1

TATE'S LINEAR ALGEBRA

In this chapter we explore linear topologies on vector spaces in order to introduce Tate spaces and their structure. Tate spaces will be central in the definition of abstract residues in Chapter 2 and the study of algebraic curves in Chapter 3. We follow definitions in [BD04] closely but not religiously.

1.1 LINEAR TOPOLOGIES

Let k be a field. From now on, a vector space will always mean a k-vector space.

Definition 1.1. A **linear topology** on a vector space V is a separated (Hausdorff) topology which is invariant under translations and which admits a base of open neighborhoods of zero consisting of vector subspaces. A vector space equipped with a linear topology will be referred as **linearly topologized**.

If we endow k with the discrete topology then V will become a topological vector space. From now on, endow k with the discrete topology.
Linear topologies behave nicely under basic topological operations.

Proposition 1.2. *Let V be a linearly topologized vector space. Then*

(a) If $W \subseteq V$ is a vector subspace then W is linearly topologized as well.

(b) If $W \subseteq V$ is a closed vector subspace then V/W is linearly topologized under its quotient topology.

(c) If $\{V_\alpha\}_\alpha$ is a collection of linearly topologized vector spaces its product $\prod_\alpha V_\alpha$ (in its product topology) and its direct sum $\bigoplus_\alpha V_\alpha$ (as a subspace of the product) are linearly topologized.

(d) If W is a vector subspace of V, then its topological closure $\overline{W}$ also is a vector subspace of V.

Proof. If $\mathscr{U}$ is a system of neighborhoods around zero consisting of vector subspaces in V then $\{U \cap W \mid U \in \mathscr{U}\}$ is a system of neighborhoods around zero consisting of vector subspaces in W. For (b), let $\pi\colon V \to V/W$ be the quotient map. Since π is open and surjective the image of a local base is a local base; moreover, the image of a vector subspace under π is a vector subspace. In addition, since W is closed it follows that V/W is Hausdorff. Now, for (c) let $\{U_{\alpha,\beta}\}_\beta$ be a local base of zero in V_α of vector subspaces, the products $U_{\alpha_1,\beta_1} \times \ldots \times U_{\alpha_n,\beta_n} \times \prod_\gamma V_\gamma$, where γ ranges over $\alpha \neq \alpha_1, \ldots, \alpha_n$, for any set $\{(\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n)\}$ form a fundamental system of neighborhoods around zero in $\prod_\alpha V_\alpha$ of open vector subspaces. Note that since $\bigoplus_\alpha V_\alpha \subseteq \prod_\alpha V_\alpha$ is a vector subspace (c) follows from (a). Finally, for (d), suppose $x, y \in \overline{W}$, then, for every open vector subspace U, $(x + U) \cap W \neq \emptyset$ and $(y + U) \cap W \neq \emptyset$, therefore for every $\alpha, \beta \in k$ we have $(\alpha x + U) \cap W \neq \emptyset$ and $(\beta y + U) \cap W \neq \emptyset$. Hence, $(\alpha x + \beta y + U) \cap W \neq \emptyset$ for every open vector subspace U and every pair $\alpha, \beta \in k$. It follows (d).□

Proposition 1.3. *If we let* LinTop_k *denote the category of linearly topologized vector spaces over* k*, where morphisms are given by continuous linear homomorphisms, the previous proposition implies existence of kernels and arbitrary products; whence, existence of limits.*

Proposition 1.4. *A finite dimensional linearly topologized vector space* V *is discrete.*

Proof. Since V is Hausdorff, it follows that

$$\bigcap_{U \in \mathscr{U}} U = \{0\}.$$

for $\mathscr{U}$ a system of neighborhoods of zero consisting of vector subspaces of V. Then, if V is finite dimensional we can choose $\mathscr{U}$ to be finite. Therefore, $\{0\}$ is open. □

Commensurability

We introduce a partial order on the set of vector subspaces of a vector space V.

Definition 1.5. For vector subspaces A and B of a vector space V we say that $A \prec B$ if the quotient $A/(A \cap B) \cong (A + B)/B$ is finite dimensional (or equivalently $A \subseteq B + W$ for some finite dimensional W). In addition, we say that A and B are **commensurable** (denoted $A \sim B$) if $A \prec B$ and $B \prec A$.

Observe that $A \sim B$ if and only if $(A+B)/(A\cap B) \cong A/(A\cap B) \oplus B/(A\cap B)$ is finite dimensional. We will constantly refer to a vector space V being finite dimensional as $V \sim 0$.

Proposition 1.6. *Let V be a vector spaces and A, B and C be vector subspaces, then:*

(a) If $A \sim B$ and $B \sim C$ then

$$\frac{A+B+C}{A\cap B\cap C} \sim 0$$

(b) If $A \prec B$ and $B \prec C$ then $A \prec C$. Moreover, commensurability is an equivalence relation.

Proof. Consider the following exact sequences

$$0 \to \frac{A\cap B}{A\cap B\cap C} \to \frac{B}{B\cap C},$$

and,

$$0 \to \frac{A\cap B}{A\cap B\cap C} \to \frac{A+B}{A\cap B\cap C} \to \frac{A+B}{A\cap B} \to 0$$

induced by inclusions. The first inclusion plus the fact that $B \sim C$ imply that $(A\cap B)/(A\cap B\cap C)$ is finite dimensional. Now, since $A \sim B$ it follows that $(A+B)/(A\cap B)$ is finite dimensional. Moreover, the second exact sequence implies that $(A+B)/(A\cap B\cap C)$. A symmetrical argument shows that $(B+C)/(A\cap B\cap C) \sim 0$. These prove (a). For (b), the inclusion

$$0 \to \frac{A+C}{A\cap C} \to \frac{A+B+C}{A\cap B\cap C}$$

plus (a) implies transitivity. □

Now, we state and prove some useful properties on the relation $\prec$.

Lemma 1.7. *Let V be a vector space and A, B vector subspaces of V. Then*

(a) If $A \subseteq B$ it follows $A \prec B$.

(b) If $A \prec B$ then $f(A) \prec f(B)$ for any k-linear map f

(c) If $\{A_i\}_{i=1}^m$ and $\{B_j\}_{j=1}^n$ be two collections of vector subspaces of V. Then,

$$\sum_{i=1}^m A_i \prec \bigcap_{j=1}^n B_j \iff A_i \prec B_j \text{ for all } i \text{ and } j.$$

Proof. First, (a) is immediate from the definition of $\prec$. Second, for (b) the map f factors as

$$A/(A\cap B) \to f(A)/(f(A)\cap f(B)) \to 0$$

Finally, for (c), if $\sum_{i=1}^{m} A_i \prec \bigcap_{j=1}^{n} B_j$ holds then by (a) above, for all i and j we have

$$A_i \prec \sum_{i=1}^{m} A_i \prec \bigcap_{j=1}^{n} B_j \prec B_j$$

On the other hand, if $A_i \prec B_j$ for all i and j then there exist finite dimensional subspaces W_{ij} such that $A_i \subseteq B_j + W_{ij}$ for all i and j. Therefore,

$$\sum_{i=1}^{m} A_i \subseteq \bigcap_{j=1}^{n} B_j + \sum_{i=1}^{m}\sum_{j=1}^{n} W_{ij}.$$

□

Next, we consider another useful lemma.

Lemma 1.8. *Let A, B, A', B' be vector subspaces of a vector space V and suppose that $A \sim A'$ and $B \sim B'$. Then $A + B \sim A' + B'$ and $A \cap B \sim A' \cap B'$.*

Proof. The following exact sequence

$$0 \to \frac{A+A'+B+B'}{A\cap A'\cap B\cap B'} \to \frac{A+A'}{A\cap A'} \oplus \frac{B+B'}{B\cap B'} \to \frac{A+A'+B+B'}{(A\cap A')+(B\cap B')} \to 0$$

plus $A \sim A'$ and $B \sim B'$ imply that both spaces

$$\frac{A+A'+B+B'}{A\cap A'\cap B\cap B'} \quad \text{and,} \quad \frac{A+A'+B+B'}{(A\cap A')+(B\cap B')}$$

are finite dimensional. Since, $(A + A' + B + B')/(A + A') \cap (B + B')$ is a quotient of the second space and $((A \cap A') + (B \cap B'))/((A \cap A') \cap (B \cap B'))$ is a subspace of the first space we can conclude $A + B \sim A' + B'$ and $A \cap B \sim A' \cap B'$. □

If we consider the set of equivalence classes of $\sim$ on a vector space V then $\prec$ is a partial order on it and by Lemma 1.8 above it inherits operations $\cap$ and $+$.

Linear compactness

Definition 1.9. Let V be a linearly topologized vector space. A closed vector subspace $L \subseteq V$ is **linearly compact** (respectively **linearly cocompact**) if for every open vector subspace U we have $L \prec U$ (respectively $V/(L+U) \sim 0$).

Remark 1.10. Linear compactness was introduced by S. Lefschetz in his influential [Lef42] using different terms. Namely, he defined a linearly topologized vector space V to be linearly compact if for every collection of closed cosets X_α; that is, $X_\alpha = W_\alpha + x_\alpha$ where W_α is a vector subspace of V and $x_\alpha \in V$, having the finite intersection property follows that $\bigcap_\alpha X_\alpha \neq \emptyset$. In these terms, linear compactness seems like a natural generalization of compactness for linearly topologized vector spaces. We extend this discussion in Remark 1.27.

Linear compactness behaves just as compactness if one uses the correct words.

Proposition 1.11. *Let V be a linearly compact vector space, then*

(a) If $A \subseteq V$ is a vector subspace satisfying $A \prec U$ for all open vector subspaces U of V then $\overline{A}$ is linearly compact.

(b) If $f\colon V \to W$ is a continuous linear homomorphism then $\overline{f(V)}$ is linearly compact.

(c) If V is discrete then $V \sim 0$.

(d) Every closed vector subspace of V is linearly compact.

(e) (Tychonov) If $\{V_\alpha\}_\alpha$ is a collection of linearly compact vector spaces then its product $\prod_\alpha V_\alpha$ and its direct sum $\bigoplus_\alpha V_\alpha$ are linearly compact.

Proof. Let U be any open vector subspace of V. Then, $A+U$ is closed, that is $A+U = \overline{A+U} \supseteq \overline{A}+U \supseteq A+U$, thus, $\overline{A}+U = A+U$. Since, $(A+U)/U \sim 0$ it follows that $(\overline{A}+U)/U \sim 0$.

For (b), since f is a continuous linear map $V \prec f^{-1}(U)$ for all U open vector subspaces of W. Hence by Lemma 1.7 $f(V) \prec U$ for all open vector subspaces U of W. The previous observation and (b) yield (a). If V is discrete, then $\{0\}$ is an open vector subspace of E; therefore, V is finite dimensional.

For (d), if $A \subseteq V$ is a closed vector subspace, and $V \prec U$ for all open vector subspaces U by Lemma 1.7 we get $A \prec U$.

Finally, for (d), it is enough proving for open vector subspaces $U = \prod_\beta U_\beta \times \prod_\gamma V_\gamma$ where β ranges over a finite set, γ ranges over $\alpha \neq \beta$ and U_β is an open vector subspace of V_β. Then, the quotient

$$\prod_\alpha V_\alpha / U \cong \prod_\beta V_\beta / U_\beta$$

where $\cong$ is a topological and algebraic isomorphism. Since V_α is linearly compact for all α and β ranges over a finite set we conclude that $\prod_\alpha V_\alpha / U$ is finite dimensional; therefore, $\prod_\alpha V_\alpha$ is linearly compact. The proof is analogous for $\bigoplus_\alpha V_\alpha$. $\square$

Completion

Definition 1.12. Let V be a linearly topologized vector space. Given a local base of open vector subspaces $\mathscr{U}$ we will say that a collection $(x_U)_{U \in \mathscr{U}}$ is a **net indexed by** $\mathscr{U}$. We will say that $(x_U)_{U \in \mathscr{U}}$ converges to x if for any $U \in \mathscr{U}$ there exists a $W \in \mathscr{U}$ such that for every $U' \subseteq W$ and $U' \in \mathscr{U}$

$$x_{U'} - x \in U.$$

Remark 1.13. The previous definition can be generalized to an arbitrary topological space and nets indexed by any directed set. In our particular case, the directed set $\mathcal{D} = \mathscr{U}^{\mathrm{op}}$.

We now introduce Cauchy nets and completion.

Definition 1.14. A net $(x_U)_{U \in \mathscr{U}}$ in a linearly topologized vector space indexed by an open local base $\mathscr{U}$ of open vector subspaces is said to be **Cauchy** if for every U open vector subspace of zero there exists $W \in \mathscr{U}$ such that for every $U_1, U_2 \in \mathscr{U}$ and $U_1, U_2 \subseteq U$ we have $x_{U_1} - x_{U_2} \in U$.

Definition 1.15. If V be a linearly topologized vector space, we say that V is said to be **complete** if

$$V \cong \varprojlim_{U \in \mathscr{U}} V/U$$

for an open local base of vector subspaces $\mathscr{U}$. We leave unproven that this definition is independent of the choice of $\mathscr{U}$. Details can be found in include reference. Moreover, V is complete if and only if every Cauchy net is convergent. One can see this by identifying the inverse limit as a subspace of the product.

Complete this! (no pun intended)

1.2 TATE SPACES

Lattices

Definition 1.16. If V is a linearly topologized vector space we say that a **c-lattice** is an open linearly compact subspace of V, *dually* a discrete linearly cocompact subspace is a **d-lattice**.

First, we prove that existence of a c-lattice in a linearly topologized vector space is equivalent to existence of a d-lattice.

Proposition 1.17. *A linearly topologized vector space V contains a c-lattice if and only if it contains a d-lattice.*

Proof. Suppose L is a c-lattice in V. Choose any direct complement D of L, that is, $V = L \oplus D$. Since L is open, then D is discrete as $D \cap L = 0$, thus 0 is open in D. Moreover, D is closed it is the kernel the projection $V \to L$ (which is continuous because L is open). Finally, we check that D is linearly cocompact: let U be any open vector subspace of V, the composition $L \hookrightarrow V \twoheadrightarrow V/(D+U)$ induces a surjection

$$L/(L \cap U) \twoheadrightarrow V/(D+U).$$

Since $\dim L/(L \cap U) < \infty$ we conclude that $\dim V/(D+U) < \infty$.
Now, suppose D is a d-lattice. Thus, there exists an open vector subspace U such that $U \cap D = 0$. This time, choose L a direct complement for D containing U. Then, the projection $V \to D$ is continuous because U is mapped to zero. Therefore, L is open. Now we prove that L is linearly compact. Let U be any open vector subspace. Then, the composition $V \twoheadrightarrow L \twoheadrightarrow L/(L \cap U)$ induces a surjection

$$V/(D + (L \cap U)) \twoheadrightarrow L/(L \cap U)$$

since both L and U are open, so is $L \cap U$. Therefore, $\dim V/(D + (L \cap U)) < \infty$. It follows that $\dim L/(L \cap U) < \infty$ and L is linearly compact. □

Remark 1.18. Note that in the proof of Proposition 1.17 it is not strictly necessary to choose a direct complement, one can choose a direct complement up to finite dimension; that is, $L + D \sim V$ and $L \cap D \sim 0$.

We now give a characterization of lattices in terms of the relation $\prec$.

Proposition 1.19. *If V admits a c-lattice, then the set of c-lattices constitutes a system of neighborhoods of zero consisting of mutually commensurable vector subspaces.*

Proof. If L and L' are two c-lattices in V then $L \prec L'$ and $L' \prec L$ because both are open; therefore, all c-lattices are commensurable. Moreover, if U is any open vector subspace and L is a c-lattice we claim that $L \cap U$ is a c-lattice as well. Indeed, let U' be any open vector subspace, then $L \cap U \prec L \prec U'$. In addition, since L and U are open, $L \cap U$ is open as well. Hence, $L \cap U \subseteq U$ is a c-lattice. This proves the statement. $\square$

We are now ready to introduce the definition of a Tate space.

Definition 1.20. A linearly topologized vector space V is a **Tate space** if it is complete and admits a c-lattice. By the previous proposition and the observation in Definition 1.15 it follows that

$$V \cong \varprojlim_{L \in \mathscr{U}} V/L$$

where $\mathscr{U}$ runs through all c-lattices of V.

Example 1.21. We give some examples of Tate spaces.

(a) Any vector space endowed with the discrete topology is a Tate space.

(b) If $\{V_\alpha\}_\alpha$ is any pro-system of finite dimensional vector spaces (thus, each one endowed with the discrete topology by Proposition 1.4), let V be their inverse limit endowed with the inverse limit topology. We claim that this is a linearly compact space. Indeed, if we realize V as a subspace of the product $\prod_\alpha V_\alpha$, then basic open vector subspaces are just restriction of finite coordinates. Hence, the quotient of V by any basic open vector subspace is a finite product of V_α, since all V_α are finite dimensional we conclude that V is linearly compact and therefore a Tate space.

(c) Let $V = k((t))$ with the topology generated by letting $t^n k[[t]]$ for $n \in \mathbb{Z}$ be a system of neighborhoods of zero. Then, $V = k[[t]] \oplus tk[t^{-1}]$ where $k[[t]]$ is the completion of $k[x]$ in the $\langle x \rangle$-adic topology, hence by the previous item linearly compact and, since it is open is a c-lattice. By the argument given in Proposition 1.17 $tk[t^{-1}]$ is a d-lattice. Therefore, V is a Tate space that is not linearly compact nor discrete.

Duality

If V is a Tate space we consider the following topology on the dual space V^* (where by dual space we mean topological dual). Open vector subspaces are given by

$$L^{\perp} = \{\phi \in E^* \colon \phi|_L = 0\}$$

where L is a linearly compact subspace. Equivalently, one can define open vector subspaces in E^* to be D^* where D a direct complement of a linearly compact vector subspace L in E (in this case $D^* \hookrightarrow E^*$ using the decomposition $L \oplus D$).
First, we prove that the word *dually* in Definition 1.9 actually makes sense.

Lemma 1.22. *Duality interchanges linearly compact with discrete spaces and vice-versa.*

Proof. If L is a linearly compact vector space, then $L^{\perp}$ is open in L^*, thus L^* is discrete. If D is discrete, then $D \cong k^{\oplus \Lambda}$ for some Λ and endowing $k^{\oplus \Lambda}$ with the discrete topology the previous isomorphism is a homeomorphism too. Moreover, since D is discrete every linear functional is continuous. Using Proposition 1.3 and the well known identity (where maps are isomorphisms in LinTop_k)

$$(k^{\oplus \Lambda})^* = \mathrm{Hom}_k(k^{\oplus \Lambda}, k) \cong \prod_{\Lambda} \mathrm{Hom}_k(k, k) \cong \prod_{\Lambda} k$$

we get the desired result by Tychonov's theorem in Proposition 1.11. ☐

Remark 1.23. A closer look at the proof of the previous lemma indicates that the dual space of a discrete space is always complete.

Proposition 1.24. *If V is a Tate space then V^* with the topology previously introduced is also a Tate space.*

Proof. If we decompose $V = L \oplus D$ where L is a c-lattice and D a d-lattice then $V^* \cong L^* \oplus D^*$ and by Lemma 1.22 L^* is discrete and D^* is linearly compact. Observe that D^* is open in V^* since it is the kernel of the projection $V^* \to V^*/L^\perp$ and $V^*/L^\perp$ is discrete by the description of our topology in the dual V^*. Since L^* is discrete, then it is complete. Moreover, by the previous remark, D^* is complete, hence V^* is complete too. □

We're ready to prove the analog of Pontryagin's duality for Tate spaces.

Theorem 1.25. *For a Tate space V the canonical map $V \to V^{**}$ is an isomorphism.*

Proof. It is enough to prove it for complete linearly compact spaces and discrete spaces, as every Tate space can be decomposed into a direct sum of a c-lattice and a d-lattice. First, we do it for discrete spaces. Suppose D is a discrete vector space. Then, the canonical map

$$\mathrm{ev}\colon D \to D^{**}$$

is open and continuous because D and D^{**} are both discrete by Lemma 1.22. Moreover, it is injective, because for every nonzero $v \in D$ there exists a linear continuous functional $\phi \in D^*$ such that $\phi(v) \neq 0$. Finally, we prove surjectivity. Let $\psi \in D^{**}$. Since $\ker \psi$ is open it contains a basic open vector subspace $A^\perp$ such that $A \subseteq D$ is a linearly compact subspace. Therefore, since D^* is linearly compact it follows that $D^* \sim A^\perp$, that is, the quotient $D^*/A^\perp$ is finite dimensional. Recall that the inclusion $\iota\colon A \to D$ induces an isomorphism $D^*/A^\perp \to A^*$ which is a homeomorphism since both spaces are discrete. We can factor ψ so that the following diagram commutes

$$\begin{tikzcd}
D^* \arrow[r, "\psi"] \arrow[d] & k \\
D^*/A^\perp \arrow[ur, dashed, "\tilde{\psi}"] \arrow[d, "\cong"'] & \\
A^* \arrow[uur, dashed, bend right, "\overline{\psi}"'] &
\end{tikzcd}$$

However, A^* is finite dimensional, therefore, there exists $a \in A$ such that $\overline{\psi} = \mathrm{ev}_a$ as maps from $A^* \to k$. Moreover, since $A^\perp \subseteq \ker \psi$ we conclude that $\psi = \mathrm{ev}_a$ as maps $D^* \to k$. This implies surjectivity. Thus $D \to D^{**}$ is an isomorphism of topological vector spaces.

Now, suppose L is a complete linearly compact space. We check first that the map

$$\mathrm{ev}\colon L \to L^{**}$$

is continuous. Let $A^{\perp}$ be an open vector subspace in L^{**} where $A \subseteq L^*$ is a linearly compact subspace. By Lemma 1.22 L^* is discrete, hence A is finite dimensional. Suppose that $A = \mathrm{span}(\phi_1, \ldots, \phi_n)$ for some $\phi_1, \ldots, \phi_n \in A$. Then, $\mathrm{ev}^{-1}(A^{\perp}) = \ker \phi_1 \cap \ldots \cap \ker \phi_n$ which is open in L. Now, we check that ev is injective. Let $v \in L$ be a nonzero vector. Choose a decomposition of $L = U \oplus F$ where U is open and F is finite dimensional containing v (this can be done because L is separated and linearly compact). Let ϕ be a linear functional such that restricted to U is zero and $\phi(v) \neq 0$. Since U is open and F discrete such ϕ exists and it is continuous. This implies injectivity of ev. Now we check that ev is surjective. Let $\mathscr{U}$ is an open local base of vector subspaces. Let $\psi\colon L^* \to k$ be a continuous linear functional. By pulling back $\pi_U\colon L \to L/U$ we get an injection $\pi_U^*\colon (L/U)^* \hookrightarrow L^*$ for every $U \in \mathscr{U}$. Since L is linearly compact, then L/U is finite dimensional, thus, there exists $v_U \in L$ such that $\psi \circ \pi_U^* = \mathrm{ev}_{x_U}$ where $\mathrm{ev}\colon L/U \to (L/U)^{**}$. In particular, observe that if $U, U' \in \mathscr{U}$ and $U' \subseteq U$ we get an induced surjection $(L/U)^* \hookrightarrow (L/U')^*$ such that the following diagram

$$\begin{tikzcd}
L^* \arrow[r, "\psi"] & k \\
(L/U')^* \arrow[u, hook] \arrow[ur, "\mathrm{ev}_{x_{U'}}"'] & \\
(L/U)^* \arrow[u, two heads] \arrow[uur, bend right, "\mathrm{ev}_{x_U}"'] \arrow[uu, bend left, hook] &
\end{tikzcd}$$

commutes. Observe that this implies that $(v_U)_{U \in \mathscr{U}}$ is a Cauchy net. Indeed, let $U \in \mathscr{U}$ and $W = U$ as in Definition 1.14. Let

$U_1, U_2 \subseteq U$ and $U_1, U_2 \in \mathscr{U}$. Notice that commutativity of the following diagram

$$\begin{tikzcd}
& (L/U_1)^* \arrow[dr, hook] \arrow[drr, bend left, "\mathrm{ev}_{x_{U_1}}"] & & \\
(L/U)^* \arrow[ur, two heads] \arrow[dr, two heads] \arrow[rr, hook] \arrow[rrr, bend left=80, "\mathrm{ev}_{x_U}"] \arrow[rrr, bend right=80, "\mathrm{ev}_{x_U}"'] & & L^* \arrow[r, "\psi"] & k \\
& (L/U_2)^* \arrow[ur, hook] \arrow[urr, bend right, "\mathrm{ev}_{x_{U_2}}"'] & &
\end{tikzcd}$$

implies that $(x_U)_{U\in\mathscr{U}}$ is Cauchy, since $\mathrm{ev}_{x_{U_1}} = \mathrm{ev}_{x_{U_2}}$ coincide in $(L/U)^*$. By completeness there exists $x \in L$ limit of $(x_U)_{U\in\mathscr{U}}$. We claim that $\psi = \mathrm{ev}_v$. Let $\phi \in L^*$. Then, $\ker\phi$ is open and there exists $U \in \mathscr{U}$ such that $U \subseteq \ker\phi$ and by linear compactness $U \sim L$. Hence, if we factor ϕ as follows

$$\begin{tikzcd}
L \arrow[r, "\phi"] \arrow[d, "\pi_U"'] & k \\
L/U \arrow[ur, "\phi_U"'] &
\end{tikzcd}$$

since L/U is discrete we conclude that ϕ_U is continuous. In other words, the image of ϕ_U under the inclusion $(L/U)^* \hookrightarrow L^*$ is ϕ. Thus, $\psi(\phi) = \phi_U(x_U + U) = \phi(x_U)$. We claim that $\psi(\phi) = \phi(x)$. Indeed, convergence of (x_U) implies that there exists a $W \in \mathscr{U}$ such that $W \subseteq U$ and $x - x_W \in U$. Thus, $\psi(\phi) = \phi_W(x_W + W) = \phi(x_W) = \phi(x)$, since $U \subseteq \ker\phi$. This implies surjectivity of $\mathrm{ev}\colon L \to L^{**}$. To conclude, we prove that ev is open. Let U be any open vector subspace in L, thus $L = U \oplus F$ for some F finite dimensional. We claim that $\mathrm{ev}(U) = (F^*)^\perp$. First, the inclusion $\mathrm{ev}(U) \subseteq (F^*)^\perp$ is immediate. Let $\psi \in (F^*)^\perp$. Let $x \in L$ such that $\mathrm{ev}_x = \psi$. Write $x = u + f$ where $u \in U$ and $f \in F$. Hence, $\mathrm{ev}_x = \mathrm{ev}_u + \mathrm{ev}_f$. Since ev is injective, it follows that there exists some $\phi \in F^*$ such that $\phi(f) \neq 0$ if f is nonzero. Therefore, $f = 0$ and $\psi \in \mathrm{ev}(U)$. This concludes the proof. □

Remark 1.26. Observe that completeness cannot be dropped in the definition of a Tate space while preserving duality. Indeed, if V is linearly compact but not complete its dual is discrete by

Lemma 1.22 and by Remark 1.23 its double dual is complete, hence $V \to V^{**}$ cannot be an isomorphism. In fact, during the proof of the duality theorem we checked that V^{**} is the completion of V.

Remark 1.27. We now discuss definitions of linearly compact spaces as given in [Lef42] and [BD04]. In [Lef42] it is proven that a linearly compact vector space is immediately complete while our definition does not imply it necessarily. However, when V is a complete space both definitions coincide. Indeed, Lefschetz proves that every linearly compact space is the dual of a discrete space which coincides with our definition of a complete linearly compact vector space by Theorem 1.25. Therefore, his definition of a **locally linearly compact vector space** (that is, a linearly topologized vector space admitting an open linearly compact vector subspace) coincides with our notion of Tate space.

Morphisms

A **morphism** of Tate spaces is a continuous linear homomorphism between Tate spaces.

Definition 1.28. A morphism $f\colon V \to W$ of Tate spaces is said to be **linearly compact** if the closure of $f(V)$ is linearly compact in W. Dually, it is **discrete** if $\ker f$ is open in V.

First, we check the duality natural property for morphisms of Tate spaces.

Proposition 1.29. *A morphism $f\colon V \to W$ of Tate spaces is linearly compact if and only if f^* is discrete.*

Proof. Suppose f^* is linearly compact, then $\ker f^* = f(V)^\perp$. However, if $\phi \in W^*$ and $\phi(f(V)) = 0$ then $\phi(\overline{f(V)}) = 0$ by continuity of ϕ. Therefore, $\ker f^* = \overline{f(V)}^\perp$ which is open because $\overline{f(V)}$ is linearly compact. Now, suppose f^* is discrete. Thus, $\ker f^*$ contains a basic open subspace $A^\perp$ such that A is linearly compact in W. Therefore, $f(V) \subseteq A$ then $\overline{f(V)} \subseteq A$ and by item (c) in Proposition 1.11 $\overline{f(V)}$ is linearly compact. □

Discrete and linearly compact operators form a 2-sided ideal in Hom; that is

Proposition 1.30. *If f is a linearly compact operator (respectively discrete) then its composition with an arbitrary morphism of Tate spaces is also linearly compact (respectively discrete).*

Proof. Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ be morphisms of Tate spaces such that g is linearly compact. Then, $g \circ f(A) \subseteq \overline{g(B)}$ which is linearly compact, thus $\overline{g \circ f(A)}$ is linearly compact too. On the other hand, note that $h(\overline{g(B)}) \subseteq \overline{h \circ g(B)}$; therefore $\overline{h(\overline{g(B)})} = \overline{h \circ g(B)}$. However, $\overline{g(B)}$ is linearly compact and by item (b) in Proposition 1.11 $h(\overline{g(B)})$ is linearly compact. In addition, the statement for discrete operators follows from the previous proposition. □

Remark 1.31. If f is a compact operator and g is a discrete operator, then gf is **finite**; that is, $\dim gf(V) < \infty$.

Proof. We have $\overline{f(V)} \prec \ker g$, therefore, $\overline{f(V)}/(\overline{f(V)} \cap \ker g)$ is finite dimensional. We have a surjection

$$\frac{\overline{f(V)}}{\overline{f(V)} \cap \ker g} \to gf(V)$$

which implies that gf is finite. □

Definition 1.32. Let V and W be Tate spaces. We denote $\mathrm{Hom}_+(V,W)$ to be the set of linearly compact morphisms and $\mathrm{Hom}_-(V,W)$ the set of discrete ones. Also, set $\mathrm{Hom}_0(V,W)$ to be $\mathrm{Hom}_+(V,W) \cap \mathrm{Hom}_-(V,W)$.

Proposition 1.33. *The sets* $\mathrm{Hom}_-(V,W), \mathrm{Hom}_+(V,W)$ *and* $\mathrm{Hom}_0(V,W)$ *are vector subspaces of* $\mathrm{Hom}(V,W)$*. Moreover,*

$$\mathrm{Hom}_-(V,W) + \mathrm{Hom}_+(V,W) = \mathrm{Hom}(V,W).$$

Proof. Let L be a c-lattice in V and consider $\pi \colon V \to L$ be a continuous linear projection. Then π realized as an element of $\mathrm{End}(V)$ satisfies $\pi \in \mathrm{End}_+(V)$ and $1 - \pi \in \mathrm{End}_-(V)$. Hence, by Proposition 1.30 for every $f \in \mathrm{Hom}(V,W)$ $f \circ \pi$ and $f \circ (1 - \pi)$ are linearly compact and discrete respectively. It follows

$$\mathrm{Hom}_-(V,W) + \mathrm{Hom}_+(V,W) = \mathrm{Hom}(V,W).$$

The other statements are immediate. □

I'll include further theory if necessary.

2

TRACE AND RESIDUE

We extend the definition of trace to a certain class of infinite dimensional operators to define an abstract residue. We follow the original structure in Tate's elegant article [Tat68] while translating his statements in the language of Tate's Linear Algebra.

2.1 FINITEPOTENT MAPS AND THEIR TRACE

Let k be a fixed ground field and V a vector space over k. In this section we will expand the notion of trace of a linear endomorphism to include certain operators even when V is infinite dimensional.

Finitepotent maps

Definition 2.1. We will say a linear map $f\colon V \to V$ is **finitepotent** if

$$\dim f^n(V) < \infty$$

for sufficiently large n.

We characterize finitepotent maps as follows.

Lemma 2.2. *A linear map $f\colon V \to V$ is finitepotent if and only if there exists a subspace $W \subseteq V$ such that*

(i) $\dim f(W) < \infty$*,*

(ii) $f(W) \subseteq W$*,*

(iii) the induced map $\bar{f}\colon V/W \to V/W$ is nilpotent.

Proof. If f is finitepotent choose $W = f^n(V)$ for sufficiently large n. The first condition follows from definition. Also, $f(W) = f^{n+1}(V) \subseteq f^n(V) = W$. In addition, $\bar{f}^n = 0$. On the other hand, if such W exists, note that condition (ii) assures that $\bar{f}$ is well defined. Moreover, as $\bar{f}$ is nilpotent, $f^n V \subseteq W$ for sufficiently large n and by condition (i) above $\dim f^n(V) < \infty$. □

Trace

If f is a finitepotent map and W is as above, $\mathrm{tr}_V(f) \in k$ may be defined as $\mathrm{tr}_W(f)$ where $\mathrm{tr}_W(f)$ is the ordinary trace of f viewed as a endomorphism of W (which is defined since f has finite-rank viewed in $\mathrm{End}(W)$). First, we will check that this definition does not depend on the choice of W. Suppose $W_1, W_2 \subseteq V$ suffice the properties on Lemma 2.2, then $W = W_1 + W_2$ suffices them too. Hence, as the induced maps on W/W_1 and W/W_2 are nilpotent, they have have zero ordinary trace and since

$$\mathrm{tr}_W(f) = \mathrm{tr}_{W_1}(f) + \mathrm{tr}_{W/W_1}(f)$$
$$\mathrm{tr}_W(f) = \mathrm{tr}_{W_2}(f) + \mathrm{tr}_{W/W_2}(f),$$

we obtain $\mathrm{tr}_{W_1}(f) = \mathrm{tr}_{W_2}(f)$, our desired result.

This definition extends some of the properties of the ordinary trace.

Lemma 2.3. *(a) If $\dim V < \infty$, any endomorphism f is finitepotent and $\mathrm{tr}_V(f)$ coincides with the ordinary trace.*

(b) If f is nilpotent, then it is finitepotent and $\mathrm{tr}_V(f) = 0$.

(c) If f is finitepotent and U is a subspace such that $f(U) \subseteq U$ then the induced maps on U and V/U are finitepotent and satisfy

$$\mathrm{tr}_V(f) = \mathrm{tr}_U(f) + \mathrm{tr}_{V/U}(f)$$

Proof. Both (a) and (b) are immediate. For (c) if W suffices the properties in Lemma 2.2 for f then $W \cap U$ and $(W+U)/U$ suffice them for the induced maps, that is, they're finitepotent. Since $W/(W \cap U) \cong W + U/U$, the diagram

$$\begin{tikzcd} W/(W\cap U) \arrow[r, "\cong"] \arrow[d, "f"] & (W+U)/U \arrow[d, "f"] \\ W/(W\cap U) \arrow[r, "\cong"] & (W+U)/U \end{tikzcd}$$

commutes and trace is invariant under conjugation, we get $\mathrm{tr}_{W/(W\cap U)}(f) = \mathrm{tr}_{(W+U)/U}(f)$. Hence

$$\mathrm{tr}_V(f) = \mathrm{tr}_W(f) = \mathrm{tr}_{W\cap U}(f) + \mathrm{tr}_{(W+U)/U}(f) = \mathrm{tr}_U(f) + \mathrm{tr}_{V/U}(f)$$

□

Definition 2.4. A subspace F of $\mathrm{End}_k(V)$ is said to be a **finitepotent subspace** if there exists an n such that for any family of n elements $f_1, \dots, f_n \in F$, the space $f_1 f_2 \cdots f_n V$ is finite dimensional. Note that every $f \in F$ is finitepotent.

The following is the natural linearity property for tr.

Proposition 2.5. *If F is a finitepotent subspace then $\mathrm{tr}_V \colon F \to k$ is k-linear*

Proof. It is enough to prove it in the case that F is finite dimensional. Let $W = F^n V$ for n as in the definition of finitepotent subspace, thus $\dim W < \infty$. Hence, for all $f \in F$, W suffices the conditions in Lemma 2.2. It follows that $\mathrm{tr}_V(f) = \mathrm{tr}_W(f)$ which is linear. □

Remark 2.6. In his paper, Tate asked if general linearity for finitepotent maps followed. His question was answered negatively in [AST07] where general linearity is reduced to the following: if the sum of two nilpotent endomorphisms is finitepotent, is the sum necessarily traceless?

Proposition 2.7. *If $f, g \in \mathrm{End}_k(V)$ and fg is finitepotent then gf is also finitepotent and*

$$\mathrm{tr}_V(fg) = \mathrm{tr}_V(gf).$$

Proof. Since fg is finitepotent let $W = (fg)^n V$ for sufficiently large n has finite dimension. On the other hand, $(gf)^{n+1} V = g(fg)^n f(V) \subseteq g(W)$, therefore, gf is also finitepotent. Let $W' = (gf)^n V$, then $g(W') \subseteq W$ and $f(W) \subseteq W'$. Thus,

$$\dim W' \leq \dim g(W) \leq \dim W,$$

and,

$$\dim W \leq \dim f(W) \leq \dim W',$$

which implies that $W \cong W'$ and that g and f induce mutually inverse isomorphisms between W and W'. Moreover, the diagram

$$\begin{tikzcd} W \arrow[r, "fg"] \arrow[d, "g"] & W \arrow[d, "g"] \\ W' \arrow[r, "gf"] & W' \end{tikzcd}$$

commutes. We conclude $\mathrm{tr}_W(fg) = \mathrm{tr}_{W'}(gf)$ and it follows $\mathrm{tr}_V(fg) = \mathrm{tr}_V(gf)$. □

Trace and Tate Spaces

Suppose that V is a Tate space and consider $E = \mathrm{End}_k(V)$ the space of continuous endomorphisms of V. Thus, by Proposition 1.30 and Proposition 1.33 we have 2-sided ideals $\mathrm{End}_0(V), \mathrm{End}_+(V)$ and $\mathrm{End}_-(V)$ of E such that $\mathrm{End}_+(V) + \mathrm{End}_-(V) = E$ and $\mathrm{End}_0(V) = \mathrm{End}_+(V) \cap \mathrm{End}_-(V)$. Moreover, Remark 1.31 implies that $\mathrm{End}_0(V)$ is a finitepotent subspace.

Lemma 2.8. *Suppose $f \in \mathrm{End}_+(V)$ and $g \in \mathrm{End}_-(V)$ or $f \in \mathrm{End}_-(V)$ and $g \in \mathrm{End}_+(V)$. Then the commutator $[f, g] = fg - gf$ belongs to $\mathrm{End}_0(V)$ and it is traceless.*

Proof. This immediate from the previous discussion and Proposition 2.7. □

2.2 DIFFERENTIAL CALCULUS

In this section we introduce the theory of derivations and differentials over an arbitrary commutative ring A. Let M be a A-module and k a commutative ring. We follow [GD64] Section 20 and [Mat86] Section 25.

Definition 2.9. A **derivation** from A to M is a map $D\colon A \to M$ satisfying properties

(i) $D(a + b) = D(a) + D(b)$ and,

(ii) (*Leibniz's Rule*) $D(ab) = aD(b) + bD(a)$

for all $a, b \in A$. The set of derivations from A to M becomes a A-module in a natural way. We will write it as $\mathrm{Der}(A, M)$. Moreover if A is a k-algebra through a map $\varphi\colon k \to A$ we say that D is a **k-derivation** if D is a derivation and $D \circ \varphi = 0$. In this case, the set of all k-derivations is denoted $\mathrm{Der}_k(A, M)$.

If $M = A$, we will denote $\mathrm{Der}_k(A, A)$ simply by $\mathrm{Der}_k(A)$. In particular, if D and D' are two k-derivations then its bracket $[D, D'] = DD' - D'D$ under composition as $A \to A$ maps is also a k-derivation. Therefore, $\mathrm{Der}_k(A)$ under this structure is a Lie Algebra.

Definition 2.10. Let B be a k-algebra and C an ideal in B with $C^2 = 0$; set $A = B/C$. In this way, C can be viewed as a A-module. In this situation we say that B is an **extension** of the

k-algebra A by the A-module C. Usually, we simply write the exact sequence

$$0 \to C \to B \xrightarrow{\pi} A \to 0$$

As usual, we will say that such sequence **splits** if there exists a retraction; that is, a k-algebra homomorphism $\rho\colon A \to B$ such that $\pi \circ \rho = 1_A$. In this case we can identify $B = C \oplus A$. Conversely, starting from any k-algebra A and any A-module C, one can always define a structure on $A \oplus C$ such that $A \oplus C$ is an extension of A by C. Namely,

$$(a, c)(a', c') = (aa', ac' + a'c)$$

for $a, a' \in A$ and $c, c' \in C$. Common notations for this algebra are $D_A(C)$ or $A * C$.

Definition 2.11. Given a commutative diagram of k-algebras

$$\begin{tikzcd} B \arrow[r, "f"] & A \\ & C \arrow[u, "g"] \arrow[ul, "h"] \end{tikzcd}$$

thinking of f as a fixed map; we say that h is a **lifting** of g to B.

Lemma 2.12. *Suppose we're given a commutative diagram as in the previous definition with an additional lifting $h'\colon C \to B$. Then, if $K = \ker f$ satisfies $K^2 = 0$ it follows that $h - h'\colon C \to K$ is a k-derivation. Conversely, if $D \in \mathrm{Der}_k(C, K)$ then $h + D$ is another lifting of g to B.*

Proof. First, observe that $(h - h')(C)$ lies in K because both h and h' are liftings of g to B. Since $K^2 = 0$, then K can be considered as $f(B)$-module and by means of g as a C-module. Then, $h - h'\colon C \to K$ is a map of C-modules. Now, let $c, c' \in C$ then

$$\begin{aligned}(h - h')(cc') &= h(c)h(c') - h'(c)h'(c') \\ &= h(c)h(c') - h'(c)h'(c') - h(c)h'(c') + h'(c')h(c)\end{aligned}$$

since $c \cdot k = h(c)k = h'(c)k$ for all $k \in K$ it follows that

$$\begin{aligned}(h - h')(cc') &= c \cdot h(c') - c' \cdot h'(c') - c \cdot h'(c') + c' \cdot h(c) \\ &= c \cdot (h - h')(c') + c' \cdot (h - h')(c)\end{aligned}$$

which implies that $h - h'$ is a k-derivation. Observe that $h + D$ is a lifting because $D(C)$ lies in K. $\square$

Theorem 2.13. *If A is a k-algebra, consider the covariant functor from the category Mod_A to itself given by $M \mapsto \mathrm{Der}_k(A, M)$. This functor is representable.*

Proof. Let $\mu \colon A \otimes_k A \to A$ be the k-algebra homomorphism given by $f \otimes g \to fg$. Set

$$I = \ker \mu, \quad \Omega_{A/k} = I/I^2, \quad \text{and,} \quad B = (A \otimes_k A)/I^2.$$

Thus, μ induces $\mu' \colon B \to A$ such that

$$0 \to \Omega_{A/k} \to B \to A \to 0$$

is an extension of A by $\Omega_{A/k}$. We claim that this extension splits. Moreover it has two splittings, by considering retractions

$$j_1 \colon A \to B \quad \text{and,} \quad j_2 \colon A \to B,$$

defined by $a \mapsto a \otimes 1 \mod I^2$ and $a \mapsto 1 \otimes a \mod I^2$. By Lemma 2.12 $d := j_2 - j_1$ is a k-derivation of A to $\Omega_{A/k}$. Now, we prove that

$$\mathrm{Der}_k(A, M) \cong \mathrm{Hom}_A(\Omega_{A/k}, M). \tag{1}$$

Let $D \in \mathrm{Der}_k(A, M)$ and define $\varphi \colon A \otimes_k A \to A * M$ by $\varphi(x \otimes y) = (xy, xD(y))$ then φ is a k-algebra homomorphism since it is compatible with the operation in $A * M$ defined in Definition 2.10. In addition, if $\sum x_i \otimes y_i$ lies in I then

$$\mu\left(\sum x_i \otimes y_i\right) = \sum x_i y_i = 0 \implies \varphi\left(\sum x_i \otimes y_i\right) = \left(0, \sum x_i D(y_i)\right)$$

whence $\varphi(I)$ lies in M. Moreover, by Leibniz's Rule φ factors through I^2 yielding a map $f \colon \Omega_{A/k} \to M$. For $a \in A$ it follows that

$$\begin{aligned} f(da) &= f(1 \otimes a - a \otimes 1 \mod I^2) = \varphi(1 \otimes a) - \varphi(a \otimes 1) \\ &= D(a) - aD(1) = D(a). \end{aligned}$$

Therefore, $D = f \circ d$. Now, we prove that such f is unique. First, observe that $\Omega_{A/k}$ has the A-module structure induced by multiplication by $a \otimes 1$ (or $1 \otimes a$ since $1 \otimes a - a \otimes 1 \in I$). Therefore, if $\xi = \sum x_i \otimes y_i \mod I^2 \in \Omega_{A/k}$ then $a\xi = \sum a x_i \otimes y_i \mod I^2$, and $f(a\xi) = \sum a x_i D(y_i) = a f(\xi)$, so that f is A-linear. We have

$$a \otimes a' = (a \otimes 1)(1 \otimes a' - a' \otimes 1) + aa' \otimes 1$$

so that if $\omega = \sum x_i \otimes y_i \in I$ then $\omega \mod I^2 = \sum x_i dy_i$ since $\sum x_i y_i = 0$. We conclude that $\{da \mid a \in A\}$ is a set of generators for the A-module $\Omega_{A/k}$. This implies uniqueness of f. Therefore, (1) holds. □

Definition 2.14. The module $\Omega_{A/k}$ introduced in the proof of the previous theorem is called **module of differentials** of A over k or **module of Kähler differentials**, and for $a \in A$ the element $da \in \Omega_{A/k}$ is called the **differential** of a.

Example 2.15. If A is generated as k-algebra by a subset $S \subseteq A$ then $\Omega_{A/k}$ is generated by $\{ds \mid s \in S\}$. Indeed, if $a \in A$ then there exist $a_i \in S$ and a polynomial $f(X) \in k[X_1, \ldots, X_n]$ such that $a = f(a_1, \ldots, a_n)$. Thus,

$$da = \sum_{i=1}^{n} f_i(a_1, \ldots, a_n) da_i \quad \text{where} \quad f_i = \partial_i f$$

In particular if $A = k[X_1, \ldots, X_n]$ then $\Omega_{A/k} = AdX_1 + \ldots AdX_n$ since $X_1, \ldots, X_n$ are linearity independent; this follows immediate from the fact that $\partial_i X_j = \delta_{ij}$.

Lemma 2.16. *Let K be a k-commutative algebra. There exists a k-linear map*

$$c\colon K \otimes_k K \to \Omega_{K/k}$$

so that $f \otimes g \mapsto f dg$ satisfying:

(i) c is surjective.

(ii) $\ker c$ is generated over k by the elements of the form

$$f \otimes gh - fg \otimes h - fh \otimes g$$

Proof. The k-bilinear map $(f, g) \mapsto f dg$ induces c. Since $\{df \mid f \in K\}$ is a generating set for $\Omega_{K/k}$ as a K-module it follows that c is surjective. finish this proof □

Further theory to be included if necessary.

2.3 ABSTRACT RESIDUE AND ITS PROPERTIES

During this section let k be a field, K a commutative k-algebra with 1, and V a K-module so that when viewed as a k-vector space is a Tate space and K acts continuously on V. Namely, for all $f \in K$ the map

$$\begin{aligned} f\colon V &\to V \\ x &\mapsto fx \end{aligned}$$

is continuous. In this way, K operates on V through E (maintaining notation from Section 2.1). We will not differentiate $f \in K$ with its induced map in E.

Lemma 2.17. *Let $f, g \in K$. Then, there are $f_+, g_+ \in \mathrm{End}_+(V)$ so that*

$$f = f_+ \mod \mathrm{End}_-(V), \quad g = g_+ \mod \mathrm{End}_-(V)$$

and, the equality

$$\mathrm{tr}([f_+, g_+]) = \mathrm{tr}([f, g_+]) = \mathrm{tr}([f_+, g])$$

holds.

Proof. The existence of f_+ and g_+ is immediate from the fact that $E = \mathrm{End}_+(V) + \mathrm{End}_-(V)$. Clearly $[f_+, g_+] \in \mathrm{End}_+(V)$ and the fact that K is commutative implies that $[f, g] = 0$. Therefore,

$$[f_+, g_+] = [f, g] \mod \mathrm{End}_-(V).$$

Hence, $[f_+, g_+] \in E_0$. Similarly, $[f, g_+]$ and $[f_+, g]$ belong to $\mathrm{End}_0(V)$. Whence, one can consider their trace. Furthermore, $f_+ \in \mathrm{End}_+(V)$ and $g_+ - g \in \mathrm{End}_-(V)$ thus $\mathrm{tr}([f_+, g_+ - g]) = 0$ by Lemma 2.8; we conclude $\mathrm{tr}([f_+, g_+]) = \mathrm{tr}([f_+, g])$. The other equality follows similarly. □

The previous lemma implies that common values of traces $[f_+, g_+]$, $[f_+, g]$ and $[f, g_+]$ depend only on f and g and not in the choice of f_+ and g_+. Therefore, one could define an assignment $(f, g) \mapsto \mathrm{tr}([f_+, g_+])$ which turns out to be k-bilinear by Proposition 2.5. Thus, there exists a map

$$\begin{aligned} r \colon K \otimes_k K &\to k \\ f \otimes g &\mapsto \mathrm{tr}([f_+, g_+]) \end{aligned}$$

With these tools at our hands we are ready to prove the existence of residue.

Theorem 2.18. *There exists a unique k-linear map*

$$\mathrm{res}_V \colon \Omega_{K/k} \to k$$

such that for each pair of elements $f, g \in K$ we have

$$\mathrm{res}_V(f dg) = \mathrm{tr}([f_+, g_+]).$$

Proof. Let $c \colon K \otimes_k K \to \Omega_{K/k}$ be as in Lemma 2.16. Then, since c is surjective, res_V it it exists can only be the unique map so that the diagram

$$\begin{tikzcd} K \otimes_k K \arrow[r, "r"] \arrow[d, two heads, "c"] & k \\ \Omega_{K/k} \arrow[ur, dotted, "\mathrm{res}_V"'] & \end{tikzcd}$$

commutes. Therefore, such map exists if and only if it vanishes on $\ker c$. To see this, let f, g and h in K and choose f_+, g_+ and h_+ in $\mathrm{End}_+(V)$ projections in $\mathrm{End}_+(V)$. Then,

$$fg = f_+g_+ + (f_+g_- + f_-g_+ + f_-g_-),$$

and $f_+g_- + f_-g_+ + f_-g_- \in \mathrm{End}_-(V)$. Whence, $(fg)_+ = f_+g_+$. Analogously $(gh)_+ = g_+h_+$ and $(fh)_+ = f_+h_+$. This fact and the identify

$$[f_+, g_+h_+] - [f_+g_+, h_+] - [f_+h_+, g_+] = 0$$

imply the desired conclusion. □

Properties of res

We prove some of the main properties of res.

Proposition 2.19. *For all $f, g \in K$ it follows that*

(a) $\mathrm{res}_V(fdg) + \mathrm{res}_V(gdf) = 0$, *and*

(b) $\mathrm{res}_V(df) = 0$.

Proof. Since $[f_+, g_+] + [g_+, f_+] = 0$ we get (a). For (b) use (a) with $g = 1$. □

Proposition 2.20. *Let W be a closed K-submodule of V. Then, for $\omega \in \Omega_{K/k}$*

$$\mathrm{res}_V(\omega) = \mathrm{res}_W(\omega) + \mathrm{res}_{V/W}(\omega)$$

holds.

Proof. It is enough to prove it for $\omega = fdg$. By Lemma 2.3 item (c) we only need to check that for all $f \in K$ the induced map $\overline{f}\colon V/W \to V/W$ and $f \circ \iota$, where ι denotes the inclusion $W \to V$, suffice

$$\begin{aligned} \overline{f} &= \overline{f_+} \mod \mathrm{End}_-(V/W), \\ f \circ \iota &= f_+ \circ \iota \mod \mathrm{End}_-(W), \\ \overline{f_+} &\in \mathrm{End}_+(V/W), \quad \text{and} \\ f_+ \circ \iota &\in \mathrm{End}_+(W), \end{aligned}$$

where $\mathrm{End}_\pm(V/W)$ and $\mathrm{End}_\pm(W)$ denote compact and discrete operators in V/W and W respectively. These statements are straightforward and we leave them as an exercise to the reader. □

As a direct consequence of the previous proposition we get the following:

Proposition 2.21. *If V is the direct sum of two closed submodules W_1 and W_2 then*

$$\mathrm{res}_V(\omega) = \mathrm{res}_{W_1}(\omega) + \mathrm{res}_{W_2}(\omega)$$

holds for all $\omega \in \Omega_{K/k}$.

If our Tate space is trivial so its residue.

Proposition 2.22. *If V is either linearly compact or discrete then* $\mathrm{res}_V(\Omega_{K/k}) = 0$.

Proof. If V is linearity compact then $\mathrm{End}_+(V) = E$ and $f_+ = f$ for all $f \in K$. Since $[f, g] = 0$ then

$$\mathrm{res}_V(fdg) = 0. \tag{2}$$

On the other hand, if V is discrete then $E = \mathrm{End}_-(V)$. Thus, $f = 0 \mod \mathrm{End}_-(V)$ for all $f \in K$. Thus, (2) holds. □

Proposition 2.23. *Let f and g belong to K. Then, if there exists a c-lattice L in V so that $fL + fgL + fg^2L \subseteq L$ it holds $\mathrm{res}_V(fdg) = 0$. In particular, when there exists L a c-lattice so that $fL \subseteq L$ and $gL \subseteq L$ then $\mathrm{res}_V(fdg) = 0$.*

Proof. Let π be a continuous projection from V to L. Then $\pi f \in \mathrm{End}_+(V)$ and $\pi f = f \mod \mathrm{End}_-(V)$. Thus, it follows that

$$\mathrm{res}_V(fdg) = \mathrm{tr}([\pi f, g])$$

by Lemma 2.17. Let $h = [\pi f, g]$ and $W = L + gL$. Let $h_{V/W}$ and h_W be the induced maps on V/W and W respectively. Then, the relation $fL + fgL + fg^2L \subseteq L$ implies that $h_{V/W} = 0$ and $h_W = 0$. By Lemma 2.3 item (c) we conclude

$$\mathrm{res}_V(fdg) = \mathrm{tr}_V(h) = \mathrm{tr}_W(h) + \mathrm{tr}_{V/W}(h) = 0. \qquad \square$$

In the following two propositions we examine the residue of a power.

Proposition 2.24. *Let $f \in K$, then $\mathrm{res}_V(f^n df) = 0$ for all $n \geq 0$. Moreover, if f is invertible the same holds for $n \leq -2$.*

Proof. First, if $f_+ = f \mod \text{End}_-(V)$ then $f_+^n = f^n \mod \text{End}_-(V)$. Therefore,

$$\text{res}_V(f^n df) = \text{tr}([f_+, f_+^n]) = 0.$$

Second, if f is invertible then

$$fd(f^{-1}) + f^{-1}df = d(ff^{-1}) = d(1) = 0.$$

which implies

$$f^{-2}df = -d(f^{-1}),$$

and multiplying by f^{-n} both sides, where $n \geq 0$, implies

$$f^{-2-n}df = -(f^{-1})^n d(f^{-1}).$$

By the preceding statement, $(f^{-1})^n d(f^{-1})$ has zero residue. □

Proposition 2.25. *If f is invertible, so that $fL \subseteq L$ for some c-lattice L, then*

$$\text{res}_V(f^{-1}df) = \dim_k(L/fL).$$

Proof. If π is a continuous projection of V into L then

$$\text{res}_V(f^{-1}df) = \text{tr}([\pi f^{-1}, f]).$$

Let $g = [\pi f^{-1}, f]$. Since $fL \subseteq L$ we obtain

$$g_{V/L} = 0, \quad g_{L/fL} = 1 \quad \text{and,} \quad g_{fL} = 0,$$

where $g_{V/L}, g_{L/fL}$ and g_{fL} denote the induced maps in V/L, L/fL and, fL respectively. Then, by Lemma 2.3 item (c) it follows that

$$\text{tr}_V(g) = \text{tr}_L(g) + \text{tr}_{V/L}(g) = \text{tr}_{fL}(g) + \text{tr}_{L/fL}(g) + \text{tr}_{V/L}(g).$$

Observe that $\dim L/fL < \infty$ since fL is open and L is linearly compact. □

Relationship of residues under extensions

Finally, we explore the case where K' is a commutative k-algebra containing K. We will examine $\Omega_{K'/k}$ and $\Omega_{K/k}$ and the relationship between their residues. In this case the injection $K \to K'$ induces a map between $\Omega_{K/k} \to \Omega_{K'/k}$ which may not be injective.

Proposition 2.26. *Let V be a Tate space such that multiplication by any $f \in K'$ induces a continuous endomorphism in* $\mathrm{End}_k(V)$. *Therefore, for all $g \in K$ multiplication by g is continuous as well. Hence, we can define*

$$\mathrm{res}_V \colon \Omega_{K/k} \to k, \quad \textit{and} \quad \mathrm{res}'_V \colon \Omega_{K'/k} \to k.$$

In this situation, the diagram

$$\begin{tikzcd} \Omega_{K/k} \arrow[r] \arrow[dr, "\mathrm{res}_V"'] & \Omega_{K'/k} \arrow[d, "\mathrm{res}'_V"] \\ & k \end{tikzcd}$$

commutes.

Proof. For $f, g \in K$ their residue symbol is independent whether $f\,dg$ is thought as an element in $\Omega_{K'/k}$ or $\Omega_{K/k}$. This observation implies the commutativity of the diagram. □

Now, assume that K' is free K-module of finite rank n and consider the tensor product $V' = K' \otimes_K V$. Since the tensor product and direct sum commute, it follows that $V' \cong K^n \otimes_K V \cong (K \otimes_K V)^n \cong V^n$. In coordinates, if (x_i) is a K-base for K' then the map $(v_1, \ldots, v_n) \mapsto x_1 \otimes v_1 + \ldots + x_n \otimes v_n$ is an isomorphism. With the topology induced by this isomorphism V' is a Tate space.

Proposition 2.27. *The space* $\mathrm{End}(V')$ *is isomorphic to the space of $n \times n$ matrices with entries in* $\mathrm{End}(V)$ *denoted* $\mathrm{M}_n(\mathrm{End}_0(V))$. *Moreover, if K acts continuously on V so does K' on V'.*

Proof. Let φ be a continuous k-endomorphism of V', then there exists a unique set $\{\varphi_{ij}\}_{i,j=1}^n$ contained in $\mathrm{End}(V)$ such that

$$\varphi\left(\sum_i x_i \otimes v_i\right) = \sum_{i,j} x_i \otimes \varphi_{ij}(v_j)$$

for all $v_1, \ldots, v_n \in V$. Now, let $f' \in K'$, then

$$f' x_i = \sum f_{ij} x_j$$

where $f_{ij} \in K$. Since $f_{ij} \in \mathrm{End}(V)$ it follows that $f' \in \mathrm{End}(V')$ by the description of our topology in V'. □

Let $\mathrm{End}_0'(V')$ be the inverse image of $\mathrm{M}_n(\mathrm{End}_0(V))$ under the isomorphism in Proposition 2.27. Note that $\mathrm{End}_0'(V') \subseteq \mathrm{End}_0(V')$. Therefore, the map

$$\mathrm{tr}_{V'}\colon \mathrm{End}_0'(V') \to k$$

is well-defined.

Proposition 2.28. *For $\varphi \in \mathrm{End}_0'(V')$ the identity*

$$\mathrm{tr}_{V'}(\varphi) = \sum_i \mathrm{tr}_V(\varphi_{ii})$$

holds.

Proof. Write (φ_{ij}) as a sum of a strictly lower triangular, strictly upper triangular and diagonal matrix. Namely,

$$\varphi = \varphi_{LT} + \varphi_{UT} + \varphi_D,$$

where φ_{LT}, φ_{UT} and φ_D have a matrix representation of a strictly lower, strictly upper and diagonal matrix respectively. Observe that φ_{LT}, φ_{UT}, φ_D belong to $\mathrm{End}_0'(V')$ and φ_{LT} and φ_{UT} are nilpotent. By Lemma 2.3 it follows that

$$\mathrm{tr}_{V'}(\varphi) = \mathrm{tr}_{V'}(\varphi_D).$$

On the other hand, by definition

$$\mathrm{tr}_{V'}(\varphi_D) = \sum \mathrm{tr}_V(\varphi_{ii}).$$

□

Theorem 2.29. *For all $f' \in K'$ and $g \in K$ the equality*

$$\mathrm{res}_V'(f'dg) = \mathrm{res}_V((\mathrm{tr}_{K'/K}(f')dg))$$

holds.

Proof. Let L be a c-lattice in V then $L' = x_1 \otimes L + \ldots + x_n \otimes L$ is a c-lattice in V'. Let $\pi\colon V \to L$ be a linear continuous projection and π' be the corresponding element to $(\delta_{ij}\pi)$ under the isomorphism $\mathrm{End}(V') \cong \mathrm{M}_n(\mathrm{End}(V))$. Therefore, $\pi'\colon V' \to L'$ is a linear continuous projection. On the other hand, let $f' \in K'$ and $g \in K$. Then, f' corresponds to $(f_{ij}) \in \mathrm{M}_n(K)$ and let g' be the corresponding element to $(\delta_{ij}g)$ in $\mathrm{End}(V')$. Hence, the commutator $[\pi' f', g']$ is mapped to $[\pi f_{ij}, g]$ by the map $\mathrm{End}(V') \to \mathrm{M}_n(\mathrm{End}(V))$. By Proposition 2.28, it follows that

$$\begin{aligned}
\mathrm{res}_{V'}(f'dg) &= \mathrm{tr}_{V'}([\pi' f', g']) \\
&= \sum \mathrm{tr}_V([\pi f_{ii}, g]) \\
&= \sum \mathrm{res}_V(f_{ii}dg) \\
&= \mathrm{res}_V\left(\left(\sum f_{ii}\right) dg\right) \\
&= \mathrm{res}_V\left(\mathrm{tr}_{K'/K}(f')dg\right).
\end{aligned}$$

3 ALGEBRAIC CURVES

In the preceding chapter we presented the "residue map" in an abstract context. In this chapter we explore residues on algebraic curves using Tate's construction. First, we recall briefly the basic theory of algebraic projective curves. We reference the reader to [BP02].

3.1 BASIC THEORY OF ALGEBRAIC CURVES

Let k be an algebraically closed field.

Definition 3.1. An **algebraic curve** is a one-dimensional non-singular projective variety.

BIBLIOGRAPHY

[AST07] Martin Argerami, Fernando Szechtman, and Ryan Tifenbach. "On Tate's trace." In: *Linear Multilinear Algebra* 55.6 (2007), pp. 515–520. ISSN: 0308-1087. DOI: 10.1080/03081080601084112. URL: https://doi.org/10.1080/03081080601084112.

[BD04] Alexander Beilinson and Vladimir Drinfeld. *Chiral algebras*. Vol. 51. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2004, pp. vi+375. ISBN: 0-8218-3528-9. DOI: 10.1090/coll/051. URL: https://doi.org/10.1090/coll/051.

[BP02] Fedor Bogomolov and Tihomir Petrov. *Algebraic curves and one-dimensional fields*. Vol. 8. Courant Lecture Notes in Mathematics. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2002, pp. xii+214. ISBN: 0-8218-2862-2. DOI: 10.1090/cln/008. URL: https://doi.org/10.1090/cln/008.

[GD64] Alexander Grothendieck and Jean Dieudonné. *Éléments de géométrie algébrique IV*. Vol. 20, 24, 28, 32. Publications Mathématiques. Institute des Hautes Études Scientifiques., 1964-1967.

[Lef42] Solomon Lefschetz. *Algebraic Topology*. American Mathematical Society Colloquium Publications, v. 27. American Mathematical Society, New York, 1942, pp. vi+389.

[Mat86] Hideyuki Matsumura. *Commutative ring theory*. Vol. 8. Cambridge Studies in Advanced Mathematics. Translated from the Japanese by M. Reid. Cambridge University Press, Cambridge, 1986, pp. xiv+320. ISBN: 0-521-25916-9.

[Tat68] John Tate. "Residues of differentials on curves." In: *Ann. Sci. École Norm. Sup. (4)* 1 (1968), pp. 149–159. ISSN: 0012-9593. URL: http://www.numdam.org/item?id=ASENS_1968_4_1_1_149_0.