

The Existence and Uniqueness Theorem for ODE's

MA 466
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Differential Equations

Way back in DE I you learned how to solve certain types of differential equations. A typical example you'd have encountered is a first order DE like

$$y'(t) = t^2 + ty^3(t),$$

with some initial condition such as $y(0) = 2$. More generally a first order DE is of the form $y'(t) = f(t, y(t))$ for some function f , with initial condition $y(t_0) = y_0$. The goal is to find $y(t)$.

Only certain classes of first order DE's can be solved explicitly. The two standard classes are linear equations and separable equations. But many equations (such as the one above) are neither linear nor separable. Nonetheless, there is a powerful theorem which asserts that many DE's, including the one above, have unique solutions, even if we can't write them down. The proof of this theorem is a beautiful application of the ideas we've developed in this course.

Contraction Mappings on a Banach Space

Let V denote a Banach space and let T denote a mapping or operator from V to V . We say that T is a *contraction* if for all $x, y \in V$ we have

$$\|T(x) - T(y)\| \leq C\|x - y\| \tag{1}$$

for some constant $0 \leq C < 1$. A contraction mapping "draws" any two points closer together.

Example 1: Let $V = C([0, 1/2])$ with the supremum norm and $T(f)(s) = \int_0^s f(u) du$. Then for all $s \in [0, 1/2]$ and $f, g \in V$ we have

$$|T(f)(s) - T(g)(s)| = \left| \int_0^s (f - g)(u) du \right| \leq \int_0^s \|f - g\|_\infty du = s\|f - g\|_\infty \leq \frac{1}{2}\|f - g\|_\infty.$$

Since this holds for all $s \in [0, 1/2]$ we clearly have $\|T(f) - T(g)\|_\infty \leq \frac{1}{2}\|f - g\|_\infty$, so T is a contraction with constant $C = 1/2$.

Example 2: The operator $T(f) = f^2$ is not a contraction on $C([0, 1/2])$, for, e.g., if $f \equiv 1$ and $g \equiv 2$ then $\|f - g\|_\infty = 1$, while $\|T(f) - T(g)\|_\infty = 3$.

Fun Exercise: Prove that contraction mappings are continuous.

In real analysis you may have seen the fact that a contraction mapping f from \mathbb{R} to \mathbb{R} (that is, a function f for which $|f(x) - f(y)| \leq c|x - y|$ where $c < 1$) has a unique fixed point, i.e., there is one and only one solution to the equation $x = f(x)$. The same fact is true in any Banach space.

Theorem (The Contraction Mapping Principle): *Let T be a contraction on a Banach space V . Then T has a unique fixed point in V , i.e., there is one and only one solution $x \in V$ to the equation*

$$x = T(x).$$

Proof: Let's do uniqueness first. Suppose we have two solutions x_1 and x_2 to $x_i = T(x_i)$. Then we have

$$\|x_1 - x_2\| = \|T(x_1) - T(x_2)\| \leq C\|x_1 - x_2\|.$$

But since $C < 1$ this gives an immediate contradiction unless $\|x_1 - x_2\| = 0$, that is $x_1 = x_2$.

Now we show the existence of a fixed point. Let x_0 be any element of V . Define a sequence recursively by $x_{n+1} = T(x_n)$. The claim is that x_n converges to a fixed point. To see this we will first show that the sequence x_n is Cauchy, and hence must converge. We have

$$\begin{aligned} \|x_{k+1} - x_k\| &= \|T(x_k) - T(x_{k-1})\| \\ &\leq c\|x_k - x_{k-1}\| \\ &= c\|T(x_{k-1}) - T(x_{k-2})\| \\ &\leq c^2\|x_{k-1} - x_{k-2}\| \\ &\quad \vdots \\ &\leq c^k\|x_1 - x_0\|, \end{aligned}$$

so $\|x_{k+1} - x_k\| \leq c^k\|x_1 - x_0\|$. As a result we have for $m < n$ that

$$\|x_m - x_n\| \leq \|x_m - x_{m+1}\| + \|x_{m+1} - x_{m+2}\| + \cdots + \|x_{n-1} - x_n\|$$

$$\begin{aligned}
&\leq \|x_1 - x_0\|(c^m + c^{m+1} + \cdots + c^{n-1}) \\
&= c^m \|x_1 - x_0\|(1 + c + c^2 + \cdots + c^{n-m-1}) \\
&< \frac{c^m}{1-c} \|x_1 - x_0\|.
\end{aligned}$$

It's clear that since $c < 1$ we have for any $\epsilon > 0$ that $c^{m-1}/(1-c) < \epsilon$ for $m \geq N$ sufficiently large. Thus if $m, n \geq N$ we have $\|x_m - x_n\| < \epsilon$. The sequence f_n is Cauchy, and since V is complete, x_n converges to some limit x .

Of course, x must be the fixed point we want. This is easy to prove, as

$$\begin{aligned}
\|x - T(x)\| &= \|x - x_n + x_n - T(x)\| \\
&\leq \|x - x_n\| + \|x_n - T(x)\| \\
&= \|x - x_n\| + \|T(x_{n-1}) - T(x)\| \\
&\leq \|x - x_n\| + \|x_{n-1} - x\|.
\end{aligned}$$

The first inequality above came from the triangle inequality, and the last inequality uses that T is a contraction. Both norms on the right above limit to zero and we conclude that $\|x - T(x)\| = 0$, so $x = T(x)$.

Example 3: Let $V = C([0, 1/2])$ and define T as

$$T(f)(s) = 1 + \int_0^s f(u) du.$$

This is very similar to an example above, and it's easy to show that T is a contraction. As a result the equation $f = T(f)$, which is just the integral equation

$$f(s) = 1 + \int_0^s f(u) du,$$

must have a unique continuous solution, at least for $0 \leq s \leq 1/2$. The integral equation above is an example of a *Volterra integral equation* (in which the upper or lower limit of the integral depends on the independent variable). Actually, we can find the solution explicitly in this case by differentiating both sides of $f(s) = 1 + \int_0^s f(u) du$ with respect to s , to obtain $f'(s) = f(s)$. Thus $f(s) = Ce^s$ for some C . Also by putting $s = 0$ in $f(s) = 1 + \int_0^s f(u) du$ we find that $f(0) = 1$, so $f(s) = e^s$.

Example 4: Let T be defined on $C([0, 1])$ as

$$T(f)(x) = x^2 + \frac{x}{2} \int_0^1 y \sin(f(y)) dy.$$

It's easy to check that T is a contraction (with respect to the sup norm), for

$$\begin{aligned} |T(f)(x) - T(g)(x)| &= \frac{x}{2} \left| \int_0^1 y (\sin(f(y)) - \sin(g(y))) dy \right| \\ &\leq \frac{1}{2} \left| \int_0^1 y |f(y) - g(y)| dy \right| \\ &\leq \frac{1}{2} \left| \int_0^1 y \|f - g\|_\infty dy \right| \\ &\leq \frac{\|f - g\|_\infty}{2} \int_0^1 y dy \\ &= \frac{1}{4} \|f - g\|_\infty \end{aligned}$$

so that $\|T(f) - T(g)\|_\infty \leq \frac{1}{4} \|f - g\|_\infty$. In doing the above estimates I used that $|\sin(a) - \sin(b)| \leq |a - b|$, easy to prove with the mean value theorem. Since T is a contraction there is thus one and only one continuous function on $[0, 1]$ which satisfies

$$f(x) = x^2 + \frac{x}{2} \int_0^1 y \sin(f(y)) dy.$$

An integral equation like this in which the limits in the integral are constant is called a *Fredholm Integral Equation*. I have no idea how to find the solution explicitly in this case—it probably can't be done.

Application to Differential Equations

The general first order differential equation looks like

$$y'(t) = f(t, y(t)), \quad (2)$$

where $f(t, y)$ is some function in two variables. For example, if $f(t, y) = t + t^2 y^2$ then the DE is $y'(t) = t + t^2 y^2(t)$. We are also given an initial condition $y(t_0) = y_0$. The goal is to find a function $y(t)$ which satisfies the differential equation and initial condition. The function should be defined

and continuous on an interval $[t_0, t_0 + \delta]$ and differentiable on the open interval $(t_0, t_0 + \delta)$ for some $\delta > 0$.

The theorem below shows that one can, under the right conditions, assert that a DE has a unique solution, even if the solution can't be written down in closed form.

In what follows we will consider only functions $f(t, y)$ which satisfy the following property: We will require that f be differentiable in y and that

$$\left| \frac{\partial f}{\partial y}(t, y) \right| \leq C \quad (3)$$

for all t in some interval $[t_0, t_0 + \delta_1]$ and all y in $[y_0 - \delta_2, y_0 + \delta_2]$, where δ_1 and δ_2 are some positive constants. The constant C must be independent of t and y . The simplest situation in which this is guaranteed is that $\frac{\partial f}{\partial y}(t, y)$ is continuous for $(t, y) \in [t_0, t_0 + \delta_1] \times [y_0 - \delta_2, y_0 + \delta_2]$ (because a continuous function is bounded on a compact set). Under these conditions we can prove

Existence-Uniqueness Theorem: *Let the function $f(t, y)$ be continuous and satisfy the bound (3). Then the differential equation (2) with initial condition $y(t_0) = y_0$ has a unique solution which is continuous on some interval $[t_0, t_0 + \delta]$ and differentiable on $(t_0, t_0 + \delta)$, where $\delta > 0$.*

Before proving the theorem let's apply it to the DE $y'(t) = t^2 + ty^3(t)$ with initial condition $y(0) = 2$ that appeared at the start of this handout. You can compute that $\partial f / \partial y = 3ty^2$, which is continuous for all t and y , and hence is bounded by a constant on any compact set of the form $0 \leq t \leq \delta_1$, $2 - \delta_2 \leq y \leq 2 + \delta_2$. Take, for example, $\delta_1 = \delta_2 = 1$ and we have $|\partial f / \partial y| \leq 27$. For some $\delta > 0$ this DE must have a unique solution for $t \in (0, \delta)$ with $y(0) = 2$.

A result like that above, in which we prove a DE has a solution on some (possibly very small) interval is called a *local* existence result, as opposed to a *global* existence result in which we prove that the DE has a solution for all time, or at least all time up to some pre-assigned limit.

Proof of Existence-Uniqueness Theorem: First recast the differential equation as an integral equation. Note that if $y(t)$ is continuous for $t_0 \leq t \leq t_0 + \delta_1$ then $f(t, y(t))$ is continuous, hence $y'(t) = f(t, y(t))$ must also be continuous for $t_0 \leq t \leq t_0 + \delta_1$. As a result we can integrate both

sides of equation (2) for $s = t_0$ to $s = t < t_0 + \delta_1$ and apply the Fundamental Theorem of Calculus to obtain

$$y(t) = y_0 + \int_{t_0}^t f(s, y(s)) ds. \quad (4)$$

Conversely, any continuous solution to equation (4) must be a solution to the original DE (2), for the right side of equation (4) must be differentiable in t (by the Fundamental Theorem of Calculus) and so we can differentiate both sides of (4) with respect to t to obtain the original DE. Also, plugging $t = t_0$ into equation (4) produces $y(t_0) = y_0$. Any differentiable solution $y(t)$ to (2) with initial condition $y(t_0) = y_0$ is necessarily a continuous solution to equation (4) and vice-versa. We will show that equation (4) has a unique continuous solution on some interval $t_0 \leq t < t_0 + \delta$.

Let's look at a proof of the theorem in a simple special case, just to get the idea down. Suppose that the bound (3) is satisfied with " $\delta_2 = \infty$ ", i.e., for all y . For any real numbers a_1 and a_2 it then follows from the mean value theorem that

$$\frac{f(s, a_1) - f(s, a_2)}{a_1 - a_2} = \frac{\partial f}{\partial y}(s, z)$$

for some z between a_1 and a_2 . Taking the absolute values of both sides of the above equation and using the bound (3) shows that

$$|f(s, a_1) - f(s, a_2)| \leq C|a_1 - a_2| \quad (5)$$

for ALL real numbers a_1, a_2 and all $s \in [t_0, t_0 + \delta_1]$.

Define an operator $T(y)$ on $C([t_0, t_0 + \delta_1])$ as

$$T(y)(t) = y_0 + \int_{t_0}^t f(s, y(s)) ds.$$

Note that by the Fundamental Theorem of Calculus, T really does turn continuous functions into continuous (in fact, differentiable) functions. From the discussion showing the equivalence of equations (2) and (4) any fixed point for T yields a solution to $y' = f(t, y)$ with $y(t_0) = y_0$. We'll prove the existence of such a fixed point.

By using equation (5) we can estimate that for any two functions y_1 and y_2 in $C([t_0, t_0 + \delta_1])$ we have

$$|T(y_1)(t) - T(y_2)(s)| = \left| \int_{t_0}^t (f(s, y_1(s)) - f(s, y_2(s))) ds \right|$$

$$\begin{aligned}
&\leq \int_{t_0}^t |f(s, y_1(s)) - f(s, y_2(s))| ds \\
&\leq C \int_{t_0}^t |y_1(s) - y_2(s)| ds \text{ by equation (5)} \\
&\leq C \|y_1 - y_2\|_\infty \int_{t_0}^t ds \\
&= C \|y_1 - y_2\|_\infty (t - t_0).
\end{aligned}$$

If we choose t close enough to t_0 so that $C(t - t_0) < 1$, i.e., $t < t_0 + 1/C$, then T becomes a contraction. Thus let $\delta = \min(\delta_1, 1/C)$, and let $I = [t_0, t_0 + \delta]$. On $C(I)$ the operator T is a contraction and hence equation (4) must have a unique fixed point $y(t)$. By the Fundamental Theorem of Calculus the right of equation (4) is differentiable in t , hence $y(t)$ is also differentiable. By putting $t = t_0$ into the integral equation version you easily see that $y(t_0) = y_0$ and differentiating both sides shows that $y'(t) = f(t, y(t))$. This proves the theorem if the bound (3) holds for all y .

What do we do in the more general case in which (3) holds only for $y \in [y_0 - \delta_2, y_0 + \delta_2]$? First, we define a new function $g(s, y)$ for $s \in [t_0, t_0 + \delta_1]$ and $y \in \mathbb{R}$ as

$$g(s, y) = \begin{cases} f(s, y), & y_0 - \delta_2 < y < y_0 + \delta_2 \\ f(s, y_0 - \delta_2), & y \leq y_0 - \delta_2 \\ f(s, y_0 + \delta_2), & y \geq y_0 + \delta_2 \end{cases}$$

The function g extends f continuously outside the rectangle $[t_0, t_0 + \delta_1] \times [y_0 - \delta_2, y_0 + \delta_2]$ in such a way that the bound (5) holds. We're going to replace the DE $y' = f(t, y)$ with the DE $y' = g(t, y)$, prove solvability of the latter, and use this to get (local) solvability of $y' = f(t, y)$.

Exercise:

- Verify that the function g as defined above is continuous (although it may not be differentiable) and satisfies

$$|g(s, a_1) - g(s, a_2)| \leq C|a_1 - a_2| \quad (6)$$

where C is whatever constant works for f in (3), a_1, a_2 are any real numbers, and $s \in [t_0, t_0 + \delta_1]$.

Given that g satisfies equation (6), we can apply exactly the same argument as above to conclude that the equation

$$y(t) = y_0 + \int_{t_0}^t g(s, y(s)) ds$$

has a unique solution on $C(I)$ where $I = [t_0, t_0 + \delta]$ for some $\delta > 0$. As before this shows that $y(t_0) = y_0$ and $y'(t) = g(t, y(t))$. BUT, since $y(t)$ is continuous and $y(t_0) = y_0$ there must be some interval $[t_0, t_0 + \delta_3]$ with $\delta_3 > 0$ on which $|y(t) - y_0| < \delta_2$, i.e., $y_0 - \delta_2 < y(t) < y_0 + \delta_2$. But for $y(t)$ on such an interval the functions $f(t, y(t))$ and $g(t, y(t))$ are identical. As a result we have that $y(t)$ satisfies $y(t_0) = y_0$ and $y'(t) = f(t, y(t))$ for $t \in [t_0, t_0 + \delta_3]$. This completes the proof of the existence-uniqueness theorem.

By the way, although a bounded like (3) isn't the most general condition possible for existence and uniqueness, we can't get away with dropping all conditions on the function $f(t, y)$, for then the DE may not have a unique solution. For example, consider the DE $y'(t) = \sqrt[3]{y(t)}$ with initial condition $y(0) = 0$. You can check this DE and initial condition has THREE solutions, $y(t) = \pm 2\sqrt{6}t^{3/2}/9$ and $y(t) \equiv 0$. Of course this f doesn't satisfy the bound (3) near $y = 0$.

This proof may look rather abstract, but in fact it is quite constructive, for it not only tells us that a solution exists, but also gives a procedure for approximating the solution. The solution is a fixed point for a contraction and we proved such points exist by making an initial guess y_0 , then iterating at $y_{k+1} = T(y_k)$; the iterates converge to the solution. It's interesting to look at what happens when this procedure is applied to a specific DE.

Suppose we want to solve $y'(t) = y^2(t) + t$ with initial condition $y(0) = 1$. The relevant integral operator is

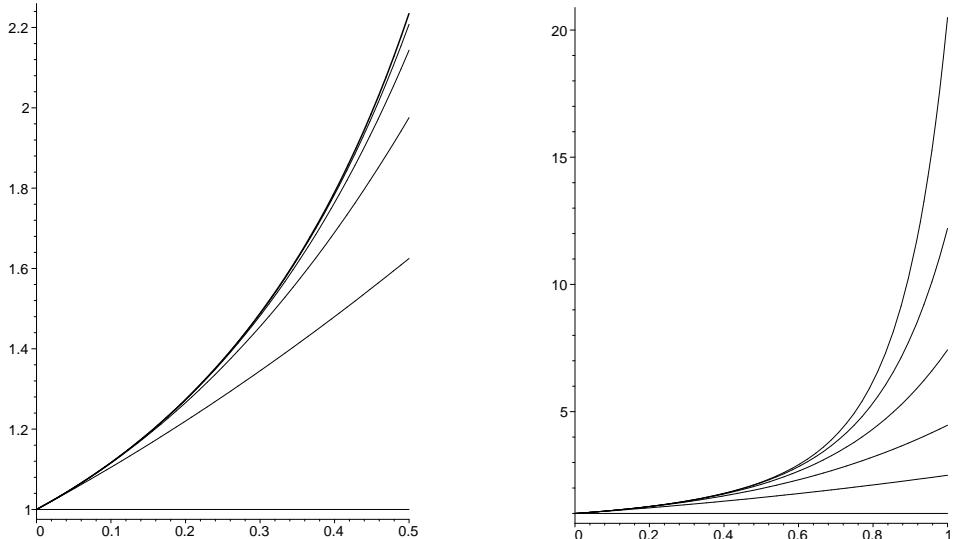
$$T(y)(t) = 1 + \int_0^t (y^2(s) + s) ds.$$

The solution we seek is a fixed point of T . Let's make an initial guess $y_0(t) \equiv 1$. Then you can easily compute

$$\begin{aligned} y_1(t) &= T(y_0)(t) = 1 + \int_0^t (y_0^2(s) + s) ds = 1 + t + \frac{1}{2}t^2 \\ y_2(t) &= T(y_1)(t) = 1 + \int_0^t (y_1^2(s) + s) ds = 1 + t + \frac{3}{2}t^2 + \frac{2}{3}t^3 + \frac{1}{4}t^4 + \frac{1}{20}t^5, \end{aligned}$$

and so on—the higher iterates get progressively messier.

On the left below is a plot of iterates y_0, y_1, y_2, y_3 , and the “true” solution (computed numerically).



On the interval shown on the left, $(0, 0.5)$, the operator T is a contraction and the iterates converge to the solution. But if the interval is increased to $(0, 1)$ as on the right, the operator T is no longer be a contraction, for the actual solution to this DE has a vertical asymptote at about $t = 0.93$. The iterates on $(0, 1)$ are shown on the right above. It’s pretty obvious that at the right end of the interval the sequence y_k is no longer Cauchy—the interval between successive iterates is getting larger, not smaller.