IV STATIC AND DYNAMIC FORCE ANALYSIS OF MECHANISMS

4.0 Overview

In this chapter, force analysis will be introduced. This is followed by the mechanical advantage of a mechanism. Then, the static force analysis will be presented and solved through a graphical method: force polygons. Based on the equations of motion, the dynamic force analysis will be carried out. Through the use of inertia forces and inertia moments, dynamic problems can be treated in the same manner as static problems yielding a graphical method known as the inertia circle. Finally, the force analysis will be solved analytically.

4.1 Force Analysis

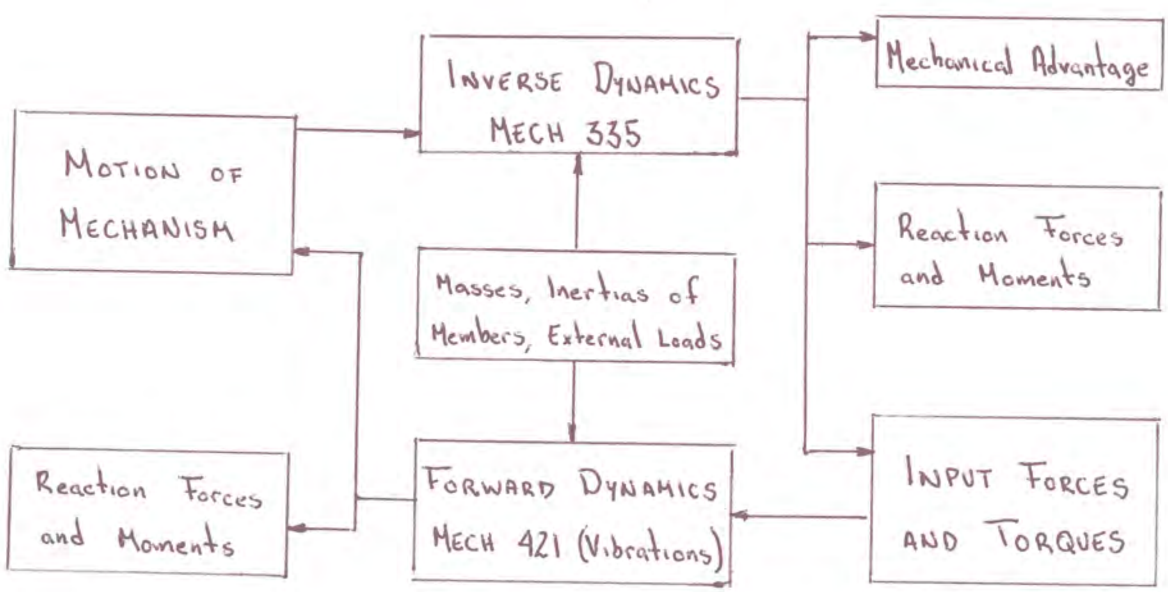
The general function of a mechanism is to transmit motion and forces from an actuator to the components that perform the desired task.

The force analysis may be divided into Static Force and Dynamic Force analyses.

Static Force Analysis deals with mechanisms without accelerations, or where the accelerations are significantly small that they can be neglected. This condition is termed "static equilibrium"

The objective of the static force analysis is to determine the force or torque required by the actuator to maintain the mechanism in static equilibrium. We can also determine the forces acting on every joint, because a critical task in the design of machines is to ensure that the strength of the links and joints is sufficient to support the forces excerted on them.

Dynamic Force Analysis deals with mechanisms having significant accelerations. In many high-speed machines, the inertial forces created by the motion of a machine exceed the forces required to perform the intended task. The inclusion of accelerations yields a condition termed "dynamic equilibrium". The dynamic force analysis may be divided into Inverse and Forward Dynamic Analyses.



4.2 Mechanical Advantage

In order to judge the quality of a linkage for an intended application, it is necessary to analyze the ability of the mechanism to transmit torque or force, i.e., we have to determine the relationship between the input force/torque and the output force/torque

For instance, with a slider-crank mechanism we may have either the crank as an input (mechanical press) or the slider as an input (internal combustion engine). In the first case, we have an input torque and an output force, while in the second case, we have an input force and an output torque.

If we assume that a mechanism is a conservative system, i.e., the energy lost by friction, heat, etc., is negligible compared to the total energy transmitted; power in (P_{IN}) equals power out (P_{OUT}), i.e.,

$$P_{IN} = P_{OUT}$$

$$\tau_{IN}\,\omega_{IN} = \tau_{OUT}\,\omega_{OUT} \qquad \ldots \quad 4.1$$

$$F_{IN} V_{IN} = F_{OUT} V_{OUT}$$

where $\quad FV = \bar{F}\cdot\bar{V} = F_x V_x + F_y V_y$

By definition, the mechanical advantage (M.A.) of a mechanism is the ratio of the output force exerted by the driven link to the necessary force required at the driver, i.e.,

$$M.A. = \frac{F_{OUT}}{F_{IN}} \qquad \ldots 4.2$$

Since $\tau = F \cdot r$, where r is the distance between the centre of rotation and the point where the force is being applied, we can express the mechanical advantage in terms of torques, i.e.,

$$F_{IN} = \tau_{IN}/r_{IN} \qquad \text{and} \qquad F_{OUT} = \tau_{OUT}/r_{OUT}$$

thus,

$$M.A. = \frac{\tau_{OUT}/r_{OUT}}{\tau_{IN}/r_{in}} = \frac{\tau_{OUT}\, r_{IN}}{\tau_{IN}\, r_{OUT}} \qquad \ldots 4.3$$

From conservation of power, we can write the mechanical advantage in terms of angular velocities, i.e.,

$$\frac{\tau_{OUT}}{\tau_{IN}} = \frac{\omega_{IN}}{\omega_{OUT}} \qquad \ldots 4.4$$

Substituting eq. (4.4) in eq (4.3) yields

$$M.A. = \frac{r_{IN}\,\omega_{IN}}{r_{OUT}\,\omega_{OUT}} \qquad \ldots 4.5$$

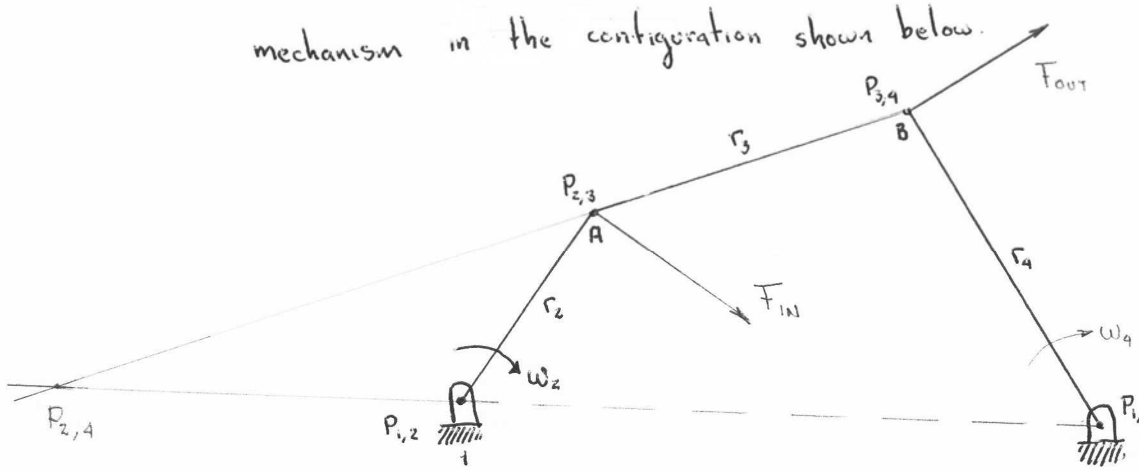
Thus, the mechanical advantage is a product of two factors

i) A ratio of distances that depend on the placement of the input and output forces

ii) An angular velocity ratio

The velocity analysis based on instant centres is a convenient method to use because it explicitly determines the angular velocity ratio between two links.

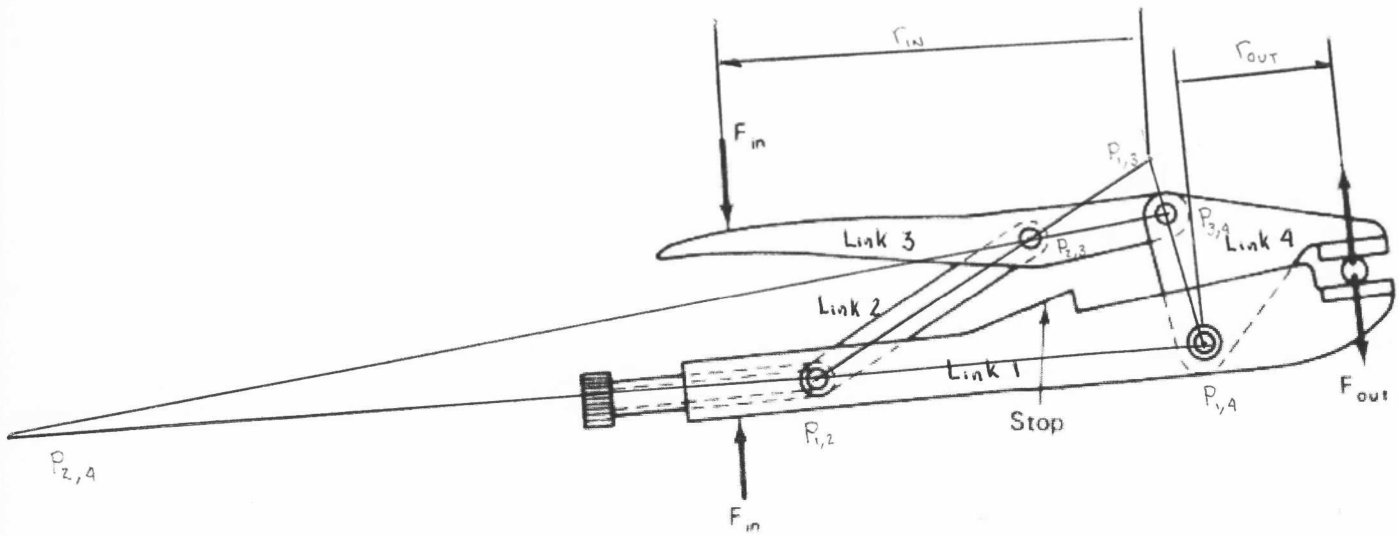
<u>Example 4.1</u> Determine the mechanical advantage of the four-bar mechanism in the configuration shown below.



- The input force is perpendicular to the input link and applied at point A.
- The output force is perpendicular to the follower and applied at point B.
- The shortest distance between the applied forces and the centres of rotation are r_2 for F_{IN} and r_4 for F_{OUT}, where $r_2 = 3cm$ and $r_4 = 4.8cm$.

$$\frac{\omega_2}{\omega_4} = \frac{\dot{\theta}_2}{\dot{\theta}_4} = \frac{r_{P_{1,4}P_{2,4}}}{r_{P_{1,2}P_{2,4}}} = \frac{13.7}{5.3} \qquad M.A = \frac{r_2\,\omega_2}{r_4\,\omega_4} = \frac{3(13.7)}{4.8(5.3)} = 1.615$$

<u>Example 4.2</u> Determine the mechanical advantage of the adjustable toggle pliers and explain their design.



The input force is applied on link 3, the centre of rotation is $P_{3,4}$, and the shortest distance between F_{in} and the absolute instant centre $P_{1,3}$ is r_{IN}. Whereas, the output force is applied by the mechanism at the jaws of the pliers and it is separated by a distance r_{OUT} from the absolute instant centre $P_{1,4}$.

These distances are measured $r_{IN} = 5.8$ cm and $r_{OUT} = 2.1$ cm. Using instant centres, we can determine the angular velocity ratio ω_{IN}/ω_{OUT}.

$$\frac{\omega_{IN}}{\omega_{OUT}} = \frac{\omega_3}{\omega_4} = \frac{r_{P_{1,4}P_{3,4}}}{r_{P_{1,3}P_{3,4}}} = \frac{1.75}{0.75}$$

Thus,

$$M.A. = \frac{r_{IN}\,\omega_{IN}}{r_{OUT}\,\omega_{OUT}} = \frac{(5.8)(1.75)}{(2.1)(0.75)} = 6.4$$

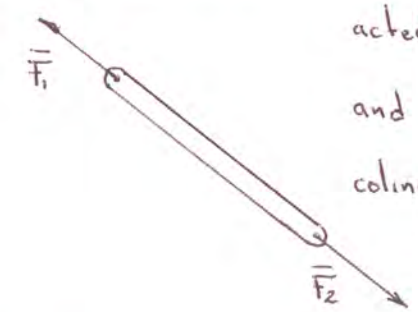
As we clamp down on the pliers, $P_{2,4}$ approaches $P_{1,2}$ and $P_{1,3}$ approaches $P_{3,4}$. Thus, the mechanical advantage approaches infinity. The screw adjustment should be set so that the maximum mechanical advantage occurs at the required distance between the jaws of the pliers. The stop prevents excessive overtravel beyond the toggle position.

4.3 Static Force

In this section, the analysis of a mechanism in static equilibrium, i.e., zero accelerations, is presented.

Described below are three cases for which links of a mechanism are in static equilibrium.

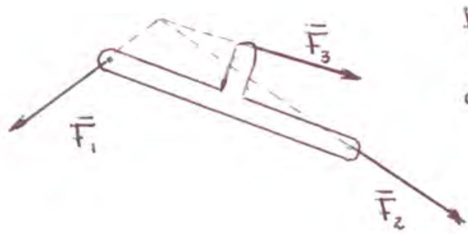
<u>Two-Force Member</u>



In order for a link to be in equilibrium when acted on by two forces: no moments can be applied, and the two forces must be equal in magnitude, colinear, and in opposite directions. Thus,

$$\bar{F}_1 + \bar{F}_2 = \bar{0}$$

<u>Three-Force Member</u>



In order for a link to be in equilibrium when acted on by three forces: no moments can be applied, and the forces must add zero and must intersect at a common point, i.e.,

$$\bar{F}_1 + \bar{F}_2 + \bar{F}_3 = \bar{0}$$

<u>Two-Couple Member</u>

In order for a link to be in equilibrium when acted on by two couples (moments), both couples must be equal but in opposite direction

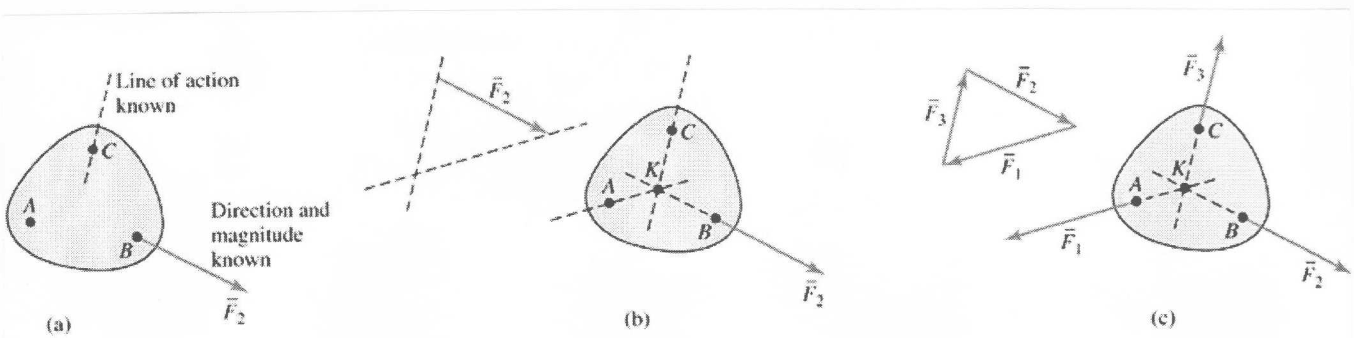
$$\sum M_c = M_1 + M_2 = 0$$

$$M_2 = F_2 d$$

4.3.1 Force Polygon Method

The described static force cases and the assumtion of neglecting both friction in joints and gravity (weightless links) will allow us to represent forces through polygons.

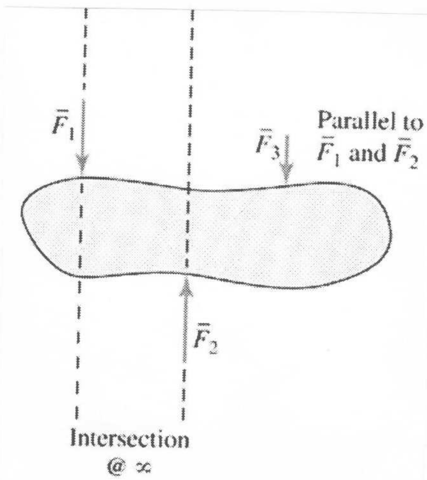
Assume a three-force member within the mechanism as shown in the figure below. The direction and magnitude of the force applied at point A are unknown, at point B are known, and at point C just the direction is known. The member is in static equilibrium if and only if the three forces intersect at a common point and sum to zero.



Construction of a Force Polygon

Once the lines of action are known, one proceeds to construct the force polygon, starting from the known force (at point B), scale it, and the rest of the force vectors are attached head to tail.

A special case occurs when two of the forces acting on a three-force member are parallel. The third force is also parallel, the three forces intersect at a common point: infinity. A polygon cannot be drawn and the problem is solved using the equations of static equilibrium.



$$\sum F = 0 \qquad \ldots \; 4.9$$

$$\sum M_G = 0$$

The force polygon method is suitable for only one configuration.

Steps for the construction of a Force Polygon

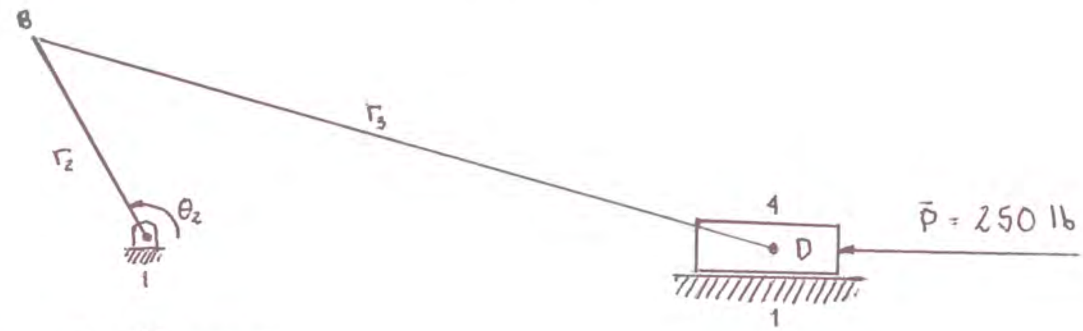
a) Draw the mechanism to scale in the configuration under analysis.

b) Break the mechanism in subsystems, either individual links or groups of links. This break is carried out depending on the number of forces acting on each subsystem, with two and three-force members employed to generate the force polygon.

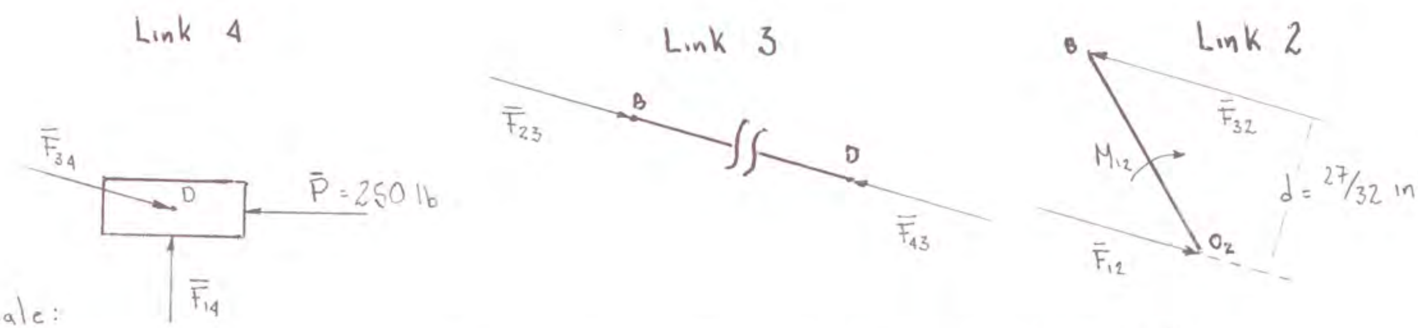
c) Determine the force polygons of those subsystems which are subjected to known external forces.

d) Use the obtained force polygons to examine the force transmitted between subsystems and complete the remaining force polygons.

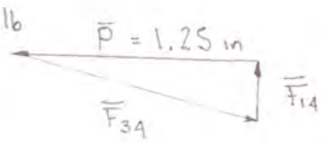
<u>Example 4.3</u> Consider the slider-crank mechanism shown below, where $\theta_2 = 120°$, $r_2 = 1.2$ in, and $r_3 = 4.0$ in. A 250 lb force is applied to the slider. Determine the torque, τ_m, required by a motor located at O_2 to maintain the mechanism in static equilibrium.



Free Body Diagrams



Scale:
1 in = 200 lb



Measurments $\bar{F}_{34} = 1\ 9/32$ in $\equiv$ 256.25 lb
$\bar{F}_{14} = 9/32$ in $\equiv$ 56.25 lb

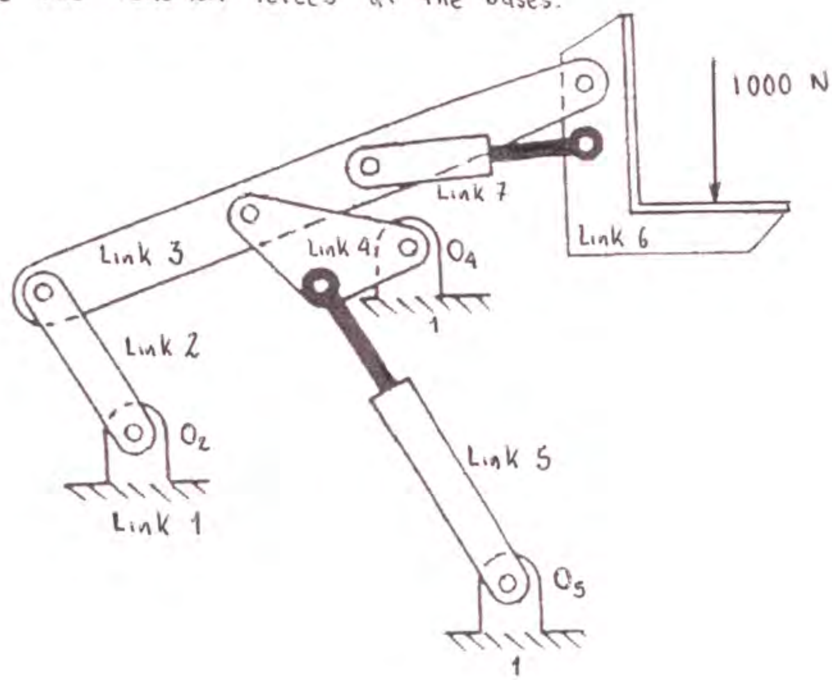
Link 4 is a three-force member. All forces intersect at point D and sum to zero. The force polygon is drawn to scale and forces are measured.

Link 3 is a two-force member and based on Newton's third law $\bar{F}_{43} = -\bar{F}_{34}$. Since link 2 is a two-force member $\bar{F}_{23} = -\bar{F}_{43}$ or $\bar{F}_{23} = \bar{F}_{34}$.

Link 2 is a two-couple member, where $\bar{F}_{32} = -\bar{F}_{23}$ and $\bar{F}_{12} = -\bar{F}_{32}$. To keep this member in static equilibrium, we take the moment about O_2

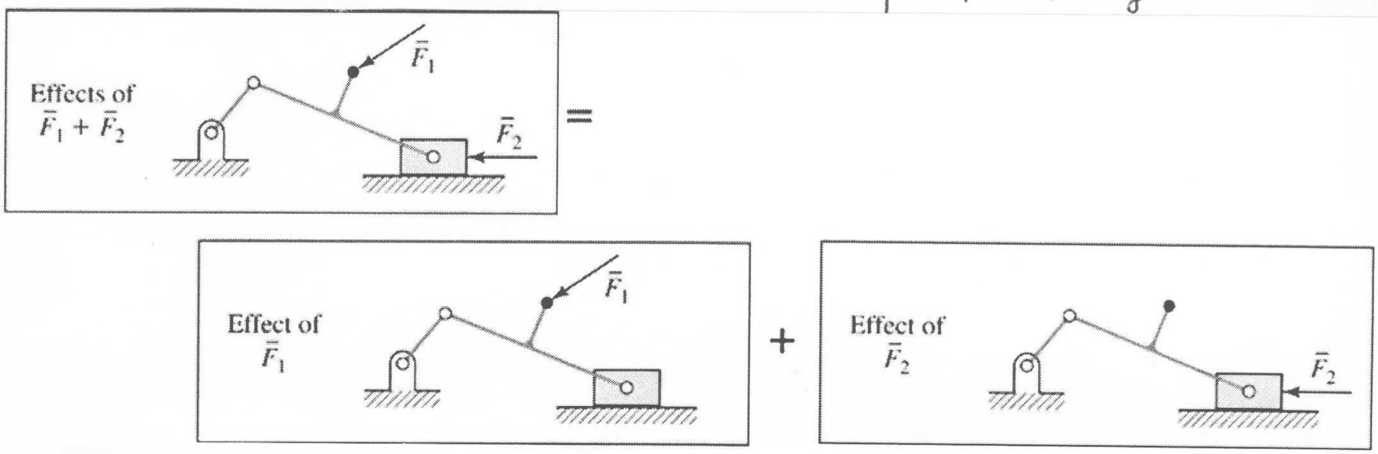
$$\tau_m = M_{12} = F_{32}\, d = -256\left(27/32\right) = 216 \text{ in} \cdot \text{lb CW}$$

Example 4.4 Shown below is a front-end loader mechanism. Its mobility is 2 dof, with the prismatic joints being actuated. For static analysis, the prismatic joints act as two-force members. Find the required force in the actuators to maintain the static equilibrium and the reaction forces at the bases.



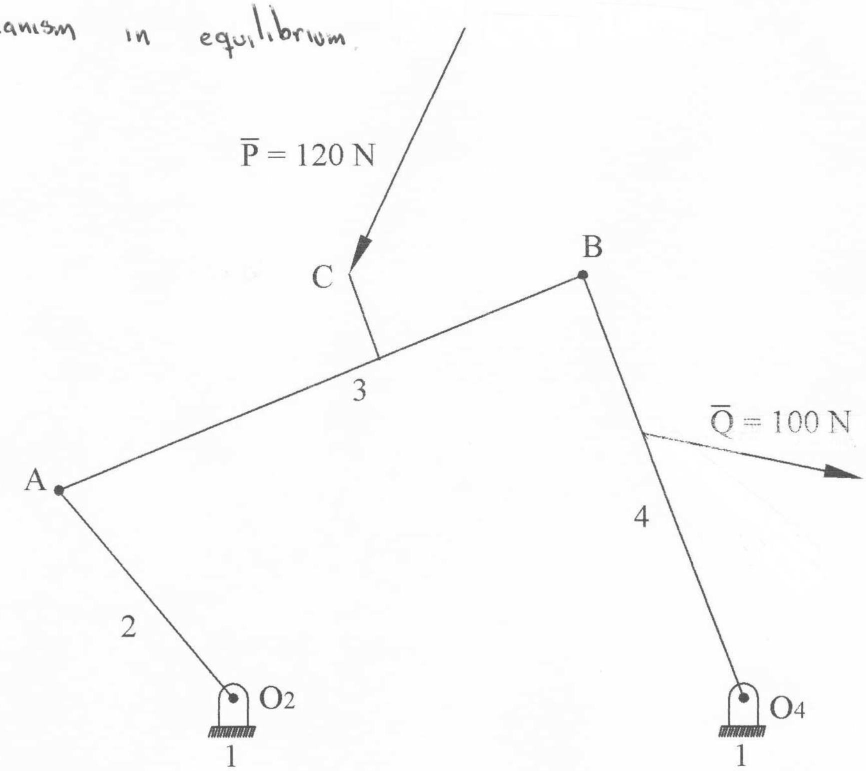
4.3.2 Principle of Superposition

The principle of superposition is used when the mechanism is subjected to more than one force simultaneously. For every force an analysis is performed. The combined result is found by superimposing each analysis:

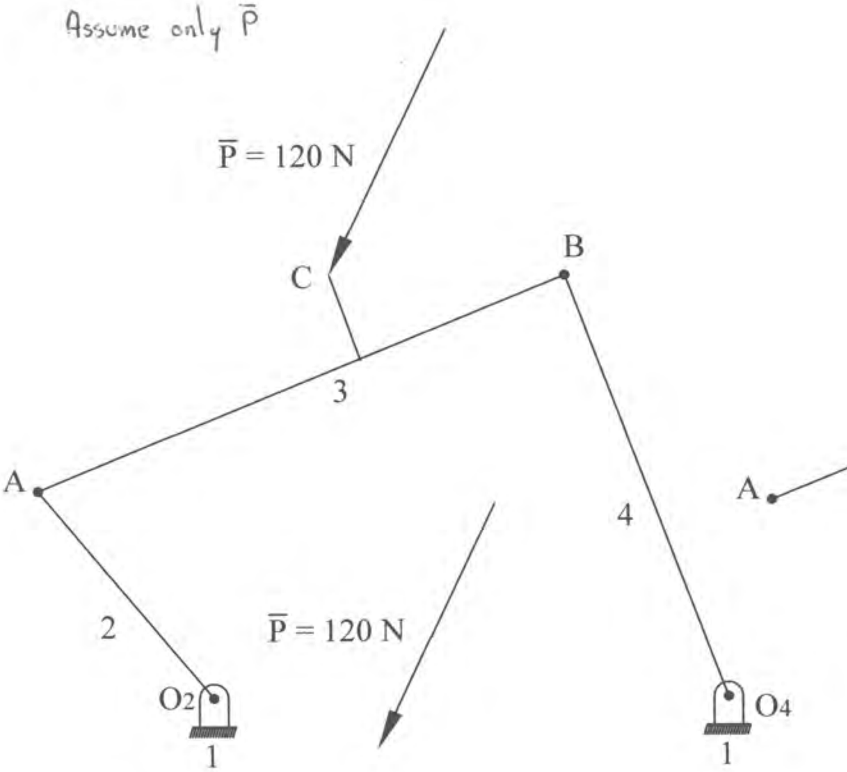


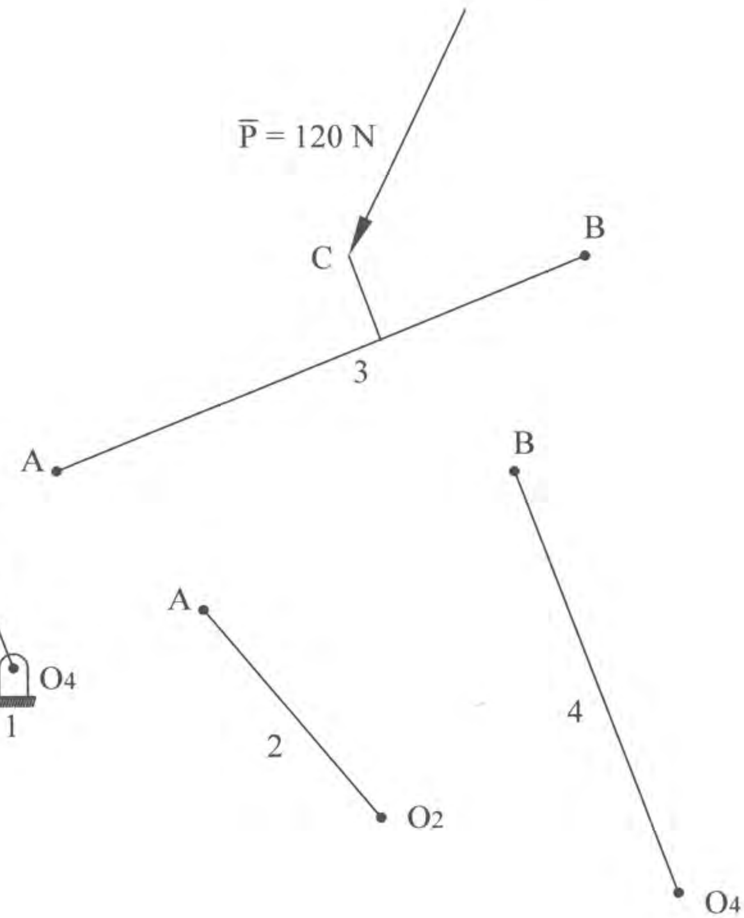
Example 4.5

Consider the four-bar mechanism shown below, which is subjected to two forces $\bar{P}$ and $\bar{Q}$. Determine the required torque (τ_2) to maintain the mechanism in equilibrium.

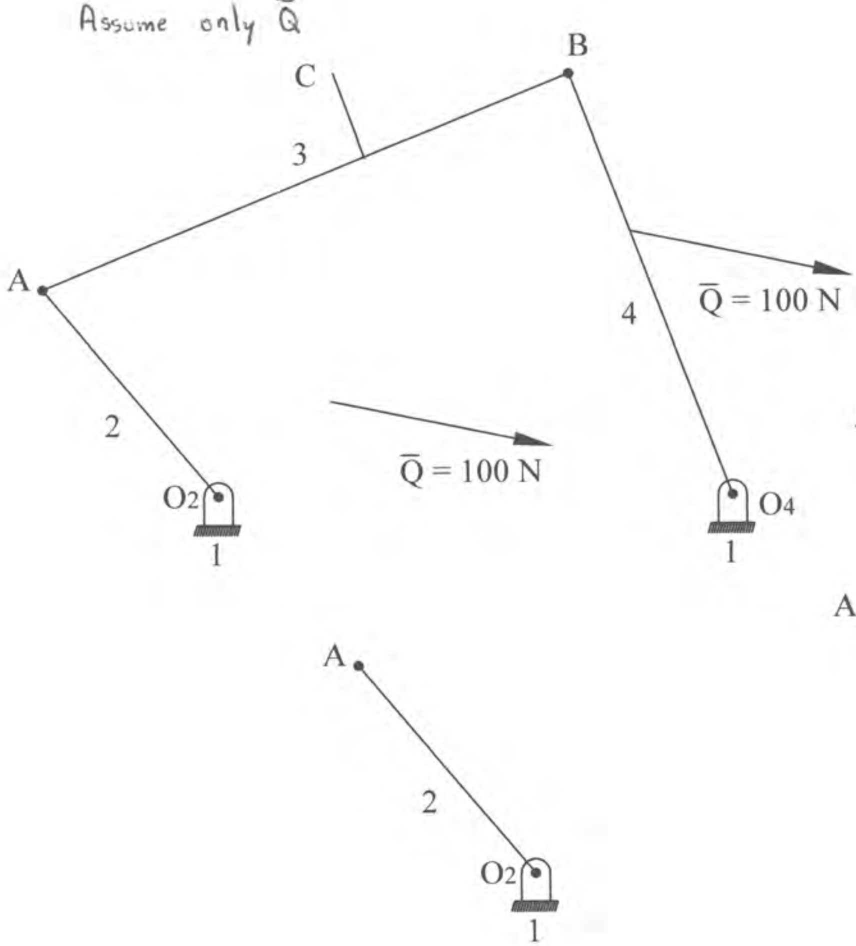


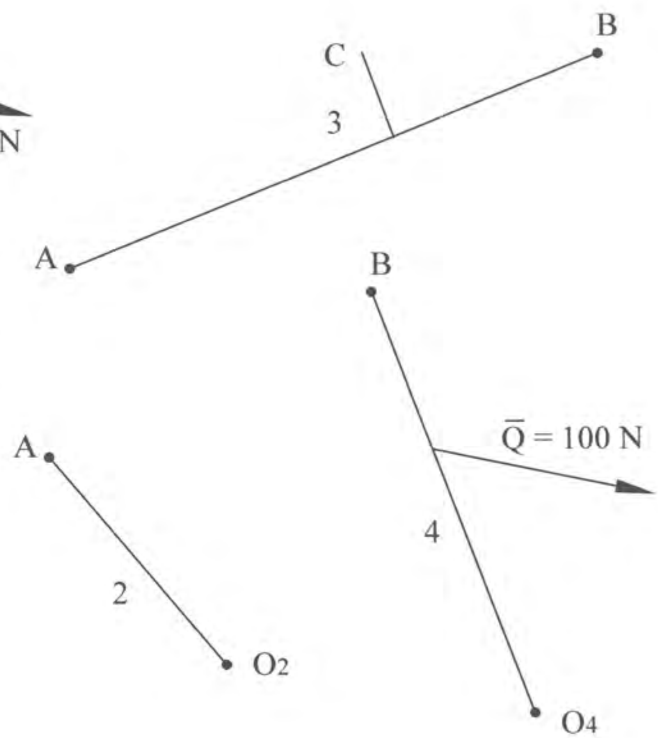
Assume only $\bar{P}$





Assume only $\bar{Q}$





4.4 Dynamic Force Analysis

Assume a mechanism under dynamic conditions, i.e., linear and angular accelerations are considered. The existance of accelerations implies that the mechanism is subjected to inertial forces. In this section, the desired motion of the mechanism is known and the resultant forces and moments are to be found.

From the design point of view, it is critical to know how these forces are acting on the mechanism, thus, a selection of proper joints and actuators can be made.

4.4.1 Kinetics

In this subsection, the governing equations of motion of a rigid body (link) are presented.

Linear Motion

The governing equation for linear motion is Newton's Second Law

$$\bar{F} = m\bar{a}_G \qquad \ldots\ 4.10$$

where

$\bar{F}$ = net external force applied to the rigid body

m = mass of the body

$\bar{a}_G$ = acceleration of the centre of mass

The mass of a rigid body may be described as the resistance of a rigid body to linear acceleration; i.e., it is more difficult to "speed up" or "slow down" an object with large mass.

The centre of mass is the point of balance of that rigid body; i.e., the rigid body's weight can be held and be in balance in all directions.

For simple symmetrical shapes, the centre of mass can be easily identified. For more complex shapes, we can break the rigid body into several (n) simple shapes. The centre of mass of the entire rigid body can be determined from a weighted average of the coordinates of every individual centre of mass with respect to a defined reference frame, i.e.,

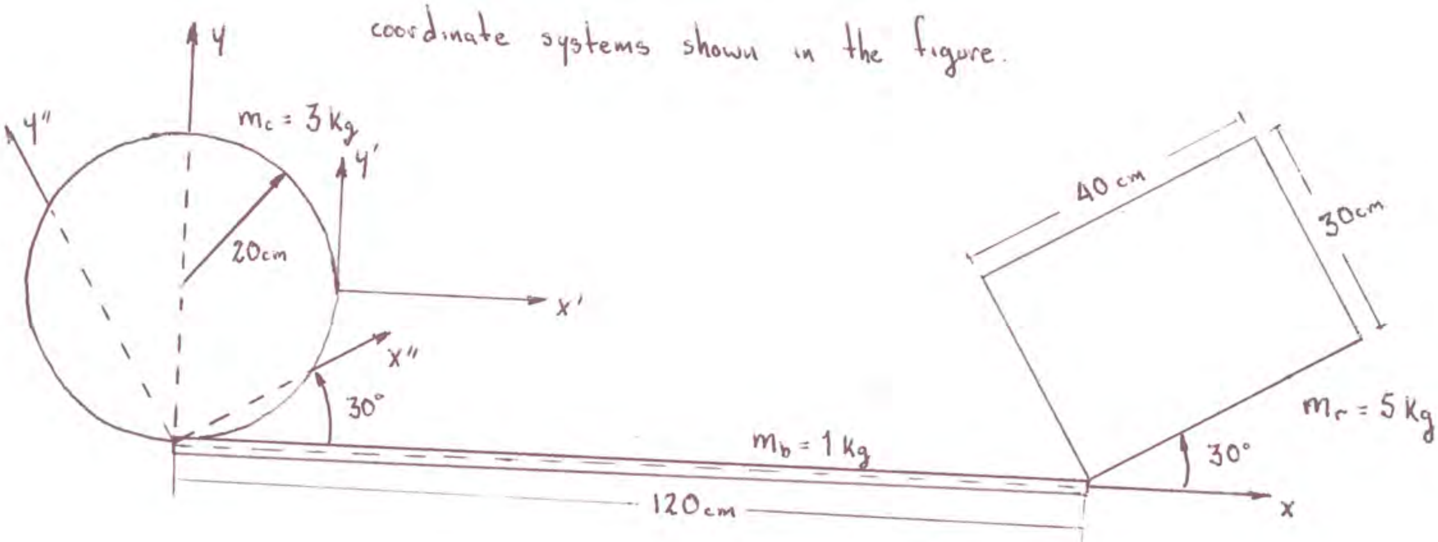
$$x_{G_T} = \frac{\sum_{i=1}^{n} m_i x_{G_i}}{\sum_{i=1}^{n} m_i} \quad \text{and} \quad y_{G_T} = \frac{\sum_{i=1}^{n} m_i y_{G_i}}{\sum_{i=1}^{n} m_i} \qquad \ldots 4.11$$

where x_{G_T} and y_{G_T} are the coordinates of the centre of mass of the entire rigid body with respect to a reference frame.

x_{G_i} and y_{G_i} are the coordinates of the centre of mass of the i^{th} individual component of the rigid body, with respect to a reference frame.

m_i is the mass of the i^{th} element.

<u>Example 4.6</u> Determine the location of the centre of mass of the following rigid body with respect to the three coordinate systems shown in the figure.



Angular Motion

Based on Newton's second law, the governing equation for angular motion is

$$M_o = I_o \alpha \qquad \ldots 4.12$$

where

M_o = net external moment applied to the rigid body about point O.

α = angular acceleration of the rigid body, also denoted as $\ddot{\theta}$

I_o = polar mass moment of inertia of the rigid body with respect to point O.

The moment of inertia may be described as the resistance of a rigid body to angular acceleration.

Determining I_o may be complicated because it must be calculated for every configuration of the mechanism. In order to simplify this problem the Parallel Axis Theorem can be employed. That is, instead of determining the moment of inertia with respect to a coordinate system at point O, we can refer it with respect to a coordinate system at the centre of mass (G).

The moments of inertia of common area elements and homogeneous solids are given in the Appendix at the end of this chapter.

For planar mechanisms, we are going to use the moment of inertia relative to the axis of rotation

By knowing the moment of inertia of a rigid body about its centre of mass, we can determine the moment of inertia of the rigid body about any other point whose coordinate system is parallel to the one attached at the centre of mass, i.e.,

$$I_o = I_G + m\left(x_G^2 + y_G^2\right) \qquad \ldots 4.13$$

where

x_G and y_G are the coordinates of the centre of mass with respect to frame $\{0\}$

m is the mass of the rigid body.

Substituting eq. (4.13) in eq. (4.12) yields

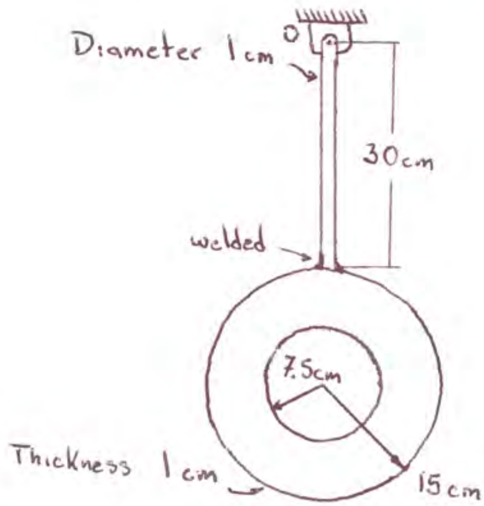
$$M_o = \left(I_G + m\left(x_G^2 + y_G^2\right)\right)\alpha \qquad \ldots 4.14$$

Another way to determine the moment of inertia is through the radius of gyration, k. This quantity is often found in handbooks. Conceptually, the radius of gyration is the distance from the centre of mass to a point where the entire mass could be concentrated and have the same moment of inertia, i.e.,

$$I_o = mk^2 \qquad \ldots 4.15$$

For composite bodies, the moment of inertia is found by adding or subtracting the moments of inertia of all the individual parts.

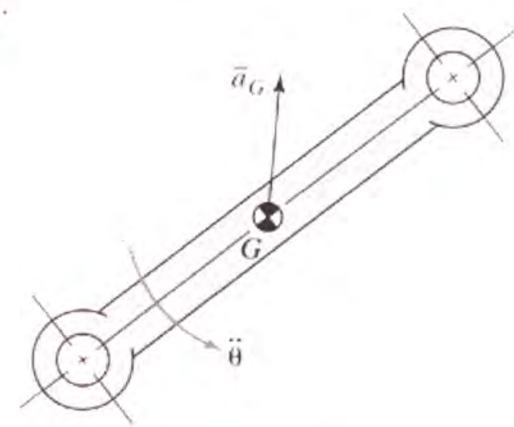
Example 4.7 The pendulum shown in the figure below is suspended from point O. it consists of a slender rod connected to a disk with a hole in the middle. Both elements are made of the same material, $\rho = 8000\ kg/m^3$. Determine the required torque by the motor attached at O to achieve an angular acceleration of $\alpha = 50\ rad/s^2$.



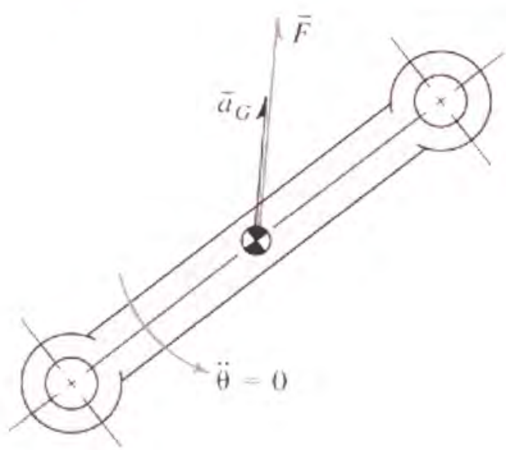
4.4.2 Inertia Circle

By means of Inertia Forces and Inertia Moments, dynamic problems can be treated in the same manner as static problems.

Shown below is an isolated link of a mechanism, with a mass m, and with a moment of inertia about its centre of mass, I_G.
In addition, for the given position, we know the linear and angular accelerations of the link.



The linear acceleration can be modeled using an external force $\bar{F}$. If the force is applied at the centre of mass, it will not provide a moment about G, i.e., the link will have zero angular acceleration, $M_G = I_G \alpha = 0$.



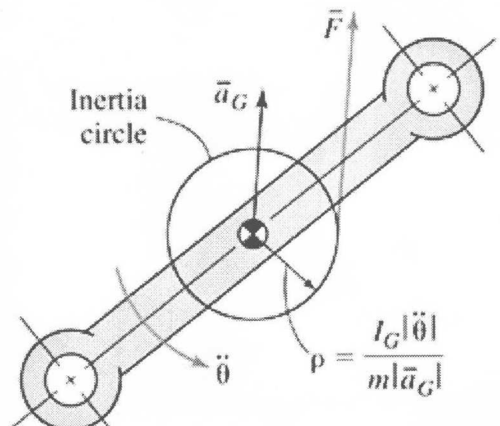
$$\bar{F} = m\,\bar{a}_G$$

Given that the acceleration of the centre of mass is achieved regardless of the location of the applied force, we can offset the location of the applied force yielding a moment

$$|M_G| = |\bar{F}|\,\rho \qquad \ldots\ 4.16$$

where ρ is the shortest distance between the line of action of the force and the centre of mass of the link. This distance ρ is referred to as the radius of the inertia circle, and it can be calculated as follows:

$$\rho = \frac{|M_G|}{|\bar{F}|} = \frac{I_G|\alpha|}{m|\bar{a}_G|} \qquad \ldots\ 4.17$$



Given that the acceleration of the link is not caused by $\bar{F}$, but by forces applied through the joints, we introduce an inertia force F_{in}, which has the same magnitude and direction as $\bar{F}$ but in opposite sense.

$$\bar{F}_{in} + \bar{F} = \bar{0} \qquad \ldots\ 4.18$$

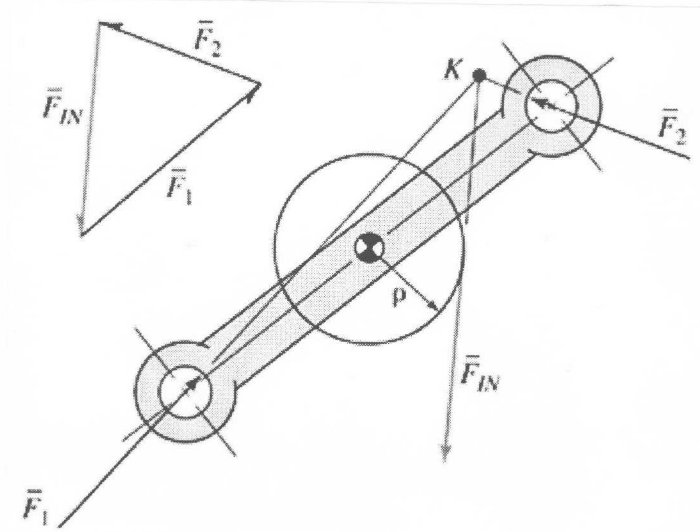
The inertia force is the resistance to linear acceleration yielding a kinetostatic system.

Also, the inertia force produces an inertia moment in the opposite direction to the one produced by the actual forces

$$M_{IN} + M_G = 0 \quad \ldots 4.19$$

The problem is then solved as if it was a static problem. The assumed link may be considered as a three-force member, in which one of the forces is the inertia force.

$$\bar{F}_{IN} + \bar{F}_1 + \bar{F}_2 = \bar{0}$$



In the case that there is more than one inertia force acting on the mechanism, the Principle of Superposition can be employed.

Steps for the analysis of a dynamic force problem.

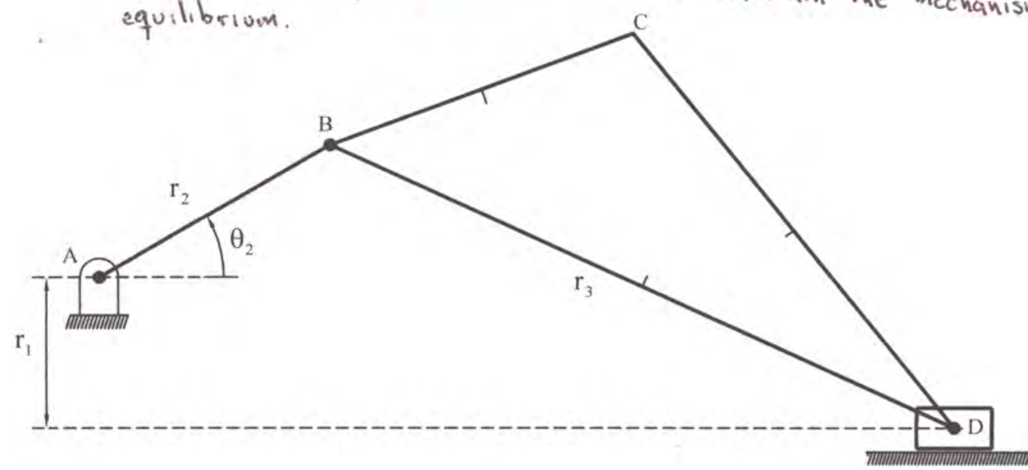
a) Draw the mechanism to scale in the configuration under analysis.

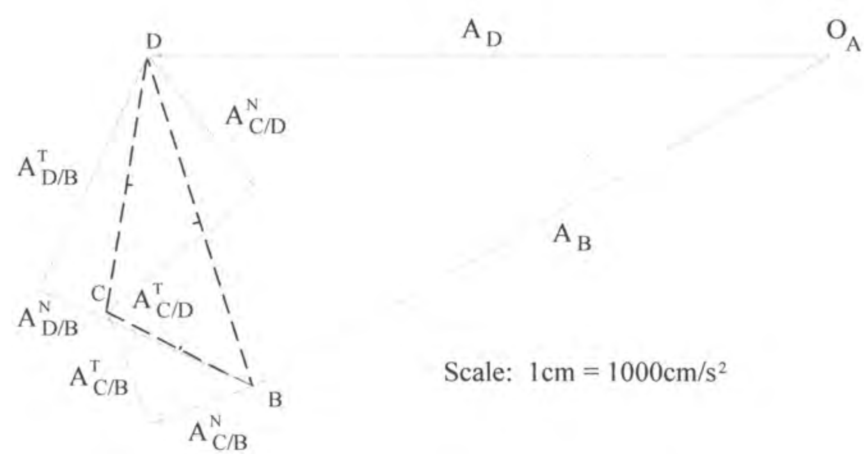
b) Construct an acceleration polygon, and determine the linear accelerations of the centres of mass and the angular accelerations of the links.

c) Determine the inertia forces and inertia moments and include them in the mechanism's diagram.

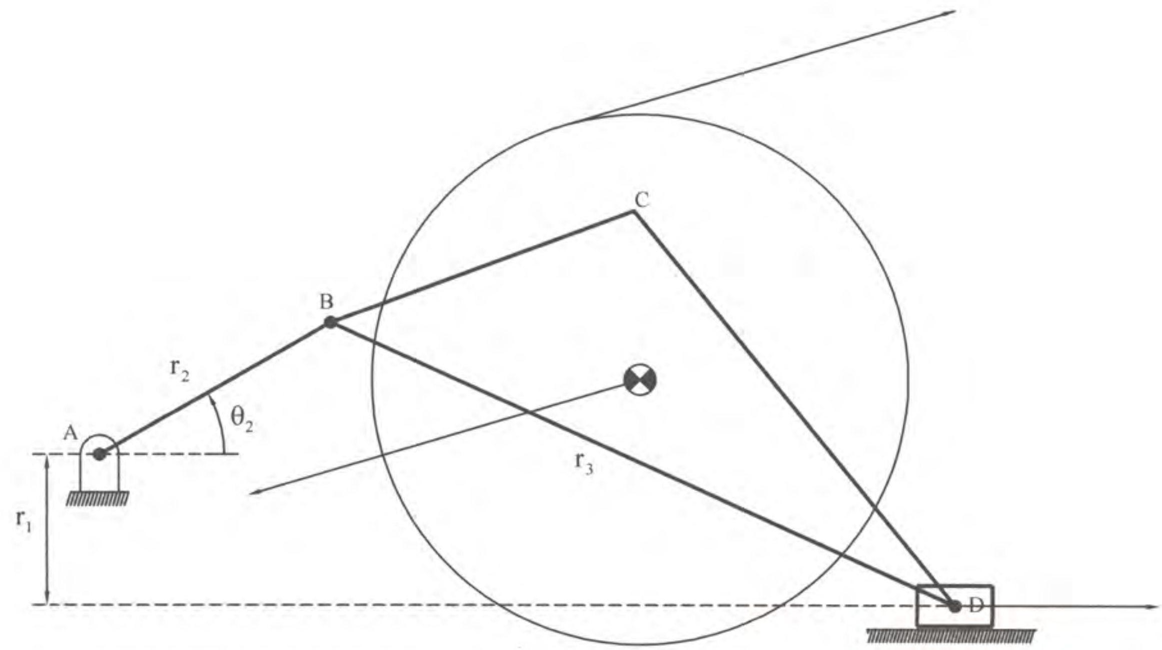
d) Treat the inertia force as an external load, employ the principle of superposition if required.

Example 4.8 Consider the slider-crank mechanism of examples 3.6 (Velocity), 3.8 (Acceleration), and 3.11 (Analytical), where $r_1 = 2cm$, $r_2 = 3.5cm$, $r_3 = 9cm$, $\theta_2 = 30°$, and $\dot{\theta}_2 = 50\ rad/s$ (constant). The acceleration polygon is provided and $\bar{a}_D = 8960\ cm/s^2$ and $\ddot{\theta}_3 = 378\ rad/s^2$ are calculated. Assume the inertia caused by link 2 very small, while $m_3 = 300gr$, $m = 150gr$, and $I_{G3} = 2.5 \times 10^{-3}\ Kg.m^2$, are the properties of links 3 and 4. Determine the torque required at link 2 to maintain the mechanism in static equilibrium.

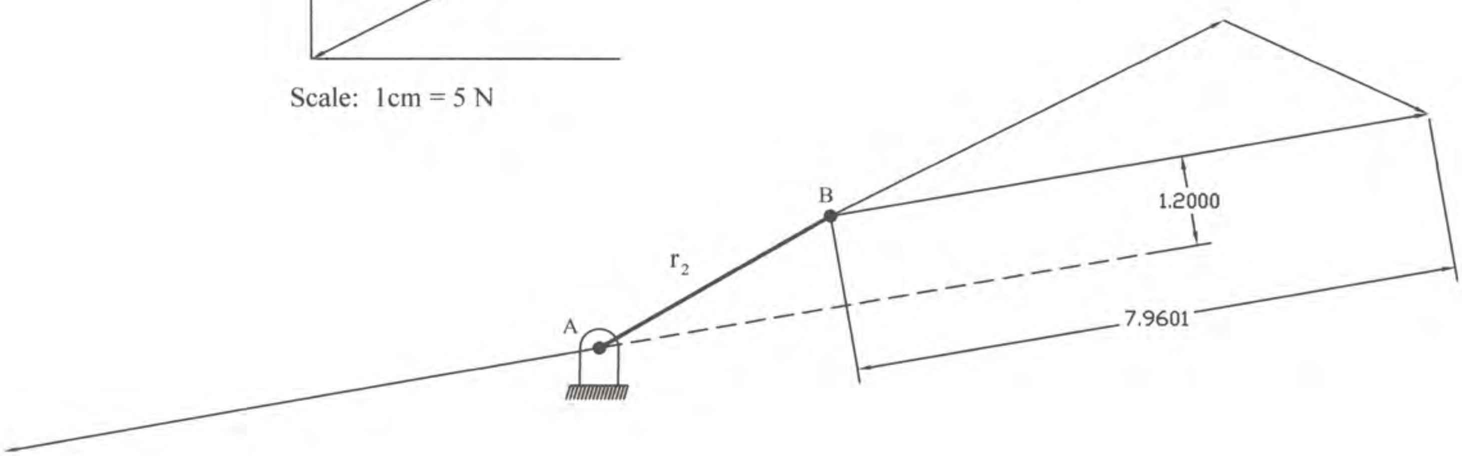




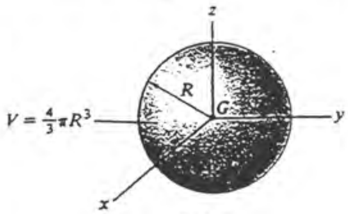
Using the Inertia Circle



Scale: 1cm = 5 N

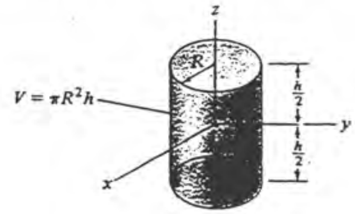


Center of Gravity and Mass Moment of Inertia of Homogeneous Solids



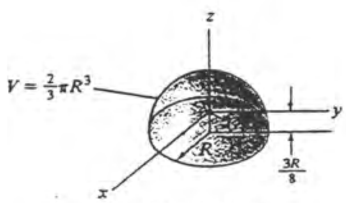
Sphere

$$I_{xx} = I_{yy} = I_{zz} = \tfrac{2}{5}mR^2$$



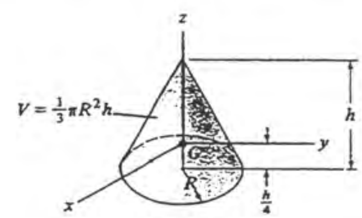
Cylinder

$$I_{xx} = I_{yy} = \tfrac{1}{12}m(3R^2 + h^2) \qquad I_{zz} = \tfrac{1}{2}mR^2$$



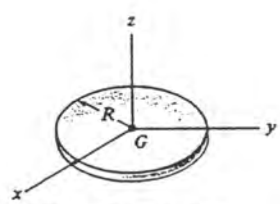
Hemisphere

$$I_{xx} = I_{yy} = 0.259mR^2 \qquad I_{zz} = \tfrac{2}{5}mR^2$$



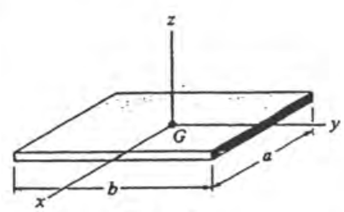
Cone

$$I_{xx} = I_{yy} = \tfrac{3}{80}m(4R^2 + h^2) \qquad I_{zz} = \tfrac{3}{10}mR^2$$



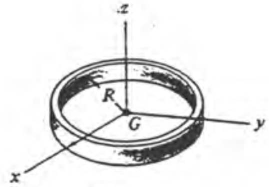
Thin circular disk

$$I_{xx} = I_{yy} = \tfrac{1}{4}mR^2 \qquad I_{zz} = \tfrac{1}{2}mR^2$$



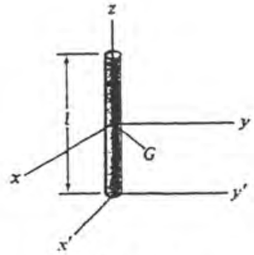
Thin plate

$$I_{xx} = \tfrac{1}{12}mb^2 \qquad I_{yy} = \tfrac{1}{12}ma^2 \qquad I_{zz} = \tfrac{1}{12}m(a^2 + b^2)$$



Thin ring

$$I_{xx} = I_{yy} = \tfrac{1}{2}mR^2 \qquad I_{zz} = mR^2$$



Slender rod

$$I_{xx} = I_{yy} = \tfrac{1}{12}ml^2 \qquad I_{x'x'} = I_{y'y'} = \tfrac{1}{3}ml^2 \qquad I_{zz} = 0$$

4.5 Analytical Force Analysis

In this section, three analytical methods for solving the force analysis of a mechanism will be presented.

- Analytical Superposition Method
- Matrix Method
- Virtual Work Method

4.5.1 Analytical Superposition Method

The effects of the individual inertia forces can be treated separately and then superimposed to determine their combined effect.

The equations of motion are based on individual free-body diagrams, one for each link.

For a mechanism in a certain configuration with known velocity and acceleration conditions, the equations of motion that describe each link are modeled with an inertial force and an inertial moment.

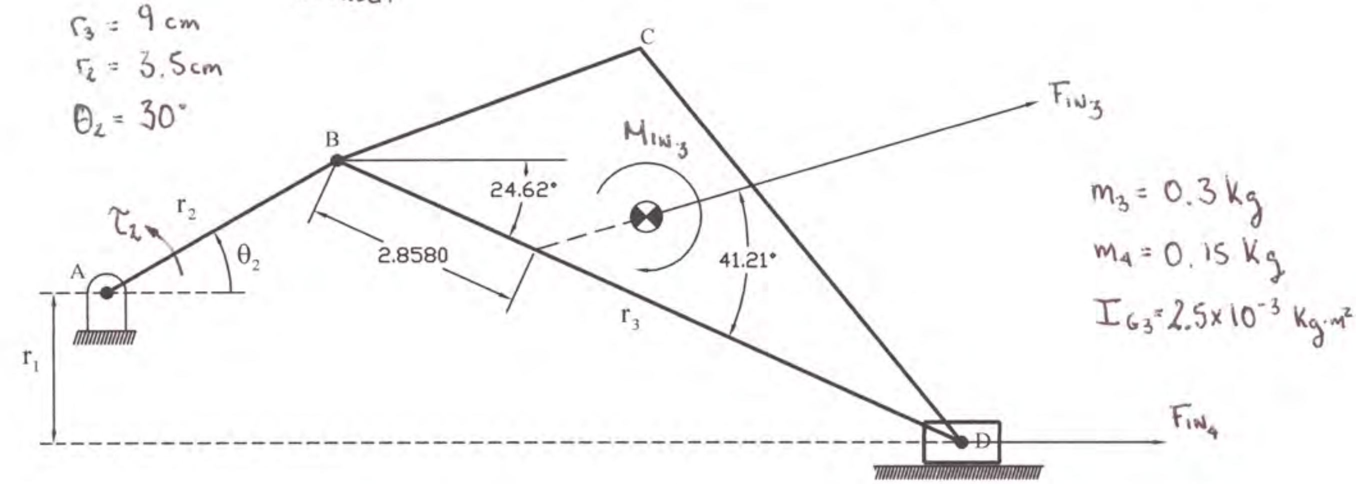
$$\sum \bar{F}_{IN_i} = -m_i \bar{a}_{G_i}$$
$$\sum M_{IN_i} = -I_{G_i} \ddot{\theta}_i \qquad \ldots \; 4.20$$

where i denotes the i^{th} link.

<u>Example 4.9</u> Consider the slider-crank mechanism of example 4.8

Determine the required torque τ_2 using the analytical superposition method.

$r_3 = 9$ cm
$r_2 = 3.5$ cm
$\theta_2 = 30°$



$m_3 = 0.3$ Kg
$m_4 = 0.15$ Kg
$I_{G_3} = 2.5 \times 10^{-3}$ Kg·m²

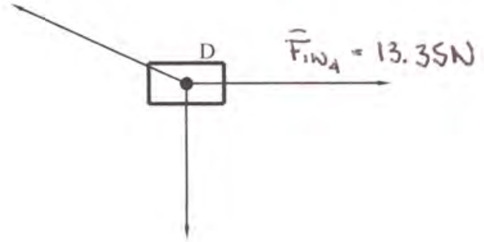
From the acceleration analysis, we know $\bar{a}_{G_3} = 90\ m/s^2 \swarrow$, $\bar{a}_D = 89\ m/s^2 \leftarrow$, and $\ddot{\theta}_3 = 378\ rad/s^2$ CCW. Thus, the equations of motion are:

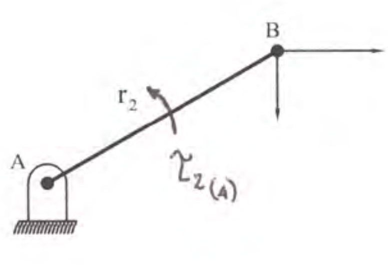
$$\bar{F}_{IN_3} = -m_3\,\bar{a}_{G_3} = -0.3(-90) = 27\text{ N} \nearrow$$

$$M_{IN_3} = -I_{G_3}\,\ddot{\theta}_3 = -2.5\times10^{-3}(378) = 0.945\text{ N·m}$$

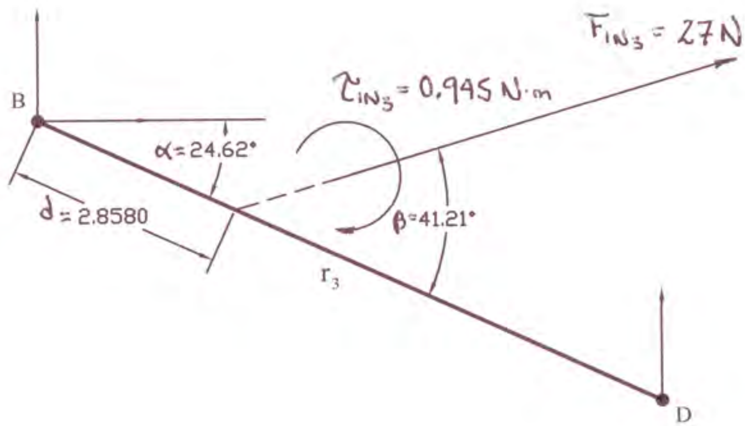
$$\bar{F}_{IN_4} = -m_4\,\bar{a}_{G_4} = -0.15(-89) = 13.35\text{ N} \longrightarrow$$

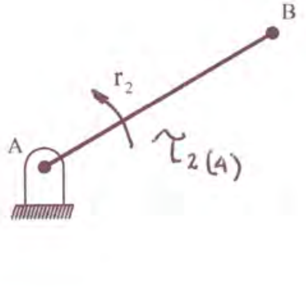
Free-body diagram of Link 4





Free-body diagram of Link 3





4.5.2 Matrix Method

The matrix method is another form for solving the force analysis of a mechanism. It is based on solving the equations of motion of all the links simultaneously, i.e., all the equations will be combine together to form a matrix.

For planar mechanisms, each link can be represented by three equations of motion, i.e.,

$$\sum F_{ix} = m_i a_{Gix}$$

$$\sum F_{iy} = m_i a_{Giy} \qquad \ldots 4.21$$

$$\sum M_{G_i} = I_{G_i} \ddot{\theta}_i$$

where x and y denote the projections of the forces and accelerations.

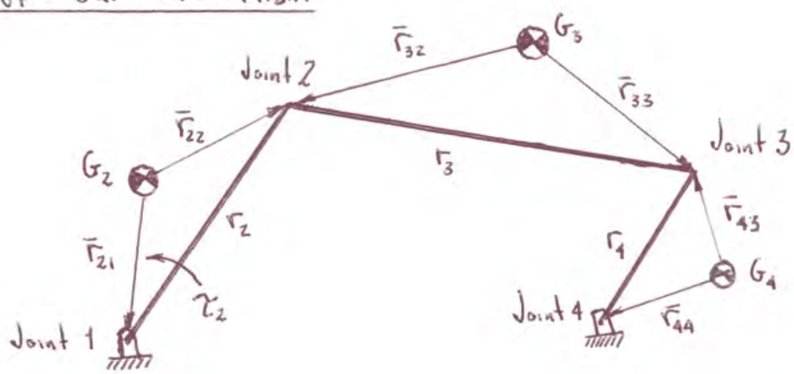
The reaction forces at the joints are equal in magnitude but opposite in direction, i.e.,

$$\overline{F}_{ij} = -\overline{F}_{ji} \qquad \ldots 4.22$$

The moment caused by a force about the centre of mass of a link requires the identification of the direction of the moment. This is achieved using cross products

$$M = \overline{r} \times \overline{F} = r_x F_y - r_y F_x \qquad \ldots 4.23$$

Four - Bar Mechanism

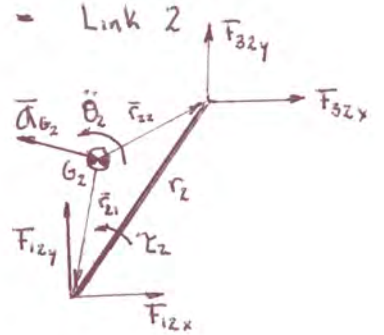


$\bar{r}_{ij}$ position vector from G_i to joint j

$\bar{r}_{ij} = r_{ijx} + r_{ijy}$

From the kinematic analysis, the accelerations of the centre of masses are known

- Link 2



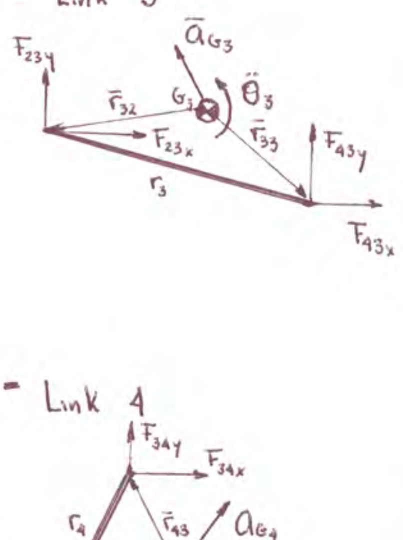
$\sum F_{2x} = m_2 a_{G2x} \rightarrow F_{12x} + F_{32x} = m_2 a_{G2x}$ $\quad \boxed{F_{12x} - F_{23x} = m_2 a_{G2x}}$

$\sum F_{2y} = m_2 a_{G2y} \rightarrow F_{12y} + F_{32y} = m_2 a_{G2y}$ $\quad \boxed{F_{12y} - F_{23y} = m_2 a_{G2y}}$

$\sum M_{G2} = I_{G2} \ddot{\theta}_2 \rightarrow r_{21x} F_{12y} - r_{21y} F_{12x} + r_{22x} F_{32y} - r_{22y} F_{32x} + \tau_2 = I_{G_2} \ddot{\theta}_2$

$$\boxed{r_{21x} F_{12y} - r_{21y} F_{12x} - r_{22x} F_{23y} + r_{22y} F_{23x} + \tau_2 = I_{G_2} \ddot{\theta}_2}$$

- Link 3



$\sum F_{3x} = m_3 a_{G3x} \rightarrow F_{23x} + F_{43x} = m_3 a_{G3x}$ $\quad \boxed{F_{23x} - F_{34x} = m_3 a_{G3x}}$

$\sum F_{3y} = m_3 a_{G3y} \rightarrow F_{23y} + F_{43y} = m_3 a_{G3y}$ $\quad \boxed{F_{23y} - F_{34y} = m_3 a_{G3y}}$

$\sum M_{G3} = I_{G3} \ddot{\theta}_3 \rightarrow r_{32x} F_{23y} - r_{32y} F_{23x} + r_{33x} F_{43y} - r_{33y} F_{43x} = I_{G_3} \ddot{\theta}_3$

$$\boxed{r_{32x} F_{23y} - r_{32y} F_{23x} - r_{33x} F_{34y} + r_{33y} F_{34x} = I_{G_3} \ddot{\theta}_3}$$

- Link 4

$\sum F_{4x} = m_4 a_{G4x} \rightarrow \boxed{F_{34x} + F_{14x} = m_4 a_{G4x}}$

$\sum F_{4y} = m_4 a_{G4y} \rightarrow \boxed{F_{34y} + F_{14y} = m_4 a_{G4y}}$

$\sum M_{G4} = I_{G4} \ddot{\theta}_4 \rightarrow \boxed{r_{43x} F_{34y} - r_{43y} F_{34x} + r_{44x} F_{14y} - r_{44y} F_{14x} = I_{G_4} \ddot{\theta}_4}$

The nine equations derived in the previous page results in a linear system of equations in nine unknowns (τ_2, $\bar{\bar{F}}_{12}$, $\bar{\bar{F}}_{23}$, and $\bar{\bar{F}}_{34}$). Thus,

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -r_{21y} & r_{21x} & r_{22y} & -r_{22x} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -r_{32y} & r_{32x} & r_{33y} & -r_{33x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -r_{43y} & r_{43x} & -r_{44y} & r_{44x} & 0 \end{bmatrix}}_{\substack{\text{Coefficient Matrix} \\ \text{(Known)}}} \underbrace{\begin{bmatrix} F_{12x} \\ F_{12y} \\ F_{23x} \\ F_{23y} \\ F_{34x} \\ F_{34y} \\ F_{14x} \\ F_{14y} \\ \tau_2 \end{bmatrix}}_{\text{Unknowns}} = \underbrace{\begin{bmatrix} m_2 a_{G2x} \\ m_2 a_{G2y} \\ I_{G2} \ddot{\theta}_2 \\ m_3 a_{G3x} \\ m_3 a_{G3y} \\ I_{G3} \ddot{\theta}_3 \\ m_4 a_{G4x} \\ m_4 a_{G4y} \\ I_{G4} \ddot{\theta}_4 \end{bmatrix}}_{\substack{\text{Kinetic Terms} \\ \text{(Known)}}}$$

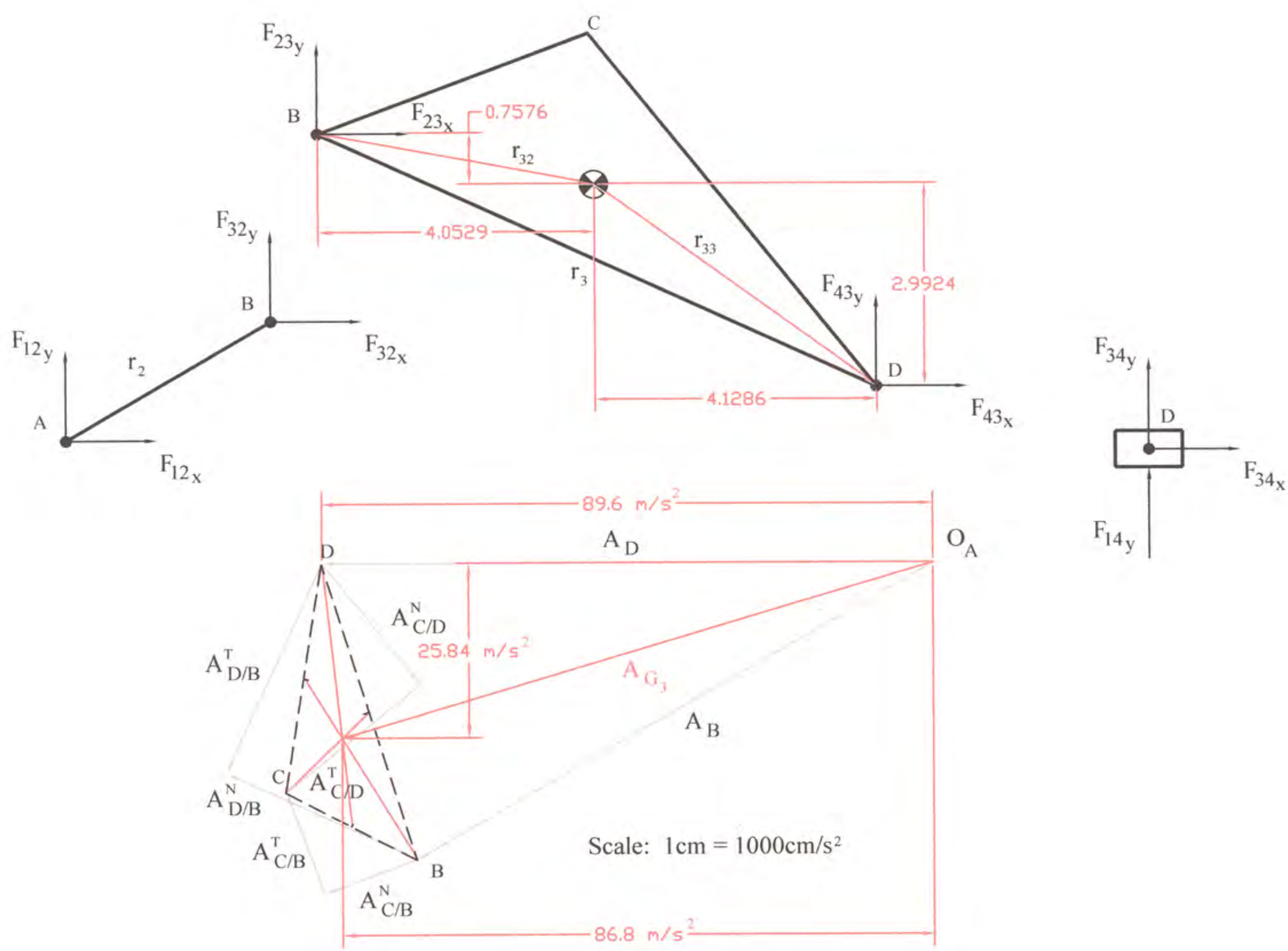
The problem above is of the form $Ax = b$, which can be solved by inverting matrix A, i.e.,

$$x = A^{-1} b$$

The solution x involves all the reactions at the joints and the required torque at the input link.

It is important to mention that the Coefficient Matrix must be square, otherwise, it is not possible to invert it.

Example 4.10 Consider the slider-crank mechanism of examples 4.8 and 4.9. Determine the required torque τ_2 using the matrix method.



4.5.3 Virtual Work (Conservation of Power)

This is an efficient method to determine the required input force/torque. This method, however, does not determine the reaction forces in the joints. It is based on conservation of power (rate of work), i.e.,

Power in + Power out = 0

The rate of work of the input forces/torques (Power in) must equal the summation of the rate of work of all the external forces and moments and the inertial related forces and moments.

$$\left.\begin{matrix}\dot{\theta}_{input} \cdot \tau_{input} \\ \text{or} \\ \bar{V}_{input} \cdot \bar{F}_{input}\end{matrix}\right. + \sum_j \dot{\theta}_j \cdot M_j + \sum_k \bar{V}_k \cdot \bar{F}_k + \sum_i \dot{\theta}_i \cdot M_{IN_i} + \sum_i \bar{V}_{G_i} \cdot \bar{F}_{IN_i} = 0 \qquad \ldots 4.24$$

where M_j is an external moment applied on link j, $\dot{\theta}_j$ angular velocity of link j

$\bar{F}_k$ is an external force applied on point k, $\bar{V}_k$ linear velocity of point k.

M_{IN_i} is the moment of inertia of the i^{th} link, $M_{IN_i} = -I_{G_i} \ddot{\theta}_i$

$\bar{F}_{IN_i}$ is the force of inertia of the centre of mass, $\bar{F}_{IN_i} = -m_i \bar{a}_{G_i}$

This method requires the velocity and acceleration conditions of the mechanism.

Example 4.11 Consider the slider-crank mechanism of examples 4.8 - 4.10. Determine the required torque τ_z using the virtual work method. From previous examples, $\dot{\theta}_3 = -18.5 \text{ rad/s}$ and $\ddot{\theta}_3 = 378 \text{ rad/s}^2$