This is what CMT-nek solves:

$\boxed{\text{Mass}}$ $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$$

$\boxed{\text{Momentum}}$ $$\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j + P\delta_{ij}) = 0$$

$\boxed{\text{Energy:}}$ $$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_j}(u_j \rho E + u_j P) = 0$$

From Ling (2016):

$\boxed{\text{Mass}}$ $$\frac{\partial (\rho_f \phi_f)}{\partial t} + \frac{\partial}{\partial x_j}(\rho_f \phi_f u_j) = 0$$

$\boxed{\text{Momentum}}$ $$\rho_f \phi_f \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) + \frac{\partial P}{\partial x_i} = f_i$$

$\boxed{\text{Energy}}$ $$\rho_f \phi_f \left(\frac{\partial E_f}{\partial t} + u_j \frac{\partial E_f}{\partial x_j} \right) - \frac{\partial}{\partial x_j}\left(-P\delta_{ij} (\phi_f u_i + \phi_p V_i)\right) = g$$

Can we go from Ling to CMT-nek?

$\boxed{\text{Mass}}$

$$\phi_f \frac{\partial \rho_f}{\partial t} + \rho_f \frac{\partial \phi_f}{\partial t} + \phi_f \frac{\partial}{\partial x_j}(u_j \rho_f) + u_j \rho_f \frac{\partial \phi_f}{\partial x_j} = 0$$

$$\boxed{\frac{\partial \rho_f}{\partial t} + \frac{\partial}{\partial x_j}(u_j \rho_f) = -\frac{\rho_f}{\phi_f}\left(\frac{\partial \phi_f}{\partial t} + u_j \frac{\partial \phi_f}{\partial x_j} \right)}$$

$\boxed{\text{Momentum}}$

$$\phi_f \left(\frac{\partial (\rho_f u_i)}{\partial t} - u_i \frac{\partial \rho_f}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j \rho_f) - u_i \frac{\partial}{\partial x_j}(\rho_f u_j) \right)$$

$$+ (\phi_f + \phi_p) \frac{\partial P}{\partial x_i} = f_i$$

$$\phi_f \left(\frac{\partial(\rho_f u_i)}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j \rho_f + P) \right) - \phi_f u_i \underbrace{\left(\frac{\partial \rho_f}{\partial t} + \frac{\partial}{\partial x_j}(\rho_f u_j) \right)}_{\text{mass RHS}} + \phi_p \frac{\partial P}{\partial x_i} = f_i$$

$$\frac{\partial(\rho_f u_i)}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j \rho_f + P) = -\frac{\rho_f u_i}{\phi_f}\left(\frac{\partial \phi_f}{\partial t} + u_j \frac{\partial \phi_f}{\partial x_j} \right) + \frac{\left(f_i - \phi_p \frac{\partial P}{\partial x_i} \right)}{\phi_f}$$

Note, when projecting f_i, pressure gradient evaluates to:

$$-\left(-\phi_p \frac{\partial P}{\partial x_i}\right) = +\phi_p \frac{\partial P}{\partial x_i}$$

Thus, if we don't project/couple PG force, we have

$$\boxed{\frac{\partial(\rho_f u_i)}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j \rho_f + P) = -\frac{\rho_f u_i}{\phi_f}\left(\frac{\partial \phi_f}{\partial t} + u_j \frac{\partial \phi_f}{\partial x_j} \right) + \frac{f_i^*}{\phi_f}}$$

Energy:

$$\phi_f \left(\frac{\partial(\rho_f E_f)}{\partial t} - E_f \frac{\partial \rho_f}{\partial t} + \frac{\partial}{\partial x_j}(E_f \rho_f u_j) - E_f \frac{\partial}{\partial x_j}(\rho_f u_j) \right) + \frac{\partial}{\partial x_j}(P \phi_f u_j) + \frac{\partial}{\partial x_j}(P \phi_p V_j) = g$$

$$\phi_f \left(\frac{\partial(\rho_f E_f)}{\partial t} + \frac{\partial}{\partial x_j}(E_f \rho_f u_j) \right) + \phi_f \frac{\partial}{\partial x_j}(P u_j) + P u_j \frac{\partial \phi_f}{\partial x_j} + \phi_p V_i \frac{\partial P}{\partial x_i} + P \frac{\partial(\phi_p V_i)}{\partial x_i} = E_f \underbrace{\left(\frac{\partial \rho_f}{\partial t} + \frac{\partial}{\partial x_j}(\rho_f u_j) \right)}_{\text{mass RHS}} + g$$

$$\frac{\partial}{\partial t}(\rho_f E_f) + \frac{\partial}{\partial x_j}(E_f \rho_f u_j + P u_j) = -\frac{E_f \rho_f}{\phi_f}\left(\frac{\partial \phi_f}{\partial t} + u_j \frac{\partial \phi_f}{\partial x_j} \right) - \frac{P u_j}{\phi_f}\frac{\partial \phi_f}{\partial x_j} - \frac{P}{\phi_f}\frac{\partial}{\partial x_j}(\phi_p V_j) + \frac{1}{\phi_f}\left(g - V_i \phi_p \frac{\partial P}{\partial x_i} \right) \quad *$$

Note, the energy projected/coupled by the pressure gradient force is:

$$-\left(-\phi_p \frac{\partial P}{\partial x_j} V_j\right) = \phi_p V_j \frac{\partial P}{\partial x_j}$$

Thus, if we don't project/couple PG work, we have

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}(\rho_f E_f) + \frac{\partial}{\partial x_j}(E_f \rho_f u_j + P u_j) = -\frac{E_f \rho_f}{\phi_f}\left(\frac{\partial \phi_f}{\partial t} + u_j \frac{\partial \phi_f}{\partial x_j}\right) \\ - \frac{P u_j}{\phi_f}\frac{\partial \phi_f}{\partial x_j} - \frac{P}{\phi_f}\frac{\partial}{\partial x_j}(\phi_p V_j) + \frac{g^*}{\phi_f}\end{aligned}}$$

In total then, we have:

Mass:
$$\boxed{\frac{\partial \rho_f}{\partial t} + \frac{\partial}{\partial x_j}(u_j \rho_f) = \frac{A}{\phi_f}}$$

Momentum:
$$\boxed{\frac{\partial(\rho_f u_i)}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j \rho_f + P) = u_i \frac{A}{\phi_f} + \frac{f_i^*}{\phi_f}}$$

Energy:
$$\boxed{\begin{aligned}\frac{\partial(\rho_f E_f)}{\partial t} + \frac{\partial}{\partial x_j}(E_f \rho_f u_j + P u_j) = E_f \frac{A}{\phi_f} - \frac{P u_j}{\phi_f}\frac{\partial \phi_f}{\partial x_j} \\ - \frac{P}{\phi_f}\frac{\partial}{\partial x_j}(\phi_p V_j) + \frac{g^*}{\phi_f}\end{aligned}}$$

where: $A = -\rho_f\left(\frac{\partial \phi_f}{\partial t} + u_j \frac{\partial \phi_f}{\partial x_j}\right)$

f^* = Projected particle hydro force (w/o PG force)

g^* = Projected particle hydro work (w/o PG work)