Parcial #2

Señales y sistemas

Juan Esteban Hernandez Garcia 1002635781

2.1 Encuentre la expresion del espectro de fourier (forma exponencial y trigonometrica) para la señal
$$x(t) = (6\sin(3t+\frac{\pi}{4}))^2 \quad \text{con } t \in [-\pi, \pi].$$

Presente la simulaciones respectivas para graficar el espectro y la reconstruccion de la señal en funcion del numero de armonicos y el error relativo.

Solucion.

- Tenemos la señal $= x(t) = (6\sin(3t+\frac{\pi}{4}))^2$

$$x(t) = (6\,\text{sen}(3t+\frac{\pi}{4}))^2 = 6^2\sin^2(3t+\frac{\pi}{4})$$

- Por Propiedad trigonometrica:

$$\text{sen}^2(\theta) = \frac{1}{2} - \frac{\cos(2\theta)}{2}$$

- Remplazando tenemos que:

$$x(t) = 36\left(\frac{1}{2} - \frac{\cos(6t+\pi/2)}{2}\right) = \frac{\overset{18}{\cancel{36}}}{\cancel{2}} - \frac{\overset{18}{\cancel{36}}}{\cancel{2}}\cos(6t+\pi/2)$$

$$x(t) = 18 + 18\cos(6t+\pi/2)$$

Ahora: $\cos(\theta+\pi/2) = -\sin(\theta)$.

$$x(t) = 18 + 18\,\text{sen}(6t)$$

- De la forma trigonometrica

$$x(t) = a_0 + \sum_{n=-N}^{N} a_n\cos(n\omega_0 t) + b_n\,\text{sen}(n\omega_0 t)$$

- Dado que $x(t)$ corresponde a una funcion seno, seno tiene simetria impar, entonces $x(t) = -x(t)$

Por lo tanto: $\boxed{a_n = 0}$

- Finalmente la funcion nos queda como:

$$x(t) = 18 + 18\,\text{sen}(6t) = a_0 + \sum_{n=-N}^{N} b_n\sin(n\omega_0 t)$$

- Procedemos a calcular los coeficientes a_0, b_n

• Para a_0 =

$$a_0 = C_0 = \frac{1}{t_f - t_i} \int_{t_i}^{t_f} x(t)\,dt$$

remplazamos

$$a_0 = \frac{1}{\pi-(-\pi)} \cdot \int_{-\pi}^{\pi} 18 + 18\,sen(6t)$$

$$a_0 = \frac{18}{2\pi}\int_{-\pi}^{\pi} dt + \frac{18}{2\pi}\int_{-\pi}^{\pi} sen(6t)$$

$$= \frac{18}{2\pi}\int_{-\pi}^{\pi} sen(6t)\,dt = \frac{18}{12\pi} - cos(6t)\Big|_{-\pi}^{\pi} = \frac{18}{12\pi}\left[\overbrace{-cos(6(\pi)) + cos(6(-\pi))}^{0}\right]$$

$$= \frac{18}{12\pi}[0] = 0$$

$$= \frac{18}{2\pi}\int_{-\pi}^{\pi} dt = \frac{18}{2\pi}(\pi) - \frac{18}{2\pi} = 9-(-9) = 18$$

$$\boxed{a_0 = 18}$$

• ahora calculamos el coeficiente b_n:

$$b_n = \frac{2}{t_f - t_i} \cdot \int_{t_i}^{t_f} x(t)\,sen(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{\pi-(-\pi)}\int_{-\pi}^{\pi} 18 + 18\,sen(6t)\,sen(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{2\pi}\left[\int_{-\pi}^{\pi} 18\,sen(n\omega_0 t)\,dt + \int_{-\pi} 18\,sen(6t)\,sen(n\omega_0 t)\,dt\right]$$

• Por la identidad trigonometrica

$$sen(\theta)\,sen(\alpha) = \frac{cos(\theta-\alpha) - cos(\theta+\alpha)}{2}$$

donde: $sen(\theta) = sen(6t)$

$sen(\alpha) = sen(n\omega_0 t)$

• remplazando nos queda:

$$= \frac{cos(6t - n\omega_0 t) - cos(6t + n\omega_0 t)}{2} = \frac{cos((6-n\omega_0)t) - cos(6+n\omega_0)t)}{2}$$

$$\omega_0 = \frac{2\pi}{T} \quad , \quad T = 2\pi \quad , \quad \omega_0 = \frac{2\pi}{2\pi} = 1 \text{ [rad/s]}$$

$$b_n = \frac{2}{2\pi}\left[\int_{-\pi}^{\pi} 18\ \text{sen}(nt)\,dt + \int_{-\pi}^{\pi} 18\ \frac{\cos(6-n)t) - \cos(6+n)t}{2}\,dt\right]$$

(1) (2)

• Solucion integral (1)

$$\frac{2}{2\pi}\int_{-\pi}^{\pi} 18\ \text{sen}(nt)\,dt = \frac{9}{\pi}\int_{-\pi}^{\pi} \text{sen}(nt)\,dt = -\frac{9}{\pi}\cos(nt)\Big/_{-\pi}^{\pi}$$

$$= -\frac{9}{\pi}\left[\cos(n\pi) - \cos(-n\pi)\right] = 0$$

• Solucion integral (2)

$$\frac{2}{2\pi}\int_{-\pi}^{\pi} 18\ \frac{\cos((6-n)t) - \cos((6+n)t)}{2}\,dt = \frac{18}{2\pi}\int_{-\pi}^{\pi} \cos((6-n)t) - \cos((6+n)t)\,dt$$

$$= \frac{18}{2\pi}\left[\int_{-\pi}^{\pi} \cos((6-n)t)\,dt - \int_{-\pi}^{\pi} \cos((6+n)t)\,dt\right]$$

$$= \frac{18}{2\pi}\left[\frac{\text{sen}((6-n)t)}{6-n}\Big/_{-\pi}^{\pi} - \frac{\text{sen}((6+n)t)}{6+n}\Big/_{-\pi}^{\pi}\right]$$

$$= \frac{18}{2\pi}\left[\left[\frac{\text{sen}((6-n)\pi)}{6-n} - \frac{\text{sen}((6-n)-\pi)}{6-n}\right] - \left[\frac{\text{sen}((6+n)\pi)}{6+n} - \frac{\text{sen}((6+n)-\pi)}{6+n}\right]\right]$$

$$= \frac{18}{2\pi}\left[\left[\frac{\text{sen}((6-n)\pi - \text{sen}((6-n)-\pi)}{6-n}\right] - \left[\frac{\text{sen}((6+n)\pi) - \text{sen}((6+n)-\pi)}{6+n}\right]\right]$$

$$b_n = 18 \cdot \frac{\text{sen}((6-n)\pi) - \text{sen}((6-n)-\pi)}{2\pi(6-n)} - 18 \cdot \frac{\text{sen}((6+n)\pi) - \text{sen}((6+n)-\pi)}{2\pi(6+n)}$$

Para $n \neq 6$, $6 = n$. No obstante, para $n = 6$ debemos calcular el limite y aproximar la indeterminacion $\frac{0}{0}$:

$$b_6 = 18 \lim_{n \to 6} \frac{\frac{d}{dn}\text{sen}((6-n)\pi) - \text{sen}((6-n)-\pi)}{\frac{d}{dn}2\pi(6-n)}$$

$$b_6 = 18 \lim_{n \to 6} \frac{\cos((6-n)\pi)(-\pi) - \cos(-(6-n)\pi)\pi)}{-2\pi}$$

$$b_6 = 18\ \frac{\cos(0)(-\pi) - \cos(0)\pi}{-2\pi} = 18\ \frac{(\cancel{-2\pi})}{\cancel{-2\pi}} = 18 \quad \boxed{b_n = 18}$$

Por lo tanto:

$$\boxed{a_n \begin{cases} 18 & n=0 \\ 0 & \forall n \setminus \{0\} \end{cases}}$$

$$\boxed{b_n \begin{cases} 18 & n=6 \\ -18 & n=-6 \\ 0 & \forall n \setminus \{6\} \end{cases}}$$

• De la Forma Exponencial

$$C_0 = a_0 = 18$$

$$C_n = \frac{a_n - j b_n}{2}$$

$$C_n = \frac{0 - j18}{2} = \frac{-j18}{2} = -j9$$

Entonces:

$$C_n \begin{cases} 18 & n=0 \\ -j9 & n=\{6,-6\} \\ 0 & \forall n \setminus \{0,6,-6\} \end{cases}$$

• Para Construir la señal

$$x(t) = \sum_{n=-N}^{N} C_n e^{jnt}$$

$$x(t) = C_{-6} e^{-j6t} + C_0 e^{0} + C_6 e^{j6t} =$$

$$= [9j(\cos(6t) - 9j^2 \operatorname{sen}(6t) + 18 - 9j\cos(6t) - 9j^2 \operatorname{sen}(6t))$$

$$x(t) = 18 + 18\operatorname{sen}(6t)$$

• Error relativo

$$Er[\%] = \left[1 - \frac{1}{P_x} \sum_{n=-N}^{n} |C_n|^2\right] 100\%$$

en nuestro caso

$$P_x = \frac{1}{T} \int_{t_i}^{t_f} |x(t)|^2 dt$$

$$P_x = \frac{1}{2\pi} \int_{-\pi}^{\pi} (18 + 18\operatorname{sen}(6t))^2 dt$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} 18^2 dt + \int_{-\pi}^{\pi} 2(18)^2 \operatorname{sen}(6t)dt + \int_{-\pi}^{\pi} 18^2 \operatorname{sen}^2(6t)dt\right.$$

$$= \frac{1}{2\pi}\left[18^2 t \Big|_{-\pi}^{\pi} \frac{-2(18)^2\cos(6t)dt}{6}\Big|_{-\pi}^{\pi} + 18^2\int_{-\pi}^{\pi}\frac{1}{2}(1-\cos(12t))dt\right]$$

$$= \frac{1}{2\pi}\left[18^2 t\Big|_{-\pi}^{\pi} - \frac{18^2}{3}\cos(6t)\Big|_{-\pi}^{\pi} + \frac{18^2}{2}t\Big|_{-\pi}^{\pi} \frac{-1}{12}\operatorname{sen}(12t)\Big|_{-\pi}^{\pi}\right]$$

$$= \frac{162}{\pi}(2\pi) - 54\left[\cos(6\pi) - \cos(-6\pi)\right] - \frac{1}{2\pi}\left[\cancel{\operatorname{sen}(12\pi)}^{0} - \cancel{\operatorname{sen}(-12\pi)}^{0}\right]$$

$$= 324 - 0 + 162 - 0 = 486$$

$$Er[\%] = \left[\left[1 - \frac{(-9)^2+(18)^2+(9)^2}{486}\right] * 100\%\right] = 0\%$$

2.2 Sea la señal portadora $C(t) = A_c \cos(2\pi f_c t)$ con $A_c, F_c, \in \mathbb{R}$, y la señal mensaje $m(t) \in \mathbb{R}$ encuentre el espectro en frecuencia de la señal modulada en amplitud (AM) $y(t) = (1 + \frac{m(t)}{A_c})$

Solucion.

$C(t) = A_c \cos(2\pi F_c t)$, $A_c, F_c \in \mathbb{R}$, $y(t) = \left(1 + \frac{m(t)}{A_c}\right) C(t)$

$$\mathcal{F}\{C(t)\} + \mathcal{F}\left\{\frac{m(t)C(t)}{A_c}\right\}$$

$$\mathcal{F}\{A_c \cos(2\pi f_c t)\} = A_c \cdot \mathcal{F}\left\{\frac{e^{j2\pi F_c t} + e^{-j2\pi F_c t}}{2}\right\}$$

$$A_c \cdot \left[\mathcal{F}\left\{\frac{e^{j2\pi F_c t}}{2}\right\} + \mathcal{F}\left\{\frac{e^{j2\pi F_c t}}{2}\right\}\right]$$

$$\frac{A_c}{\cancel{2}}\left[\cancel{2}\pi\delta(\omega - 2\pi F_c) + \cancel{2}\pi\delta(\omega + 2\pi F_c)\right]$$

$$A_c\pi\delta(\omega - 2\pi F_c) + A_c\pi\delta(\omega + 2\pi F_c)$$

$$C(\omega) = A_c\pi\delta\left[(\omega - 2\pi F_c) + (\omega + 2\pi F_c)\right]$$

$$\mathcal{F}\left\{\frac{m(t)(A_c \cos(2\pi F_c t))}{A_c}\right\} = \mathcal{F}\{\cos(2\pi F_c t) m(t)\}$$

$$= \mathcal{F}\left\{\frac{m(t)e^{j2\pi F_c t}}{2}\right\} + \mathcal{F}\left\{\frac{m(t)e^{-j2\pi F_c t}}{2}\right\}$$

$$\frac{M(\omega - 2\pi F_c t)}{2} + \frac{M(\omega + 2\pi F_c t)}{2} = \frac{1}{2} M\left[(\omega \cdot 2\pi F_c t) + (\omega + 2\pi F_c t)\right]$$

$$Y(\omega) = A_c\pi\delta\left[(\omega - 2\pi F_c t) + (\omega + 2\pi F_c t)\right] + \frac{1}{2} M\left[(\omega - 2\pi F_c t) + [\omega + 2\pi F_c t)\right]$$