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Homological Stability of Symmetric Groups

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Nous, les mathématiciens, on ne coûte pas cher. Pour travailler, il suffit d'un crayon, de papier et d'un ordinateur. Ah, j'oubliais, d'une corbeille à papier. C'est fou ce que l'on peut écrire comme bêtises.

Jean-Pierre Serre

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Abstract

Homological stability is a topological property of a sequence of groups. We would like to determine whether the group homomorphisms from the sequence induce isomorphisms in homology, and if so, in which range. After recalling some basic algebraic notions we explain the concept of spectral sequence and how it is relevant in the homological stability framework. Then we study methods by Andrew Putman for proving homological stability. We focus on the homology of symmetric groups and give all details to find a stable range.

Contents

Introduction

Group homology is a mathematical construction, using techniques from algebraic topology and commutative algebra, that associates a sequence of groups to a given group. We begin in Chapter 1 presenting the basic definitions on this subject. We take advantage of our double point of view and explore topological and algebraic intuitions to gain a better understanding of these notions, mostly taken from [2].

In Chapter 2 we present spectral sequences. These are mathematical objects first introduced by Leray in [7] that are powerful tools in homological algebra, they are a higher analogue of long exact sequences. We explore different ways to get spectral sequences and state classical results about the convergence of spectral sequences. The reader who wants to see the proof of these statements can find them in [9]. And we give a simple yet interesting example of an argument using spectral sequences to prove that the Tor functor is symmetric. Then, we present some of the work of Jean-Pierre Serre whose contributions on this subject are ground-braking [13]. We present the Lyndon-Hochschild-Serre spectral sequence [8], [6] and state and prove some results regarding this spectral sequence.

Chapter 3 focuses on homological stability. This subject was pioneered by Quillen [12], and many other mathematicians have made significant contributions. Since we are mainly interested in symmetric groups we look into the work of Nakaoka [10]. Rather than exploring a general case we exploit our knowledge on spectral sequences presented in Chapter 2. Although we only give proofs for homology with trivial coefficients, we set up a background fit for proofs with twisted coefficients.

It is in Section 3.4 that we attack the main objective of this project. Our goal is to give all details and clarify all the steps to prove homological stability for symmetric groups.

Theorem 3.4.1 *Consider the increasing sequence of groups given by the symmetric groups* $\{\mathfrak{S}_n\}_{n\geq 1}$. *Then for all* $k \geq 0$ *the map*

$$H_k(\mathfrak{S}_n) \to H_k(\mathfrak{S}_{n+1})$$

is an isomorphism for $n \geq 2k+2$ *and a surjection for* $n = 2k+1$.

We start by studying an article about twisted homological stability from Putman [11]. We recover and explain all tools needed to prove this theorem. This includes many results from other papers that are admitted in Putman's article. We set a clear background with some definitions and carefully add more advanced results. The most important result we prove again is the so called classical stability theorem, taken from a paper by Hatcher [5].

Additionally, Appendix A presents interesting results about the cohomology ring of dihedral groups. We show how spectral sequences can be used to compute cohomology rings. This is a taste of the power of spectral sequences, even with few tools we can prove strong results. The one we present is the following computation exhibiting an infinite family of finite groups having the same mod 2 cohomology.

Theorem A.0.4 *Let $n \in \mathbb{N}$. For the dihedral group D_{2^n} we have*

$$H^*(D_{2^n}; \mathbb{F}_2) \cong \mathbb{F}_2[x, y, w]/(xy)$$

where x, y are one dimensional and w is two dimensional.

Using the Lyndon-Hochschild-Serre spectral sequence and comparing group extensions in a clever way we are able to identify all differentials in a spectral sequence, which allow us to find the cohomology ring of these dihedral groups.

Chapter 1

Some homological algebra

Let G be a group and consider the group ring $\mathbb{Z}G$. We would like to find a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. There are many methods to do it, we will present two of them. One that uses topology and another that is purely algebraic. This will allow us to define and calculate the homology of G.

Furthermore we introduce some algebraic notions that give us more general ways to define the homology of a group and an algebraic/topological setup on which the homology of groups appears. Most of these concepts will be used and explored in Chapter 3.

1.1 Topological free resolutions

Topology has provided many tools to solve algebraic problems. Here we study how topology gives us methods to build projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$.

Definition 1.1.1 (G-complex)**.** Let G be a group. We say that a CW-complex X is a *G-complex* if there is an action of G on the cells of X.

Remark 1.1.2. Notice that if X is a G-complex then its cellular chain complex $C_*^{cell}(X)$ is a chain complex of $\mathbb{Z}G$-modules. Additionally, if the action of G on X is free, one can see that $C_*^{cell}(X)$ is chain complex of free $\mathbb{Z}G$-modules.

From these definitions the next proposition follows immediately.

Proposition 1.1.3. *Let X be a contractible free G-complex. Then its augmented cellular chain complex*

$$C_*^{cell}(X) \to \mathbb{Z} \to 0$$

is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

Now it is natural to consider Eilenberg-Mac Lane spaces since $K(G,1)$ is connected, its fundamental group is isomorphic to G and its universal cover X is a contractible free G-complex.

Proposition 1.1.4. *The augmented cellular chain complex of the universal cover of $K(G,1)$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

Example 1.1.5. Let $G \cong \mathbb{Z}^2$. It is known that the torus T^2 has fundamental group isomorphic to $\mathbb{Z}^2$. Then there is a free action of $\mathbb{Z}^2$ on its universal cover $\mathbb{R}^2$.

Since the set orbit of n-cells on the universal cover forms a basis for $C_n^{cell}(\mathbb{R}^2)$ seen as a free $\mathbb{Z}^2$-module. We will give the details of a cellular structure on $\mathbb{R}^2$ on which $\mathbb{Z}^2$ acts freely and transitively.

- The 0-cells are given by $Z^2 \subset \mathbb{R}^2$.
- The 1-cells are the segments of the form $[(a,b),(a+1,b)]$ and $[(a,b),(a+1,b)]$ for all $a, b \in \mathbb{Z}$.
- The 2-cells are given by the squares with vertices $(a,b),(a+1,b),(a,b+1),(a+1,b+1)$ for all $a, b \in \mathbb{Z}$.

Here $\mathbb{Z}^2$ acts by translation on $\mathbb{R}^2$. It is clear that the $\mathbb{Z}^2$ action on this cellular structure is free and transitive. Now we note that there is only one G-orbit of 0-cells, two orbits of 1-cells (vertical and horizontal segments), and one orbit of 2-cells.

We get the following free resolution.

$$0 \longrightarrow \mathbb{Z}G \longrightarrow (\mathbb{Z}G)^{\oplus 2} \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$$

Remark 1.1.6. Note that this example also gives us the tools to compute a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ where $G = \mathbb{Z}^n, n \in \mathbb{N}$.

1.2 Standard resolution

We would like to have a functorial way to find a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

Definition 1.2.1 (Standard resolution)**.** Let G be a group. Define a simplicial complex X, where simplices correspond to finite subsets of G. The corresponding free resolution $F_* = C'_*(X)$ is called the *standard free resolution* of $\mathbb{Z}$ over $\mathbb{Z}G$. More precisely F_n is a free $\mathbb{Z}$-module generated by $(n+1)$-tuples of elements of G, where the G-action is given by the multiplication of G. The boundary map $\partial : F_n \to F_{n-1}$ is given by $\partial = \sum_{i=0}^{n} (-1)^i d_i$, and

$$d_i(g_0, \dots, g_n) = (g_0, \dots, \hat{g}_i, \dots, g_n)$$

In Remark 1.1.6 we presented a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}^n]$, i.e. when G is a free $\mathbb{Z}$-module. Now we would like to find a resolution when G is not torsion-free. We consider the case when G is a finite cyclic group.

Example 1.2.2. Let $n \geq 2$. We would like to find a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[C_n]$. Let t be a generator of C_n and denote $T = \sum_{i=0}^{n-1} t^i$. We get the following resolution of infinite length.

$$\cdots \xrightarrow{T} \mathbb{Z}[C_n] \xrightarrow{t-1} \mathbb{Z}[C_n] \xrightarrow{T} \mathbb{Z}[C_n] \xrightarrow{t-1} \mathbb{Z}[C_n] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

1.3 Co-invariants and homology

Recall that a $\mathbb{Z}G$-module is just an abelian group equipped with an action of G. It seems reasonable to study the G-orbits in G-modules. It turns out that this gives rise to an additive functor. Then we can study the failure of exactness of this functor, i.e. the homology.

Definition 1.3.1 (Group of co-invariants)**.** Let G be a group and M a G-module. We define

$$M_G = M/\langle g \cdot m - m | m \in M, g \in G \rangle$$

the *group of co-invariants* of M.

Remark 1.3.2. Notice that M_G is the largest quotient of M on which G acts trivially. We can also see that

$$M_G \cong \operatorname{colim}_G M$$

where M is seen as a functor

$$M : BG \to Set$$

Proposition 1.3.3. *Let X be a G-set, then*

$$(\mathbb{Z}X)_G \cong \mathbb{Z}[X_G]$$

Proof. If we see X as a functor $X : BG \to Set$ then we have the following diagram

$$\begin{tikzcd} BG \arrow[r, "X"] & Set \arrow[r, bend left, "F_{Ab}"] \arrow[r, phantom, "\perp"] & Ab \arrow[l, bend left, "U"] \end{tikzcd}$$

Using the previous remark and the fact that left adjoints preserve colimits we get

$$\mathbb{Z}[X_G] \cong F_[\operatorname{colim}_{BG} X] \cong \operatorname{colim}_{BG}(F_{Ab} \circ X) \cong (\mathbb{Z}X)_G$$

□

Remark 1.3.4. This defines an additive co-invariants functor

$$(-)_G : \mathrm{Mod}_G \to Ab$$

There is an equivalent definition of the group of co-invariants.

Proposition 1.3.5 ([2], Proposition II.2.4)**.** *Let M be a G-module, then*

$$M_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} M$$

Corollary 1.3.6. *The co-invariants functor $(-)_G$ is right exact. And if M is a free-G-module of rank k then M_G is a free $\mathbb{Z}$-module of rank k.*

Now that we have an additive right exact functor we can study its homology. The next definition follows naturally.

Definition 1.3.7 (Homology of a group)**.** Let G be a group and $\varepsilon : F \to \mathbb{Z}$ a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Then the *homology groups* of G are the following

$$H_i G = H_i(F_G)$$

Remark 1.3.8. The homology of a group is independent of the choice of projective resolution. This is a classical result in homological algebra; to see a proof of this statement refer to [2].

Let us look again at topology. Let X be the universal cover of a $K(G,1)$ space. Then its cellular chain complex is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ since G acts freely on the cells of X. Notice that $C_*^{cell}(X) \otimes_{\mathbb{Z}G} \mathbb{Z} \cong C_*^{cell}(K(G,1))$, then the following result is immediate.

Proposition 1.3.9. *Let G be a group, then*

$$H_* G \cong H_* K(G,1)$$

Remark 1.3.10. More generally, if Y is a path connected space with a contractible covering space X with covering group G we get

$$C_*^{sing}(X)_G \cong C_*^{sing}(Y)$$

Since $C_*^{sing}(X)$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ it follows that

$$H_* G = H_*(C_*^{sing}(X)_G) \cong H_*(C_*^{sing}(Y)) = H_* Y$$

There are many interesting results that give tools to compute the homology of a group. There is an analogous result to that of a Mayer-Vietoris sequence on homology.

Proposition 1.3.11 ([2], Corollary II.7.7)**.** *Let $G = G_1 *_A G_2$ be such that the maps from A to G_1 and G_2 are both injective. Then there is an exact sequence*

$$\cdots \longrightarrow H_n A \longrightarrow H_n G_1 \oplus H_n G_2 \longrightarrow H_n G \longrightarrow H_{n-1} A \longrightarrow \cdots$$

Up to this point we have defined the homology of a group with coefficients $\mathbb{Z}$. We generalize our definition of homology to have coefficients in M, where M is any G-module.

Definition 1.3.12 (Homology with coefficients)**.** Let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and M a G-module. Then the *homology of G with coefficients* in M is defined as follows

$$H_*(G; M) = H_*(F \otimes_G M)$$

1.4 Induced modules

Given a ring homomorphism $R \to S$, which gives S the structure of an R-module, we would like to find a way to obtain S-modules from R-modules. For an R-module M consider the tensor product $S \otimes_R M$. The multiplication by elements of S endows this R-module with an S-module structure. This function from R-modules to S-modules is called the extension by scalars from R to S.

Now we apply this construction to a homomorphism of group rings $\mathbb{Z}H \to \mathbb{Z}G$ induced by an inclusion $H \subset G$.

Definition 1.4.1 (Induced module)**.** Let M be an $\mathbb{Z}H$-module. Then its *induced* $\mathbb{Z}G$*-module* is

$$\operatorname{Ind}_H^G M = \mathbb{Z}G \otimes_{\mathbb{Z}H} M$$

Note that the induced module as abelian group admits the following direct sum decomposition.

$$\operatorname{Ind}_H^G M = \bigoplus_{g \in G/H} gM$$

Now we state two useful facts about induced modules.

Proposition 1.4.2. *([2], Proposition III.5.3) Let $N = \oplus_{i \in I} M_i$ be a G-module such that the G-action transitively permutes the summands. Let M be one of the summands M_i, and let $H \subset G$ be the isotropy group of i. Then M is an H-module and*

$$N \cong \operatorname{Ind}_H^G M$$

From this proposition the next corollary follows.

Corollary 1.4.3. *([2], Corollary III.5.4) Let $N = \oplus_{i \in I} M_i$ be a G-module such that the G-action permutes the summands according to a G-action on I. Let i be the isotropy group of I and let E be a set of representatives of I/G. Then M_i is a G_i-module and there is a G-isomorphism*

$$N \cong \bigoplus_{i \in E} \operatorname{Ind}_{G_i}^G M_i$$

There are many interesting properties of induced modules. One that will be fundamental in Chapter 2 is the so called Shapiro lemma.

Lemma 1.4.4 (Shapiro's lemma, [2] Proposition III.6.2)**.** *Let $H \subset G$ and M be an H-module, then*

$$H_*(H, M) \cong H_*(G, \operatorname{Ind}_H^G M)$$

Proof. Let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Then it can also be seen as a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$. Thus,

$$\begin{aligned} H_*(H, M) &= H_*(F \otimes_{\mathbb{Z}H} M) \\ &= H_*(F \otimes_{\mathbb{Z}G} (\mathbb{Z}G \otimes_{\mathbb{Z}H} M)) \\ &\cong H_*(G, \operatorname{Ind}_H^G M) \end{aligned}$$

□

We provide one last result that will help us in Section 2.4.

Proposition 1.4.5. *[[2] Corollary III.5.7] Let M be a G-module. Then,*

$$\mathbb{Z}G \otimes M \cong \mathbb{Z}G \otimes M^{triv}$$

where M^{triv} is the underlying abelian group structure of M.

Proof. We build an explicit isomorphism

$$\phi : \mathbb{Z}G \otimes M \to \mathbb{Z}G \otimes M^{triv} : g \otimes m \mapsto g \otimes g^{-1}m$$

with inverse

$$\phi^{-1} : \mathbb{Z}G \otimes M^{triv} \to \mathbb{Z}G \otimes M : g \otimes m \mapsto g \otimes gm$$

One can easily check that these are G-module homomorphisms and mutually inverses. $\square$

1.5 FI-modules

In 2014, a paper from Church-Ellenberg-Farb[3] introduced the theory of FI-modules as a tool for cohomological problems. Putman recalls the main definitions on FI-modules in ([11] §1.3 & §1.4).

Denote by FI the category whose objects are finite sets and whose morphisms are injections. This category is equivalent to the category with objects $\overline{n} = \{1, \ldots, n\}$ and morphisms the injections $\overline{m} \hookrightarrow \overline{n}$.

Definition 1.5.1 (FI-module)**.** Let R be a commutative ring. An *FI-module* over R is a functor M from FI to the category of R-modules. Then an FI-module $M \in \mathrm{Fun}(\mathrm{FI}, \mathrm{Mod}_R)$ consists of an R-module $M(\overline{n})$ for all $n \in \mathbb{N}$. And for every injection $f : \overline{m} \hookrightarrow \overline{n}$ an induced R-module homomorphism $f_* : M(\overline{m}) \to M(\overline{n})$.

Note that there is a sequence induced by the inclusions $\overline{n} \hookrightarrow \overline{n+1}$

$$M(\overline{0}) \to M(\overline{1}) \to \cdots \to M(\overline{n-1}) \to M(\overline{n}) \to M(\overline{n+1}) \to \cdots$$

Moreover, $\mathfrak{S}_n$ is the group of FI-automorphisms of $\overline{n}$. Therefore $M(\overline{n})$ has a natural $\mathfrak{S}_n$ action, making it a $R[\mathfrak{S}_n]$-module.

We can also see that the maps from the FI-module yield maps in homology

$$H_k(\mathfrak{S}_n; M(\overline{0})) \to H_k(\mathfrak{S}_n; M(\overline{1})) \to H_k(\mathfrak{S}_n; M(\overline{2})) \to \cdots$$

for all $k \in \mathbb{N}$. We would like to understand under which conditions this chain stabilizes. This is the principal objective Chapter 3.

1.6 Semisimplicial sets

Previous constructions suggest that we should focus on categories with only injective morphisms. It seems reasonable to consider a subcategory of the simplex category removing non-injective morphisms, which modifies the notion of simplicial sets. In ([11], §3), Putman gives an overview of the basics on this objects.

Here we will note Δ the subcategory of the simplex category with the same objects, but with only injective morphisms.

Then we define semisimplicial sets from this restricted category just as simplicial sets are defined from the simplex category.

Definition 1.6.1 (Semisimplicial set)**.** Define the category of semisimplicial sets as the functor category $\mathrm{Fun}(\Delta^{op}, \mathrm{Set})$. Unwinding this definition, a *semisimplicial set* $\mathbb{X} : \Delta^{op} \to \mathrm{Set}$ consists of:

- Sets of k-simplices $\mathbb{X}^k = \mathbb{X}([k])$ for all $k \geq 0$.
- For each map $\iota : [l] \to [k]$ in Δ, a map $\iota^* : \mathbb{X}^k \to \mathbb{X}^l$ called a face map.

Notice that any simplicial set defines a semisimplicial set by restriction.

In most cases, the spaces we encounter are simplicial complexes. This is undesirable because we cannot always quotient a simplicial complex by a group action. So we want to work with some special kind of simplicial complexes that are stable under quotient. This motivates the following definition.

Definition 1.6.2 (Ordered simplicial complex)**.** An *ordered simplicial complex* is a semisimplicial set satisfying some additional conditions.

- It has an arbitrary set of vertices $\mathbb{X}^0$;
- The k-simplices are ordered $(k+1)$-tuples of distinct elements in $\mathbb{X}^0$, i.e.

$$\mathbb{X}^k \subset \{(v_0, \dots, v_k) : \text{the } v_i\text{'s are distinct vertices}\} \subset (\mathbb{X}^0)$$

- An order-preserving injection $\iota : [l] \to [k]$ induced a face map

$$\iota^* : \mathbb{X}^k \to \mathbb{X}^l : (v_0, \dots, v_k) \mapsto (v_{\iota(0)}, \dots, v_{\iota(l)})$$

  Observe that after removing any vertex from a simplex we still have a simplex.

Now we need a way to associate an ordered simplicial complex to any simplicial complex.

Definition 1.6.3 (Large ordering)**.** For a simplicial complex X we define its *large ordering* X_{ord} as the ordered simplicial complex with same vertex set than X and such that its k-simplices are all ordered $(k+1)$-tuples of distinct vertices of X.

Note that if there is a group G with an action on X then it also has an action on X_{ord}.

Example 1.6.4. One classical example that will be very useful in other chapters is the large ordering of the standard n-simplex Δ_n. Let us denote it by $\mathbb{S}\mathrm{im}_n = (\Delta_n)_{ord}$. The k-simplices of $\mathbb{S}\mathrm{im}_n$ are all ordered sequences of length $k+1$ of distinct elements of $[n]$. It is clear that $\mathfrak{S}_{n+1}$ acts on Δ_n, so it acts also on $\mathbb{S}\mathrm{im}_n$.

In our example we can give an explicit decomposition of our large ordering. Frank D. Farmer proved in 1978 that $\mathbb{S}\mathrm{im}_n$ is $(n-1)$-connected ([4], Theorem 5).

Example 1.6.5 ($\mathbb{S}\mathrm{im}_1$). In Fig. 1.1 we can see Δ_1 and $\mathbb{S}\mathrm{im}_1$.

Example 1.6.6 ($\mathbb{S}\mathrm{im}_2$). In Fig. 1.2 we can see Δ_2 and the 2-skeleton of $\mathbb{S}\mathrm{im}_2$, which has six additional 2-simplices.

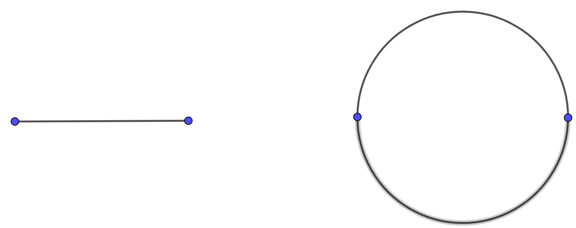


Figure 1.1: $\mathbb{S}\mathrm{im}_1$

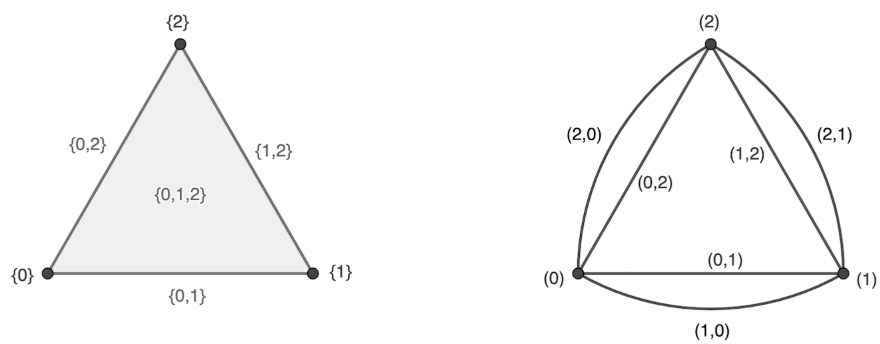


Figure 1.2: 2-skeleton of $\mathbb{S}\mathrm{im}_2$

Chapter 2

Spectral Sequences

We would like to have algebraic tools to compute the homology of groups when it can not be done easily with the methods explained in previous chapters. Spectral sequences are algebraic objects that allow us to compute the algebraic invariants we want.

In the 1940's Jean Leary introduced what we now call spectral sequences as computational techniques to solve problems in algebraic topology on his paper [7], although he had not named these objects yet. Some years later, Jean-Pierre Serre used Leray's ideas and developed further the theory of spectral sequences, finding many applications and examples. Some of Serre's breakthroughs on spectral sequences are presented on his thesis [13].

2.1 General setting

Before defining spectral sequence one must be comfortable with many algebraic objects such as (bi)graded modules, (bi)graded (commutative) algebras, (stable) filtrations, differentials and so on. Let us introduce some of these objects.

Definition 2.1.1 (Differential bigraded module)**.** A *differential bigraded* R-module is a collection of R-modules $\{E^{p,q}\}_{p,q\in\mathbb{Z}}$, equipped with a linear map $d : E^{*,*} \to E^{*,*}$ of bidegree $(s, 1-s)$ or $(-s, s-1)$ for some $s \in \mathbb{Z}$, such that $d \circ d = 0$.

Example 2.1.2. A simple example of bigraded module is a collection of chain complexes $E^{*,q}$ for $q \in \mathbb{N}$. We see that we can stack these sequences in a grid where the q-th row is the q-th chain complex.

Definition 2.1.3 (Spectral sequence)**.** A *spectral sequence* is a collection of differential bigraded modules $\{E_r^{*,*}, d_r\}$ for $r = 1, 2, \dots,$ where all differentials are of bidegree $(r, 1-r)$ or all of bidegree $(-r, r-1)$. And for all p, q, r we have

$$E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*})$$

Now we would like to define the limit of a spectral sequence and study its convergence.

Start with $E_2^{*,*}$. We will drop the heavy $(-)^{*,*}$ notation but we still consider bigraded objects. Define $Z_2 = \ker d_2$ and $B_2 = \operatorname{im} d_2$. By definition we have

$$B_2 \subset Z_2 \subset E_2$$

and

$$E_3 \cong Z_2/B_2$$

Similarly now define $\overline{Z}_3 = \ker d_3$ and $\overline{B}_3 = \operatorname{im} d_3$. Let Z_3 and B_3 be its preimages by the quotient map $E_2 \to E_2/B_2$, i.e.

$$\overline{Z}_3 \cong Z_3/B_2 \quad \overline{B}_3 \cong B_3/B_2$$

This gives us $E_4 \cong Z_3/B_3$. After iterating this process we obtain the following inclusions

$$B_2 \subset B_3 \subset \cdots \subset B_n \subset \cdots \quad \cdots \subset Z_n \subset \cdots \subset Z_3 \subset Z_2 \subset E_2$$

where $E_{n+1} \cong Z_n/B_n$ and the following short exact sequence arises

$$0 \longrightarrow Z_{n+1}/B_n \longrightarrow Z_n/B_n \longrightarrow B_{n+1}/B_n \longrightarrow 0$$

Therefore

$$Z_n/Z_{n+1} \cong B_{n+1}/B_n$$

Now we are prepared to define the target of the spectral sequence. Define

$$Z_\infty = \bigcap_n Z_n \subset E_2$$

the submodule of elements that survive all homology stages. And

$$B_\infty = \bigcup_n B_n \subset E_2$$

the submodule of elements that eventually bound.

It is clear that $B_\infty \subset Z_\infty$, then we set

$$E_\infty = Z_\infty/B_\infty$$

the bigraded module remaining after the infinite sequence of homologies. This is the target of the spectral sequence.

Now we introduce some useful terminology for spectral sequences. Given a spectral sequence $\{E_r^{*,*}, d_r\}$, in the literature we refer to $E_r^{*,*}$ as the E_r-page/term or r^{th} page/term.

Definition 2.1.4 (Collapsing spectral sequence)**.** A spectral sequence $\{E_r^{*,*}, d_r\}$ *collapses* at the N^{th} page if $d_r = 0$ for $r \geq N$. Then we have $E_N \cong E_\infty$.

2.2 Filtrations and exact couples

To understand how spectral sequences arise, it is crucial to be comfortable with the notion of filtration. Intuitively a filtration is just a family of subspaces such that each one of them is characterized by some property. A simple example is the filtration of a simplicial complex given by its skeletons.

Definition 2.2.1 (Filtration)**.** A *filtration* F^* on an R-module A is a family of submodules $\{F^pA\}_{p\in\mathbb{Z}}$ such that

$$\cdots F^{p+1}A \subset F^pA \subset F^{p-1}A \cdots \subset A$$

in this case we say the filtration is decreasing, or

$$\cdots F^{p-1}A \subset F^pA \subset F^{p+1}A \cdots \subset A$$

and we way this filtration is increasing.

To a filtered module we can associate a graded module E_0^* determined by

$$E_0^p(A) = \begin{cases} F^pA/F^{p+1}A, \text{ if } F \text{ is decreasing} \\ F^pA/F^{p-1}A, \text{ if } F \text{ is increasing} \end{cases}$$

Now if there is a filtration F on a graded module H^*, then $E_0^*(H^*)$ is bigraded. Using the degree of H^* we define

$$F^pH^r = F^pH^* \cap A^r$$
$$E_0^{p,q}(H^*) = F^pH^{p+q}/F^{p+1}H^{p+q}$$

Definition 2.2.2 (Convergence of spectral sequences)**.** Let $\{E_r^{*,*}, d_r\}$ be a spectral sequence. We say it *converges* to a graded R-module H^* if there is a filtration F on H such that

$$E_\infty^{p,q} \cong E_0^{p,q}(H^*, F)$$

Definition 2.2.3 (Filtered differential graded module)**.** Let A be an R-module. We say A is *filtered differential and graded* if

1. A is a direct sum of submodules
$$A = \bigoplus_{n=o}^{\infty} A_n$$
2. There is an R-linear map $d : A \to A$ of degree ± 1 satisfying $d \circ d = 0$.
3. A has a filtration F and the differential respects the filtration,
$$d : F^pA \to F^pA$$

Now we state a fundamental theorem concerning the convergence of spectral sequences. We concentrate on decreasing filtrations and differentials of degree 1. Note that dual statements can be made and proved similarly.

Theorem 2.2.4 ([9], Theorem 2.6)**.** *Each filtered differential graded module (A, d, F^*) determines a spectral sequence $\{E_r^{*,*}, d_r\}_{r\geq 1}$ with d_r of bidegree $(r, 1-r)$ and*

$$E_1^{p,q} \cong H^{p+q}(F^pA/F^{p+1}A)$$

Suppose further that the filtration is bounded. Then the spectral sequence converges to $H(A, d)$, i.e.

$$E_\infty^{p,q} \cong F^pH^{p+q}(A, d)/F^{p+1}H^{p+q}(A, d)$$

It turns out that spectral sequences arise in many different situations without a filtration. Here we present how a so called exact couple gives rise to a spectral sequence. Loosely speaking, an exact couple is a triangle of objects with all maps making a cycle such that the triangle is exact at each vertex.

Definition 2.2.5 (Exact couple)**.** Let D, E be R-modules and consider the following diagram where arrows are module homomorphisms.

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

We say that $\mathcal{C} = \{D, E, i, j, k\}$ is an *exact couple* if the diagram is exact at each module.

Remark 2.2.6. From this definition it follows that E is a differential R-module with $d = j \circ i : E \to E$.

Similar to spectral sequences, we would like to have a sequence of exact couples somehow related. This motivates the following definition.

Definition 2.2.7 (Derived couple)**.** Let $\mathcal{C} = \{D, E, i, j, k\}$ be an exact couple. Define

$$\begin{aligned} E' &= H(E, d) \\ D' &= i(D) = \operatorname{im} i = \ker j \end{aligned}$$

Then define the new homomorphisms

$$\begin{aligned} i' &= i_{D'} : D' \to D' \\ j' &: D' \to E' : i(x) \mapsto j(x) + dE \\ k' &: E' \to D' : e + dE \mapsto k(e) \end{aligned}$$

We call $\mathcal{C}' = \{D', E', i', j', k'\}$ the *derived couple* of $\mathcal{C}$.

Proposition 2.2.8 ([9], Proposition 2.7)**.** *The derived couple $\mathcal{C}' = \{D', E', i', j', k'\}$ is an exact couple.*

Remark 2.2.9. We can iterate the process of building the derived couple. This gives us a sequence of exact couples where the n-th derived couple $\mathcal{C}^{(n)}$ is $\{D^{(n)}, E^{(n)}, i^{(n)}, j^{(n)}, k^{(n)}\}$. Note that for all $n \in \mathbb{N}$ we have

$$E^{(n+1)} = H(E^{(n)}, d^{(n)})$$

This suggest there is a strong connection between exact couples and spectral sequences.

Theorem 2.2.10 ([9], Theorem 2.8)**.** *Let $D^{*,*}, E^{*,*}$ be bigraded R-modules equipped with homomorphisms i of bidegree (-1,1), j of bidegree (0,0), and k of bidegree (1,0).*

$$\begin{tikzcd} D^{*,*} \arrow[rr, "i"] & & D^{*,*} \arrow[dl, "j"] \\ & E^{*,*} \arrow[ul, "k"] & \end{tikzcd}$$

This determines a spectral sequence $\{E_r^{*,*}, d_r\}$ where d_r has bidegree $(r, 1-r)$, $E_r = (E^{*,*})^{(r-1)}$ and $d_r = j^{(n)} \circ k^{(n)}$.

An exact couple can be represented in a diagram as follows:

$$\begin{tikzcd} & \vdots \arrow[d, "i"] & & \vdots \arrow[d, "i"] & \\ \cdots \arrow[r, "k"] & D^{p+2,q-1} \arrow[r, "j"] \arrow[d, "i"] & E^{p+2,q-1} \arrow[r, "k"] & D^{p+3,q-1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\ \cdots \arrow[r, "k"] & D^{p+1,q} \arrow[r, "j"] \arrow[d, "i"] & E^{p+1,q} \arrow[r, "k"] & D^{p+2,q} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\ \cdots \arrow[r, "k"] & D^{p,q+1} \arrow[r, "j"] \arrow[d, "i"] & E^{p,q+1} \arrow[r, "k"] & D^{p+1,q+1} \arrow[r, "j"] \arrow[d, "i"] & \cdots \\ & \vdots & & \vdots & \end{tikzcd}$$

where the complex formed by one vertical and two horizontal maps is exact.

This information is also displayed on the form of a so called unrolled exact couple:

$$\begin{tikzcd} \cdots \arrow[r, "i"] & D^{p+1,*} \arrow[rr, "i"] & & D^{p,*} \arrow[rr, "i"] \arrow[dl, "j"] & & D^{p-1,*} \arrow[r, "i"] \arrow[dl, "j"] & \cdots \\ & & E^{p,*} \arrow[ul, "k"] & & E^{p-1,*} \arrow[ul, "k"] & & \end{tikzcd}$$

Now that we have seen two approaches to spectral sequences, we claim that they are equivalent. From a filtered differential graded module (A, d, F) we can get an exact couple yielding the same associated spectral sequence [9]. Note that there is a long exact sequence

$$\cdots \xrightarrow{k} H^{p+q}(F^{p+1}A) \xrightarrow{i} H^{p+q}(F^pA) \xrightarrow{j} H^{p+q}(F^pA/F^{p+1}A)$$
$$\xrightarrow{k} H^{p+q}(F^{p+1}A) \xrightarrow{i} H^{p+q}(F^pA) \xrightarrow{j}$$

with k the connecting homomorphism. Now we can define bigraded modules

$$\begin{aligned} E^{p,q} &= H^{p+q}(F^pA/F^{p+1}A) \\ D^{p,q} &= H^{p+q}(F^pA) \end{aligned}$$

This yields a an exact couple

$$\begin{tikzcd} & D^{p+1,q-1} \arrow[rr, "i"] & & D^{p,q} \arrow[dl, "j"] \\ D^{p+1,q} & & E^{p,q} \arrow[ll, "k"] & \end{tikzcd}$$

Note that the diagram is not a triangle because when we add the bidegrees of i, j, k we do not get (0,0), we obtain (0,1). This gives a triangle of bigraded modules but not of modules.

Proposition 2.2.11 ([9], Proposition 2.11)**.** *Let (A, d, F) be a filtered differential graded module. Then the spectral sequence associated to its (decreasing) filtration and the spectral sequence associated to its exact couple are the same.*

Theorem 2.2.12 (The Künneth theorem, [9] Theorem 2.12)**.** *Let (A, d_A) and (B, d_B) be differential graded R-modules and, for each n,*

$$Z^n(A) = \ker d_A : A^n \to A^{n+1}$$
$$B^n(A) = \operatorname{im} d_A A^{n-1} \to A^n$$

are flat R-modules, then there is a short exact sequence

$$0 \longrightarrow \bigoplus_{r+s=n} H^r(A) \otimes_R H^s(B) \longrightarrow H^n(A \otimes_R B)$$
$$\longrightarrow \bigoplus_{r+s=n} \operatorname{Tor}_1^R(H^r(A), H^s(B)) \longrightarrow 0$$

2.3 Double complexes

Now we introduce the objects that appear the most in our context.

Definition 2.3.1 (Double complex)**.** A *double complex* $\{M^{*,*}, d', d''\}$ is a bigraded R-module $M^{*,*}$ with two differentials $d', d'' : M^{*,*} \to M^{*,*}$ of bi-degrees (1,0) and (0,1) respectively satisfying:

$$d' \circ d'' + d'' \circ d' = 0$$

A double complex can be represented as follows

$$\begin{tikzcd} & \vdots & \vdots & \\ \cdots \arrow[r, "d'"] & M^{n,m+1} \arrow[u, "d''"] \arrow[r, "d'"] & M^{n+1,m+1} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \\ \cdots \arrow[r, "d'"] & M^{n,m} \arrow[u, "d''"] \arrow[r, "d'"] & M^{n+1,m} \arrow[u, "d''"] \arrow[r, "d'"] & \cdots \\ & \vdots \arrow[u, "d''"] & \vdots \arrow[u, "d''"] & \end{tikzcd}$$

From any double complex we can define a differential graded module called total complex where

$$\mathrm{total}(M)^n = \bigoplus_{p+q=n} M^{p,q}$$

with differential

$$d = d' + d'' : \mathrm{total}(M)^n \to \mathrm{total}(M)^{n+1}$$

The condition on the definition of double complex makes $d \circ d = 0$.

Now we would like to build a spectral sequence from a double complex exploiting all its structure. First note that we can take homology in two directions. We define

$$H_I^{m,n}(M) = H^{m,n}(M^{*,*}, d')$$
$$H_{II}^{m,n}(M) = H^{m,n}(M^{*,*}, d'')$$

The condition on the differentials of double complexes implies that $H_I^{*,*}(M)$ and $H_{II}^{*,*}(M)$ are each differential bigraded modules with differentials induced by the differentials of M.

Theorem 2.3.2 ([9], Theorem 2.15)**.** *Let $\{M^{*,*}, d', d''\}$ be a double complex. There are two spectral sequences $\{{}_IE_r^{*,*}, {}_Id_r\}$ and $\{{}_{II}E_r^{*,*}, {}_{II}d_r\}$ with*

$${}_IE_2^{*,*} \cong H_I^{*,*}(H_{II}(M))$$
$${}_{II}E_2^{*,*} \cong H_{II}^{*,*}(H_I^{*,*}(M))$$

Furthermore, if $M^{p,q} = 0$ if $p < 0$ or $q < 0$ then both spectral sequences converge to $H^*(\mathrm{total}(M), d)$.

Although we will not give the proof of the theorem, we will present the ideas used in the proof since it is a constructive proof.

Consider two filtrations on $(\mathrm{total}(M), d)$:

$$F_I^p(\mathrm{total}(M))^t = \bigoplus_{r \geq p} M^{r,t-r}$$
$$F_{II}^p(\mathrm{total}(M))^t = \bigoplus_{r \geq p} M^{t-r,r}$$

The first one is called the column-wise filtration and the second one is the row-wise filtration. Note that both filtrations are decreasing, stable and bounded. Then by theorem 2.2.4 we conclude that the spectral sequences associated to $(\mathrm{total}(M), d)$ with these filtrations both converge to $H^*(\mathrm{total}(M), d)$.

Example 2.3.3 (Symmetry in Tor functor). Let M, N be R-modules with projective resolutions $P_\bullet \to M \to 0$ and $Q_\bullet \to N \to 0$.

Consider the double complex $(K_{*,*}, d', d'')$ defined as follows

$$K_{i,j} = P_i \otimes Q_j$$
$$d' = d_P \otimes 1$$
$$d'' = \pm 1 \otimes d_Q$$

We have two filtrations

$$F_p^I(\text{total}\, K)_t = \bigoplus_{r \le p} K_{r,t-r}$$
$$F_p^{II}(\text{total}\, K)_t = \bigoplus_{r \le p} K_{t-r,r}$$

The previous theorem tell us that the spectral sequences associated to these filtrations converge to $H(P_\bullet \otimes Q_\bullet, d_P \otimes 1 + \pm 1 \otimes d_Q)$. We compute the E_2 pages of both spectral sequences. Using the Künneth theorem 2.2.12 and the fact that the P_i's and Q_j's are projective we get

$$\begin{aligned} H^{II}(K) &= H(K_{*,*}, d'') \\ &= H(P_\bullet \otimes Q_\bullet, \pm 1 \otimes d_Q) \\ &= P_\bullet \otimes H(Q_\bullet, d_Q) \\ &= P_\bullet \otimes N \end{aligned}$$

Similarly we get $H^I(K) = M \otimes Q_\bullet$.

Note that these pages are concentrated in one row and one column respectively. Therefore we obtain

$$\begin{aligned} H^I_{*,*} H^{II}(K) &= H(P_\bullet \otimes N, d_P \otimes 1) \\ H^{II}_{*,*} H^I(K) &= H(M \otimes Q_\bullet, 1 \otimes d_Q) \end{aligned}$$

and both sequences collapse on this page.

By definition we have $H(P_\bullet \otimes N, d_P \otimes 1) = \mathrm{Tor}^R_*(M, N)$. It follows that

$$\begin{aligned} H(P_\bullet \otimes N, d_P \otimes 1) &= \mathrm{Tor}^R_*(M, N) \\ H(M \otimes Q_\bullet, 1 \otimes d_Q) &= \mathrm{Tor}^R_*(M, N) \\ H(P_\bullet \otimes Q_\bullet, d) &= \mathrm{Tor}^R_*(M, N) \end{aligned}$$

Moreover it is clear that $P_\bullet \otimes Q_\bullet \cong Q_\bullet \otimes P_\bullet$. Then we conclude

$$\begin{aligned} \mathrm{Tor}^R_*(M, N) &\cong H(P_\bullet \otimes Q_\bullet, d) \\ &\cong H(Q_\bullet \otimes P_\bullet, d) \\ &\cong \mathrm{Tor}^R_*(N, M) \end{aligned}$$

2.4 Lyndon-Hochschild-Serre spectral sequence

In 1951, Jean-Pierre Serre introduced the Leray-Serre spectral sequence in his thesis [13]. This spectral sequence relates the total space of a fibration and its base and fibre. Later, Serre and Hochschild [6], using some results from Lyndon [8], introduced another spectral sequence giving analogous relations in the realm of groups. Instead of taking fibrations on spaces we consider group extensions.

Definition 2.4.1 (Group extension)**.** Let G, H, Q be groups. We say that G is a *group extension* of Q by H if there is a short exact sequence

$$1 \longrightarrow H \longrightarrow G \longrightarrow Q \longrightarrow 1$$

Group extensions are used to describe a group G in terms of a normal subgroup and the corresponding quotient.

Now we have the language to introduce the Lyndon-Hochschild-Serre spectral sequence.

Theorem 2.4.2 (The Lyndon-Hochschild-Serre spectral sequence, [9] Theorem 8^{bis}.12)**.** *Let*

$$1 \longrightarrow H \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be a group extension. Suppose M is a G-module. Then there is a first quadrant spectral sequence with

$$E^2_{p,q} \cong H_p(Q, H_q(H, M))$$

and converging strongly to $H_*(G, M)$.

Proof. We will not give details about the module theory in this proof, these details are found in [9] theorem 8^{bis}.12.

Observe that taking the coinvariants of a G-module M with respect to the action of H, makes the action of H trivial on M_H. Therefore Q acts on M_H.

Notice that $\mathbb{Z}Q \otimes_{\mathbb{Z}G} M \cong M_H$. So we obtain the following relation

$$(M_H)_Q = M_G$$

Now suppose that $F_\bullet \to \mathbb{Z} \to 0$ is a free right resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Let $\tilde{F}_\bullet \to \mathbb{Z} \to 0$ be a free right resolution of $\mathbb{Z}$ over $\mathbb{Z}Q$. We can build a double complex

$$C_{p,q} = \tilde{F}_p \otimes_{\mathbb{Z}Q} (F_q \otimes M)_H$$

where Q acts on $(F_q \otimes M)_H$ by the diagonal action.

Remember that double complexes yield two spectral sequences, Theorem 2.3.2.

The vertical filtration: the first page of this spectral sequence is given by

$$E^1_{p,q} \cong H_p(Q; (F_q \otimes_{\mathbb{Z}H} M))$$

Recall that F_q is a free G-module. Thus we can write $F_q \cong \mathbb{Z}G \otimes A_q$ where A_q is a free $\mathbb{Z}$-module. Using Proposition 1.4.5 we get that

$$\begin{aligned}
\tilde{F}_\bullet \otimes_{\mathbb{Z}Q} (\mathbb{Z}G \otimes A_q \otimes M)_H &\cong \tilde{F}_\bullet \otimes_{\mathbb{Z}Q} \mathbb{Z} \otimes_{\mathbb{Z}H} \mathbb{Z}G \otimes A_q \otimes M \\
&\cong \tilde{F}_\bullet \otimes_{\mathbb{Z}Q} \mathbb{Z}Q \otimes A_q \otimes M \\
&\cong \tilde{F}_\bullet \otimes A_q \otimes M
\end{aligned}$$

Here $\tilde{F}_\bullet \otimes A_q$ is a $\mathbb{Z}$-free resolution of A_q, which is a free $\mathbb{Z}$-module over $\mathbb{Z}$. Thus taking homology is equivalent to compute the modules $\mathrm{Tor}^p_{\mathbb{Z}}(A_q, M)$ which we know are zero for $p > 0$ because A_q is free. It follows that $E^1_{p,q} = 0$ for $p > 0$ and

$$E^1_{0,q} \cong F_q \otimes_{\mathbb{Z}G} M$$

From here we can conclude that this spectral sequence collapses in the second page $E^2_{0,q} = H_q(G; M)$ and then both spectral sequences converge to

$$H_\bullet(G; M)$$

The horizontal filtration: since $F_\bullet$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}Q$ and $(F_q \otimes M)_H$ is a Q-module we obtain

$$E^1_{p,q} \cong H_q(\tilde{F}_p \otimes_{\mathbb{Z}Q} (F_q \otimes M)_H, 1 \otimes d) \cong \tilde{F}_p \otimes_{\mathbb{Z}Q} H_q(H; M)$$
$$E^2_{p,q} \cong H_p(Q; H_q(H; M))$$

This is the spectral sequence that we were looking for. □

Remark 2.4.3. In fact this spectral sequence is just a particular case of the Leray-Serre spectral sequence. It is the spectral sequence given by the fibration obtained from the classifying spaces of the groups in the extension.

The Lyndon-Hochschild-Serre spectral sequence is a powerful tool to compute the homology and cohomology of groups of a group G. One classical computation is the (co)homology groups with coefficients $\mathbb{F}_p$ where $p \in \mathbb{Z}$ is a prime. The key component to find such results is an appropriate choice of group extension. As we are interested in $\mathbb{F}_p$ one can think of using the p-Sylow subgroups of G. If there is a unique p-Sylow subgroup of G then we get a group extension and we can use our spectral sequence.

Proposition 2.4.4 ([9], Proposition 8^{bis}.7)**.** *Let G be a finite group, and let M be a G-module. Then the order of every element of $H_i(G, M)$ for $i > 0$ is a divisor of $|G|$.*

Proposition 2.4.5 ([9], Proposition 8^{bis}.13)**.** *Suppose G is a finite group and P is a normal Sylow p-subgroup. Then*

$$H_i(G, \mathbb{F}_p) \cong H_i(P, \mathbb{F}_p)_Q$$

where G acts trivially on $\mathbb{F}_p$ and $Q = G/P$.

Proof. There is a spectral sequence $\{E^r_{*,*}\}$ given by Theorem 2.4.2 converging to $H_*(G, \mathbb{F}_P)$ such that

$$E^2_{p,q} \cong H_p(Q, H_q(P, \mathbb{F}_p))$$

Note that the order of Q is relatively prime to p, and the multiplication by p map in $H_q(P, \mathbb{F}_p)$ is the zero map. By Proposition 2.4.4 we can conclude that $E^2_{p,q} = 0$ for $p > 0$. Therefore the spectral sequence collapses to the first column

$$E^2_{0,q} \cong H_q(P, \mathbb{F}_p)_Q$$

As the target of this spectral sequence is $H_*(G, \mathbb{F}_P)$ the result follows. □

To obtain more results one can think of different group extensions. For instance, we can take the so called central extension. Denote by $Z(G) \lhd G$ the centre of G, recall that it is a normal subgroup of G. Then we can study the spectral sequence associated to the extension

$$1 \longrightarrow Z(G) \longrightarrow G \longrightarrow G/Z(G) \longrightarrow 1$$

More results using the Lyndon-Hochschild-Serre spectral sequence can be found in section IV.1 from the book of Adem and Milgram [1].

Chapter 3

Homological stability

Consider a family of groups $\{G_n\}_{n\in\mathbb{N}}$, we say it is an increasing sequence of groups if there is an inclusion $G_n \hookrightarrow G_{n+1}$ for all $n \in \mathbb{N}$. These inclusions induce maps in homology $H_k(G_n) \to H_k(G_{n+1})$. We are interested when there is a lower bound on n depending on k for which these induced maps are isomorphisms. We call this phenomenon homological stability. In this case, for all k and $n \gg k$ the homology group $H_k(G_n)$ is independent of n.

3.1 Coefficient systems

We already know how to compute homology groups with trivial coefficients $\mathbb{Z}$. It seems reasonable to define a setup allowing more general coefficients than the constant ones considered in Chapter 1.

Consider a semisimplicial set $\mathbb{X}$, define its simplex category $Simp(\mathbb{X})$ by

- The objects of $Simp(\mathbb{X})$ are the simplices of $\mathbb{X}$.
- For $\sigma \in \mathbb{X}^k$ and $\sigma' \in \mathbb{X}^l$ we have

$$\mathrm{Mor}(\sigma, \sigma') = \{\iota : [l] \to [k] : \iota^*(\sigma) = \sigma'\}$$

  This set of morphisms is non-empty when σ' is a face of σ. In fact all morphisms are generated by face maps.

We can add a terminal object $*$ as a (-1)-simplex, which gives us the augmented simplex category $\widetilde{Simp}(\mathbb{X})$.

Now we can define coefficient systems on semisimplicial sets.

Definition 3.1.1 (Coefficient system)**.** A *coefficient system* over a commutative ring R on a semisimplicial set $\mathbb{X}$ is a covariant functor

$$\mathcal{F} : Simp(\mathbb{X}) \to \mathrm{Mod}_R$$

Let us unwind this definition. Such a functor consists of two pieces of data:

1. For each simplex σ of $\mathbb{X}$, an R-module $\mathcal{F}(\sigma)$
2. For $\iota \in \mathrm{Mor}(\sigma, \sigma')$, a R-module homomorphism $\iota^* : \mathcal{F}(\sigma) \to \mathcal{F}(\sigma')$.

Note that we can extend this definition to the augmented simplex category, which yields what we call an augmented coefficient system.

Now consider a group G and a semisimplicial set $\mathbb{X}$ equipped with a G-action. We would like to have coefficient systems on $\mathbb{X}$ also equipped with a G-action that is compatible with the one of $\mathbb{X}$. Let $\mathcal{F}$ be a coefficient system on $\mathbb{X}$. Such a G-action on $\mathcal{F}$ must include for all $g, h \in G$ isomorphisms

$$\mathcal{F}(\sigma) \to \mathcal{F}(h \cdot \sigma)$$
$$\mathcal{F}(h \cdot \sigma) \to \mathcal{F}(gh \cdot \sigma)$$

In other words, if for all $h \in G$ we note $\mathcal{F}_h$ the coefficient system on $\mathbb{X}$ defined by $\mathcal{F}_h(\sigma) = \mathcal{F}(h \cdot \sigma)$, then for all $g, h \in G$ there are natural transformations

$$\Phi_{g,h} : \mathcal{F}_h \to \mathcal{F}_{gh}$$

such that

- For all $h \in G$, $\Phi_{1,h} = Id_{\mathcal{F}_h}$
- For all $g_1, g_2 \in G$, $\Phi_{g_1,g_2h} \circ \Phi_{g_2,h} = \Phi_{g_1g_2,h}$

If such natural transformations exist, we say that $\mathcal{F}$ is a G-equivariant coefficient system on $\mathbb{X}$.

Remark 3.1.2. For a simplex σ of $\mathbb{X}$, the stabilizer group G_σ acts on $\mathcal{F}(\sigma)$, equipping it with a $R[G_\sigma]$-module structure.

From the G-equivariant structure we can also deduce that for all $k \in \mathbb{N}$ the direct sum

$$\bigoplus_{\sigma \in \mathbb{X}^k} \mathcal{F}(\sigma)$$

is an $R[G]$-module, where the G-action restricts to the G_σ-action on $\mathcal{F}(\sigma)$. Furthermore, if $\tau \in \mathbb{X}^k/G$ and $\tilde{\tau} \in \mathbb{X}^k$ is a lift we get

$$\bigoplus_{\sigma \in \mathbb{X}^k} \mathcal{F}(\sigma) = \bigoplus_{\tau \in \mathbb{X}^k/G} \left(\bigoplus_{\sigma \in G\tilde{\tau}} \mathcal{F}(\sigma) \right)$$

By corollary 1.4.3 we deduce

$$\bigoplus_{\sigma \in G\tilde{\tau}} \mathcal{F}(\sigma) \cong \mathrm{Ind}_{G_{\tilde{\tau}}}^{G} \mathcal{F}(\tilde{\tau})$$

Remark 3.1.3. Putting all this together we finaly get

$$\bigoplus_{\sigma \in \mathbb{X}^k} \mathcal{F}(\sigma) \cong \bigoplus_{\tau \in \mathbb{X}^k/G} \mathrm{Ind}_{G_{\tilde{\tau}}}^{G} \mathcal{F}(\tilde{\tau})$$

Example 3.1.4 ($\mathfrak{S}_{n+1}$-equivariant coefficient system on $\mathbb{S}\mathrm{im}_n$). Let M be an FI-module. We define a coefficient system $\mathcal{F}_{M,n}$ on $\mathbb{S}\mathrm{im}_n$ via

$$\mathcal{F}_{M,n}(i_1,\ldots,i_k) = M([n]\backslash\{i_1,\ldots,i_k\})$$

Given an order preserving map $\iota : [l] \to [k]$ the map

$$\mathcal{F}_{M,n}(\iota) : \mathcal{F}_{M,n}(i_1,\ldots,i_k) \to \mathcal{F}_{M,n}(i_{\iota(0)},\ldots,i_{\iota(l)})$$

is induced by the inclusion $[n]\backslash\{i_1,\ldots,i_k\} \hookrightarrow [n]\backslash\{i_{\iota(0)},\ldots,i_{\iota(l)}\}$. We can extend this to an augmented coefficient system by setting $\mathcal{F}_{M,n}(*) = M([n])$.

Notice that $\mathfrak{S}_{n+1}$ acts on $\mathbb{S}\mathrm{im}_n$, so we equip $\mathcal{F}_{M,n}$ with a $\mathfrak{S}_{n+1}$-equivariant augmented coefficient system structure. For $g,h \in \mathfrak{S}_{n+1}$ define the natural transformation $\Phi_{g,h}$ by

$$\Phi_{g,h} : \mathcal{F}_{M,n}(h(i_0),\ldots,h(i_k)) \to \mathcal{F}_{M,n}(gh(i_0),\ldots,gh(i_k))$$

as the map induced by the bijection g on $[n]$ restricted to

$$[n]\backslash\{h(i_0),\ldots,h(i_k)\} \to [n]\backslash\{gh(i_0),\ldots,gh(i_k)\}$$

Finally, we are ready to define the homology of a semisimplicial set $\mathbb{X}$ with coefficient system $\mathcal{F}$. First define a chain complex $C_\bullet(\mathbb{X};\mathcal{F})$ as follows:

- For $k \in \mathbb{N}$, $C_k(\mathbb{X};\mathcal{F}) = \bigoplus_{\sigma\in\mathbb{X}^k} \mathcal{F}(\sigma)$
- The boundary map
$$d : C_k(\mathbb{X};\mathcal{F}) \to C_{k-1}(\mathbb{X},\mathcal{F})$$
is $d = \sum_{i=0}^{k}(-1)^i d_i$, where
$$d_i : C_k(\mathbb{X};\mathcal{F}) \to C_{k-1}(\mathbb{X};\mathcal{F})$$
is described as follows. For $\sigma \in \mathbb{X}^k$, $\delta^i : [k-1] \to [k]$, on the $\mathcal{F}(\sigma)$ factor of $C_k(\mathbb{X},\mathcal{F})$, d_i is the composition
$$\mathcal{F}(\sigma) \xrightarrow{\iota^*} \mathcal{F}(\iota^*(\sigma)) \hookrightarrow \bigoplus_{\sigma'\in\mathbb{X}^{k-1}} \mathcal{F}(\sigma) = C_{k-1}(\mathbb{X};\mathcal{F})$$
In other words these maps are induced by the face maps of $\mathbb{X}$.

Definition 3.1.5 (Homology of a semisimplicial set)**.** We define the *homology* of $\mathbb{X}$ with coefficients $\mathcal{F}$ as the homology of this chain complex:

$$H_k(\mathbb{X},\mathcal{F}) = H_k(C_\bullet(\mathbb{X};\mathcal{F})$$

We can also define the reduced homology taking the augmented chain complex, setting $\tilde{C}_{-1}(\mathbb{X};\mathcal{F}) = \mathcal{F}(*)$.

If we consider a G-semisimplicial set $\mathbb{X}$ together with a G-equivariant coefficient system $\mathcal{F}$ then $\tilde{C}_\bullet(\mathbb{X};\mathcal{F})$ is a chain complex of $R[G]$-modules.

In this setup we can define an augmented coefficient system $\mathcal{H}_q(\mathcal{F})$ on $\mathbb{X}/G$. Let $\sigma \in \mathbb{X}/G$ be a simplex and $\tilde{\sigma}$ a lift of σ in $\mathbb{X}$, define

$$\mathcal{H}_q(\mathcal{F})(\sigma) = H_q(G_{\tilde{\sigma}};\mathcal{F}(\tilde{\sigma}))$$

which we can prove to be well defined ([11], §5.3), i.e. independent of the choice of lift $\tilde{\sigma}$.

3.2 Spectral sequence

Now we will benefit from the background on spectral sequences presented in chapter 2. It turns out that a group G which acts on a semisimplicial set $\mathbb{X}$ and a G-equivariant coefficient system $\mathcal{F}$ yield a spectral sequence. This spectral sequence is presented by Putman in ([11], §5.6).

Consider a free resolution $F_\bullet(G) \to \mathbb{Z}$ of R over $\mathbb{Z}[G]$. Then we define a double complex

$$C_{p,q} = \tilde{C}_p(\mathbb{X};\mathcal{F}) \otimes_G F_q(G)$$

This yields two spectral sequences as we have seen in Theorem 2.3.2. First we will focus on the spectral sequence obtained from the vertical filtration. We have

$$E^1_{p,q} \cong H_q(\tilde{C}_p(\mathbb{X};\mathcal{F}) \otimes_G F_\bullet(G))$$

Since $F_\bullet(G)$ is a free resolution of R over $R[G]$we have by definition

$$E^1_{p,q} \cong H_q(G;\tilde{C}_p(\mathbb{X};\mathcal{F}))$$

Notice that remark 3.1.3 gives us

$$\tilde{C}_p(\mathbb{X};\mathcal{F}) \cong \bigoplus_{\sigma\in\mathbb{X}^p/G} \operatorname{Ind}^G_{G_{\tilde{\sigma}}} \mathcal{F}(\tilde{\sigma})$$

Introducing this into our spectral sequence we obtain

$$E^1_{p,q} \cong \bigoplus_{\sigma\in\mathbb{X}^p/G} H_q(G;\operatorname{Ind}^G_{G_{\tilde{\sigma}}} \mathcal{F}(\tilde{\sigma}))$$

And using Shapiro's lemma 1.4.4 we conclude

$$E^1_{p,q} \cong \bigoplus_{\sigma\in\mathbb{X}^p/G} H_q(G_{\tilde{\sigma}};\mathcal{F}(\tilde{\sigma})) = \tilde{C}_p(\mathbb{X}/G;\mathcal{H}_q(\mathcal{F}))$$

Therefore

$$E^2_{p,q} \cong H_p(\tilde{C}_\bullet(\mathbb{X}/G;\mathcal{H}_q(\mathcal{F})))$$

If we add some reasonable hypothesis to our semisimplicial set and coefficient system we get nicer results. Suppose that G acts transitively on the simplices of each dimension of $\mathbb{X}$. This implies that $\mathbb{X}/G$ is a semisimplicial complex with only one k-simplex for $0 \le k \le \dim\mathbb{X}$. Then the expression for the E^1-page simplifies to

$$E^1_{p,q} \cong H_q(G_{\tilde{\sigma}};\mathcal{F}(\tilde{\sigma})) \tag{3.1}$$

where $\sigma \in \mathbb{X}^p/G$ is the only p-simplex of $\mathbb{X}/G$.

Proposition 3.2.1. *Let $\mathbb{X}$ be a semisimplicial set and G a group. Suppose that G acts transitively on $\mathbb{X}$. Then $\mathbb{X}/G$ is* $(\dim\mathbb{X}-1)$*-connected.*

Proof. The chain complex $\tilde{C}_\bullet(\mathbb{X}/G)$ consists in one copy of $\mathbb{Z}$ at each dimension. It is enough to prove that the boundary maps alternate between isomorphisms and zero. Recall that the boundary maps are induced by the face maps of $\mathbb{X}$. Let $\sigma_k \in \mathbb{X}^k$ be the only k-simplex of $\mathbb{X}/G$. Then

$$d(\sigma_k) = \sum_{i=0}^{k}(-1)^i d_i(\sigma_k) = \sigma_{k-1}\sum_{i=0}^{k}(-1)^i = \begin{cases} 0 \text{ if } k \text{ is odd} \\ \sigma_{k-1} \text{ if } k \text{ is even} \end{cases}$$

Since $\tilde{C}_k(\mathbb{X}/G)$ is generated by σ_k it follows that

$$d_k = \begin{cases} 0 \text{ if } k \text{ is odd} \\ Id_{\mathbb{Z}} \text{ if } k \text{ is even} \end{cases}$$

Therefore the reduced homology groups are zero below $\dim \mathbb{X}$ and

$$H_{\dim \mathbb{X}}(\tilde{C}_k(\mathbb{X}/G) = \begin{cases} \mathbb{Z} \text{ if } k \text{ is odd} \\ 0 \text{ if } k \text{ is even} \end{cases}$$

Now we show that $\mathbb{X}/G$ is 1-connected. Note that it has a single 0-simplex, 1-simplex and 2-simplex. Then its 1-skeleton is homotopic to S^1.

It remains to show that the attaching map of the 2-simplex is the identity. Let σ be the only 2-simplex of $\mathbb{X}/G$. We want to understand the attaching map

$$f : S^1 \to (\mathbb{X}/G)^{(1)}$$

which induces a map

$$f_* : \pi_1 S^1 = \mathbb{Z} \to \pi_1(\mathbb{X}/G)^{(1)} = \mathbb{Z} : 1 \mapsto [f]$$

We can describe $[f]$ using a lift of σ in $\mathbb{X}$. Such lift is of the form (i, j, k) where $i, j, k \leq n$. Then we have

$$[f] = [(i, j)] \star [(j, k)] \star [(k, i)]$$

where $\star$ denotes concatenation of paths. Since there is only one 1-simplex in $\mathbb{X}/G$ and the boundary of (i, j, k) is formed by the 1-simplices $\{(i, j), (j, k), (i, k)\}$ we get that $[(i, j)] = [(j, k)] = 1$ and $[(k, i)] = -1$. Therefore we obtain $[f] = 1$. It follows that $f_* = Id_{\mathbb{Z}}$, so

$$(\mathbb{X}/G)^{(2)} = D^2$$

Hence, $\mathbb{X}/G$ is simply connected.

Using Hurewicz theorem we conclude that $\mathbb{X}/G$ is $(n-1)$-connected. □

Note that we can generalize the proof of Proposition 3.2.1 to get the following result for twisted coefficients.

Lemma 3.2.2. *Let $\mathbb{X}$ be a semisimplicial set and G a group. Suppose that G acts transitively on $\mathbb{X}$ and that there is a G-equivariant coefficient system $\mathcal{M}$ on $\mathbb{X}$. Then for $k \leq \dim \mathbb{X} - 1$ we have $H_k(\mathbb{X}_n; \mathcal{M}_n) = 0$.*

Using Proposition 3.2.1 we can simplify the expression for the E_2-page

$$E^2_{p,q} \cong H_p(\tilde{C}_\bullet(\mathbb{X}/G; \mathcal{H}_q(\mathcal{F}))) = 0 \quad \forall p < \dim \mathbb{X} \tag{3.2}$$

Recall from Theorem 2.3.2 that a double complex gives rise to two spectral sequences. It remains to study the spectral sequence given by the horizontal filtration. This is only interesting when we suppose that $\mathbb{X}$ is $c(\mathbb{X})$-connected, the bigger $c(\mathbb{X})$ the more information we can get. This implies that the chain complex $\tilde{C}_\bullet(\mathbb{X}; \mathcal{F})$ is exact up to degree $c(\mathbb{X})$. As the modules $F_q(G)$ are projective it follows that $\tilde{C}_\bullet(\mathbb{X}; \mathcal{F}) \otimes_G F_q(G)$ is still exact up to degree $c(\mathbb{X})$. Therefore its homology vanishes up to degree $c(\mathbb{X})$. Hence

$$H_r(\mathrm{total}(\tilde{C}_\bullet(\mathbb{X}; \mathcal{F}) \otimes_G F_\bullet(G))) = 0 \tag{3.3}$$

for $r \leq c(\mathbb{X})$.

3.3 Stability machine

We will use the spectral sequence studied in Section 3.2 to state and prove a first result on homological stability. In this section we will use constant coefficients.

Before stating the theorem we give a more precise setup. Let $\{G_n\}_{n\in\mathbb{N}}$ be an increasing sequence of groups and let $\{\mathbb{X}_n\}_{n\in\mathbb{N}}$ be semisimplicial sets such that G_n acts on $\mathbb{X}_n$.

1. Suppose that $\dim \mathbb{X}_n = n$ and that the G_n-action on $\mathbb{X}_n$ is transitive. By this we mean that the action of G_n is transitive on the simplices of each dimension of $\mathbb{X}_n$, just as in Section 3.2.
2. Suppose that the stabilizer of a p-simplex $\sigma \in \mathbb{X}_n^p$ is isomorphic to G_{n-p-1}, i.e. $(G_n)_\sigma \cong G_{n-p-1}$.
3. For all one simplices $e = (v, w) \in \mathbb{X}_n^1$ there is some $\lambda \in G_n$ such that $\lambda(v) = w$ and λ commutes with $(G_n)_e$.
4. Suppose there exists some integer $c \geq 2$ such that $\mathbb{X}_n$ is $\lfloor \frac{n-2}{c} \rfloor$-connected, we denote $\chi_n = \lfloor \frac{n-2}{c} \rfloor$.

Theorem 3.3.1 (Classical homological stability ([5], Theorem 1.1))**.** *Let $\{G_n\}_{n\in\mathbb{N}}$ be an increasing sequence of groups and let $\{\mathbb{X}_n\}_{n\in\mathbb{N}}$ be semisimplicial sets such that G_n acts on $\mathbb{X}_n$. Suppose that the conditions (1)-(4) hold for all n. Then for all k the map $H_k(G_{n-1}) \to H_k(G_n)$ is an isomorphism for $n \geq ck + 2$ and a surjection for $ck + 1$.*

Remark 3.3.2. To start, we want to find a linear function $\varphi(k)$ of positive slope that give us a lower bound for stabilization. Then we will prove that $\varphi(k) = ck + 2$.

Proof. This proof consists of a spectral sequence argument. A double complex gives rise to two spectral sequences converging to the same bigraded module, if we know how one of these spectral sequence behaves we can deduce some information on the other.

Consider the double complex

$$C_{p,q} = \tilde{C}_p(\mathbb{X}_n) \otimes_G F_q(G)$$

as in Section 3.2 with trivial coefficient system. This yields two spectral sequences, both converging to
$$H_\bullet(\text{total}(\tilde{C}_\bullet(\mathbb{X}_n) \otimes_G F_\bullet(G)))$$

Let us start doing some observations on the spectral sequence given by the horizontal filtration. Since $\mathbb{X}_n$ is χ_n-connected, we can use Eq. (3.3) and we obtain that
$$H_r(\text{total}(\tilde{C}_\bullet(\mathbb{X}_n) \otimes_G F_\bullet(G))) = 0 \qquad \forall r \leq \chi_n \tag{3.4}$$
Now that we know the target of both spectral sequences is zero below some range, we can deduce that some of the differentials appearing in the other spectral sequence must be isomorphisms.

Denote by $\{E^r_{*,*}\}$ the spectral sequence that arises from the vertical filtration. Let $\sigma \in \mathbb{X}^p/G$, using our hypothesis and Eq. (3.1) we get
$$E^1_{p,q} \cong H_q((G_n)_{\tilde{\sigma}}) \cong H_q(G_{n-p-1})$$
with differentials
$$d^1_{p,q} : H_q((G_n)_{\tilde{\sigma}}) \to H_q((G_n)_{\tilde{\tau}})$$
where $\tau \in \mathbb{X}^{p-1}/G$. Here $d^1_{p,q}$ is the alternating sum of partial boundary maps
$$(d^1_{p,q})_i : H_q((G_n)_{\tilde{\sigma}}) \to H_q((G_n)_{\tilde{\tau}})$$
induced by the inclusion $(G_n)_{\tilde{\sigma}} \to (G_n)_{\partial_i\tilde{\sigma}}$ followed by the conjugation taking it to $(G_n)_{\tilde{\tau}}$. We summarize this information in the following diagram.
$$\begin{tikzcd}
H_q((G_n)_{\tilde{\sigma}}) \arrow[r, "(d^1_{p,q})_i"] \arrow[d] & H_q((G_n)_{\tilde{\tau}}) \arrow[d] \\
H_q(G_n) \arrow[r] & H_q(G_n)
\end{tikzcd}$$
The vertical maps from this diagram are induced by inclusion and the lower map is induced by conjugation in G_n.

We want to show that above a bound $\varphi(k)$ the maps
$$d : E^1_{0,k} \cong H_k(G_{n-1}) \to E^1_{-1,k} \cong H_k(G_n)$$
are isomorphisms. To prove it we must understand what are these maps. The map d is induced by the inclusion $G_{n-1} \to G_n$. More specifically, we include a vertex stabilizer into the whole group. In Fig. 3.1 we can visualize the E^1-page described in our situation.

We will proceed by induction on k. The case $k = 0$ being trivial since we have reduced homology, we assume our hypothesis of induction for $1, \ldots, k-1$, i.e. the map $H_i(G_{n-1}) \to H_i(G_n)$ is an isomorphism for $n \geq ci + 2$ and a surjection for $n = ci + 1$ where $i = 0, 1, \ldots, k-1$. Now we start the induction step to prove these isomorphisms for k.

Step 1: Surjectivity. By definition of the spectral sequence we have
$$E^2_{-1,k} = \text{coker}(d : H_k(G_{n-1}) \to H_k(G_n))$$

$$\begin{tikzcd}
0 & H_k(G_n) \arrow[l] & H_k(G_{n-1}) \arrow[l, "d"'] & H_k(G_{n-2}) \arrow[l, "d_1"'] & \cdots \arrow[l] \\
0 & H_{k-1}(G_n) \arrow[l] & H_{k-1}(G_{n-1}) \arrow[l] & H_{k-1}(G_{n-2}) \arrow[l] \arrow[ull, dotted] & H_{k-1}(G_{n-3}) \arrow[l] & \cdots \arrow[l] \\
 & & \cdots & H_{k-2}(G_{n-2}) \arrow[l] & H_{k-2}(G_{n-3}) \arrow[l] \arrow[uulll, dotted] & \cdots \arrow[l] \\
0 & H_0(G_n) \arrow[l] & H_0(G_{n-1}) \arrow[l] & H_0(G_{n-2}) \arrow[l] & H_0(G_{n-3}) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Figure 3.1: E^1-page

so d is surjective if and only if $E^2_{-1,k} = 0$.

To show that d is surjective it suffices to prove that the domain of any differential d_r with target $E^r_{-1,k}$ is zero. In this case d would be the only map that can kill $E^*_{-1,k}$.

Let us take a closer look at Fig. 3.1. If

$$H_q(G_{n-p-1}) \cong H_q(G_{n-p-2}) \quad \forall p+q=k, \quad q>0 \tag{3.5}$$

then all differentials d_r with target $E^r_{-1,k}$ will be zero.

So it is enough to show that $E^2_{p,q} = 0$ for $p+q \leq k$, and $q < k$ under the assumption $k \leq \chi_n$. Putting together Proposition 3.2.1 and Eq. (3.2) we get that $E^2_{p,q} = 0$ for $p \leq n-1$. Recall that Eq. (3.4) tells us that $E^\infty_{p,q} = 0$ for $p+q \leq \chi_n - 1$.

Note that Eq. (3.5) encodes the following relation:

$$\varphi(k) \geq \varphi(k-1) + 1, \quad k \leq \min\{\chi_n + 1, n-1\}.$$

We can ask for a much stronger condition, which is still sufficient but not necessary. We want

$$\varphi(k) \geq \varphi(k-1) + c, \quad k \leq \min\{\chi_n + 1, n-1\}.$$

Note that our bound function

$$\varphi(k) = ck + 1$$

satisfies all this conditions since for all k we have

$$k - 1 \leq \chi_n = \lfloor \frac{n-2}{c} \rfloor \implies ck+1 \leq n$$
$$k \leq n-1 \iff k+1 \leq n$$

By hypothesis of induction this implies that all differentials $d^1_{p,q}$ on the E^1-page such that $p+q \leq k$ and $q > 0$ are isomorphisms, in particular Eq. (3.5) holds. Hence $d : H_k(G_{n-1}) \to H_k(G_n)$ is surjective for $n \geq ck+1$.

Step 2: Injectivity. We use a similar argument as before. For $i \leq \chi_n$ the term $E^\infty_{0,k}$ is zero. We want to prove that all differentials with target $E^r_{0,k}$ are zero, this way the differential d must be injective.

The additional step that we need to prove is that the differential

$$d_1 : E^1_{1,k} \cong H_k(G_{n-2}) \to E^1_{0,k} \cong H_k(G_{n-1})$$

is the zero map. We prove this using hypothesis (3). Let u be the vertex chosen to represent the orbit of vertices. Take $e = (v, w)$ and λ as they are described in hypothesis (3). We have $h_v, h_w \in G_n$ such that $h_v v = u$ and $h_w w = u$. Denote by $c_g : G_n \to G_n$ the conjugation map by $g \in G_n$. This yields the next commutative diagram.

$$\begin{tikzcd}
 & (G_n)_v \arrow[r, "c_{h_v}"] \arrow[d, "c_\lambda"] & (G_n)_u \arrow[d, "c_{h_w \lambda h_v^{-1}}"] \\
(G_n)_e \arrow[ur, "i"] \arrow[dr, "i"] & (G_n)_w \arrow[r, "c_{h_w}"] & (G_n)_u
\end{tikzcd}$$

Notice that $h_w \lambda h_v^{-1} \in (G_n)_u$, therefore $c_{h_w \lambda h_v^{-1}}$ induces the identity on homology. Thus we have the following commutative diagram.

$$\begin{tikzcd}
 & H_\bullet((G_n)_v) \arrow[dd] \arrow[dr] & \\
H_\bullet(G_{n-2}) = H_\bullet((G_n)_e) \arrow[ur] \arrow[dr] & & H_\bullet((G_n)_u) = H_\bullet(G_{n-1}) \\
 & H_\bullet((G_n)_w) \arrow[ur] &
\end{tikzcd}$$

As the map d_1 is the alternating sum between the two rows in this diagram it follows that it is zero.

It remains to study the bounds. Looking at Fig. 3.1 we can see that the function φ must satisfy

$$\varphi(k) \geq \varphi(k-1) + c, \quad k + 1 \leq n - 1$$

Note that the last condition is stronger that the one we found for surjectivity. Putting all this together we obtain

$$\varphi(k) = ck + 2$$

This conclues the proof. □

Now we are interested in a similar result but using twisted coefficients. In order to do it we need to introduce some more notation and terminology.

Fix a commutative ring R. An increasing sequence of groups and modules is a sequence $\{(G_n, M_n)\}_{n \in \mathbb{N}}$ such that:

- $\{G_n\}_{n \in \mathbb{N}}$ is an increasing sequence of groups;
- For all $n \in \mathbb{N}$, M_n is a $R[G_n]$-module;
- For all $n \in \mathbb{N}$ there is an inclusion $M_n \subset M_{n+1}$ as abelian groups;
- This inclusion $M_n \hookrightarrow M_{n+1}$ is G_n-equivariant, where G_n acts on M_{n+1} via $G_n \hookrightarrow G_{n+1}$.

This yields maps

$$\cdots \longrightarrow H_k(G_{n-1}; M_{n-1}) \longrightarrow H_k(G_n; M_n) \longrightarrow H_k(G_{n+1}; M_{n+1}) \longrightarrow \cdots$$

and we would like to determine their stabilization.

We incorporate M_n using G_n-equivariant augmented coefficient systems $\mathcal{M}_n$ on $\mathbb{X}_n$ such that $\mathcal{M}_n(*) = M_n$.

Then we must adapt the conditions 1-4 above to this new language:

1. Suppose that $\dim \mathbb{X}_n = n$ and that the G_n-action on $\mathbb{X}_n$ is transitive. By this we mean that the action of G_n is transitive on the simplices of each dimension of $\mathbb{X}_n$, just as in Section 3.2.
2. Suppose that the stabilizer of a p-simplex $\sigma \in \mathbb{X}_n^p$ is isomorphic to G_{n-p-1}, i.e. $(G_n)_\sigma \cong G_{n-p-1}$, with $\mathcal{M}_n(\sigma) = M_{n-k-1}$. In particular $\mathcal{M}_n(*) = M_n$.
3. For all one simplices $e = (v, w) \in \mathbb{X}_n^1$ there is some $\lambda \in G_n$ such that $\lambda(v) = w$ and λ commutes with $(G_n)_e$ and fixes all elements of $\mathcal{M}_n(e)$.
4. There exists some integer $c \geq 2$ such that for any $k \leq \chi_n = \lfloor \frac{n-2}{c} \rfloor$ we have $H_k(\mathbb{X}_n; \mathcal{M}_n) = 0$.

Now we can state a much more general theorem.

Theorem 3.3.3 (Twisted homological stability ([11], Theorem 5.2))**.** *Let $\{G_n, M_n\}_{n\in\mathbb{N}}$ be an increasing sequence of groups and modules, let $\{\mathbb{X}_n\}_{n\in\mathbb{N}}$ be semisimplicial sets such that G_n acts on $\mathbb{X}_n$ and let $\mathcal{M}_n$ be a G_n-equivariant augmented coefficient system on $\mathbb{X}_n$. Suppose that the conditions (1)-(4) hold for all n. Then for all k the map $H_k(G_{n-1}; M_{n-1}) \to H_k(G_n; M_n)$ is an isomorphism for $n \geq ck + 2$ and a surjection for $ck + 1$.*

The proof if this theorem is similar to the one of Theorem 3.3.1.

3.4 Stability for symmetric groups

An important increasing sequence of groups is the one given by symmetric groups $\{\mathfrak{S}_n\}_{n\geq 1}$. Since any finite group is a subgroup of a symmetric group, then understanding symmetric groups might give us an insight for any other sequence of groups.

In 1960, Nakaoka stated some of the first results on homological stability for symmetric groups on his paper [10].

Theorem 3.4.1. *Consider the increasing sequence of groups given by the symmetric groups $\{\mathfrak{S}_n\}_{n\geq 1}$. Then for all $k \geq 0$ the map*

$$H_k(\mathfrak{S}_n) \to H_k(\mathfrak{S}_{n+1})$$

is an isomorphism for $n \geq 2k + 2$ and a surjection for $n = 2k + 1$.

Remark 3.4.2. Here it is understood that we work with untwisted coefficients $\mathbb{Z}$.

Proof. First we want a semisimplicial set on which $\mathfrak{S}_{n+1}$ acts. Using Example 1.6.4 we know that $\mathfrak{S}_n$ acts on $\mathbb{S}\mathrm{im}_n$.

To prove the theorem it suffices to show that the hypothesis of Theorem 3.3.1 are satisfied.

By definition of $\mathbb{S}\mathrm{im}_n$ we can see that it is n-dimensional and that the $\mathfrak{S}_{n+1}$-action is transitive on simplices. Moreover for an ordered k-simplex $\sigma \in \mathbb{S}\mathrm{im}_n^k$ it is not hard to see that

$$(\mathfrak{S}_{n+1})_\sigma \cong \mathfrak{S}_{n-k}$$

since it must fix all vertices of σ.

For $(i,j) \in \mathbb{S}\mathrm{im}_n^1$ its stabilizer is the subgroup of $\mathfrak{S}_{n+1}$ of permutations of $\{1, \dots, \hat{i}, \dots, \hat{j}, \dots, n+1\}$, which is isomorphic to $\mathfrak{S}_{n-1}$. Using standard properties of symmetric groups we can see that it commutes with the transposition τ that only permutes i and j.

Finally, we use a result of Farmer ([4], Theorem 5) which proves that $\mathbb{S}\mathrm{im}_n$ is $(n-1)$-connected. So we can take $c = 1$.

All hypothesis of Theorem 3.3.1 are satisfied so we can use it and conclude this proof. $\square$

Remark 3.4.3. The proof of Farmer ([4], Theorem 5) is very technical. It requires many computations and it is not trivial at all. The case $n = 1$ is easy, we can see that $\mathbb{S}\mathrm{im}_1$ is 0-connected in Example 1.6.5. However the case $n = 2$ needs some more thought. In Example 1.6.6 we get the feeling that it is not that simple.

It is possible to generalize the result from Theorem 3.4.1 to twisted coefficients. In order to do it one must introduce the twisted coefficients using FI-modules. In Example 3.1.4 we explained that an FI-module yields a $\mathfrak{S}_{n+1}$-equivariant coefficient system on $\mathbb{S}\mathrm{im}_n$. This provides most of the background needed to prove the general case. A complete proof can be found in section 7 from an article by Putman [11].

Appendix A

Dihedral groups

We would like to compute the cohomology ring of dihedral groups. In order to do this we will apply our knowledge of spectral sequences explained in Chapter 2, specially the Lyndon-Hochschild-Serre spectral sequence presented in Section 2.4. Using a central group extension we will be able to compute explicitly $H^*(D_{2^n};\mathbb{F}_2)$ for all $n \geq 3$. We will argue by induction. To start we will need a result about the spectral sequence associated to the central extension of D_{2^n}.

Before stating any theorem we will explain carefully the base case, which has the core argument using spectral sequences.

Proposition A.0.1. *Let $n \geq 2$. Then we have*

$$H^i(D_{2^n};\mathbb{F}_2) \cong \mathbb{F}_2^{i+1}$$

Proof. For the first part of the proposition we consider the following presentation of D_{2^n}.

$$D_{2^n} = \langle \sigma, \rho \mid \sigma^2, \rho^{2^{n-1}}, (\sigma\rho)^2 \rangle$$

Note that $D_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ whose cohomology we know $H^*(D_4;\mathbb{F}_2) \cong \mathbb{F}_2[x,y]$ for generators x, y of degree 1. Denote $\Sigma_\rho = 1 + \rho + \cdots + \rho^{2^{n-1}-1}$. Let $B_\bullet \to \mathbb{Z}$ be a free $\mathbb{Z}[C_2]$-resolution of $\mathbb{Z}$. Explicitly $B_n = \mathbb{Z}[C_2]$, see Example 1.2.2. If we denote by σ the generator of C_2 the homomorphisms in $B_\bullet$

alternate between $\sigma - 1$ and $\sigma + 1$. We have the following diagram.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
B_4 \arrow[d, "\sigma-1"] & \mathbb{Z}(D_{2^n}) \arrow[l, "\epsilon_3"'] \arrow[d, "\sigma-1"] & \mathbb{Z}(D_{2^n}) \arrow[l, "\rho-1"'] \arrow[d, "-(\rho\sigma+1)"] & \mathbb{Z}(D_{2^n}) \arrow[l, "\Sigma_\rho"'] \arrow[d, "-(\sigma+1)"] & \mathbb{Z}(D_{2^n}) \arrow[l, "\rho-1"'] \arrow[d, "\rho\sigma-1"] & \cdots \arrow[l, "\Sigma_\rho"'] \\
B_3 \arrow[d, "\sigma+1"] & \mathbb{Z}(D_{2^n}) \arrow[l, "\epsilon_2"'] \arrow[d, "\sigma+1"] & \mathbb{Z}(D_{2^n}) \arrow[l, "\rho-1"'] \arrow[d, "-(\rho\sigma-1)"] & \mathbb{Z}(D_{2^n}) \arrow[l, "\Sigma_\rho"'] \arrow[d, "-(\sigma-1)"] & \mathbb{Z}(D_{2^n}) \arrow[l, "\rho-1"'] \arrow[d, "\rho\sigma+1"] & \cdots \arrow[l, "\Sigma_\rho"'] \\
B_1 \arrow[d, "\sigma-1"] & \mathbb{Z}(D_{2^n}) \arrow[l, "\epsilon_1"'] \arrow[d, "\sigma-1"] & \mathbb{Z}(D_{2^n}) \arrow[l, "\rho-1"'] \arrow[d, "-(\rho\sigma+1)"] & \mathbb{Z}(D_{2^n}) \arrow[l, "\Sigma_\rho"'] \arrow[d, "-(\sigma+1)"] & \mathbb{Z}(D_{2^n}) \arrow[l, "\rho-1"'] \arrow[d, "\rho\sigma-1"] & \cdots \arrow[l, "\Sigma_\rho"'] \\
B_0 \arrow[d, "\epsilon"] & \mathbb{Z}(D_{2^n}) \arrow[l, "\epsilon_0"'] & \mathbb{Z}(D_{2^n}) \arrow[l, "\rho-1"'] & \mathbb{Z}(D_{2^n}) \arrow[l, "\Sigma_\rho"'] & \mathbb{Z}(D_{2^n}) \arrow[l, "\rho-1"'] & \cdots \arrow[l, "\Sigma_\rho"'] \\
\mathbb{Z} & & & & &
\end{tikzcd}$$

Where the i-th row is a $\mathbb{Z}(D_{2^n})$ free resolution of B_i. Consider the total complex of this diagram, Theorem IV.2.4 from [1] indicates a way to use the vertical and horizontal differentials to obtain a total complex.

$$\mathbb{Z} \longleftarrow \mathbb{Z}(D_{2^n}) \longleftarrow \mathbb{Z}(D_{2^n})^2 \longleftarrow \mathbb{Z}(D_{2^n})^3 \longleftarrow \cdots$$

If we see this modulo 2, all differentials are zero. To see all details of this proof see section IV.2 from [1].

$$\mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2(D_{2^n}) \xleftarrow{0} \mathbb{F}_2(D_{2^n})^2 \xleftarrow{0} \mathbb{F}_2(D_{2^n})^3 \xleftarrow{0} \cdots$$

After tensoring by $- \otimes_{\mathbb{F}_2[D_{2^n}]} \mathbb{F}_2$ and taking homology we get

$$H^i(D_{2^n}; \mathbb{F}_2) \cong \mathbb{F}_2^{i+1}$$

$\square$

Proposition A.0.2. *Consider the dihedral group D_8. The cohomology ring of D_8 is*

$$H^*(D_8; \mathbb{F}_2) \cong \mathbb{F}_2[x, y, w]/(xy)$$

where x, y are one dimensional and w is two dimensional.

Proof. First we fix the following presentation of D_8.

$$D_8 = \langle \sigma, \rho \mid \sigma^2, \rho^4, (\sigma\rho)^2 \rangle$$

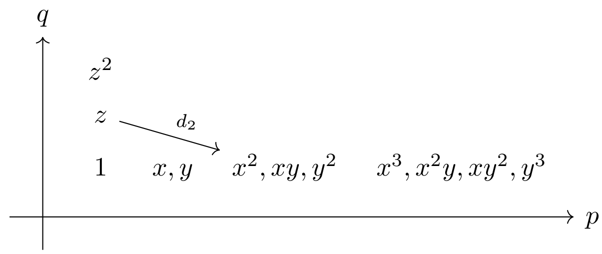
This presentation gives us the following central extension

$$1 \longrightarrow \langle \rho^2 \rangle = \mathbb{Z}/2 \xrightarrow{\triangleleft} D_8 \xrightarrow{\pi} \mathbb{Z}/2 \times \mathbb{Z}/2 = \langle \overline{\sigma} \rangle \times \langle \overline{\rho} \rangle \longrightarrow 1$$

Now we look at the Lyndon-Hochschild-Serre spectral sequence associated to this central extension. The E_2 page is given by

$$E_2^{p,q} \cong H^p(\mathbb{Z}/2 \times \mathbb{Z}/2; H^q(\mathbb{Z}/2; \mathbb{F}_2)) \cong H^p(\mathbb{Z}/2 \times \mathbb{Z}/2; \mathbb{F}_2)$$

Therefore $E_2^{*,q} \cong \mathbb{F}_2[x,y]$ and $E_2^{0,*} \cong \mathbb{F}_2[z]$. We can represent the E_2 page as follows



As the cohomology ring of D_8 is the target of this spectral sequence we know it is a quotient of $\mathbb{F}_2[x,y,z]$. In order to determine its cohomology we must compute $d_2 z$. For starters we have

$$d_2 z = ax^2 + bxy + cy^2, \quad a,b,c \in \mathbb{F}_2.$$

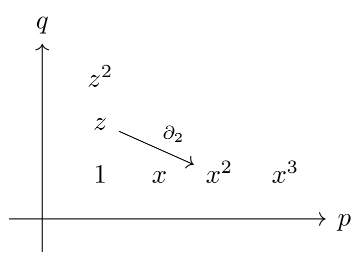
To find a,b,c we use maps in cohomology induced by morphisms between group extensions. Our setup is the following

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[r] \arrow[d, equal] & \bullet \arrow[r, dash] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Z}/2 \arrow[d, "i_1,\Delta,i_2"] \\
\mathbb{Z}/2 \arrow[r] & D_8 \arrow[r] & \mathbb{Z}/2 \times \mathbb{Z}/2
\end{tikzcd}$$

Let us do these computation when we take i_1 the inclusion on the first factor as the map on the right side of the diagram. After computing the pullback we get

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[r] \arrow[d, equal] & \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r, dash] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Z}/2 \arrow[d, "i_1"] \\
\mathbb{Z}/2 \arrow[r] & D_8 \arrow[r] & \mathbb{Z}/2 \times \mathbb{Z}/2
\end{tikzcd}$$

Now we compute the E_2 page associated to the upper group extension. Arguing as before we obtain



and as $H^*(\mathbb{Z}/2 \times \mathbb{Z}/2) \cong \mathbb{F}_2[x,z]$ we must have $\partial_2 z = 0$. Now it remains to study the induced maps in cohomology. We have the following commutative diagram

$$\begin{tikzcd}
H^*(\mathbb{Z}/2;\mathbb{F}_2) \arrow[rr, "0"] \arrow[dr, "(p_1)^*"'] & & H^*(\mathbb{Z}/2;\mathbb{F}_2) \\
& H^*(\mathbb{Z}/2\times\mathbb{Z}/2;\mathbb{F}_2) \arrow[ur, "(i_2)^*"'] \arrow[dr, "(i_1)^*"'] & \\
H^*(\mathbb{Z}/2;\mathbb{F}_2) \arrow[rr, "0"'] \arrow[ur, "(p_2)^*"'] & & H^*(\mathbb{Z}/2;\mathbb{F}_2)
\end{tikzcd}$$

where diagonal compositions are the identity. And we have

$$\partial_2 z = (i_1)^* d_2 z$$

It follows that

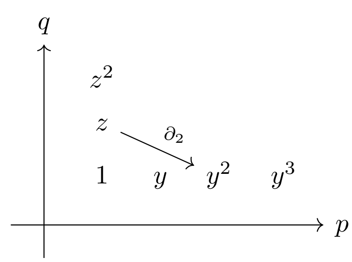
$$0 = (i_1)^*(ax^2 + bxy + cy^2) = ax^2$$

so $a = 0$.

Now we study the group extension that we obtain by choosing i_2, inclusion on the second factor.

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[r] \arrow[d, equal] & \mathbb{Z}/4 \arrow[r, dash] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{Z}/2 \arrow[d, "i_2"] \\
\mathbb{Z}/2 \arrow[r] & D_8 \arrow[r] & \mathbb{Z}/2\times\mathbb{Z}/2
\end{tikzcd}$$

We repeat the same kind of argument. We get a spectral sequence with E_2 page



but now it converges to $H^*(\mathbb{Z}/4;\mathbb{F}_2) \cong \mathbb{F}[w] \otimes E(y)$, with $|y| = 1, |w| = 2$. Therefore $\partial_2 z = y^2$, which gives us

$$y^2 = \partial_2 z = (i_2)^*(d_2 z) = cy^2$$

so $c = 1$.

It remains to compute b. To do it we study the last group extension, the one using the diagonal inclusion Δ. We have

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[r] \arrow[d, equal] & \mathbb{Z}/2\times\mathbb{Z}/2 \arrow[r, dash] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{Z}/2 \arrow[d, "\Delta"] \\
\mathbb{Z}/2 \arrow[r] & D_8 \arrow[r] & \mathbb{Z}/2\times\mathbb{Z}/2
\end{tikzcd}$$

Recall we already computed the E_2 page of the top group extension. It gives us $\partial_2 z = 0$ since the extension is trivial. Additionally we have a diagram in cohomology

$$\begin{tikzcd}
H^*(\mathbb{Z}/2;\mathbb{F}_2) \arrow[dr, "(p_1)^*"] \arrow[drr, bend left, "Id"] & & \\
& H^*(\mathbb{Z}/2\times\mathbb{Z}/2;\mathbb{F}_2) \arrow[r, "\Delta^*"] & H^*(\mathbb{Z}/2;\mathbb{F}_2) \\
H^*(\mathbb{Z}/2;\mathbb{F}_2) \arrow[ur, "(p_2)^*"'] \arrow[urr, bend right, "Id"'] & &
\end{tikzcd}$$

with

$$\begin{aligned}\Delta^* : H^*(\mathbb{Z}/2\times\mathbb{Z}/2;\mathbb{F}_2) &\longrightarrow H^*(\mathbb{Z}/2;\mathbb{F}_2)\\ x &\longmapsto w\\ y &\longmapsto w\end{aligned}$$

where $w \in H^1(\mathbb{Z}/2;\mathbb{F}_2)$ is a generator. Then we get

$$0 = \partial_2 z = \Delta^*(d_2 z) = bw^2 + w^2 = (b+1)w^2$$

so we must have $b = 1$.

Finally we have $d_2 z = xy + y^2$. Which means that z does not survives to E_∞ because it is not a cycle. To determine $H^*(D_8;\mathbb{F}_2)$ we wonder if z^2 survives to E_∞. Now we prove that z^2 is a cycle in E_2. As the differential of every page is a derivation we get

$$d_2(z^2) = 2d_2(z) = 0$$

Therefore z^2 survives to E_3. Let us write $w = z^2$. We get

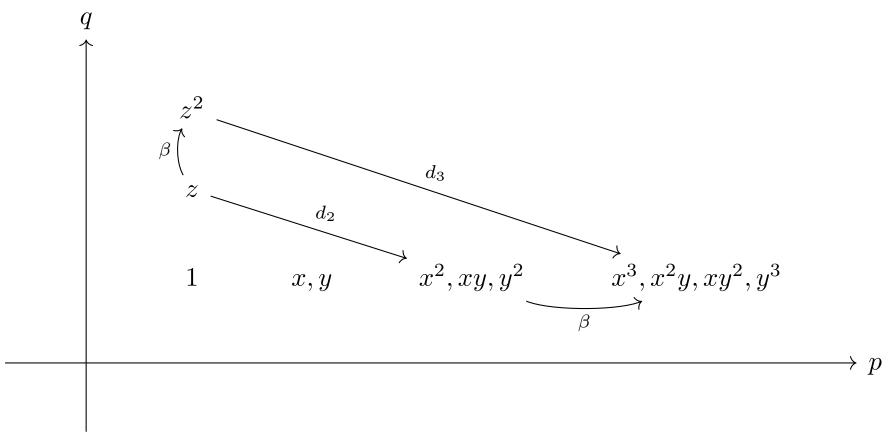
$$E_3 \cong \mathbb{F}_2[x,y,w]/((x+y)y)$$

After a linear change of variables ($x' = x + y$) finally have

$$E_3 \cong \mathbb{F}_2[x,y,w]/(xy)$$

where x, y are one dimensional and w is two dimensional. We claim that this spectral sequence collapses at the E_3-page. Using Proposition A.0.1 and counting the dimension of total$(E_3)^n$ it follows that this spectral sequence must collapse at the E_3-page. $\square$

Remark A.0.3. Notice that we can do explicit calculations of the E_3-page using Bockstein and transgression. We use this tools as a black box. In a few words, the Bockstein is an operator β on graded algebras of degree 1. It appears in the total of the pages from our spectral sequence. We can visualize it in the following commutative diagram



It is known that $\beta(z) = z^2$ and $\beta(x) = x^2$, $\beta(y) = y^2$. This gives us

$$\begin{aligned} d_3(z^2) &= d_3(\beta(z)) = \beta(d_2(z)) = \beta(xy + y^2) \\ &= \beta(x)y + x\beta(y) + 2y\beta(y) = x^2y + xy^2 \\ &= x(xy + y^2) = xd_2(z) = 0 \in E_3 \end{aligned}$$

Therefore z^2 survives to E_∞.

Now we are ready to prove a more general result.

Theorem A.0.4. *Let $n \in \mathbb{N}$. For the dihedral group D_{2^n} we have*

$$H^*(D_{2^n}; \mathbb{F}_2) \cong \mathbb{F}_2[x, y, w]/(xy)$$

where x, y are one dimensional and w is two dimensional.

Proof. We proceed by induction on n. We already have the basis case. Suppose we have the result for $k = 1, \ldots, n-1$.

Here we fix the following presentation for D_{2^n}.

$$D_{2^n} = \langle x, y \mid x^2, y^2, (xy)^{2^{n-1}} \rangle$$

The central extension of D_{2^n} is

$$\langle (xy)^{2^{n-2}} \rangle = \mathbb{Z}/2 \longrightarrow D_{2^n} \longrightarrow D_{2^{n-1}}$$

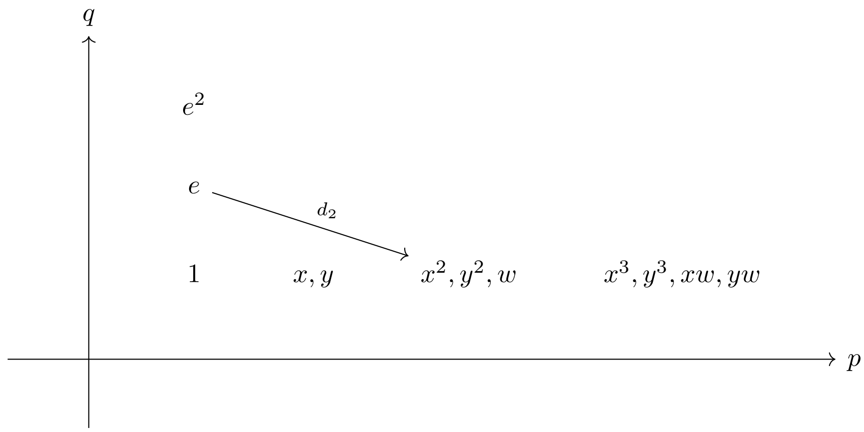
The spectral sequence associated to this extension converges to the cohomology of D_{2^n}. Its E_2 page verifies

$$E_2^{p,q} = H^p(D_{2^{n-1}}; H^q(\mathbb{Z}/2; \mathbb{F}_2)) \cong H^p(D_{2^{n-1}}; \mathbb{F}_2)$$

Then, using our hypothesis of induction we get

$$E_2^{*,q} = \mathbb{F}_2[x, y, w]/(xy)$$

Then, the E_2 page can be represented as follows.



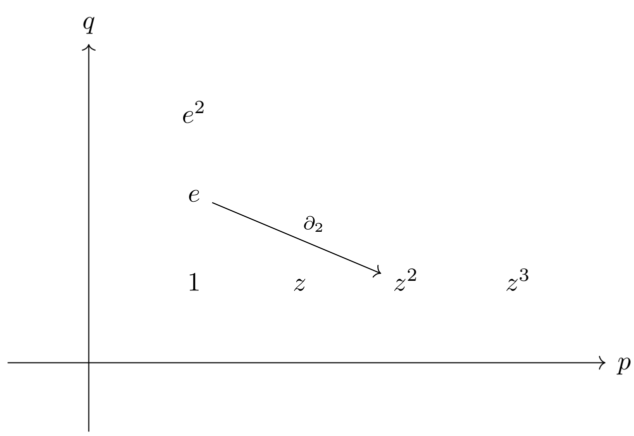
Now we proceed as in Proposition A.0.2. We would like to prove that e does not survives to E_∞ and that e^2 survives. Suppose that

$$d_2 e = ax^2 + by^2 + cw$$

where $a, b, c \in \mathbb{F}_2$. To determine a, b, c we build group extensions using pullbacks to compare spectral sequences.

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[r] \arrow[d] & \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Z}/2 \arrow[d, "i_x, i_y"] \\
\mathbb{Z}/2 \arrow[r] & D_{2^n} \arrow[r] & D_{2^{n-1}}
\end{tikzcd}$$

The spectral sequence of the top group extension has a E_2 page of the form



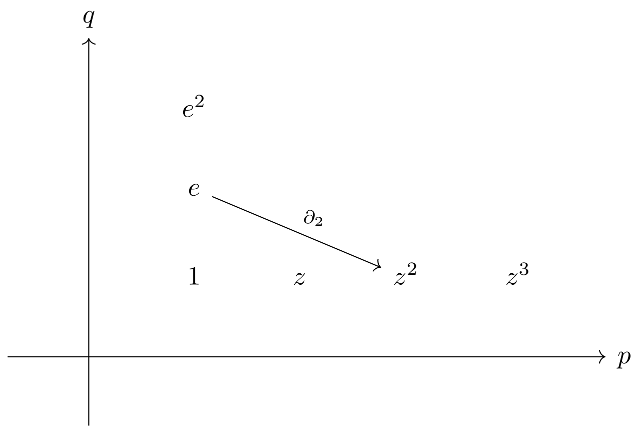
where $z = x$ if we chose i_x, or $z = y$ if we chose i_y. In both cases the target of the spectral sequence is $\mathbb{F}_2[e, z]$ therefore $\partial_2 e = 0$. It follows that

$$0 = \partial_2 e = (i_z)^*(d_2 e)$$

Thus $a = b = 0$. It remains to compute c. Here we must use the last inclusion of $\mathbb{Z}/2$ into $D_{2^{n-1}}$, we send 1 to $(xy)^{2^{n-3}}$. For simplicity let us call this map Δ. This choice gives us the following group extension.

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[r] \arrow[d] & \mathbb{Z}/4 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{Z}/2 \arrow[d, "\Delta"] \\
\mathbb{Z}/2 \arrow[r] & D_{2^n} \arrow[r] & D_{2^{n-1}}
\end{tikzcd}$$

The E_2 page of the top extension is as before.



However this spectral sequence converges to $H^*(\mathbb{Z}/4; \mathbb{F}_2)$ instead of $H^*(\mathbb{Z}/2 \times \mathbb{Z}/2; \mathbb{F}_2)$. Therefore $\partial_2 e = z^2$ and we deduce that $c = 1$. This gives us $d_2 e = w$, and since d_2 is a derivation we get $d_2 e^2 = 0$. So we have an expression for the E_3 page

$$E_3 = \mathbb{F}_2[x, y]/(xy) \otimes \mathbb{F}[e^2]$$

To finish the proof we must show that this spectral sequence collapses at the E_3-page.

The E_3-page looks as follows.

$$
\begin{array}{c|ccccc}
q & & & & & \\
 & 0 & 0 & 0 & 0 & \cdots \\
 & z^2 & xz^2, yz^2 & x^2z, y^2z & x^3z, y^3z & \cdots \\
 & 0 & 0 & 0 & 0 & \cdots \\
 & 1 & x, y & x^2, y^2 & x^3, y^3 & \cdots \\
\hline
 & & & & & p
\end{array}
$$

We know that all odd rows of E_3 are zero. A counting argument tells us that there are $i+1$ elements in dimension i in E_3. By Proposition A.0.1, all these elements must survive to E_∞. So all differentials are zero. It follows that $E_3 = E_\infty$. Now we rename e^2, call it v of degree 2. It follows that

$$H^*(D_{2^n}; \mathbb{F}_2) \cong \mathbb{F}_2[x, y, v]/(xy)$$

This finishes the proof. □

Remark A.0.5. As in Proposition A.0.2 we can identify the Bockstein of w. This way w is uniquely determined. Consider the appropriate diagram of this spectral sequence.

$$
\begin{tikzcd}
e^2 \arrow[ddrrr, "d_3"] & & & \\
e \arrow[u, bend left, "\beta"] \arrow[drr, "d_2"] & & & \\
1 & x, y & x^2, y^2, w \arrow[r, bend right, "\beta"'] & x^3, y^3, xw, yw
\end{tikzcd}
$$

To compute $\beta(w)$ we must use spectral sequence arguments as the ones to find d_2e. Suppose $\beta(w) = ax^3 + by^3 + cxw + dyw$ with $a, b, c, d \in \mathbb{F}_2$. Notice that $\beta(w) = d_3 \circ \beta(e)$, so for $z \in \{x, y\}$

we have

$$(i_z)^* \circ \beta(w) = (i_z)^* \circ d_3 \circ \beta(e) = \partial_3 \circ \beta(e) = \partial_3(e^2) = \beta \circ \partial_2(e) = \beta(0) = 0$$

This tells us that $a = b = 0$ because $(i_z)^*(w) = 0$. Now we do the same computation with Δ.

$$\Delta^* \circ \beta(w) = \partial_3(e^2) = \beta \circ \partial_2(e) = \beta(x^2) = 0$$

Therefore $c = d$, and as $\beta(w) \neq 0 \in E_2$ we must have $c = d = 1$. Putting all this together we obtain

$$\beta(w) = (x + y)w$$

Remark A.0.6. This result is surprising. We found a non-trivial family of groups $\{D_{2^n}\}_{n\in\mathbb{N}}$ such that they all have the same cohomology ring with coefficients $\mathbb{F}_2$.

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