeed, let $D = -(g_i)$ and define $q_i(f) = h$ as above. Then $(hg_i) = -D' \leq n_i A$ for some $n_i \in \mathbb{N}$. Let $n_0 = \max n_i$. Then $f^j q_i(f) g_i \in L(nA)$ for each $1 \leq i \leq d$ and $0 \leq j \leq n - n_0$. Moreover, these functions are linearly independent over $k(f)$ because $g_1, \dots, g_d$ are linearly independent over $k(f)$. So,

$$\ell(nA) \geq (n - n_0 + 1)d = \deg nA - (n_0 - 1)d,$$

that is, $\deg nA - \ell(nA) \leq N$ where $N := (n_0 - 1)d$.

Combining our results, if D and D' are linearly equivalent and $D' \leq nA$ then

$$\deg D - \ell(D) = \deg D' - \ell(D') \leq \deg nA - \ell(nA) \leq N. \qquad \square$$

Theorem 3.27. *For all divisors D on X we have $\dim_k I(D) < \infty$.*

Proof. By Lemma 3.26, there exists a divisor D_0 on X such that $\deg D_0 - \ell(D_0)$ is maximal. We claim that $I(D_0) = 0$. Indeed, otherwise let $(\overline{f}_p)_p \in I(D)$ nonzero. Pick $D' \geq D_0$ $D' = \sum_p n'_p p$ such that $\nu_p(f_p) \geq -n'_p$ for all $p \in X$, then $(f_p)_p$ lies in the kernel of the surjection $I(D_0) \to I(D')$. So $\deg D' - \ell(D') > \deg(D_0) - \ell(D_0)$ by Lemma 3.25, a contradiction.

If $D \leq D'$ we have a surjection $I(D) \to I(D')$ with finite-dimensional kernel by Lemma 3.25. Thus $I(D) \sim I(D')$. Since $I(D_0)$ is finite dimensional, we deduce that $I(D)$ is finite dimensional for every divisor D. $\square$

Corollary 3.28. *$\widetilde{K}/K$ is linearly compact.*

Proof. For every divisor D on X the exact sequence

$$0 \to (\widetilde{K}(D) + K)/K \to \widetilde{K}/K \to I(D) \to 0,$$

the fact that the collection $\widetilde{K}(D)$ is a system of neighborhoods around zero of vector subspaces in $\widetilde{K}$ and Theorem 3.27 yield the result. $\square$

Corollary 3.29. *For all $p \in X$*

$$\mathrm{res}_{\widetilde{K}}(\Omega_{K/k}) = 0.$$

Proof. The statement follows from Proposition 2.23, Proposition 2.25, Proposition 3.23 and Corollary 3.28. □

Now, we are ready to prove the residue theorem.

Theorem 3.30 (Residue Theorem). *For X a smooth algebraic curve and $\omega \in \Omega_{K/k}$, the identity*

$$\sum_{p \in X} \mathrm{res}_p(\omega) = 0$$

holds.

Proof. First, observe that the expression $\sum_{p \in X} \mathrm{res}_p(\omega) = 0$ makes sense by Proposition 3.15 and the fact that ω has a finite amount of poles. Now, it is enough to prove the statement for $\omega = f dg$ for $f, g \in K$. Let $p_1, \dots, p_n$ be the collection of poles of f and g combined. Then, the product $K_{p_1} \oplus K_{p_2} \oplus \cdots K_{p_n}$ is a closed K-submodule of $\widetilde{K}$ (it is the kernel of the projection on the coordinates different than p_i). Choose M a complementary subspace. Then

$$\widetilde{K} = K_{p_1} \oplus K_{p_2} \oplus \cdots K_{p_n} \oplus M.$$

Let $U = \widetilde{L} \cap M$. Then U is a c-lattice in M and $fU \subseteq U$ and $gU \subseteq U$. Hence, Proposition 2.26 yields

$$\mathrm{res}_M(\omega) = 0.$$

Therefore, by Proposition 2.24 and Corollary 3.29 it follows that

$$0 = \mathrm{res}_{\widetilde{K}}(\omega) = \sum_{p \in X} \mathrm{res}_p(\omega).$$

□

3.3 RIEMANN-ROCH FORMULA

We finish up this chapter by proving the Riemann-Roch formula by means of adèlic methods.

3.3.1 *Differential pairing and the adèlic complex*

Let X be a smooth algebraic curve, K its function field and $\widetilde{K}$ it associated adèle ring. In the previous section we introduced the **adèlic complex** of X, that is, for every divisor D one has the complex $\mathcal{A}(D)$

$$K \oplus \widetilde{K}(D) \to \widetilde{K}, \quad (\alpha, \beta) \mapsto \alpha - \beta.$$

This complex is quasi-isomorphic to the complexes

$$\widetilde{K}(D) \to \widetilde{K}/K, \quad K \to \widetilde{K}/\widetilde{K}(D)$$

and we have $H^0(\mathcal{A}(D)) = K \cap \widetilde{K}(D)$ and $H^1(\mathcal{A}(D)) = \widetilde{K}/(\widetilde{K}(D) + K)$. Since $\widetilde{K}(D)$ is linearly compact and K is discrete it follows that $\dim_k H^0(\mathcal{A}(D)) < \infty$. Moreover, we proved in Theorem 3.27 that $H^1(\mathcal{A}(D))$ is finite-dimensional. To study these cohomology groups we will introduce the notion of canonical divisor and the differential pairing.

First, we will extend the valuation ν_p for all $p \in X$ for differentials $\omega \in \Omega_{K/k}$. Let t_p be a uniformizing parameter at p. Then, write $\omega = f dt_p$ for $f \in K$ and set $\nu_p(\omega) = \nu_p(f)$. This formula does not depend on the choice of uniformizing parameter t_p.

Definition 3.31. Let $\omega \in \Omega_{K/k}$. We define the **canonical divisor** W of ω to be the principal divisor associated to ω.

Remark 3.32. The fact (see the proof of Theorem 2.15) that $\dim_K \Omega_{K/k} = 1$ implies that all canonical divisors are linearly equivalent.

Definition 3.33. For a differential $\omega \in \Omega_{K/k}$ we define the differential pairing $\langle -, - \rangle$ on $\widetilde{K} \times \widetilde{K}$ by the formula

$$\langle (f_p)_p, (g_p)_p \rangle = ((f_p)_p, (g_p)_p) \mapsto \sum_{p \in X} \operatorname{res}_p(f_p g_p \omega).$$

Observe that $\langle -, - \rangle$ is continuous. Indeed, multiplication and the assignment $(f_p)_p \mapsto \operatorname{res}_p(f_p \omega)$ are continuous in $\widetilde{K}(X)$.

For a k-vector subspace $V \subseteq \widetilde{K}$ let $V^\perp = \{(f_p)_p \in \widetilde{K} \colon \langle V, (f_p)_p \rangle = 0\}$.

Lemma 3.34. *$K \subseteq K^\perp$ and $K^\perp$ is a K-vector space.*

Proof. The Residue Theorem (Theorem 3.30) implies that $K \subseteq K^\perp$. The fact that $K^\perp$ is a K-space is obvious. □

In fact, one can prove a stronger statement.

Lemma 3.35. $K = K^{\perp}$.

Proof. Consider the quotient $K^{\perp}/K$. It is a closed subspace of $\widetilde{K}/K$ because the pairing $\langle -, - \rangle$ is continuous. Hence, it is linearly compact. Moreover, the pairing map yields an injection $K^{\perp} \hookrightarrow (\widetilde{K}/K)^*$. Since $(\widetilde{K}/K)^*$ is a discrete space by Lemma 1.22, $K^{\perp}/K$ is discrete. It follows that $\dim_k K^{\perp}/K < \infty$. Then, the fact that $\dim_k K$ is infinite and $K^{\perp}$ is a K-space yields that $K^{\perp} = K$. □

Now, the complement of $\widetilde{K}(D)$ for a divisor D is characterized by the canonical divisor. Indeed, from the definition of W it follows that for all divisors D

$$\widetilde{K}(D)^{\perp} = \widetilde{K}(W - D).$$

Lemma 3.36. *For a divisor D*

$$H^0(\mathcal{A}(D)) \cong H^1(\mathcal{A}(W - D)).$$

Proof. We will prove that $H^0(\mathcal{A}(D))^* \cong \widetilde{K}/H^0(\mathcal{A}(D))^{\perp}$. This yields the desired isomorphism, since

$$\begin{aligned} H^0(\mathcal{A}(D)) &= (K \cap \widetilde{K}(D))^{\perp} \\ &= K^{\perp} + \widetilde{K}(D)^{\perp} \\ &= K + \widetilde{K}(W - D) \quad (\text{By } \textit{Lemma } 3.35) \end{aligned}$$

Hence, $\widetilde{K}/H^0(\mathcal{A}(D))^{\perp} \cong H^1(\mathcal{A}(W - D))$. Now, the pairing restricted to $H^0(\mathcal{A}(D)) \times \widetilde{K}/H^0(\mathcal{A}(D))^{\perp}$ is perfect, therefore

$$\widetilde{K}/H^0(\mathcal{A}(D))^{\perp} \cong H^0(\mathcal{A}(D))^*.$$ □

Now, Lemma 3.36 yields $\dim_k(H^0(\mathcal{A}(D))) = \dim_k H^1(\mathcal{A}(W - D))$. So, for the Euler characteristic of the complex $\mathcal{A}(D)$ we obtain

$$\chi_{\mathcal{A}(D)} = \dim_k H^0(\mathcal{A}(D)) - \dim_k H^1(\mathcal{A}(D)) = \chi_{\mathcal{A}(W-D)}.$$

Our next objective is to prove that

$$\deg D = \chi_{\mathcal{A}(D)} - \chi_{\mathcal{A}(0)}.$$

Definition 3.37. A linear map between two k-vector spaces $f\colon V \to W$ is a **Fredholm map** if $\ker f$ and $\operatorname{coker} f$ are finite-dimensional. Then, the **index** of f is the Euler Characteristic of the complex

$$V \xrightarrow{f} W$$

i.e,

$$\operatorname{ind}(f) = \dim_k \ker f - \dim_k \operatorname{coker} f$$

In our case, the natural maps

$$K \oplus \widetilde{K}(D) \to \widetilde{K}, \quad \widetilde{K}(D) \to \widetilde{K}/K, \quad K \to \widetilde{K}/\widetilde{K}(D)$$

are all Fredholm with index equal to $\chi_{\mathcal{A}(D)}$. The following is an important property of Fredholm maps.

Lemma 3.38. *Let $U \xrightarrow{f} V \xrightarrow{g} W$ be Fredholm maps. Then $g \circ f$ is Fredholm and*

$$\operatorname{ind}(g \circ f) = \operatorname{ind}(f) + \operatorname{ind}(g).$$

Proof. Consider the following commutative diagram with exact rows

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & U \arrow[r, "1"] \arrow[d, "f"] & U \arrow[r] \arrow[d, "g\circ f"] & 0 \\
0 \arrow[r] & \ker g \arrow[r] & V \arrow[r, "g"] & W &
\end{tikzcd}$$

By the Snake Lemma we get an exact sequence

$$0 \to \ker f \to \ker(g \circ f) \to \ker g \to \operatorname{coker} f \to \operatorname{coker}(g \circ f).$$

Now, consider this time the diagram

$$\begin{tikzcd}
 & U \arrow[r, "f"] \arrow[d, "g\circ f"] & V \arrow[r] \arrow[d, "g"] & \operatorname{coker} f \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W \arrow[r, "1"] & W \arrow[r] & 0 &
\end{tikzcd}$$

and again, an application of the Snake Lemma yields an exact sequence

$$\ker(g \circ f) \to \ker g \to \operatorname{coker} f \to \operatorname{coker}(g \circ f) \to \operatorname{coker} g \to 0.$$

Gluing these two exact sequences one gets

$$0 \to \ker f \to \ker(g \circ f) \to \ker g \to \operatorname{coker} f \\ \to \operatorname{coker}(g \circ f) \to \operatorname{coker} g \to 0. \quad (4)$$

Then, $g \circ f$ is Fredholm. For the index, since (4) is an exact sequence of finite-dimensional vector spaces, it follows that

$$\dim_k \ker f - \dim_k \ker(g \circ f) + \dim_k \ker g \\ - \dim_k \operatorname{coker} f + \dim_k \operatorname{coker}(g \circ f) - \dim_k \operatorname{coker} g = 0$$

which yields the desired formula. □

Let D be an effective divisor. Then, we have an injection $\widetilde{K}(0) \to \widetilde{K}(D)$ such that the following diagram commutes

$$\begin{tikzcd} \widetilde{K}(0) \arrow[r] \arrow[d] & \widetilde{K}/K \\ \widetilde{K}(D) \arrow[ur] & \end{tikzcd}$$

Lemma 3.39. *If D is a divisor, then*

$$\deg D = \chi_{\mathcal{A}(D)} - \chi_{\mathcal{A}(0)}$$

Proof. Suppose D is effective. By Lemma 3.38 it follows that

$$\chi_{\mathcal{A}(0)} = -\dim_k \operatorname{coker}(\widetilde{K}(0) \to \widetilde{K}(D)) + \chi_{\mathcal{A}(D)}.$$

Observe that $\dim_k \operatorname{coker}(\widetilde{K}(0) \to \widetilde{K}(D)) = \deg D$. This yields the desired conclusion. The general case follows by considering D' effective such that $D + D'$ is effective. □

From these relations it follows the well-known Riemann-Roch formula in its adèlic form.

Theorem 3.40 (adèlic Riemann-Roch).

$$\dim_k H^0(\mathcal{A}(D)) = \dim_k H^0(\mathcal{A}(W - D)) + \deg D + \chi_{\mathcal{A}(0)}$$

Proof. Using the two formulas

$$\deg D = \chi_{\mathcal{A}(D)} - \chi_{\mathcal{A}(0)}, \quad \dim_k H^0(\mathcal{A}(D)) = \dim_k H^1(\mathcal{A}(W-D))$$

we get

$$\dim_k H^0(\mathcal{A}(D)) = \dim_k H^0(\mathcal{A}(W-D)) + \deg D + \chi_{\mathcal{A}(0)}.$$

In the notation of the previous section and putting $g = \dim_k H^1(\mathcal{A}(0))$

$$\ell(D) = \ell(W-D) + \deg D + 1 - g. \qquad \square$$

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