TEMA 5

PROPIEDADES DE LA TRANSFORMADA DE FOURIER DE SEÑALES CONTINUAS

PROPIEDAD 1: LINEALIDAD

$$z(t) = \alpha x(t) + \beta y(t) \implies Z(\omega) = \alpha X(\omega) + \beta Y(\omega)$$

Solución:

$$\begin{aligned} Z(\omega) &= \int_{-\infty}^{\infty} z(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \big(\alpha x(t) + \beta y(t)\big) e^{-j\omega t} dt = \\ &= \alpha \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt + \beta \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \\ &= \alpha X(\omega) + \beta Y(\omega) \end{aligned}$$

PROPIEDAD 2: DESPLAZAMIENTO EN EL TIEMPO

$$y(t) = x(t \pm t_0) \implies Y(\omega) = X(\omega)e^{\pm j\omega t_0}$$

Solución:

Probamos primero con $y(t) = x(t - t_0)$:

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}dt = \left\{ \begin{array}{lcl} \tau = t - t_0 & \longrightarrow & d\tau = dt \\ \tau = -\infty & \longrightarrow & t = -\infty \\ \tau = \infty & \longrightarrow & t = \infty \end{array} \right\} =$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau+t_0)}d\tau = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}e^{-j\omega t_0}d\tau = e^{-j\omega t_0}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau =$$

$$= X(\omega)e^{-j\omega t_0}$$

Probamos a continuación con $y(t) = x(t + t_0)$:

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t+t_0)e^{-j\omega t}dt = \left\{ \begin{array}{lcl} \tau = t + t_0 & \longrightarrow & d\tau = dt \\ \tau = -\infty & \longrightarrow & t = -\infty \\ \tau = \infty & \longrightarrow & t = \infty \end{array} \right\} =$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau-t_0)}d\tau = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}e^{j\omega t_0}d\tau = e^{j\omega t_0}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau =$$

$$= X(\omega)e^{j\omega t_0}$$

PROPIEDAD 3: INVERSIÓN EN EL TIEMPO

$$y(t) = x(-t) \implies Y(\omega) = X(-\omega)$$

Solución:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \left\{ \begin{array}{lcl} t = -\tau & \longrightarrow & dt = -d\tau \\ t = -\infty & \longrightarrow & \tau = \infty \\ t = \infty & \longrightarrow & \tau = -\infty \end{array} \right\} =$$

$$= -\int_{\infty}^{-\infty} x(-\tau)e^{j\omega\tau}d\tau = \int_{-\infty}^{\infty} x(-\tau)e^{-j(-\omega)\tau}d\tau \implies$$

$$X(-\omega) = \int_{-\infty}^{\infty} x(-\tau)e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty} y(\tau)e^{-j\omega\tau}d\tau = Y(\omega) \implies$$

$$Y(\omega) = X(-\omega)$$

PROPIEDAD 4: COMPRESIÓN EN EL TIEMPO (ESCALADO)

$$y(t) = x(\alpha t) \implies Y(\omega) = \frac{1}{|\alpha|} X\left(\frac{\omega}{\alpha}\right)$$

Solución:

Primero vamos a probarlo para el caso $\alpha > 0$:

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(\alpha t) e^{-j\omega t} dt = \left\{ \begin{array}{lcl} \tau = \alpha t & \longrightarrow & d\tau = \alpha dt \\ t = -\infty & \longrightarrow & \tau = -\infty \\ t = \infty & \longrightarrow & \tau = \infty \end{array} \right\} =$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega \frac{\tau}{\alpha}} \frac{d\tau}{\alpha} = \frac{1}{\alpha} \int_{-\infty}^{\infty} x(\tau) e^{-j\left(\frac{\omega}{\alpha}\right)\tau} d\tau = \frac{1}{\alpha} X\left(\frac{\omega}{\alpha}\right) =$$

$$= \frac{1}{|\alpha|} X\left(\frac{\omega}{\alpha}\right)$$

Probamos a continuación para el caso $\alpha < 0$:

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(-|\alpha| t) e^{-j\omega t} dt = \left\{ \begin{array}{lcl} \tau = -|\alpha| t & \longrightarrow & d\tau = -|\alpha| dt \\ t = -\infty & \longrightarrow & \tau = \infty \\ t = \infty & \longrightarrow & \tau = -\infty \end{array} \right\} =$$

$$= -\int_{\infty}^{-\infty} x(\tau) e^{-j\omega\left(-\frac{\tau}{|\alpha|}\right)} \frac{d\tau}{|\alpha|} = \frac{1}{|\alpha|} \int_{-\infty}^{\infty} x(\tau) e^{-j\left(-\frac{\omega}{|\alpha|}\right)\tau} d\tau = \frac{1}{|\alpha|} X\left(-\frac{\omega}{|\alpha|}\right) =$$

$$= \frac{1}{|\alpha|} X\left(\frac{\omega}{\alpha}\right)$$

PROPIEDAD 5: CONJUGACIÓN

$$y(t) = x^*(t) \implies Y(\omega) = X^*(-\omega)$$

Solución:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \implies$$

$$\implies X^*(\omega) = \left(\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right)^* = \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}dt \implies$$

$$\implies X^*(-\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt = Y(\omega) \implies$$

$$\implies Y(\omega) = X(-\omega)$$

PROPIEDAD 6: MODULACIÓN (DESPLAZAMIENTO EN LA FRECUENCIA)

$$y(t) = x(t)e^{\pm j\omega_0 t} \implies Y(\omega) = X(\omega \mp \omega_0)$$

Solución:

Probamos primero con $y(t) = x(t)e^{j\omega_0 t}$:

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt = X(\omega - \omega_0)$$

Probamos a continuación con $y(t) = x(t)e^{-j\omega_0 t}$:

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j\omega_0 t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega+\omega_0)t}dt = X(\omega + \omega_0)$$

PROPIEDAD 7: MULTIPLICACIÓN EN EL TIEMPO

$$z(t) = x(t)y(t) \implies Z(\omega) = \frac{1}{2\pi}X(\omega) * Y(\omega)$$

Solución:

En lugar de demostrar que la transformada de Fourier de $z(t)$ es $Z(\omega)$, vamos a demostrar que la transformada inversa de $Z(\omega)$ es igual a $z(t)$:

$$z(t) = Z^{-1}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi}\big(X(\omega) * Y(\omega)\big)e^{j\omega t}d\omega =$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} X(\tau)Y(\omega-\tau)d\tau\right)e^{j\omega t}d\omega =$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} X(\tau)Y(\omega-\tau)d\tau\right)e^{j\omega t}\frac{e^{j\tau t}}{e^{j\tau t}}d\omega =$$

$$= \left(\frac{1}{2\pi}\right)^2 \left(\int_{-\infty}^{\infty} X(\tau)e^{j\tau t}d\tau\right)\left(\int_{-\infty}^{\infty} Y(\omega-\tau)e^{j\omega t}e^{-j\tau t}d\omega\right) =$$

$$= \left(\frac{1}{2\pi}\right)^2 \left(\int_{-\infty}^{\infty} X(\tau)e^{j\tau t}d\tau\right)\left(\int_{-\infty}^{\infty} Y(\omega-\tau)e^{j(\omega-\tau)t}d\omega\right) =$$

$$= \left\{\begin{array}{lcl} \lambda = \omega - \tau & \longrightarrow & d\lambda = d\omega \\ \omega = -\infty & \longrightarrow & \lambda = -\infty \\ \omega = \infty & \longrightarrow & \lambda = \infty \end{array}\right\} =$$

$$= \left(\frac{1}{2\pi}\right)^2 \left(\int_{-\infty}^{\infty} X(\tau)e^{j\tau t}d\tau\right)\left(\int_{-\infty}^{\infty} Y(\lambda)e^{j\lambda t}d\lambda\right) =$$

$$= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\tau)e^{j\tau t}d\tau\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\lambda)e^{j\lambda t}d\lambda\right) =$$

$$= x(t)y(t)$$

PROPIEDAD 8: CONVOLUCIÓN EN EL TIEMPO

$$z(t) = x(t) * y(t) \implies Z(\omega) = X(\omega)Y(\omega)$$

Solución:

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} (x(t) * y(t))\, e^{-j\omega t}dt =$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau\right) e^{-j\omega t}dt = \left\{ \begin{array}{lll} z = t - \tau & \longrightarrow & dz = dt \\ \tau = -\infty & \longrightarrow & z = -\infty \\ \tau = \infty & \longrightarrow & z = \infty \end{array} \right\} =$$

$$= \int_{-\infty}^{\infty} \left(\int_{\infty}^{-\infty} x(\tau)y(z)d\tau\right) e^{-j\omega(z+\tau)}dz =$$

$$= \left(\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau\right) \left(\int_{\infty}^{-\infty} y(z)e^{-j\omega z}dz\right) =$$

$$= X(\omega)Y(\omega)$$

PROPIEDAD 9: DERIVACIÓN EN EL TIEMPO

$$y(t) = \frac{d^n x(t)}{dt^n} \implies Y(\omega) = (j\omega)^n X(\omega)$$

Solución:

De nuevo vamos a utilizar la transformada inversa en la demostración:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \implies$$

$$\implies \frac{dx(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{d}{dt}\left(e^{j\omega t}\right)d\omega \implies$$

$$\implies \frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)j\omega e^{j\omega t}d\omega \implies y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{X(\omega)j\omega}_{Y(\omega)} e^{j\omega t}d\omega$$

Luego se comprueba que la transformada de $y(t) = \dfrac{dx(t)}{dt}$ es $Y(\omega) = j\omega X(\omega)$.

Este resultado es fácilmente generalizable para $y(t) = \dfrac{d^n x(t)}{dt^n}$:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \implies$$

$$\implies \frac{d^n x(t)}{dt^n} = \frac{d^n}{dt^n}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{d^n}{dt^n}\left(e^{j\omega t}\right)d\omega \implies$$

$$\implies \frac{d^n x(t)}{dt^n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)(j\omega)^n e^{j\omega t}d\omega \implies y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{X(\omega)(j\omega)^n}_{Y(\omega)} e^{j\omega t}d\omega$$

De esta forma, se se comprueba que la transformada de $y(t) = \dfrac{d^n x(t)}{dt^n}$ es $Y(\omega) = (j\omega)^n X(\omega)$.

PROPIEDAD 10: INTEGRACIÓN EN EL TIEMPO

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau \implies Y(\omega) = \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

Solución:

Sabemos que $x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\cdot u(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau)\cdot 1 \cdot d\tau + \int_{t}^{\infty} x(\tau)\cdot 0 \cdot d\tau = \int_{-\infty}^{t} x(\tau)d\tau$.

Por otra parte, la transformada de $u(t)$ es $U(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$.

Finalmente, vamos a aprovechar la propiedad ya demostrada de la transformada de una convolución:

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t) \implies Y(\omega) = X(\omega)U(\omega) \implies =$$
$$\implies Y(\omega) = X(\omega)\left(\frac{1}{j\omega} + \pi\delta(\omega)\right) = \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$