TEMA 5

PROPIEDADES DE LAS SERIES DE FOURIER DE SEÑALES CONTINUAS

PROPIEDAD 1: LINEALIDAD

$$y(t) \rightleftharpoons d_k \, , z(t) \rightleftharpoons e_k \implies x(t) = \alpha y(t) + \beta z(t) \rightleftharpoons c_k = \alpha d_k + \beta e_k$$

Solución:

Sean dos señales $y(t)$ y $z(t)$ periódicas de período fundamental T_0.

$$\left.\begin{array}{l} y(t) \rightleftharpoons d_k \\ z(t) \rightleftharpoons e_k \end{array}\right] \implies x(t) = \alpha y(t) + \beta z(t) \rightleftharpoons c_k = \alpha d_k + \beta e_k$$

$$d_k = \frac{1}{T_0}\int_0^{T_0} y(t)e^{-jk\omega_0 t}dt \qquad e_k = \frac{1}{T_0}\int_0^{T_0} z(t)e^{-jk\omega_0 t}dt$$

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_0^{T_0} \big(\alpha y(t) + \beta z(t)\big)e^{-jk\omega_0 t}dt =$$

$$= \alpha\left(\frac{1}{T_0}\int_0^{T_0} y(t)e^{-jk\omega_0 t}dt\right) + \beta\left(\frac{1}{T_0}\int_0^{T_0} z(t)e^{-j)^{k}t}dt\right) =$$

$$= \alpha d_k + \beta e_k$$

PROPIEDAD 2: CONJUGACIÓN

$$y(t) \rightleftharpoons d_k \implies x(t) = (y(t))^* \rightleftharpoons c_k = \left(d_{-k}\right)^*$$

Solución:

$$y(t) \rightleftharpoons d_k \implies (y(t))^* \rightleftharpoons \left(d_{-k}\right)^*$$

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt \implies (c_k)^* = \left(\frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega t}dt\right)^* \implies$$

$$\implies c_k^* = \frac{1}{T_0}\int_0^{T_0} (x(t))^* \left(e^{-jk\omega_0 t}\right)^* dt \implies c_k^* = \frac{1}{T_0}\int_0^{T_0} y(t)e^{+jk\omega_0 t}dt \overset{k=-n}{\implies}$$

$$\overset{k=-n}{\implies} c_{-n}^* = \frac{1}{T_0}\int_0^{T_0} y(t)e^{-jn\omega_0 t}dt = d_n \overset{k=-n}{\implies} c_k^* = d_{-k} \implies$$

$$\implies c_k = d_{-k}^*$$

PROPIEDAD 3: INVERSIÓN EN EL TIEMPO

$$y(t) \rightleftharpoons d_k \implies x(t) = y(-t) \rightleftharpoons c_k = d_{-k}$$

Solución:

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)\cdot e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_0^{T_0} y(-t)e^{-j\omega_0 t}dt =$$

$$= \left\{\begin{array}{lll} \tau = -t & \longrightarrow & d\tau = -dt \\ t = 0 & \longrightarrow & \tau = 0 \\ t = T_0 & \longrightarrow & \tau = -T_0 \end{array}\right\} = \frac{1}{T_0}\int_0^{-T_0} x_1(\tau)\cdot e^{+jk\omega_0\tau}(-d\tau) =$$

$$= \frac{1}{T_0}\int_{-\tau_0}^{0} x(\tau)e^{-j(-k)\omega_0\tau}d\tau = \frac{1}{T_0}\int_{<T_0>} x(\tau)e^{-j(-k)\omega_0\tau}d\tau =$$

$$= d_{-k}$$

PROPIEDAD 4: COMPRESIÓN EN EL TIEMPO (ESCALADO)

$$y(t) \rightleftharpoons d_k \implies x(t) = y(at) \rightleftharpoons c_k = d_k$$

Solución:

Si $|a| > 1$, hay una compresión de la señal. Es importante notar que, si $y(t)$ es una señal periódica de período T_0, entonces $y(at)$ es una señal periódica de período $\frac{T_0}{a}$.

$$T_0' = \frac{T_0}{a} \longrightarrow \omega_0' = \frac{2n}{T_0'} = \frac{2n}{T_0}a = a\omega_0$$

$$c_k = \frac{1}{T_0'}\int_0^{T_0'} x(t)e^{-jk\omega_0' t}dt = \frac{1}{T_0/a}\int_0^{T_0/a} y(at)e^{-jka\omega_0 t}dt =$$

$$= \left\{\begin{array}{lll} \tau = at & \longrightarrow & d\tau = a\,dt \implies dt = \dfrac{d\tau}{a} \\ t = 0 & \longrightarrow & \tau = 0 \\ t = T_0/a & \longrightarrow & \tau = T_0 \end{array}\right\} =$$

$$= \frac{a}{T_0}\int_0^{T_0} x(\tau)e^{-jk\omega_0\tau}\frac{d\tau}{a} = \frac{1}{T_0}\int_0^{T_0} x(\tau)e^{-jk\omega_0\tau}d\tau =$$

$$= d_k$$

PROPIEDAD 5: DESPLAZAMIENTO EN EL TIEMPO

$$y(t) \rightleftharpoons d_k \implies x(t) = y(t \pm T) \rightleftharpoons c_k = e^{\pm jk\omega_0 T} d_k$$

Solución:

Demostración para $x(t) = y(t-T)$:

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_0^{T_0} y(t-T)e^{-jk\omega_0 t}dt =$$

$$= \left\{ \begin{array}{lll} \tau = t - T & \longrightarrow & d\tau = dt \\ t = 0 & \longrightarrow & \tau = -T \\ t = T_0 & \longrightarrow & \tau = T_0 - T \end{array} \right\} = \frac{1}{T_0}\int_{-T}^{-T+T_0} x(\tau)e^{-jk\omega_0(\tau+T)}d\tau =$$

$$= \frac{1}{T_0}\int_{-T}^{-T+T_0} x(\tau)e^{-jk\omega_0\tau}e^{-jk\omega_0 T}d\tau = e^{-jk\omega_0 T}\frac{1}{T_0}\int_{-T}^{-T+T_0} x(\tau)e^{-jk\omega_0\tau}d\tau =$$

$$= e^{-jk\omega_0 T} d_k$$

Demostración para $x(t) = y(t+T)$:

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_0^{T_0} y(t+T)e^{-jk\omega_0 t}dt =$$

$$= \left\{ \begin{array}{lll} \tau = t + T & \longrightarrow & d\tau = dt \\ t = 0 & \longrightarrow & \tau = T \\ t = T_0 & \longrightarrow & \tau = T_0 + T \end{array} \right\} = \frac{1}{T_0}\int_{T}^{T+T_0} x(\tau)e^{-jk\omega_0(\tau-T)}d\tau =$$

$$= \frac{1}{T_0}\int_{T}^{T+T_0} x(\tau)e^{-jk\omega_0\tau}e^{jk\omega_0 T}d\tau = e^{jk\omega_0 T}\frac{1}{T_0}\int_{-T}^{-T+T_0} x(\tau)e^{-jk\omega_0\tau}d\tau =$$

$$= e^{jk\omega_0 T} d_k$$

PROPIEDAD 6: DESPLAZAMIENTO EN LA FRECUENCIA

$$y(t) \rightleftharpoons d_k \implies x(t) = y(t)e^{jm\omega_0 t} \rightleftharpoons c_k = d_{k-m}$$

Solución:

$$d_{k-m} = \frac{1}{T_0}\int_0^{T_0} y(t)e^{-j(k-m)\omega_0 t}dt = \frac{1}{T_0}\int_0^{T_0} y(t)e^{-jk\omega_0 t}e^{jm\omega_0 t}dt =$$
$$= \frac{1}{T_0}\int_0^{T_0} y(t)e^{jk\omega_0 t}e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk\omega_0 t}dt =$$
$$= c_k$$

PROPIEDAD 7: CONVOLUCIÓN EN EL TIEMPO

$$y(t) \rightleftharpoons d_k,\ z(t) \rightleftharpoons e_k \implies x(t) = y(t) * z(t) \rightleftharpoons c_k = T_0 d_k e_k$$

Solución:

Sean $y(t), z(t)$ dos señales períodicas de período T_0.

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jk_0 t}dt = \frac{1}{T_0}\int_0^{T_0} \left(y(t) * z(t)\right)e^{-jk\omega_0 t}dt =$$
$$= \frac{1}{T_0}\int_0^{T_0}\left(\int_0^{T_0} y(\tau)z(t-\tau)d\tau\right)e^{-j\omega_0 t}dt =$$
$$= \frac{1}{T_0}\int_0^{T_0}\left(\int_0^{T_0} x_1(z)x_2(t-z)dz\right)e^{-jk\omega_0 t}\frac{e^{-jk\omega_0 \tau}}{e^{-jk\omega_0 \tau}}dt =$$
$$= \left(\frac{1}{T_0}\int_0^{T_0} y(z)e^{-jk\omega_0 \tau}d\tau\right)\cdot\left(\int_0^{T_0} z(t-z)e^{-jk\omega_0(t-\tau)}dt\right) =$$
$$= \left\{\begin{array}{lll} t-\tau=\alpha & \longrightarrow & dt = d\alpha \\ t=0 & \longrightarrow & \alpha = -\tau \\ t = T_0 & \longrightarrow & \alpha = T_0-\tau \end{array}\right\} = d_k T_0\left(\frac{1}{T_0}\int_{-\tau}^{-\tau+T_0} z(\alpha)e^{-jk\omega_0\alpha}d\alpha\right) =$$
$$= T_0 d_k e_k$$

Es importante notar que, en el caso de la convolución, se ha utilizado la convolución periódica, de forma que la convolución completa es la repetición períodica de la convolución en un solo período.

PROPIEDAD 8: MULTIPLICACIÓN EN EL TIEMPO

$$y(t) \rightleftharpoons d_k,\; z(t) \rightleftharpoons e_k \implies x(t) = y(t)z(t) \rightleftharpoons c_k = d_k * e_k$$

Solución:

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t} \qquad z(t) = \sum_{k=-\infty}^{\infty} e_k e^{jk\omega_0 t}$$

$$x(t) = y(t)\cdot z(t) = \left(\sum_{i=-\infty}^{\infty} d_i e^{j\omega_0 t}\right)\cdot\left(\sum_{k=-\infty}^{\infty} e_k e^{jk\omega_0 t}\right) =$$

$$= \sum_{i=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} d_i e_k \cdot e^{j(i+k)\omega_0 t} = \left\{\begin{array}{lcl} m = i+k & \longrightarrow & k = m-i \\ k = -\infty & \longrightarrow & m = -\infty \\ k = \infty & \longrightarrow & m = \infty \end{array}\right\} =$$

$$= = \sum_{i=-\infty}^{\infty} \underbrace{\boxed{\sum_{m=-\infty}^{\infty} d_i e_{m-i}}}_{c_k} e^{jm\omega_0 t} \implies c_k = \sum_{m=-\infty}^{\infty} d_i e_{m-i} \implies$$

$$\implies c_k = d_k * e_k$$

PROPIEDAD 9: DERIVACIÓN EN EL TIEMPO

$$y(t) \rightleftharpoons d_k \implies x(t) = \frac{dy(t)}{dt} \rightleftharpoons c_k = jk\omega_0 d_k$$

Solución:

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t} \implies \frac{dy(t)}{dt} = \frac{d}{dt}\sum_{k=-\infty}^{\infty}\left(d_k e^{jk\omega_0 t}\right) = \sum_{k=-\infty}^{\infty}\frac{d}{dt}\left(d_k e^{jk\omega_0 t}\right)$$

$$\implies x(t) = \frac{dy(t)}{dt} = \sum_{k=-\infty}^{\infty}\frac{d}{dt}\left(d_k e^{jk\omega_0 t}\right) = \sum_{k=-\infty}^{\infty} \underbrace{\boxed{d_k jk\omega_0}}_{c_k} e^{jk\omega_0 t} \implies$$

$$\implies c_k = jk\omega_0 d_k$$

<u>Nota</u>: La generalización de esta fórmula es:

$$x(t) = \frac{d^n y(t)}{dt^n} \longleftrightarrow c_k = (jkw_0)^n\, d_k$$

PROPIEDAD 10: INTEGRACIÓN EN EL TIEMPO

$$y(t) \rightleftharpoons d_k \implies x(t) = \int_{-\infty}^{t} y(\tau)d\tau \rightleftharpoons c_k = \frac{d_k}{jk\omega_0}$$

Solución:

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t} \implies y(\tau) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 \tau} \implies$$

$$\implies \int_{-\infty}^{t} y(\tau)d\tau = \int_{-\infty}^{t} \left(\sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 \tau} \right) d\tau =$$

$$= \sum_{k=-\infty}^{\infty} \left(d_k \left(\int_{-\infty}^{t} e^{jk\omega_0 \tau} d\tau \right) \right) = \sum_{k=-\infty}^{\infty} d_k \left[\frac{e^{jk\omega_0 \tau}}{jk\omega_0} \right]_{-\infty}^{t} =$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{\boxed{\frac{d_k}{jk\omega_0}}}_{c_k} e^{jk\omega_0 t} \implies c_k = \frac{d_k}{jk\omega_0}$$

Aunque esta es la demostración de algunas fuentes, la realidad es que no queda justificada la anulación de la exponencial compleja en $\tau = -\infty$.

Por ello, para demostrar esta propiedad podemos utilizar la anterior:

$$x(t) = \int_{-\infty}^{t} y(\tau)d\tau \implies y(t) = \frac{dx(t)}{dt}$$

Puesto que, según la anterior propiedad, $d_k = jk\omega_0 c_k$, debe ocurrir que $c_k = \dfrac{d_k}{jk\omega_0}$.