École polytechnique fédérale de Lausanne

Master Semester Project

Laboratory for Topology and Neuroscience

Homological Stability

Author:
Juan Felipe Celis Rojas

Supervisor:
Jérôme Scherer

Taught properly, mathematics enables the student to think clearly and independently within the limits of his aptitude

Max Dehn

Abstract

We give a survey on different methods to prove homological stability for a family of groups. We mainly focus on the categorical framework of Oscar Randal-Williams and Nathalie Wahl and use it on illuminating examples. We briefly compare the isomorphism range of their method to the one of Andrew Putman. Then we explore the generalization of Randal-Williams and Wahl's work presented by Manuel Krannich. We explain how these methods are related and how Krannich's approach is more flexible. We follow the new proof of homological stability for mapping class groups of surfaces by Oscar Harr, Max Vistrup and Nathalie Wahl using Krannich's machinery. To conclude we give details on different techniques to prove the connectivity of simplicial complexes, which is needed to use the homological stability framework.

Introduction

A family of groups

$$\cdots \hookrightarrow G_{n-1} \hookrightarrow G_n \hookrightarrow G_{n+1} \cdots$$

satisfies *homological stability* if the induced maps in homology, with some compatible coefficient system $\{F_n\}_{n \in \mathbb{N}}$,

$$H_i(G_n; F_n) \to H_i(G_{n+1}; F_{n+1})$$

are isomorphisms in a range $0 \leq i \leq f(n)$ increasing with n. This phenomenon occurs in many known families of groups such as symmetric groups [38], general linear groups [28, 42], unitary groups [35], braid groups [1], mapping class groups of surfaces [18] and 3-manifolds [21].

Homological stability is useful to compute the homology of a given group in a certain range, but it can even more importantly be applied to the computation of the stable homology. One important example is the Mumford conjecture on the stable rational cohomology of moduli spaces of Riemann surfaces. The proof of Madsen and Weiss [33] used the isomorphism range for mapping class groups of orientable surfaces with respect to genus, proven by Harer [18], to compute the desired cohomology ring, and even better to identify its homotopy type. The computation of stable (co)homology is a hard task, it requires many different techniques, such as group completion [34, 43, 3] and infinite loop spaces. After this first proof of the Mumford conjecture, some mathematicians have improved and generalized this approach. For example Galatius, Madsen, Tillman and Weiss [15] give significant simplifications. Some expository accounts of this proof generalized to the moduli spaces of algebraic curves have been written, we recommend [48] by Tillman, and [51] by Wahl to explore the role of homological stability in the proof.

In this project we follow mostly the approach of Randal-Williams and Wahl [45]. We give some preliminaries on braid groups, mapping class groups and decorated surfaces in Chapter 1 that will be crucial to develop the examples of our homological stability machinery. In Chapter 2 we introduce monoidal categories and we explain how they fit in the homological stability framework. In Chapter 3 we associate a semi-simplicial set to each group of the family for which we want to establish homological stability. These semi-simplicial sets have a fundamental role in this approach. Their connectivity determines namely the slope of the isomorphism range.

Then we present the actual homological stability theorems in Chapter 4. We prove again the results of Randal-Williams and Wahl [45] giving more details in certain steps of their proofs than in the original article. We carefully explain the spectral sequence argument to prove their general framework for homological stability. Under some reasonable conditions, the ambient category $\mathcal{C}$ is supposed to be pre-braided homogeneous, and for two objects $A, X \in \mathcal{C}$ we build a family of semi-simplicial sets $W_n(A,X)_\bullet$, described in Chapter 3, with a group action by $\mathrm{Aut}(A \oplus X^{\oplus n})$. This setup allows us to state the following theorem.

Theorem 4.3.5 *Let $(\mathcal{C}, \oplus, 0)$ be a pre-braided homogeneous category, and A, X two objects in $\mathcal{C}$. Assume there is an integer $k \geq 2$ such that for all $n \geq 1$, the semi-simplicial set $W_n(A,X)_\bullet$ is at least $(\frac{n-2}{k})$-connected. Let $F : \mathcal{C} \to \mathrm{Mod}_{\mathbb{Z}}$ be a coefficient system of degree r at N, and $G_n = \mathrm{Aut}(A \oplus X^{\oplus n})$ and $F_n = F(A \oplus X^{\oplus n})$. Then*

$$H_i(G_n; F_n) \to H_i(G_{n+1}; F_{n+1})$$

is an epimorphism for $i \leq \frac{n}{k} - r$ and an isomorphism for $i \leq \frac{n}{k} - r - 1$ for all $n > N$.

After obtaining the isomorphism range for constant, abelian and twisted coefficients we compare these results with the work of Putman [40] for symmetric groups. To obtain even better isomorphism ranges we explore Krannich's approach [30], which is a generalization of Randal-Williams and Wahl's setup, as it relies only on the existence of a so-called Yang-Baxter element instead of a full braided structure.

We present some examples in Chapter 5 that illustrate the usefulness of Randal-Williams and Wahl's machinery. First we prove homological stability for symmetric groups and then for braid groups and mapping class groups, using Chapter 1. Later, in Chapter 6 we focus on homological stability for mapping class groups of orientable surfaces with respect to genus. We study the proof of Harr, Vistrup and Wahl [19] of the best known isomorphism range using the techniques of Krannich. The biggest challenge to use these techniques is to prove the connectivity of the semi-simplicial sets associated to the sequence of groups. We follow the arguments of Wahl in [50] to prove the connectivity of the desired semisimplicial sets. We present full details and enlightening illustrations on three methods to prove connectivity: suspension, pushing and surgery. Combining all these results we obtain the next isomorphism range for the homology of mapping class groups of surfaces.

Theorem 6.4.2 *Let $S^s_{g,r}$ be a connected surface of genus g with b boundary components and s punctures. Denote by $\Gamma(S^s_{g,r})$ its mapping class group. The map*

$$H_i(\Gamma(S^s_{g,r}); \mathbb{Z}) \to H_i(\Gamma(S^s_{g,r+1}); \mathbb{Z})$$

induced by gluing a pair of pants along one boundary component is always injective, and an isomorphism for $i \leq \frac{2g}{3}$. The map

$$H_i(\Gamma(S^s_{g,r+1}); \mathbb{Z}) \to H_i(\Gamma(S^s_{g+1,r}); \mathbb{Z})$$

induced by gluing a pair of pants along two boundary component is an epimorphism for $i \leq \frac{2g+1}{3}$, and an isomorphism for $i \leq \frac{2g-2}{3}$.

Contents

Chapter 1

Braid groups and mapping class group

This projects is a continuation of my bachelor project on homological stability of symmetric groups [6]. So we assume that the reader is already familiar with category theory, semi-simplicial sets and spectral sequences. In this project we approach homological stability with much wider lens so we need some background in other topics. Let us start by describing this new prerequisites.

1.1 Isotopies

Isotopies were introduced to study deformations of objects. We are interested on compact connected surfaces with boundary components, isotopies is a fundamental concept to study these objects. However this is not the only reason why isotopies are so important. They are also a key concept in knot theory, among other interesting subjects in mathematics.

Definition 1.1.1. Let S be a surface and f, g two simple closed curves on S. We say f, g are *isotopic* if there is a homotopy

$$H : S^1 \times [0, 1] \to S$$

from f to g such that for any $t \in [0, 1]$ the closed curve $H(S^1, \{t\})$ is simple.

Moreover, an *isotopy of a surface* S is a homotopy

$$H : S \times [0, 1] \to S$$

such that

$$H(-, t) : S \to S$$

is a homeomorphism for all $t \in [0, 1]$.

Next theorem was proved by Munkres in 1960, see Theorem 6.3 in [37]. Later, it was also proved by Smale and Whitehead, see Corollary 1.18 in [52]. We state it without giving a proof.

Theorem 1.1.2 (Theorem 1.13 in [14]). *Let S be a compact surface. Then every homeomorphism of S is isotopic to a diffeomorphism of S.*

There is an other crucial result on isotopies, again we will only give the statement.

Theorem 1.1.3 (Isotopy Extension Theorem, Corollary 1.4 [12]). *Let N, M be two smooth manifolds, and $h_t : N \to M, t \in [0,1]$ a locally flat proper isotopy. Then h_t can be covered by an ambient isotopy on M. That is $H_t : M \to M, t \in [0,1]$ such that $H_0 = Id_M$ and $H_t h_0 = h_t$ for all $t \in [0,1]$.*

1.2 Braid groups

In this section we will do a small detour to get a first example of braided monoidal category. This category will in fact come back later in an interesting case of homological stability. We start by the definitions of Artin [2] and then we continue with the approach of Mac Lane in Chapter XI from [31].

Definition 1.2.1. Let x, y, z be Cartesian coordinates in $\mathbb{R}^3$. A *braid string* is a curve in $\mathbb{R}^3$ that intersects each plane $z = a$ for all $a \in \mathbb{R}$ exactly once, so we may use z as parameter. Such a string can be described by a function $X : \mathbb{R} \to \mathbb{R}^2; z \mapsto X(z)$. The curve in $\mathbb{R}^3$ is the graph of the function X, that is $\{(X(z), z), z \in \mathbb{Z}\}$. We assume that for each braid there are constants $a < b \in \mathbb{R}$ such that X is constant on $(-\infty, a)$ and (b, ∞).

A *braid* on n strings, or n-braid, is formed by a set of n braid strings $X_i(z)$ for $i = 1, \dots, n$ without intersections, i.e.

$$X_i(z) \neq X_j(z), \quad \forall i \neq j. \tag{1.1}$$

Loosely speaking, this is equivalent to n braid strings around each other without cutting or tying them.

We say that two n-braids are equivalent if they are isotopic. In other words, this means that two braids are equivalent if one can be continuously deformed into the other without twisting nor cutting strings.

We stated before that braids actually form a group, so let us define the *braid multiplication*. One braid can be multiplied by an other one by attaching the top ends of the first one to the bottom ends of the second one. This operation is associative, has an identity (braid of straight lines, no twisting) and an inverse, all up to equivalence. This yields the *braid group* β_n on n strings.

Now that we have a geometric definition of the braid group, let us give an algebraic one, see Theorem 16 in Artin's paper [2]. That is, we want to find generators of the braid groups and relations defining them. For $0 < i < n$ let σ_i be the braid on n strings that twists the i-th string under the $(i+1)$-th string exactly once, see fig. 1.1. These braids form a generator set for β_n and satisfy the following relations:

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \forall i = 1, \dots, n-1 \\ \sigma_i \sigma_j &= \sigma_j \sigma_i, \quad \forall |i - j| \neq 1. \end{aligned}$$

Artin proved in [2] Theorem 16 that this is a complete set of defining relations for the braid groups. We can visualize this relations in fig. 1.2 by drawing so called braid diagrams.

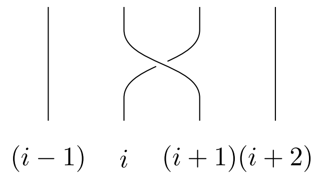

Figure 1.1: Braid generator σ_i

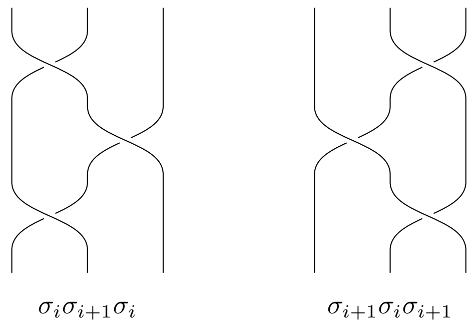

Figure 1.2: Braid relations

Example 1.2.2. A typical example is that of a classical braid, i.e. made of hair. Start with three strings and apply $\sigma_1^{-1}\sigma_2$ as many times as possible. In fig. 1.3 we repeat this operation three times. Curiously enough, if we identify the end points of the same color in fig. 1.3 we obtain a space homeomorphic to the Borromean rings. That is, three rings interconnected such that if we cut one, the other two are no longer entangled.

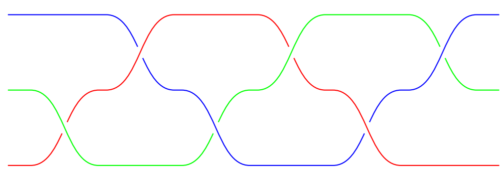

Figure 1.3: Braid $(\sigma_1^{-1}\sigma_2)^3$, Borromean rings

Note that each braid on n strings defines a permutation on the n end points. This actually describes a surjective group homomorphism $\beta_n \to \Sigma_n$. There are many ways to see that this map cannot be injective, for starters β_n is infinite and Σ_n is finite. The kernel of this map are all braids that after all crossings strings come back to their original place. For example the braid in fig. 1.3.

1.3 Braid category

Now we can build the so called *braid category* β. Let the objects of β be the natural numbers $\mathbb{N}$ with zero included. The morphisms in β are defined as follows:

$$\mathrm{Hom}_\beta(n,m) = \begin{cases} \beta_n, & \text{if } n = m \\ \emptyset, & \text{otherwise.} \end{cases}$$

This yields a well defined category. In fact, this is a groupoid because all morphisms are isomorphisms. Later, in Section 2.1, we will see that the braid category plays an important role in monoidal categories and in some examples of homological stability phenomena.

1.4 Configuration spaces

Definition 1.4.1. Let S be a surface and $n \geq 1$. The *configuration space* of n distinct ordered points in S is

$$\mathrm{Conf}^{ord}(S,n) = S^{\times n} \backslash \{(x_1,\dots,x_n) \in S^{\times n} : \exists i \neq j, x_i = x_j\}.$$

Note that the symmetric group Σ_n acts freely on $\mathrm{Conf}^{ord}(S,n)$. Then define the *configuration space* of n distinct unordered points in S by

$$\mathrm{Conf}(S,n) = \mathrm{Conf}^{ord}(S,n)/\Sigma_n.$$

Theorem 1.4.2. *We have an isomorphism*

$$\beta_n \cong \pi_1(\mathrm{Conf}(\mathbb{C},n)). \tag{1.2}$$

Sketch of proof. By definition, a braid $\sigma \in \beta_n$ on n strings can be described by a set of n distinct braid strings $\{f_i\}_{i=0}^n$ so that

$$\sigma = (f_1,\dots,f_n) : I \to \mathbb{C} \times I.$$

Each braid string is a function

$$f_i : I \to \mathbb{C} \times I$$

such that for all $i \neq j$ we have $f_i(t) \neq f_j(t)$ by eq. (1.1). Then σ describes a loop in $\mathrm{Conf}(\mathbb{C},n)$, so an element of $\pi_1(\mathrm{Conf}(\mathbb{C},n))$.

This identification yields the desired isomorphism. Indeed, under this map any loop can be described as a braid, and a loop corresponding to a braid is trivial if and only if the braid is trivial as well. □

Theorem 1.4.3. *Let n be a positive integer. The configuration space* $\mathrm{Conf}(\mathbb{C},n)$ *is an Eilenberg-MacLane space $K(\beta_n,1)$.*

Proof. We prove this statement by induction. The case $n = 1$ is clear since $\mathrm{Conf}(\mathbb{C},1) \simeq \mathbb{C}$ is contractible. Now let $n > 1$ and consider the map

$$\psi_n : \mathrm{Conf}^{ord}(\mathbb{C},n) \to \mathrm{Conf}^{ord}(\mathbb{C},n-1)$$

defined by forgetting the last point. Observe that

$$\psi_n^{-1}(x_1, \dots, x_{n-1}) \simeq \mathbb{C}\backslash\{x_1, \dots, x_{n-1}\}$$

is homotopic to a wedge of circles, a $K(*_{n-1}\mathbb{Z}, 1)$. Fadell and Neuwirth showed that

$$\mathbb{C}\backslash\{x_1, \dots, x_{n-1}\} \to \mathrm{Conf}^{ord}(\mathbb{C}, n) \to \mathrm{Conf}^{ord}(\mathbb{C}, n-1)$$

is a fibration with section, see Theorem 1 in [13]. Since all higher homotopy groups of $\mathbb{C}\backslash\{x_1, \dots, x_{n-1}\}$ vanish, the homotopy long exact sequence of this fibration yields that

$$\pi_k(\mathrm{Conf}^{ord}(\mathbb{C}, n)) \cong \pi_k \mathrm{Conf}^{ord}(\mathbb{C}, n-1))$$

for all $k > 1$. By inductive hypothesis we conclude that all higher homotopy groups of $\mathrm{Conf}^{ord}(\mathbb{C}, n)$ vanish. The quotient map

$$\mathrm{Conf}^{ord}(\mathbb{C}, n) \to \mathrm{Conf}(\mathbb{C}, n)$$

yields that all higher homotopy groups of $\mathrm{Conf}^{ord}(\mathbb{C}, n)$ vanish as well. Hence

$$\mathrm{Conf}(\mathbb{C}, n) \simeq K(\beta_n, 1).$$

□

1.5 Mapping class group of surfaces

In this section, we denote by S a surface which is the connected sum of $g \geq 0$ tori, with $b \geq 0$ open disks removed and $n \geq 0$ points removed from the interior.

Definition 1.5.1. The *mapping class group* of S is the group

$$\pi_0(\mathrm{Homeo}^+(S, \partial S)) \tag{1.3}$$

where $\mathrm{Homeo}^+(S, \partial S)$ is the group of orientation-preserving homeomorphisms of S that restrict to the identity on a neighborhood of ∂S.

Remark 1.5.2. There are many ways to define the mapping class group. For instance, using Theorem 1.1.2, we can choose diffeomorphisms instead of homeomorphisms. This yields

$$\pi_0(\mathrm{Homeo}^+(S, \partial S)) \cong \pi_0(\mathrm{Diff}^+(S \,\mathrm{rel}\, \partial S))$$

where $\mathrm{Diff}^+(S \,\mathrm{rel}\, \partial S)$ is the group of orientation-preserving diffeomorphisms of S that restrict to the identity on a neighborhood of ∂S.

Theorem 1.5.3 (Generalized Birman exact sequence)**.** *Let S be a surface with no marked points and assume that*

$$\pi_1(\mathrm{Homeo}^+(S, \partial S)) = 1.$$

Denote by S_n the surface obtained from S by marking n points in its interior. Then the sequence

$$1 \longrightarrow \pi_1(\mathrm{Conf}(S, n)) \xrightarrow{push} \pi_0(\mathrm{Homeo}^+(S_n, \partial S_n)) \xrightarrow{forget} \pi_0(\mathrm{Homeo}^+(S, \partial S)) \longrightarrow 1$$

is exact.

Proof. See Theorem 9.1 in [14]. □

Theorem 1.5.4. *Let D_n be a disk D^2 with n punctures. Then*

$$\pi_1(\mathrm{Conf}(\mathbb{C}, n)) \cong \pi_0(\mathrm{Homeo}^+(D_n, \partial D_n)) \tag{1.4}$$

Proof. Use Theorem 1.5.3 with $S = D^2$ to get the following exact sequence

$$1 \longrightarrow \pi_1(\mathrm{Conf}(D^2, n)) \longrightarrow \pi_0(\mathrm{Homeo}^+(D_n, \partial D_n)) \longrightarrow \pi_0(\mathrm{Homeo}^+(D^2, \partial D^2)) \longrightarrow 1.$$

Since $\pi_0(\mathrm{Homeo}^+(D^2, \partial D^2))$ is trivial we get

$$\pi_1(\mathrm{Conf}(D^2, n)) \cong \pi_0(\mathrm{Homeo}^+(D_n, \partial D_n)).$$

Since

$$\pi_1(\mathrm{Conf}(D^2, n)) \cong \pi_1(\mathrm{Conf}(\mathbb{C}, n))$$

we conclude the proof. □

Remark 1.5.5. Theorems 1.5.3 and 1.5.4 do not imply that β_n is isomorphic to the mapping class group of $\mathbb{C}\backslash\{x_1, \ldots, x_n\}$. Indeed the mapping class group of $\mathbb{C}\backslash\{x_1, \ldots, x_n\}$ is not isomorphic to the mapping class group of D_n because these spaces do not have the same boundary.

Theorem 1.5.6. *Let D_n be a disk D^2 with n punctures. There is an isomorphism*

$$\beta_n \cong \pi_0(\mathrm{Homeo}^+(D_n, \partial D_n)). \tag{1.5}$$

Proof. This follows from Theorems 1.4.2 and 1.5.4. □

1.6 Decorated surfaces

One of our main goal for this project is to understand the homological stability for mapping class groups of surfaces with respect to genus. To do so, we first need to take a small detour on the *category of decorated surfaces*.

Definition 1.6.1. Let $\mathbf{M}_1$ be the groupoid of *decorated surfaces*. Objects are pairs (S, I) where S is a compact connected surface with at least one boundary component and

$$I : [-1, 1] \to \partial S$$

is a parametrized interval in the boundary of S. Morphisms in $\mathbf{M}_1$ are the isotopy classes of diffeomorphisms restricting to the identity on a neighborhood of I.

Remark 1.6.2. Up to isotopy, fixing an interval I in a boundary component $\partial_0 S$ is equivalent to fixing the whole boundary component. Hence the endomorphisms of a decorated surface (S, I) identify with the mapping class group of S relative to $\partial_0 S$. That is

$$\mathrm{End}_{\mathbf{M}_1}((S, I)) \cong \pi_0(\mathrm{Homeo}^+(S, \partial_0 S)). \tag{1.6}$$

The boundary connected sum is a widely known operation in topology defined by

$$\begin{tikzcd}
I \arrow[r] \arrow[d] & S_1 \arrow[d] \\
S_2 \arrow[r] & S_1 \natural S_2 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where the maps from the interval to the surfaces factor through the boundary of the surfaces. We want to adapt this operation so that it is well defined in our groupoid of decorated surfaces.

Definition 1.6.3. Let $(S_1, I_1), (S_2, I_2) \in \mathbf{M}_1$ be two decorated surfaces. Denote by $I_i^{\pm}$ the right, respectively left, half interval of I_i, $i = 1, 2$. Define the *boundary connected sum* $(S_1 \natural S_2, I_1 \natural I_2)$ by

$$\begin{tikzcd}
I \arrow[r] \arrow[d] & I_1^+ \arrow[r] & S_1 \arrow[dd] \\
I_2^- \arrow[d] & & \\
S_2 \arrow[rr] & & S_1 \natural S_2 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

and $I_1 \natural I_2 = I_1^- \cup I_2^+$. Moreover we impose that

$$(S_1 \natural D^2, I_1 \natural I) = (S_1, I_1) \quad \& \quad (D^2 \natural S_2, I \natural I_2) = (S_2, I_2).$$

Remark 1.6.4. This boundary connected sum in $\mathbf{M}_1$ indeed extends to morphisms in $\mathbf{M}_1$ because (D^2, I) has no non-trivial automorphisms. By definition (D^2, I) is a strict unit for this operation.

Proposition 1.6.5. *The boundary connected sum is associative.*

Proof. Let $(S_1, I_1), (S_2, I_2), (S_3, I_3) \in \mathbf{M}_1$ be three decorated surfaces. We want to compare $((S_1 \natural S_2) \natural S_3, (I_1 \natural I_2) \natural I_3)$ and $(S_1 \natural (S_2 \natural S_3), I_1 \natural (I_2 \natural I_3))$. It is clear that

$$(I_1 \natural I_2) \natural I_3) = I_1^- \cup I_3^+ = I_1 \natural (I_2 \natural I_3).$$

Let us show that

$$(S_1 \natural S_2) \natural S_3 \simeq S_1 \natural (S_2 \natural S_3).$$

Observe that we have a map $S_2 \natural S_3 \to (S_1 \natural S_2) \natural S_3$ coming from

$$\begin{tikzcd}
I \arrow[r] \arrow[d] & I_2^+ = (I_1 \natural I_2)^+ \arrow[r] & S_2 \arrow[r] \arrow[dd] & S_1 \natural S_2 \arrow[ddd] \\
I_3^- \arrow[d] & & & \\
S_3 \arrow[rr] \arrow[d, equal] & & S_2 \natural S_3 \arrow[dr, dashed, "\exists!"] & \\
S_3 \arrow[rrr] & & & (S_1 \natural S_2) \natural S_3
\end{tikzcd}$$

and a map $S_1 \to (S_1 \natural S_2)\natural S_3$ coming from $S_1 \to S_1 \natural S_2$. These maps yield the a commutative square

$$\begin{tikzcd}
I \arrow[r] \arrow[d] & I_1^+ \arrow[r] & S_1 \arrow[dd] \\
I_2^- = (I_2 \natural I_3)^- \arrow[d] & & \\
S_2 \natural S_3 \arrow[rr] & & (S_1 \natural S_2)\natural S_3
\end{tikzcd}$$

because it factors through

$$\begin{tikzcd}
I \arrow[r] \arrow[d] & I_1^+ \arrow[r] & S_1 \arrow[dd] \\
I_2^- = (I_2 \natural I_3)^- \arrow[d] & & \\
S_2 \arrow[rr] \arrow[d] & \ulcorner & (S_1 \natural S_2) \arrow[d] \\
S_2 \natural S_3 \arrow[rr] & & (S_1 \natural S_2)\natural S_3.
\end{tikzcd}$$

Therefore the universal property of push-outs gives us a map

$$\begin{tikzcd}
I \arrow[r] \arrow[d] & I_1^+ \arrow[r] & S_1 \arrow[dd] \arrow[ddr, bend left] & \\
I_2^- = (I_2 \natural I_3)^- \arrow[d] & & & \\
S_2 \natural S_3 \arrow[rr] \arrow[drrr, bend right] & \ulcorner & S_1(\natural S_2 \natural S_3) \arrow[dr, dashed, "\exists!"] & \\
 & & & (S_1 \natural S_2)\natural S_3.
\end{tikzcd}$$

We can do the same trick to get a map $(S_1 \natural S_2)\natural S_3 \to S_1(\natural S_2 \natural S_3)$. The uniqueness from the universal property yields that these two maps are mutually inverse of each other. Hence we can conclude

$$(S_1 \natural S_2)\natural S_3 \simeq S_1 \natural (S_2 \natural S_3).$$

The boundary connected sum is indeed associative. $\square$

We want more structure on the category of decorated surfaces. For this we define the Dehn twist on its objects. Dehn worked with this operation on [9] for mapping class groups. Translations of his work in english can be found in [10].

Definition 1.6.6. Let $(S, I) \in \mathbf{M}_1$, and denote by $\partial_0 S$ the boundary component of S containing I. Without loss of generality, we may assume there is a small cylinder $A \simeq S^1 \times [0,1]$ with base $\partial_0 S$ in S. A *θ-Dehn twist* on (S, I) is the homeomorphism on S fixing $S \backslash A$ and defined on A by

$$S^1 \times [0,1] \to S^1 \times [0,1] : (e^{i\phi}, t) \mapsto (e^{i(\phi+(1-t)\theta)}, t).$$

We will focus on full Dehn twists and half Dehn twists, that is $\theta = 2\pi$ and $\theta = \pi$ respectively.

Proposition 1.6.7. *Let $(S_1, I_1), (S_2, I_2) \in \mathbf{M}_1$. Half Dehn twists define a map:*

$$(S_1 \natural S_2, I_1 \natural I_2) \to (S_2 \natural S_1, I_2 \natural I_1)$$

such that its square is not the identity.

Proof. We define the homeomorphism on the cylinder with base $\partial_0 = \partial_0(S_1 \natural S_2)$ by half a Dehn twist, that is $\theta = \pi$. Outside this cylinder we define it as the identity. Schematically the map is defined as in fig. 1.4.

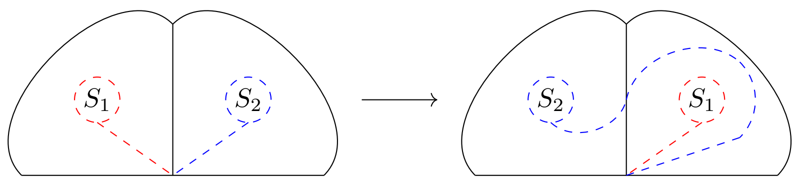

Figure 1.4: Half Dehn Twist

The full Dehn twist is the square of the half Dehn twist. By definition, the Dehn twist is an automorphism which is not the identity. In fig. 1.5 we can see it schematically. □

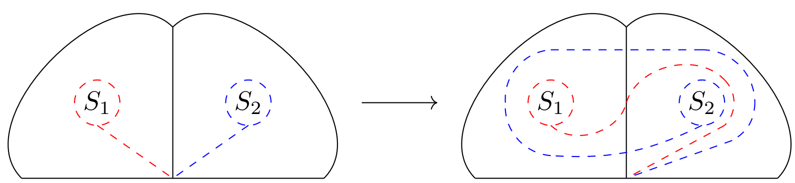

Figure 1.5: Dehn Twist

In [39], Palais proved the following result, in a larger context. This result was also proved by Cerf, see 2.2.2 Corollaire 2 in [7].

Proposition 1.6.8. *Let $(S_1, I_1), (S_2, I_2) \in \mathbf{M}_1$ be two decorated surfaces. Then the restriction map*

$$\mathrm{Diff}(S_1 \natural S_2 \operatorname{rel} \partial_0) \to \mathrm{Emb}((S_2, I_2^+), (S_1 \natural S_2, I_2^+)) : f \mapsto f|S_2$$

is a locally trivial fibre bundle.

In Théorème 5 from [16] Gramain proved a contractibility result on the group of embeddings of surfaces.

Proposition 1.6.9. *Under the same assumptions the previous proposition. The connected components of* $\mathrm{Emb}((S_2, I_2^+), (S_1 \natural S_2, I_2^+))$ *are contractible.*

The previous two propositions yield the following nice result.

Corollary 1.6.10. *Let* $(S_1, I_1), (S_2, I_2) \in \mathbf{M}_1$ *be two decorated surfaces, denote by* ∂_0 *the boundary component of* $(S_1 \natural S_2, I_1 \natural I_2)$ *containing* $I_1 \natural I_2$*. There is a fibration sequence*

$$\mathrm{Diff}(S_1 \natural S_2 \,\mathrm{rel}\, S_2 \cup \partial_0) \to \mathrm{Diff}(S_1 \natural S_2 \,\mathrm{rel}\, \partial_0) \to \mathrm{Emb}((S_2, I_2^+), (S_1 \natural S_2, I_2^+)).$$

whose base has contractible components.

Chapter 2

Monoidal categories

Monoids are fundamental algebraic structures in mathematics. As good mathematicians we want to explore this definition through some wider lens, so it is natural to think about category theory. In this chapter we discover how monoids an be understood in a categorical framework and give us tools that will be crucial for homological stability results. We define monoidal categories and describe additional properties that will be of use later chapters. All along we will give meaningful examples to help comprehension and set up the foundations of many stability results.

2.1 Categories

Monoidal categories are motivated by operators such as the tensor product and direct sum on vector spaces. We want to study categories with a sort of tensor product, so let's start by giving some precise definitions. We follow chapters VII and XI from Mac Lane's book [31] to introduce this categorical framework.

Definition 2.1.1. A *monoidal category* is a category $\mathcal{C}$ equipped with a bifunctor

$$\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

which is associative up to isomorphism α, i.e. there is an isomorphism

$$\alpha : (A \oplus B) \oplus C \xrightarrow{\sim} A \oplus (B \oplus C),$$

and which has an object e which is a left unit up to isomorphism λ and a right unit up to ρ, i.e.

$$\lambda : e \oplus A \xrightarrow{\sim} A, \quad \rho : A \oplus e \xrightarrow{\sim} A.$$

Moreover, the pentagonal diagram

$$\begin{tikzcd}
A \oplus (B \oplus (C \oplus D)) \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & (A \oplus B) \oplus (C \oplus D) \arrow[r, "\alpha"] & ((A \oplus B) \oplus C) \oplus D \\
A \oplus ((B \oplus C) \oplus D) \arrow[rr, "\alpha"'] & & (A \oplus (B \oplus C)) \oplus D \arrow[u, "\alpha \oplus id_D"']
\end{tikzcd}$$

must commute for all objects $A, B, C, D \in \mathcal{C}$. And the following triangle

$$\begin{tikzcd} A \oplus (e \oplus B) \arrow[r, "\alpha"] \arrow[dr, "id_A \oplus \lambda"'] & (A \oplus e) \oplus B \arrow[d, "\rho \oplus id_B"] \\ & A \oplus B \end{tikzcd}$$

commutes for all $A, B \in \mathcal{C}$.

We say that $(\mathcal{C}, \oplus, e, \alpha, \lambda, \rho)$ is *strict* monoidal if α, λ, ρ are the identity.

Definition 2.1.2. Let $(\mathcal{C}, \oplus, e, \alpha, \lambda, \rho)$ be monoidal category equipped with a natural isomorphism

$$\gamma : A \oplus B \xrightarrow{\sim} B \oplus A$$

for all $A, B \in \mathcal{C}$ called *twist*. If $\gamma^2 = id$ then we say that $(\mathcal{C}, \oplus, e, \alpha, \lambda, \rho)$ is *symmetric*. Otherwise we say that $(\mathcal{C}, \oplus, e, \alpha, \lambda, \rho)$ is *braided*.

From now on $(\mathcal{C}, \oplus, 0)$ will always denote a strict monoidal category where the unit 0 is initial in $\mathcal{C}$. Moreover, for any object $A \in \mathcal{C}$ let ι_A be the unique morphism from 0 to A. We will often abuse notation and write $A = id_A$ for any object $A \in \mathcal{C}$.

Example 2.1.3. Let FI be the category of finite sets with injections. Together with the disjoint union of sets and the empty set this forms a strict monoidal category $(FI, \sqcup, \emptyset)$. Clearly the empty set is a unit for disjoint union, and this operation is associative and commutative. So this category is in fact symmetric monoidal.

Example 2.1.4. Recall the braid category β from Section 1.3. We build a monoidal structure on β. Define the *braid addition* as a bifunctor

$$+ : \beta \times \beta \to \beta$$

which is defined as expected on objects (natural numbers), while addition of braids consists on laying braids side by side. Notice that braid addition is strictly associative with the empty braid, i.e. $\emptyset \in \mathrm{Hom}_\beta(0, 0)$, as unit. Hence $(\beta, +, \emptyset)$ is a strict monoidal category. It is not symmetric because addition is commutative on objects but not on morphisms. We define a braiding

$$\gamma_{m,n} : m + n \to n + m$$

by crossing m strings over n strings. This definition is indeed natural in m and n. It is cleat that $\gamma^2 \neq Id$. Therefore $(\beta, +, \emptyset)$ is a braided monoidal category. The braid category is our first example of a braided monoidal category.

Example 2.1.5. Recall the category of decorated surfaces $\mathbf{M}_1$ from Definition 1.6.1. We claim $(\mathbf{M}, \natural, (D^2, I))$ is a braided monoidal category. Proposition 1.6.5 and remark 1.6.4 imply that $(\mathbf{M}, \natural, (D^2, I))$ is monoidal. Now consider the operation of doing half a Dehn twist. By Proposition 1.6.7 this defines a braiding in $(\mathbf{M}, \natural, (D^2, I))$ making it braided monoidal.

Definition 2.1.6. A monoidal category $\mathcal{C}, \oplus, 0$ is *locally homogeneous* at a pair of objects (A, X) if the following conditions are satisfied:

- **LH1**: For all $0 \leq p < n$, the post-composition action $\mathrm{Aut}(A \oplus X^{\oplus n}) \curvearrowright \mathrm{Hom}(X^{\oplus p+1}, A \oplus X^{\oplus n})$ is transitive;

- **LH2**: For all $0 \leq p < n$, the map

$$\mathrm{Aut}(A \oplus X^{\oplus n-p-1}) \to \mathrm{Aut}(A \oplus X^{\oplus n}) : f \mapsto f \oplus X^{\oplus p+1}$$

  is injective with image the fixed points of $\iota_{A \oplus X^{\oplus n-p-1}} \oplus X^{\oplus p+1}$ under the post-composition action.

Moreover, we say that $\mathcal{C}, \oplus, 0$ is *homogeneous* if for any objects $A, B \in \mathcal{C}$:

- **H1**: The post-composition action of $\mathrm{Aut}(B)$ on $\mathrm{Hom}(A, B)$ is transitive;
- **H2**: The map

$$\mathrm{Aut}(A) \to \mathrm{Aut}(A \oplus B) : f \mapsto f \oplus B$$

  is injective with image the fixed points of $\iota_A \oplus B$.

Remark 2.1.7. Notice that (H) implies (LH) at every pair of objects $(A, X) \in \mathcal{C}$.

Example 2.1.8. The monoidal category $(FI, \sqcup, \emptyset)$ is homogeneous. Since morphisms in FI are injective set functions, then one can see that (H1) holds. To see that (H2) holds, we first observe that the map $\mathrm{Aut}(A) \to \mathrm{Aut}(A \sqcup B) : f \mapsto f \sqcup B$ is clearly injective. Now let us study its image. Let $f \in \mathrm{Aut}(A)$, then

$$(f \sqcup B) \circ (\iota_A \sqcup B) = (f \circ \iota_A) \sqcup B = \iota_A \sqcup B$$

by unicity of ι_A. Now let $g \in \mathrm{Aut}(A \sqcup B)$ be such that $g \circ (\iota_A \sqcup B) = \iota_A \sqcup B$. Then we can see that g restricts to the identity on B, so $g = g_A \sqcup B$. This proves that (H2) holds. Hence $(FI, \sqcup, \emptyset)$ is homogeneous.

Definition 2.1.9. A monoidal category $\mathcal{C}, \oplus, 0$ is *pre-braided* if its underlying groupoid is braided and for each pair of objects $A, B \in \mathcal{C}$, the groupoid braiding $\gamma_{A,B} : A \oplus B \xrightarrow{\sim} B \oplus A$ satisfies

$$\gamma_{A,B} \circ (A \oplus \iota_B) = \iota_B \oplus A.$$

2.2 Monoidal categories and homological stability

Now that we have a wide categorical framework we may ask: how does it fit into homological stability? See Remark 1.4 from [45]. Consider a monoidal category $(\mathcal{C}, \oplus, 0)$ that is locally homogeneous at (A, X). Define a sequence of groups

$$G_n = \mathrm{Aut}(A \oplus X^{\oplus n}). \tag{2.1}$$

Observe that condition (LH2, 2.1.6) yields injective maps

$$\Sigma^X : G_n \to G_{n+1}; f \to f \oplus X.$$

Then we have a sequence of groups

$$G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow \cdots .$$

The category $\mathcal{C}$ encodes information about how this groups are related.

Example 2.2.1. Recall the monoidal category $(FI, \sqcup, \emptyset)$ from example 2.1.8. Let $A = \emptyset$ and $X = \{*\}$. Then we have

$$\mathrm{Aut}(A \sqcup X^{\sqcup n}) = \mathrm{Aut}(\{*\}^{\sqcup n}) \cong \mathrm{Aut}(\{1, 2, \dots, n\}) = \Sigma_n$$

which yields the sequence of symmetric groups

$$\Sigma_0 \hookrightarrow \Sigma_1 \hookrightarrow \Sigma_2 \hookrightarrow \cdots .$$

This will allow us to study the stable range of symmetric groups.

2.3 Groupoids and Quillen's construction

In order to work with eq. (2.1) we need (locally) homogeneous monoidal categories. Since not all categories are (locally) homogeneous, we would like to construct a (locally) homogeneous category from any category satisfying some reasonable conditions. We will use a particular case of Quillen's construction presented by Grayson in [17]. Then we will follow Wahl and Randall-Williams approach from section 1 in [45] to determine under which conditions Quillen's construction is (locally) homogeneous.

Definition 2.3.1. A (small) *groupoid* $\mathcal{G}$ is a (small) category which all morphisms are isomorphisms. That is, for any two objects $A, B \in \mathcal{G}$ we have $\mathrm{Hom}_{\mathcal{G}}(A, B) = \mathrm{Iso}_{\mathcal{G}}(A, B)$. Then, a *monoidal groupoid* $(\mathcal{G}, \oplus, 0)$ is a monoidal category whose underlying category $\mathcal{G}$ is a groupoid.

Definition 2.3.2. Let $(\mathcal{G}, \oplus, 0)$ be a monoidal groupoid. *Quillen's construction* on $\mathcal{G}$ is the category $U\mathcal{G}$ whose objects are the same than the objects of $\mathcal{G}$, and morphisms are equivalence classes of pairs $(X, f) \in \mathcal{G} \times \mathrm{Hom}_{\mathcal{G}}(X \oplus A, B)$ where $(X, f) \sim (X', f')$ if there is an isomorphism $g : X \to X'$ such that the diagram

$$\begin{tikzcd}
X \oplus A \arrow[r, "f"] \arrow[d, "g \oplus A"'] & B \\
X \oplus A \arrow[ur, "f'"'] &
\end{tikzcd}$$

commutes.

Notice that if $\mathcal{G}$ is braided with no zero divisors, then $U\mathcal{G}$ is a pre-braided monoidal category.

Now we need to find conditions on $\mathcal{G}$ under which $U\mathcal{G}$ is (locally) homogeneous. It turns out to be the following cancellation condition.

Definition 2.3.3. Let (A, X) be a pair of objects of a monoidal groupoid $\mathcal{G}$. We say that $\mathcal{G}$ satisfies *local cancellation* at (A, X) if

- **LC**: For all $0 \leq p < n$, if $Y \in \mathcal{G}$ is such that $Y \oplus X^{\oplus p+1} \cong A \oplus X^{\oplus n}$ then $Y \cong A \oplus X^{\oplus n-p-1}$.

Moreover we say that $\mathcal{G}$ satisfies *cancellation* if

- **C**: For all objects $A, B, C \in \mathcal{G}$, if $A \oplus C \cong B \oplus C$ then $A \cong B$.

Around the 1970's non-cancellation phenomena were hot topics in group theory and topology. It started with non-cancellation for factors in products of manifolds [24]. Then it was related with localization theory of nilpotent spaces by Hilton, Mislin and Roitberg [23]. Now we will see how relevant is the cancellation property in the context of homological stability.

Theorem 2.3.4. *Let $(\mathcal{G}, \oplus, 0)$ be a braided monoidal groupoid with no zero divisors. Then*

1. *$U\mathcal{G}$ satisfies (LH1, 2.1.6) at (A, X) if and only if $\mathcal{G}$ satisfies (LC, 2.3.3) at (A, X).*
2. *If $\mathrm{Aut}_{\mathcal{G}}(A \oplus X^{\oplus n-p-1}) \to \mathrm{Aut}_{\mathcal{G}}(A \oplus X^{\oplus n}); f \mapsto f \oplus X^{\oplus p+1}$ is injective for all $0 \leq p < n$, then $U\mathcal{G}$ satisfies (LH2, 2.1.6) at (A, X).*

In particular, if (1) and (2) are satisfied then $U\mathcal{G}$ is locally homogeneous at (A, X).

3. *$U\mathcal{G}$ satisfies (H1, 2.1.6) if and only if $\mathcal{G}$ satisfies (C, 2.3.3).*
4. *If $\mathrm{Aut}_{\mathcal{G}}(A) \to \mathrm{Aut}_{\mathcal{G}}(A \oplus B); f \mapsto f \oplus B$ is injective for all $A, B \in \mathcal{G}$, then $U\mathcal{G}$ satisfies (H2, 2.1.6).*

If both (3) and (4) hold then $U\mathcal{G}$ is homogeneous.

Proof. We will prove (1) and (2). The proof for (3) and (4) is analogous.

Assume that $\mathcal{G}$ satisfies (LC) at (A, X). Let $[B, f], [C, g] \in \mathrm{Hom}_{U\mathcal{G}}(X^{\oplus p+1}, A \oplus X^{\oplus n})$. Since $\mathcal{G}$ is a groupoid we get that $B \oplus X^{\oplus p+1} \cong C \oplus X^{\oplus p+1}$. By local cancellation we obtain an isomorphism $\phi : B \xrightarrow{\sim} C$. In particular we get that $[C, g] = [B, g \circ (\phi \oplus X^{\oplus p+1})]$. Now pick $[0, f \circ (g \circ (\phi \oplus X^{\oplus p+1}))^{-1}] \in \mathrm{Aut}_{U\mathcal{G}}(A \oplus X^{\oplus n})$ and observe that

$$[0, f \circ (g \circ (\phi \oplus X^{\oplus p+1}))^{-1}] \circ [C, g] = [0, f \circ (g \circ (\phi \oplus X^{\oplus p+1}))^{-1}] \circ [B, g \circ (\phi \oplus X^{\oplus p+1})] = [B, f].$$

Therefore the post-composition action of $\mathrm{Aut}_{U\mathcal{G}}(A \oplus X^{\oplus n})$ on $\mathrm{Hom}_{U\mathcal{G}}(X^{\oplus p+1}, A \oplus X^{\oplus n})$ is transitive. We have proven that $U\mathcal{G}$ satisfies (LH1) at (A, X).

Now suppose that $U\mathcal{G}$ satisfies (LH1) at (A, X) and let $[Y, f] \in \mathrm{Hom}_{U\mathcal{G}}(X^{\oplus p+1}, A \oplus X^{\oplus n})$. By transitivity of the post-composition action we know there is $[Z, \psi] \in \mathrm{Aut}_{U\mathcal{G}}(A \oplus X^{\oplus n})$ such that $[Z, \psi] \circ [Y, f] = [A \oplus X^{\oplus n-p-1}, id]$. Since $\mathcal{G}$ has no zero divisors and ψ is an isomorphism we may assume that $Z = 0$. Hence $[Y, \psi \circ f] = [A \oplus X^{\oplus n-p-1}, id]$, which implies that $Y \cong A \oplus X^{\oplus n-p-1}$. This proves local cancellation at (A, X).

Now we give a proof of (2). Let $[Y, f] \in \ker(- \oplus X^{\oplus p+1})$, that is $[Y, f] \oplus X^{\oplus p+1} = [0, id]$. So there is an isomorphism $\phi : Y \to 0$ such that $f \oplus X^{\oplus p+1} = \phi \oplus A \oplus X^{\oplus n}$. By assumption we know that $f = \phi \oplus A \oplus X^{\oplus n-p-1}$. In this case $[Y, f] = [0, id]$, so we have proven that $- \oplus X^{\oplus p+1}$ is injective. It remains to identify its image.

Let $B = A \oplus X^{\oplus n-p-1}$ and $C = X^{\oplus p+1}$- Now we describe explicitly the fixed points of $\iota_B \oplus C$ using the no zero divisors assumption for simplicity.

$$\begin{aligned} Fix(\iota_B \oplus C) &= \{[0, \phi] \in \mathrm{Aut}_{U\mathcal{G}}(A \oplus X^{\oplus n}) : [0, \phi] \circ (\iota_B \oplus C) = \iota_B \oplus C\} \\ &= \{[0, \phi] \in \mathrm{Aut}_{U\mathcal{G}}(A \oplus X^{\oplus n}) : [B, \phi] = [B, id_{B \oplus C}]\} \\ &= \mathrm{im}(- \oplus X^{\oplus p+1}) \end{aligned}$$

where the second equality holds because $[B, id_{B\oplus C}] = [0, \iota_B \oplus C]$. Thus

$$[B, id_{B\oplus C}] = [0, \phi] \circ [B, id_{B\oplus C}] = [B, \phi].$$

This finishes the proof of (2). $\square$

Example 2.3.5. Let Σ be the groupoid of finite sets with bijections. This is symmetric monoidal where the sum operation is the disjoint union of sets. The unit of Σ is the empty set. It is clear that finite sets have no zero-divisors and they have the cancellation property (C,2.3.3). Then Theorem 2.3.4 says that $U\Sigma$ is homogeneous.

We claim that $U\Sigma$ can be identified with the category of finite sets with injections from example 2.1.8. Let A, B be finite sets then

$$\mathrm{Hom}_{U\Sigma}(A, B) = \{[X, f], \quad f : X \sqcup A \to B\}$$

where we identify $(X, f) \simeq (Y, g)$ if there is a bijection $h : X \to Y$ such that the diagram

$$\begin{tikzcd} X \sqcup A \arrow[d, "h\sqcup A"'] \arrow[dr, "f"] & \\ & B \\ Y \sqcup A \arrow[ur, "g"'] & \end{tikzcd}$$

commutes. In other words $f|_A = g|_A$, so $[X, f]$ is fully determined by the injection $f|_A$. This proves our claim.

Proposition 2.3.6. *Let $\mathbf{M}_1$ be the groupoid of decorated surfaces from Definition 1.6.1. Then $\mathbf{M}_1$ is pre-braided with underlying groupoid $\mathbf{M}_1$ and locally homogeneous at (A, X) for any orientable surface X.*

Proof. Let X be an orientable decorated surface. The classification of surfaces yields that $\mathbf{M}_1$ satisfies local cancellation at (A, X) for any A.

Now let $(S_1, I_1), (S_2, I_2) \in \mathbf{M}_1$. We claim that the map

$$\mathrm{Aut}_{\mathbf{M}_1}((S_1, I_1)) \to \mathrm{Aut}_{\mathbf{M}_1}((S_1 \natural S_2, I_1 \natural I_2)) \tag{2.2}$$

is injective. Indeed, this follows from the long exact sequence of homotopy groups associated to the fibration sequence from Corollary 1.6.10

$$\mathrm{Diff}^+(S_1 \natural S_2 \,\mathrm{rel}\, S_2 \cup \partial_0) \longrightarrow \mathrm{Diff}^+(S_1 \natural S_2 \,\mathrm{rel}\, \partial_0) \longrightarrow \mathrm{Emb}((S_2, I_2^+), (S_1 \natural S_2, I_2^+))$$

where $\mathrm{Diff}^+(S_1 \,\mathrm{rel}\, \partial_0) \cong \mathrm{Diff}^+(S_1 \natural S_2 \,\mathrm{rel}\, S_2 \cup \partial_0)$. The base of this fibration is contractible. So the map

$$\mathrm{Aut}_{\mathbf{M}_1}((S_1, I_1)) \cong \pi_0(\mathrm{Diff}^+(S_1 \,\mathrm{rel}\, \partial_0)) \to \mathrm{Diff}^+(S_1 \natural S_2 \,\mathrm{rel}\, \partial_0) \cong \mathrm{Aut}_{\mathbf{M}_1}((S_1 \natural S_2, I_1 \natural I_2)$$

is injective.

By Theorem 2.3.4 it follows that $U\mathbf{M}_1$ is locally homogeneous at (A, X) for any X orientable. $\square$

Chapter 3

Connectivity of spaces

The strategy to prove our stability results is to find suitable spaces with high connectivity on which the relevant group acts transitively, see Section 2 from [45]. In this chapter we explore the connectivity of simplicial complexes and semi-simplicial sets. More precisely we give some criterion, Theorem 2.10 in [45], to estimate the connectivity of semi-simplicial sets, which plays a crucial role in the stability results. We will give some examples of semi-simplicial sets that satisfy the connectivity criterion.

3.1 Destabilization spaces

To a given pair of objects in a monoidal category we want to associate a sequence of semi-simplicial sets and then study their connectivity. The construction, presented in section 2 from [45], uses the objects we introduced in chapter 2, in particular Quillen's construction from definition 2.3.2.

Definition 3.1.1. Let (A, X) be a pair of objects in a monoidal category $(\mathcal{C}, \oplus, 0)$. For an integer n, the *space of destabilizations* associated to (A, X) is the semi-simplicial set $W_n(A, X)_\bullet$, whose p-simplices are defined by

$$W_n(A, X)_p = \mathrm{Hom}_{\mathcal{C}}(X^{\oplus p+1}, A \oplus X^{\oplus n})$$

with face maps

$$d_i : \mathrm{Hom}_{\mathcal{C}}(X^{\oplus p+1}, A \oplus X^{\oplus n}) \to \mathrm{Hom}_{\mathcal{C}}(X^{\oplus p}, A \oplus X^{\oplus n}); f \mapsto f \circ (X^{\oplus i} \oplus \iota_X \oplus X^{\oplus p-i}).$$

Post-composition in $\mathcal{C}$ defines a simplicial action of $\mathrm{Aut}(A \oplus X^{\oplus n})$ on $W_n(A, X)_\bullet$.

For different values of n, this yields the sequence we were looking for. We are interested in the connectivity of these semi-simplicial sets. To study it we first need to give the following definitions.

Definition 3.1.2. We say that a monoidal category $\mathcal{C}$ satisfies **LH3** at (A, X) with *slope k* if for all $n \geq 1$, $|W_n(A, X)_\bullet|$ is $(\frac{n-2}{k})$-connected.

Definition 3.1.3. Let $(\mathcal{C}, \oplus, 0)$ be a homogeneous monoidal category. We say that $\mathcal{C}$ is *locally standard* at a pair (A, X) if

- **LS1**: The morphisms $\iota_A \oplus X \oplus \iota_X$ and $\iota_{A\oplus X} \oplus X$ are distinct in $\mathrm{Hom}_{\mathcal{C}}(X, A \oplus X^{\oplus 2})$;
- **LS2**: For all $n \geq 1$, the map

$$\mathrm{Hom}_{\mathcal{C}}(X, A \oplus X^{\oplus n-1}) \to \mathrm{Hom}_{\mathcal{C}}(X, A \oplus X^{\oplus n}); f \mapsto f \oplus \iota_X$$

is injective.

Example 3.1.4. The category $(FI, \sqcup, \emptyset)$ is locally standard at any pair of finite sets (A, X). Indeed the inclusion of X into the first X-factor in $A \sqcup X \sqcup X$ is different than the inclusion into the second X-factor. And for any two $f, g \in \mathrm{Hom}(X, A \sqcup X^{\sqcup n-1})$, if $f \sqcup \iota_X = g \sqcup \iota_X$ one can see that $f = g$. Hence $(FI, \sqcup, \emptyset)$ is locally standard at (A, X).

We claim that we could have stated condition (LS2) in a different way. In the next lemma we provide equivalent formulations of (LS2).

Lemma 3.1.5. *Let $(\mathcal{C}, \oplus, 0)$ be locally homogeneous at (A, X). The following are equivalent:*

1. *$\mathcal{C}$ satisfies (LS2) at (A, X);*
2. *For every n and every edge $f \in W_n(A, X)_1$, $Stab(f) = Stab(d_0 f) \cap Stab(d_1 f)$;*
3. *For every n and p if $f, f' \in W_n(A, X)_p$ have the same ordered set of vertices then $f = f'$.*

Proof. Assume that (3) holds and let us prove (2). Let $f \in W_n(A, X)_1$. Recall that

$$d_0 f = f \circ (\iota_X \oplus X), \quad d_1 f = f \circ (X \oplus \iota_X).$$

By definition of d_0 and d_1 we know that $Stab(f) \subseteq Stab(d_0 f) \cap Stab(d_1 f)$. It remains to prove the other inclusion. Let $\phi \in Stab(d_0 f) \cap Stab(d_1 f)$. Then $\phi \circ f$ has the same vertices as f. By (3) it follows that $\phi \circ f = f$, which proves (2).

Now we suppose (2) holds and we prove (1). Let $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, A \oplus X^{\oplus n-1})$ be such that $f \oplus \iota_X = g \oplus \iota_X$. By (LH1, definition 2.1.6) we may assume that $f = \iota_{A\oplus X^{\oplus n-2}} \oplus X$ and $g = \phi \circ f$ for some $\phi \in \mathrm{Aut}(A \oplus X^{\oplus n-1})$. Then

$$f \oplus \iota_X = (\phi \circ f) \oplus \iota_X = (\phi \oplus X) \circ (f \oplus \iota_X).$$

Therefore

$$\phi \oplus X \in Stab(\iota_{A\oplus X^{\oplus n-2}} \oplus X \oplus \iota_X) \cap Stab(\iota_{A\oplus X^{\oplus n-1}} \oplus X)$$

since the map out of the initial object $\iota_{A\oplus X^{\oplus n-1}}$ is unique, i.e. the diagram

$$\begin{tikzcd} 0 \arrow[r, "\iota_{A\oplus X^{\oplus n-1}}"] \arrow[dr, "\iota_{A\oplus X^{\oplus n-1}}"'] & A \oplus X^{\oplus n-1} \arrow[d, "\phi"] \\ & A \oplus X^{\oplus n-1} \end{tikzcd}$$

commutes. By (2) we conclude that $\phi \oplus X$ stabilizes the edge $\iota_{A\oplus X^{\oplus n-2}} \oplus X^{\oplus 2}$. Then (LH2, definition 2.1.6) implies that $\phi = \phi' \oplus X$. In this case ϕ fixes f so $g = f$.

Finally assume that (1) holds. Let $f, g \in W_n(A,X)_p$ be two p-simplices with the same ordered set of vertices. By (LH1, definition 2.1.6) we may assume that $f = \iota_{A\oplus X^{\oplus n-p-1}} \oplus X^{\oplus p+1}$, and there is $\phi \in \mathrm{Aut}(A \oplus X^{\oplus n})$ such that $\phi \circ f = g$. Recall that f, g have the same vertices, in particular

$$f \circ (\iota_{X^{\oplus p}} \oplus X) = \phi \circ f \circ (\iota_{X^{\oplus p}} \oplus X).$$

Then $\phi \in Stab(\iota_{A\oplus X^{\oplus n-1}} \oplus X)$ and (LH2, definition 2.1.6) yields $\phi = \phi' \oplus X$. Now iterate this process by applying d_i to f and g (i.e. precomposing by $X^{\oplus i} \oplus \iota_X \oplus X^{\oplus p-i}$) from $i = p-1$ until $i = 0$. With this process we find that $\phi = \psi \oplus X^{\oplus p+1}$. Hence

$$g = (\psi \oplus X^{\oplus p+1}) \circ (\iota_{A\oplus X^{\oplus n-p-1}} \oplus X^{\oplus p+1}) = \iota_{A\oplus X^{\oplus n-p-1}} \oplus X^{\oplus p+1} = f$$

again by unicity of the map from the initial object, $\iota_{A\oplus X^{\oplus n-p-1}} = \psi \circ \iota_{A\oplus X^{\oplus n-p-1}}$. This proves (3) as desired. □

Proposition 3.1.6. *Let $(\mathcal{C}, \oplus, 0)$ be locally homogeneous at (A,X). Then $\mathcal{C}$ is locally standard at (A,X) if and only if all simplices of $W_n(A,X)_\bullet$ for all n are determined by their vertices and their vertices are all distinct.*

Proof. Suppose that all simplices of $W_n(A,X)_\bullet$ for all n are determined by their (pairwise distinct) vertices. For $n = 1$ this is exactly condition (LS1, 3.1.3). Then Lemma 3.1.5 implies that (LS2, 3.1.3) holds.

Conversely, if $W_n(A,X)_\bullet$ is locally standard, then Lemma 3.1.5 says that all simplices are determined by their set of vertices. Now we must prove that the vertices of an arbitrary simplex are all distinct. It suffices to verify this pairwise, i.e. on the edges of $W_n(A,X)_\bullet$. By (LH1, 2.1.6) it suffices to prove in on the edge $\iota_A \oplus X^{\oplus 2} \oplus \iota_{X^{\oplus n-2}}$. Then we see that (LS2, 3.1.3) says that this holds if and only if (LS1, 3.1.3) holds. This finishes the proof. □

This last proposition allows us to associate the semi-simplicial sets $W_n(A,X)_\bullet$ to a sequence of simplicial complexes.

Definition 3.1.7. For a pair of objects (A,X) in a monoidal category $(\mathcal{C}, \oplus, 0)$, we define a sequence of simplicial complexes $S_n(A,X)$ specified as follows. The vertex set of $S_n(A,X)$ are the 0-simplices of $W_n(A,X)_\bullet$, i.e. $W_n(A,X)_0$. A set of vertices spans a simplex if there is a simplex of $W_n(A,X)_\bullet$ having them as vertices.

Note that this definition yields a family of maps

$$\pi_p : W_n(A,X)_p \to S_n(A,X)_p, \quad p \in \mathbb{N}$$

that induce on geometric realizations

$$\pi : |W_n(A,X)_\bullet| \to |S_n(A,X)|. \tag{3.1}$$

Definition 3.1.8. Coming back to our semi-simplicial sets and simplicial complexes we distinguish two cases to study their connectivity:

- **(A)**: For any $\{v_0, \ldots, v_p\} \in S_n(A,X)_p$ and any $\lambda \in \Sigma_{p+1}$ we have $(v_{\lambda(0)}, \ldots, v_{\lambda(p)}) \in W_n(A,X)_p$.

- **(B)**: For any $\sigma \in S_n(A,X)_p$ there is a single element in $\pi^{-1}(\{\sigma\})$.

Remark 3.1.9. Observe that if (B, 3.1.8) and (LS, 3.1.3) hold then the map π from eq. (3.1) is a homeomorphism. However, case (A,3.1.8) is more common, it happens whenever the category is symmetric monoidal.

Proposition 3.1.10. *Let $(\mathcal{C}, \oplus, 0)$ be a symmetric monoidal category satisfying (LH, 2.1.6) and (LS, 3.1.3 at (A, X). Then $W_n(A,X)_\bullet$ satisfies condition (A, 3.1.8).*

Proof. Let $\lambda \in \Sigma_{p+1}$ be a permutation of $\{0,\dots,p\}$. The twist of $\mathcal{C}$ defines a morphism $\lambda : X^{\oplus p+1} \to X^{\oplus p+1}$ such that $\lambda \circ i_j = i_{\lambda(j)}$ where

$$i_j = \iota_{X^{\oplus j}} \oplus X \oplus \iota_{X^{\oplus p-j}}$$

is the inclusion $X \to X^{\oplus p+1}$ in the i-th factor. For any $f \in W_n(A,X)_p$ observe that the vertices of $f \circ \lambda$ are the vertices of f permuted by λ. So we can get any permutation of the vertices of f. Local standardness and Proposition 3.1.6 make (A) hold because the pre-image under the projection map eq. (3.1) of any p-simplex of $S_n(A,X)$ is non-empty. □

3.2 Cohen-Macaulay

In the beginning of the 20th century, Macaulay's book [32] made significant contributions to commutative algebra. Later, his work was extended to the so called Cohen-Macaulay rings and modules. Inspired by these developments Stanley [47], Hochster [25] and Reisner [46], among others, studied Cohen-Macaulay spaces. Later on, Quillen [41] worked on a stronger version of Cohen-Macaulay spaces, changing a vanishing homological condition by a connectivity condition. We follow the description of Randal-Williams and Wahl in section 2 from [45] which coincides with Quillen's setup.

Definition 3.2.1. A simplicial complex X is *weakly Cohen-Macaulay* of dimension n if

1. X is $(n-1)$-connected, and
2. for every $p \geq 0$, the link of each p-simplex in X is $(n-p-2)$-connected.

If X only satisfies (2) we call it *locally weakly Cohen-Macaulay* of dimension n.

Proposition 3.2.2. *Let $(\mathcal{C}, \oplus, 0)$ be a monoidal category satisfying (LH, 2.1.6) and (LS, 3.1.3) at (A, X) and assume that each $W_n(A,X)_\bullet$ satisfies condition (A, 3.1.8). If $\sigma \in S_n(A,X)_p$ with $p \leq n-1$ then $Link(\sigma) \cong S_{n-p-1}(A,X)$.*

Proof. (Sketch): Let $\sigma = \{v_0,\dots,v_p\} \in S_n(A,X)_p$ be a p-simplex. Our strategy is to build a map $\alpha : S_{n-p-1}(A,X) \to Link(\sigma)$ and then show it is an isomorphism. Due to local homogeneity (LH1,2.1.6) we may assume that

$$v_i = \iota_{A \oplus X^{\oplus n-p-1+i}} \oplus X \oplus \iota_{A \oplus X^{\oplus p-i}}.$$

Then define α by $\alpha(\{g_0,\dots,g_k\}) = \{\hat{g_0},\dots,\hat{g_k}\}$ where $k = n-p-1$ and

$$\hat{g_i} = (A \oplus X^{\oplus n-p-1}\iota_{X^{\oplus p+1}}) \circ g_i.$$

See Proposition 2.12 from [45] to see the detail on why this map is well-defined, simplicial and an isomorphism. ☐

Corollary 3.2.3. *Under the same assumptions, suppose there are $a, k \geq 1$ such that for all $n \geq 0$, $S_n(A,X)$ is $(\frac{n-a}{k})$-connected. Then each $S_n(A,X)$ is weakly Cohen-Macaulay of dimension $\frac{n-a+k}{k}$.*

Definition 3.2.4. Let Y be a simplicial complex. Its associated *ordered semi-simplicial set* $Y_\bullet^{ord}$ is defined by the same vertex set than Y and its p-simplices are all ordered $(p+1)$-tuples of distinct vertices of Y.

Remark 3.2.5. Under condition (A,3.1.8) and (LS,3.1.3), $W_n(A,X)_\bullet$ is isomorphic to $S_n(A,X)_\bullet^{ord}$.

Now we state and prove the main result of this chapter.

Theorem 3.2.6. *Let $(\mathcal{C},\oplus,0)$ be a monoidal category satisfying (LH, 2.1.6) and (LS, 3.1.3 at (A,X). Suppose $W_n(A,X)_\bullet$ satisfies condition (A, 3.1.8) for all $n \geq 0$. Let $a,k \geq 1$. Then $S_n(A,X)$ is $(\frac{n-a}{k})$-connected for all $n \geq 0$ if and only if $W_n(A,X)_\bullet$ is $(\frac{n-a}{k})$-connected for all $n \geq 0$.*

Proof. Note that Remark 3.2.5 yields that $W_n(A,X)_\bullet$ is isomorphic to $S_n(A,X)_\bullet^{ord}$. A total ordering of vertices $S_n(A,X)$ gives a section $|S_n(A,X)| \to \left|S_n(A,X)_\bullet^{ord}\right|$ of the projection map eq. (3.1). Hence, if $|W_n(A,X)_\bullet|$ is l-connected so is $|S_n(A,X)|$, proving one implication of the theorem.

For the other implication see that Corollary 3.2.3 yields that $S_n(A,X)$ is weakly Cohen-Macaulay of dimension $\frac{n-a+k}{k}$. Since $W_n(A,X)_\bullet = S_n(A,X)_\bullet^{ord}$, we use the fact that $Y_\bullet^{ord}$ is $(n-1)$-connected whenever Y is weakly Cohen-Macaulay of dimension n, see Proposition 2.14 from [45]. Therefore $W_n(A,X)_\bullet$ is $(\frac{n-a}{k})$-connected, finishing the proof. ☐

Example 3.2.7. Recall the categories Σ and FI from example 2.3.5 and example 2.1.8 respectively. We know that $U\Sigma = FI$ is symmetric monoidal and locally standard at all pairs. Then Proposition 3.1.10 yields that for any pair of finite sets (A,B) and any $n \geq 0$, $W_n(A,X)_\bullet$ satisfies condition (A, 3.1.8). Now Theorem 3.2.6 says that the semi-simplicial sets $W_n(A,X)_\bullet$ are $(\frac{n-a}{k})$-connected if and only if the simplicial complexes $S_n(A,X)$ are $(\frac{n-a}{k})$-connected.

Now we focus on a particular case: $A = \emptyset$ and $X = \{*\}$. Denote $S_n = S_n(\emptyset,\{*\})$ and observe that its vertices can be identified with $\{1,\dots,n\}$ and that any subset of vertices corresponds to an inclusion $[k] \to [n]$ for $k \leq n$, so it defines a simplex. Hence S_n can be identified with Δ^{n-1}, which is contractible for $n \geq 2$, so at least $(n-2)$-connected for all $n \geq 0$. By Theorem 3.2.6 we conclude that $W_n(\emptyset,\{*\})_\bullet$ is also $(n-2)$-connected.

Finally recall from example 2.2.1 that

$$\mathrm{Aut}(\emptyset \sqcup \{*\}^{\sqcup n}) = \Sigma_n$$

so we have build up the setting for homological stability for symmetric groups.

Remark 3.2.8. The semi-simplicial complexes $W_n(\emptyset,\{*\})_\bullet$ coincide with the semi-simplicial complexes used by Putman in [40] to prove homological stability for symmetric groups.

Chapter 4

Stability theorems

Using the framework established in chapter 2 and chapter 3 we present the stability machinery from sections 3 and 4 from [45]. First we will restrict ourselves to the constant and abelian coefficient case. Then, we will go into the details of the twisted coefficient case.

4.1 Constant and abelian coefficients

We follow Randal-Williams and Wahl's approach that is based on Quillen's argument from his unpublished notes [42] on the stability problem of general linear groups. First let us state the classical homological stability result with constant coefficients.

Theorem 4.1.1. *Let $(\mathcal{C}, \oplus, 0)$ be a pre-braided category which is locally homogeneous at a pair (A, X). Suppose that $\mathcal{C}$ satisfies (LH3) at (A, X) with slope $k \geq 2$. Then the map*

$$H_i(\mathrm{Aut}(A \oplus X^{\oplus n}); \mathbb{Z}) \to H_i(\mathrm{Aut}(A \oplus X^{\oplus n+1}); \mathbb{Z})$$

is an epimorphism if $i \leq \frac{n}{k}$ and an isomorphism if $i \leq \frac{n-1}{k}$.

There are various ways to generalize this statement for other coefficients. In Section 4.2 we generalize this result to a much broader kind of coefficients, but for now we focus on so called abelian coefficients.

Definition 4.1.2. Let $(\mathcal{C}, \oplus, 0)$ be a pre-braided category which is locally homogeneous at a pair (A, X). The *stable group* of the pair (A, X) is the colimit of the diagram

$$\cdots \xrightarrow{-\oplus X} \mathrm{Aut}(A \oplus X^{\oplus n}) \xrightarrow{-\oplus X} \mathrm{Aut}(A \oplus X^{\oplus n+1}) \xrightarrow{-\oplus X} \mathrm{Aut}(A \oplus X^{\oplus n+2}) \xrightarrow{-\oplus X} \cdots$$

and it is denoted by $\mathrm{Aut}(A \oplus X^{\oplus \infty})$.

Definition 4.1.3. We say that an $\mathrm{Aut}(A \oplus X^{\oplus \infty})$-module M is *abelian* if the action of $\mathrm{Aut}(A \oplus X^{\oplus \infty})$ on M factors through the abelianisation of $\mathrm{Aut}(A \oplus X^{\oplus \infty})$.

Observe that an $\mathrm{Aut}(A \oplus X^{\oplus\infty})$-module M is also an $\mathrm{Aut}(A \oplus X^{\oplus n})$-module for any n by restriction.

Now we state Theorem 4.1.1 adapted to abelian coefficients, meaning that the coefficient is an abelian $\mathrm{Aut}(A \oplus X^{\oplus\infty})$-module.

Theorem 4.1.4. *Let $(\mathcal{C}, \oplus, 0)$ be a pre-braided category which is locally homogeneous at a pair (A, X). Suppose that $\mathcal{C}$ satisfies (LH3) at (A, X) with slope $k \geq 3$. Then for any abelian $\mathrm{Aut}(A \oplus X^{\oplus\infty})$-module M the map*

$$H_i(\mathrm{Aut}(A \oplus X^{\oplus n}); M) \to H_i(\mathrm{Aut}(A \oplus X^{\oplus n+1}); M)$$

is an epimorphism if $i \leq \frac{n-k+2}{k}$ and an isomorphism if $i \leq \frac{n-k}{k}$.

Proof of Theorems 4.1.1 and 4.1.4. Let us write for short $G_n = \mathrm{Aut}(A \oplus X^{\oplus n})$ and $W_n = W_n(A, X)_\bullet$, and let M be an abelian $\mathrm{Aut}(A \oplus X^{\oplus\infty})$-module. Fix $E_\bullet G_{n+1}$ a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[G_{n+1}]$, and write $\tilde{C}_*(W_{n+1})$ for the augmented cellular chain complex of W_{n+1}. Consider the double complex

$$E_\bullet G_{n+1} \otimes_{G_{n+1}} (\tilde{C}_*(W_{n+1}) \otimes_{\mathbb{Z}} M) \tag{4.1}$$

where G_{n+1} acts diagonally on $\tilde{C}_*(W_{n+1}) \otimes_{\mathbb{Z}} M$. We clarify that there are no differences between $\bullet$ and $*$, we use different notation to distinguish more easily the two filtrations associated with this double complex. Think about $\bullet$ as horizontal and $*$ as vertical. Each filtration yields a spectral sequence converging to the totalization of eq. (4.1), i.e.

$$H_*(\mathrm{Tot}(E_\bullet G_{n+1} \otimes_{G_{n+1}} (\tilde{C}_*(W_{n+1}) \otimes_{\mathbb{Z}} M))).$$

Consider the spectral sequence obtained from the double complex eq. (4.1) by first taking the homology on $\tilde{C}_*(W_{n+1}) \otimes_{\mathbb{Z}} M$. Our (LH3, 3.1.2) assumption and the Universal Coefficient theorem yield that

$$H_i(\tilde{C}_*(W_{n+1}) \otimes_{\mathbb{Z}} M) = 0 \quad \forall i \leq \frac{n-1}{k}.$$

Therefore both spectral sequences coming from eq. (4.1) converge to zero in degrees below $\frac{n-1}{k}$. More explicitly, the E^1-page of this vertical spectral sequence is already zero below $\frac{n-1}{k}$ and hence

$$H_i(\mathrm{Tot}(E_\bullet G_{n+1} \otimes_{G_{n+1}} (\tilde{C}_*(W_{n+1}) \otimes_{\mathbb{Z}} M))) = 0 \quad \forall i \leq \frac{n-1}{k}. \tag{4.2}$$

Now consider the spectral sequence coming from the other filtration, i.e. taking homology first on $E_\bullet G_{n+1}$. We claim that its first page is

$$E^1_{p,q} = H_q(G_{n-p}; M) \tag{4.3}$$

for $-1 \leq p \leq n$.

Observe that (LH1, 2.1.6) implies that the G_{n+1}-action on W_{n+1} is transitive on p-simplices for all $p \leq n$. Now define

$$\sigma_p = \iota_{A \oplus X^{\oplus n-p}} \oplus X^{\oplus p+1} : X^{\oplus p+1} \to A \oplus X^{\oplus n} \tag{4.4}$$

for all $p \leq n$, the standard p-simplex of W_{n+1}. Then (LH2, 2.1.6) implies that the map $- \oplus X^{\oplus p+1} : G_{n-p} \to G_{n+1}$ is an isomorphism onto the stabilizer of σ_p. Hence $W_{n+1} \cong G_{n+1}/G_{n-p}$ and this isomorphism is compatible with the G_{n+1}-action. Therefore

$$\tilde{C}_p(W_{n+1}) \cong \mathbb{Z}[G_{n+1}/G_{n-p}]$$

as $\mathbb{Z}[G_{n+1}]$-modules. By Shapiro's lemma (Proposition III.6.2 in [5]) we conclude that eq. (4.3) is true. For more details on this the reader can refer to Section 3.2 in [6]. We illustrate the first page of this spectral sequence in fig. 4.1. Our goal is to find a range for which the map

$$d^1_{0,i} : E^1_{0,i} = H_i(G_n; M) \to E^1_{-1,i} = H_i(G_{n+1}; M) \tag{4.5}$$

is an isomorphism.

$$\begin{tikzcd}
0 & H_i(G_{n+1};M) \arrow[l] & H_i(G_n;M) \arrow[l, "d^1_{0,i}"'] & H_i(G_{n-1};M) \arrow[l, "d^1_{1,i}"'] & H_i(G_{n-2};M) \arrow[l] & \cdots \arrow[l] \\
0 & H_{i-1}(G_{n+1};M) \arrow[l] & H_{i-1}(G_n;M) \arrow[l] & H_{i-1}(G_{n-1};M) \arrow[l] & H_{i-1}(G_{n-2};M) \arrow[l] & \cdots \arrow[l] \\
0 & H_{i-2}(G_{n+1};M) \arrow[l] & H_{i-2}(G_n;M) \arrow[l] & H_{i-2}(G_{n-1};M) \arrow[l] & H_{i-2}(G_{n-2};M) \arrow[l] & \cdots \arrow[l] \\
 & \vdots & \vdots & \vdots & \vdots & \\
0 & H_0(G_{n+1};M) \arrow[l] & H_0(G_n;M) \arrow[l] & H_0(G_{n-1};M) \arrow[l] & H_0(G_{n-2};M) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Figure 4.1: E^1-page

To do this we must understand where the differential $d^1 : E^1_{p,q} \to E^1_{p-1,q}$ does come from. It is induced by the differential of the cellular chain complex $\tilde{C}_*(W_{n+1})$, so it is the alternating sum of maps

$$H_q(Stab(\sigma_p); M) \xrightarrow{(inc, id_M)_*} H_q(Stab(d_i\sigma_p); M) \xrightarrow{c_{h_i}} H_q(Stab(\sigma_{p-1}); M)$$

where c_{h_i} is induced in homology by

$$(g, m) \mapsto (h_i g h_i^{-1}, h_i m) : (Stab(d_i\sigma_p), M) \to (Stab(\sigma_{p-1}), M)$$

for any element $h_i \in G_{n+1}$ taking $d_i\sigma_p$ to σ_{p-1}. All along this proof we denote by *inc* the inclusion of stabilizers.

Notice that we can pick h_i where $h_i \circ d_i\sigma_p = \sigma_{p-1}$ such that h_i centralizes $Stab(\sigma_p)$. Indeed, let

$$h_i = A \oplus X^{\oplus n-p} \oplus b_{X^{\oplus i}, X} \oplus X^{\oplus p-i} \in G_{n+1}$$

where b is the underlying groupoid braiding. Then

$$\begin{aligned} h_i \circ d_i\sigma_p &= (A \oplus X^{\oplus n-p} \oplus b_{X^{\oplus i}, X} \oplus X^{\oplus p-i}) \circ (\iota_{A \oplus X^{\oplus n-p}} \oplus X^{\oplus i} \oplus \iota_X \oplus X^{\oplus p-i}) \\ &= \iota_{A \oplus X^{\oplus n-p}} \oplus \iota_X \oplus X^{\oplus i} \oplus X^{\oplus p-i}) = \sigma_{p-1} \end{aligned}$$

as desired. Moreover h_i commutes with $Stab(\sigma_p)$ because h_i acts trivially on $A \oplus X^{\oplus n-p}$. Under this choice of h_i for all i we get

$$c_{h_i} \circ (inc, id_M) : (Stab(\sigma_p), M) \to (Stab(\sigma_{p-1}), M); (g, m) \mapsto (g, h_i m)$$

and then

$$d^1 = \sum_{i=0}^{p} (-1)^i (inc, h_i \cdot -) : H_*(Stab(\sigma_p); M) \to H_*(Stab(\sigma_{p-1}); M). \tag{4.6}$$

Observe that if all h_i act trivially on M then the differential is zero whenever p is odd, and it is identified with the stabilization map when p is even. So far all we have discussed applies for both trivial and abelian modules.

To prove that $d^1_{0,i} : H_i(G_n; M) \to H_i(G_{n+1}; M)$ is an isomorphism we proceed by strong induction on i and prove separately surjectivity and injectivity. We must distinguish the two cases: constant coefficients and abelian coefficients. Clearly constant coefficients are also abelian coefficients, however the proof for constant coefficients is needed for the proof with abelian coefficients.

Constant coefficients: Observe that for $i = 0$ the differential $d^1_{0,0} : H_0(G_n; \mathbb{Z}) \to H_0(G_{n+1}; \mathbb{Z})$ is always an isomorphism because the 0-homology is trivial, this initializes our induction. Our goal is to prove the following two statements: (E_I) the map $d^1_{0,i}$ is an epimorphism for all $i \leq I$ and all n such that $i \leq \frac{n}{k}$; and (I_I) the map $d^1_{0,i}$ is an isomorphism for all $i \leq I$ and all n such that $i \leq \frac{n-1}{k}$.

The induction hypothesis is to assume that (E_{I-1}) and (I_{I-1}) hold. Now we prove that (E_I) and (I_I) hold. Notice that (E_I) follows from the statements:

- (E_I1): $E^\infty_{-1,i} = 0$;
- (E_I2): $E^2_{p,q} = 0$ for $p + q = i$ and $q < i$.

Statement (E_I1) makes sure that $E^1_{-1,i}$ is in the vanishing range, so the differentials $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ with $p + q = i$ must kill the term $E^1_{-1,i}$ before getting to the infinity page. In fig. 4.1 we can view these differentials as the dashed arrows with target $E^1_{-1,i}$. The blue arrow comes from the second page, i.e. $r = 2$, and the red one from the third page, i.e. $r = 3$. Statement (E_I2) says that the domain of all these differentials $d^r_{p,q}$ with $p + q = i$ is zero except for $r = 1$. Hence $d^1_{0,i}$ is the only differential that can kill $E^1_{-1,i}$ so it must be surjective.

To prove (I_I) it is enough to prove the following:

- (I_I1): $E^\infty_{0,i} = 0$;
- (I_I2): $E^2_{p,q} = 0$ for $p + q = i + 1$ and $q < i$;
- (I_I3): The map $d^1_{1,i} : H_i(G_{n-1}; M) \to H_i(G_n; M)$ is the zero map.

Just as above for surjectivity, statement (I_I1) makes sure that $E^1_{0,i}$ is killed before the infinity page by a differential $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ with $p+q = i+1$, which are represented by dashed arrows with target $E^1_{0,i}$ in fig. 4.1. Then (I_I2) says that for all $r \geq 2$ $d^r_{p,q}$ is the zero map for $p+q = i+1$.

Statement (I_I3) says that also $d^1_{1,i}$ is zero so the only differential that can kill $E^1_{0,i}$ is $d^1_{0,i}$ by being injective, proving (I_I).

Now let us prove these five statements. First observe that (E_I1) and (I_I1) hold because $E^\infty_{-1,i}$ and $E^\infty_{0,j}$ for $i \leq \frac{n}{k}$ and $j \leq \frac{n-1}{k}$ are in the vanishing range of the spectral sequence eq. (4.2).

The proof for (E_I2) and (I_I2) is similar, only the ranges are different. We prove only (E_I2). First let us show that the map induced by the inclusion of stabilizers

$$E^1_{p',q} = H_q(G_{n-p'};\mathbb{Z}) \to H_q(G_{n+1};\mathbb{Z})$$

is an isomorphism for $p' \leq p$ and a surjection for $p' = p+1$. This map is the composition

$$H_q(G_{n-p'};\mathbb{Z}) \longrightarrow H_q(G_{n-p'+1};\mathbb{Z}) \longrightarrow H_q(G_{n-p'+2};\mathbb{Z}) \longrightarrow \cdots \longrightarrow H_q(G_{n+1};\mathbb{Z})$$

of stabilisation maps. Now (E_{I-1}) and (I_{I-1}) imply all of these maps are surjective for $n - p' \geq qk$ and isomorphisms if $n - p' \geq qk + 1$. These conditions hold because $p + q \leq i \leq \frac{n}{k}$. Consider the diagram

$$\begin{tikzcd}
\cdots & H_q(Stab(\sigma_{p-1});\mathbb{Z}) \arrow[l] \arrow[d, "f_{p-1}"] & H_q(Stab(\sigma_p);\mathbb{Z}) \arrow[l, "d^1"'] \arrow[d, "f_p"] & H_q(Stab(\sigma_{p+1});\mathbb{Z}) \arrow[l, "d^1"'] \arrow[d, "f_{p+1}"] & \cdots \arrow[l] \\
\cdots & H_q(G_{n+1};\mathbb{Z}) \arrow[l] & H_q(G_{n+1};\mathbb{Z}) \arrow[l] & H_q(G_{n+1};\mathbb{Z}) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

where the maps on the top row are the differentials d^1 and on the bottom row we alternate 0 and id. By eq. (4.6) we know that the maps on top row alternate between the zero map and the stabilization maps. By the above we know that f_{p-1} and f_p are isomorphisms and f_{p+1} is surjective when $p + q \leq i$. If the first condition holds then the second one holds as well because $p \geq 1$ and $k \geq 2$. Then the exactness of the bottom row implies the exactness of the top row in this range. This shows that

$$E^2_{p,q} = ker(d^1_{p,q})/im(d^1_{p+1,q}) = 0$$

for $p + q \leq i$ as desired.

Finally recall the description of d^1 given in eq. (4.6). Then

$$d^1_{1,i} = (-1)^0(inc, h_0 \cdot -) + (-1)^1(inc, h_1 \cdot -) = 0$$

because G_{n+1} acts trivially on $\mathbb{Z}$. So (I_I3) holds, which concludes the prooffor trivial coefficients.

Abelian coefficients: Here we must modify the ranges from our statements for constant coefficients, see the statements in Theorems 4.1.1 and 4.1.4. Consider

- (E_I) the map $d^1_{0,i}$ is an epimorphism for all $i \leq I$ and all n such that $i \leq \frac{n-k+2}{k}$;
- (I_I) the map $d^1_{0,i}$ is an isomorphism for all $i \leq I$ and all n such that $i \leq \frac{n-k}{k}$.

The only difference between these conditions and the ones for trivial coefficients are the bounds on i. Now assume (E_{I-1}) and (I_{I-1}). We prove (E_I) and (I_I) by showing that (E_I1), (E_I2), (I_I1), (I_I2) and (I_I3) hold.

Let us start initializing our induction. For $i = 0$ the differential

$$d^1_{0,0} : H_0(G_n; M) = M_{G_n} = M_{H_1(G_n;\mathbb{Z})} \to H_0(G_{n+1}; M) = M_{G_{n+1}} = M_{H_1(G_{n+1};\mathbb{Z})}$$

is always surjective. By Theorem 4.1.1 we know that $H_1(G_n; \mathbb{Z}) \cong H_1(G_\infty; \mathbb{Z})$ for $n \geq k$, which makes $d^1_{0,0}$ an isomorphisms in that range. This proves (E_0) and (I_0).

The strategy to prove the induction step is the same as above except for $(I_I 3)$. First observe that $(E_I 1)$ and $(I_I 1)$ hold because $E^\infty_{-1,i}$ and $E^\infty_{0,j}$ are in the vanishing range of the spectral sequence eq. (4.2) for $i - 1 \leq \frac{n-k+2}{k} - 1 \leq \frac{n-1}{k}$ and $j \leq \frac{n-k}{k} \leq \frac{n-1}{k}$, assuming $k \geq 2$.

To prove (E_2) we proceed as for constant coefficients. Using that conjugation by h_i on $H_i(G_i; M)$ is the identity we get the following commutative diagram

$$\begin{array}{ccccccccc}
\cdots \longleftarrow & H_q(Stab(\sigma_{p-1}); M) & \xleftarrow{d^1} & H_q(Stab(\sigma_p); M) & \xleftarrow{d^1} & H_q(Stab(\sigma_{p+1}); M) & \longleftarrow \cdots \\
 & \big\downarrow{\scriptstyle f_{p-1}} & & \big\downarrow{\scriptstyle f_p} & & \big\downarrow{\scriptstyle f_{p+1}} & \\
\cdots \longleftarrow & H_q(G_{n+1}; M) & \longleftarrow & H_q(G_{n+1}; M) & \longleftarrow & H_q(G_{n+1}; M) & \longleftarrow \cdots
\end{array}$$

where the maps in the top row are differentials, and on the bottom row we alternate between zero and identity. By (I_{I-1}) and (E_{I-1}) we know that f_{p-1} and f_p are isomorphisms, and f_{p+1} is surjective as long as $n - p - 1 \geq k + kq$. Our assumption yields that $n \geq k - 2 + kp + kq$ so $n - p \geq k - 2 + (k-1)p + kq$. Since $p \geq 1$ and we assumed that $k \geq 3$ (only for abelian coefficients) we get that $(k-1)p - 2 \geq 0$. Hence the condition is satisfied. It follows that the upper row is exact and then $E^2_{p,q} = 0$. Condition (I_2) is proved exactly as (E_2), only the ranges change.

Finally we prove that (I_3) holds. Recall eq. (4.6) so

$$d^1_{1,i} = (-1)^0(inc, id_M -) + (-1)^1(inc, h_1 \cdot -) = d_0 - d_1.$$

Here we cannot conclude as above because h_1 may not act trivially on M.

Now let $g = A \oplus X^{\oplus n-2} \oplus b^{-1}_{X,X} \oplus X \in Fix(\sigma_0)$. Recall that h_1 centralizes $Stab(\sigma_1)$ and sends $d_1\sigma_1$ to σ_0. Then

$$\sigma_0 = g \circ \sigma_0 = g \circ h_1 \circ d_1\sigma_1.$$

So g does not centralize $Stab(\sigma_1)$ but it centralizes $Stab(\sigma_2)$, as does $g \circ h_1$. Now observe that the braid relation

$$(b_{X,X} \oplus X) \circ (X \oplus b_{X,X}) \circ (b_{X,X} \oplus X) = (X \oplus b_{X,X}) \circ (b_{X,X} \oplus X) \circ (X \oplus b_{X,X})$$

abelianises to

$$[X \oplus b_{X,X}] = [b_{X,X} \oplus X] \in H_1(\mathrm{Aut}(X^{\oplus 3}); \mathbb{Z})$$

so it follows that $[g \circ h_1] = 0 \in H_1(G_{n+1}; \mathbb{Z})$. Therefore the two compositions

$$H_i(Stab(\sigma_2); M) \xrightarrow{(inc, id_M)_*} H_i(Stab(\sigma_1); M) \underset{c_{g \circ h_1} \circ (inc, id_M)_*}{\overset{(inc, id_M)_*}{\rightrightarrows}} H_i(Stab(\sigma_0); M)$$

are equal because $g \circ h_1$ acts trivially on M. Statement (E_I) implies that the left-hand side map is surjective for $i \leq \frac{n-k}{k}$. Hence the two right-hand side maps are equal, so their difference $d^1_{1,i}$ is zero. This proves $(I_I 3)$ which concludes the proof of the theorems. $\square$

Corollary 4.1.5. *Let $a \geq 1$ be an integer. In the setup of Theorem 4.1.4 with $A = X^{\oplus a}$ the action of $\mathrm{Aut}(X^{\oplus \infty})$ on $H_i(\mathrm{Aut}(X^{\oplus a+n}); M)$ is trivial in degrees $i \leq \frac{n-k}{k}$.*

4.2 Coefficient systems

From now on, denote by $\mathcal{C}_{A,X}$ the full subcategory of $\mathcal{C}$ whose objects are $A \oplus X^{\oplus n}$ for all $n \in \mathbb{N}$. We will always write $G_n = \mathrm{Aut}(A \oplus X^{\oplus n})$ and $G_\infty = \mathrm{Aut}(A \oplus X^{\oplus \infty})$ the colimit of the sequence $\{G_n\}_{n\in\mathbb{N}}$.

So far we have proven a homological stability result for coefficients M that are fixed for all groups G_n. Now we would like to use more general coefficients M_n for G_n and study maps $H_i(G_n; M_n) \to H_i(G_{n+1}; M_{n+1})$. Let us describe a framework where coefficients of this type are well defined and allow us to get some interesting results.

Definition 4.2.1. We call the functor

$$\Sigma^X = - \oplus X : \mathcal{C}_{A,X} \to \mathcal{C}_{A,X}$$

the *upper suspension functor*. And we define the *upper suspension map* by

$$\sigma^X = A \oplus X^{\oplus n} \oplus \iota_X : A \oplus X^{\oplus n} \to A \oplus X^{\oplus n+1}.$$

Notice that this map defines a natural transformation $\sigma^X : Id \to \Sigma^X$.

Definition 4.2.2. A *coefficient system* for $\mathcal{C}$ at (A, X) is a functor

$$F : \mathcal{C}_{A,X} \to \mathcal{A}$$

where $\mathcal{A}$ is an abelian category.

Remark 4.2.3. Notice that F associates to the sequence $\{G_n\}$ a sequence $\{F_n\}$ such that F_n is a G_n-module with G_n-equivariant maps $F_n \to F_{n+1}$ induced by the upper suspension functor Σ^X.

Definition 4.2.4. The *lower suspension* by X is the map

$$\sigma_X = (b_{X,A} \oplus X^{\oplus n}) \circ (\iota_X \oplus A \oplus X^{\oplus n}) : A \oplus X^{\oplus n} \to A \oplus X^{\oplus n+1}.$$

Now we can define a *lower suspension functor* as we did before for the upper suspension. Define $\Sigma_X : \mathcal{C}_{A,X} \to \mathcal{C}_{A,X}$ by

$$\begin{aligned} \Sigma_X : \mathrm{Hom}(A \oplus X^{\oplus n}, A \oplus X^{\oplus k}) &\to \mathrm{Hom}(A \oplus X^{\oplus n+1}, A \oplus X^{\oplus k+1}) \\ f &\mapsto (A \oplus b_{X,X^{\oplus k}}) \circ (X \oplus f) \circ (A \oplus b_{X,X^{\oplus n}}). \end{aligned}$$

Note that the braid relation in $\mathcal{C}$ yields

$$\sigma_X = (b_{X,A} \oplus X^{\oplus n}) \circ (b_{A \oplus X^{\oplus n},X}) \circ \sigma^X$$

so we may write

$$\Sigma_X(f) = (b_{A,X} \oplus X^{\oplus k}) \circ \Sigma^X(f) \circ (b^{-1}_{X,A} \oplus X^{\oplus n}).$$

As before we can see that $\sigma_X : Id \to \Sigma_X$ is a natural transformation.

Remark 4.2.5. Observe that these two suspension functors commute and together with the natural transformations fit into the diagram

$$\begin{tikzcd} id \arrow[r, "\sigma_X"] \arrow[d, "\sigma^X"'] & \Sigma_X \arrow[d, "\sigma^X(\Sigma_X)"] \\ \Sigma^X \arrow[r, "\sigma_X(\Sigma^X)"'] & \Sigma^X\Sigma_X = \Sigma_X\Sigma^X \end{tikzcd}$$

which commutes. Indeed for $f \in \mathrm{Hom}(A \oplus X^{\oplus n}, A \oplus X^{\oplus k})$ we have

$$\begin{aligned} \Sigma_X\Sigma^X(f) &= (b_{A,X} \oplus X^{\oplus k+1}) \circ \Sigma^X(\Sigma^X(f)) \circ (b_{X,A}^{-1} \oplus X^{\oplus n+1}) \\ &= (b_{A,X} \oplus X^{\oplus k+1}) \circ (f \oplus X^{\oplus 2}) \circ (b_{X,A}^{-1} \oplus X^{\oplus n+1}) \\ &= [(b_{A,X} \oplus X^{\oplus k}) \circ (f \oplus X) \circ (b_{X,A}^{-1} \oplus X^{\oplus n})] \oplus X \\ &= \Sigma^X((b_{A,X} \oplus X^{\oplus k}) \circ \Sigma^X(f) \circ (b_{X,A}^{-1} \oplus X^{\oplus n})) \\ &= \Sigma^X\Sigma_X(f) \end{aligned}$$

and

$$\begin{aligned} \sigma^X(\Sigma_X) \circ \sigma_X &= \sigma^X(\Sigma_X) \circ (b_{X,A} \oplus X^{\oplus n}) \circ (b_{A\oplus X^{\oplus n},X}) \circ \sigma^X \\ &= (A \oplus X^{\oplus n+1} \oplus \iota_X) \circ (b_{X,A} \oplus X^{\oplus n}) \circ (b_{A\oplus X^{\oplus n},X}) \circ (A \oplus X^{\oplus n} \oplus \iota_X) \\ &= (b_{X,A} \oplus X^{\oplus n} \oplus \iota_X) \circ (b_{A\oplus X^{\oplus n},X}) \circ (A \oplus X^{\oplus n} \oplus \iota_X) \\ &= (b_{X,A} \oplus X^{\oplus n+1}) \circ (b_{A\oplus X^{\oplus n},X}) \oplus \iota_X \circ (A \oplus X^{\oplus n} \oplus \iota_X) \\ &= (b_{X,A} \oplus X^{\oplus n+1}) \circ (b_{A\oplus X^{\oplus n+1},X}) \circ (A \oplus X^{\oplus n+1} \oplus \iota_X) \\ &= (b_{X,A} \oplus X^{\oplus n+1}) \circ (b_{A\oplus X^{\oplus n+1},X}) \circ (A \oplus X^{\oplus n+1} \oplus \iota_X) \circ (A \oplus X^{\oplus n} \oplus \iota_X) \\ &= \sigma_X(\Sigma^X) \circ \sigma^X. \end{aligned}$$

Now we are ready to define the suspension functor Σ.

Definition 4.2.6. Let $F : \mathcal{C}_{A,X} \to \mathcal{A}$ be a coefficient system. Then its suspension is

$$\Sigma F = F \circ \Sigma_X.$$

Definition 4.2.7. Let $F : \mathcal{C}_{A,X} \to \mathcal{A}$ be a coefficient system. Then define $\ker F$ and $\operatorname{coker} F$ to be the kernel and cokernel respectively of the natural transformation

$$F(\sigma_X) : F \to \Sigma F.$$

The *kernel* and *cokernel* of a coefficent system are again cooefficient systems.

Definition 4.2.8. A *coefficient system of G_∞^{ab}-modules* is a coefficient system

$$F : \mathcal{C}_{A,X} \to G_\infty^{ab}\text{-}\operatorname{Mod}.$$

That is, the G_n-action on F_n factorizes through G_n^{ab} and it has a G_∞^{ab}-action coming from the upper suspension $\Sigma_\infty^X : G_n \to G_\infty$.

Definition 4.2.9. Let $F : \mathcal{C}_{A,X} \to G_\infty^{ab}\text{-}\operatorname{Mod}$ be a coefficient system of G_∞^{ab}-modules. We can endow F_n with a new G_n-action described by

$$G_n \times F_n \to F_n : (g, x) \to g \cdot (\Sigma_\infty^X(g) * x)$$

where $\cdot$ is the G_n-action coming from F, and $*$ is the G_∞^{ab}-action induced by Σ_∞^X. We say that F_n equipped with this action is the *internalized* G_n-module of F_n and we denote it by F_n°.

Definition 4.2.10. A coefficient system $F : \mathcal{C}_{A,X} \to \mathcal{A}$ has *degree* $r < 0$ at $N \in \mathbb{Z}$ with respect to X if $F_n = 0$ for all $n \geq N$.

Now we define it inductively for $r \geq 0$. We say F has degree $r \geq 0$ at $N \in \mathbb{Z}$ if

1. $\ker F$ has degree -1 at N; and
2. $\operatorname{coker} F$ has degree $r - 1$ at $N - 1$.

Furthermore F is *split* if

1. $F \to \Sigma F$ is split injective in the category of coefficient systems; and
2. $\operatorname{coker} F$ is split of degree $r - 1$ at $N - 1$.

where all coefficient systems of degree $r < 0$ are split by convention.

Lemma 4.2.11. *Let $F : \mathcal{C}_{A,X} \to \mathcal{A}$ be a coefficient system of degree r at N, then for any $j \geq 1$, $\Sigma^j F$ has degree r at $(N - j)$.*

Example 4.2.12. In past examples 2.1.8 and 2.2.1 we have built the homogeneous category for the homological stability of symmetric groups. Here we look at the coefficient systems. A functor

$$F : FI \to R\text{-}\operatorname{Mod}$$

is a coefficient system for the symmetric groups and it is called an *FI-module.* These FI-modules were introduced by Church, Ellenberg and Farb in [8].

Around 1980, van der Kallen was working on the homology of linear groups [28]. He realized that some results could be generalized to relative homology, so he defined the following setup to formalize the relative version of his results. This setup was later used by Randal-Williams and Wahl [45] to prove their twisted coefficient theorem on homological stability.

Definition 4.2.13. Let $\mathcal{R}ep$ be the category whose objects are pairs (G, M) where G is a group and M is a G-module. Morphisms in $\mathcal{R}ep$ are pairs (ϕ, f) where ϕ is a group morphism and f is a ϕ-linear map. Then we define $\mathcal{R}el\mathcal{R}ep$ to be the arrow category of $\mathcal{R}ep$: its objects are the morphisms of $\mathcal{R}ep$, and its morphisms are commuting squares in $\mathcal{R}ep$.

Definition 4.2.14. Consider $(\phi, f) : (G, M) \to (G', M')$ a morphism in $\mathcal{R}ep$, and let $P_\bullet$ and $P'_\bullet$ be projective resolutions of M and M' respectively, seen as G- and G'-modules. There is a ϕ-linear map of resolutions $P_\bullet \to P'_\bullet$ covering f. Define the *relative homology groups* $H_*(G', G; M', M)$ by the homology of the mapping cone of the chain map

$$\mathbb{Z} \otimes_{\mathbb{Z}G} P_\bullet \to \mathbb{Z} \otimes_{\mathbb{Z}G'} P'_\bullet.$$

Note that this fits into a long exact sequence

$$\cdots \to H_i(G;M) \to H_i(G';M') \to H_i(G',G;M',M) \to H_{i-1}(G;M) \to \cdots \tag{4.7}$$

and defines a functor

$$H_i(-) : \mathcal{R}el\mathcal{R}ep \to \mathrm{Mod}_{\mathbb{Z}}\,.$$

Recall that we are focused on $H_*(G;F_n)$ where F_n comes from a coefficient system $F : \mathcal{C}_{A,X} \to \mathcal{A}$. Using the definition of relative homology, we obtain the *relative groups associated to* Σ^X and σ^X setting

$$Rel_*^F(A,n) = H_*(G_{n+1}, G_n; F_{n+1}, F_n).$$

Similarly, if $F : \mathcal{C}_{A,X} \to \mathbb{Z}[G_\infty^{ab}]$-Mod we can do an analogous construction for the internalized coefficient system F°:

$$Rel_*^{F^\circ}(A,n) = H_*(G_{n+1}, G_n; F^\circ_{n+1}, F^\circ_n).$$

Let F be an arbitrary coefficient system, Remark 4.2.5 yields two commuting squares

$$\begin{array}{ccc} G_n & \xrightarrow{\Sigma_X} & G_{n+1} \\ {\scriptstyle \Sigma^X}\downarrow & & \downarrow{\scriptstyle \Sigma^X} \\ G_{n+1} & \xrightarrow[\Sigma_X]{} & G_{n+2} \end{array} \qquad\qquad \begin{array}{ccc} F_n & \xrightarrow{F(\sigma_X)} & F_{n+1} \\ {\scriptstyle F(\sigma^X)}\downarrow & & \downarrow{\scriptstyle F(\sigma^X)} \\ F_{n+1} & \xrightarrow[F(\sigma_X)]{} & F_{n+2} \end{array}$$

for any $n \geq 0$. This defines in fact a morphism in $\mathcal{R}el\mathcal{R}ep$. So applying the functor H_i we obtain a map in relative homology

$$s_n = (\Sigma_X, F(\sigma_X)) : Rel_i^F(A,n) \to Rel_i^F(A,n+1)$$

and we also get a map in relative homology for the internalized coefficient system

$$s_n : Rel_i^{F^\circ}(A,n) \to Rel_i^{F^\circ}(A,n+1).$$

Proposition 4.2.15. *The composition*

$$Rel_i^F(A,n) \xrightarrow{(id,F(\sigma_X))} Rel_i^{\Sigma F}(A,n) \xrightarrow{(\Sigma_X,id)} Rel_i^F(A,n+1)$$

is a factorization of s_n. And there is a similar factorization for the map on relative homology groups of internalized coefficients $Rel_i^{F^\circ}(A,n)$.

Proposition 4.2.16. *For any coefficient system F the maps $s_n : Rel_i^F(A,n) \to Rel_i^F(A,n+1)$ define a chain complex $(Rel_i^F(A,n), s_n)_{n\geq 0}$. Moreover, if $F : \mathcal{C}_{A,X} \to G_\infty^{ab}$-Mod we have a chain complex on the relative homology groups with internalized modules F°, i.e. $(Rel_i^{F^\circ}(A,n), s_n)_{n\geq 0}$.*

4.3 Twisted coefficients

Now that we have proved our results for constant and abelian coefficients, and we have defined coefficient systems, we are ready to prove similar statements for twisted coefficients. Dwyer [11] and van der Kallen [28] were among the first to prove homological stability with twisted coefficients for general linear groups. Around the same time, 1980, Quillen also worked on this problem in his unpublished notes [42]. Later, Ivanov worked on homological stability for Teichmüller modular groups in [26].

The main theorem from [45] is the following.

Theorem 4.3.1. *Let $(\mathcal{C}, \oplus, 0)$ be a pre-braided category which is locally homogeneous at a pair (A, X). Suppose that $\mathcal{C}$ satisfies (LH3) at (A, X) with slope $k \geq 2$. If $F : \mathcal{C}_{A,X} \to \mathrm{Mod}_{\mathbb{Z}}$ is a coefficient system of degree r at N, then*

1. *$Rel_i^F(A, n)$ vanishes for $n \geq \max(N + 1, k(i + r))$, and*
2. *if F is split then $Rel_i^F(A, n)$ vanishes for $n \geq \max(N + 1, ki + r)$.*

If $F : \mathcal{C}_{A,X} \to \mathbb{Z}[G_\infty^{ab}]\text{-}\mathrm{Mod}$ is a coefficient system of degree r at N and $k \geq 3$, then

3. *$Rel_i^{F^\circ}(A, n)$ vanishes for $n \geq \max(2N + 1, k(i + r) + k - 2)$, and*
4. *if F is split then $Rel_i^{F^\circ}(A, n)$ vanishes for $n \geq \max(2N + 1, ki + 2r + k - 2)$.*

Inductive Hypothesis 4.3.2. *All four statements of Theorem 4.3.1 hold for all coefficient systems of degree strictly less than r at any $N \geq 0$ in all homological degrees, and for all coefficient systems of degree r at any $N \geq 0$ in homological degrees strictly less than i.*

Our strategy is to prove that the map $s_n = (\Sigma_X, F(\sigma_X))$ is injective in a range. By Proposition 4.2.16 $s_{n+1} \circ s_n = 0$ so we must have $Rel_i^F(A, n) = 0$ in some range. To prove the injectivity of s_n we recall its factorization from Proposition 4.2.15 and we prove that both maps are injective in a range.

Proposition 4.3.3. *Let $F : \mathcal{C}_{A,X} \to \mathrm{Mod}_{\mathbb{Z}}$ be a coefficient system of degree r at N. Suppose that Inductive Hypothesis 4.3.2 holds. Then the map*

$$(id, F(\sigma_X)) : Rel_i^F(A, n) \to Rel_i^{\Sigma F}(A, n)$$

is

1. *surjective if $n \geq \max(N, k(k + r - 1))$;*
2. *injective if $n \geq \max(N, k(i + r))$;*
3. *surjective if $n \geq \max(N, ki + (r - 1))$ and F is split;*
4. *split injective if the coefficient system is split.*

If $F : \mathcal{C}_{A,X} \to \mathbb{Z}[G_\infty^{ab}]\text{-}\mathrm{Mod}$, the analogous map

$$(id, F^\circ(\sigma_X)) : Rel_i^{F^\circ}(A, n) \to Rel_i^{\Sigma F^\circ}(A, n)$$

is

5. *surjective if* $n \geq \max(2N-1, k(i+r)-2)$*;*

6. *injective if* $n \geq \max(2N-1, k(i+r)+k-2)$*;*

7. *surjective if* $n \geq \max(2N-1, ki+2r+k-4)$ *and* F *is split;*

8. *split injective if the coefficient system is split.*

Proof. If F is split there is a retraction $\Sigma F \to F$ of coefficient systems. This induces a map on $Rel_i^{(-)}(A,n)$ splitting the map $(id, F(\sigma_X))$. If F is a coefficient system of $\mathbb{Z}[G_\infty^{ab}]$-modules, the same retraction yields splittings $(\Sigma F)_n^\circ \to F_n^\circ$. This proves (4) and (8).

Now take any coefficient system F, not necessarily split. There are short exact sequences of coefficient systems

$$0 \longrightarrow \ker F \longrightarrow F \longrightarrow \sigma_X(F) \longrightarrow 0$$

$$0 \longrightarrow \sigma_X(F) \longrightarrow \Sigma F \longrightarrow \operatorname{coker} F \longrightarrow 0$$

yielding long exact sequences on $Rel_i^{(-)}(A,n)$. Recall from definition 4.2.10 that $\ker F$ has degree -1 at N.

The long exact sequence coming from the top extension yields that $Rel_i^F(A,n) \cong Rel_i^{\sigma_X(F)}(A,n)$ for $n \geq N$. In this range, the long exact sequence from the bottom extension is

$$\cdots \longrightarrow Rel_{i+1}^{\operatorname{coker} F}(A,n) \longrightarrow Rel_i^F(A,n) \longrightarrow Rel_i^{\Sigma F}(A,n) \longrightarrow Rel_i^{\operatorname{coker} F}(A,n) \longrightarrow \cdots.$$

Hence the map $(id, F(\sigma_X))$ is injective whenever $Rel_{i+1}^{\operatorname{coker} F}(A,n) = 0$, and surjective whenever $Rel_i^{\operatorname{coker} F}(A,n) = 0$. Since $\operatorname{coker} F$ is of degree $r-1$ at $N-1$ we conclude using the Inductive Hypothesis 4.3.2. $\square$

Proposition 4.3.4. *Let* $F : \mathcal{C}_{A,X} \to \mathrm{Mod}_{\mathbb{Z}}$ *be a coefficient system of degree* r *at* N*. Suppose that Inductive Hypothesis 4.3.2 holds. Then the map*

$$(\Sigma_X, id) : Rel_i^{\Sigma F}(A,n) \to Rel_i^F(A,n+1)$$

is

1. *surjective for* $n \geq \max(N, k(i+r)-(k-1))$*;*

2. *injective for* $n \geq \max(N+1, k(i+r))$*;*

3. *surjective for* $n \geq \max(N, ki+r-(k-1))$ *if the coefficient system is split;*

4. *injective for* $n \geq \max(N+1, ki+r)$ *if the coefficient system is split.*

Similarly, for $F : \mathcal{C}_{A,X} \to \mathbb{Z}[G_\infty^{ab}]\text{-Mod}$*, the analogous map*

$$(\Sigma_X, id) : Rel_i^{\Sigma F^\circ}(A,n) \to Rel_i^{F^\circ}(A,n+1)$$

is

5. *surjective for* $n \geq \max(2N-2, k(i+r)-1)$*;*
6. *injective for* $n \geq \max(2N+1, k(i+r)+1)$*;*
7. *surjective for* $n \geq \max(2N-2, ki+2r-1)$ *if the coefficient system is split;*
8. *injective for* $n \geq \max(2N+1, ki+2r+k-2)$ *if the coefficient system is split.*

Proof. As in Theorem 4.1.1 we use the notation $G_n = \mathrm{Aut}(A \oplus X^{\oplus n})$ and $W_n = W_n(A, X)_\bullet$. We want to build a relative version of the spectral sequence from the proof of Theorem 4.1.1. Our assumptions imply that G_n acts transitively on p-simplices of W_n for $p < n$ and the stabilizer of a p-simplex is isomorphic to G_{n-p-1}. We focus on homological degrees smaller than $\frac{n-1}{k}$. The assumptions also yield that W_{n+1} and W_{n+2} are $\frac{n-1}{k}$-connected.

Let $F_n = F(A \oplus X^{\oplus n})$ and recall that the upper suspension induces a G_{n+1}-equivariant map $F(\sigma^X) : F_{n+1} \to F_{n+2}$. Let $P_\bullet$ and $Q_\bullet$ be projective resolutions of F_{n+1} and F_{n+2} as G_{n+1}- and G_{n+2}-modules. Respectively there is a map of projective resolutions $P_\bullet \to Q_\bullet$ compatible with the suspension. The cone of the chain map

$$\mathbb{Z} \otimes_{\mathbb{Z}G_{n+1}} P_\bullet \to \mathbb{Z} \otimes_{\mathbb{Z}G_{n+2}} Q_\bullet$$

computes the relative homology $H_*(G_{n+2}, G_{n+1}; F_{n+2}, F_{n+1})$.

Let $\widetilde{C}_*(W_n)$ be the augmented cellular chain complex of W_n. Post-composition by the upper suspension yields a map of double complexes

$$\mathbb{Z} \otimes_{\mathbb{Z}G_{n+1}} (\widetilde{C}_*(W_{n+1}) \otimes_{\mathbb{Z}} P_\bullet) \to \mathbb{Z} \otimes_{\mathbb{Z}G_{n+2}} (\widetilde{C}_*(W_{n+2}) \otimes_{\mathbb{Z}} Q_\bullet)$$

whose level cone in the $\bullet$ direction is the double complex

$$C_{p,q} = \mathbb{Z} \otimes_{\mathbb{Z}G_{n+1}} (\widetilde{C}_p(W_{n+1}) \otimes_{\mathbb{Z}} P_{q-1}) \bigoplus \mathbb{Z} \otimes_{\mathbb{Z}G_{n+2}} (\widetilde{C}_p(W_{n+2}) \otimes_{\mathbb{Z}} Q_q). \tag{4.8}$$

We proceed as in Theorem 4.1.1. We look at both spectral sequences associated to this double complex. Start with the filtration obtained by taking homology in the p-direction. This yields a spectral sequence with first page

$$E^1_{p,q} = H_p(\mathbb{Z} \otimes_{\mathbb{Z}G_{n+1}} (\widetilde{C}_*(W_{n+1}) \otimes_{\mathbb{Z}} P_{q-1})) \bigoplus H_p(\mathbb{Z} \otimes_{\mathbb{Z}G_{n+2}} (\widetilde{C}_*(W_{n+2}) \otimes_{\mathbb{Z}} Q_q)).$$

The Universal Coefficient Theorem gives

$$E^1_{p,q} = \mathbb{Z} \otimes_{\mathbb{Z}G_{n+1}} (H_p(W_{n+1}; \mathbb{Z}) \otimes_{\mathbb{Z}} P_{q-1}) \bigoplus \mathbb{Z} \otimes_{\mathbb{Z}G_{n+2}} (H_p(W_{n+2}; \mathbb{Z}) \otimes_{\mathbb{Z}} Q_q).$$

Recall our connectivity assumption: W_{n+1} is $\frac{n-1}{k}$-connected and W_{n+2} is $\frac{n}{k}$-connected. This implies that $E^1_{p,q} = 0$ for all $p + q \leq \frac{n}{k}$. So both spectral sequences coming from eq. (4.8) converge to zero in this range.

Now look at the spectral sequence obtained by first taking homology in the q-direction. As in Theorem 4.1.1 we use Shapiro's Lemma to describe the E^1-page as follows:

$$E^1_{p,q} = H_q(Stab(\sigma^X \circ \sigma_p), Stab(\sigma_p); F_{n+2}, F_{n+1})$$

where σ_p is any p-simplex of W_{n+1}. To actually make computations we fix σ_p to be the standard p-simplex

$$\sigma_p = \iota_A \oplus X^{\oplus p+1} \oplus \iota_{X^{\oplus n-p}}.$$

For $p = -1$ the stabilizers are the whole groups, then

$$E^1_{-1,i} = Rel^F_i(A, n+1).$$

For $p \geq 0$ we have an isomorphism

$$(\Sigma_X)^{p+1} : G_{n-p} \to Stab(\sigma_p)$$

which pulls back F_{n+1} to $(\Sigma^{p+1}F)_{n-p}$.

Analogously for W_{n+2}, $\sigma^X \circ \sigma_p$ is its standard p-simplex and $(\Sigma_X)^{p+1}$ pulls back F_{n+2} to $(\Sigma^{p+1}F)_{n+1-p}$.

All these identifications commute with the upper suspension. Therefore

$$E^1_{p,q} \cong Rel^{\Sigma^{p+1}F}_q(A, n-p).$$

Note that all this can be used for internalized modules as well.

Observe that for $p = 0$

$$E^1_{0,i} \cong Rel^{\Sigma F}_i(A, n)$$

and the differential

$$d^1_{0,i} : Rel^{\Sigma F}_i(A, n) \to Rel^F_i(A, n+1)$$

identifies with (Σ_X, id). Indeed $d^1_{0,i}$ is induced by the inclusion of stabilizers into the groups followed by the lower suspension, which is exactly (Σ_X, id). We can see the E^1-page in fig. 4.2.

$$\begin{array}{ccccccccccc}
0 & \longleftarrow & Rel^F_i(A,n+1) & \xleftarrow{d^1_{0,i}} & Rel^{\Sigma F}_i(A,n) & \xleftarrow{d^1_{1,i}} & Rel^{\Sigma^2 F}_i(A,n-1) & \longleftarrow & Rel^{\Sigma^3 F}_i(A,n-2) & \longleftarrow & \cdots \\
0 & \longleftarrow & Rel^F_{i-1}(A,n+1) & \longleftarrow & Rel^{\Sigma F}_{i-1}(A,n) & \longleftarrow & Rel^{\Sigma^2 F}_{i-1}(A,n-1) & \longleftarrow & Rel^{\Sigma^3 F}_{i-1}(A,n-2) & \longleftarrow & \cdots \\
0 & \longleftarrow & Rel^F_{i-2}(A,n+1) & \longleftarrow & Rel^{\Sigma F}_{i-2}(A,n) & \longleftarrow & Rel^{\Sigma^2 F}_{i-2}(A,n-1) & \longleftarrow & Rel^{\Sigma^3 F}_{i-2}(A,n-2) & \longleftarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
0 & \longleftarrow & Rel^F_0(A,n+1) & \longleftarrow & Rel^{\Sigma F}_0(A,n) & \longleftarrow & Rel^{\Sigma^2 F}_0(A,n-1) & \longleftarrow & Rel^{\Sigma^3 F}_0(A,n-2) & \longleftarrow & \cdots
\end{array}$$

Figure 4.2: E^1-page for twisted coefficients

Split case: Lemma 4.2.11 says that $\Sigma^{p+1}F$ has degree r at $N-p-1$. Our Inductive Hypothesis 4.3.2 yields that $Rel^{\Sigma^{p+1}F}_q(A, n-p) = 0$ for $n-p \geq \max(N-p, kq+r)$ and $q < i$. Following similar arguments as in the proof of Theorem 4.1.1 we conclude that $d^1_{0,i}$ is surjective if

$n \geq \max(N, k(i-1)+1+r)$. In this case we get $E^{\infty}_{0,i} = 0$ and $Rel_q^{\Sigma^{p+1}F}(A, n-p) = 0$ for all $p+q = i$, $q < i$. So all differentials d^r with codomain $E^r_{0,i}$ are zero. Therefore $d^r_{0,i}$ is the only differential that can kill $E^1_{0,i}$ before the E^{∞}-page. Hence (Σ_X, id) is surjective in this range.

For injectivity we need that $Rel_q^{\Sigma^{p+1}F}(A, n-p) = 0$ for all $p+q = i+1$, $q < i$ and $d^1_{1,i} = 0$. The first condition is verified for $n \geq \max(N, ki+r)$. In this case the sequence

$$Rel_i^F(A, n+1) \xleftarrow{d^1_{0,i}} Rel_i^F(A, n) \xleftarrow{d^1_{1,i}} Rel_i^F(A, n-1)$$

is exact in the middle because it is the only way that $E^{\infty}_{0,i} = 0$, all higher differentials have trivial domain. We claim that $d^1_{1,i} = 0$ in the same range. This differential is the difference between two maps coming from the faces of the 1-simplex, i.e. $d_0\sigma_1$ and $d_1\sigma_1$. Observe that $d_1\sigma_1 = \sigma_0$ and $d_0\sigma_1 = h\sigma_0$ where $h \in G_{n+1}$. The first map defining $d^1_{1,i}$ is induced on homology by the composition of

$$(Stab(\sigma^X \circ \sigma_1), Stab(\sigma_1); F_{n+2}, F_{n+1}) \to (Stab(\sigma^X \circ d_0\sigma_1), Stab(d_0\sigma_1); F_{n+2}, F_{n+1}) \tag{4.9}$$

induced by the inclusion of stabilizers, with the map

$$(Stab(\sigma^X \circ d_0\sigma_1), Stab(d_0\sigma_1); F_{n+2}, F_{n+1}) \to (Stab(\sigma^X \circ \sigma_0), Stab(\sigma_0); F_{n+2}, F_{n+1}) \tag{4.10}$$

induced by $(c_{\Sigma^X h}, c_h)$, where c_h denotes the conjugation by h as in the proof of Theorem 4.1.1, i.e.

$$c_h : (Stab(d_0\sigma_1), F_{n+1}) \to (Stab(\sigma_0), F_{n+1}); (g, f) \mapsto (hgh^{-1}, hf).$$

The second map defining $d^1_{1,i}$ is induced on homology by

$$(Stab(\sigma^X \circ \sigma_1), Stab(\sigma_1); F_{n+2}, F_{n+1}) \to (Stab(\sigma^X \circ \sigma_0), Stab(\sigma_0); F_{n+2}, F_{n+1})$$

the inclusion of stabilizers. Now, if we choose $h = A \oplus b_{X,X} \oplus X^{\oplus n-1}$, then h centralizes the stabilizer of σ_1, and $\Sigma^X h$ centralizes $Stab(\sigma^X \circ \sigma_1)$.

Hence these two maps

$$Rel_i^{\Sigma^2 F}(A, n-1) \to Rel_i^{\Sigma F}(A, n)$$

only differ by the automorphism in $Rel_i^{\Sigma^2 F}(A, n-1)$ induced by the action of $(F(\Sigma^X h), F(h))$ on (F_{n+2}, F_{n+1}). This automorphism is trivial on

$$(F(\sigma_X^2)(F_n), F(\sigma_X^2)(F_{n-1})) \subset (F_{n+2}, F_{n+1}) \tag{4.11}$$

because $h\sigma_X^2 = \sigma_X^2$ and $\Sigma^X h\sigma_X^2 = \sigma_X^2$. By Proposition 4.3.3(3) the map

$$Rel_i^F(A, n-1) \to Rel_i^{\Sigma F}(A, n-1) \to Rel_i^{\Sigma^2 F}(A, n-1)$$

is surjective for $n-1 \geq \max(N, ki+r-1)$. So the inclusion in eq. (4.11) is an equality in that range. Hence the two maps are equal and their difference $d^1_{1,i}$ is zero for $n \geq \max(N+1, ki+r)$.

Non-split case: Same argument than for split case but with different ranges.

Internalized split case: The arguments for the surjectivity of $d^1_{0,i}$ are the same than above, only the ranges change. By Lemma 4.2.11 and Inductive Hypothesis 4.3.2, $Rel_q^{\Sigma^{p+1}F^{\circ}}(A, n-p) = 0$

for $n-p \geq \max(2(N-p)-1, kq+2r+k-2)$ and $q < i$. In particular if $n \geq \max(2N-2, ki+2r-1)$ then $Rel_q^{\Sigma^{p+1}F^\circ}(A, n-p) = 0$ for $p+q = i$, $q < i$, and $E^\infty_{0,i} = 0$. Hence $d^1_{0,1}$ is surjective in this range.

If $n \geq \max(2N-3, ki+2r)$ then $Rel_q^{\Sigma^{p+1}F^\circ}(A, n-p) = 0$ for $p+q = i+1$, $q < i$. This implies that the sequence

$$Rel_i^{F^\circ}(A, n+1) \xleftarrow{d^1_{0,i}} Rel_i^{F^\circ}(A, n) \xleftarrow{d^1_{1,i}} Rel_i^{F^\circ}(A, n-1)$$

is exact in the middle in this range. Again because this is the only way that $E^\infty_{0,1} = 0$ that we know to be true for $n \geq \max(2N-3, ki+2r)$.

Now we show that $d^1_{1,i} = 0$ for $n \geq \max(2N-3, ki+2r)$. Our description of $d^1_{1,i}$ as a difference of two maps still holds, but one of these maps is different. The map from eq. (4.10) remains the same, but the map from eq. (4.9) is now induced by the composition of the inclusion of stabilizers with $(c_{\Sigma^X h}, c_h)$, where

$$(c_{\Sigma^X h}, c_h) = (\Sigma^X h(-)\Sigma^X h^{-1}, h(-)h^{-1}; [\Sigma^X_\infty h] \otimes F(\Sigma^X h), [\Sigma^X_\infty h] \otimes F(h)).$$

Here the map $[\Sigma^X_\infty h] \otimes F(h)$ is induced on F°_{n+1} by multiplication by $\Sigma^X_\infty h$ on the first factor, and multiplication by $F(h)$ on the second factor. For the choice

$$h = (A \oplus X \oplus b^{-1}_{X,X} \oplus X^{\oplus n-2}) \circ (A \oplus b_{X,X} \oplus X^{\oplus n-1})$$

sends $d_0\sigma_1$ to σ_0 and centralizes $Stab(\sigma_2)$. Similarly, $\Sigma^X h$ centralizes $Stab(\sigma^X \circ \sigma_2)$. Moreover $[\Sigma^X_\infty h] = 0 \in G^{ab}_\infty$. Just as int the previous cases, this implies that the two maps defining $d^1_{1,i}$ only differ by the automorphism in $Rel_i^{F^\circ}(A, n-1)$ induced by the action of $(F(\Sigma^X h), F(h))$. We know that the map $(F(\Sigma^X h), F(h))$ restricted to

$$(F(\sigma^3_X)(F_{n-1}), F(\sigma^3_X)(F_{n-2})) \subset (F_{n+2}, F_{n+1})$$

is the identity.

Now consider the map

$$\phi : (Stab(\sigma^X \circ \sigma_2), Stab(\sigma_2); F^\circ_{n-1}, F^\circ_{n-2}) \to (Stab(\sigma^X \circ \sigma_1), Stab(\sigma_1); F^\circ_{n+2}, F^\circ_{n+1}) \tag{4.12}$$

given by the inclusion of stabilizers and

$$id \otimes F(\sigma^3_X) : F^\circ_{n-1} \to F^\circ_{n+2}.$$

It follows that $d^1_{1,i} \circ H(\phi) = 0$ because the two maps defining $d^1_{1,i}$ are equal on the image of $H(\phi)$, which is exactly $(F(\sigma^3_X)(F_{n-1}), F(\sigma^3_X)(F_{n-2}))$. So $d^1_{1,i}$ is zero whenever $H(\phi)$ is surjective. Observe that

$$\begin{tikzcd} Rel_i^{F^\circ}(A, n-2) \arrow[r, "{(id, id\otimes F(\sigma^3_X))}"] \arrow[rr, bend right, "H(\phi)"'] & Rel_i^{\Sigma^3 F^\circ}(A, n-2) \arrow[r, "{(\Sigma_X, id)}"] & Rel_i^{\Sigma^2 F^\circ}(A, n-1) \end{tikzcd}$$

commutes. So Propositions 4.3.3 and 4.3.4 yields that this map is surjective whenever $n \geq \max(2N+1, ki+2r+k-2)$. To get this range we assumed that $k \geq 3$. Therefore $d^1_{0,i}$ is injective in this range.

Internalized non-split case: Again, we use the same argument than for the split case but with different ranges. $\square$

Proof of Theorem 4.3.1. By Proposition 4.2.15 and Proposition 4.2.16 the diagram

$$\begin{tikzcd}
Rel_i^F(A,n) \arrow[rr, "s_n"] \arrow[dr, "{(id,F(\sigma_X))}"'] \arrow[rrrr, bend left, "0"] & & Rel_i^F(A,n+1) \arrow[rr, "s_{n+1}"] \arrow[dr, "{(id,F(\sigma_X))}"'] & & Rel_i^F(A,n+2) \\
& Rel_i^{\Sigma F}(A,n) \arrow[ur, "{(\Sigma_X,id)}"'] & & Rel_i^{\Sigma F}(A,n+1) \arrow[ur, "{(\Sigma_X,id)}"'] &
\end{tikzcd}$$

commutes. For a coefficient system of G_∞^{ab}-modules there is a similar diagram with internalized G_∞^{ab}-module coefficients. Therefore $Rel_i^F(A,n)$ must be zero whenever the four bottom maps are injective. The ranges for the first and second maps are included in the ranges of the third and fourth maps. So we only consider the first two maps.

Split case: The first map is injective because the coefficient system is split. The second map is injective for $n \geq \max(N+1, ki+r)$ by Proposition 4.3.4(4). Whence $Rel_i^F(A,n) = 0$ for $n \geq \max(N+1, ki+r)$.

Non-split case: The first map is injective for $n \geq \max(N, k(i+r))$ by Proposition 4.3.3(2). The second map is injective for $n \geq \max(N+1, k(i+r))$ by Proposition 4.3.4(2). Therefore $Rel_i^F(A,n) = 0$ for $n \geq \max(N+1, k(i+r))$.

Now for an internalised coefficient system of G_∞^{ab}-modules we obtain the following ranges.

Split case: The first map is injective because the coefficient system is split. The second map is injective for $n \geq \max(2N+1, ki+2r+k-2)$ by Proposition 4.3.4(8). Whence $Rel_i^{F^\circ}(A,n) = 0$ in that range.

Non-split case: The first map is injective for $n \geq \max(2N-1, k(i+r)+k-2)$ by Proposition 4.3.3(6). The second map is injective for $n \geq \max(2N+1, k(i+r)+1)$ by Proposition 4.3.4(6). Therefore $Rel_i^{F^\circ}(A,n) = 0$ for $n \geq \max(2N+1, k(i+r)+k-2)$.

We have proven the four statements of Theorem 4.3.1 for any coefficient system of degree r at any $N \geq 0$ in all homological degrees. By induction our proof is completed. □

Theorem 4.3.5. *Let $(\mathcal{C}, \oplus, 0)$ be a pre-braided homogeneous category, and A, X two objects in $\mathcal{C}$. Assume there is an integer $k \geq 2$ such that for all $n \geq 1$, the semi-simplicial set $W_n(A,X)_\bullet$ is at least $(\frac{n-2}{k})$-connected. Let $F : \mathcal{C} \to \mathrm{Mod}_\mathbb{Z}$ be a coefficient system of degree r at N, and $G_n = \mathrm{Aut}(A \oplus X^{\oplus n})$ and $F_n = F(A \oplus X^{\oplus n})$. Then*

$$H_i(G_n; F_n) \to H_i(G_{n+1}; F_{n+1}) \tag{4.13}$$

is an epimorphism for $i \leq \frac{n}{k} - r$ and an isomorphism for $i \leq \frac{n}{k} - r - 1$ for all $n > N$. Moreover, for $F : \mathcal{C} \to \mathbb{Z}[G_\infty^{ab}]\text{-}\mathrm{Mod}$ and assuming now that $k \geq 3$, we have that

$$H_i(G_n; F_n^\circ) \to H_i(G_{n+1}; F_{n+1}^\circ) \tag{4.14}$$

is an epimorphism for $i \leq \frac{n+2}{k} - r - 1$ and an isomorphism for $i \leq \frac{n+2}{k} - r - 2$ for all $n > 2N$.

Proof. Recall there is a long exact sequence eq. (4.7)

$$\cdots \to Rel_{k+1}^F(A,n) \to H_k(G_n; F_n) \to H_k(G_{n+1}; F_{n+1}) \to Rel_k^F(A,n) \to H_{k-1}(G_n; F_n) \to \cdots$$

by definition of relative homology groups. The vanishing of $Rel_k^F(A,n)$ yields a range on which the maps
$$H_k(G_n;F_n) \to H_k(G_{n+1};F_{n+1})$$
are epimorphisms and isomorphisms. The same holds for the internalised coefficients. $\square$

Corollary 4.3.6. *Suppose $\mathcal{C}, A, X$ and $F : \mathcal{C} \to \mathrm{Mod}_{\mathbb{Z}}$ are as in Theorem 4.3.5. Assume that $W_n(A,X)_\bullet$ is at least $(\frac{n-2}{k})$-connected for some $k \geq 3$. Let G'_n denote the commutator subgroup of $G_n = \mathrm{Aut}(A \oplus X^{\oplus n})$. Then*
$$H_i(G'_n;F_n) \to H_i(G'_{n+1};F_{n+1}) \tag{4.15}$$
is an epimorphism for $i \leq \frac{n+2}{k} - r - 1$ and an isomorphism for $i \leq \frac{n+2}{k} - r - 2$ for all $n > 2N$.

Proof. Consider the coefficient system
$$\overline{F} = F \otimes_{\mathbb{Z}} \mathbb{Z}[G_\infty^{ab}] : \mathcal{C} \to \mathbb{Z}[G_\infty^{ab}]\text{- Mod}$$
and observe that Shapiro's lemma implies
$$H_*(G'_n; Res_{G'_n}^{G_n} F_n) \cong H_*(G_n; Ind_{G'_n}^{G_n} Res_{G'_n}^{G_n} F_n) \cong H_*(G_n; \mathbb{Z}[G_n^{ab}] \otimes_{\mathbb{Z}} F_n)$$
where G_n acts diagonally on $\mathbb{Z}[G_n^{ab}] \otimes_{\mathbb{Z}} F_n$. Now, Theorem 4.1.1 yields that
$$G_n^{ab} = H_1(G_n;\mathbb{Z}) \cong H_1(G_\infty;\mathbb{Z}) = G_\infty^{ab}$$
for $n \geq k+1$. Therefore, in this range we obtain
$$H_*(G'_n;F_n) \cong H_*(G_n;\mathbb{Z}[G_\infty^{ab}] \otimes_{\mathbb{Z}} F_n).$$
It remains to study the G_n-module $\mathbb{Z}[G_\infty^{ab}] \otimes_{\mathbb{Z}} F_n$. We claim that
$$\mathbb{Z}[G_\infty^{ab}] \otimes_{\mathbb{Z}} F_n \cong \overline{F}_n^\circ.$$
Indeed, we first notice that they have the same $\mathbb{Z}$-module structure because
$$\overline{F}_n^\circ = F_n \otimes_{\mathbb{Z}} \mathbb{Z}[G_\infty^{ab}]$$
with the action
$$G_n \times F_n \otimes_{\mathbb{Z}} \mathbb{Z}[G_\infty^{ab}] \to F_n \otimes_{\mathbb{Z}} \mathbb{Z}[G_\infty^{ab}] : (g, x \otimes h) \mapsto g(\Sigma_\infty^X(g)(x \otimes h)) = gx \otimes \Sigma_\infty^X(g)h$$
Moreover the G_n-action on $\mathbb{Z}[G_\infty^{ab}] \otimes_{\mathbb{Z}} F_n$ is described by
$$G_n \times (\mathbb{Z}[G_\infty^{ab}] \otimes_{\mathbb{Z}} F_n) \to \mathbb{Z}[G_\infty^{ab}] \otimes_{\mathbb{Z}} F_n : (g, h \otimes x) \mapsto \Sigma_\infty^X(g)h \otimes gx$$
which coincides with the G_n-action on $\overline{F}_n^\circ$. It follows that
$$H_*(G_n;\overline{F}_n^\circ) \cong H_*(G'_n;F_n)$$
and now use Theorem 4.3.5 to conclude. $\square$

Remark 4.3.7. Let us compare the results for twisted coefficients obtained above by Randal-Williams and Wahlwith those provided by Putman in [40]. We follow the ideas of Putman presented in [40].

Here he forgets about pre-braided locally homogeneous categories. And views twisted coefficients in a different way that can be translated to our definitions from Section 4.2. Putman developed a framework focusing on symmetric groups and general linear groups. In some cases the ranges for twisted coefficients are better than the ones we proved in this chapter.

In both frameworks the proof for homological stability for constant coefficients is the same. However, Putman for twisted coefficients the stability range is slightly different. There is no general rule to say which one is better. It depends on each specific case.

Also, Putman does not give different bounds for split twisted coefficients. So sometimes he gets better results for twisted coefficients in general, but Randal-Williams and Wahl get better results for split twisted coefficients.

4.4 Homological stability for topological moduli spaces

There have been many applications of Randal-Williams and Wahl's paper [45]. Their framework focuses on families of discrete groups, however we know that homological stability phenomena occur in other setups, for example configuration spaces and moduli spaces. Krannich generalized this framework so that it could be extended to more examples exhibiting homological stability phenomena. We will compare these two frameworks for the homological stability of mapping class groups of surfaces with respect to genus and see that Krannich's theorem give the best known bounds in this case.

We will not give all details of Krannich's work but rather the intuition behind it and how it is related to Randal-Williams and Wahl. For details on this new framework we refer to Krannich's paper [30].

Instead of considering a sequence of automorphism groups

$$\cdots \to \mathrm{Aut}(A \oplus X^{\oplus n-1}) \to \mathrm{Aut}(A \oplus X^{\oplus n}) \to \mathrm{Aut}(A \oplus X^{\oplus n+1}) \cdots$$

we consider a sequence of spaces

$$\cdots \to \mathcal{M}_{n-1} \to \mathcal{M}_n \to \mathcal{M}_{n+1} \to \cdots$$

which we may think of as the classifying spaces of our groups. Then we will see that it is much more advantageous to treat them as a single space

$$\mathcal{M} = \bigsqcup_n \mathcal{M}_n \tag{4.16}$$

with a grading

$$g_{\mathcal{M}} : \mathcal{M} \to \mathbb{N} \tag{4.17}$$

and a stabilization map

$$s : \mathcal{M} \to \mathcal{M} \tag{4.18}$$

that increases the degree by one. That means it restricts to maps

$$s : \mathcal{M}_n \to \mathcal{M}_{n+1}. \tag{4.19}$$

Many of the spaces $\mathcal{M}$ coming from families of groups which we know satisfy homological stability share a common homotopy theoretic characteristic: they form a graded E_1-module over an E_2-algebra. This is the analogue in Krannich's setup of a module over a braided monoidal category in Randal-Williams and Wahl.

Loosely speaking we think of a *graded E_1-module $\mathcal{M}$ over an E_2-algebra $\mathcal{A}$* as a pair of spaces $(\mathcal{M}, \mathcal{A})$ with gradings $g_{\mathcal{M}} : \mathcal{M} \to \mathbb{N}$, $g_{\mathcal{A}} : \mathcal{A} \to \mathbb{N}$, a multiplication $\oplus : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which is commutative up to homotopy, and an action map $\oplus : \mathcal{M} \times \mathcal{A} \to \mathcal{M}$ which is associative up to homotopy. These need to satisfy various conditions which we will not specify, refer to Definition 2.3 in [30]. To rigorously define a graded E_1-module over an E_2-algebra we need to know about colored operads and the Swiss-Cheese operad.

Once we have such a pair $(\mathcal{M}, \mathcal{A})$ we must choose a *stabilizing object* $X \in \mathcal{A}$, which is an element of degree 1. This means that if $\mathcal{A} = \bigsqcup_n \mathcal{A}_n$ then $X \in \mathcal{A}_1$. This choice yields a stabilization map

$$s = (- \oplus X) : \mathcal{M} \to \mathcal{M} \tag{4.20}$$

increasing the degree by 1.

The main construction of Krannich is the *canonical resolution* of $\mathcal{M}$,

$$R_\bullet(\mathcal{M}) \to \mathcal{M}, \tag{4.21}$$

which is a fibrant augmented semi-simplicial space up to higher coherent homotopy. The fibre $W_\bullet(A)$ of the canonical resolution at $A \in \mathcal{M}$ is the destabilization space in this context. It is also an augmented semi-simplicial space up to higher coherent homotopy whose space of p-simplices is the homotopy fibre at A of s^{p+1}.

The connectivity conditions on destabilization spaces is now a single connectivity condition on the canonical resolution. We say that the canonical resolution of $\mathcal{M}$ is *graded $\varphi(g_{\mathcal{M}})$-connected in degrees greater than or equal to m* for a function $\varphi : \mathbb{N} \to \mathbb{Q}$ if the restriction

$$|R_\bullet(\mathcal{M})|_n \to \mathcal{M}_n \tag{4.22}$$

of the geometric realization is $\lfloor \varphi(n) \rfloor$-connected for all $n \geq m$.

Remark 4.4.1. Let us unwrap this connectivity condition. Since $R_\bullet(\mathcal{M}) \to \mathcal{M}$ is fibrant we have

$$W_\bullet(A) \simeq \operatorname{hofib}_A(R_\bullet(\mathcal{M})).$$

And by Lemma 1.4 in [30] the realization $|W_\bullet(A)|$ is homotopy equivalent to the homotopy fibre of the realization $|R_\bullet(\mathcal{M})| \to \mathcal{M}$. Consider a function $\varphi : \overline{\mathbb{N}} \to \mathbb{Q} \cup \{\infty\}$ with $\varphi(\infty) = \infty$. By the above we conclude that the canonical resolution of $\mathcal{M}$ is graded $\varphi(g_{\mathcal{M}})$-connected in degree greater or equal to m if and only if the spaces of destabilization $W_\bullet(A)$ are $(\lfloor \varphi(g_{\mathcal{M}}(A)) \rfloor - 1)$-connected for all $A \in \mathcal{M}$ with finite degree $g_{\mathcal{M}}(A) \geq m$. Since all points in the same connected component have equivalent homotopy fibre it suffices to check only one point in each component.

For homological stability with constant and abelian coefficients we get the following result.

Theorem 4.4.2 (Theorem A in [30]). *Let $\mathcal{M}$ be a graded E_1-module over an E_2-algebra with stabilizing object X and L a local system on $\mathcal{M}$. If the canonical resolution of $\mathcal{M}$ is graded $\left(\frac{g_{\mathcal{M}}-2+k}{k}\right)$-connected in degrees at least 1 for some $k \geq 2$, then*

$$s_* : H_i(\mathcal{M}_n; s^* L) \to H_i(\mathcal{M}_{n+1}; L) \tag{4.23}$$

1. *is an isomorphisms for $i \leq \frac{n-1}{k}$ and an epimorphism for $i \leq \frac{n-2+k}{k}$, if L is constant, and*
2. *is an isomorphisms for $i \leq \frac{n-1-k}{k}$ and an epimorphism for $i \leq \frac{n}{k}$, if L is abelian and $k \geq 3$.*

Remark 4.4.3. We can improve the stable range given by this theorem. If $g_{\mathcal{M}}$ is a grading then so is $g_{\mathcal{M}} + m$ for any $m \geq 0$. In particular if the canonical resolution of $\mathcal{M}$ is graded $\left(\frac{g_{\mathcal{M}}-m+k}{k}\right)$-connected for $m \geq 2$ we can apply Theorem 4.4.2 to $\mathcal{M}$ graded by $g_{\mathcal{M}} + (m-2)$ which shifts the stable range. Summing this up we get the following improved statements, see Remark 3.3 in [30]:

1. If the canonical resolution is graded $\left(\frac{g_{\mathcal{M}}-m+k}{k}\right)$-connected for $m \geq 3$, the surjectivity range for constant coefficients in Theorem 4.4.2 can be improved from $i \leq \frac{n-m+k}{k}$ to $i \leq \frac{n-m+k+1}{k}$, and for abelian coefficients from $i \leq \frac{n-m+2}{k}$ to $i \leq \frac{n-m+3}{k}$.
2. If the canonical resolution is graded $(g_{\mathcal{M}} - 1)$-connected in degrees at least 1, then the isomorphism range in Theorem 4.4.2 can be improved from $i \leq \frac{n-1}{2}$ to $i \leq \frac{n}{2}$.

Now we need to adapt our definition of coefficient system to this framework. In fact, Krannich's definition generalizes the one of Randal-Williams and Wahl.

Observe that the fundamental groupoid of an E_2-algebra $\mathcal{A}$ is a braided monoidal category $(\Pi(\mathcal{A}), \oplus, b, 0)$, and the groupoid of an E_1-module $\mathcal{M}$ over $\mathcal{A}$ is a right module $(\Pi(\mathcal{M}), \oplus)$ over $(\Pi(\mathcal{A}), \oplus, b, 0)$.

A *coefficient system* $\mathcal{F}$ for $\mathcal{M}$ with stabilizing object X is a functor

$$\mathcal{F} : \Pi(\mathcal{M}) \to \mathrm{Mod}_{\mathbb{Z}} \tag{4.24}$$

with a natural transformation $\sigma^F : F \to \Sigma F$ where $\Sigma F = F(- \oplus X)$ is the *suspension* of F, coming with the suspension map $F \to \Sigma F$. Such a functor enhances the stabilization map to

$$(s; \sigma^F) : (\mathcal{M}_n; \mathcal{F}) \to (\mathcal{M}_{n+1}; \mathcal{F}). \tag{4.25}$$

Moreover, we inductively say that $\mathcal{F}$ has *degree r at $N \geq 0$* if $\ker(\mathcal{F})$ has (split) degree -1 at N and the $\mathrm{coker}(\mathcal{F})$ has (split) degree $r-1$ at $(N-1)$; where $\mathcal{F}$ has (split) degree -1 at N if $F_n = 0$ for all $n \geq N$.

Theorem 4.4.4 (Theorem C in [30]). *Let $\mathcal{M}$ be a graded E_1-module over an E_2-algebra with stabilizing object X and $\mathcal{F}$ a coefficient system on $\mathcal{M}$ of degree r at $N \geq 0$. If the canonical resolution of $\mathcal{M}$ is graded $\left(\frac{g_{\mathcal{M}}-2+k}{k}\right)$-connected in degrees at least 1 for some $k \geq 2$, then*

$$(s; \sigma^F)_* : H_i(\mathcal{M}_n; s^* L) \to H_i(\mathcal{M}_{n+1}; L) \tag{4.26}$$

is an isomorphisms for $i \leq \frac{n-rk-k}{k}$ and an epimorphism for $i \leq \frac{n-rk}{k}$ when $n > N$. If $\mathcal{F}$ is of split degree r at $N \geq 0$ then $(s; \sigma^F)_*$ is an isomorphisms for $i \leq \frac{n-r-k}{k}$ and an epimorphism for $i \leq \frac{n-r}{k}$ when $n > N$.

Chapter 5

Classical Examples

5.1 Symmetric groups

Nakaoka proved the first result on the homological stability with constant coefficients in [38]. Our proof of homological stability for symmetric groups is a generalized version of Kerz's proof in [29]. Although we get the same isomorphism range than Nakaoka, he also proves that the morphism

$$H_i(\Sigma_n; F(n)) \to H_i(\Sigma_{n+1}; F(n+1))$$

is always injective.

Throughout the previous chapters we have given many examples that build the set up for the symmetric groups. Let us start with the groupoid Σ of finite sets with bijections, see Example 2.3.5. The Quillen construction associated to this groupoid is the homogeneous category $(FI, \sqcup, \emptyset)$ of finite sets with injections from Examples 2.1.8 and 2.2.1. Moreover $(FI, \sqcup, \emptyset)$ is locally standard, see Example 3.1.4. Then Proposition 3.1.10 implies that the semi-simplicial sets $W_n(A, X)_\bullet$ satisfy condition (A, 3.1.8).

From now on we fix $A = \emptyset$ and $X = \{*\}$ as in Example 2.2.1 and let $W_n = W_n(\emptyset, \{*\})_\bullet$. The simplicial complex $S_n = S_n(\emptyset, \{*\})$ can be identified with Δ^{n-1}, which is contractible for all $n \geq 1$. Therefore S_n is at least $(n-2)$-connected for all $n \geq 0$. By Theorem 3.2.6 we deduce that W_n is also $(n-2)$-connected, in particular it is at least $(\frac{n-2}{2})$-connected for all $n \geq 1$.

Let us use Theorem 4.1.1, Theorem 4.1.4 and Theorem 4.3.5 together with FI-modules Example 4.2.12 to give results about the homological stability of symmetric groups. This theorem in presented by Randal-Williams and Wahl in Theorem 5.1 from [45].

Theorem 5.1.1. *Let $F : FI \to R\text{-}\mathrm{Mod}$ be a coefficient system. Then the map*

$$H_i(\Sigma_n; F(n)) \to H_i(\Sigma_{n+1}; F(n+1))$$

is:

1. *an epimorphism for $i \leq \frac{n}{2}$ and an isomorphism for $i \leq \frac{n-1}{2}$, if F is constant;*

2. *an epimorphism for* $i \leq \frac{n-r}{2}$ *and an isomorphism for* $i \leq \frac{n-r-2}{2}$*, if* F *is split of degree* r *at* N *for* $ngeqN+1$*;*

3. *an epimorphism for* $i \leq \frac{n}{2} - r$ *and an isomorphism for* $i \leq \frac{n-2}{2} - r$*, if* F *is of degree* r *at* N *for* $ngeqN+1$*.*

Now let $A_n \subset \Sigma_n$ *be the alternating groups. Then the map*

$$H_i(A_n; F(n)) \to H_i(A_{n+1}; F(n+1))$$

is:

1. *an epimorphism for* $i \leq \frac{n-1}{3}$ *and an isomorphism for* $i \leq \frac{n-3}{3}$*, if* F *is constant;*

2. *an epimorphism for* $i \leq \frac{n-2r-1}{3}$ *and an isomorphism for* $i \leq \frac{n-2r-4}{3}$*, if* F *is split of degree* r *at* N *for* $ngeq2N+1$*;*

3. *an epimorphism for* $i \leq \frac{n-1}{3} - r$ *and an isomorphism for* $i \leq \frac{n-4}{3} - r$*, if* F *is of degree* r *at* N *for* $ngeq2N+1$*.*

In both cases, if F *is of degree* r *at* $N > 0$*, then (2) holds for all* $n \geq N+1$ *and (3) for all* $n \geq 2N+1$*.*

Remark 5.1.2. Be careful with a misleading typo in Theorem 5.1 in [45].

Proof. Let us start with symmetric groups. Statement (1) follows from Theorem 4.1.1 with $k = 2$. Note that Theorem 4.3.1 yields that

- $Rel_i^F(A, n)$ vanishes for $n \geq \max(N+1, 2(i+r))$, and
- if F is split then $Rel_i^F(A, n)$ vanishes for $n \geq \max(N+1, 2i+r)$.

After rewriting these bounds and following the proof of Theorem 4.3.5 we get the ranges claimed in (2) and (3).

Now let us prove the statements for alternating groups. The bounds for constant coefficients follow from corollary 4.3.6 with $k = 3$. For twisted coefficients one must look at Theorem 4.3.1:

- $Rel_i^{F^\circ}(A, n)$ vanishes for $n \geq \max(2N+1, 3(i+r)+1)$, and
- if F is split then $Rel_i^{F^\circ}(A, n)$ vanishes for $n \geq \max(2N+1, 3i+2r+1)$.

We rewrite this bounds and follow the proof of Theorem 4.3.5. Finally, applying corollary 4.3.6 proves the ranges claimed in the theorem. □

5.2 Decorated surfaces with boundary

[45] Recall the groupoid of decorated surfaces $\mathbf{M}_1$ from Definition 1.6.1. From Example 2.1.5 we know that $(\mathbf{M}_1, \natural, (D^2, I))$ is a braided monoidal groupoid. Additionally, Proposition 2.3.6 says that $U\mathbf{M}_1$ is a pre-braided and locally homogeneous at (A, X) for any orientable surface X.

5.2.1 Braid groups

To prove homological stability for braid groups we follow Randal-Williams and Wahl's approach [45]. Let $X = (S^1 \times [0,1], I)$ be the cylinder.

Lemma 5.2.1 (Proposition 7.2, [21])**.** *For $X = (S^1 \times [0,1], I)$ the cylinder and A any object in $U\mathbf{M}_1$ the simplicial complex $S_n^{U\mathbf{M}_1}(A, X)$ is $(n-2)$-connected.*

Remark 5.2.2. Let $X = (S^1 \times [0,1], I)$ be the cylinder and $A = (S, J)$ be any decorated surface. Then we have

$$A \natural X^{\natural n} \simeq (S \backslash (\sqcup_n intD^2), \partial_0 S).$$

Therefore

$$\mathrm{Aut}_{U\mathbf{M}_1}(A \natural X^{\natural n}) \cong \pi_0 \, \mathrm{Diff}(S \backslash (\sqcup_n intD^2) \, \mathrm{rel} \, \partial_0 S) \cong \pi_0 \, \mathrm{Diff}(S_{(n)} \, \mathrm{rel} \, \partial_0 S)$$

where $S_{(n)}$ is the surface obtained from S after doing n punctures.

Theorem 5.2.3. *Let $F : (U\mathbf{M}_1)_{S, S^1 \times [0,1]} \to \mathrm{Mod}_{\mathbb{Z}}$ be a coefficient system. Then the map*

$$H_i(\pi_0 \, \mathrm{Diff}(S_{(n)} \, \mathrm{rel} \, \partial_0 S); F(S_{(n)})) \to H_i(\pi_0 \, \mathrm{Diff}(S_{(n+1)} \, \mathrm{rel} \, \partial_0 S); F(S_{(n+1)}))$$

is

- *an epimorphism for $i \leq \frac{n}{2}$ and an isomorphism for $i \leq \frac{n-1}{2}$ if F is constant;*
- *an epimorphism for $i \leq \frac{n-r}{2}$ and an isomorphism for $i \leq \frac{n-r-2}{2}$ if F is split of degree r;*
- *an epimorphism for $i \leq \frac{n}{2} - r$ and an isomorphism for $i \leq \frac{n-2}{2} - r$ if F is of degree r.*

Moreover, for a coefficient system $G : (U\mathcal{B}_2)_{S, S^1 \times [0,1]} \to \mathrm{Mod}_{\mathbb{Z}}$ the same holds for the map

$$H_i(\pi_1 \, \mathrm{Conf}(intS, n); G(S_{(n)})) \to H_i(\pi_1 \, \mathrm{Conf}(intS, n+1); G(S_{(n+1)}))$$

under the additional assumption that S has a single boundary component.

Proof. In this theorem $A = (S, J)$ is any decorated surface and $X = (S^1 \times [0,1], I)$ is the cylinder. The statement for constant coefficients follows from Theorem 4.1.1 with $k = 2$ coming from Lemma 5.2.1. For twisted coefficients of degree r it follows directly from Theorem 4.3.5. And for split twisted coefficients of degree r one must modify the proof of Theorem 4.3.5 using the adequate bounds from Theorem 4.3.1. $\square$

Remark 5.2.4. When $S = D^2$, Theorem 1.5.6 yields that the mapping class group $\pi_0 \, \mathrm{Diff}(S_{(n)} rel \partial_0 S)$ is isomorphic to the braid group β_n. This yields the homological stability for braid groups. For constant coefficients this result is attributed to Arnold [1].

5.2.2 Orientable surfaces

Homological stability for surfaces with respect to genus is one of the most important example of this machinery. First proved by Harer in [18], and later improved by Ivanov, Bolsen, Randall-Williams and Wahl [26, 4, 44, 50] among others. This result was used in the proof of the Mumford conjecture by Madsen and Weiss [33].

To prove this result using our machinery we follow again Randal-Williams and Wahl [45]. Let $X = ((S^1 \times S^1)\backslash \mathrm{int} D^2, I)$ be the torus with one boundary component.

Lemma 5.2.5 (Proposition 5.5, [22])**.** *Let A be any orientable surface and $X = ((S^1 \times S^1)\backslash intD^2, I)$ the torus with one boundary component. Then the complex $S_n(A, X)$ is $\frac{n-3}{2}$-connected.*

Now pick $A = (S_{g,b}, J)$ an orientable surface of genus g with b boundary components. Then

$$A \natural X^{\natural n} = S_{g,b} \natural (S^1 \times S^1)\backslash \mathrm{int} D^2 \cong S_{g+n,b}$$

is an orientable surface of genus $g + n$ with b boundary components. Recall that Remark 1.6.2 yields

$$\mathrm{Aut}_{U\mathbf{M}_1}(A \natural X^{\natural n}) \cong \pi_0 \operatorname{Diff}(S_{g+n,b} \operatorname{rel} \partial_0 S).$$

Theorem 5.2.6. *Let X be the torus with one boundary component, and $A = S_{0,b}$ a genus 0 orientable surface with $b \geq 1$ boundary components. Let $F : (U\mathbf{M}_1)_{A,X} \to \mathrm{Mod}_{\mathbb{Z}}$ be a coefficient system. Then the map*

$$H_i(\pi_0 \operatorname{Diff}(S_{g,b} \operatorname{rel} \partial_0 S); F(S_{g,b})) \to H_i(\pi_0 \operatorname{Diff}(S_{g+1,b} \ \operatorname{rel} \partial_0 S); F(S_{g+1,b}))$$

is

- *an epimorphism for $i \leq \frac{g-1}{2}$ and an isomorphism for $i \leq \frac{g-2}{2}$ if F is constant;*
- *an epimorphism for $i \leq \frac{g-r-1}{2}$ and an isomorphism for $i \leq \frac{g-r-3}{2}$ if F is split of degree r;*
- *an epimorphism for $i \leq \frac{g-1}{2} - r$ and an isomorphism for $i \leq \frac{g-3}{2} - r$ if F is of degree r.*

Proof. This proof is completely analogous to the proof of Theorem 5.2.3 with $n = g - 1$ and $k = 2$ due to Lemma 5.2.5. $\square$

Remark 5.2.7. The theorem just above for constant coefficients gives a better range than Harer's classical stability theorem for mapping class groups [18]. However this is not the best known range.

Chapter 6

Disordered arcs

In Section 5.2.2 we showed homological stability for mapping class groups of surfaces with respect to genus. Our isomorphism range is better that the original one proved by Harer in [18] but it is not the best known range. In this chapter we follow Harr, Vistrup and Wahl's paper [19] giving a new proof of the best known isomorphism range. We want to show how powerful the homological stability machinery of Randall-Williams and Wahl [45] is. In this chapter we will also use the machinery of Krannich [30], but we will not go deep into the details since it is a generalization of Randall-Williams and Wahl's setup.

6.1 Bidecorated surfaces

In Sections 1.6 and 5.2 we worked on the category of decorated surfaces to get homological stability on the mapping class group surfaces with respect to genus. But as we have already mentioned in Remark 5.2.7 this result does not give the best known isomorphism range. To improve our range we will use the category of bidecorated surfaces.

Informally, a *bidecorated surface* is a triple (S, I, J) where S is a compact connected surface with boundary components, and I and J are two parametrized intervals in the boundary of S. We will give a precise definition that will be more convenient for the monoidal structure. Let us start by constructing a sequence of bidecorated surfaces X_m inductively.

Let $X_1 = D^2$ be the unit disk in the complex plane, with two embeddings

$$\iota_1^0 : I \to X_1; t \mapsto e^{i(\pi/4 + t\pi/2)}$$
$$\iota_1^1 : I \to X_1; t \mapsto e^{i(5\pi/4 + t\pi/2)}.$$

For $i = 0, 1$ we denote by $\overline{\iota_1^i}$ the reversed map, that means $\overline{\iota_1^i}(t) = \iota_1^i(1-t)$.

Now we inductively build $(X_{m+1}, \iota_{m+1}^0, \iota_{m+1}^1)$ from $(X_m, \iota_m^0, \iota_m^1)$. Define

$$X_{m+1} = \frac{X_m \sqcup X_1}{\iota_m^i(t) \sim \overline{\iota_1^i}(t), t \in [1/2, 1]}$$

and

$$\iota^i_{m+1}(t) = \begin{cases} \iota^i_m(t), & \text{if } t \le 1/2, \\ \iota^i_1(t), & \text{else.} \end{cases}$$

for $i = 0, 1$.

Remark 6.1.1. Observe that we can define the surface X_{m+1} as a push-out. Indeed, it fits into the diagram

$$\begin{tikzcd} I \sqcup I \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_m \arrow[r] & X_{m+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where the maps are chosen accordingly to the quotient relation.

Example 6.1.2. The bidecorated surface X_2 is a cylinder. See Figure 6.1

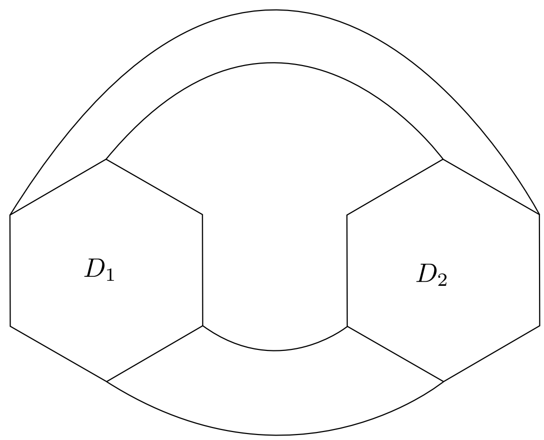

Figure 6.1: Cylinder X_2

Lemma 6.1.3. *Let $m \geq 1$. Then $X_m \simeq S_{g,b}$ is an orinetable surface of genus g with b boundary components. More explicitly*

$$(g, b) = \begin{cases} (\frac{m}{2} - 1, 2), & \textit{if } m \textit{ is even,} \\ (\frac{m-1}{2}, 1), & \textit{if } m \textit{ is odd.} \end{cases}$$

Proof. It is clear that X_m is a connected surface for all $m \geq 1$. The classification of oriantable connected surfaces yields that $X_m \simeq S_{g,b}$ for some g, b. To find them, we first compute the Euler characteristic of X_m. Additionally, we can compute its Euler characteristic using the push-out diagram from Remark 6.1.1 to get

$$\chi(X_{m+1}) = \chi(X_m) + \chi(X_1) - \chi(I \sqcup I) = \chi(X_m) - 1 = 1 - m. \tag{6.1}$$

Moreover, we know that

$$\chi(S_{g,b}) = 2 - 2g - b \tag{6.2}$$

so it is enough to find b.

Observe that gluing one disk along two intervals in the same boundary component we obtain a bidecorated surface whose marked intervals lie in two distinct boundary components. Conversely, if we glue a disk along two intervals in distinct boundary components we obtain a new bidecorated surface whose marked intervals are in the same boundary component. Since X_1 is the disk with a single boundary component we get that

$$b = \begin{cases} 1, & \text{if } m \text{ is odd,} \\ 2, & \text{if } m \text{ is even.} \end{cases} \tag{6.3}$$

Combining eqs. (6.1) to (6.3) we conclude that

$$(g, b) = \begin{cases} (\frac{m}{2} - 1, 2), & \text{if } m \text{ is even,} \\ (\frac{m-1}{2}, 1), & \text{if } m \text{ is odd.} \end{cases}$$

$\square$

Definition 6.1.4. A *bidecorated surface* is a triple (S, m, φ) where S is a compact connected surface with boundary components, $m \geq 1$ is an integer, and

$$\phi : \partial X_m \sqcup (\sqcup_k S^1) \to \partial S$$

is an homeomorphism giving a parametrization of the boundary of S. The two parametrized intervals in S are

$$I_i = \phi \circ \iota_m^i$$

for $i = 0, 1$. There are no morphisms between (S, m, φ) and (S', m', φ') unless $S \simeq S'$ and $m = m'$. In this case we set

$$\mathrm{Hom}_{\mathbf{M}_2}((S, m, \varphi), (S', m', \varphi')) = \pi_0 \,\mathrm{Homeo}_\partial(S, S').$$

As we did for decorated surfaces, we want to define a monoidal structure on bidecorated surfaces. The monoidal product will be again the boundary connected sum, but this time it will be along the two parametrized intervals.

Definition 6.1.5. Let (S, m, φ) and (S', m', φ') be two bidecorated surfaces. Define the monoidal product $\sharp$ by

$$(S, m, \varphi) \sharp (S', m', \varphi') = \left(\frac{S \sqcup S'}{I_i(t) \sim \overline{I_i}(t), t \in [1/2, 1]}, m + m', \varphi \sharp \varphi' \right),$$

where $i = 0, 1$, and

$$\varphi \sharp \varphi' : \partial X_{m+m'} \sqcup (\sqcup_{k+k'} S^1) \hookrightarrow \partial (S \sharp S')$$

is obtained through the identification

$$\partial X_{m+m'} \cong (\partial X_m \backslash \iota_m(1/2, 1)) \cup (\partial X_{m'} \backslash \iota_{m'}(0, 1/2)).$$

Definition 6.1.6. We denote by $(\mathbf{M}_2, \sharp, U)$ the *monoidal groupoid of bidecorated surfaces*. Its objects are bidecorated surfaces plus a formal unit U. The only morphism involving U is the identity.

Proposition 6.1.7 (Proposition 3.4 in [19]). *Let $(S, m, \varphi) \in \mathbf{M}_2$, and $p \geq 0$. The map*

$$\sharp D^{\sharp p+1} : \mathrm{Aut}_{\mathbf{M}_2}(S) \to \mathrm{Aut}_{\mathbf{M}_2}(S \sharp D^{\sharp p+1})$$

is injective, where $D = (X_1, 1, id)$ is the disk.

6.2 Yang-Baxter

Instead of using the framework of Randal-Williams and Wahl [45] we want to use an improved version by Krannich [30]. The advantage is that we no longer have to provide the structure of a braiding on the monoidal category on whose automorphism groups we are interested in. It is enough to provide a single morphism satisfying a simple equation.

Definition 6.2.1. Let $(\mathcal{C}, \oplus, 0)$ be a monoidal category. A *Yang-Baxter operator* in $\mathcal{C}$ is a pair (X, τ) consisting on an object $X \in \mathcal{C}$ and a morphism $\tau \in \mathrm{Aut}(X \oplus X)$, satisfying the Yang-Baxter equation

$$(\tau \oplus X)(X \oplus \tau)(\tau \oplus X) = (X \oplus \tau)(\tau \oplus X)(X \oplus \tau) \in \mathrm{Aut}_{\mathcal{C}}(X^{\oplus 3}). \tag{6.4}$$

Remark 6.2.2. Proposition 2.2 in [27] establishes a equivalence between Yang-Baxter operators (X, τ) on $(\mathcal{C}, \oplus, 0)$ and strong monoidal functors

$$\Phi_{X,\tau} : \beta \to \mathcal{C}$$

where β is the braid category. This functor is defined on objects by

$$\Phi_{X,\tau}(n) = X^{\oplus n}$$

and on morphisms by

$$\Phi_{X,\tau}(\sigma_i) = X^{\oplus i-1} \oplus \tau \oplus X^{\oplus n-i-1}$$

for σ_i the elementary i-th braid in β_n.

Example 6.2.3. Let $(\mathcal{C}, \oplus, 0)$ be a braided monoidal structure with braiding $b_{X,X} \in \mathrm{Aut}_{\mathcal{C}}(X \oplus X)$. Then $(X, b_{X,X})$ is a Yang-Baxter operator for any object $X \in \mathcal{C}$.

Proposition 6.2.4. *Let $(\mathcal{C}, \oplus, 0)$ be a monoidal category, and (X, τ) a Yang-Baxter operator in $\mathcal{C}$. Suppose that $\mathcal{C}$ acts on a category $\mathbf{M}$. Then there is an acting of the braid groupoid*

$$\alpha_\tau : \mathbf{M} \times \beta \to \mathbf{M}$$

given on objects by

$$\alpha_\tau(A, n) = A \oplus X^{\oplus n}$$

and on morphisms by

$$\alpha_\tau(f, \sigma_i) = f \oplus X^{\oplus i-1} \oplus \tau \oplus X^{\oplus n-i-1}$$

for σ_i the elementary i-th braid in β_n. Furthermore taking classifying spaces, this action endows $B\mathbf{M}$ with the structure of an E_1-module over the E_2-algebra $B\beta$.

Sketch of proof. The functor α_τ is defined by

$$\alpha_\tau(-,-) = (-) \oplus \Phi_{X,\tau}(-)$$

so it clearly defines an action. $\square$

Remark 6.2.5. It is often useful to restrict this action to the full subcategory $\mathbf{M}_{A,X}$, see Section 4.2. Moreover, if the cancellation property 2.3.3 is satisfied or if we replace $\mathbf{M}_{A,X}$ by a category whose objects are the natural numbers and setting $\mathrm{Aut}(n) = \mathrm{Aut}_{\mathbf{M}}(A \oplus X^{\oplus n})$, then the E_1-module structure over the E_2-algebra $B\beta$ is graded.

In this framework we can apply the homological stability theorems of Krannich [30]. So let us find a Yang-Baxter operator in the monoidal groupoid of bidecorated surfaces. Let $D = (X_1, 1, id) \in \mathbf{M}_2$ and write

$$D^{\sharp m} = D_1 \sharp D_2 \sharp \cdots \sharp (D_i \sharp D_{i+1}) \sharp \cdots \sharp D_m$$

where all $D_i = (X_1, 1, \varphi)$ is a disk. The indices are just to keep track of the disks glued to obtain X_m, defined in Section 6.1.

Definition 6.2.6. For $1 \leq i < m$ define a_i to be the isotopy class of a curve in the interior of $D_i \sharp D_{i+1}$ that is parallel to the boundary components. Moreover, set $T_i \in \mathrm{Aut}_{\mathbf{M}_2}(D^{\sharp m})$ to be the Dehn twist (by cutting and gluing) along a_i in $D^{\sharp m}$.

Lemma 6.2.7 (Lemma 3.5 in [19])**.** *The curves $a_1, \ldots, a_{m-1}$ satisfy the following:*

1. *for all $i < m$ the intersection number of a_i and a_{i+1} is 1, and*
2. *if $|i-j| > 1$ then $a_i \cap a_j = \emptyset$.*

Proof. Since a_i is in the interior of $D_i \natural D_{i+1}$ it can only intersect a_{i-1} and a_{i+1}. To see the intersection number of a_i and a_{i+1} see Figure 6.2 $\square$

Proposition 6.2.8 (Fact 3.9 and Proposition 3.11 in [14])**.** *Let S be a surface and a, b two isotopy classes of simple closed curves. Denote by T_a the Dehn twist along a, and similarly for b. If the intersection number of a and b is 1 then*

$$T_a T_b T_a = T_b T_a T_b. \tag{6.5}$$

If the intersection number of a and b is 0 then

$$T_a T_b = T_b T_a. \tag{6.6}$$

Proof. Suppose that the intersection number of a and b is 1. Then Equation (6.5) is equivalent to

$$(T_a T_b) T_a (T_a T_b)^{-1} = T_b.$$

Next, observe that for any f in the mapping class group of S

$$T_{f(a)} = f T_a f^{-1}.$$

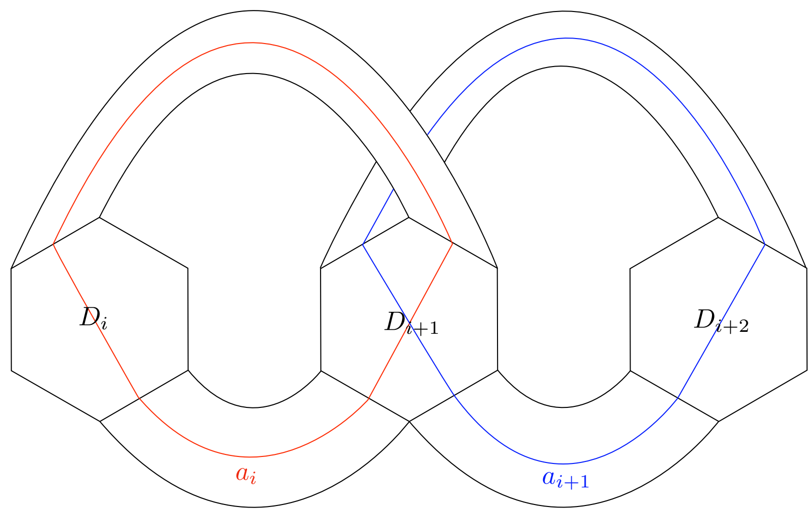

Figure 6.2: Intersection between a_i and a_{i+1}

Indeed, pick ϕ a representative of f and α a representative of a. The right hand side of this equation takes a neighborhood of $\phi(\alpha)$ to a neighborhood of α, performs a twist around α and then maps the neighborhood of α to the initial neighborhood of $\phi(\alpha)$. This is the same than twisting around $\phi(\alpha)$, see Fact 3.7 in [14]. Then Equation (6.5) is equivalent to

$$T_{T_aT_b(a)} = T_b.$$

This equation holds because $T_aT_b(a) = b$ whenever the intersection number of a and b is 1. We can visualize this fact in Figure 6.3. Now suppose that the intersection number of a and b is 0. Then

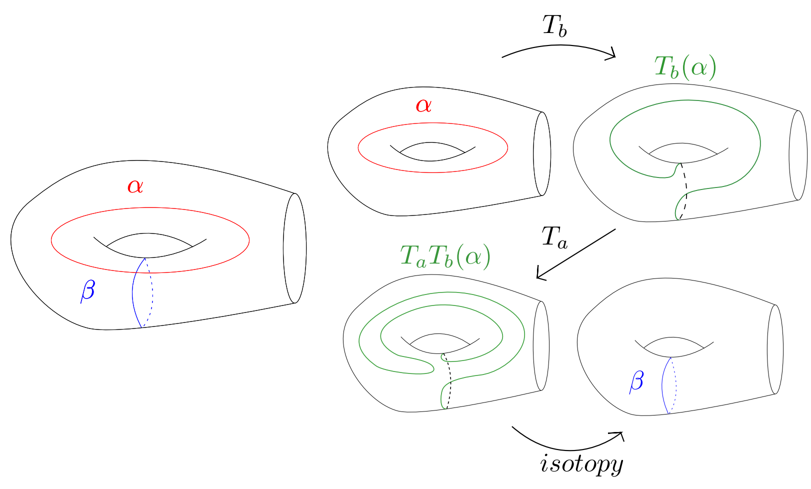

Figure 6.3: Relation $T_aT_b(a) = b$

by definition of the Dehn twist we can see that Equation (6.6) holds. □

Remark 6.2.9. The same relations hold for the inverse Dehn twist.

Corollary 6.2.10. *Let $D = (X_1, 1, id) \in \mathbf{M}_2$. Then (D, T_1^{-1}) is a Yang Baxter operator.*

Proof. First observe that

$$T_i \sharp id_D = T_i \quad \text{and} \quad id_D \sharp T_i = T_{i+1}.$$

Then combining Lemma 6.2.7, Proposition 6.2.8 and Remark 6.2.9 we obtain

$$(T_1^{-1} \sharp id_D)(id_D \sharp T_1^{-1})(T_1^{-1} \sharp id_D) = (id_D \sharp T_1^{-1})(T_1^{-1} \sharp id_D)(id_D \sharp T_1^{-1})$$

which is exactly the Yang-Baxter equation. □

Remark 6.2.11. Note that in the proofs of Theorems 4.1.1, 4.1.4 and 4.3.5 we only used braidings of the form $b_{A,X^{\oplus n}}$ and $b_{X,X^{\oplus n}}$. And we only needed that $b_{X,X}$ satisfies the Yang-Baxter Equation (6.4). In the monoidal category $\mathbf{M}_2$ we are interested in the homological stability for $A = X = D$. Hence a Yang-Baxter operator suffices to do the proofs from Chapter 4.

6.3 Disordered arc complex

Just as in the framework of Randal-Williams and Wahl [45], to apply the results of Krannich [30] we also need a highly connected destabilization space.

Definition 6.3.1. Let S be an orientable surface with at least one boundary component, and let b_0, b_1 be distinct points in ∂S. The *complex of non-separating arcs* $\mathcal{B}^\nu(S, b_0, b_1)$ is the simplicial complex whose p-simplices are collections of $p+1$ distinct isotopy classes of arc between b_0 and b_1 that admit representatives $a_0, \ldots, a_p$ such that

1. $a_i \cap a_j = \{b_0, b_1\}$ for each $i \neq j$, and
2. $S \backslash (a_0 \cup \cdots \cup a_p)$ is connected.

The superscript ν is added for convenience and defined by $\nu = 1$ if b_0 and b_1 lie on the same boundary component, and $\nu = 1$ otherwise.

Definition 6.3.2. Let S, b_0, b_1 be as before. The *disordered arc complex* is the subcomplex $\mathcal{D}^\nu(S_{g,r}, b_0, b_1) \subseteq \mathcal{B}^\nu(S, b_0, b_1)$ consisting of those simplices σ that admit arc representatives disordered arc complex $a_0, \ldots, a_p$ subject to the conditions to be in the non-separating arcs complex, and

3. the ordering of the arcs at b_0 agrees with the ordering of the arcs at b_1.

Theorem 6.3.3 (Theorem 2.4 in [19])**.** *The disordered arc complex $\mathcal{D}^\nu(S_{g,r}, b_0, b_1)$ is $\left(\frac{2g+\nu-5}{3}\right)$-connected.*

Proposition 6.3.4 (Proposition 2.2 in [19])**.** *Let $S = (S, m, \varphi) \in \mathbf{M}_2$. There is an isomorphism of semi-simplicial sets*

$$W_n^K(S, D)_\bullet \cong \mathcal{D}^\nu(S \sharp D^{\sharp n})_\bullet$$

where the marked points b_0 and b_1 are the midpoints of the intervals I_0 and I_1 in $S \sharp D^{\sharp n}$ and with $\nu = parity(n+m)$. That is $\nu = 1$ is 1 if I_0 and I_1 lie on the same boundary component of $S \sharp D^{\sharp n}$, and $\nu = 2$ otherwise.

Remark 6.3.5. The destabilization space $W_n^K(A,X)_\bullet$ defined by Krannich in [30] is slightly different from the one of Randal-Williams and Wahl which we defined in Definition 3.1.1. However, under some additional conditions these two spaces coincide. For more details see Section 7.3 in [30].

6.4 Homological stability

Now we have everything necessary to apply Krannich's results. The following theorem follows directly from Theorems 4.4.2 and 4.4.4.

Theorem 6.4.1 (Theorem 4.9 in [19])**.** *Let $S = (S, m, \varphi) \in \mathbf{M}_2$ with m odd, so that I_0 and I_1 are in the same boundary component. Let $F : \mathbf{M}_2|_{S,D} \to \mathrm{Mod}_{\mathbb{Z}}$ be a coefficient system. Write $F_n = F(S\sharp D^{\sharp n})$. The map*

$$H_i(\mathrm{Aut}_{\mathbf{M}_2}(S\sharp D^{\sharp n}); F_n) \to H_i(\mathrm{Aut}_{\mathbf{M}_2}(S\sharp D^{\sharp n+1}); F_{n+1})$$

is

1. *an epimorphism for $i \leq \frac{n}{3}$ and an isomorphism for $i \leq \frac{n-3}{3}$ if F is constant;*
2. *an epimorphism for $i \leq \frac{n-3k-2}{3}$ and an isomorphism for $i \leq \frac{n-3k-5}{3}$ if F has degree k at N and $n \geq N$;*
3. *an epimorphism for $i \leq \frac{n-k-2}{3}$ and an isomorphism for $i \leq \frac{n-k-5}{3}$ if F has split degree k at N and $n \geq N$.*

Sketch of proof. By Proposition 6.2.4 $\mathbf{M}_2$ is an E_1-module over $X = \bigsqcup_n X_n$ with stabilizing object $D = (X_1, 1, id)$. We have a grading

$$g_{\mathbf{M}_2} : \mathbf{M}_2|_{S,D} \to \mathbb{N}; S\sharp D^{\sharp n} \mapsto n-2.$$

Remark 4.4.1 yields that the canonical resolution is $\varphi(g_{\mathbf{M}_2})$-connected if and only if $\mathcal{D}^\nu(S\sharp D^{\sharp n})_\bullet$ is $(\varphi(g_{\mathbf{M}_2}(S\sharp D^{\sharp n})) - 1)$-connected. The connectivity of the disordered arc complex yields that the canonical resolution of $\mathbf{M}_2$ is $\left(\frac{n-1}{3}\right)$-connected, so $k = 3$.

The improvement of Theorem 4.4.2 by Remark 4.4.3(1) with $m = 4$, and Theorem 4.4.4 yield the desired bounds. □

Theorem 6.4.2. *Let $S^s_{g,r}$ be a connected surface of genus g with r boundary components and s punctures. Denote by $\Gamma(S^s_{g,r})$ its mapping class group. The map*

$$H_i(\Gamma(S^s_{g,r}); \mathbb{Z}) \to H_i(\Gamma(S^s_{g,r+1}); \mathbb{Z})$$

induced by gluing a pair of pants along one boundary component is always injective, and an isomorphism for $i \leq \frac{2g}{3}$. The map

$$H_i(\Gamma(S^s_{g,r+1}); \mathbb{Z}) \to H_i(\Gamma(S^s_{g+1,r}); \mathbb{Z})$$

induced by gluing a pair of pants along two boundary component is an epimorphism for $i \leq \frac{2g+1}{3}$, and an isomorphism for $i \leq \frac{2g-2}{3}$.

Proof. Let $S = (S^s_{0,r}, 1, \varphi)$. By Lemma 6.1.3 it follows that

$$S\sharp S^{\sharp 2g} \cong (S^s_{g,r}, 1+2g, \varphi) \quad \text{and} \quad S\sharp S^{\sharp 2g+1} \cong (S^s_{g,r+1}, 2+2g, \varphi).$$

Moreover the maps

$$S\sharp S^{\sharp 2g} \to S\sharp S^{\sharp 2g+1} \to S\sharp S^{\sharp 2g+2}$$

induce on automorphisms the two maps from the statement.

The first map is always injective on homology because postcomposing $\sharp D : S^s_{g,r} \to S^s_{g,r+1}$ with $S^s_{g,r+1} \to S^s_{g,r+1} \cup_{S^1} D^2 \simeq S^s_{g,r}$ filling in one new boundary component, is homotopic to the identity. Theorem 6.4.1 yields that

$$H_i(\mathrm{Aut}_{\mathbf{M}_2}(S\sharp D^{\sharp n}); \mathbb{Z}) \to H_i(\mathrm{Aut}_{\mathbf{M}_2}(S\sharp D^{\sharp n+1}); \mathbb{Z})$$

is surjective for $i \leq \frac{2g}{3}$. Since it is always injective, the isomorphism range is the epimorphism range. This proves the statement for the first map.

For the second map use again Theorem 6.4.1 to see that it is surjective for $i \leq \frac{2g+1}{3}$ and an isomorphism for $i \leq \frac{2g-2}{3}$. $\square$

Corollary 6.4.3. *Combining the two results from Theorem 6.4.2 we obtain that*

$$H_i(\Gamma(S^s_{g,r}); \mathbb{Z}) \to H_i(\Gamma(S^s_{g,r+1}); \mathbb{Z})$$

is an epimorphism for $i \leq \frac{2g}{3}$, and an isomorphism for $i \leq \frac{2g-2}{3}$. $\square$

Remark 6.4.4. By computations of Morita in [36] we know that the slope $\frac{2}{3}$ is the optimal slope for the isomorphism range. For instance

$$\mathbb{Z}/12 \cong H_i(\Gamma(S_{2,r}); \mathbb{Z}) \to H_i(\Gamma(S_{3,r}); \mathbb{Z}) \cong 0$$

is not injective.

6.5 Connectivity argument

The goal of this section is to prove Theorem 6.3.3. Instead of giving a full detailed proof, we will present different arguments to prove connectivity in a broader context. We follow Section 4 in Wahl's paper [50]. The arguments are collected from Harer, Hatcher, Ivanov, Randal-Williams and Wahl [18, 20, 26, 44, 49].

Let us start by defining the complexes we are interested in. Denote by S an arbitrary surface with boundary and $\Delta \subset \partial S$ a non-empty set of points. We consider isotopy classes of arcs in S with boundary in Δ. In particular, an arc a is trivial if it separates S into two components, one of which is a disc intersecting Δ only in the boundary of a. Let $\mathcal{A}(S, \Delta)$ be the simplicial complex whose vertices are the isotopy classes of non-trivial arcs in S with boundary in Δ. A p-simplex of $\mathcal{A}(S, \Delta)$ is a $(p+1)$-tuple of distinct isotopy classes of arcs $\langle a_0, \ldots, a_p \rangle$ representable by arcs with disjoint interiors.

Now we generalize Definition 6.3.1 of the non-separating arcs complex. Given two disjoint sets of points $\Delta_0, \Delta_1 \subset \partial S$ let $\mathcal{B}(S, \Delta_0, \Delta_1) \subset \mathcal{A}(S, \Delta_0 \cup \Delta_1)$ be the subcomplex of arcs which have one boundary point in Δ_0 and the other in Δ_1. Then define the non-separating arc complex $\mathcal{B}_0(S, \Delta_0, \Delta_1) \subset \mathcal{B}(S, \Delta_0, \Delta_1)$ whose simplices $\sigma = \langle a_0, \ldots, a_p \rangle$ satisfy that $S \backslash \{a_0, \ldots, a_p\}$ is connected (this is condition (2) in Definition 6.3.1).

The ordered arc complex $\mathcal{O}(S, b_0, b_1)$ is the subcomplex of $\mathcal{B}_0(S, \{b_0\}, \{b_1\})$ of simplices $\sigma = \langle a_0, \ldots, a_p \rangle$ such that the ordering of the arcs at b_0 is the opposite to the ordering at b_1.

The proof for the ordered arc complex can be adapted to the disordered arc complex. This is done in Section 2 from [19].

We start by explaining the arguments we use. There are three main types of connectivity arguments:

1. direct calculation;
2. suspension, that is showing that a complex is the suspension of a previous complex;
3. induction from the connectivity of a larger complex.

We prove connectivity along the inclusions

$$\mathcal{A}(S, \Delta_0 \cup \Delta_1) \hookleftarrow \mathcal{B}(S, \Delta_0, \Delta_1) \hookleftarrow \mathcal{B}_0(S, \Delta_0, \Delta_1) \hookleftarrow \mathcal{O}(S, b_0, b_1). \tag{6.7}$$

6.5.1 Connectivity of the arc complex

Theorem 6.5.1 (Theorem 4.1 in [50])**.** *The complex $\mathcal{A}(S, \Delta)$ is contractible, unless S is a disc or an annulus with Δ included in a single boundary component, in which case it is $(q+2r-7)$-connected, where $q = |\Delta|$ and r is the number of boundary components of S.*

To prove this theorem we use arguments (1) and (2) above. We use a suspension argument.

Proposition 6.5.2 (See Proposition in [20])**.** *Let Δ' be obtained from Δ by adding an extra point in a component of ∂S already containing a point of Δ. Then $\mathcal{A}(S, \Delta')$ is isomorphic to the suspension of $\mathcal{A}(S, \Delta)$. In particular if $\mathcal{A}(S, \Delta)$ is d-connected then $\mathcal{A}(S, \Delta')$ is $(d+1)$-connected.*

Sketch of proof. Let $\Delta' = \Delta \cup \{p'\}$ and $p \in \Delta$ the closest element to p' in the boundary component. Denote by I the arc in S joining p' to the next vertex of Δ' on the other side of p. Similarly I' is the arcs defined by exchanging the roles of p and p'. Now let X be the subcomplex of $\mathcal{A}(S, \Delta')$ consisting of collections of arcs not containing I or I'. Then

$$\mathcal{A}(S, \Delta') = Star(I) \cup_X Star(I').$$

We have a similar decomposition for the suspension $\Sigma\mathcal{A}(S, \Delta)$ through the embedding

$$X \to \Sigma\mathcal{A}(S, \Delta); x \to \left(\overline{x}, \frac{\theta(x) - \theta'(x)}{2}\right)$$

where $\overline{x}$ is obtained from x by collapsing the segment of the boundary joining p and p', and θ, θ' are the weights of the simplex at p and p' respectively. This yields a piecewise linear homeomorphism. $\square$

Remark 6.5.3. This proposition allows us to restrict the proof of the theorem to the case where there is at most one point on each boundary component.

Sketch of proof of Theorem 6.5.1. We first prove the statement in the special cases. Assume that S is a disk or a cylinder with all points on Δ in the same boundary component. Note that $\mathcal{A}(D^2, \Delta)$ in non-empty as soon as $q \geq 4$. Similarly, $\mathcal{A}(S^1 \times I, \Delta)$ in non-empty as soon as $q \geq 2$. The cases $(q = 4, r = 1)$ for D^2, and $(q = 2, r = 2)$ for $S^1 \times I$ hold because being (-1)-connected means non-empty. Proposition 6.5.2 proves by induction that $\mathcal{A}(S, \Delta)$ is $(q + 2r - 7)$-connected for larger values of q.

Now let us prove the general case. Using *Proposition* 6.5.2 we may assume that there is at most one point of Δ in each boundary component. First we prove that $\mathcal{A}(S, \Delta)$ is non-empty. If Δ has at least two points we are good because any arc between these two points is non-trivial. If Δ has a single point, then either S has genus at least 1 or there is at least three boundary components. In both cases we can find non-trivial arcs so $\mathcal{A}(S, \Delta)$ is non-empty.

Then fix a point $p \in \Delta$ and an arc $a \in \mathcal{A}(S, \Delta)$ with at least one of its endpoints at p. The idea is to show that there is a retraction $\mathcal{A}(S, \Delta) \to Star(a)$. □

6.5.2 Connectivity of the non-separating arc complex

Before stating any result we need new terminology. Let Δ_0, Δ_1 be two disjoint non-empty sets of points in ∂S. This defines a decomposition of ∂S into vertices (that is points in $\Delta_0 \cup \Delta_1$), edges between vertices and circles without vertices. An edge is *pure* if both endpoints are in the same set Δ_0 or Δ_1. Otherwise, we say that the edge is *impure*. Note there is always an even number of impure edges.

Our strategy will be to eliminate some cases by "pushing and surgery" and then use an inductive argument, type (3), for the general case. In the following lemmas we prove the desired result for some specific cases. After this we will prove the general case.

Lemma 6.5.4 (Lemma 4.4 in [50]). *If S has at least one pure boundary component then $\mathcal{B}(S, \Delta_0, \Delta_1)$ is contractible.*

Proof. Let $a \in \mathcal{B}(S, \Delta_0, \Delta_1)$ be an arc with one boundary point p in a pure boundary component. We build a retraction $\mathcal{B}(S, \Delta_0, \Delta_1) \to Star(a)$ by performing surgery on all arcs intersecting a. We need a map $f : I \times \mathcal{B}(S, \Delta_0, \Delta_1) \to \mathcal{B}(S, \Delta_0, \Delta_1)$ such that $f(s, Star(a)) = id_{Star(a)}$ for all $s \in I$, and $f(1, \mathcal{B}(S, \Delta_0, \Delta_1)) \subseteq Star(a)$.

Before defining the retraction we need to define *surgery on arcs.* Let b be an arcs intersecting a at x. Define $L(b)$ and $R(b)$ to be the arcs obtained from b by cutting at x and joining the new endpoints at p along a, then push them slightly so they become disjoint from a and b. We call $L(b)$ the arc running to the left of a towards p. Similarly, we call $R(b)$ the arc running to the right of a towards p. Observe that there is always exactly one of these arcs that is non-trivial in $\mathcal{B}(S, \Delta_0, \Delta_1)$ because p is in a pure boundary component (i.e. no arcs from p to other point in the same boundary component), and there is only one endpoint of b which is not is the same set Δ_0 or Δ_1 than p. Denote by $C(b)$ the only non-trivial arc among $L(b)$ and $R(b)$. After surgery on b, we keep $C(b)$ and forget the other arcs. This is represented in Figure 6.4.

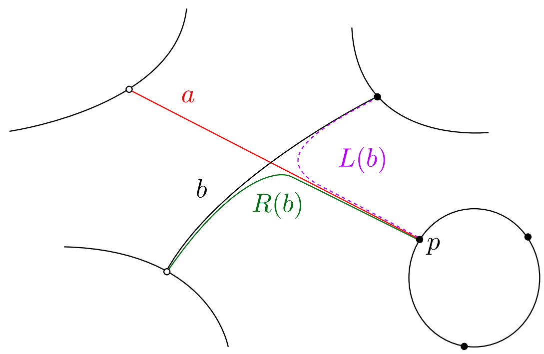

Figure 6.4: Surgery on one arc

Let $\sigma \in \mathcal{B}(S, \Delta_0, \Delta_1)$ and represent it by $\langle a_0 \dots, a_q \rangle$ with minimal and transverse intersection with a. Assume there are k germs $\gamma_1, \dots, \gamma_k$ intersecting a ordered from the closest to p to the furthest away from p. Define a sequence of k $(q+1)$-simplices $r_1(\sigma), \dots, r_k(\sigma)$ where $r_i(\sigma)$ is obtained from σ by performing surgery on the first i germs and keeping the last $q-i+1$ arcs. Observe that in $r_i(\sigma)$ we have γ_i and $C(\gamma_i)$.

A point in a simplex σ can be seen as a convex combination of arcs via barycentric coordinates $(t_0, \dots, t_m)$, the arc a_i having weight t_i. To the germ γ_i assign weight $w_i = t_{j_i}/2$. Then we have $\sum_{i=1}^{k} w_i \leq 1$. For $\sum_{j=1}^{i-1} w_j \leq s \leq \sum_{j=1}^{i} w_i$ define the retraction by

$$f(s, [\sigma, (t_0, \dots, t_m)]) = [r_i(\sigma), (v_0, \dots, v_{m+1})]$$

where $v_j = t_j$ except for the pair

$$(v_{j_i}, v_{m+1}) = \left(t_j - 2\left(s - \sum_{j=1}^{i-1} w_j \right), 2\left(s - \sum_{j=1}^{i-1} w_j \right) \right).$$

This means that the weight of (b_{j_i}, b_{m+1}) goes from $(t_{j_i}, 0)$ to $(0, t_{j_i})$ as s goes from $\sum_{j=1}^{i-1} w_j$ to $\sum_{j=1}^{i} w_j$. Finally, for $\sum_{j=1}^{k} w_j \leq s \leq 1$ define the retraction to be constant equal to

$$f(s, [\sigma, (t_0, \dots, t_m)]) = f\left(\sum_{j=1}^{k} w_j, [\sigma, (t_0, \dots, t_m)] \right).$$

Clearly $f(s, Star(a)) = id_{Star(a)}$ for all $s \in I$. It remains to prove that $f(1, [\sigma, (t_0, \dots, t_m)])$ is in $Star(a)$. Indeed, $f(1, [\sigma, (t_0, \dots, t_m)])$ lies in the face of $r_k(\sigma)$ which is in $Star(a)$, that is the face whose all vertices are arcs that do not intersect a. Hence our retraction is well defined. Moreover this retraction is continuous because going to a face of σ corresponds to a t_i (and corresponding w_{j_i} if it exists) going to zero. $\square$

Lemma 6.5.5 (Lemma 4.5 in [50]). *If S has at least one pure edge between a pure and an impure one in one boundary component, then $\mathcal{B}(S, \Delta_0, \Delta_1)$ is contractible.*

Proof. Let $q \in \Delta_1$ and $p, p', p'' \in \Delta_0$ such that they are all in the same boundary component and ordered that way counterclockwise. Let I the arc from q to p' passing around p. We want to build a retraction $\mathcal{B}(S, \Delta_0, \Delta_1) \to Star(I)$. The intuition behind this retraction is to move arcs from p to p'. For a non-trivial arc a with one endpoint at p, we want to move it along I until its endpoint reaches p'. This situation is represented in Figure 6.5. For a collection of arcs we push

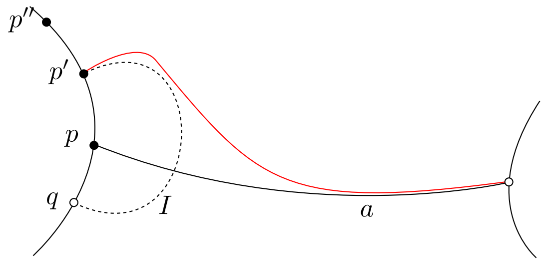

Figure 6.5: Pushing one arc

them one by one in a similar way, see Figure 6.6. Now let us formalize this retraction. We need

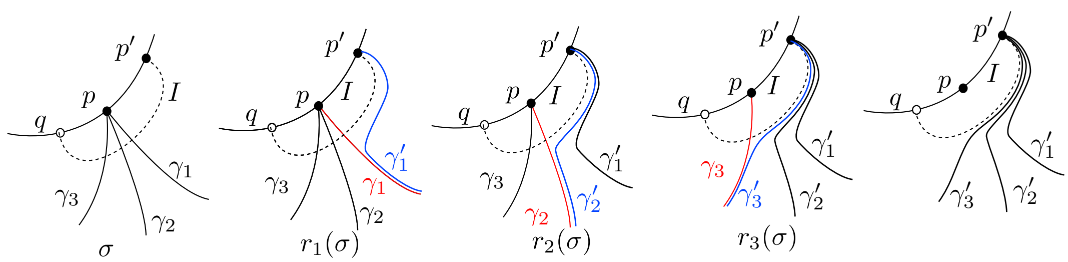

Figure 6.6: Pushing multiple arcs

a map $f : I \times \mathcal{B}(S, \Delta_0, \Delta_1) \to \mathcal{B}(S, \Delta_0, \Delta_1)$ such that $f(s, Star(I)) = id_{Star(I)}$ for all $s \in I$, and $f(1, \mathcal{B}(S, \Delta_0, \Delta_1)) \subseteq Star(I)$.

For a simplex $\sigma = \langle a_0, \dots, a_m \rangle$ of $\mathcal{B}(S, \Delta_0, \Delta_1)$, consider the arcs attached at p. Suppose there are k germs of arcs $\gamma_1, \dots, \gamma_k$ attached at p in that order, with γ_i a germ of the arc a_{j_i}. It is possible that $j_i = j_{i'}$ for $i \neq i'$ if a_{j_i} is a loop at p. Define a sequence of k $(l+1)$-simplices $r_1(\sigma), \dots, r_k(\sigma)$ where $r_i(\sigma)$ is obtained from σ by moving the first i germs of arcs at p to p', and keeping the last $m - i + 1$ germs. Note that γ_i has one copy at p and another at p', these are the red and blue arcs at each step of Figure 6.6.

Denote by L the operator that moves the first germ of the arc at p to p'. If we write $r_i(\sigma) = \langle b_0, \dots, b_{m+1}$ then $b_l = L^{\epsilon_i(l)}(a_l)$ for $l \leq m$, and $b_{m+1} = L^{\epsilon_{i+1}(j_i)}(a_{j_i})$, where $\epsilon_i(l)$ is the number of $j < i$ such that γ_j is a germ of a_l.

As in Lemma 6.5.4 we view points in σ via barycentric coordinates $(t_0, \dots, t_m)$. To γ_i assign the weight $w_i = t_{j_i}/2$. And define the same exactly as in Lemma 6.5.4, geometrically is different but we use the same notation. For $\sum_{j=1}^{i-1} w_j \leq s \leq \sum_{j=1}^{i} w_i$ define the retraction by

$$f(s, [\sigma, (t_0, \dots, t_m)]) = [r_i(\sigma), (v_0, \dots, v_{m+1})]$$

where $v_j = t_j$ except for the pair

$$(v_{j_i}, v_{m+1}) = \left(t_j - 2\left(s - \sum_{j=1}^{i-1} w_j\right), 2\left(s - \sum_{j=1}^{i-1} w_j\right)\right).$$

Finally, for $\sum_{j=1}^{k} w_j \leq s \leq 1$ define the retraction to be constant equal to

$$f(s, [\sigma, (t_0, \ldots, t_m)]) = f\left(\sum_{j=1}^{k} w_j, [\sigma, (t_0, \ldots, t_m)]\right).$$

It is clear that $f(s, Star(I)) = id_{Star(I)}$ for all $s \in I$. Moreover observe that $f(1, [\sigma, (t_0, \ldots, t_m)])$ lies in the face of $r_k(\sigma)$ which is in $Star(I)$. Hence our retraction is well defined and continuous by the same argument than Lemma 6.5.4. □

Lemma 6.5.6 (Lemma 4.6 in [50])**.** *Let $\mathcal{B}(S, \Delta_0, \Delta_1)$ be non-empty. Then adding a pure edge between two impure edges increases the connectivity by one.*

Sketch of proof. The argument we use is similar to the argument of Proposition 6.5.2. We show that the complex $\mathcal{B}(S, \Delta_0', \Delta_1')$ obtained from $\mathcal{B}(S, \Delta_0, \Delta_1)$ after adding a pure edge between two impure edges, is the suspension of $\mathcal{B}(S, \Delta_0, \Delta_1)$. Note that to get $\mathcal{B}(S, \Delta_0', \Delta_1')$ we added a point p' to Δ_0 or Δ_1 in a boundary component between two impure edges. This situation is represented in Figure 6.7.

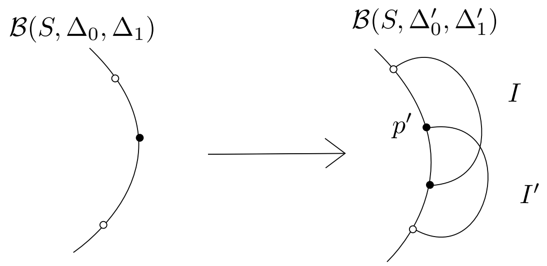

Figure 6.7: Suspension of the complex of non-intersecting arcs

Then we have

$$\mathcal{B}(S, \Delta_0', \Delta_1') = Star(I) \cup_X Star(I')$$

where X is the subcomplex of $\mathcal{B}(S, \Delta_0', \Delta_1')$ consisting of collections of arcs not containing I or I'. Now we use an embedding

$$X \to \mathcal{B}(S, \Delta_0, \Delta_1)$$

as in Proposition 6.5.2 to get a piecewise linear homeomorphism

$$Star(I) \cup_X Star(I') \to \Sigma\mathcal{B}(S, \Delta_0, \Delta_1).$$

□

Lemma 6.5.7 (Lemma 4.7 in [50])**.** *Let S be such that it has at least one impure edge, and $\mathcal{B}(S, \Delta_0, \Delta_1)$ be non-empty. Adding a boundary component to S disjoint from Δ increases the connectivity of $\mathcal{B}(S, \Delta_0, \Delta_1)$ by one.*

Theorem 6.5.8 (Theorem 4.3 in [50])**.** *Let Δ_0, Δ_1 be two disjoint non-empty sets of points in ∂S. Then $\mathcal{B}(S, \Delta_0, \Delta_1)$ is $(4g+r+r'+l+m-6)$-connected, where g is the genus of S, r is the number of connected components of ∂S, r' is the number of boundary components of ∂S containing $\Delta_0 \cup \Delta_1$, l is half the number of impure edges and m is the number of pure edges.*

Theorem 6.5.9. *Let Δ_0, Δ_1 be two disjoint non-empty sets of points in ∂S. Then $\mathcal{B}_0(S, \Delta_0, \Delta_1)$ is $(2g + r' - 3)$-connected, where g is the genus of S and r' the number of boundary components of ∂S containing $\Delta_0 \cup \Delta_1$.*

Proof. We proceed by induction on (g, r, q) for $q = |\Delta_0 \cup \Delta_1| \geq 2$.

First note that the statement is true for $g = 0$ and $r' \leq 2$ and any $r \geq r'$ and any q. And the complex is non-empty whenever $g \geq 1$ or $r' \geq 2$.

Take S, Δ_0, Δ_1 such that $(g, r, q) \geq (0, 3, 2)$. Then

$$2g + r' - 3 \leq 4g + r + r' + l + m - 6$$

because $r \geq 1$ and $l + m \geq 1$. Now let $k \leq 2g + r' - 3$ and consider a map

$$f : S^k \to \mathcal{B}_0(S, \Delta_0, \Delta_1)$$

which we assume to be simplicial for some triangulation of S^k. We can extend this to a simplicial map

$$\hat{f} : D^{k+1} \to \mathcal{B}(S, \Delta_0, \Delta_1).$$

We want a lift making (only) the upper triangle commute in the square

$$\begin{tikzcd} S^k \arrow[r, "f"] \arrow[d, hook] & \mathcal{B}_0(S, \Delta_0, \Delta_1) \arrow[d, hook] \\ D^{k+1} \arrow[r, "\hat{f}"'] \arrow[ur, dashed] & \mathcal{B}(S, \Delta_0, \Delta_1) \end{tikzcd}$$

to prove that f is nullhomotopic. We call a simplex σ of D^{k+1} *regular bad* if $\hat{f}(\sigma) = \langle a_0, \ldots, a_p \rangle$ and each a_j separates $S \backslash (\cup_{i \neq j} a_i)$. Let σ be a regular bad simplex of maximal dimension p. Then we have a decomposition

$$S \backslash \hat{f}(\sigma) = \bigsqcup_{i=1}^{c} X_i$$

of $S \backslash \hat{f}(\sigma)$ into connected components. The maximality of σ yields a restriction of $\hat{f}$ to a map $Link(\sigma) \to J_\sigma$ where

$$J_\sigma = \mathcal{B}_0(X_1, \Delta_0^1, \Delta_1^1) * \cdots * \mathcal{B}_0(X_c, \Delta_0^c, \Delta_1^c),$$

where all Δ_ϵ^i come from Δ_ϵ and they are non-empty because the arcs of $\hat{f}$ are impure.

We can see that each X_i has $(g_i, r_i, q_i) < (g, r, q)$. Our induction hypothesis yields that $\mathcal{B}_0(X_i, \Delta_0^i, \Delta_1^i)$ is $(2g_i + r_i' - 3)$-connected.

The Euler characteristic of $S \backslash \hat{f}(\sigma)$ gives

$$\sum_i 2 - 2g_i - r_i = 2 - g - r + p' + 1$$

where $p' + 1$ is the number of arcs is $\hat{f}(\sigma)$, so $p' < p$. Now adding the equation $\sum_i r_i - r_i' = r - r'$ to the previous one we obtain

$$\sum_i 2g_i + r_i' = 2g + r' - p' + 2c - 3.$$

We know that connectivity behaves well under joins, in our case we get that J_σ is $(2g+r'-p'+c-5)$-connected, see Proposition 6.1 in [50].

Recall that $c \geq 2$ and $p' \leq p$, so $k - p \leq 2g + r' - p' + c - 5$. Hence we can extend the restriction of $\hat{f}$ to $Link(\sigma) \simeq S^{k-p}$ to a map $F : K \to J_\sigma$ where K is a $(k - p + 1)$-disc whose boundary is $Link(\sigma)$. Now modify $\hat{f}$ on the interior of $Star(\sigma) \simeq \partial\sigma * K$ using $\hat{f} * F$, that means that we replace $\hat{f}$ by $\hat{f} * F$ on $Star(\sigma)$. This can be done because $\hat{f}$ and $\hat{f} * F$ agree on $\partial Star(\sigma)$. Now we show that the situation has improved, this means that we reduced the number of regular bad simplices of maximal dimension. If a simplex $\alpha * \beta$ is regular bad then β is trivial because β cannot separate $S \backslash \hat{f}(\sigma)$ by maximality of σ. Thus $\alpha * \beta = \alpha$ is a face of sigma. Hence the new bad simplices in the image of $\hat{f}$ are faces of σ, in particular they are of strictly smaller dimension.

This reduces the number of regular bad simplices of maximal dimension. After removing all regular bad simplices our map has image inside $\mathcal{B}_0(S, \Delta_0, \Delta_1)$. So the map is null homotopic, proving the connectivity. Our result follows by induction. □

6.5.3 Connectivity of the (dis)ordered arc complex

In this subsection we use arguments of type (3), that is inductive deduction from a larger complex.

Theorem 6.5.10. *The ordered arc complex $\mathcal{O}(S, b_0, b_1)$ is $(g-2)$-connected.*

Proof. If S has genus 0 or 1 then the result is trivial. Now we proceed by induction assuming $g \geq 2$.

Let $k \leq g - 2$ and consider a simplicial map

$$f : S^k \to \mathcal{O}(S, b_0, b_1).$$

Since $g - 2 \leq 2g + r' - 3$, Theorem 6.5.9 implies that there is an extension

$$\hat{f} : D^{k+1} \to \mathcal{B}_0(S, \{b_0\}, \{b_1\})$$

of f.

We want to modify $\hat{f}$ so that its image lies in $\mathcal{O}(S, b_0, b_1)$. That is, we want a lift making (only) the upper triangle commute in the square

$$\begin{tikzcd}
S^k \arrow[r, "f"] \arrow[d, hook] & \mathcal{O}(S, b_0, b_1) \arrow[d, hook] \\
D^{k+1} \arrow[r, "\hat{f}"'] \arrow[ur, dashed] & \mathcal{B}_0(S, \{b_0\}, \{b_1\})
\end{tikzcd}$$

which proves that f is null-homotopic. To do so, we proceed as in Theorem 6.5.9. Let $\sigma = \langle a_0, \ldots, a_p \rangle$ be a p-simplex of $\mathcal{B}_0(S, \{b_0\}, \{b_1\})$ such that the arcs are ordered $a_0 < \cdots < a_p$ in the anti-clockwise ordering at b_0. There is an unique decomposition $\sigma = \sigma^g * \sigma_b$ where $\sigma^g = \langle a_0, \ldots, a_i \rangle$ with i maximal such that the clockwise ordering at b_i starts with $a_0 < \cdots < a_i$. Note that if $\sigma = \sigma^g$ then $\sigma \in \mathcal{O}(S, b_0, b_1)$. On the contrary, if $\sigma^g = \emptyset$ then we say that σ is *purely bad*. We want to get rid of all purely bad simplices in the image of $\hat{f}$.

Let $\sigma = \langle a_0, \ldots, a_p \rangle$ be a purely bad p-simplex, then $S \backslash \sigma$ has genus at least $g - p$. If b_0 and b_1 lie on different boundary components then cutting along a_0 reduces by one the number of boundary components and does not affect the genus. Cutting along any remaining arcs can reduce the genus at most by one. Then cutting along all p of them can reduce the genus at most by p. Now if b_0 and b_1 lie on the same boundary component, pick two arcs a_i, a_j such that they are ordered clockwise at b_0 and at b_1. Then the complement $S \backslash (a_i \cup a_j)$ has the same number of boundary components than S. Using the Euler characteristic we conclude that the genus decreased by 1. We can visualize this case in Figure 6.8. The $p - 1$ arcs left can each reduce the genus by 1.

Finally, let σ be a simplex of D^{k+1} such that $\hat{f}(\sigma)$ is purely bad of maximal dimension. The link of σ is mapped to $\mathcal{O}(S \backslash \hat{f}(\sigma), b_0', b_1')$ where b_ϵ' are copies of b_ϵ living between the boundary containing b_0 and the first (anti-clockwise) arc of σ at b_0, and the first (clockwise) arc of σ at b_1. Note that $\hat{f}(\sigma)$ is a p'-simplex with $p' \leq p$ so $S \backslash \hat{f}(\sigma)$ has genus at least $g - p$. Recall that $Link(\sigma) \simeq S^{k-p}$ and that $k - p \leq g - p - 2$. Since $p' \geq 2$ and $\hat{f}(\sigma)$ is bad then $S \backslash \hat{f}(\sigma)$ has genus $\tilde{g} < g$. Inductively, the restriction of $\hat{f}$ to $Link(\sigma)$ extends to F a map on $K \simeq D^{k+1-p}$. As in Theorem 6.5.9, replace $\hat{f}$ by $\hat{f} * F$ on $Star(\sigma) \simeq \partial\sigma * K$. In this process we got rid of one purely bad simplex of maximal dimension. This way our map ha image inside $\mathcal{O}(S, b_0, b_1)$ and it is null homotopic, proving the connectivity of the ordered arc complex when S has genus g. The result follows by induction. $\square$

Similarly we can prove the Theorem 6.3.3 on the connectivity of the disordered arc complex.

Idea of proof of Theorem 6.3.3. Just as in Theorem 6.5.10 we want build the dashed arrow in the square

$$\begin{tikzcd}
S^k \arrow[r, "f"] \arrow[d, hook] & \mathcal{D}^\nu(S, b_0, b_1) \arrow[d, hook] \\
D^{k+1} \arrow[r, "\hat{f}"'] \arrow[ur, dashed] & \mathcal{B}_0(S, \{b_0\}, \{b_1\})
\end{tikzcd}$$

making the upper triangle commute. For this we define accordingly *purely bad simplices* and we do surgery to remove regular bad simplices of maximal dimension. Repeating this process finitely many times we eliminate all regular bad simplices, yielding the desired dashed arrow. This implies

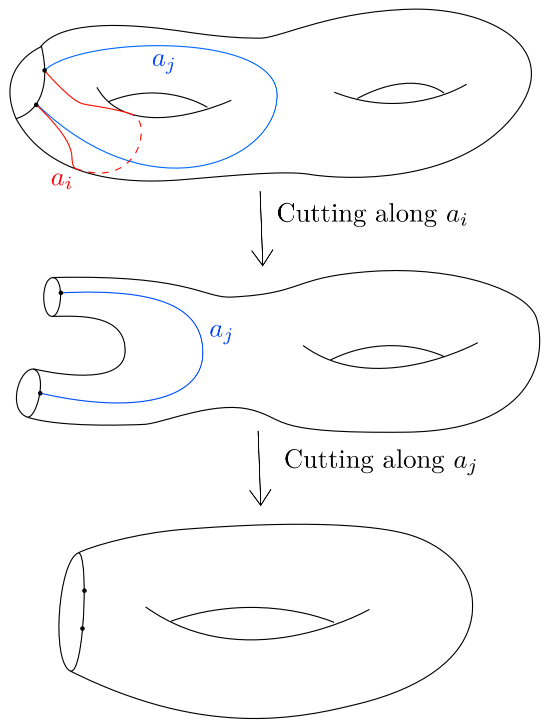

Figure 6.8: Removing two arcs with same ordering at b_0 and b_1

that f is null-homotopic, in a range that we do not specify here. For detail look at Theorem 2.4 in [19]. □

Bibliography

[1] V. I. Arnold, Certain topological invariants of algebraic functions, Trudy Moskov. Mat. Obšč. **21** (1970), 27–46

[2] E. Artin, Theory of braids, Ann. of Math. (2) **48** (1947), 101–126

[3] M. Barratt and S. B. Priddy, On the homology of non-connected monoids and their associated groups, Comment. Math. Helv. **47** (1972), 1–14

[4] S. K. Boldsen, Improved homological stability for the mapping class group with integral or twisted coefficients, Math. Z. **270** (2012), no. 1-2, 297–329

[5] K. S. Brown, *Cohomology of groups*, corrected reprint of the 1982 original, Graduate Texts in Mathematics, 87, Springer, New York, 1994

[6] J. F. Celis-Rojas, Homological Stability of Symmetric Groups, 2023

[7] J. Cerf, Topologie de certains espaces de plongements, Bull. Soc. Math. France **89** (1961), 227–380

[8] T. Church, J. S. Ellenberg and B. Farb, FI-modules and stability for representations of symmetric groups, Duke Math. J. **164** (2015), no. 9, 1833–1910

[9] M. Dehn, Die Gruppe der Abbildungsklassen, Acta Math. **69** (1938), no. 1, 135–206

[10] M. Dehn, *Papers on group theory and topology*, translated from the German and with introductions and an appendix by John Stillwell, Springer, New York, 1987

[11] W. G. Dwyer, Twisted homological stability for general linear groups, Ann. of Math. (2) **111** (1980), no. 2, 239–251

[12] R. D. Edwards and R. C. Kirby, Deformations of spaces of imbeddings, Ann. of Math. (2) **93** (1971), 63–88

[13] E. R. Fadell and L. P. Neuwirth, Configuration spaces, Math. Scand. **10** (1962), 111–118

[14] B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, 49, Princeton Univ. Press, Princeton, NJ, 2012

[15] S. Galatius et al., The homotopy type of the cobordism category, Acta Math. **202** (2009), no. 2, 195–239

[16] A. Gramain, Le type d'homotopie du groupe des difféomorphismes d'une surface compacte, Ann. Sci. École Norm. Sup. (4) **6** (1973), 53–66

[17] D. R. Grayson, Higher algebraic K-theory. II (after Daniel Quillen), in *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*, pp. 217–240, Lecture Notes in Math., Vol. 551, Springer, Berlin-New York

[18] J. L. Harer, Stability of the homology of the mapping class groups of orientable surfaces, Ann. of Math. (2) **121** (1985), no. 2, 215–249

[19] O. Harr, M. Vistrup and N. Wahl, Disordered arcs and Harer stability, High. Struct. **8** (2024), no. 1, 193–223

[20] A. E. Hatcher, On triangulations of surfaces, Topology Appl. **40** (1991), no. 2, 189–194; MR1123262

[21] A. E. Hatcher and N. Wahl, Stabilization for mapping class groups of 3-manifolds, Duke Math. J. **155** (2010), no. 2, 205–269

[22] A. E. Hatcher and K. Vogtmann, Tethers and homology stability for surfaces, Algebr. Geom. Topol. **17** (2017), no. 3, 1871–1916

[23] P. J. Hilton, G. Mislin and J. Roitberg, *Localization of nilpotent groups and spaces*, North-Holland Mathematics Studies Notas de Matemática, No. 15 No. 55, North-Holland, Amsterdam-Oxford, 1975 American Elsevier Publishing Co., Inc., New York, 1975

[24] P. J. Hilton, G. Mislin and J. Roitberg, Sphere bundles over spheres and non-cancellation phenomena, in *Symposium on Algebraic Topology (Battelle Seattle Res. Center, Seattle, Wash., 1971)*, pp. 34–46, Lecture Notes in Math., Vol. 249, Springer, Berlin-New York

[25] M. Hochster, Cohen-Macaulay varieties, geometric complexes, and combinatorics, in *The mathematical legacy of Richard P. Stanley*, 219–229, Amer. Math. Soc., Providence, RI

[26] N. V. Ivanov, On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients, in *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*, 149–194, Contemp. Math., 150, Amer. Math. Soc., Providence, RI,

[27] A. Joyal and R. H. Street, Braided tensor categories, Adv. Math. **102** (1993), no. 1, 20–78

[28] W. van der Kallen, Homology stability for linear groups, Invent. Math. **60** (1980), no. 3, 269–295

[29] M. C. Kerz, The complex of words and Nakaoka stability, Homology Homotopy Appl. **7** (2005), no. 1, 77–85

[30] M. Krannich, Homological stability of topological moduli spaces, Geom. Topol. **23** (2019), no. 5, 2397–2474

[31] S. Mac Lane, *Categories for the working mathematician*, second edition, Graduate Texts in Mathematics, 5, Springer, New York, 1998

[32] F. S. Macaulay, *The algebraic theory of modular systems*, revised reprint of the 1916 original, Cambridge Mathematical Library, Cambridge Univ. Press, Cambridge, 1994

[33] I. H. Madsen and M. S. Weiss, The stable moduli space of Riemann surfaces: Mumford's conjecture, Ann. of Math. (2) **165** (2007), no. 3, 843–941

[34] D. McDuff and G. B. Segal, Homology fibrations and the "group-completion" theorem, Invent. Math. **31** (1975/76), no. 3, 279–284

[35] B. Mirzaii and W. van der Kallen, Homology stability for unitary groups, Doc. Math. **7** (2002), 143–166

[36] S. Morita, Generators for the tautological algebra of the moduli space of curves, Topology **42** (2003), no. 4, 787–819

[37] J. R. Munkres, Obstructions to the smoothing of piecewise-differentiable homeomorphisms, Ann. of Math. (2) **72** (1960), 521–554

[38] M. Nakaoka, Decomposition theorem for homology groups of symmetric groups, Ann. of Math. (2) **71** (1960), 16–42

[39] R. S. Palais, Local triviality of the restriction map for embeddings, Comment. Math. Helv. **34** (1960), 305–312

[40] A. Putman, A new approach to twisted homological stability with applications to congruence subgroups, J. Topol. **16** (2023), no. 4, 1315–1388

[41] D. G. Quillen, Homotopy properties of the poset of nontrivial p-subgroups of a group, Adv. in Math. **28** (1978), no. 2, 101–128

[42] D. Quillen, August 8, 1970: The stability problem, 1971

[43] O. Randal-Williams, 'Group-completion', local coefficient systems and perfection, Q. J. Math. **64** (2013), no. 3, 795–803

[44] O. Randal-Williams, Resolutions of moduli spaces and homological stability, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 1, 1–81

[45] O. Randal-Williams and N. Wahl, Homological stability for automorphism groups, Adv. Math. **318** (2017), 534–626

[46] G. A. Reisner, Cohen-Macaulay quotients of polynomial rings, Advances in Math. **21** (1976), no. 1, 30–49

[47] R. P. Stanley, Cohen-Macaulay complexes, in *Higher combinatorics (Proc. NATO Advanced Study Inst., Berlin, 1976)*, pp. 51–62, NATO Adv. Study Inst. Ser. C: Math. Phys. Sci., 31, Reidel, Dordrecht-Boston, Mass.

[48] U. Tillmann, Mumford's conjecture—a topological outlook, in *Handbook of moduli. Vol. III*, 399–429, Adv. Lect. Math. (ALM), 26, Int. Press, Somerville, MA

[49] N. Wahl, Homological stability for the mapping class groups of non-orientable surfaces, Invent. Math. **171** (2008), no. 2, 389–424

[50] N. Wahl, Homological stability for mapping class groups of surfaces, in *Handbook of moduli. Vol. III*, 547–583, Adv. Lect. Math. (ALM), 26, Int. Press, Somerville, MA,

[51] N. Wahl, The Mumford conjecture, Madsen-Weiss and homological stability for mapping class groups of surfaces, in *Moduli spaces of Riemann surfaces*, 109–138, IAS/Park City Math. Ser., 20, Amer. Math. Soc., Providence, RI,

[52] J. H. C. Whitehead, Manifolds with transverse fields in euclidean space, Ann. of Math. (2) **73** (1961), 154–212