Complex Fans: A Representation for Vectors in Polar Form with Interval Attributes

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If we allow the magnitude and angle of a complex number (expressed in polar form) to range over an interval, it describes a semicircular region, similar to a fan; these regions are what we call Complex Fans. Complex numbers are a special case of Complex Fans, where the magnitude and angle are point intervals.

Operations (specially addition) with complex numbers in polar form are complicated. What most applications do is to convert them to rectangular form, perform operations, and return the result to polar form. However, if the complex number is a Complex Fan, that transformation increases ambiguity in the result. That is, the resulting Fan is not the smallest Fan that contains all possible results. The need for minimal results took us to develop algorithms to perform the basic arithmetic operations with complex fans, ensuring the result will always be the smallest possible complex fan. We have developed the arithmetic operations of addition, negation, subtraction, product, and division of complex fans. The algorithms presented in this paper are written in pseudo-code, and the programs in Common Lisp, making use of CLOS (Common Lisp Object System). Translation to any other high level programming language should be straightforward.

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1. INTRODUCTION

Vectors can be represented in two forms: rectangular and polar. In either representation, we need two numbers to express a vector. If we choose the rectangular representation, those two numbers are the vector components in the x and y axes. In polar representation, a vector is expressed by its magnitude and angle.

Traditionally, in Engineering, vectors are represented in rectangular form, because arithmetic operations are simpler in this representation. Nevertheless, there are some applications where it makes more sense to talk about the vector's angle and magnitude, than its orthogonal components. For instance, when solving linear circuits in sinusoidal steady state, we often talk about the phase angle and magnitude of phasors (a kind of vector used in electrical engineering to represent sinusoids [Kerr 1977; Lancaster 1974; Walton 1987; Gönen 1988; Grainger and Stevenson 1994]).

On the other hand, when dealing with uncertainty, we need to allow the magnitude and angle of a vector to range over intervals. The resulting objects, instead of denoting a point in a plane, now span over circular segments. We call those circular segments *complex fans*. For example, fan V of Figure 1 is a phasor whose magnitude ranges from a to b, and whose angle ranges from α_1 to α_2.

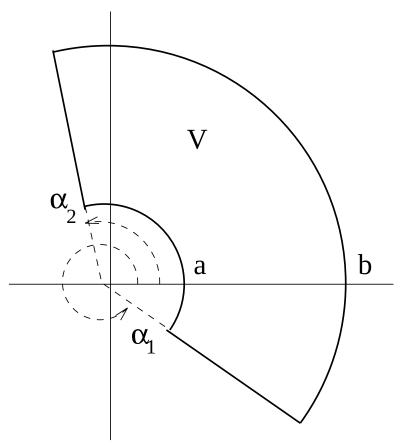

Fig. 1. A Complex Fan

In this paper, we develop the complex fan arithmetic that will be needed to perform operations on that domain. In developing the complex fan arithmetic, we pay special attention to providing minimality in the results. That is, the results of an operation is the smallest complex fan that encloses all possible results of that operation.

We provide algorithms to compute the basic arithmetic operations; that is, $+$, $-$, $*$, and $/$. Other operations can be defined in terms of the basic ones. The formulae for product, division, and negation (i.e. unary $-$), operate independently on magnitudes and angles. So the formulae for phasors can be used, performing interval operations [Alefeld and Herzberger 1983; Moore 1966; Struss 1990; Kearfott 1996], instead of real ones. The formulae for addition (subtraction is defined in terms of addition and negation) are more complicated, and we cannot just apply interval operators in a straightforward manner if we want to provide minimal results.

The rest of the paper is organized as follows: section 2 establishes the notation used throughout the paper; section 3 presents a brief introduction to interval arithmetic; sections 4 and 5 present algorithms to perform the basic arithmetic operations with complex fans, yielding the minimum complex fan that encloses all possible results; section 6 presents an application; section 7 presents a series of tests to the implementation of the algorithms; finally, section 8 concludes the work.

2. NOTATION

The convention used throughout the paper is that capital letters represent variables that contain intervals or complex fans. The context should suffice to distinguish among them, and we will not make it explicit unless an ambiguous situation arises. Small case letters denote interval limits, and Greek letters are used to represent angles.

An interval, being a continuous set, can be represented by its extremes. For instance,

$$A = (a,\ b] = \{x \ s.t. \ a < x \leq b\} \tag{1}$$

(,), [, and] indicate whether the extremes are include in the set or not. They are called open and closed ends, respectively.

The complex fan shown in Figure 1 can be represented as

$$V = V_m \angle V_\alpha \tag{2}$$

where V_m represents V's magnitude, and V_α its angle. V_m ranges over the interval $a \leq V_m \leq b$, and V_α ranges over the interval $\alpha_1 \leq V_\alpha \leq \alpha_2$. Then, the complex fan V can also be expressed as

$$V = [a, b]\angle[\alpha_1, \alpha_2] \tag{3}$$

Note that if $a = b$ and $\alpha_1 = \alpha_2$, the complex fan reduces to a phasor (i.e. a point in the complex plane).

Throughout the rest of the chapter, when we deal with two fans, we assume

$$\begin{aligned} V_1 &= V_{1m}\angle V_{1\alpha} = [a, b]\angle[\alpha_1, \alpha_2] \\ V_2 &= V_{2m}\angle V_{1\alpha} = [c, d]\angle[\alpha_3, \alpha_4] \end{aligned} \tag{4}$$

In the figures of section 5, we (arbitrarily) number the corners of the fans as indicated in Figure 2. Also, the addition of two corners i and j will be indicated by point ij. For example, addition of points 2 and 5 yields point 25, as shown in the same figure.

3. INTERVALS

There are several comprehensive treaties on interval computation. Among others, [Alefeld and Herzberger 1983] defines interval arithmetic, as well as other functions. [Moore 1966; Struss 1990; Kearfott 1996] present more advanced results on interval computation.

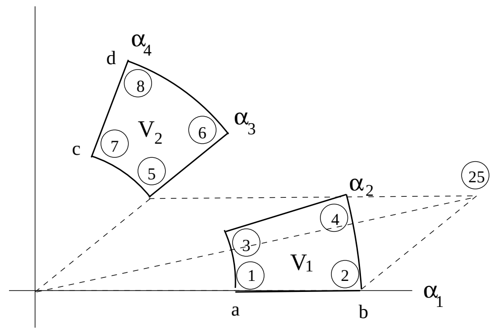

Fig. 2. Notation: Corner Numbering for Complex Fans

If A and B are intervals, a binary operation *op* can be defined as follows:

$$A \; op \; B = \{a \; op \; b \; s.t. \; a \in A, \;\; b \in B\} \tag{5}$$

Since an interval is a set, any binary operation is defined as the operation applied to all pairs that belong to the Cartesian product of both intervals.

Although that is non-constructive definition (i.e. there is an infinite number of operations for any set other than punctual sets), it can be proven that for monotonic operations the extremes of the operand intervals can be used to compute the extremes of the resulting interval. Given the above, the arithmetic operations can be defined as follows:

$$\begin{aligned} -(a,b) &= (-b,-a) \\ (a,b)+(c,d) &= (a+c,b+d) \\ (a,b)*(c,d) &= (a*c,b*d) \\ (a,b)-(c,d) &= (a-d,b-c) \\ (a,b)/(c,d) &= (a/d,b/c) \end{aligned} \tag{6}$$

In the case of intervals with any combination of open and closed ends, the rule is that an extreme of the result is closed only when both of the extremes that form it are closed. Since these formulae only hold for positive numbers, all the algorithms presented here were developed to handle positive magnitudes as well.

In the implementation of complex fans, we find that we have two kinds of intervals: magnitude and angle intervals. Magnitude intervals and their operations were described above. For angle intervals we have another complication, they are cycle intervals, modulo 360. Angle additions have to be performed in modulo 360 arithmetic. Angle additions are used in product, division, and negation operations (see equations 7-9), and for fan rotations in addition (see section 5).

Any interval, let us say [15, 45] may have two interpretations, the angle that goes from 15 to 45, and the angle that goes from 45, all the way to 15, passing through 0. A solution is to standardize the interpretation, always representing the extremes in counterclockwise order, such that in the above example only the first interpretation is valid. Under this scheme, empty intervals can be represented as point intervals with open ends, i.e. $(a, \; a) \;=\; \{x \; s.t. \; a < x < a\}$.

4. PRODUCT, DIVISION, AND NEGATION

The formulae for product, division, and negation (i.e. unary $-$) of vectors operate independently on magnitudes and angles. So the formulae for vectors can be used to perform complex fan operations, using interval operations instead of real ones.

4.1 Product

The product of two complex fans, $V = V_1 * V_2$, is given by the formula

$$V = ([a, b] * [c, d])\angle([\alpha_1, \alpha_2] + [\alpha_3, \alpha_4]) \tag{7}$$

4.2 Division

The quotient of two complex fans, $V = V_1/V_2$, is analogous to the product

$$V = ([a, b]/[c, d])\angle([\alpha_1, \alpha_2] - [\alpha_3, \alpha_4]) \tag{8}$$

4.3 Negation

The negation of a complex fan is another complex fan, with the same magnitude, and whose angle is the complement of the angle of the original fan.

$$V = -V_1 = [a, b]\angle([\alpha_1, \alpha_2] + [180, 180]) \tag{9}$$

5. SUBTRACTION, ADDITION

We do not have a simple formula for complex fan addition. We cannot just apply interval operators to vector formulae in a straightforward manner.

In this section, we present an algorithm to evaluate complex fan addition, yielding the minimum complex fan that encloses all possible results. The derivation of the algorithm is justified by the mathematics for deriving the formulas presented here.

5.1 Subtraction

Subtraction can be defined in terms of addition and negation as follows

$$V = V_1 - V_2 = V_1 + (-V_2) \tag{10}$$

so, solving the problem of addition, we solve subtraction as well.

5.2 Addition

Addition presents a complication: there is no formula to add two vectors in polar form. In one approach, the addends are converted to rectangular form, the addition is performed and then the result is transformed back into polar form. When we generalize this idea to interval magnitude and angle, the smallest rectangle that encloses a fan includes some points not present in the original fan. Then, when we convert the resulting rectangle to a fan, more points are included than are needed. This is illustrated in Figure 3.

In this section we present an algorithm that computes the smallest possible fan that encloses the result of $V_1 + V_2$, where

$$V_1 + V_2 = \{v + w \mid (v, w) \in V_1 \times V_2\} \tag{11}$$

The result of the addition must be complete and minimal. To be complete, it must contain all possible results of the addition, as defined above. To be minimal,

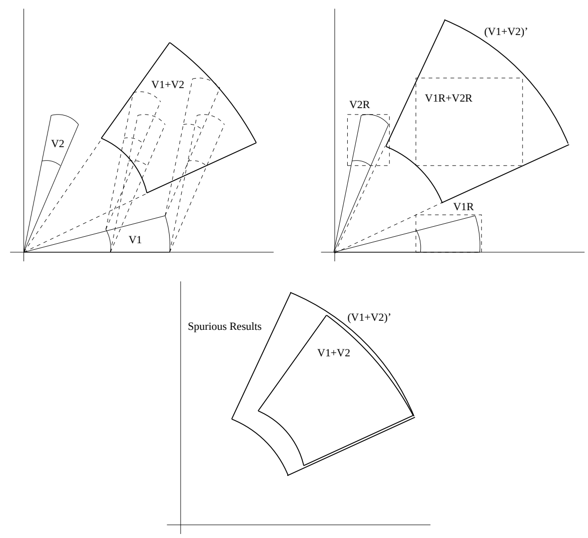

Fig. 3. Imprecision Added by Representation Changes

its boundaries (both, magnitude and phase angle) must be such that the intervals they represent cannot shrink without leaving possible results out. This implies that at least one result must fall on each of the boundaries of the resulting complex fan.

Clearly, one possible value can be obtained by adding the corners of the fans. Nevertheless, these extreme values do not necessarily produce the extreme values of the result. The only case when the corners produce the extreme results is when the fans span less than 90 degrees. So, we decompose V_1 and V_2 into four fans, representing their intersection with each quadrant, taking as reference the smallest angle of the first fan (i.e. V_1 always starts at zero). Ensuring this assumption might require a rotation, in which case, the final results will need to be corrected back to the original reference. For instance, V_1 will be

$$V_1 = \bigcup_{i=1\ldots4} V_{1i} \tag{12}$$

where

$$V_{1i} = (0, \infty)\angle[90(i-1), 90i] \cap V_1 \tag{13}$$

Thus, addition can be expressed as

$$\begin{aligned} V &= V_1 + V_2 \\ &= (V_{11} \cup V_{12} \cup V_{13} \cup V_{14}) + (V_{21} \cup V_{22} \cup V_{23} \cup V_{24}) \\ &= \{v + w \mid (v, w) \in (V_{11} \cup V_{12} \cup V_{13} \cup V_{14}) \times (V_{21} \cup V_{22} \cup V_{23} \cup V_{24})\} \end{aligned} \tag{14}$$

and since Cartesian product distributes over union,

$$\begin{aligned} V &= \{v + w \mid (v, w) \in (V_{11} \times V_{21} \cup V_{11} \times V_{22} \ldots)\} \\ &= \bigcup_{i,j=i\ldots4} V_{1i} + V_{2j} \end{aligned} \tag{15}$$

Each partial addition is performed with fans that do not extend more than 90 degrees. We can categorize each partial addition into one of three cases: when the two addends are in the same, adjacent, or opposite quadrants. We analyze each case and determine a procedure to compute the partial results. The final result is the union of all the partial results. For each case considered, we will demonstrate that the minimality condition is met.

5.2.1 *Case 1: Same Quadrant.* In this part, we consider the case when both fans are in the same quadrant. The analysis is made for the first quadrant; for other quadrants, we just apply a rotation, which will be corrected at the end of the process.

Consider the addition $V = V_1 + V_2$, where $\alpha_1 = 0$, since we take V_1 as our reference. Vector algebra states that

$$\begin{aligned} V_m &= \sqrt{V_{1m}^2 + V_{2m}^2 + 2V_{1m}V_{2m}\cos\theta} \\ V_\alpha &= \tan^{-1}\left(\frac{V_{1m}\sin V_{1\alpha} + V_{2m}\sin V_{2\alpha}}{V_{1m}\cos V_{1\alpha} + V_{2m}\cos V_{2\alpha}}\right) \end{aligned} \tag{16}$$

where $\theta = V_{1\alpha} - V_{2\alpha}$.

Since square root is a monotonic function, V_m reaches an extreme when V_m^2 does. Since addition monotonically increases with its addends, and cosine is monotonically decreasing in θ's domain (i.e. $0 \leq \theta \leq 90$), we have the following,

$$\begin{aligned} {V_m}_{max}^2 &= {V_{1m}}_{max}^2 + {V_{2m}}_{max}^2 + 2V_{1m\,max}V_{2m\,max}\cos\theta_{min} \\ &= b^2 + d^2 + 2bd\cos\theta_{min} \\ \\ {V_m}_{min}^2 &= {V_{1m}}_{min}^2 + {V_{2m}}_{min}^2 + 2V_{1m\,min}V_{2m\,min}\cos\theta_{max} \\ &= a^2 + c^2 + 2ac\cos\theta_{max} \end{aligned} \tag{17}$$

The final computation for the extremes of θ and V_m (see Figure 4), can be expressed as follows

$$\begin{aligned} \theta_{max} &= \max(\alpha_4 - \alpha_1, \alpha_2 - \alpha_3) \\ \theta_{min} &= \begin{cases} 0 & V_{1\alpha} \cap V_{2\alpha} \neq \phi \quad (\alpha_2 \geq \alpha_3) \\ \alpha_3 - \alpha_2 & otherwise \end{cases} \\ V_{m\,max} &= \sqrt{{V_m}_{max}^2} \\ V_{m\,min} &= \sqrt{{V_m}_{min}^2} \end{aligned} \tag{18}$$

For the computation of V_α, $\tan^{-1}$ is a monotonic function. To find the extremes of the function, all we need to do is find the extrema of its argument. A change of variables simplifies the expressions,

$$f(x, y, u, v) = \frac{x\sin u + y\sin v}{x\cos u + y\cos v} \tag{19}$$

The parallelogram rule for vector addition [Swokowski 1975] suggests that the smallest possible angle occurs adding points 2 and 5, denoted by point 25 in Fig-

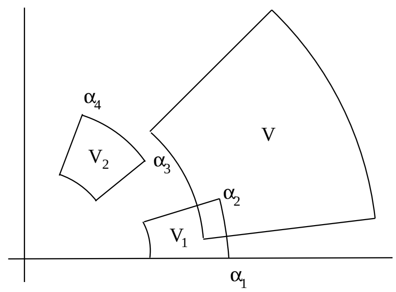

Fig. 4. Vector Addition, Case 1

ure 5, and the largest angle occurs adding points 3 and 8, denoted by point 38 in the same figure.

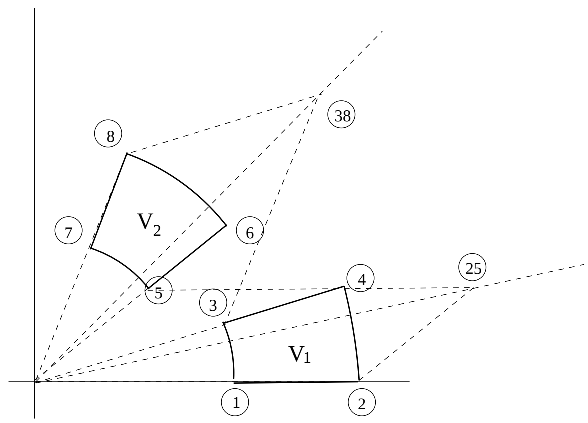

Fig. 5. Maximum and Minimum Angles in Vector Addition

To prove our intuition, we first show that the minimum angle has to occur when adding some points in the boundaries of minimum angles. For a given x_0 and y_0 (i.e. fixing the magnitudes of both fans),

$$g_1(u, v) = \frac{x_0 \sin u + y_0 \sin v}{x_0 \cos u + y_0 \cos v} \tag{20}$$

The extremes of g_1 are found where its gradient is zero,

$$\begin{aligned} \nabla g_1 &= 0 \qquad (21) \\ \frac{\partial g_1}{\partial u} &= \frac{x_0 \cos u (x_0 cosu + y_0 \cos v) - (-x_0 \sin u)(x_0 \sin u + y_0 \sin v)}{(x_0 \cos u + y_0 \cos v)^2} = 0 \\ \frac{\partial g_1}{\partial v} &= \frac{y_0 \cos v (x_0 cosu + y_0 \cos v) - (-y_0 \sin v)(x_0 \sin u + y_0 \sin v)}{(x_0 \cos u + y_0 \cos v)^2} = 0 \end{aligned}$$

$$\begin{aligned} x_0 + y_0 \cos(u - v) &= 0 \qquad (22) \\ x_0 &= -y_0 \cos(u - v) \end{aligned}$$

Since x_0, y_0, and $\cos(u-v)$ are positive, there is no solution to equation 22. Therefore, g_1 has no extremes in the given domain. So, we analyze how the function behaves on each variable. For a given u_0 (i.e. fixing the angle of the second fan),

$$\begin{aligned} h_1(v) &= g_1(u_0, v) \\ h_1'(v) &= \frac{x_0^2 + x_0 y_0 \cos(u_0 - v)}{(x_0 \cos u_0 + y_0 \cos v)^2} \end{aligned} \tag{23}$$

and for a given v_0,

$$\begin{aligned} h_2(u) &= g_1(u, v_0) \\ h_2'(u) &= \frac{y_0^2 + x_0 y_0 \cos(u - v_0)}{(x_0 \cos u + y_0 \cos v_0)^2} \end{aligned} \tag{24}$$

Since

$$\begin{gathered} -90 \le u_0 - v \le 90 \\ 0 \le \cos(u_0 - v) \le 1 \end{gathered} \tag{25}$$

therefore

$$\begin{gathered} h_1'(v) > 0 \\ h_2'(u) > 0 \end{gathered} \tag{26}$$

This indicates that f (the angle of the result) grows with u and v (the angles of the addends), so it reaches its maximum (minimum) value in the domain, when u and v are maxima (minima).

We proceed to determine f's behavior with respect to x and y (the magnitudes of the addends).

$$\begin{aligned} g_2(x,y) &= f(x,y,u_0,v_0) \\ &= \frac{x \sin u_0 + y \sin v_0}{x \cos u_0 + y \cos v_0} \end{aligned} \tag{27}$$

$$\begin{aligned} \nabla g_2(x,y) &= 0 \\ \frac{\partial g_2}{\partial x} &= \frac{\sin u_0 (x \cos u_0 + y \cos v_0) - (\cos u_0)(x \sin u_0 + y \sin v_0)}{(x \cos u_0 + y \cos v_0)^2} \\ \frac{\partial g_2}{\partial y} &= \frac{\sin v_0 (x \cos u_0 + y \cos v_0) - (\cos v_0)(x \sin u_0 + y \sin v_0)}{(x \cos u_0 + y \cos v_0)^2} \end{aligned}$$

$$\begin{aligned} y \sin(u_0 - v_0) &= 0 \\ x \sin(v_0 - u_0) &= 0 \end{aligned} \tag{28}$$

Since x and y are both positive, solving equations 28 reduces to solving $\sin(u_0 - v_0) = 0$ and $\sin(v_0 - u_0) = 0$. The solution to the first is found for $u_0 - v_0 = k\pi$, $k = 0, 1, 2 \ldots$, which is located outside the domain of the angle difference for the case we are analyzing. We analyze how the function behaves with respect to x and y (magnitudes). For a given y_0 (i.e. fixing the magnitude of the second fan),

$$h_3(x) = g(x, y_0) = \frac{x \sin u_0 + y_0 \sin v_0}{x \cos u_0 + y_0 \cos v_0}$$

$$h'_3(x) = \frac{y_0 \sin(u_0 - v_0)}{(x \cos u_0 + y_0 \cos v_0)^2} \quad (29)$$

and for a given x_0

$$h_4(y) = g(x_0, y) = \frac{x_0 \sin u_0 + y \sin v_0}{x_0 \cos u_0 + y \cos v_0}$$
$$h'_4(y) = \frac{x_0 \sin(v_0 - u_0)}{(x_0 \cos u_0 + y \cos v_0)^2} \quad (30)$$

From equations 26, we see that to compute the maximum angle, we need to use α_2 and α_4. Assuming $u_0 = \alpha_2$ and $v_0 = \alpha_4$, we can distinguish three cases. When $\alpha_2 > \alpha_4$, $\sin(u_0 - v_0) > 0$, which makes $h'_3(x) > 0$ and $h'_4(y) < 0$, we compute the maximum angle using $x = b$ (the maximum) and $y = c$ (the minimum). When $\alpha_2 < \alpha_4$, which makes $h'_3(x) < 0$ and $h'_4(y) > 0$. We compute the maximum angle using $x = a$ and $y = d$. When $\alpha_2 = \alpha_4$, the maximum angle is equal to $\alpha_2 = \alpha_4$, i.e. it does not depend on the values of of x and y.

To compute the minimum angle, we assume $u_0 = \alpha_1$ and $v_0 = \alpha_3$. Here, we have only one case, $\alpha_1 \leq \alpha_3$, and the minimum angle is computed using $x = b$ and $y = c$.

Figure 6 shows a summary of the partial computation of fan addition for case 1.

```
AddCase1(V_1, V_2)
    if α_2 ≥ α_3
        θ_min = 0
    else
        θ_min = α_2 − α_3
    θ_max = max(α_4 − α_1, α_2 − α_3)

    V_m min = sqrt(a^2 + c^2 + 2ac cos θ_max)
    V_m max = sqrt(b^2 + d^2 + 2bd cos θ_min)

    V_α min = tan^-1 ((b sin α_1 + c sin α_3) / (b cos α_1 + c cos α_3))

    if α_2 < α_4
        V_α max = tan^-1 ((a sin α_2 + d sin α_4) / (a cos α_2 + d cos α_4))
    else if α_2 > α_4
        V_α max = tan^-1 ((b sin α_2 + c sin α_4) / (b cos α_2 + c cos α_4))
    else if α_2 = α_4
        V_α max = α_2

return([V_m min, V_m max], ∠[V_α min, V_α max])
```

Fig. 6. Pseudo-Code for Case 1

5.2.2 *Case 2: Adjacent Quadrants.* We can assume that fan V_1 will be in the first quadrant, and fan V_2 will be in the second. If that is not the case, we can make a rotation, and the results will be corrected at the end. So, we have the following conditions for case 2:

$$\begin{aligned} 0 \leq \alpha_1 \leq \alpha_2 \leq 90 \\ 90 \leq \alpha_3 \leq \alpha_4 \leq 180 \\ and\ (\alpha_4 - \alpha_1) > 90 \end{aligned} \tag{31}$$

If the last condition $(\alpha_4 - \alpha_1) > 90$ is not satisfied, the problem can be reduced to Case 1 by a simple rotation.

From equations 16, if we fix the magnitudes, V_m depends solely on θ.

$$V_m = f_2(x, y, \theta) = x^2 + y^2 + 2xy \cos\theta \tag{32}$$

Since $0 \leq \theta \leq 180$, and cosine is monotonically decreasing with respect to θ, V_m reaches a maximum when θ is at its minimum value. When $\theta \leq 90,\ 0 \leq \cos\theta \leq 1$, and f_2 reaches a maximum when x and y reach their maxima. When $\theta > 90,\ -1 \leq \cos\theta < 0$, we have a more complex situation. Analyzing how f_2 changes with θ,

$$\begin{aligned} \frac{\partial f_2}{\partial \theta} &= 2xy(-\sin\theta) \\ 90 &< \theta \leq 180 \\ \frac{\partial f_2}{\partial \theta} &\leq 0 \end{aligned} \tag{33}$$

we have a minimum where the derivative is zero (i.e. $\theta = 180$). If that point does not belong to θ's domain, we know that the function is monotonically decreasing with respect to θ. That is, $V_{m\,max}$ occurs for θ_{min}, and $V_{m\,min}$ occurs for θ_{max}.

Now, let us see how V_m behaves with respect to x and y, for a given θ.

$$\begin{aligned} V_m &= f_3(x, y) = f(x, y, \theta_0) \\ \nabla f_3 &= 0 \qquad (34) \\ \frac{\partial f_3}{\partial x} &= 2x + 2y\cos\theta_0 = 0 \\ \frac{\partial f_3}{\partial y} &= 2y + 2x\cos\theta_0 = 0 \end{aligned}$$

$$\begin{aligned} x &= -y\cos\theta_0 \\ y &= -x\cos\theta_0 \end{aligned} \tag{35}$$

To compute $V_{m\,max}$ we consider $\theta_0 = \theta|_{min}$. We distinguish three regions.

$$\theta_0 \begin{cases} < 90 \\ = 90 \\ > 90 \end{cases} \tag{36}$$

For the case when $\theta_0 \leq 90,\ \cos\theta_0 \geq 0$, so we see that

$$\begin{aligned} \frac{\partial f_3}{\partial x} > 0 \\ \frac{\partial f_3}{\partial y} > 0 \end{aligned} \tag{37}$$

therefore, V_m reaches a maximum at $x_{max},\ y_{max}$. That is,

$$V_{m\,max} = \sqrt{b^2 + d^2 + 2bd\cos\theta_{min}} \tag{38}$$

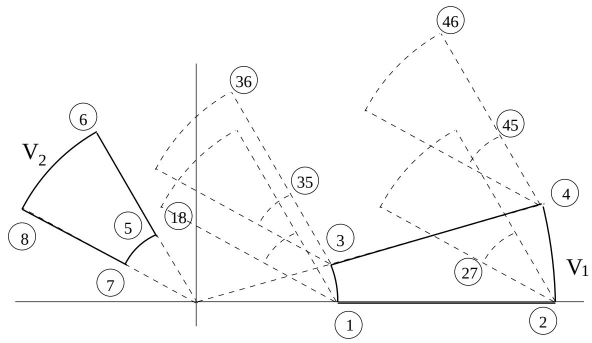

Fig. 7. Fan Addition in the Second Case

For the case when $90 < \theta_0 < 180$, $-1 \leq \cos\theta_0 < 0$,

$$\begin{aligned} \frac{\partial f_3}{\partial x} &= 2x + 2y\cos\theta_0 = 0 \\ x &= -y\cos\theta_0 \end{aligned} \tag{39}$$

We note that $\frac{\partial^2 f_3}{\partial x^2} = 2$, which implies the extrema above is a minimum. This in turn implies that the maximum of V_m has to be in one of the corners of $V_{1m} \times V_{2m}$, when the angle is minimum. That is (see Figure 7),

$$V_{m\,max} = \max(p35_m, p36_m, p45_m, p46_m) \tag{40}$$

To compute $V_{m\,min}$ we consider $\theta_0 = \theta_{max} = \alpha_4 - \alpha_1$. We find only one case, $90 < \theta_0 \leq 180$ (see the case definition, equation 31), and the minimum is not necessarily in one of the corners. From equation 39, if x and y are reals, we find a minimum at $x = -y\cos\theta_0$. In general, x and y are real intervals, so if $I_x = x \cap -y\cos\theta_0 \neq \emptyset$, we have a minimum at $x_m = I_{x\,min}$. If they do not intersect, depending on whether x falls to the left or right of $-y\cos\theta_{max}$, we can use x_{max} or x_{min}, respectively. Figure 8 shows the (negation of the) projection of y on the direction of x; when x falls to the left of $-y\cos\theta$, a minimum is found at x_{max}; when x intersects $-y\cos\theta$, a minimum is found at $I_{x\,min}$; when x falls to the right of $-y\cos\theta$, a minimum is found at x_{min}. The situation is symmetric for y, depending on whether y intersects, or falls to the left or right of $-x\cos\theta_{max}$, we can use $y_m = I_{y\,min}$, y_{max}, or y_{min}, respectively. Figure 9 shows the algorithm to compute the magnitude for fan addition, for Case 2.

We now proceed to derive the minimum and maximum values for the angle. From equation 26, we have that $h_1'(v) > 0$ and $h_2'(u) > 0$. This indicates that the maximum angle has to be computed using α_2 and α_4.

From equation 35, we have that $x_0 = -y_0\cos\theta$, where $0 \leq \theta \leq 180$. For $\theta = 90$ there is no solution, since $x_0 > 0$, $y_0 > 0$, and $\cos\theta = 0$. For $\theta < 90$, $0 < \cos\theta \leq 1$, so there is no solution in the domain either, since $x_0 > 0$, $y_0 > 0$, and $\cos\theta > 0$. Later on, we will show that point 38 yields the maximum angle.

For the case when $\theta > 90$, $-1 \leq \cos\theta < 0$, from equation 35, we can determine

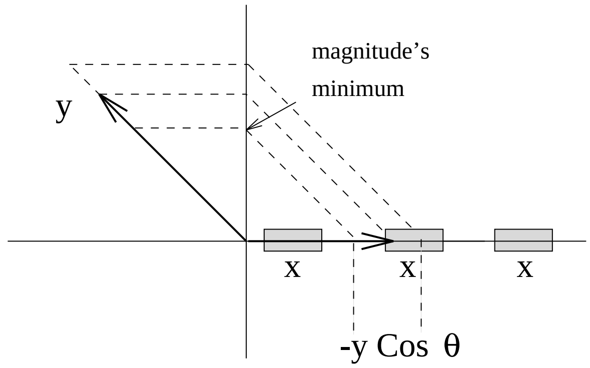

Fig. 8. Minimum Magnitude of Vector Addition; x and y Are Intervals

```
MagnitudeCase2(V_1, V_2)
    θ_min = α_3 − α_2
    θ_max = α_4 − α_1
    if θ_min ≤ 90
        V_mmax = sqrt(b^2 + d^2 + 2bd cos θ_min)
    else
        V_mmax = max(p35_m, p36_m, p45_m, p46_m)

    I_x = V_1m ∩ (−V_2m cos θ_max)
    if I_x ≠ ∅
        x_m = I_xmin
    else if a > −d cos θ_max
        x_m = a
    else if b < −c cos θ_max
        x_m = b

    I_y = V_2m ∩ (−V_1m cos θ_max)
    if I_y ≠ ∅
        y_m = I_ymin
    else if c > −b cos θ_max
        y_m = c
    else if d < −a cos θ_max
        y_m = d

    V_mmin = sqrt(x_m^2 + y_m^2 + 2 x_m y_m cos θ_max)
return([V_mmin, V_mmax])
```

Fig. 9. Pseudo-Code for Magnitude, Case 2

the angle for which the extreme happens.

$$\theta = \cos^{-1}\left(-\frac{y_0}{x_0}\right) \tag{41}$$

This is the value of θ that produces the maximum angle γ of the resulting fan. Figure 10 shows a case where the extremes for the angle cannot be computed using the corners of the fans. The same figure shows a diagram of the computation of γ and ϕ. Depending on the values of ϕ, we can determine the maximum magnitude

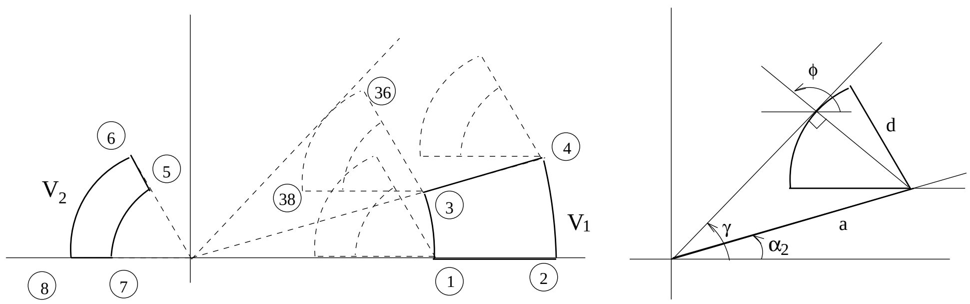

Fig. 10. Case where Correction Is Needed, and Computation of ϕ

as follows

$$\phi \begin{cases} \in \; V_2 V_{\alpha max} = \gamma \\ < \; V_2 V_{\alpha max} = p_{38\,\alpha} \\ > \; V_2 V_{\alpha max} = p_{36\,\alpha} \end{cases} \tag{42}$$

To prove that points 38 and 36 yield the maximum angle, we have to analyze how the angle behaves with respect to x and y. From equation 28, we find a solution for $k = 1$, $\theta = 180$. This is a special case, when angle does not change with x or y.

From equations 29 and 30, with $0 \leq \theta \leq 180$, we have that

$$\begin{aligned} h_3'(x) &< 0 \\ h_4'(y) &> 0 \end{aligned} \tag{43}$$

which shows that V_α reaches a minimum for x_{min} and y_{max}. Similarly, V_α reaches a maximum for x_{max} and y_{min}.

The algorithm to compute the minimum angle is the mirror image of the one for the maximum angle. Summarizing, the algorithm to compute the angle of fan addition for the second case is shown in Figure 11.

5.2.3 *Case 3: Opposite Quadrants.* This section covers the partial computation of $V = V_1 + V_2$, for the case where V_1 and V_2 are in opposite quadrants. Again, we can assume that fan V_1 will be in the first quadrant, and fan V_2 in the third one. If that is not the case, we can make a rotation, and the results will be corrected at the end. So, the following conditions define this case:

$$\begin{aligned} 0 \leq \alpha_1 \leq \alpha_2 \leq 90, \\ 180 \leq \alpha_3 \leq V_{2a} \leq \alpha_4 \leq 270, \\ 90 \leq (\alpha_3 - \alpha_2) \leq 180, \\ 90 \leq (\alpha_4 - \alpha_1) \leq 180 \end{aligned} \tag{44}$$

```
AngleCase2(V_1, V_2)
    V_{α min} = min p25_a, p27_a
    V_{α max} = max p18_a, p38_a
    if V_{α max} < 90
        γ = α_2 + sin^{-1}(d/a)
        φ = γ + 90
        if φ ∩ V_{2a}
            V_{α max} = γ
        else if φ < α_3
            V_{α max} = p36_α
        else if φ > α_4
            V_{α max} = p38_a)
    if V_{α min} > 90
        γ = α_3 − sin^{-1}(b/c)
        φ = γ − 90
        if φ ∩ V_{1α}
            V_{α min} = φ
        else if φ < α_1
            V_{α min} = p25_α
        else if φ > α_2
            V_{α min} = p45_α
return([V_{α min}, V_{α max}])
```

Fig. 11. Pseudo-Code for Angle, Case 2

If any of the last two conditions ($90 \leq (\alpha_3 - \alpha_2) \leq 180$ or $90 \leq (\alpha_4 - \alpha_1) \leq 180$) is not satisfied, we can make a rotation and reduce the problem to case 2. A consequence of those two last conditions is that the maximum angle $\theta_{max} = 180$. An example of case 3 is shown in Figure 12.

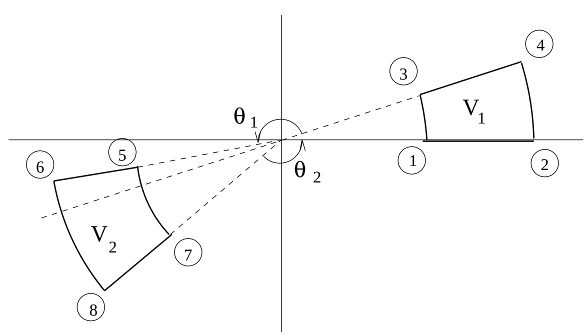

Fig. 12. Fan Addition, Case 3

To compute the extremes of the magnitude, considering $90 \leq \theta_1 < 180$, we can compute $V_{m\,max}$ following the procedure for case 2 ($\alpha_3 - \alpha_2 < 180$), using the minimum of angles $\theta_1 = \alpha_3 - \alpha_2$ and $\theta_2 = \alpha_4 - \alpha_1$. $V_{m\,min}$ can also be computed using the procedure for case 2, with $\theta_{\max} = 180$. Figure 13 shows the algorithm to compute the magnitude for case 3.

```
MagnitudeCase3(V_1, V_2)
    θ_1 = α_3 − α_2
    θ_2 = 360 − (α_4 − α_1)
    if θ_1 < θ_2
        V_{m max} = max(p35_m, p36_m, p45_m, p46_m)
    else
        V_{m max} = max(p17_m, p18_m, p27_m, p28_m)
    I = V_{1m} ∩ V_{2m}
    if I ≠ ∅
        x_m = I_{min}
        y_m = I_{min}
    else if a > d
        x_m = a
        y_m = d
    else if b < c
        x_m = b
        y_m = c
    V_{m min} = sqrt(x_m^2 + y_m^2 − 2 x_m y_m) = x_m − y_m
return([V_{m min}, V_{m max}])
```

Fig. 13. Pseudo-Code for Magnitude, Case 3

For the determination of the extremes of the resulting angles, from equations 19 to 22, and using interval computation, the solution to equation 22 turns to

$$0 \in x - y \cos \theta_{max} \tag{45}$$

Since $\theta_{max} = 180$, we distinguish three sub-cases here.

$$x + y \begin{cases} > 0 \\ \ni 0 \\ < 0 \end{cases} \tag{46}$$

In the first sub-case, the magnitude of one of the fans is large enough to *pull* the resultant to *its side* (the third sub-case is symmetrical to this one). In the second sub-case none of the magnitudes dominate over the other one; the resulting fan has points scattered at all angles. So, the resulting angle can be anything, i.e. $[0, 360]$. Figure 14 illustrates these cases.

Angle correction for sub-cases 1 and 3 (and the angle computation) is similar to that performed in case 2, except that, in these cases, we might need to perform angle correction in both extremes. See Figure 15.

In summary, Figure 16 shows the pseudo-code for the computation of the angle for case 3.

The preceding algorithm was derived considering all extremes of the fans as closed. If the fans have open extremes, we still have to do the computation as if they were closed. At the end, we have to take into consideration what points are producing the extreme of the intervals and verify whether those points have to be included in the result or not.

For example, in computing the maximum angle for the second case (see Figure 11), we use point $p36$ (in one of the cases), which is the result of adding points 2 and 6. We can say that a point is included in the fan if both boundaries are

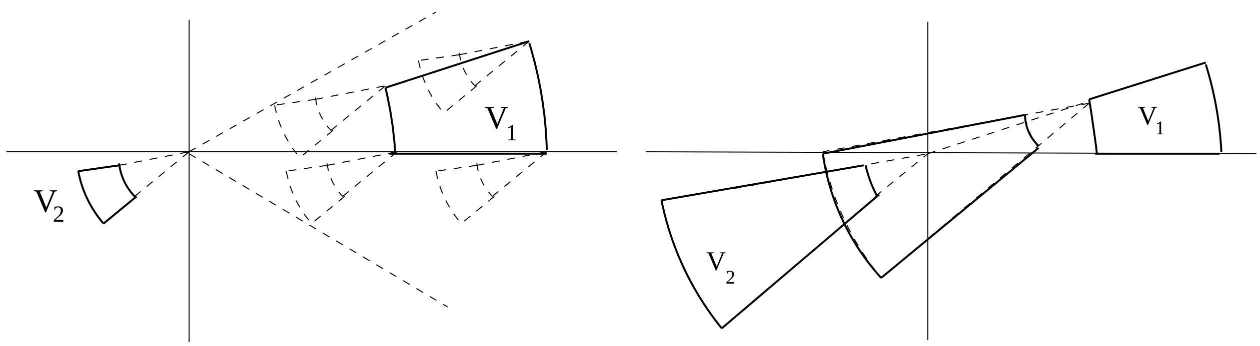

Fig. 14. Angle Computation for Case 3

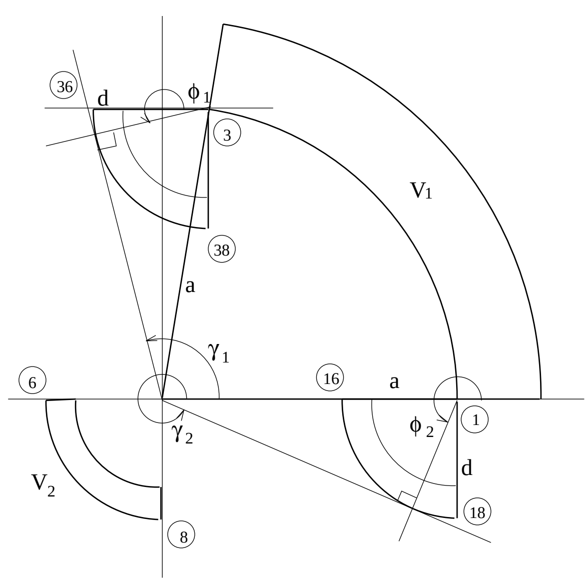

Fig. 15. Angle Correction for Case 3

```
AngleCase3(V_1, V_2)
    θ_max = 180
    MagDiff = V_1m − V_2m cos θ_max
    if 0 ∈ MagDiff
        return([0, 360])
    else if MagDiff > 0
        V_α max = p36_α
        if V_α max > 90
            γ_1 = sin^{-1} (d/a) + α_2
            φ_1 = γ_1 + 90
            if φ_1 ∈ V_2a
                V_α max = γ_1
            else if φ_1 < α_3
                V_α max = p36_α
            else if φ_1 > α_4
                V_α max = p38_α
        V_α min = p18_α
        if V_α min > 270
            γ_2 = 360 − sin^{-1} (d/a)
            φ_2 = γ_2 − 90
            if φ_2 ∈ V_2a
                V_α min = γ_2
            else if φ_2 < α_3
                V_α min = p16_α
            else if φ_2 > α_4
                V_α min = p18_α
    else
        ... case MagDiff< 0 is symmetrical
return([V_α min, V_α max])
```

Fig. 16. Pseudo-Code for Angle, Case 3

closed; in the case of point 3, if the minimum magnitude and maximum angle of fan V_1 are closed, point $p3$ is included. If both points, $p3$ and $p6$ are included in fans V_1 and V_2, respectively, the right extreme of the angle of the result will be closed.

5.2.4 *Completeness and Minimality.* Completeness and minimality are the two desired properties of our algorithm. Completeness refers to the fact that the solution fan V encloses all possible points in the operation $V_1 + V_2$. By minimality we understand that the result V cannot be reduced and still comprise all possible results.

Equation 15 guarantees completeness if each partial result is complete. For each case we have shown completeness, since the resulting fan includes the minimum and maximum of magnitude and angle of all possible results of the complex fan addition. Therefore, all possible results are included in the answer.

Given that each case has been proven to guarantee completeness, all we need to guarantee completeness of the algorithm is an adequate definition for complex fan union. Let $V_1 = [a, b]\angle[\alpha_1, \alpha_2]$ and $V_2 = [c, d]\angle[\alpha_3, \alpha_4]$ be complex fans. We define complex fan union as follows[1]:

$$V = V_1 \cup V_2 = [a, b] \cup [c, d] \angle [\alpha_1, \alpha_2] \cup [\alpha_3, \alpha_4]. \quad (47)$$

If we see a vector as a pair (magnitude, angle), a complex fan is the set of all vectors in the Cartesian product of the magnitude and phase angle of that complex fan. Given that definition, it is clear that

$$([a, b] \cup [c, d]) \times ([\alpha_1, \alpha_2] \cup [\alpha_3, \alpha_4]) \supseteq ([a, b] \times [\alpha_1, \alpha_2]) \cup ([c, d] \times [\alpha_3, \alpha_4]). \quad (48)$$

so, complex fan union guarantees completeness and so does our algorithm.

Each partial case of complex fan addition guarantees that the partial result has at least one of the possible results on each of its boundaries. The union of two complex fans is defined as the independent union of their magnitudes and phase angles, and computed by interval union, which is set union. This makes sure that the final result still contains at least one point on each of its boundaries. Therefore, those partial results cannot be reduced without missing some of the possible results.

There is an important issue regarding the evaluation of expressions involving more than one addition. The above results guarantee minimality with respect to a addition in the sense that there is no smaller fan that encloses all the results of the operation. Even if the result is expressed with the minimum fan possible, it contains, most of the time, spurious results. That is, there are regions that are enclosed by the resulting fan, and do not form part of the actual result. Figure 17 shows two complex fans, its real addition (a very irregular shape), and the computed complex fan. All points not in the irregular shape are spurious results, produced by complex fan addition.

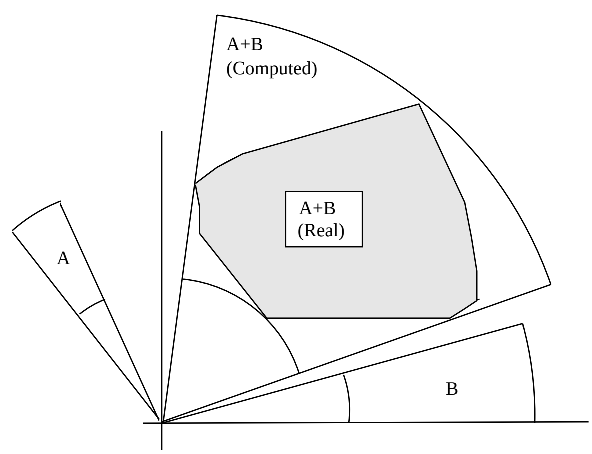

Fig. 17. Spurious Results in Complex Fan Addition

In most cases, the extremes of the resulting case are computed using the corners of the addend fans, and that is precisely where the spurious result lie in the resulting

[1]Interval union is defined differently than set union; if the intersection is empty, the convex hull is taken as the union. This operation may produce some spurious results, due to the inclusion of the holes in non-intersecting intervals.

fan of an addition. When the result of an addition is added to another fan, spurious results (lying in the corners) will be used to compute the new result. This problem is caused by the uncertainty contained in the representation, which propagates through arithmetic operations. How much uncertainty is being generated depends on the order of evaluation. In other words, complex fan addition is commutative, but not associative. Figure 18 shows an example of how the order of evaluation alters the result; we first define three complex fans equidistant in angles; we evaluate $a+b+c$ in all possible ordering combinations. Figure 18 proves that the evaluation order may affect the results.

$$
\begin{array}{l}
a = [10, 20]\angle[10, 20] \\
b = [10, 20]\angle[130, 140] \\
c = [1, 2]\angle[250, 260] \\
\\
(a{+}b){+}c \rightarrow [6.4524, 22.1928]\angle[28.8175, 121.1825] \\
(a{+}c){+}b \rightarrow [6.3229, 22.1141]\angle[26.3283, 117.1285] \\
(b{+}a){+}c \rightarrow [6.4524, 22.1928]\angle[28.8175, 121.1825] \\
(b{+}c){+}a \rightarrow [6.3229, 22.1141]\angle[32.8715, 123.6717] \\
(c{+}a){+}b \rightarrow [6.3229, 22.1141]\angle[26.3283, 117.1285] \\
(c{+}b){+}a \rightarrow [6.3229, 22.1141]\angle[32.8715, 123.6717]
\end{array}
$$

Fig. 18. Complex Fan Addition Is Not Associative

From our minimality results, we have that Figure 18 contains all the possible results of $a + b + c$. Therefore, the intersection of all of them must as well enclose all the results, and it is at least as small as any of the results for the different evaluation orders. Taking the intersection in this example, reveals that none of the evaluation orders provides a minimal result. This indicates that there is not a straightforward method to determine an evaluation ordering which provides optimal results for addition expressions of complex fans. More work needs to be done in this direction.

6. APPLICATIONS

Qualitative Reasoning(QR), a sub-field of Artificial Intelligence, tries to derive descriptions of behavior of complex systems based on a qualitative description of the system's structure. The main idea is to reason about a complex system in the same terms as an engineer or scientist would. Systems that use QR techniques are, in general, capable of dealing with incomplete or uncertain information, both in the formulation of the model, and in the values of the quantities involved.

In [Flores 1997] Flores presents a framework, called QPA (Qualitative Phasor Analysis), for reasoning about linear electrical circuits in sinusoidal steady state. The reasoning process relies on a constraint-based model of the circuit, derived from electro-magnetic theory and generated automatically from the structure of the circuit. In a linear circuit operating in steady state, all quantities are sinusoidals of the same frequency as the source. Since any sinusoidal can be expressed as the real part of a complex exponential, we use the complex form, which simplifies computations; this complex form, characterized by magnitude and angle, is called a phasor. In order to capture magnitude and phase angle information in the model, all constraints operate on phasor variables.

Constraint propagation (CP) is the main inference mechanism. QPA's CP module reasons with as much information and precision as the user provides, ranging from qualitative to quantitative values. Complex Fans provide a general mechanism to represent phasors at different levels of precision. When the user does not know much about a quantity, it can be represented by intervals. The presence of uncertainty in phasor quantities gives place to complex fans. On the other hand, if a quantity is known precisely, it can be expressed by a real (a floating point in a computer representation), or a point phasor in the complex case. Any combination of reals and intervals are allowed in a complex fan.

QPA is capable of performing the following reasoning tasks: analysis, parameter design, diagnosis, control design, and structure simplification. All of those reasoning tasks are based on the results of circuit analysis, which in turn relies on the computation of quantities represented as complex fans. The computation process in circuit analysis can be seen as solving the set of constraints that govern the circuit behavior. We start with a basic set of constraints which represents the set of possible behaviors the given circuit may exhibit.

Figure 19, shows an example circuit, and Table 1 shows how the circuit's structure

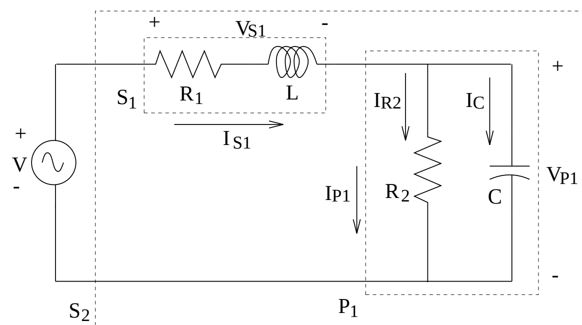

Fig. 19. A Simple Circuit

is represented as a set of constraints. Those constraints are derived from basic circuit theory; for instance, in series cluster S_1, we know the current is the same for both elements, and their voltages add up to form the voltage of the cluster. We also know that the impedance of an inductor, Z_L, can be computed as the product of its inductance and the frequency of the system, and that its current is behind its voltage by exactly 90 degrees.

Every time the user posts a new constraint (e.g. a parameter value), the set of expected behaviors is possibly reduced by constraint propagation. The user-provided constraints (normally) represent the operating conditions of the circuit. QPA provides functions to assert and verify constraints related to a given circuit. Asserting constraints consists of passing the same constraint to the model of the circuit and running propagation. To verify if a given constraint holds in a model, we check if it is explicitly recorded; if it is, the constraint holds. If the constraint does not exist in the model, it can be verified via constraint propagation. The

Region	Algebraic	Phase Angle
Series S_2	$I_{S_2} = I_{S_1}$ $I_{S_2} = I_{P_1}$ $V_{S_2} = V_{S_1} + V_{P_1}$ $V_{S_2} = Z_{S_2} I_{S_2}$ $Z_{S_2} = Z_{S_1} + Z_{P_1}$	
Series S_1	$I_{S_1} = I_{R_1}$ $I_{S_1} = I_L$ $V_{S_1} = V_{R_1} + V_L$ $V_{S_1} = Z_{S_1} I_{S_1}$ $Z_{S_1} = Z_{R_1} + Z_L$	
R_1	$Z_{R_1} = R_1$ $V_{R_1} = Z_{R_1} I_{R_1}$	$\angle Z_{R_1} = 0$ In-Phase (I_{R_1}, V_{R_1})
L	$Z_L = \omega L$ $V_L = Z_L I_L$	$\angle Z_L = 90$ Behind (I_L, V_L)
Parallel P_1	$V_{P_1} = V_{R_2}$ $V_{P_1} = V_C$ $I_{P_1} = I_{R_2} + I_C$ $V_{P_1} = Z_{P_1} I_{P_1}$ $Z_{P_1} = \frac{Z_{R_2} Z_C}{Z_{R_2} + Z_C}$	
R_2	$Z_{R_2} = R_2$ $V_{R_2} = Z_{R_2} I_{R_2}$	$\angle Z_{R_2} = 0$ In-Phase (I_{R_2}, V_{R_2})
C	$Z_C = \frac{1}{\omega C}$ $V_C = Z_C I_C$	$\angle Z_C = -90$ Ahead (I_C, V_C)

Table 1. Set of Constraints for Example Circuit

constraint is asserted to the model, and propagation is run. If propagation detects an inconsistency, the constraint clearly does not hold. If propagation does not fail, we can say the constraint holds.

Figure 20 shows part of the set of constraints after asserting values for some of the circuit parameters and variables, and running constraint propagation.

QPA shows how complex fans can be used to perform circuit analysis where the values of the variables can range from qualitative to quantitative, in an intermixed form. That is, some variables may be precisely specified (real numbers), while others may be partially specified by the use of intervals, and yet others be left totally unspecified (we know all values must be positive). This characteristic allows the user to provide the system with as much information as available at a given time. The system produces results as specific as its knowledge about the circuit being analyzed. Of course, if all parameters are precisely specified, the result is precise and coincides, in the numeric results, with any conventional circuit analyzer [Conant 1993].

This section presents an application in the area of electrical engineering. There are other fields of physics that use vectors as well. Vectors are normally expressed in complex form, but sometimes their expression in polar form is much more natural to the user than their rectangular counterpart. Complex Fans is a mathematical tool that enables those fields to deal with uncertain and incomplete information.

```
Component1:
  Single VOLTAGE-SOURCE: V
  . . .
Component2:
  SERIES cluster: S2
  nodes: (1, 0)
  ZS2 = ([20.3477, 22.3792] ∠ [19.7045, 29.6023])
  VS2 = 100.0000
  IS2 = ([4.4684, 4.9146] ∠ [330.3977, 340.2955])
  OMC = ((= VS2 VV) (= IS2 IV) (= IS2 IP1) (= IS2 IS1))
  Component1:
     SERIES cluster: S1
     . . .
  Component2:
     PARALLEL cluster: P1
     . . .
     Component1:
        Single RESISTOR: R2
        nodes: (3, 0)
        R2 = [20.0000, 21.0000]
        ZR2 = [20.0000, 21.0000]
        VR2 = ([26.3706, 40.7957] ∠ [258.0103, 272.9154])
        IR2 = ([1.2557, 2.0398] ∠ [258.0103, 272.9154])
        OMC = ((= VR2 VP1))
     Component2:
        Single CAPACITOR: C
        nodes: (3, 0)
        C = [0.0020, 0.0025]
        ZC = ([6.6667, 8.3333] ∠ 270.0000)
        VC = ([26.3706, 40.7957] ∠ [258.0103, 272.9154])
        IC = ([3.7853, 5.0618] ∠ [348.0103, 2.9154])
        OMC = ((= VP1 VC))
```

Fig. 20. Printout of a Circuit Model

7. IMPLEMENTATION

The algorithms presented in this paper were implemented in Common Lisp, using CLOS (Common Lisp Object System), and run in a SPARC workstation under the Unix operating system. Our implementation and the nature of Lisp makes unnecessary a test driver module. The Abstract Data Type (ADT) we developed for Complex Fans provides:

—Implementation of constructor functions for Complex Fan objects

—Implementation for all arithmetic functions

—Debugging functions with the appropriate flags to enable/disable scaffolding

—Overloading, which allows function calls with mixed arguments

Figure 21 shows how to create a complex fan. The function polar creates a complex number, expressed in polar form, where the attributes can be reals or intervals. This feature enables us to handle mixed objects. The basic arithmetic operations are provided. The function names are the arithmetic operators preceded by *general*.

```
(setq cf1 (polar (interval [ 10 25 ] ) (interval [ 0 135 ] angle)))
(setq cf2 (polar (interval [ 1 3 ] ) (interval [ 150 285 ] angle)))

(general+ cf1 cf2)

([7, 27.908] ∠ [342.542, 152.457])
```

Fig. 21. Complex Fan Addition

The example of figure 21 was chosen to show all possible cases present in addition. The following figures show the results of the scaffolding, which reflects the ideas expressed in the algorithms presented in section 5. Figure 22 shows the decomposition of the addends in their components (intersecting the different quadrants), and the formation of all possible pairs (the Cartesian product of the components of the addends). The addition of those pairs are all operations we need to ensure completeness of the process. Figure 23 shows the scaffolding produced by the ad-

```
V1= ([10, 25] ∠ [ 0, 135])
V2= ([ 1, 3] ∠ [150, 285])
      A segments: (([10, 25] ∠ [ 0, 90])
                   ([10, 25] ∠ [ 90, 135]))
      B segments: (([ 1, 3] ∠ [150, 180])
                   ([ 1, 3] ∠ [180, 270])
                   ([ 1, 3] ∠ [270, 285]))
      All pairs: (((([10, 25] ∠ [ 0, 90]) + ([ 1, 3] ∠ [150, 180]))
                   (([10, 25] ∠ [ 0, 90]) + ([ 1, 3] ∠ [180, 270]))
                   . . . )
```

Fig. 22. Decomposition of Addends for Addition of Figure 21

dition function for the first pair of addends. If the scaffolding flag is set to true, the different parameters computed by each case are displayed. Figure 24 shows the

```
Polar+case2 ...
qV1 = 1,                              qV2 = 2
V1 = ([10, 25] ∠ [ 0, 90]),           V2 = ([ 1, 3] ∠ [150, 180])
Rotation = 0
A = ([10, 25] ∠ [ 0, 90]),            B = ([ 1, 3] ∠ [150, 180])
a = 10,                               b = 25
theta1 = 0,                           theta2 = 90
c = 1,                                b = 25
theta1 = 0,                           theta2 = 90
c = 1,                                d = 3
theta3 = 150,                         theta4 = 180
Alphamin = 60,                        Alphamax = 180
p1 = (10 ∠ 0),                        p2 = (25 ∠ 0)
p3 = (10 ∠ 90),                       p4 = (25 ∠ 90)
p5 = (1 ∠ 150),                       p6 = (3 ∠ 150)
p7 = (1 ∠ 180),                       p8 = (3 ∠ 180)
p18 = (7 ∠ 0),                        p25 = (24.139 ∠ 1.186)
p27 = (24 ∠ 0),                       p35 = (10.535 ∠ 94.715)
p36 = (11.789 ∠ 102.730),             p38 = (10.440 ∠ 106.699)
p45 = (25.514 ∠ 91.945),              p46 = (26.627 ∠ 95.599)
Bproj = [1, 3],                       Ia = EMPTY
Aproj = [10, 25],                     Ib = EMPTY
am = 10,                              bm = 3
Vmmin = 7,                            Vmmax = 26.627
Vamin = 0,                            Vamax = 106.699
Vamin = 0,                            Vamax = 106.6990
Returning ([7, 26.627] ∠ [0, 106.699])
```

Fig. 23. Scaffolding for Addition of Figure 21

results of all partial operations. The final result is the union of all partial results.

```
Add list: (([7, 26.627] ∠ [0, 106.699])
           ([7, 25.179] ∠ [342.542, 90])
           ([7, 25.938] ∠ [343.3, 90])
           ([10.049 27.908] ∠ [91.945, 144.926])
           ([7, 27.204] ∠ [90, 152.457])
           ([7, 24.303] ∠ [83.760, 150.069]))

(general+ cf1 cf2)→ ([7, 27.908] ∠ [342.542, 152.457])
```

Fig. 24. Partial Additions and Final Result

8. CONCLUSIONS

In this article, we define Complex Fans, a data type that results when we allow the magnitude and angle of a complex number (expressed in polar form) to range over an interval. We also define the basic arithmetic operations, which we proved to be minimal in the sense that the resulting complex fan is the smallest complex fan that contains all possible results of the operation.

Complex Fans were introduced by [Klatte and Ullrich 1980]; they called them "Complex Sectors". They proposed sector arithmetic as an alternative to rectangular and circular arithmetics in the complex interval space. The Complex Fans algorithms, proposed in this paper, constitute what they refer to as the "direct method", which was avoided in their paper. They define six different alternatives; all of them are based on mapping Complex Fans to rectangles or circles, performing the operations on the chosen domain, and then returning to the original representation.

Although it is not formally proven, it is shown in figure 3, that our algorithms perform better (with respect to minimality of spurious results) than the best of the six arithmetics they propose (i.e. k_1 – mapping to rectangles).

From all operations with complex numbers in polar form, addition is the most complicated, since there are no direct formulas to express the results. In this case, conversion to rectangular and back to polar would not provide the minimality that we claim here. Thus, special algorithms had to be developed.

All algorithms were proven mathematically, and verified empirically. The algorithms were presented in pseudo-code, and the programs were written in Common Lisp, making use of CLOS (Common Lisp Object System). Since all programs are written in Common Lisp, the implementation is highly portable; the programs can be run in any platform that supports common lisp. The programs have been tested in Allegro Common Lisp for SPARC workstations and PCs (under Linux and Windows95); they have also been tested in Golden Common Lisp for PCs. Translation of the code to any other high level programming language should be straightforward, specially if it is object-oriented and supports polymorphism.

A section with applications to circuit analysis in electrical engineering was presented. Nevertheless, any other application involving vectors can find this package of use.

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