Solución Problemas Números Complejos

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October 16, 2024

Apuntes

- $i^2 = -1$
- $i^3 = i^2 \cdot i = -i$
- $i^4 = i^2 \cdot i^2 = 1$
- $(a + bi) \cdot (c + di) = (ac - bd + adi + bci)$
- $\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(a+bi)\cdot(c-di)}{c^2+d^2}$
- $x = e^{Ln(x)} \leftrightarrow x = Ln(e^x)$

1 Ejercicio 1

Calcular parte real e imaginaria de los siguientes números complejos:
a) $z_1 = \frac{3-2i}{1+4i}$ b) $z_2 = \frac{1}{i} + \frac{1}{1+i}$ c) $z_3 = cos(i)$ d) $z_4 = sen(2 + i)$

1.1 Solución

$$z_1 = \frac{3-2i}{1+4i} = \frac{3-2i}{1+4i} \cdot \frac{1-4i}{1-4i} = \frac{3-8-12i-2i}{17} = \boxed{-\frac{5}{17} - \frac{14}{17}i}$$

$$z_2 = \frac{1}{i} + \frac{1}{1+i} = \frac{1+2i}{-1+i} = \frac{1+2i}{-1+i} \cdot \frac{-1-i}{-1-i} = \frac{-1+2-1i-2i}{2} = \boxed{\frac{1}{2} - \frac{3}{2}i}$$

$$z_3 = cos(i) = \frac{e^{-1}+e}{2} = \frac{\frac{1}{e}+e}{2} = \boxed{\frac{1}{2e} + \frac{e}{2}}$$

$$z_4 = sen(2+i) = \frac{e^{i(2+i)} - e^{-i(2+i)}}{2i} = \frac{e^{-1+2i} - e^{1-2i}}{2i} = -\frac{i}{2}(\frac{1}{e}\cdot(cos(2)+isen(2)) - e\cdot(cos(-2)+isen(-2)))$$

$$= -\frac{i}{2}(\frac{1}{e}\cdot(cos(2)+isen(2)) - e\cdot(cos(2)-isen(2))) = \frac{1}{2e}(sen(2)-icos(2)) - \frac{e}{2}(-sen(2)-icos(2))$$

$$= \boxed{(\frac{1}{2e} + \frac{e}{2})sen(2) + i(-\frac{1}{2e} + \frac{e}{2})cos(2)}$$

2 Ejercicio 2

Calcular el modulo y argumento de:
a) $z_1 = 3^i$ b) $z_2 = i^i$ c) $z_3 = i^{3+i}$ d) $z_4 = (1+i)^{2+i}$

2.1 Solución

$$z_1 = 3^i = e^{Ln(3^i)} = e^{iLn(3)} = \boxed{r = 1, \sigma = Ln(3)}$$

$$z_2 = (e^{i\frac{\pi}{2}})^i = e^{-\frac{\pi}{2}} = \boxed{r = e^{-\frac{\pi}{2}}, \sigma = 0}$$

$$z_3 = i^{3+i} = i^3 \cdot i^i = (-i) \cdot e^{-\frac{\pi}{2}} = e^{-\frac{\pi}{2}} \cdot e^{i\frac{3\pi}{2}} = \boxed{r = e^{-\frac{\pi}{2}}, \sigma = \frac{\pi}{2}}$$

$$z_4 = (1+i)^{2+i} = (1+i)^2 \cdot (1+i)^i = (\sqrt{2} \cdot e^{i\frac{\pi}{4}})^2 \cdot (\sqrt{2} \cdot e^{i\frac{\pi}{4}})^i = (2 \cdot e^{i\frac{\pi}{2}}) \cdot ((\sqrt{2})^i \cdot e^{-\frac{\pi}{4}})$$

$$= 2e^{-\frac{\pi}{4}} \cdot e^{i\frac{\pi}{2}} \cdot e^{iLn(\sqrt{2})} = 2e^{-\frac{\pi}{4}} \cdot e^{i(\frac{\pi}{2}+Ln(\sqrt{2}))} = \boxed{r = 2e^{-\frac{\pi}{4}}, \sigma = \frac{\pi}{2} + Ln(\sqrt{2})}$$

3 Ejercicio 3

Calcular las raíces cuartas de la unidad

3.1 Solución

$$z^4 = 1 \rightarrow z = \sqrt[4]{1} e^{i(\frac{2k\pi}{4})} || k = 0, 1, 2, 3$$

$$k = 0 \rightarrow e^0 = \boxed{1}$$

$$k = 1 \rightarrow e^{i\frac{\pi}{2}} = \boxed{i}$$

$$k = 2 \rightarrow e^{i\pi} = \boxed{-1}$$

$$k = 3 \rightarrow e^{i3\frac{\pi}{2}} = \boxed{-i}$$

4 Ejercicio 4

Calcular las raíces cúbicas de -8

4.1 Solución

$$z^3 = -8 \rightarrow \sqrt[3]{8} e^{i(\frac{\pi+2k\pi}{3})} || k = 0, 1, 2$$

$$k = 0 \rightarrow 2e^{i\frac{\pi}{3}} = 2(cos(\frac{\pi}{3}) + isen(\frac{\pi}{3})) = \boxed{1 + i\sqrt{3}}$$

$$k = 1 \rightarrow 2e^{i\pi} = \boxed{-2}$$

$$k = 2 \rightarrow 2e^{i\frac{5\pi}{3}} = 2(cos(\frac{5\pi}{3}) + isen(\frac{5\pi}{3})) = \boxed{1 - i\sqrt{3}}$$

5 Ejercicio 5

Determina la forma binómica de $e^{\sqrt{i}}$

5.1 Solución

$\sqrt{i} = \sqrt{e^{i\frac{\pi}{2}}} = e^{i(\frac{\frac{\pi}{2}+2k\pi}{2})}||k = 0,1$

$k = 0 \rightarrow e^{i\frac{\pi}{4}} = cos(\frac{\pi}{4}) + sen(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$k = 1 \rightarrow e^{i\frac{5\pi}{4}} = cos(\frac{5\pi}{4}) + sen(5\frac{\pi}{4}) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

Opción 1

$$e^{\sqrt{i}} = e^{\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}} = e^{\frac{\sqrt{2}}{2}}(cos(\frac{\sqrt{2}}{2})+isen(\frac{\sqrt{2}}{2})) = \boxed{e^{\frac{\sqrt{2}}{2}}cos(\frac{\sqrt{2}}{2}) + ie^{\frac{\sqrt{2}}{2}}sen(\frac{\sqrt{2}}{2})}$$

Opción 2

$$e^{\sqrt{i}} = e^{\frac{-\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}} = e^{\frac{-\sqrt{2}}{2}}(cos(-\frac{\sqrt{2}}{2})+isen(-\frac{\sqrt{2}}{2})) = \boxed{e^{-\frac{\sqrt{2}}{2}}cos(\frac{\sqrt{2}}{2}) - ie^{-\frac{\sqrt{2}}{2}}sen(\frac{\sqrt{2}}{2})}$$

6 Ejercicio 6

Expresa en forma binómica el número $(-\frac{\sqrt{3}}{2} + \frac{i}{2})^6$

6.1 Solución

$(-\frac{\sqrt{3}}{2} + \frac{i}{2})^6 = (e^{i\frac{5\pi}{6}})^6 = e^{i5\pi} = e^{i\pi} = \boxed{-1}$

7 Ejercicio 7

Calcula $\sqrt{-16-30i}$

7.1 Solución

$\sqrt{-16-30i} = (x+iy) \rightarrow (\sqrt{-16-30i})^2 = (x+iy)^2 \rightarrow -16-30i = x^2+2ixy-y^2 \rightarrow$

$x^2 - y^2 = -16$

$2xy = -30 \rightarrow x = -\frac{15}{y}$

$(-\frac{15}{y})^2 - y^2 = -16 \rightarrow \frac{225}{y^2} - y^2 = -16 \rightarrow y^4 + 16y^2 - 225 = 0$

$\xrightarrow{y^2 = t}$

$t^2 + 16t - 225 = 0 \to (t-25)(t+9) = 0 \to y = \pm 5, x = \mp 3$

$\boxed{z_1 = -3 + 5i, z_2 = 3 - 5i}$

8 Ejercicio 8

Determinar $m \in \mathbb{R}$ de modo que $(2e^{i\sqrt{2}})^m$ sea un número real negativo

8.1 Solución

$(2e^{i\sqrt{2}})^m = 2^m \cdot \underline{e^{im\sqrt{2}}}$

Nota: Nos fijamos en la parte imaginaria ya que 2^m siempre sera un valor positivo

$2^m(cos(m\sqrt{2}) + isen(m\sqrt{2}))$

Nota: Para que el número sea real negativo, el coseno debe ser negativo y el seno nulo. Por lo tanto, $\forall k \in \mathbb{Z} \to cos((2k+1)\pi) = -1, sen((2k+1)\pi) = 0$

$$m\sqrt{2} = (2k+1)\pi \to \boxed{m = \frac{(2k+1)\pi}{\sqrt{2}} || k \in \mathbb{Z}}$$

9 Ejercicio 9

Determinar los números complejos no nulos tal que su quinta potencia, z^5, sea igual a su conjugado, es decir, $\overline{z}$

9.1 Solución

$z = re^{i\sigma}, \overline{z} = z = re^{-i\sigma}$

$(re^{i\sigma})^5 = re^{-i\sigma} \to r^5 e^{i\sigma 5} = re^{-i\sigma}$

$r^5 = r \to r(r^4 - 1) = 0 \to r = 0X, r = -1X, r = 1\checkmark$

$$e^{i\sigma 5} = e^{-i\sigma} \to 5\sigma = -\sigma + 2k\pi \to \sigma = \frac{k\pi}{3}$$

$k = 0 \to z = 1, \boxed{1^5 = \overline{1}}$

$k = 1 \to z = e^{i\frac{\pi}{3}}, \boxed{e^{i\frac{5\pi}{3}} = e^{-i\frac{\pi}{3}}}$

$\boxed{\forall k \in \mathbb{N} \to (e^{i\frac{k\pi}{3}})^5 = e^{-i\frac{k\pi}{3}}}$