Canonical quantization of symplectic manifolds

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— CHAPTER I —

DQ-ALGEBRAS AND DQ-ALGEBROIDS

1 Star products

some words on history...

Definition 1.1 ([Bay+78], [KS12, Definition 2.2.2.]). An associative operation $\star$ on $\mathscr{O}_X[[\hbar]]$ is a *star product* if it is $\mathbb{C}[[\hbar]]$-bilinear and satisfies

$$f \star g = \sum_{i \geq 0} P_i(f, g)\hbar^i \quad \text{for } f, g \in \mathscr{O}_X \tag{1.2}$$

where the P_i's are bi-differential operators such that $P_0(f, g) = fg$ and $P_i(f, 1) = P_i(1, f) = 0$ for all $f \in \mathscr{O}_X$ and $i > 0$. We call $(\mathscr{O}_X[[\hbar]], \star)$ a *star algebra*. A star-product defines a Poisson structure on X by the rule

$$\{f, g\} := P_1(f, g) - P_1(f, g) = \frac{1}{\hbar}[f, g]_\star \mod \hbar\mathscr{O}_X \tag{1.3}$$

If X is a Poisson manifold and the induced Poisson bracket by $\star$ coincides with the Poisson structure of X, we say that $\star$ is a *deformation* of the Poisson structure (bracket) on X. Moreover, if (X, ω) is symplectic and $\star$ deforms the standard Poisson bracket induced by ω, we say that $\star$ is a *symplectic star product*.

the previous probably needs some background on differential operators...

Moyal product

Definition 1.4. Let $X = \mathbb{R}^{2n}$ endowed with its standard symplectic structure. We define the *Moyal product* by the rule

$$\begin{aligned} f \star g &= \operatorname{prod}\left(\exp\left(\frac{\hbar}{2}\Pi\right)(f \otimes g)\right) \qquad (1.5) \\ &= fg + \frac{\hbar}{2}\sum_{i,j}\Pi^{ij}(\partial_i f)(\partial_j g) + \frac{\hbar^2}{8}\sum_{i,j,k,m}\Pi^{ij}\Pi^{km}(\partial_i\partial_k g)(\partial_j\partial_m g) + \dots, \end{aligned}$$

where Π is the Poisson bi-vector. One calls $\mathscr{O}_X[[\hbar]]$ equipped with the Moyal product, the *Weyl algebra*.

Remark 1.6. The notation Weyl algebra in Definition 1.4 is supported in the following observation. Consider the subalgebra $(\mathbb{C}[p,q],\star)$[1] of the algebra $(\mathscr{O}_{\mathbb{R}^{2n}}(\mathbb{R}^{2n}),\star)$. An easy computation yields

$$p_r \star q_s = p_s q_s + \delta_{rs}\frac{\hbar}{2} \quad \text{and,} \quad q_s \star p_r = p_r q_s - \delta_{sr}\frac{\hbar}{2}.$$

Therefore,

$$[p_r, q_s]_\star = \delta_{rs}\hbar.$$

Particularly, this presents an isomorphism between $(\mathbb{C}[p,q],\star)$ and[2] $\mathbb{C}\{x,\partial\}/\langle x\partial - \partial x - 1\rangle$, also known as the Weyl algebra. This will be further explained by the identification of $\mathbb{R}^{2n}$ with $T^*\mathbb{R}^n$ and deformation quantization of the cotangent bundle.

Lemma 1.7. *The center of the algebra* $(\mathscr{O}_{\mathbb{R}^{2n}},\star)$ *is* $\mathbb{C}[[\hbar]]$.

Proof. Choose coordinates (p_i, q_i) and let $f \in (\mathscr{O}_{\mathbb{R}^{2n}},\star)$ be central. Then

$$0 = [f, q_i]_\star = -\hbar\frac{\partial f}{\partial q_i} \quad \text{and,} \quad \hbar\frac{\partial f}{\partial q_i} = [f, p_i]_\star = 0,$$

so that f is constant. $\square$

[1]Here p and q are shorthand for coordinates $(p_1,\dots,p_n,q_1,\dots,q_n)$

[2]We mantain the same shorthand as before, and we extend it correctly to $x\partial - \partial x - 1$

2 DQ-ALGEBRAS

Definition 1.8 ([KS12, Definition 2.2.5.]). A *DQ-algebra* $\mathscr{A}$ on X is a sheaf of $\mathbb{C}[[\hbar]]$-algebras locally isomorphic to a star-algebra $(\mathscr{O}_X[[\hbar]], \star)$ as $\mathbb{C}[[\hbar]]$-algebras.

A DQ-algebra induces a natural Poisson structure on X as follows: Let $f, g \in \mathscr{O}_X$ and denote by σ_0 the composition

$$\mathscr{A} \twoheadrightarrow \mathscr{A}/\hbar\mathscr{A} \xrightarrow{\sim} \mathscr{O}_X.$$

Choose $a, b \in \mathscr{A}$ such that $\sigma_0(a) = f$ and $\sigma_0(b) = g$. Since $ab - ba \in \hbar\mathscr{A}$, define

$$\{f, g\} := \sigma_0\left(\frac{ab - ba}{\hbar}\right). \tag{1.9}$$

Since $\sigma_0(\hbar\mathscr{A}) = 0$, it follows that (1.9) is independent of the choice of liftings. Moreover, any two locally isomorphic DQ-algebras induce the same Poisson structure. Whenever X is symplectic, we say that $\mathscr{A}$ is *symplectic* if it is locally isomorphic to a symplectic star-algebra.

3 DQ-ALGEBROIDS

Symplectic DQ-algebroids

Definition 1.10.

— CHAPTER II —

DILATION EQUIVARIANCE

Let (X, ω) be a symplectic manifold and $\mathscr{A}$ a symplectic DQ-algebra on X. Denote by $Z(\mathscr{A})$ the center of $\mathscr{A}$. From Lemma 1.7 and the natural map $\mathbb{C}[[\hbar]] \to \mathscr{A}$ is injective onto the center.

include reference of DQ-Darboux

— Appendix i —

ABSTRACT NONSENSE

Throughout this appendix we assume familiarity with basic category theory, basic sheaf theory and the definition of a (strict) 2-category. We denote by $\circ$ horizontal composition and by $\diamond$ vertical composition. The main reference will be [Bre94].

1 Stacks

Let X be a topological space. We do not provide proofs, but reference the reader to the source [SGA 1, Exposé VI].

Fibred categories

Definition A.1. A *fibred category* $\mathcal{C}$ over X is an assignment

(i) for every open set $U \subseteq X$ a category $\mathcal{C}_U$;

(ii) for every inclusion $i\colon V \hookrightarrow U$ an inverse image functor $i^*\colon \mathcal{C}_U \to \mathcal{C}_V$, which may be taken to be the identity functor whenever $f = \mathrm{id}_U$;

(iii) a natural isomorphism $\tau_{i,j}\colon (i \circ j)^* \Rightarrow j^* \circ i^*$ for every pair of inclusions $W \overset{j}{\hookrightarrow} V \overset{i}{\hookrightarrow} U$.

Subject to the condition that the diagram

$$\begin{tikzcd}
(i \circ j \circ k)^* \arrow[r, Rightarrow, "\tau_{i,j,k}"] \arrow[d, Rightarrow, "\tau_{i,j\circ k}"] & k^* \circ (i \circ j)^* \arrow[d, Rightarrow, "k^*\circ\tau_{i,j}"] \\
(j \circ k)^* \circ i^* \arrow[r, Rightarrow, "\tau_{j,k}\circ i^*"] & (k^* \circ j^* \circ i^*),
\end{tikzcd} \tag{A.2}$$

commutes for any three composable arrows $N \overset{k}{\hookrightarrow} W \overset{j}{\hookrightarrow} V \overset{i}{\hookrightarrow} U$. For an object $x \in \mathcal{C}_U$, we denote by $x|_V$ the inverse image $i^*(x)$ for an inclusion $i\colon V \hookrightarrow U$.

Definition A.3. Let $\mathcal{C}$ and $\mathcal{D}$ be two fibred categories over X. A *morphism* $(F, \alpha)\colon \mathcal{C} \to \mathcal{D}$ *of fibred categories* consists of

(i) for every open set $U \subseteq X$ a functor $F_U\colon \mathcal{C}_U \to \mathcal{D}_U$

(ii) for every inclusion $i\colon V \hookrightarrow V$ a natural isomorphism $\alpha_i\colon F_V \circ i^* \Rightarrow i^* \circ F_U$

subject to the compatibility condition that the diagram

$$\begin{tikzcd}
& F_W \circ (i \circ j)^* \arrow[rr, Rightarrow, "\alpha_{i\circ j}"] \arrow[dl, Rightarrow, "F_W\circ\tau"] & & (i \circ j)^* \circ F_U \arrow[dr, Rightarrow, "\tau\circ F_U"] & \\
F_W \circ j^* \circ i^* \arrow[drr, Rightarrow, "\alpha_j\circ i^*"] & & & & j^* \circ i^* \circ F_U \\
& & j^* \circ F_V, \arrow[urr, Rightarrow, "j^*\circ\alpha_i"] & &
\end{tikzcd} \tag{A.4}$$

where τ denotes the corresponding natural isomorphism for $\mathcal{C}$ and $\mathcal{D}$, commutes for all inclusions $W \overset{j}{\hookrightarrow} V \overset{i}{\hookrightarrow} U$.

Definition A.5. A fibred morphism $(F, \alpha)\colon \mathcal{C} \to \mathcal{D}$ is called a *weak equivalence* if every F_U is fully faithful and *locally surjective*; that is, for every U open subset of X, y of $\mathcal{D}_U$, and $p \in X$ there exists an object x of $\mathcal{C}_U$ and an open neighborhood V of p contained in U such that $F_V(x) \cong y|_V$.

Definition A.6. Given two fibred morphisms $(F, \alpha), (G, \beta)\colon \mathcal{C} \to \mathcal{D}$, a *fibred transformation* (or simply *2-morphism*) $\Psi\colon F \Rightarrow G$ is a collection of natural transformations $\Psi_U\colon F_U \Rightarrow G_U$ indexed by open sets $U \subseteq X$ subject to the following compatibility condition: for any inclusion $i\colon V \hookrightarrow U$, the diagram of natural transformations

$$\begin{tikzcd} F_V \circ i^* \arrow[r, Rightarrow, "\alpha_i"] \arrow[d, Rightarrow, "\Psi_V \circ i^*"] & i^* \circ F_U \arrow[d, Rightarrow, "i^* \circ \Psi_U"] \\ G_V \circ i^* \arrow[r, Rightarrow, "\beta_i"] & i^* \circ G_U \end{tikzcd} \tag{A.7}$$

commutes.

Remark A.8. Fibred categories over X form a 2-category with objects as in Definition A.1, 1-morphisms as in Definition A.3, and 2-morphisms as in Definition A.6. We denote this 2-category by Fibred_X.

Remark A.9. Let U be an open subset of X and x, y objects of $\mathcal{C}_U$. The assignment $V \to \mathcal{C}_V(x|_V, y|_V)$ defines a presheaf on U. Moreover, every morphism $F\colon \mathcal{C} \to \mathcal{D}$ induces a morphism at the level of presheaves.

Definition A.10. A fibred category $\mathcal{C}$ over X is a *prestack on* X if for every U and every pair x, y of objects of $\mathcal{C}_U$ the presheaf on Remark A.9 is a sheaf. The full 2-subcategory of Fibred_X made of prestacks is denoted by $\mathsf{Prestacks}_X$.

Remark A.11. Every fibred category $\mathcal{C}$ admits an *associated prestack*, in the sense of a left 2-adjoint for

$$\mathsf{Prestacks}_X \hookrightarrow \mathsf{Fibred}_X.$$

Indeed, consider the associated sheaf (also known as sheafification) to the presheaf in Remark A.9. The usual adjointness associated sheaf $\dashv$ presheaf extends to the desired 2-adjointness.

Definition A.12. Let $\mathcal{C}$ be a fibred category over X and $\mathscr{U} = \{U_\alpha\}_{\alpha\in A}$ be an open cover of an open set $U \subseteq X$. The category $\mathrm{Desc}(\mathscr{U}, \mathcal{C})$ of *descent data* consists of

(i) as objects: pairs of collections $(x, \varphi) = (\{x_\alpha\}_{\alpha\in A}, \{\varphi_{\alpha\beta}\}_{\alpha,\beta\in A})$ where x_α is an object of $\mathcal{C}_\alpha$ and $\varphi_{\alpha\beta}\colon x_\beta|_{U_{\alpha\beta}} \xrightarrow{\sim} x_\alpha|_{U_{\alpha\beta}}$ an isomorphism. These are subject to the cocycle condition

$$\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma} \tag{A.13}$$

in $\mathcal{C}_{U_{\alpha\beta\gamma}}$, for each α, β and γ in A.

(ii) as arrows: $(x,\varphi) \xrightarrow{f} (y,\psi)$ a set of arrows $\{f_\alpha \colon x_\alpha \to y_\alpha\}_{\alpha\in A}$ such that the diagram

$$\begin{tikzcd} x_\beta|_{U_{\alpha\beta}} \arrow[r, "f_\beta"] \arrow[d, "\varphi_{\alpha\beta}"] & y_\beta|_{U_{\alpha\beta}} \arrow[d, "\psi_{\alpha\beta}"] \\ x_\alpha|_{U_{\alpha\beta}} \arrow[r, "f_\alpha"] & x_\alpha|_{U_{\alpha\beta}} \end{tikzcd} \tag{A.14}$$

Remark A.15. Let $\mathcal{C}$ be a fibred category over X, U an open set of X, and $\mathscr{U}$ a cover of U. There is a natural functor $\mathcal{C}_U \to \mathrm{Desc}(\mathscr{U},\mathcal{C})$ sending $x \mapsto (\{x|_{U_\alpha}\}_{\alpha\in A}, \{\mathrm{id}_x|_{U_{\alpha\beta}}\}_{\alpha,\beta\in A})$ and $f\colon x \to y \mapsto \{f|_{U_\alpha}\colon x|_{U_\alpha} \to y|_{U_\alpha}\}_{\alpha\in A}$. Then $\mathcal{C}$ is a prestack if and only if this functor is fully faithful for all open sets U and all covers $\mathscr{U}$ of U.

Definition A.16. A fibred category $\mathcal{C}$ over X is a *stack on* X if for every open subset U of X and every cover $\mathscr{U}$ of U the functor $\mathcal{C}_U \to \mathrm{Desc}(\mathscr{U},\mathcal{C})$ on Remark A.15 is an equivalence of categories. If in addition each category $\mathcal{C}_U$ is a groupoid, we say that $\mathcal{C}$ is *stack in groupoids.* We denote the full 2-subcategory of stacks on X by Stacks_X.

Definition A.17. Let $\mathcal{C}$ be a prestack on X. The *associated stack* is a stack $\mathcal{C}$, endowed with a weak equivalence $\mathcal{C} \to \mathcal{C}^+$ such that for every open subset U of X and any pair of objects x and y of $\mathcal{C}_U$ the map

$$\mathcal{C}_U(x,y) \to \mathcal{C}^+_U(F_U(x), F_U(y))$$

is a bijection. If $\mathcal{C}^+$ exists it is determined up to unique 2-isomorphism.

Proposition A.18. *Let $\mathcal{C}$ be a prestack on X, then $\mathcal{C}$ admits an associated stack.*

Proof. Let $\mathcal{C}^+_U := \mathrm{colim}_{\mathscr{U}} \mathrm{Desc}(\mathscr{U},\mathcal{C})$ understood as a pseudo-colimit of categories. For details see [Stacks, Tag 02ZN]. □

2 Algebroids

This section tries to follow [DP05, Section 1]. Let $\mathbb{K}$ be a commutative unital ring.

Definition A.19. A $\mathbb{K}$-*linear category* (which we shorthand as $\mathbb{K}$-category) is a category whose Hom sets are endowed with a $\mathbb{K}$-module structure, so that composition is bilinear. A $\mathbb{K}$-*functor* is a functor between $\mathbb{K}$-categories which is linear at the level of morphisms.

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