# Canonical quantization of symplectic manifolds

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— CHAPTER I —

# DQ-ALGEBRAS AND DQ-ALGEBROIDS

## 1 Star products

some words on history...

**Definition 1.1** ([Bay+78], [KS12, Definition 2.2.2.]). An associative operation $\star$ on $\mathscr{O}_X[[\hbar]]$ is a *star product* if it is $\mathbb{C}[[\hbar]]$-bilinear and satisfies

$$f \star g = \sum_{i \geq 0} P_i(f,g)\hbar^i \quad \text{for } f, g \in \mathscr{O}_X \tag{1.2}$$

where the $P_i$'s are bi-differential operators such that $P_0(f,g) = fg$ and $P_i(f,1) = P_i(1,f) = 0$ for all $f \in \mathscr{O}_X$ and $i > 0$. We call $(\mathscr{O}_X[[\hbar]], \star)$ a *star algebra*.

the previous probably needs some background on differential operators...

### Moyal-Weyl star product

**Definition 1.3.** Let $X = \mathbb{R}^{2n}$ endowed with its standard symplectic structure. We define the *Moyal-Weyl $\star$-product* by the rule

$$f \star g = \operatorname{prod}\left(\exp\left(\frac{i\hbar}{2}\Pi\right)(f \otimes g)\right) \tag{1.4}$$

where $\Pi$ is the Poisson bi-vector. One calls $\mathscr{O}_X[[\hbar]]$ equipped with the Moyal-Weyl star product, the *Weyl algebra*.

*Remark* 1.5. The notation Weyl algebra in Definition 1.3 is supported in the following observation. Consider the subalgebra $(\mathbb{C}[p,q],\star)$[1] of the algebra $(\mathscr{O}_{\mathbb{R}^{2n}}(\mathbb{R}^{2n}),\star)$. An easy computation yields

$$p_r \star q_s = p_s q_s + \delta_{rs}\frac{i\hbar}{2} \quad \text{and,} \quad q_s \star p_r = p_r q_s - \delta_{sr}\frac{i\hbar}{2}.$$

Therefore,

$$[p_r, q_s] = \delta_{rs} i\hbar,$$

which is *Planck's law*. Particularly, this presents an isomorphism between $(\mathbb{C}[p,q],\star)$ and[2] $\mathbb{C}\{x,\partial\}/\left\langle x\partial - \partial x - 1\right\rangle$, also known as the Weyl algebra. This will be further explained by the identification of $\mathbb{R}^{2n}$ with $T^*\mathbb{R}^n$ and deformation quantization of the cotangent bundle. In the meantime, this identification is proven useful in the following Lemma.

write reference when cotangent bundle is explained.

**Lemma 1.6.** *The center of the algebra $\mathbb{C}\{x,\partial\}/\left\langle x\partial - \partial x - 1\right\rangle$ is $\mathbb{C}$. In particular, from Remark 1.5, it follows that the center of $(\mathscr{O}_{\mathbb{R}^{2n}},\star)$ is $\mathbb{C}[[t]]$.*

*Proof.* We carry out the case $n = 1$. Let $P = \sum_a f_a(x)\partial^a$ be a central element. An easy calculation, using the fact that $[\partial, f] = f'$, yields

$$[\partial, P] = \sum_a f'_a(x)\partial^a.$$

Since $P$ is central, we conclude that $f'_a = 0$, so that $f_a$ is a constant for all $a$. Therefore, $P = g(\partial)$ for some polynomial $g$. Again, we have

$$[x, g(\partial)] = g'(\partial)$$

and centrality yields that $P$ is constant. The previous proof only uses that the ground ring of the Weyl algebra is of characteristic zero, so that one carries inductively using $\mathbb{C}[x,\partial_x][y_1,\dots,y_m,\partial_1,\dots,\partial_m] \cong \mathbb{C}[x,y_1,\dots,y_m,\partial_x,\partial_1,\dots,\partial_m]$. □

## 2 DQ-algebras

See Kashiwara-Schapira.

[1] Here $p$ and $q$ are shorthand for coordinates $(p_1,\dots,p_n,q_1,\dots,q_n)$

[2] We mantain the same shorthand as before, and we extend it correctly to $x\partial - \partial x - 1$

## 3 DQ-algebroids

See Kashiwara-Schapira. Important here to include some motivation.

— CHAPTER II —

# DILATION EQUIVARIANCE

Let $(X, \omega)$ be a symplectic manifold and $\mathscr{A}$ a symplectic DQ-algebra on $X$. Denote by $Z(\mathscr{A})$ the center of $\mathscr{A}$. From Lemma 1.6 and the natural map $\mathbb{C}[[\hbar]] \to \mathscr{A}$ is injective onto the center.

include reference of DQ-Darboux

## — Appendix I —

# ABSTRACT NONSENSE

Once and for all fix a strongly inaccesible cardinal $\kappa$, so that $V_\kappa$ is a Grothendieck universe. (hahaha)

# REFERENCES

[Bay+78] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. “Deformation theory and quantization. I, II. Physical applications”. In: *Ann. Physics* 111.1 (1978), pp. 61–110, 111–151. ISSN: 0003-4916. DOI: 10.1016/0003-4916(78)90225-7 (cit. on p. 2).

[KS12] Masaki Kashiwara and Pierre Schapira. “Deformation quantization modules”. In: *Astérisque* 345 (2012), pp. xii+147. ISSN: 0303-1179. DOI: 10.24033/ast.902 (cit. on p. 2).