Canonical quantization of symplectic manifolds

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A thesis submitted in partial fulfilment of the requirements for the degree of
Master of Arts in Mathematics,
Universidad de los Andes, 2020.

— CHAPTER I —

DQ-ALGEBRAS AND DQ-ALGEBROIDS

1 Star products

some words on history...

Definition 1.1 ([Bay+78], [KS12, Definition 2.2.2.]). An associative operation $\star$ on $\mathscr{O}_X[[\hbar]]$ is a *star product* if it is $\mathbb{C}[[\hbar]]$-bilinear and satisfies

$$f \star g = \sum_{i \geq 0} P_i(f,g)\hbar^i \quad \text{for } f,g \in \mathscr{O}_X \tag{1.2}$$

where the P_i's are bi-differential operators such that $P_0(f,g) = fg$ and $P_i(f,1) = P_i(1,f) = 0$ for all $f \in \mathscr{O}_X$ and $i > 0$. We call $(\mathscr{O}_X[[\hbar]], \star)$ a *star algebra*. A star-product defines a Poisson structure on X by the rule

$$\{f,g\} := P_1(f,g) - P_1(f,g) = \frac{1}{\hbar}[f,g]_\star \mod \hbar\mathscr{O}_X \tag{1.3}$$

If X is a Poisson manifold and the induced Poisson bracket by $\star$ coincides with the Poisson structure of X, we say that $\star$ is a *deformation* of the Poisson structure (bracket) on X. Moreover, if (X,ω) is symplectic and $\star$ deforms the standard Poisson bracket induced by ω, we say that $\star$ is a *symplectic star product*.

the previous probably needs some background on differential operators...

Moyal product

Definition 1.4. Let $X = \mathbb{R}^{2n}$ endowed with its standard symplectic structure. We define the *Moyal product* by the rule

$$f \star g = \mathrm{prod}\left(\exp\left(\frac{\hbar}{2}\Pi\right)(f \otimes g)\right) \tag{1.5}$$
$$= fg + \frac{\hbar}{2}\sum_{i,j}\Pi^{ij}(\partial_i f)(\partial_j g) + \frac{\hbar^2}{8}\sum_{i,j,k,m}\Pi^{ij}\Pi^{km}(\partial_i\partial_k g)(\partial_j\partial_m g) + \dots,$$

where Π is the Poisson bi-vector. One calls $\mathscr{O}_X[[\hbar]]$ equipped with the Moyal product, the *Weyl algebra*.

Lemma 1.6. *The center of the algebra* $(\mathscr{O}_{\mathbb{R}^{2n}}, \star)$ *is* $\mathbb{C}[[\hbar]]$.

Proof. Choose coordinates (p_i, q_i) and let $f \in (\mathscr{O}_{\mathbb{R}^{2n}}, \star)$ be central. Then

$$0 = [f, q_i]_\star = -\hbar\frac{\partial f}{\partial q_i} \quad \text{and,} \quad \hbar\frac{\partial f}{\partial q_i} = [f, p_i]_\star = 0,$$

so that f is constant. $\square$

2 DQ-algebras

Definition 1.7 ([KS12, Definition 2.2.5.]). A *DQ-algebra* $\mathscr{A}$ on X is a sheaf of $\mathbb{C}[[\hbar]]$-algebras locally isomorphic to a star-algebra $(\mathscr{O}_X[[\hbar]], \star)$ as $\mathbb{C}[[\hbar]]$-algebras.

A DQ-algebra induces a natural Poisson structure on X as follows: Let $f, g \in \mathscr{O}_X$ and denote by σ_0 the composition

$$\mathscr{A} \twoheadrightarrow \mathscr{A}/\hbar\mathscr{A} \xrightarrow{\sim} \mathscr{O}_X.$$

Choose $a, b \in \mathscr{A}$ such that $\sigma_0(a) = f$ and $\sigma_0(b) = g$. Since $ab - ba \in \hbar\mathscr{A}$, define

$$\{f, g\} := \sigma_0\left(\frac{ab - ba}{\hbar}\right). \tag{1.8}$$

Since $\sigma_0(\hbar\mathscr{A}) = 0$, it follows that (1.8) is independent of the choice of liftings. Moreover, any two locally isomorphic DQ-algebras induce the same Poisson structure. Whenever X is symplectic, we say that $\mathscr{A}$ is *symplectic* if it is locally isomorphic to a symplectic star-algebra.

3 DQ-algebroids

Definition 1.9 ([KS12, Definition 2.3.1])**.** A *DQ-algebroid* $\mathcal{C}$ on X is a $\mathbb{C}[[\hbar]]$-algebroid stack (see Definition A.31) such that for each open set $U \subseteq X$ and each object L of $\mathcal{C}(U)$, the sheaf of $\mathbb{C}[[\hbar]]$-algebras $\underline{\mathrm{End}}_{\mathcal{C}_U}(L)$ is a DQ-algebra on U.

Via (1.8) and local conectedness (see Definition A.30) every DQ-algebroid $\mathcal{C}$ induces a Poisson structure on X.

Symplectic DQ-algebroids

Definition 1.10.

— CHAPTER II —

DILATION EQUIVARIANCE

Let (X, ω) be a symplectic manifold and $\mathscr{A}$ a symplectic DQ-algebra on X. Denote by $Z(\mathscr{A})$ the center of $\mathscr{A}$. From Lemma 1.6 and the natural map $\mathbb{C}[[\hbar]] \to \mathscr{A}$ is injective onto the center.

include reference of DQ-Darboux

— Appendix i —

ABSTRACT NONSENSE

Throughout this appendix we assume familiarity with basic category theory and basic sheaf theory.

1 Stacks

We do not provide proofs, but reference the reader to the source [SGA 1, Exposé VI].

Definition A.1. A *fibred category* $\mathcal{C}$ over X is an assignment

(i) for every open set $U \subseteq X$ a category $\mathcal{C}(U)$;

(ii) for every inclusion $i\colon V \hookrightarrow U$ an inverse image functor $i^*\colon \mathcal{C}(U) \to \mathcal{C}(V)$, which may be taken to be the identity functor whenever $f = \mathrm{id}_U$;

(iii) a natural isomorphism $\tau_{i,j}\colon (i \circ j)^* \Rightarrow j^* \circ i^*$ for every pair of inclusions $W \overset{j}{\hookrightarrow} V \overset{i}{\hookrightarrow} U$.

Subject to the condition that the diagram

$$\begin{tikzcd} (i \circ j \circ k)^* \arrow[r, Rightarrow, "\tau_{i\circ j,k}"] \arrow[d, Rightarrow, "\tau_{i,j\circ k}"] & k^* \circ (i \circ j)^* \arrow[d, Rightarrow, "k^*\circ\tau_{i,j}"] \\ (j \circ k)^* \circ i^* \arrow[r, Rightarrow, "\tau_{j,k}\circ i^*"] & k^* \circ j^* \circ i^*, \end{tikzcd} \tag{A.2}$$

commutes for any three composable arrows $N \overset{k}{\hookrightarrow} W \overset{j}{\hookrightarrow} V \overset{i}{\hookrightarrow} U$. For an object x of $\mathcal{C}(U)$, we denote by $x|_V$ the inverse image $i^*(x)$ for an inclusion $i\colon V \hookrightarrow U$.

Definition A.3. Let $\mathcal{C}$ and $\mathcal{D}$ be two fibred categories over X. A *morphism* $(F,\alpha)\colon \mathcal{C} \to \mathcal{D}$ *of fibred categories* consists of

(i) for every open set $U \subseteq X$ a functor $F_U\colon \mathcal{C}(U) \to \mathcal{D}(U)$

(ii) for every inclusion $i\colon V \hookrightarrow V$ a natural isomorphism $\alpha_i\colon F_V \circ i^* \Rightarrow i^* \circ F_U$

subject to the compatibility condition that the diagram

$$\begin{tikzcd}
 & F_W \circ (i \circ j)^* \arrow[rr, Rightarrow, "\alpha_{i\circ j}"] \arrow[dl, Rightarrow, "F_W \circ \tau"'] & & (i \circ j)^* \circ F_U \arrow[dr, Rightarrow, "\tau \circ F_U"] & \\
F_W \circ j^* \circ i^* \arrow[drr, Rightarrow, "\alpha_j \circ i^*"] & & & & j^* \circ i^* \circ F_U \\
 & & j^* \circ F_V, \arrow[urr, Rightarrow, "j^* \circ \alpha_i"] & &
\end{tikzcd} \tag{A.4}$$

where τ denotes the corresponding natural isomorphism for $\mathcal{C}$ and $\mathcal{D}$, commutes for all inclusions $W \overset{j}{\hookrightarrow} V \overset{i}{\hookrightarrow} U$.

Definition A.5. A fibred morphism $(F,\alpha)\colon \mathcal{C} \to \mathcal{D}$ is called a *weak equivalence* if every F_U is fully faithful and *locally surjective*; that is, for every U open subset of X, y of $\mathcal{D}(U)$, and $p \in X$ there exists an object x of $\mathcal{C}(U)$ and an open neighborhood V of p contained in U such that $F_V(x) \cong y|_V$.

Definition A.6. Given two fibred morphisms $(F,\alpha),(G,\beta)\colon \mathcal{C} \to \mathcal{D}$, a *fibred transformation* (or simply *2-morphism*) $\Psi\colon F \Rightarrow G$ is a collection of natural transformations $\Psi_U\colon F_U \Rightarrow G_U$ indexed by open sets $U \subseteq X$ subject to the following compatibility condition: for any inclusion $i\colon V \hookrightarrow U$, the diagram of natural transformations

$$\begin{tikzcd}
F_V \circ i^* \arrow[r, Rightarrow, "\alpha_i"] \arrow[d, Rightarrow, "\Psi_V \circ i^*"] & i^* \circ F_U \arrow[d, Rightarrow, "i^* \circ \Psi_U"] \\
G_V \circ i^* \arrow[r, Rightarrow, "\beta_i"] & i^* \circ G_U
\end{tikzcd} \tag{A.7}$$

commutes.

Remark A.8. Fibred categories over X form a 2-category with objects as in Definition A.1, 1-morphisms as in Definition A.3, and 2-morphisms as in Definition A.6. We denote this 2-category by Fibred_X.

Remark A.9. Let U be an open subset of X and x, y objects of $\mathcal{C}(U)$. The assignment $V \to \mathcal{C}(V)(x|_V, y|_V)$ defines a presheaf on U. We denote this presheaf by $\underline{\mathrm{Hom}}_{\mathcal{C}}(x, y)$. Moreover, every morphism $F\colon \mathcal{C} \to \mathcal{D}$ induces a morphism at the level of presheaves.

Definition A.10. A fibred category $\mathcal{C}$ over X is a *prestack on* X if for every U and every pair x, y of objects of $\mathcal{C}(U)$ the presheaf $\underline{\mathrm{Hom}}_{\mathcal{C}}(x, y)$ is a sheaf. The full 2-subcategory of Fibred_X made of prestacks is denoted by $\mathsf{Prestacks}_X$.

Remark A.11. Every fibred category $\mathcal{C}$ admits an *associated prestack,* in the sense of a left 2-adjoint for

$$\mathsf{Prestacks}_X \hookrightarrow \mathsf{Fibred}_X.$$

Indeed, consider the associated sheaf (also known as sheafification) to the presheaf in Remark A.9. The usual adjointness associated sheaf $\dashv$ presheaf extends to the desired 2-adjointness.

Definition A.12. Let $\mathcal{C}$ be a fibred category over X and $\mathscr{U} = \{U_\alpha\}_{\alpha\in A}$ be an open cover of an open set $U \subseteq X$. The category $\mathrm{Desc}(\mathscr{U}, \mathcal{C})$ of *descent data* consists of

(i) as objects: pairs of collections $(x, \varphi) = (\{x_\alpha\}_{\alpha\in A}, \{\varphi_{\alpha\beta}\}_{\alpha,\beta\in A})$ where x_α is an object of $\mathcal{C}_\alpha$ and $\varphi_{\alpha\beta}\colon x_\beta|_{U_{\alpha\beta}} \xrightarrow{\sim} x_\alpha|_{U_{\alpha\beta}}$ an isomorphism. These are subject to the cocycle condition

$$\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma} \tag{A.13}$$

in $\mathcal{C}_{U_{\alpha\beta\gamma}}$, for each α, β and γ in A.

(ii) as arrows: $(x, \varphi) \xrightarrow{f} (y, \psi)$ a set of arrows $\{f_\alpha\colon x_\alpha \to y_\alpha\}_{\alpha\in A}$ such that the diagram

$$\begin{tikzcd}
x_\beta|_{U_{\alpha\beta}} \arrow[r, "f_\beta"] \arrow[d, "\varphi_{\alpha\beta}"] & y_\beta|_{U_{\alpha\beta}} \arrow[d, "\psi_{\alpha\beta}"] \\
x_\alpha|_{U_{\alpha\beta}} \arrow[r, "f_\alpha"] & x_\alpha|_{U_{\alpha\beta}}
\end{tikzcd} \tag{A.14}$$

Remark A.15. Let $\mathcal{C}$ be a fibred category over X, U an open set of X, and $\mathscr{U}$ a cover of U. There is a natural functor $\mathcal{C}(U) \to \mathrm{Desc}(\mathscr{U}, \mathcal{C})$ sending $x \mapsto (\{x|_{U_\alpha}\}_{\alpha\in A}, \{\mathrm{id}_x|_{U_{\alpha\beta}}\}_{\alpha,\beta\in A})$ and $f\colon x \to y \mapsto \{f|_{U_\alpha}\colon x|_{U_\alpha} \to y|_{U_\alpha}\}_{\alpha\in A}$. Then $\mathcal{C}$ is a prestack if and only if this functor is fully faithful for all open sets U and all covers $\mathscr{U}$ of U.

Definition A.16. A fibred category $\mathcal{C}$ over X is a *stack on* X if for every open subset U of X and every cover $\mathscr{U}$ of U the functor $\mathcal{C}(U) \to \mathrm{Desc}(\mathscr{U}, \mathcal{C})$ on Remark A.15 is an equivalence of categories. If in addition each category $\mathcal{C}(U)$ is a groupoid, we say that $\mathcal{C}$ is *stack in groupoids.* We denote the full 2-subcategory of stacks on X by Stacks_X.

Definition A.17. Let $\mathcal{C}$ be a prestack on X. The *associated stack* is a stack $\mathcal{C}$, endowed with a weak equivalence $F\colon \mathcal{C} \to \mathcal{C}'$ such that for every open subset U of X and any pair of objects x and y of $\mathcal{C}(U)$ the map

$$\mathcal{C}(U)(x,y) \to \mathcal{C}'(U)(F_U(x), F_U(y))$$

is a bijection. If $\mathcal{C}'$ exists it is determined up to unique 2-isomorphism.

Proposition A.18. *Let $\mathcal{C}$ be a prestack on X, then $\mathcal{C}$ admits an associated stack.*

Proof. Let $\mathcal{C}'(U) := \mathrm{colim}_{\mathscr{U}} \mathrm{Desc}(\mathscr{U}, \mathcal{C})$ understood as a pseudo-colimit of categories. For details see [Stacks, Tag 02ZN]. □

Remark A.19. From the definition of associated stack, the associated stack of a stack is equivalent to itself. Therefore, from general nonsense, we conclude that stackification is faithful. Moreover, since $\mathcal{C} \to \mathcal{C}'$ is fully faithful and locally surjective, then stackification is locally full.

Write the correct adjunction or 2-adjunction

2 Algebroid stacks

This section tries to follow [DP05, Section 1]. Let $\mathbb{K}$ be a commutative unital ring.

Definition A.20. A $\mathbb{K}$-*linear category* is a category whose Hom sets are endowed with a $\mathbb{K}$-module structure, so that composition is bilinear. A $\mathbb{K}$-*functor* is a functor between $\mathbb{K}$-categories which is linear at the level of morphisms.

write in terms of enriched categories might kill two birds with one shot... linear

Remark A.21. Any $\mathbb{K}$-linear category admits a $\mathbb{K}$-Yoneda embedding, that is, one embeds it on its category of $\mathsf{Mod}(\mathbb{K})$-valued $\mathbb{K}$-functors.

Example A.22. Let A be a $\mathbb{K}$-algebra, then the category of left A-modules $\mathsf{Mod}(A)$ is a $\mathbb{K}$-category.

Example A.23. Let A be a $\mathbb{K}$-algebra, we denote by A^+ the category of one object $\bullet$ such that $A^+(\bullet, \bullet) = A$. Given B another $\mathbb{K}$-algebra, any linear map $f\colon A \to B$ induces a $\mathbb{K}$-functor $f^+\colon A^+ \to B^+$. This defines a fully faithful functor between $\mathbb{K}$-algebras and (small) $\mathbb{K}$-categories. Moreover, any transformation $f^+ \Rightarrow g^+$ correspond to elements $b \in B$ such that $bf(a) = g(a)b$ for all $a \in A$.

These two examples are related by the following proposition.

Proposition A.24. *Let A be a $\mathbb{K}$-algebra. The category of left A-modules $\mathsf{Mod}(A)$ is equivalent to the category of $\mathbb{K}$-functors $\mathsf{Hom}_{\mathbb{K}}(A^+, \mathsf{Mod}(\mathbb{K}))$ from A^+ to the category of left $\mathbb{K}$-modules $\mathsf{Mod}(\mathbb{K})$. Moreover, via the $\mathbb{K}$-Yoneda embedding*

$$A^+ \to \mathsf{Hom}_{\mathbb{K}}((A^+)^{\mathrm{op}}, \mathsf{Mod}(\mathbb{K})) \approx_{\mathbb{K}} \mathsf{Mod}(A^{\mathrm{op}})$$

A^+ is identified with the full subcategory of right A-modules which are free of rank one.

Proof. Given an A-module M we may consider the $\mathbb{K}$-functor $A^+ \to \mathsf{Mod}(\mathbb{K})$ which sends $\bullet \mapsto M_{\mathbb{K}}$, that is M as a $\mathbb{K}$-module, and at the level of morphisms $A \to \mathrm{End}_{\mathbb{K}}(M)$ maps a to multiplication by a. On the other hand, given a $\mathbb{K}$-functor $F\colon A^+ \to \mathsf{Mod}(\mathbb{K})$, the $\mathbb{K}$-module $F(\bullet)$ admits an A-structure via the map $A \to \mathrm{End}_{\mathbb{K}}(M)$ given by F at the level of morphisms. These two processes are strictly invertible, and *a fortiori* form an equivalence. The second part is obvious if one notices $(A^+)^{\mathrm{op}} = (A^{\mathrm{op}})^+$. □

Let X be a topological space.

Definition A.25. A $\mathbb{K}$-*fibred category* over X is a fibred category $\mathcal{C}$ such that

(i) $\mathcal{C}(U)$ is a $\mathbb{K}$-linear category for all U;

(ii) $i^*\colon \mathcal{C}(U) \to \mathcal{C}(V)$ is a $\mathbb{K}$-functor for all inclusions $V \xhookrightarrow{i} U$.

Analogously, we extend it to $\mathbb{K}$-*(pre)stacks.* A morphism $(F, \alpha)\colon \mathcal{C} \to \mathcal{D}$ between $\mathbb{K}$-fibred categories is a $\mathbb{K}$-*morphism* if F induces $\mathbb{K}$-functors $F_U\colon \mathcal{C}(U) \to \mathcal{D}_U$ for all U.

Example A.26. Let $\mathscr{A}$ be a sheaf of $\mathbb{K}$-algebras. The assignment $U \mapsto \mathscr{A}(U)^+$ defines a $\mathbb{K}$-prestack. Indeed, if $\mathscr{A}$ is just a presheaf, then $\mathscr{A}^+$ defines a $\mathbb{K}$-fibred category and the 2-morphisms $\tau_{i,j}$ in Definition A.1 may be chosen to be the identity natural transformation in a such a way that (A.2) translates that $\mathscr{A}$ defines functor. Moreover, the presheaf on Remark A.9 for $\mathscr{A}(U)^+$ is a sheaf for all U if and only if $\mathscr{A}$ is a sheaf. In addition, a map $f\colon \mathscr{A} \to \mathscr{B}$ of sheaves induces $\mathbb{K}$-morphisms of the corresponding prestacks. In this case, we take α_i in Definition A.3 to be the identity natural transformation so that In order to achieve a compact data, we may assume that(A.4) translates that f is a morphism of presheaves. The associated stack to $U \mapsto \mathscr{A}(U)^+$ is denoted by $\mathscr{A}^+$. From Remark A.19 any $\mathbb{K}$-functor $\Phi\colon \mathscr{A}^+ \to \mathscr{B}^+$ is locally induced by a map of $\mathbb{K}$-algebras. More precisely, there exists a cover $\mathscr{U} = \{U_\alpha\}_\alpha$ of X such that $\Phi|_{U_\alpha} = (f_\alpha)^I nordertoachieveacompactdata, wemayassumethat+$ for some morphism $f_\alpha\colon \mathscr{A}|_{U_\alpha} \to \mathscr{B}|_{U_\alpha}$ of sheaves of $\mathbb{K}$-algebras.

Example A.27. Let $\mathscr{A}$ be a sheaf of $\mathbb{K}$-algebras. The assignment $U \mapsto \mathsf{Mod}(\mathscr{A}|_U)$ defines a $\mathbb{K}$-stack on X which we denote by $\mathfrak{Mod}(\mathscr{A})$.

These two examples are related by the analogue of Proposition A.24.

Proposition A.28. *Let $\mathscr{A}$ be a sheaf of $\mathbb{K}$-algebras. The stack of left $\mathscr{A}$-modules $\mathfrak{Mod}(\mathscr{A})$ is equivalent to the stack of $\mathbb{K}$-functors $\mathfrak{Hom}_{\mathbb{K}}(\mathscr{A}^+, \mathfrak{Mod}(\mathbb{K}_X))$ from $\mathscr{A}^+$ to the category of left $\mathbb{K}_X$-modules $\mathfrak{Mod}(\mathbb{K}_X)$. Moreover, via the $\mathbb{K}$-Yoneda embedding of stacks*

$$\mathscr{A}^+ \to \mathfrak{Hom}_{\mathbb{K}}((\mathscr{A}^+)^{\mathrm{op}}, \mathfrak{Mod}(\mathbb{K}_X)) \approx_{\mathbb{K}} \mathfrak{Mod}(\mathscr{A}^{\mathrm{op}})$$

$\mathscr{A}^+$ is identified with the full substack of right $\mathscr{A}$-modules which are locally free of rank one.

Need to state correctly what does "stack of $\mathbb{K}$-functors" for large objects mean

Definition A.29. Let $\mathcal{C}$ be fibred category over X. We say that $\mathcal{C}$ is *non-empty* if $\mathcal{C}(X)$ has at least one object. It is *locally non-empty* if there exists an open covering $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ of X such that $\mathcal{C}|_{U_\alpha}$ is non-empty.

Definition A.30. Let $\mathcal{C}$ be a fibred category. We say that $\mathcal{C}$ is *locally connected* if for every open subset $U \subseteq X$ and any pair of objects x, y of $\mathcal{C}(U)$ there exists an open cover $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ of U such that $x|_{U_\alpha} \simeq y|_{U_\alpha}$ in $\mathcal{C}(U_\alpha)$.

Definition A.31. A $\mathbb{K}$-*algebroid stack* is a $\mathbb{K}$-stack which is locally non-empty and locally connected.

Example A.32. Let $\mathscr{A}$ be a sheaf of $\mathbb{K}$-algebras. The stack $\mathscr{A}^+$ is an algebroid stack. Indeed, from Proposition A.28 $\mathscr{A}^+$ is $\mathbb{K}$-equivalent to the stack of locally free right $\mathscr{A}$-modules, which is evidently non-empty and locally connected.

The following proposition shows that one cannot strengthen local non-emptiness in Definition A.31 to global non-emptiness and expect a non-trivial answer, in the sense of Example A.32.

Proposition A.33 ([DP05, Lemma 1.1.]). *Let $\mathcal{C}$ be a non-empty and locally connected $\mathbb{K}$-stack. For x an object of $\mathcal{C}(X)$, $\mathcal{C}$ is $\mathbb{K}$-equivalent to $\underline{\mathrm{End}}_{\mathcal{C}}(x)^+$.*

Proof. Let $\mathscr{A}$ denote the sheaf of $\mathbb{K}$-algebras $\underline{\mathrm{End}}_{\mathcal{C}}(x)$. For $U \subseteq X$ an open subset, consider the assignment $y \mapsto \underline{\mathrm{Hom}}_{\mathcal{C}|_U}(x|_U, y)$ between $\mathcal{C}(U)$ and right $\mathscr{A}|_U$-modules, which are locally free of rank one, defines a $\mathbb{K}$-equivalence $\mathcal{C} \to \mathscr{A}^+$ via Proposition A.28. Indeed, for the quasi-inverse, send the unique object of $\mathscr{A}(U)^+$ to $x|_U$ and at the level of morphisms $\mathscr{A}(U) \to \mathcal{C}(U)(x|_U, x|_U)$ is the identity. This induces a map between the associated stack $\mathscr{A}^+ \to \mathcal{C}$ which is the desired quasi-inverse. $\square$

Now, we describe algebroid stacks via local data. Let $\mathcal{C}$ be a $\mathbb{K}$-algebroid stack. By definition, there exists a covering $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ such that $\mathcal{C}|_{U_\alpha}$ is non-empty and locally connected. Let x_α be an object of $\mathcal{C}(U_\alpha)$ and $\mathscr{A}_\alpha := \underline{\mathrm{End}}_{\mathcal{C}|_{U_\alpha}}(x_\alpha)$, from Proposition A.33 we obtain a $\mathbb{K}$-equivalence $\Phi_\alpha \colon \mathcal{C}|_{U_\alpha} \to \mathscr{A}_\alpha^+$ with quasi-inverse $\Psi_\alpha \colon \mathscr{A}_\alpha^+ \to \mathcal{C}|_{U_\alpha}$. On double intersections $U_{\alpha\beta}$ denote by $\Phi_{\alpha\beta}$ the composition $\Phi_\alpha \circ \Psi_\beta \colon \mathscr{A}_\beta^+|_{U_{\alpha\beta}} \to \mathscr{A}_\alpha^+|_{U_{\alpha\beta}}$. On triple intersections $U_{\alpha\beta\gamma}$ the natural isomorphism $\Psi_\beta \circ \Phi_\beta \Rightarrow \mathrm{id}_{\mathcal{C}|_{U_{\alpha\beta}}}$ induces 2-isomorphisms

$\theta_{\alpha\beta\gamma}\colon \Phi_{\alpha\beta}\circ\Phi_{\beta\gamma} \Rightarrow \Phi_{\alpha\gamma}$ such that on quadruple intersections $U_{\alpha\beta\gamma\delta}$ the diagram

$$\begin{tikzcd} \Phi_{\alpha\beta} \circ \Phi_{\beta\gamma} \circ \Phi_{\gamma\delta} \arrow[r, Rightarrow, "\theta_{\alpha\beta\gamma}\circ \mathrm{id}_{\Phi_{\gamma\delta}}"] \arrow[d, Rightarrow, "\mathrm{id}_{\Phi_{\alpha\beta}}\circ\theta_{\beta\gamma\delta}"] & \Phi_{\alpha\gamma} \circ \Phi_{\gamma\delta} \arrow[d, Rightarrow, "\theta_{\alpha\gamma\delta}"] \\ \Phi_{\alpha\beta} \circ \Phi_{\beta\delta} \arrow[r, Rightarrow, "\theta_{\alpha\beta\delta}"] & \Phi_{\alpha\delta} \end{tikzcd} \tag{A.34}$$

commutes. From Remark A.19 there exists a cover $\mathscr{U}_{\alpha\beta} = \{U^i_{\alpha\beta}\}_{i\in I}$ such that $\Phi_{\alpha\beta}|_{U^i_{\alpha\beta}} = (f^i_{\alpha\beta})^+$ for some isomorphisms of $\mathbb{K}$-algebras $f^i_{\alpha\beta}\colon \mathscr{A}_\beta|_{U^i_{\alpha\beta}} \to \mathscr{A}_\alpha|_{U^i_{\alpha\beta}}$. On triple intersections $U^{ijk}_{\alpha\beta\gamma} = U^i_{\alpha\beta} \cap U^j_{\alpha\gamma} \cap U^k_{\beta\gamma}$ we have 2-isomorphisms $\theta_{\alpha\beta\gamma}|_{U^{ijk}_{\alpha\beta\gamma}} : (f^i_{\alpha\beta})^+ \circ (f^k_{\beta\gamma})^+ \Rightarrow (f^j_{\alpha\gamma})^+$. Therefore, there are sections $a^{ijk}_{\alpha\beta\gamma} \in \mathscr{A}_\alpha^\times(U^{ijk}_{\alpha\beta\gamma})$ such that

$$f^i_{\alpha\beta} \circ f^k_{\beta\gamma} = \mathrm{ad}(a^{ijk}_{\alpha\beta\gamma}) f^j_{\alpha\gamma}.$$

On quadruple intersections $U^{ijklmn}_{\alpha\beta\gamma\delta} = U^{ijk}_{\alpha\beta\gamma} \cap U^{ilm}_{\alpha\beta\delta} \cap U^{jln}_{\alpha\gamma\delta} \cap U^{kmn}_{\beta\gamma\delta}$ the diagram (A.34) translates to the equation

$$a^{ijk}_{\alpha\beta\gamma} a^{jln}_{\alpha\beta\gamma} = f^i_{\alpha\beta}(a^{kmn}_{\beta\gamma\delta}) a^{iln}_{\alpha\beta\delta}$$

In order to achieve a compact local description, we may assume that X is paracompact and use fact that hypercoverings are cofinal among coverings [God58, Chapitre II, Lemme 3.8.1.]. Therefore, we get the following local characterization

Proposition A.35 ([DP05, Proposition 2.1.], [KS12, Proposition 2.1.3.]). *Assume X is paracompact. For gluing datum consisting of*

(i) an open cover $\mathscr{U} = \{U_\alpha\}_{\alpha\in A}$;
(ii) $\mathbb{K}$-algebras $\mathscr{A}_\alpha$ on U_α;
(iii) isomorphisms of $\mathbb{K}$-algebras $f_{\alpha\beta}\colon \mathscr{A}_\beta \to \mathscr{A}_\alpha$ on $U_{\alpha\beta}$ and;
(iv) sections $a_{\alpha\beta\gamma} \in \mathscr{A}^\times(U_{\alpha\beta\gamma})$

subject to the conditions on morphisms

$$f_{\alpha\beta} \circ f_{\beta\gamma} = \mathrm{ad}(a_{\alpha\beta\gamma}) f_{\alpha\gamma} \tag{A.36}$$

and on sections

$$a_{\alpha\beta\gamma} a_{\alpha\beta\gamma} = f_{\alpha\beta}(a_{\beta\gamma\delta}) a_{\alpha\beta\delta} \tag{A.37}$$

there exists a $\mathbb{K}$-algebroid on X to which this gluing datum is associated. Moreover, the data is unique up to equivalence of stacks, with this equivalence unique up to unique isomorphism.

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