Problema 1 (segundo parcial)

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Dado el siguiente sistema de ecuaciones diferenciales, obtén la solución general del sistema completo:

$$\left.\begin{array}{rcl} x' &=& 2x - y - z + e^{-t} \\ y' &=& x - z \\ z' &=& x - y + 2e^{-t} \end{array}\right\}$$

$$Y' = A \cdot Y + B \Longrightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} e^{-t} \\ 0 \\ 2e^{-t} \end{pmatrix}$$

a) SGSH:

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \Longrightarrow |A - \lambda I| = \begin{vmatrix} (2-\lambda) & -1 & -1 \\ 1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} = -\lambda^3 + 2\lambda^2 - \lambda = 0 \Longrightarrow \left| \begin{array}{l} \lambda = 0 \\ \lambda = 1 \text{ (doble)} \end{array} \right.$$

- Autovectores asociados a $\lambda = 0$: $S_{\lambda=0}$

$$(A - (0) \cdot I) X = \theta \Longrightarrow \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right) \Longrightarrow \left| \begin{array}{l} x = \alpha \\ y = \alpha \\ z = \alpha \end{array} \right. \Longrightarrow B_{S_{\lambda=0}} = \{(1,1,1)\}$$

- Autovectores asociados a $\lambda = 1$: $S_{\lambda=1}$

$$(A - (1) I) X = \theta \Longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right) \Longrightarrow \left| \begin{array}{l} x = \alpha + \beta \\ y = \alpha \\ z = \beta \end{array} \right. \Longrightarrow B_{S_{\lambda=1}} = \{(1,1,0), (1,0,1)\}$$

$$Y_{SGSH} = C_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{0 \cdot t} + C_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{t} + C_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{t}$$

b) SGSC / SPSC: método de coeficientes indeterminados.

$$S_2(t) = e^{-t} \longrightarrow \left| \begin{array}{l} \left.\begin{array}{l} \alpha = -1 \\ \beta = 0 \end{array}\right] \rightarrow t = 0 \\ \left.\begin{array}{l} P_m(t) = 0 \rightarrow m = 0 \\ Q_n(t) = 1 \rightarrow n = 0 \end{array}\right] K = 0 \end{array} \right. \Longrightarrow Y_p(t) = A \cdot e^{-t}$$

Probaremos $\quad Y_p = \begin{pmatrix} a_1 e^{-t} \\ a_2 e^{-t} \\ a_3 \cdot e^{-t} \end{pmatrix} \longrightarrow Y_p' = \begin{pmatrix} -a_1 e^{-t} \\ -a_2 e^{-t} \\ -a_3 e^{-t} \end{pmatrix}$

$$Y' = A \cdot Y + B \Longrightarrow \begin{pmatrix} -a_1 e^{-t} \\ -a_2 e^{-t} \\ -a_3 e^{-t} \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} a_1 e^{-t} \\ a_2 e^{-t} \\ a_3 \cdot e^{-t} \end{pmatrix} + \begin{pmatrix} e^{-t} \\ 0 \\ 2e^{-t} \end{pmatrix}$$

$$\Longrightarrow \left| \begin{aligned} -a_1 e^{-t} &= (2a_1 - a_2 - a_3) e^{-t} + e^{-t} \\ -a_2 e^{-t} &= (a_1 \quad - a_3) e^{-t} + 0 \\ -a_3 e^{-t} &= (a_1 - a_2) e^{-t} + 2e^{-t} \end{aligned} \right. \Longrightarrow \left(\begin{aligned} -a_1 &= 2a_1 - a_2 - a_3 + 1 \\ -a_2 &= a_1 \quad - a_3 \\ -a_3 &= a_1 - a_2 \quad + 2 \end{aligned} \right. \Longrightarrow$$

$$\Longrightarrow \left. \begin{aligned} 3a_1 - a_2 - a_3 &= -1 \\ a_1 + a_2 - a_3 &= 0 \\ a_1 - a_2 + a_3 &= -2 \end{aligned} \right| \Longrightarrow \begin{aligned} a_1 &= -1 \\ a_2 &= -1/2 \\ a_3 &= -3/2 \end{aligned} \Rightarrow Y_p = \begin{pmatrix} -e^{-t} \\ -1/2\, e^{-t} \\ -3/2\, e^{-t} \end{pmatrix}$$

Solución:

$$Y_{SGSH} = C_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{0 \cdot t} + C_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{t} + C_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} -e^{-t} \\ -1/2\, e^{-t} \\ -3/2\, e^{-t} \end{pmatrix} \Longrightarrow$$

$$\Rightarrow \boxed{\begin{aligned} x(t) &= C_1 + C_2 e^{t} + C_3 e^{t} - e^{-t} \\ y(t) &= C_1 + C_2 e^{t} \qquad\quad -\tfrac{1}{2} e^{-t} \\ z(t) &= C_1 \qquad\quad + C_3 e^{t} - \tfrac{3}{2} e^{-t} \end{aligned}}$$

Problema 2 (segundo parcial)

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Determina los puntos críticos del siguiente sistema y analiza su tipo para todos ellos utilizando el método de linealización, indicando junto con el tipo si son estables, asintóticamente estables o inestables.

$$\left.\begin{aligned} x' &= (x-1)^2 + y^2 - 10 \\ y' &= (x+1)^2 + y^2 - 10 \end{aligned}\right\}$$

- Vamos a identificar los puntos críticos:

$$\left.\begin{aligned} x' = 0 &\Rightarrow (x-1)^2 + y^2 - 10 = 0 \Rightarrow y^2 = 10 - (x-1)^2 \\ y' = 0 &\Rightarrow (x+1)^2 + y^2 - 10 = 0 \Rightarrow y^2 = 10 - (x+1)^2 \end{aligned}\right| \Rightarrow \cancel{10} - (x-1)^2 = \cancel{10} - (x+1)^2 \Rightarrow$$

$$\Rightarrow \cancel{x^2} - 2x + \cancel{1} = \cancel{x^2} + 2x + \cancel{1} \Rightarrow x = 0 \Rightarrow y^2 = 10 - 1 = 9 \Rightarrow y = \pm 3$$

Luego los puntos críticos son $(0,3)$ y $(0,-3)$.

- Calculamos ahora la matriz Jacobiana:

$g_1(x,y) = (x-1)^2 + y^2 - 10$ $\qquad$ $g_2(x) = (x+1)^2 + y^2 - 10$

$$J_g(x,y) = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2(x-1) & 2y \\ 2(x+1) & 2y \end{pmatrix}$$

$\boxed{(0,3)}$

$$J_g(0,3) = \begin{pmatrix} -2 & 6 \\ 2 & 6 \end{pmatrix} = A$$

$$|A - \lambda I| = \begin{vmatrix} (-2-\lambda) & 6 \\ 2 & (6-\lambda) \end{vmatrix} = \lambda^2 - 4\lambda - 24 = 0 \Rightarrow \left| \begin{aligned} \lambda &= 2 + 2\sqrt{7} > 0 \\ \lambda &= 2 - 2\sqrt{7} < 0 \end{aligned}\right.$$

Luego se trata de un punto silla.

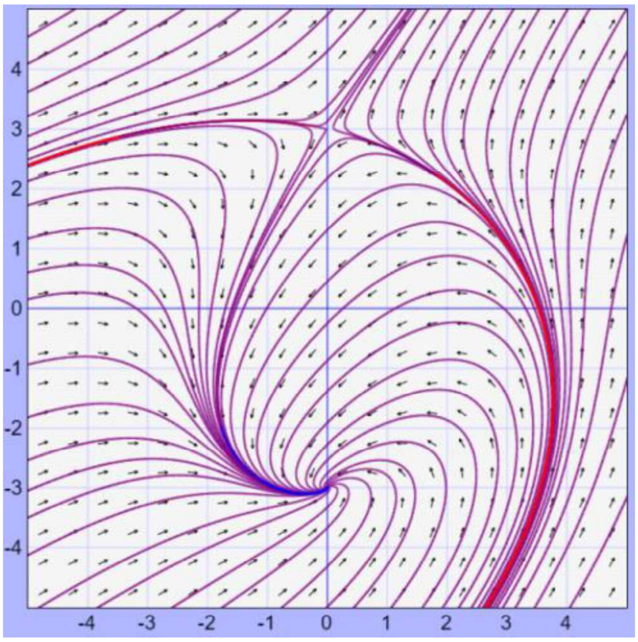
Se trata de un punto silla (inestable).

$\boxed{(0,-3)}$

$$J_g(0,-3) = \begin{pmatrix} -2 & -6 \\ 2 & -6 \end{pmatrix} = B$$

$$|B - \lambda I| = \begin{vmatrix} (-2-\lambda) & -6 \\ 2 & (-6-\lambda) \end{vmatrix} = \lambda^2 + 8\lambda + 24 = 0 \Rightarrow \left| \begin{aligned} \lambda &= -4 + i\,2\sqrt{2} \\ \lambda &= -4 - i\,2\sqrt{2} \end{aligned}\right.$$

Se trata de una espiral estable (asintóticamente estable).



4
3
2
1
0
-1
-2
-3
-4
-4 -3 -2 -1 0 1 2 3 4

Problema 3 segundo parcial)

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Resuelve el problema de valor inicial $\begin{cases} y'' - 6y' + 5y = 3e^{2x} \\ y(0) = 2 \\ y'(0) = 3 \end{cases}$ utilizando transformadas de Laplace.

$$y'' - 6y' + 5y = 3e^{2x} \Longrightarrow \mathcal{L}\{y'' - 6y' + 5y\} = \mathcal{L}\{3 \cdot e^{2x}\} \Longrightarrow$$

$$\Longrightarrow \mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = 3 \cdot \mathcal{L}\{e^{2x}\} \Longrightarrow$$

$$\Longrightarrow \left(s^2 \cdot Y(s) - s \cdot y(0) - y'(0)\right) - 6\left(s \cdot Y(s) - y(0)\right) + 5 \cdot Y(s) = 3 \cdot \frac{1}{s-2} \Longrightarrow$$

$$\Longrightarrow \left(s^2 \cdot Y(s) - 2s - 3\right) - 6\left(s \cdot Y(s) - 2\right) + 5 \cdot Y(s) = \frac{3}{s-2} \Longrightarrow$$

$$\Longrightarrow (s^2 - 6s + 5) \cdot Y(s) = \frac{3}{s-2} + 2s + 3 - 12 \Longrightarrow$$

$$\Longrightarrow (s^2 - 6s + 5) \cdot Y(s) = \frac{3 + (2s-9) \cdot (s-2)}{s-2} \Longrightarrow$$

$$\Longrightarrow Y(s) = \frac{2s^2 - 13s + 21}{(s-2)(s^2 - 6s + 5)} = \frac{2s^2 - 13s + 21}{(s-1)(s-2)(s-5)}$$

$$\frac{2s^2 - 13s + 21}{(s-1)(s-2)(s-5)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-5} = \frac{5/2}{s-1} + \frac{(-1)}{s-2} + \frac{(1/2)}{s-5}$$

$$Y(s) = \frac{5}{2} \cdot \frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{2} \frac{1}{s-5} \Longrightarrow$$

$$\Longrightarrow y(x) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left\{\frac{5}{2} \frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{2} \frac{1}{s-5}\right\} \Longrightarrow$$

$$\Longrightarrow \boxed{y(x) = \frac{5}{2} e^{x} - e^{2x} + \frac{1}{2} e^{5x}}$$

Problema 4 (segundo parcial)

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Calcula los primeros cuatro términos de las dos soluciones linealmente independientes de la ecuación $(3+x)y'' + (1+x)y' + (1-4x)y = 0$ utilizando series de potencias.

$$(3+x)y'' + (1+x)y' + (1-4x)y = 0 \Longrightarrow y'' + \underbrace{\boxed{\frac{(1+x)}{3+x}}}_{b_1(x)} y' + \underbrace{\boxed{\frac{(1-4x)}{3+x}}}_{b_0(x)} y = 0$$

$x=0$ es un punto ordinario de la ecuación, ya que $b_1(x)$ y $b_0(x)$ son analíticas en $x=0$.

$$y = \sum_{n=0}^{\infty} C_n \cdot x^n \longrightarrow y' = \sum_{n=1}^{\infty} C_n \cdot n \cdot x^{n-1} \longrightarrow y'' = \sum_{n=2}^{\infty} C_n \cdot n \cdot (n-1) \cdot x^{n-2}$$

$$(3+x)y'' + (1+x)y' + (1-4x)y = 0 \Longrightarrow$$

$$\Longrightarrow (3+x)\sum_{n=2}^{\infty} C_n \cdot n \cdot (n-1) \cdot x^{n-2} + (1+x)\sum_{n=1}^{\infty} C_n \cdot n \cdot x^{n-1} + (1-4x)\sum_{n=0}^{\infty} C_n \cdot x^n = 0 \Longrightarrow$$

$$\Longrightarrow 3\sum_{n=2}^{\infty} C_n \cdot n \cdot (n-1)\, \underset{\uparrow\, x^0}{x^{n-2}} + \sum_{n=2}^{\infty} C_n \cdot n \cdot (n-1)\, \underset{\uparrow\, x^1}{x^{n-1}} + \sum_{n=1}^{\infty} C_n \cdot n \cdot \underset{\uparrow\, x^0}{x^{n-1}} +$$

$$+ \sum_{n=1}^{\infty} C_n \cdot n \cdot \underset{\uparrow\, x^1}{x^n} + \sum_{n=0}^{\infty} C_n \cdot \underset{\uparrow\, x^0}{x^n} - 4\sum_{n=0}^{\infty} C_n \cdot \underset{\uparrow\, x^1}{x^{n+1}} = 0 \Longrightarrow$$

$$\Longrightarrow 3 \cdot C_2 \cdot 2(2-1) \cdot x^0 + 3\sum_{n=3}^{\infty} C_n \cdot n \cdot (n-1)\, x^{\boxed{n-2}} + \sum_{n=2}^{\infty} C_n \cdot n \cdot (n-1)\, x^{\boxed{n-1}} +$$

$\to K=n-2 \to n=K+2$ $\qquad \to K=n-1 \to n=K+1$

$$+ C_1 \cdot 1 \cdot x^0 + \sum_{n=2}^{\infty} C_n \cdot n \cdot x^{\boxed{n-1}} + \sum_{n=1}^{\infty} C_n \cdot n \cdot x^{\boxed{n}} + C_0 \cdot x^0 + \sum_{n=1}^{\infty} C_n \cdot x^{\boxed{n}} - 4\sum_{n=0}^{\infty} C_n \cdot x^{\boxed{n+1}} = 0 \Longrightarrow$$

$\to K=n-1 \to n=K+1$ $\qquad \to K=n$ $\qquad \to K=n$ $\qquad \to K=n+1 \to n=K-1$

$$\Longrightarrow 6C_2 + 3\sum_{K=1}^{\infty} C_{K+2} \cdot (K+2) \cdot (K+1) \cdot x^K + \sum_{K=1}^{\infty} C_{K+1} \cdot (K+1) \cdot K \cdot x^K + C_1 +$$

$$+ \sum_{K=1}^{\infty} C_{K+1} \cdot (K+1) \cdot x^K + \sum_{K=1}^{\infty} C_K \cdot K \cdot x^K + C_0 + \sum_{K=1}^{\infty} C_K \cdot x^K - 4 \cdot \sum_{K=1}^{\infty} C_{K-1} \cdot x^K = 0 \Longrightarrow$$

$$\Longrightarrow (C_0 + C_1 + 6C_2) +$$

$$+ \sum_{K=1}^{\infty} \Big(3(K+1)(K+2)C_{K+2} + \big(K(K+1) + (K+1)\big)C_{K+1} + (K+1)C_K - 4C_{K-1} \Big) x^K = 0 \Longrightarrow$$

$$\Rightarrow \begin{cases} c_0 + c_1 + 6c_2 = 0 \\ 3(k+1)(k+2)c_{k+2} + (k+1)^2 c_{k+1} + (k+1)c_k - 4c_{k-1} = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} c_2 = \dfrac{-(c_0+c_1)}{6} \\ c_{k+2} = \dfrac{-(k+1)^2 c_{k+1} - (k+1)c_k + 4c_{k-1}}{3(k+1)(k+2)} \quad (k \geq 1) \end{cases}$$

$$k=1 \longrightarrow c_3 = \frac{-4c_2 - 2c_1 + 4c_0}{3\cdot 2\cdot 3} = \frac{2}{9}\left(\frac{c_0+c_1}{6}\right) - \frac{1}{9}c_1 + \frac{2}{9}c_0 = \frac{7}{27}c_0 - \frac{2}{27}c_1$$

Solución:

$$y(x) = \sum_{n=0}^{\infty} c_n \cdot x^n = c_0 + c_1 x + c_2 \cdot x^2 + c_3 x^3 + \cdots =$$

$$= c_0 + c_1 x + \left(-\frac{1}{6}c_0 - \frac{1}{6}c_1\right)x^2 + \left(\frac{7}{27}c_0 - \frac{2}{27}c_1\right)x^3 + \cdots \Rightarrow$$

$$\boxed{y(x) = c_0\left(1 + 0\cdot x - \frac{1}{6}x^2 + \frac{7}{27}x^3 + \cdots\right) + c_1\left(0 + x - \frac{1}{6}x^2 - \frac{2}{27}\right)x^3 + \cdots}$$